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REVIEW

The Semiclassical Einstein Equations in Cosmological Spacetimes

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Abstract

The semiclassical Einstein equation describes the backreaction of quantum matter fields on classical background spacetimes. This conference proceeding reviews some recent results obtained by N. Pinamonti, D. Siemssen and the author [Ann. Henri Poincaré **22**, 3965–4015, 2021] on the initial-value problem of the semiclassical Einstein equation coupled to a quantum, massive, scalar field with arbitrary coupling to the scalar curvature in cosmological spacetimes. The central issue of the problem arises from the fact that the linearized expectation value of the renormalized stress-energy tensor of the quantum matter field hides a nonlocal contribution depending on the highest derivative. This is encoded in an unbounded, tame operator which lose derivatives, and thus it prevents a direct analysis of the dynamical equation. The system can nevertheless be reformulated as an inverse problem, allowing one to isolate and invert the highest-derivative contribution. In this form, existence and uniqueness of solutions can be established using Banach fixed-point methods. This proceeding is based on the talk given by the author at the *IQSA2025 Intermediate conference* (Tropea, Italy).

Keywords Semiclassical einstein equations · Semiclassical gravity · Cosmology · Stochastic gravity · Inflationary universe · Initial-value problem

1 Introduction to Semiclassical Gravity

1.1 The Semiclassical Einstein–Klein–Gordon System

In Semiclassical Gravity, dynamics of four-dimensional, classical, curved, globally hyperbolic spacetimes $(\mathcal{M}, g_{\mu\nu})$ are governed by the propagation of a quantum matter field through the so-called *semiclassical Einstein equation*, in which the classical Einstein tensor of the spacetime is sourced by the renormalized quantum stress-energy tensor of the quantum matter field, evaluated in a suitable physical state ω. In units $\hbar = c = 1$ and metric signature $(-, +, +, +)$,

$$G_{\mu\nu} = \kappa \langle :T_{\mu\nu}: \rangle_\omega \qquad \mu, \nu = 0, \dots, 3, \tag{1}$$

where $\kappa \doteq 8\pi G$, see [1–4] and references therein.

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Solutions $(g_{\mu\nu}, \omega)$ of those equations incorporate the quantum effects due to the backreaction of the quantum field upon the spacetime, i.e., the modification of the background geometry induced by its propagation. The full dynamical system of equations satisfied by $(g_{\mu\nu}, \omega)$ is

$$\begin{cases} G_{\mu\nu}(x) = \kappa\langle :T_{\mu\nu}:(x)\rangle_\omega \\ P_x\omega_2(x, y) = 0, \\ P_y\omega_2(x, y) = 0, \end{cases} \tag{2}$$

where $\omega_2(x, y)$ denotes the two-point function of the quantum state, and $P_{x/y}$ are the differential operators that realize the equation of motion of the quantum matter field acting respectively on $x, y \in \mathcal{M}$. For instance, if the quantum matter field is a free, real, scalar field ϕ descending from the Klein-Gordon action

$$S_0(\phi, g_{\mu\nu}) \doteq -\frac{1}{2}\int_{\mathcal{M}} \left(\nabla_\rho\phi\nabla^\rho\phi + m^2\phi^2 + \xi R\phi^2\right)\sqrt{g}d^4x, \tag{3}$$

then it fulfills the Klein-Gordon equation

$$P\phi = \left(\Box_g - m^2 - \xi R\right)\phi = 0, \tag{4}$$

where P denotes the Klein-Gordon operator on $\mathcal{M}$, $\Box_g \doteq g^{\varrho\sigma}\nabla_\varrho\nabla_\sigma$, $m > 0$ is the mass of the field, and $\xi \in \mathbb{R}$ the coupling to the background curvature. In this case, (2) describes the *semiclassical Einstein–Klein–Gordon* coupled system.

Despite the effective nature of this formulation, which is far from playing the role of a fundamental theory, it may constitute a low-energy approximation of an overall quantum gravity; thus, a useful framework where investigating those phenomena arising from the interplay between quantum matter and gravity that cannot be captured within classical General Relativity alone. From this viewpoint, Semiclassical Gravity has found its most concrete and influential applications in Cosmology and Black Hole Physics [4, 5].

On the one hand, the semiclassical Einstein equations in cosmological spacetimes provide the backbone for modeling quantum field backreaction in highly symmetric, expanding geometries such as de Sitter and Friedmann-Lemaître-Robertson-Walker spacetimes, which should model the evolution of our Universe at large scales, according to the cosmological principle and recent cosmological observations. The motivations to take into account quantum backreaction in expanding cosmological spacetimes is founded on the idea that the interplay between quantum matter-energy content and gravity drove the evolution of our Universe in the early stages, in particular its accelerated expansion during the Inflationary phase. Hence, quantum expectation values of the stress-energy tensor generate effective energy densities and pressures that can modify expansion dynamics, contributing to scenarios such as trace-anomaly-driven inflation, vacuum polarization effects, and quantum corrections to early-Universe evolution. For example, one of the oldest and most influential Semiclassical Gravity model for Inflationary Universe was given by Starobinsky [6] (see also [7]), based on the observation that the renormalized expectation value of the stress-energy tensor of conformal quantum fields in curved spacetime acquires a non-vanishing, anomalous trace; such a quantum trace anomaly may induce higher-curvature corrections in the effective gravitational dynamics, such as R^2, which drastically modify the Einstein equations when coupled with the spacetime curvature. The resulting model, called Starobinsky Inflation, produces a quasi-de Sitter phase, with the accelerated expansion emerging from vacuum polarization of quantum fields, and, furthermore, yields a nearly scale-invariant spectrum of scalar perturbations, in good agreement with current observations [8, 9].

