Modified Gravity as an Alternative to Dark Energy

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Abstract

The late time accelerated expansion of the Universe may indicate that General Relativity (GR) fails on cosmological scales. In this review, we study structure formation in modified gravity models and explain how large scale structure of the Universe can be used to distinguish between modified gravity models and dark energy models in GR. An emphasize is made on the necessity to obtain the non-linear power spectrum of dark matter perturbations by properly taking into a mechanism to recover GR on small scales, which is essential to evade stringent constraints on deviations from GR at solar system scales.

1 Introduction

The late-time acceleration of the Universe is surely the most challenging problem in cosmology. Within the framework of general relativity (GR), the acceleration originates from dark energy. The simplest option is the cosmological constat. However, in order to explain the current acceleration of the Universe, the required value of the cosmological constant must be incredibly small. Alternatively, there could be no dark energy, but a large distance modification of GR may account for the late-time acceleration of the Unverse. Recently considerable efforts have been made to construct models for modified gravity as an alternative to dark energy and distinguish them from dark energy models by observations (see [1–4] for reviews). Although fully consistent models have not been constructed yet, some indications of the nature of the modified gravity models have been obtained. In general, there are three regimes of gravity in modified gravity models [2, 5]. On the largest scales, gravity must be modified significantly in order to explain the late time acceleration without introducing dark energy. On the smallest scales, the theory must approach GR because there exist stringent constraints on the deviation from GR at solar system scales. On intermediate scales between the cosmological horizon scales and the solar system scales, there can be still a deviation from GR. In fact, it is a very common feature in modified gravity models that there is a significant deviation from GR on large scale structure scales. This is due to the fact that, once we modify GR, there arises a new scalar degree of freedom in gravity. This scalar mode changes gravity even below the length scale where the modification of the tensor sector of gravity becomes significant, which causes the cosmic acceleration.

Therefore, large scale structure of the Universe offers the best opportunity to distinguish between modified gravity models and dark energy models in GR. [6–19]. The expansion history of the Universe determined by the Friedman equation can be completely the same in modified gravity models and dark energy models. In fact, it is always possible to find a dark energy model that can mimic the expansion history of the Universe in a given modified gravity model by tuning the equation of state of dark energy. However, this degeneracy can be broken by the growth rate of structure formation. This is because the scalar degree of freedom in modified gravity models changes the strength of gravity on sub-horizon scales and thus changes the growth rate of structure formation. Thus combining the geometrical test and structure formation test, one can distinguish between dark energy models and modified gravity models.

However, there is a subtlety in testing modified gravity models using large scale structure of the Universe. In any successful modified gravity models, we should recover GR on small scales. Indeed, unless there is an additional mechanism to screen the scalar interaction which changes the growth rate of structure formation, the modification of gravity contradicts to the stringent constraints on the deviation from GR at solar system scales. This mechanism affects the non-linear clustering of dark matter. We

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expect that the power-spectrum of dark matter perturbations approaches the one in the GR dark energy model with the same expansion history of the Universe because the modification of gravity disappears on small scales. Then the difference between a modified gravity model and a dark energy model with the same expansion history becomes smaller on smaller scales. This recovery of GR has important implications for weak lensing measurements because the strongest signals in weak lensing measurements come from non-linear scales.

In this review, we use two examples, branewrold models and $f(R)$ gravity models to explain how large scale structure of the Universe can be used to distinguish between modified gravity models from dark energy models in GR. For this purpose, a general framework based on Brans-Dicke (BD) gravity is introduced to describe inhomogeneities under horizon scales. The two examples are included in this general framework. Then the mechanisms to recover GR is introduced and we study their effects on the non-linear clustering of dark matter.