On the other hand, in Black Hole Physics Semiclassical gravity represents the suitable framework for describing Hawking radiation and the associated evaporation process of a dynamical black holed. In the seminal paper by Hawking [10], it was shown that quantum fields on a fixed black hole background exhibit thermal flux at future null infinity when the backreaction is not taken into account. In particular, the renormalized stress-energy tensor develops a positive, outgoing flux at infinity, in form of radiation, which is compensated by a negative, ingoing energy flux across the event horizon. When such a ingoing flux is coupled with the Einstein tensor into the semiclassical Einstein equations, this implies a decrease of the black hole mass in time, i.e., the evaporation [11–13] (see also [14]). Based on this, semiclassical methods may give further insights about black hole thermodynamics, (generalized) second laws, and the semiclassical stability of trapping surfaces formed in collapse models.

This conference proceeding reviews recent results obtained by N. Pinamonti, D. Siemssen, and the author [15] on the Cauchy problem for the semiclassical Einstein–Klein–Gordon system for non-conformally coupled fields in cosmological spacetimes. Such a review provides a coherent and unified account of those results and places them within the broader landscape of the literature, in particular with regard to the existence of global, complete semiclassical solutions and to Hamilton's formulation of the Nash–Moser theorem.

1.2 The Locally Covariant Approach to Semiclassical Gravity

A formulation of a self-consistent semiclassical theory of gravity is anything but devoid of difficulties, based on the urgency of combining a classical dynamical metric with quantum matter in a way that is mathematically consistent and conceptually coherent. Whereas the curvature of the spacetime described by the Einstein tensor $G_{\mu\nu}(x)$ is a classical sharp quantity, the renormalized quantum stress-energy tensor $:T_{\mu\nu}:(x)$ is a stochastic source subjected to quantum fluctuations, which makes manifest the limited range of validity of the semiclassical approximation (see [4] and references therein). Moreover, the expectation value of the $:T_{\mu\nu}:(x)$ is generally a nonlinear and nonlocal functional of both the metric and of the quantum state ω, so the semiclassical Einstein equations coupled with the equations of motion of the quantum state turn into a system of nonlocal, integro-differential system of equations with potential issues of existence, uniqueness, and stability of solutions. These issues are strongly related to the appearance of runaway solutions, which seem to indicate a breakdown of the theory at large energy scales, close to the Planck scale [16, 17]. Furthermore, an initial-value problem aimed to determine the evolution of the pair $(g_{\mu\nu}, \omega)$ should incorporate the requirement that the state to be positive, causal, and sufficiently regular both at the Cauchy surface Σ where initial data are posed, and hence throughout its future development. In practice, however, it is difficult to state a priori conditions on the initial data that guarantees the quantum state to retain the necessary regularity at later times, because verifying this property generally depends on detailed knowledge of the resulting semiclassical solution itself [18, 19].

Despite those difficulties, a mathematically precise formulation of Semiclassical Gravity has been provided the recent years in light of the formulation of locally covariant quantum field theories in globally hyperbolic spacetimes, in which the definition of the quantum field theory is given independently of global symmetries and preferred choices of “vacuum states" and “particles". More precisely, in the Algebraic approach to Quantum Field Theory (AQFT), a local and covariant quantum field theory is not specified by operators on a fixed Hilbert space, but by a covariant functorial assignment of *-algebras of observables to spacetime regions

$$\mathcal{A} : \mathbf{Loc} \to \mathbf{Alg} \tag{5}$$

from the category **Loc** of globally hyperbolic spacetimes (with causal, isometric embeddings as morphisms) to the category of **Alg** of unital *-algebras. Such a formulation encodes the principles of locality and general covariance of the theory, and furthermore eliminates reliance on preferred coordinate systems or vacuum states, which are usually not at disposal in curved spacetimes. The choice of globally hyperbolic spacetimes ensures that a well-posed initial value formulations for hyperbolic field equations is achieved. For a list of references, see, e.g., [3, 20–27]

In particular, on this class of spacetimes and for free matter models it is guaranteed the existence of retarded and advanced fundamental solutions $G_{A,R}$ of $P_{x/y}$, which are defined as bi-distributions acting on the space of complex, smooth, compactly-supported (or test) functions $\mathcal{D}(\mathcal{M}, \mathbb{C})$. Hence, the retarded minus advanced fundamental solution $G \doteq G_R - G_A$, i.e., the causal propagator, guarantees a causal propagation of Cauchy data in $(\mathcal{M}, g_{\mu\nu})$ both in the past and in the future of the Cauchy surface Σ. Indeed, if $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic and u_f is a given solution of $P_{x/y}u(x, y) = 0$ with assigned some compactly-supported initial data (u_0, u_1) on Σ, then there exists $f \in C^\infty_{sp}(\mathcal{M}, \mathbb{R})$ constructed out of (u_0, u_1) such that $u_f = G(f)$, where $\operatorname{supp} f \subset J^+(\operatorname{supp} f) \cup J^-(\operatorname{supp} f)$ [28].