2 Quasi-static perturbations in modified gravity models

We consider perturbations around the Friedman-Robertson-Walker universe described in the Newtonian gauge:

$$ds^2 = -(1+2\psi)dt^2 + a^2(1+2\phi)\delta_{ij}dx^i dx^j. \tag{1}$$

We will work on the evolution of matter fluctuations inside the Hubble horizon. Then we can use the quasi-static approximation and neglect the time derivatives of the perturbed quantities compared with the spatial derivatives. As mentioned in the introduction, the large distance modification of gravity which is necessary to explain the late-time acceleration generally modifies gravity even on sub-horizon scales due to the introduction of a new scalar degree of freedom. This modification of gravity due to the scalar mode can be described by the Brans-Dicke (BD) gravity. The action of the BD theory is given by

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g_4}\left(\varphi R - \frac{\omega_{\rm BD}}{\varphi}(\nabla\varphi)^2\right). \tag{2}$$

Under the quasi-static approximations, perturbed modified Einstein equations give

$$\begin{aligned}
\phi + \psi &= -\varphi, && (3)\\
\frac{1}{a^2}\nabla^2\psi &= 4\pi G\rho_m\delta - \frac{1}{2a^2}\nabla^2\varphi, && (4)\\
(3+2\omega_{\rm BD})\frac{1}{a^2}\nabla^2\varphi &= -8\pi G\rho_m\delta, && (5)
\end{aligned}$$

where we expanded the BD scalar as $\varphi = \varphi_0 + \varphi(x,t)$, $G = 1/\varphi_0$, ρ_m is the background dark matter energy density and δ is dark matter density perturbations. In general, modified gravity models that explain the late time acceleration have $\omega_{\rm BD} \sim O(1)$ on sub-horizon scales today. This would contradict to the solar system constraints which require $\omega_{\rm BD} > 40000$. However, this constraint can be applied only when the BD scalar has no potential and no self-interactions. Thus, in order to avoid this constraint, the BD scalar should acquire some interaction terms on small scales. In general we expect that the BD scalar field equation is given by

$$(3+2\omega_{\rm BD})\frac{1}{a^2}k^2\varphi = 8\pi G\rho_m\delta - \mathcal{I}(\varphi), \tag{6}$$

in a Fourier space. Here the interaction term $\mathcal{I}$ can be expanded as

$$\begin{aligned}
\mathcal{I}(\varphi) &= M_1(k)\varphi + \frac{1}{2}\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_D(\boldsymbol{k}-\boldsymbol{k}_{12})M_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\varphi(\boldsymbol{k}_1)\varphi(\boldsymbol{k}_2)\\
&+\frac{1}{6}\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2 d^3\boldsymbol{k}_3}{(2\pi)^6}\delta_D(\boldsymbol{k}-\boldsymbol{k}_{123})M_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\varphi(\boldsymbol{k}_1)\varphi(\boldsymbol{k}_2)\varphi(\boldsymbol{k}_3) + ..., && (7)
\end{aligned}$$

where $\boldsymbol{k}_{ij} = \boldsymbol{k}_i + \boldsymbol{k}_j$ and $\boldsymbol{k}_{ijk} = \boldsymbol{k}_i + \boldsymbol{k}_j + \boldsymbol{k}_k$.

There are two known mechanisms where the non-linear interaction terms $\mathcal{I}$ are responsible for the recovery of GR on small scales. One is the chameleon mechanism [20]. In this case, the BD scalar has a non-trivial potential. The potential gives a mass to the BD scalar. Then the BD scalar mediates the Yukawa-type force and the interaction decays exponentially beyond the length scale determined by the inverse of mass, the compton wavelength. Then the scalar interaction is hidden beyond the compton wavelength and GR is recovered. The BD scalar is coupled to the trace of the energy momentum tensor. Thus the effective potential depends on the energy density of the environment. The potential is tuned so that the mass of the BD scalar becomes large for a dense environment such as the solar system. Then the compton wavelength becomes very short for a dense environment and the scalar mode is effectively hidden. In this paper, we deal with this mechanism perturbatively. M_1 determines the mass term in the cosmological background. The higher order terms $M_i, (i > 1)$ describe the change of the mass term due to the change of the energy density. If the chameleon mechanism is at work, the effective mass becomes larger when the density fluctuations become non-linear.