For a free quantum, scalar, Klein-Gordon field ϕ, the *-algebra $\mathcal{A}(\mathcal{M}, g_{\mu\nu})$ is called CCR algebra [21–24] and is generated by a unit $\mathbb{I}$ and by the set of smeared quantum fields $\{\phi(f)\}$, $f \in \mathcal{D}(\mathcal{M}, \mathbb{C})$, together with all their composite operators, in which each smeared field $\phi(f)$ is viewed as a $\mathcal{A}$-valued distribution on $U \subset \mathcal{M}$ of the form

$$\phi(f) \doteq \langle \phi, f \rangle = \int_U \phi(x) f(x) \sqrt{g} d^4x, \tag{6}$$

which satisfies the following properties. For $f_1, f_2 \in \mathcal{D}(\mathcal{M}, \mathbb{C})$

1. Linearity.
2. *-involution: $\phi^*(f_1) = \phi(\bar{f}_1)$.
3. On-shell condition $\phi(P f_1) = 0$.
4. $[\phi(f_1), \phi(f_2)] = iG(f_1, f_2)\mathbb{I}$,

where P was given in (4). The last condition given in terms of the causal propagator G encodes the commutator function of the quantum theory as direct consequence of the Einstein causality condition, which implies that elements of the algebra cannot influence each other when two regions are spacelike separated. Finally, quantum states $\omega : \mathcal{A}(\mathcal{M}, g_{\mu\nu}) \to \mathbb{C}$ are linear, positive, and normalized functionals on $\mathcal{A}(\mathcal{M}, g_{\mu\nu})$. If the state is Gaussian (or quasi-free), then its n-points correlation functions are completely characterized by its two-point function

$$\omega_2(f_1, f_2) = \mu_s(f_1, f_2) + \frac{i}{2} G(f_1, f_2), \tag{7}$$

in which the antisymmetric part is fixed by the causal propagator, while the symmetric part $\mu_s(f_1, f_2)$ completely characterizes the quantum state. Hence, the standard canonical formulation of quantum field theories in terms of Hilbert and Fock spaces is recovered via the so-called GNS construction [2, 19, 23]. Denoting with $(\mathcal{H}_\omega, \pi_\omega, |\Omega_\omega\rangle)$ the GNS representation induced by the quasi-free state ω, the Hilbert space $\mathcal{H}_\omega$ is the canonical (bosonic) Fock space constructed out of the one-particle Hilbert space $\mathfrak{H}_\omega$, the representation π_ω is determined by the canonical creation and annihilation operators on $\mathcal{H}_\omega$, and the canonical smeared field $\hat{\phi}_\omega(f)$ is the quantum field operator such that

$$\omega_2(f_1, f_2) \equiv \langle \phi(f_1)\phi(f_2)\rangle_\omega = \langle \Omega_\omega | \hat{\phi}_\omega(f_1)\hat{\phi}_\omega(f_2)|\Omega_\omega\rangle. \tag{8}$$

1.3 The Quantum Stress-Energy Tensor

Generally, there are no criteria to select the quantum state ω, i.e., by specifying μ_s in (7), in an arbitrary curved spacetime, unlike the flat case, where the preferred choice of the Minkowski vacuum state is dictated by both the Poincaré symmetry of the spacetimes and the positivity of the energy. However, there exists a class of "physically reasonable" states which generalizes the Minkowski vacuum state can be also selected in curved spacetimes, called Hadamard states. The Hadamard condition satisfied by these states replaces that missing symmetry principle with a local short-distance regularity requirement that captures the universal ultraviolet structure of physically reasonable quantum states. In the locally covariant framework, Hadamard states are precisely those whose two-point functions (7) have the correct, universal singular behavior dictated solely by the local geometry and the field equation, and fully encoded in the Hadamard singularity [1, 19]. More precisely, given a convex neighborhood $\mathcal{O} \subset \mathcal{M}$, $x, y \in \mathcal{M}$ the two-point function of a Hadamard state is given by

$$\omega_2(x, y) = \mathcal{H}_{0^+}(x, y) + \mathcal{W}(x, y), \tag{9}$$

in the sense of distributions, where $\mathcal{W} \in C^\infty(\mathcal{M}^2)$ and

$$\mathcal{H}_{0^+} \doteq \lim_{\epsilon \to 0^+} \left(\frac{\mathsf{u}}{\sigma_\epsilon} + \mathsf{v} \log(\mu^2 \sigma_\epsilon) \right), \tag{10}$$

is the Hadamard parametrix. It is constructed out of $\sigma_\epsilon(x, y) = \sigma(x, y) + i\epsilon(t(x) - t(y))$, which defined by one half of the squared geodesic distance $\sigma(x, y)$ between x, y and by an arbitrary time function t. Moreover, $\mathsf{u} \in C^\infty(\mathcal{M}^2)$ is the square root of the van Vleck-Morette determinant between the points x and y divided by $8\pi^2$, $\mathsf{v} \in C^\infty(\mathcal{M}^2)$ admits an asymptotic expansion of the form $\mathsf{v} = \sum_{n \geq 0} \mathsf{v}_n \sigma^n$. Finally, μ is a free regularization parameter having physical dimensions of a mass. More recently, the Hadamard condition has been reformulated in the framework of microlocal analysis in terms of the Wave Front set of distributions. According to Radzikowski's theorem, the Hadamard property is equivalent to a condition on the wavefront set of the two-point distribution - the so-called microlocal spectrum condition - which is the curved-spacetime analogue of the positive-frequency (energy positivity) spectrum condition in Minkowski spacetime, and it captures both the location and directional structure of singularities in phase space [29].

Hadamard states are essential to obtain a local and covariant expression for the renormalized quantum stress-energy tensor $\langle :T_{\mu\nu}:(x)\rangle_\omega$, because on this class of states a point-splitting procedure to obtain composite (Wick) observables may be provided, which generalizes the normal-ordering procedure in Minkowski spacetime to curved spacetimes [30, 31]. In this regularization procedure, the divergences contained in $\mathcal{H}_{0^+}$ are subtracted from the two-point function before computing the coinciding point limits of the prescribed composite observables. Since the bi-distribution $\mathcal{W}$ in (9) is smooth, a finite expectation value in ω of the Wick observable is finally obtained, and thus Wick observables may be promoted to local and covariant quantum observables of an enlarged Wick algebra $\mathcal{A}_{\text{Wick}}(\mathcal{M}, g_{\mu\nu})$ [3, 20, 23–25].

For example, given two bi-differential operators $\mathfrak{D}_{1,2}$ on $C^\infty(\mathcal{O})$, the expectation value of the composite observable $:\left(\mathfrak{D}'_1\phi\right)(\mathfrak{D}_2\phi):$ is

$$\langle :\left(\mathfrak{D}'_1\phi\right)(\mathfrak{D}_2\phi):\rangle_\omega = \lim_{x'\to x} \mathfrak{D}'_1\mathfrak{D}_2\left(\omega_2(x',x) - \mathcal{H}_{0^+}(x',x)\right) = \left[\mathfrak{D}'_1\mathfrak{D}_2\mathcal{W}\right], \tag{11}$$

where $\mathfrak{D}'_1$ acts on x' and is viewed as implicitly parallel-transported in the coinciding point limit prescription denoted by $[\cdot]$ (see, e.g., [3, 32, 33]).

When the Hadamard point-splitting procedure is applied to the classical stress-energy tensor of a free Klein-Gordon field obtained by varying the action given in (3), that is

$$\begin{aligned} T_{\mu\nu} &\doteq -\frac{2}{\sqrt{g}}\frac{\delta S_0(\phi, g_{\mu\nu})}{\delta g^{\mu\nu}} \\ &= \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla_\rho\phi\nabla^\rho\phi + m^2\phi^2\right) + \xi\left(G_{\mu\nu}\phi^2 - \nabla_\mu\nabla_\nu\phi^2 + g_{\mu\nu}\nabla_\rho\nabla^\rho\phi^2\right), \end{aligned} \tag{12}$$

a full representation of $\langle :T_{\mu\nu}:\rangle_\omega$ may be provided. It reads

$$\langle :T_{\mu\nu}:\rangle_\omega = [D_{\mu\nu}\mathcal{W}] + \frac{A}{3}g_{\mu\nu} + \alpha_1 m^4 g_{\mu\nu} + \alpha_2 m^2 G_{\mu\nu} + \alpha_3 J_{\mu\nu} + \alpha_4 I_{\mu\nu}. \tag{13}$$

Here, $D_{\mu\nu}$ is the bi-differential operator inferred from (12) which realizes the point-splitting (11), while

$$\frac{A}{3} = \frac{1}{4\pi^2}\left(\frac{(6\xi-1)m^2R}{24} + \frac{(6\xi-1)^2R^2}{288} + \frac{R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - R_{\mu\nu}R^{\mu\nu}}{720} + \frac{(5\xi-1)\Box_g R}{120}\right), \tag{14}$$

is the so-called trace anomaly contribution which breaks the classical conformal invariance of $T_{\mu\nu}$ and makes the quantum stress-energy tensor covariantly conserved, $\nabla^\mu\langle :T_{\mu\nu}:\rangle_\omega = 0$ [30, 31, 34]. As it arises from the singular part of the quantum (Hadamard) state, the trace anomaly does not depend on the state ω, but it is purely fixed by the geometry of the spacetime in a local and covariant way. Finally, according to, the finite number of renormalization freedoms appearing in the normal ordered field $:T_{\mu\nu}:$ are fully characterized $g_{\mu\nu}$, $G_{\mu\nu}$, $I_{\mu\nu}$, and $J_{\mu\nu}$, associated to the renormalization constants α_i, $i = 1, \ldots, 4$. They may be directly obtained by varying the local Lagrangian density

$$S_F = -\frac{1}{2}\int_{\mathcal{M}}\left(\alpha_1 m^4 + \alpha_2 m^2 R + \alpha_3 R^2 + \alpha_4 C_{\mu\nu\varrho\sigma}C^{\mu\nu\varrho\sigma}\right)\sqrt{g}d^4x, \tag{15}$$

where R is the Ricci scalar, and $C_{\mu\nu\varrho\sigma}$ is the Weyl tensor ($I_{\mu\nu}$ and $J_{\mu\nu}$ are respectively given by the variation of $\sqrt{g}C_{\mu\nu\varrho\sigma}C^{\mu\nu\varrho\sigma}$ and $\sqrt{g}R^2$).