The other mechanism relies on the existence of the non-linear derivative interactions. A typical example is the Dvali-Gabadadze-Porratti (DGP) model where we are supposed to be living on a 4D brane in a 5D Minkowski spacetime [21]. In this model, the BD scalar is identified as the brane bending mode which describes the deformation of the 4D brane in the 5D bulk spacetime. The brane bending mode has a large second-order term in the equation of motion which cannot be neglected even when the metric perturbations remain linear [22–24]. This corresponds to the existence of a large $M_2(k)$ term. it has been shown that once this second order term dominates over the linear term, the scalar mode is hidden and the solutions for metric perturbations approach GR solutions. For a static spherically symmetric source, we can identify the length scale below which the second order interaction becomes important. This length scale is known as the Vainshtein radius [25]. In the cosmological situation, it is expected that once the density perturbations become non-linear, the second order term becomes important and we recover GR. In the next section, we apply the perturbation theory to solve the equations. Thus we only keep up to the third order in the expansion of $\mathcal{I}$ which is necessary to calculate the quasi non-linear power spectrum.

The evolution equations for matter perturbations are obtained from the conservation of energy momentum tensor, the continuity equation and the Euler equation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a}\nabla \cdot [(1+\delta)\boldsymbol{v}] = 0, \tag{8}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + H\boldsymbol{v} + \frac{1}{a}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{a}\nabla\psi. \tag{9}$$

Eqs. (4), (6), (8) and (9) are the basic equations that have to be solved.

3 Linear regime

In this section, we study the behaviour of perturbations on linear scales under horizon. First we study two explicit examples and then discuss the possibility to distinguish between modified gravity models and dark energy models.

3.1 Linear growth rate

By linearizing Eqs. (4), (6), the solutions for metric perturbations are given by

$$\frac{k^2}{a^2}\Phi = 4\pi G\left(\frac{2(1+\omega_{\rm BD}) + M_1 a^2/k^2}{3+2\omega_{\rm BD}+M_1 a^2/k^2}\right)\rho_m\delta_m, \tag{10}$$

$$\frac{k^2}{a^2}\Psi = -4\pi G\left(\frac{2(2+\omega_{\rm BD}) + M_1 a^2/k^2}{3+2\omega_{\rm BD}+M_1 a^2/k^2}\right)\rho_m\delta_m, \tag{11}$$

where we only keep the linear order in $\mathcal{I}$. There are two ways to recover GR solutions at linearized level. One is to take $\omega_{\rm BD} \to \infty$. The other is to consider a large mass for the BD scalar which satisfies $M_1 \gg k/a$. However, modified gravity models that explain the late time acceleration do not satisfy these conditions in general and linearized gravity under horizon scales deviates from GR. The evolution equation for density perturbations is obtained by Eq. (8) and (9) using the solutions (10) and (11):

$$\widehat{\mathcal{L}}\delta_m = 0, \tag{12}$$

where the linear operator $\widehat{\mathcal{L}}$ is given by

$$\widehat{\mathcal{L}} \equiv \frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{\kappa^2}{2}\rho_{\rm m}\left(\frac{2(2+\omega_{\rm BD}) + M_1 a^2/k^2}{3+2\omega_{\rm BD} + M_1 a^2/k^2}\right), \tag{13}$$

and we assumed the irrotationality of the fluid. In the following, we consider two explicit examples and study the behaviour of density perturbations.