2 Cosmological Spacetimes and Backreaction

2.1 The Cosmological Einstein–Klein–Gordon system

We focus our attention on cosmological spacetimes, namely on the class of flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime $(\mathcal{M}, g_{\mu\nu})$, which should describe our Universe at large scales according to the cosmological principle and the experimental observations. In this class of homogeneous and isotropic geometries, $\mathcal{M} = I_t \times \Sigma$, $I_t \subset \mathbb{R}$ is an interval of time and Σ is a three dimensional Euclidean space. Since every flat FLRW spacetime is conformally flat with respect to conformal coordinates $(\tau, \mathbf{x})$, the line element may be written as

$$ds^2 = a(\tau)^2\Big(-d\tau^2 + \sum_{i=1}^{3} dx_i dx^i\Big), \tag{16}$$

where the conformal factor $a(\tau) > 0$ is called scale factor and describes the "history" of our Universe, as unique degree of freedom of the geometry. If one requires that cosmological solutions to be homogeneous and isotropic, as imposed in the ΛCDM model of our Universe, then some constraints on the stress-energy tensor sourcing the Friedmann equations should be imposed. It turns out that the stress–energy must have the form of a perfect fluid $T_\mu{}^\nu = \mathrm{diag}(-\varrho, p, p, p)$ with respect to a comoving observer, where ϱ and p respectively denote the energy density and the pressure.

In [15], a single-field semiclassical model was formulated by studying the initial-value problem for the backreaction of a massive scalar field with arbitrary, non-conformal coupling $\xi \in \mathbb{R}$, in which the classical stress-energy tensor of a perfect fluid is replaced by the quantum stress-energy tensor (13). In this case, it is convenient to adopt a new set of equations coming from the semiclassical Einstein equations, in order to formulate the semiclassical initial-value problem in terms of the renormalized energy density $\langle{:}\varrho{:}\rangle \doteq \langle{:}T_0{}^0{:}\rangle_\omega$ and the renormalized trace $\langle{:}T_\rho{}^\rho{:}\rangle_\omega$. For a free scalar field [3],

$$\langle{:}T_\rho{}^\rho{:}\rangle_\omega = \left(3\left(\xi - \frac{1}{6}\right)\Box - m^2\right)\langle{:}\phi^2{:}\rangle_\omega + \langle{:}T_\rho{}^\rho{:}\rangle_\omega^{(\mathrm{an})} + 4\tilde{\alpha}_1 m^4 - \tilde{\alpha}_2 m^2 R + \gamma \Box_g R, \tag{17}$$

where

$$\langle{:}T_\rho{}^\rho{:}\rangle_\omega^{(\mathrm{an})} = \frac{1}{4\pi^2}\left(\frac{(6\xi-1)^2}{8}\frac{a''}{a^6} + \frac{1}{60}\left(\frac{a'^4}{a^8} - \frac{a''a'^2}{a^7}\right)\right) \tag{18}$$

is the effective cosmological trace anomaly, and the state-depend contribution is encoded by the vacuum polarization

$$\langle{:}\phi^2{:}\rangle_\omega(x) \doteq [\omega_2(x,y) - \mathcal{H}_{0^+}(x,y)] = \mathcal{W}(x,x). \tag{19}$$

The renormalization constants $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ are respectively renormalization of the cosmological constant Λ and the Newton constant κ. Also, as FLRW spacetimes are conformally flat, the Weyl tensor vanishes, and thus the renormalization constant $\gamma \in \mathbb{R}$ is proportional to α_4 in (13). Eventually, due to the form of $\langle{:}T_\rho{}^\rho{:}\rangle_\omega$ given in (17), the semiclassical dynamics becomes in striking contrast with its classical counterpart for two-folded reasons. On the one hand, $\langle{:}\phi^2{:}\rangle_\omega$ is a nonlocal functional of the metric due to the quantum state, which is globally defined on $\mathcal{M}$; on the other hand, $\Box\langle{:}\phi^2{:}\rangle_\omega$ and $\Box R$ contain up to fourth-order derivatives of $a(\tau)$, contrary to the classical case where only second derivatives appear.

To obtain an explicit form of the Wick observables of the theory, a quasi-free, homogeneous and isotropic state ω was chosen to be sufficiently regular to evaluate $\langle{:}T_{\mu\nu}(x){:}\rangle_\omega$ in cosmological spacetimes. The Klein-Gordon operator (4) reads

$$P = \frac{1}{a^3}\left(\partial_\tau^2 - \nabla_{\mathbf{x}}^2 + a^2 m^2 + a^2\left(\xi - \frac{1}{6}\right)R\right)a, \tag{20}$$

due to the conformally flatness of the cosmological spacetime. Then, employing a decomposition in spatial Fourier modes similarly to the flat case, the two-point function of a cosmological, pure, quasi-free state may be written in the sense of distributions as

$$\omega_2(x,y)(\tau_x, \tau_y, |\mathbf{x}-\mathbf{y}|) = \lim_{\epsilon\to 0^+}\frac{1}{(2\pi)^3 a(\tau_x)a(\tau_y)}\int_{\mathbb{R}^3}\overline{\zeta}_k(\tau_x)\zeta_k(\tau_y)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}e^{-\epsilon k}d\mathbf{k}, \tag{21}$$

where the temporal modes ζ_k fulfill the equation

$$\zeta_k''(\tau) + \Omega_k^2(\tau)\zeta_k(\tau) = 0, \qquad \Omega_k^2(\tau) \doteq k^2 + a^2m^2 + \left(\xi - \frac{1}{6}\right) Ra^2, \tag{22}$$

and satisfy the normalization condition $\zeta_k'\overline{\zeta}_k - \zeta_k\overline{\zeta}_k' = i$. The modes equation (22) may be solved perturbatively after imposing some initial conditions on the quantum state in the past; thus, an exact expression for the modes $\zeta_k(\tau)$ as convergent Dyson series may be obtained in terms of retarded propagators applied to those initial conditions (see also [35, 36]). Based on this, and after requiring suitable regularity conditions for the modes, finite expressions for the Wick observables $\langle{:}\phi^2{:}\rangle$ and $\langle{:}\varrho{:}\rangle$ may be achieved by employing a generalized version of the adiabatic regularization scheme, in which the point-splitting is performed in momentum space using the decomposition in spatial Fourier modes (see [15] and references therein).

Once ω is constructed and the renormalized composite observables are obtained, the Einstein–Klein–Gordon coupled system (2) in the cosmological case is reduced to the following form

$$\begin{cases} -R(a, a'') + 4\Lambda = \kappa \langle{:}T_\rho{}^\rho{:}\rangle_\omega \left(a, a', a'', a^{(3)}, a^{(4)}\right), \\ G_{00}(\tau_0) - a^2\Lambda = \kappa \langle{:}T_{00}{:}\rangle_\omega \left(a_0, a_0', a_0'', a_0^{(3)}\right), \\ \nabla^\nu \langle{:}T_{\nu\mu}{:}\rangle_\omega = 0, \end{cases} \tag{23}$$

equipped with some initial conditions for a and ω. In particular, the fourth-order derivative contributions appearing in (17) constrain to fix four initial data $(a(\tau_0), a'(\tau_0), a''(\tau_0), a^{(3)}(\tau_0)) = (a_0, a_0', a_0'', a_0^{(3)})$ for the spacetime geometry. The first equation is the traced semiclassical Einstein equation constructed out of (17), which replaces the classical second Friedmann equation as dynamical equation for $a(\tau)$. The second equation plays the role of classical first Friedmann equation when coupled with the covariant conservation of $\langle{:}T_{\nu\mu}{:}\rangle_\omega$, and thus corresponds to an energy constraint for ω at the initial time τ_0. As shown in [15], Section 4, it is always possible to choose a sufficiently regular, quasi-free state (21) and suitable initial conditions which are compatible with that energy constraint. So, the formulation of an initial-value problem for the cosmological Einstein–Klein–Gordon coupled system (23) may be reduced to prove existence and uniqueness of solutions for the traced semiclassical Einstein equation.

2.2 The Backreaction Problem and an Inversion Solution

In [15], the initial-value problem for the traced semiclassical Einstein equation was studied for arbitrary $\xi \neq 1/6$, i.e., when the higher-order derivative contributions inside $\Box_g R$ and $\Box_g \langle{:}\phi^2{:}\rangle_\omega$ may not be avoided in the quantum trace (17) (the conformally-coupled case $\xi = 1/6$ was previously studied in [36]).

Fixed suitable initial data $(a_0, a_0', a_0'', a_0^{(3)})$ on the scale factor and chosen a quasi-free state ω according to the statements explained before, it was shown that a partial integration of the traced semiclassical Einstein equation may reduce the problem to a unique dynamical equation for the scale factor $a(\tau)$ (see [15], Theorem 4.4), which reads as

$$\partial_\tau \left(a^2 \left(\langle{:}\phi^2{:}\rangle_\omega - c_\xi R - F(a, R)\right)\right) = 0. \tag{24}$$

Here, F is the unique solution of

$$\begin{cases} (-\Box + M_c)F = S, \\ \langle{:}\phi^2{:}\rangle_\omega - c_\xi R = F, \end{cases} \tag{25}$$

for $\xi \neq 1/6$, where

$$c_\xi \doteq \frac{\tilde{\gamma}}{3(1/6-\xi)}, \qquad M_c \doteq -\frac{m^2}{3(1/6-\xi)},$$

and S is a certain function of the derivatives of a up to the second order ($\tilde{\gamma} \in \mathbb{R}$ is a new redefinition of γ in (17)).

At a first glance, (24) does not seem to contain functional issues unlike the previous form, specifically because of the apparent absence of higher-order derivative contributions. However, a "backreaction problem" still remains at this stage, and it prevents a well-posedness of the initial-value problem. More precisely, a careful analysis of the vacuum polarization discloses a problematic term at the linear order in the expansion of the modes $\zeta_k(\tau)$ given in (22), which contains the essence of the nonlocal nature of the semiclassical Einstein equations (1) in cosmological spacetimes. This issue is encoded in the following retarded operator

$$\begin{aligned} &\mathcal{T}_{\tau_0} : C_1^\infty[\tau_0,\tau] \to C^0[\tau_0,\tau] \\ &\mathcal{T}_{\tau_0}[f] \doteq -\frac{1}{8\pi^2}\int_{\tau_0}^{\tau} f'(\eta)\log(\tau-\eta)d\eta. \end{aligned} \tag{26}$$

which does not depend on the choice of the state or the initial conditions. Moreover, it is unbounded, i.e., it is not continuous in the uniform norm $\|\cdot\|_\infty$ for any $\tau > \tau_0$, but it acts on a higher-order derivative of $f \in C_1^\infty[\tau_0,\tau]$.