3.2 DGP models

In DGP models, we are supposed to be living in a 4D brane in a 5D spacetime. The model is described by the action given by

$$S = \frac{1}{4\kappa^2 r_c}\int d^4x\sqrt{-g_5}R_5 + \frac{1}{2\kappa^2}\int d^4x\sqrt{-g}(R + L_{\rm m}), \tag{14}$$

where $\kappa^2 = 8\pi G$, R_5 is the Ricci scalar in 5D and $L_{\rm m}$ stands for the matter lagrangian confined to a brane. The cross over scale r_c is the parameter in this model which is a ratio between the 5D Newton constant and the 4D Newton constant. The modified Friedman equation is given by

$$\epsilon\frac{H}{r_c} = H^2 - \frac{\kappa^2}{3}\rho, \tag{15}$$

where $\epsilon = \pm 1$ represents two distinct branches of the solutions [26]. From this modified Friedman equation, we find that the cross-over scale r_c must be fine-tuned to be the present-day horizon scales in order to modify gravity only at late times. The solution with $\epsilon = +1$ is known as the self-accelerating branch because even without the cosmological constant, the expansion of the Universe is accelerating as the Hubble parameter is constant, $H = 1/r_c$. On the other hand $\epsilon = -1$ corresponds to the normal branch. In this branch, we need a cosmological constant to realize the cosmic acceleration. However, due to the modified gravity effects, the Universe behaves as if it were filled with the Phantom dark energy with the equation of state w smaller than -1. It is known that the self-accelerating solution is plagued by the ghost instabilities (see [27] for a review). Also it gives a poor fit to the observations such as supernovae and cosmic microwave background anisotropies [28]. However, as we will see later, this model is the simplest modified gravity model where the mechanism of the recovery of GR on small scales is naturally encoded and it can be used to get insights into the effect of this mechanism on the non-linear power spectrum.

In this model, gravity becomes 5D on large scales larger than r_c. On small scales, gravity becomes 4D but it is not described by GR. The quasi-static perturbations are described by the BD theory where the BD parameter is given by

$$\omega_{\rm BD}(t) = \frac{3}{2}\Big(\beta(t) - 1\Big), \quad \beta(t) = 1 - 2\epsilon H r_c\left(1 + \frac{\dot{H}}{3H^2}\right), \tag{16}$$

where $\dot{H}$ is a cosmic time derivative of the Hubble parameter H. Note that the BD parameter depends on time in this model. The BD scalar is massless $M_1 = 0$. Then the solutions for the metric perturbations are given by [29,30]

$$\frac{k^2}{a^2}\Phi = 4\pi G\left(1 - \frac{1}{3\beta}\right)\rho_m\delta_m, \tag{17}$$

$$\frac{k^2}{a^2}\Psi = 4\pi G\left(1 + \frac{1}{3\beta}\right)\rho_m\delta_m, \tag{18}$$

and the linear growth rate is determined as

$$\widehat{\mathcal{L}}\delta_m = 0, \quad \widehat{\mathcal{L}} \equiv \frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{\kappa^2}{2}\rho_m\left(1 - \frac{1}{3\beta}\right). \tag{19}$$

In the self-accelerating branch, $\beta < 0$ and the BD parameter is negative which makes the Newton constant effectively smaller than GR. Thus the growth rate receives additional suppressions compared with dark energy models in GR. On the other hand, in the normal branch, $\beta > 0$ and the BD parameter is positive which makes the Newton constant larger than GR. Then the growth rate is enhanced. In order to demonstrate the effect of this modification, it is instructive to consider a dark energy model in GR which follows the same expansion history of the Universe in the self-accelerating universe:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_{de}), \tag{20}$$

where $\rho_{de} = H/r_c$. The equation of state of dark energy $w_{de} = p_{de}/\rho_{de}$ is given by $w_{de} = -1/(1 + \Omega_m)$ where $\Omega_m = 8\pi G\rho_m/3H^2$ [29]. Now let us consider the linear growth rate in this dark energy mode. Since gravity is not modified, the linear growth rate is given by

$$\widehat{\mathcal{L}}\delta_m = 0, \quad \widehat{\mathcal{L}} \equiv \frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{\kappa^2}{2}\rho_m. \tag{21}$$

Fig. 1 shows the comoving distance in the self-accelerating universe, the dark energy model and ΛCDM model. Clearly, we cannot distinguish between the dark energy model and the self-accelerating universe. However, if we look at the linear growth rate of density perturbations, this degeneracy is broken.