The appearance of such an unbounded operator is intimately related to the issue of "loss of derivatives" for nonlinear problems in infinite dimensions. In contrast to the finite-dimensional setting, where differentiability usually suffices to prove the existence of bounded, linearized inverses, nonlinear maps between scales of Banach or Fréchet spaces often fail to be uniformly invertible at linear order due to the deterioration of regularity. This failure manifests itself in the absence of a bounded inverse for the linearized operator on the same scale of spaces, but only on weaker ones. In our case,

$$\|\mathcal{T}_{\tau_0}[f]\|_\infty \leq C\left(\|f\|_\infty + \|f'\|_\infty\right), \qquad C \in \mathbb{R}. \tag{27}$$

In Hamilton's formulation [37], this systematic obstruction was formulated in terms of tame Fréchet spaces, according to which operators are required to satisfy smoothness and tame bounds that control higher regularity norms in terms of lower ones. Hence, it was proved that the tame bounds gain offset the derivative losses, providing the so-called Nash-Moser theorem in that tame framework. As inverse function theorem, it states that the linearized operator is invertible and its inverse is still smooth and tame.

A prototypical inverse problem can be phrased as follows: given a non-linear map $L : \mathcal{U} \subset F \to G$ between two graded Fréchet spaces, i.e., Fréchet spaces possessing an increasing sequence of seminorms ($\|f\|_n \leq \|f\|_{n+1}\ \forall n \in \mathbb{N}$), the linear inverse problem consists on solving the linearization of $L(f) = g$ around f, namely

$$\mathcal{D}L(f)h = k, \qquad k \doteq g - L(f),$$

where $\mathcal{D}L(f)$ is the linear map obtained as first functional Fréchet derivative (sometimes denoted also as Gateaux differential) of L. In Banach spaces, the implicit/inverse function theorem asks for a bounded inverse of $\mathcal{D}L(f_0)$ at a single point f_0; however, if the linearization loses derivatives (or is not uniformly bounded on a fixed Banach norm), then the Banach-space theorem does not apply. On the other hand, Nash-Moser framework allows this loss of derivatives whenever L is controlled by tame estimates of the form

$$\|L(f)\|_n \leq C\left(1 + \|f\|_{n+r}\right), \qquad C \in \mathbb{R},$$

for fixed $r, n \in \mathbb{N}$. In this viewpoint, L is a tame operator of degree r, where r measures "how many derivatives" should be controlled. Under those assumptions, Nash-Moser theorem states that, if $\mathcal{D}L(f)h = k$ has a unique left/right inverse $h = \mathcal{V}L(f)k$ that is still tame for all $f \in \mathcal{U}, k \in G$, then P is locally invertible and L^{-1} is smooth tame.[1]

In our framework, the unbounded, linearized operator (26) may be viewed as a tame operator of degree $r = 1$, and its inversion formula $h = \mathcal{T}_{\tau_0}[f]$ may be explicitly obtained in $C^0[\tau_0, \tau]$. It provides

$$f(\tau) = f(\tau_0) + \mathcal{T}_{\tau_0}^{-1}[h] = f(\tau_0) + \int_{\tau_0}^{\tau} K(\tau - \eta)h(\eta)d\eta, \tag{28}$$

where $K(x)$ is the inverse Laplace transform of $8\pi^2(\log(s) + \gamma)^{-1}$, with γ denoting the Euler-Mascheroni constant. The inverse $\mathcal{T}_{\tau_0}^{-1}$ given in (28) is unique in $[\tau_0, \tau_1]$, it is still linear and retarded. Mostly, it is bounded in $C^0[\tau_0, \tau]$, $\tau > \tau_0$, i.e.,

$$\|\mathcal{T}_{\tau_0}^{-1}[h]\|_\infty \leq C(\tau - \tau_0)\|h\|_\infty, \tag{29}$$

where the Lipschitz constant $C(\tau - \tau_0) \to 0$ as $\tau \to \tau_0$. So, $\mathcal{T}_{\tau_0}^{-1}$ gains one derivative with respect to $\mathcal{T}_{\tau_0}$, and thus may be viewed as a smoothing, regularizing, tame operator that is bounded already on C^0.

Once (29) is achieved, the traced semiclassical Einstein equation (24) may be rewritten as an equivalent fixed-point equation on $C[\tau_0, \tau_1]$, $\tau_1 > \tau_0$, for the derivative $a^{(3)}(\tau)$ as

$$X' = \mathcal{C}[X'], \qquad X \doteq \frac{a''}{a}, \tag{30}$$

where $\mathcal{C}$ is a contraction map on the Banach space $\mathcal{B}_\delta \doteq \{X' \in C[\tau_0, \tau_1] : \|X' - X'_0\| \leq \delta\}$, with $X'_0 \doteq X'(\tau_0)$. This reformulation permits the application of the Banach fixed-point theorem, which yields local existence and uniqueness of solutions $X'(\tau)$ in $[\tau_0, \tau_1]$, and hence of $X(\tau)$ by direct integration with choice of the initial data $X_0 \doteq X(\tau_0)$. As $X(\tau)$ corresponds to $a''(\tau)$ as stated in (30), the scale factor $a(\tau)$ of the semiclassical cosmological spacetime may be uniquely obtained by solving the following initial-value problem

$$\begin{cases} a'' = Xa, \\ a'(\tau_0) = a'_0, \\ a(\tau_0) = a_0, \end{cases} \tag{31}$$

by viewing $a[X]$ and its derivatives as functionals of X. In this end, such a procedure provides local existence and uniqueness of solutions of $a(\tau)$ for $\tau \in [\tau_0, \tau_1]$ after fixing the initial data $(a_0, a'_0, a''_0, a_0^{(3)})$. Although the obtained local solution is in principle only of class $C^3[\tau_0, \tau_1]$, in fact it may be obtained of class $C^4[\tau_0, \tau_1]$ by imposing sufficient regularity conditions on the modes defining the state. Furthermore, it may be made maximal by gluing two local solutions which overlap each other for arbitrary large $\tau > \tau_0$. In the end, a unique maximal solution $a_{\max}(\tau)$ may be obtained on the maximal interval I by taking the value of $a(\tau)$, $\tau \in I$, given by any local solution whose interval contains τ (see [36] in the case of a massive, conformally coupled, scalar field).

[1] The Nash–Moser theorem has found comparatively few applications in quantum field theories. To the best of author's knowledge, one of the first direct application within the framework of local and covariant interacting quantum field theory appears in [38], where the theorem was used to to prove local existence of solutions of a renormalization group equation.

3 Conclusions: Cosmological Implications of Quantum Backreaction

In recent years, the study of solutions to the semiclassical Einstein equations in cosmological spacetimes has been developed in several frameworks and through a variety of mathematical techniques [35, 36, 39–44]. Overall, these advances represent significant progress in well-posedness results for Semiclassical Gravity, because they show that the Einstein equations with quantum backreaction admit mathematically controlled solutions within this class of spacetimes. Specifically, such solutions depend continuously on the initial data, and that, under physically reasonable assumptions, they can be extended to maximal or even global solutions. The interplay between local and covariant quantum field theory and geometric analysis therefore provides a solid foundation for further investigations of early-Universe dynamics, the stability of cosmological models, and the role of quantum states in the backreaction problem.

Actually, cosmological spacetimes provide highly symmetric settings: homogeneity and isotropy drastically constrain the form of both the geometry and the admissible quantum states, which are typically chosen within a preferred class (e.g. Hadamard/adiabatic-like states) compatible with the symmetries. In such situations, the renormalized stress–energy tensor becomes a controlled functional of a small set of mode data and only finitely many geometric derivatives, yielding a comparatively clean and workable initial-value prescription.

By contrast, in generic globally hyperbolic spacetimes there is no canonical symmetry principle to single out the quantum state, and it is not obvious a priori which pieces of state data must be fixed to ensure that $\langle :T_{\mu\nu}: \rangle_\omega$ is well defined and compatible with the coupled Einstein–Klein–Gordon evolution. A proposed framework for well-posedness of initial-value problems in Semiclassical Gravity has been recently given in [45–47], where the authors clarify what inputs on an initial Cauchy surface Σ are required even to make sense of $\langle :T_{\mu\nu}: \rangle_\omega$ on Σ within local and covariant point-splitting renormalization. In particular, they propose that admissible initial data should include a suitable surface Hadamard property for the state on the free CCR algebra, together with geometric data given by the induced metric on Σ and its first three normal (time) derivatives off Σ (the higher derivatives, notably the fourth, entering the stress tensor are expected to be fixed implicitly once the semiclassical equations are imposed). However, formulating the surface Hadamard property involves a "tower of constraints" on the initial data which is, in principle, infinite, unless one settles for weaker regularity requirements, and thus very difficult to handle.

A possible way to overcome these issues is to consider a semiclassical forcing problem, in which the Hadamard property is ensured in the causal future of Σ, and only afterwards to restrict the resulting Hadamard state back to Σ, thereby recovering the corresponding surface Hadamard property. This viewpoint has been adopted in the recent paper by Galanda, Murro, Pinamonti, Schmid, and the author [48], which studies the asymptotic stability of Minkowski spacetime in linearized Semiclassical Gravity. More precisely, this work addresses the delicate problem of formulating a Cauchy problem for complete solutions of the Einstein–Klein–Gordon system (2), namely global solutions that remain well behaved at arbitrarily large times. This question is closely related to the stability properties of linearized semiclassical solutions, which have often been called into question because of the appearance of exponentially growing modes, usually referred to as "runaway solutions", around semiclassical backgrounds. The presence of these unphysical and spurious solutions is typically associated with the nonlocal and higher-derivative structure of the semiclassical equations (see, e.g., [16, 17, 49] in the case of a Minkowski background). The analysis presented in [48] builds on earlier results on semiclassical stability obtained in [50], where the authors considered a semiclassical toy model consisting of a quantum scalar field coupled to a classical scalar field, designed to reproduce the cosmological dynamics generated by the linearization of the traced semiclassical Einstein equations. Within this framework, the linearized Cauchy problem around Minkowski spacetime was studied in [48] for small perturbations $h_{\mu\nu}$ of the flat background metric sourced by the renormalized stress-energy tensor of a quantum scalar field.

In contrast with earlier claims in the literature, the resulting structural instability was shown to be controllable through a conformal rescaling of the Minkowski metric, whose conformal factor is determined by a universal cosmological constant fixed by the mass of the quantum field,

$$\Lambda = \frac{3}{32\pi^2} \left(\frac{m}{M_P} \right)^4 M_P^2. \tag{32}$$

From a physical viewpoint, this suggests that quantum backreaction may naturally drive the background geometry toward an effective de Sitter expansion, thus providing a possible semiclassical mechanism for the large-scale expansion of the Universe associated with a cosmological constant, in agreement with cosmological observations [51].

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Data Availability No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Ethical Approval This work does not involve human participants, animals, or sensitive data, and therefore no ethical approval was required.

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