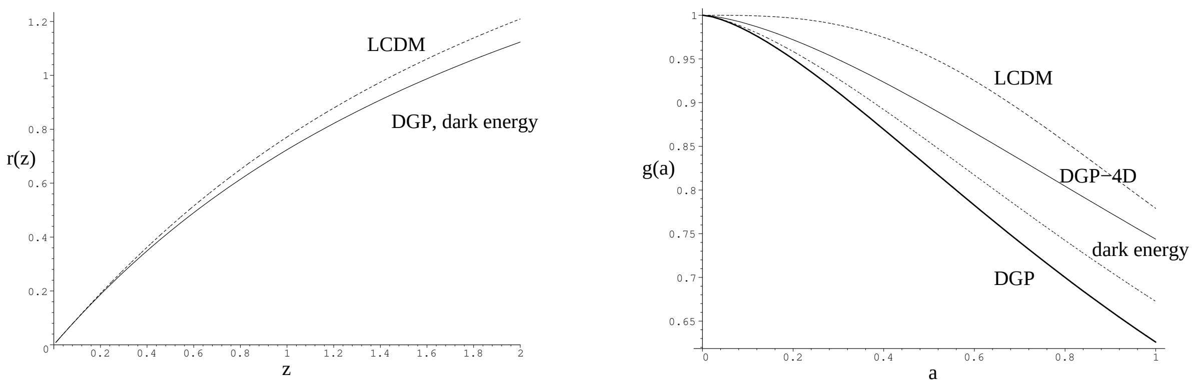

Figure 1: The comoving distance $r(z)$ and the growth rate $g(a) = \delta_m(a)/a$ are shown for the standard LCDM (long dashed), DGP (solid, thick) and the equivalent GR dark energy model. From [30].

3.3 $f(R)$ gravity models

We consider another class of modified theory of gravity that generalizes the Einstein-Hilbert action to include an arbitrary function of the scalar curvature R:

$$S = \int d^4x\sqrt{-g}\left[\frac{R + f(R)}{2\kappa^2} + L_{\rm m}\right], \tag{22}$$

where $\kappa^2 = 8\pi G$ and $L_{\rm m}$ is the Lagrangian of the ordinary matter. This theory is equivalent to the BD theory with $\omega_{\rm BD} = 0$ but there is a non-trivial potential [31]. This can be seen from the trace of modified Einstein equations:

$$3\Box f_R - R + f_R R - 2f = -\kappa^2\rho, \tag{23}$$

where $f_R = df/dR$ and $\Box$ is a Laplacian operator and we assumed matter dominated universe. We can identify f_R as the BD scalar field and its perturbations are defined as

$$\varphi = \delta f_R \equiv f_R - \overline{f}_R, \tag{24}$$

where the bar indicates that the quantity is evaluated on the cosmological background. In this paper, we assume $|f_R| \ll 1$ and $|f/R| \ll 1$. These conditions are necessary to have the background which is close to ΛCDM cosmology. Then the BD scalar perturbations satisfy

$$3\frac{1}{a^2}\nabla^2\varphi = -\kappa^2\bar{\rho}_m\delta + \delta R, \quad \delta R \equiv R(f_R) - R(\overline{f}_R). \tag{25}$$

This is noting but the equation for the BD scalar perturbations with $\omega_{\rm BD} = 0$ and the potential gives the non-linear interaction term

$$\mathcal{I}(\varphi) = \delta R(\varphi). \tag{26}$$

By linearizing the potential we find

$$M_1 = \overline{R}_f(t) \equiv \frac{d\overline{R}(f_R)}{df_R}. \tag{27}$$

The solutions for the metric perturbations are then given by

$$\frac{k^2}{a^2}\Phi = 4\pi G\left(\frac{2 + M_1a^2/k^2}{3 + M_1a^2/k^2}\right)\rho_m\delta_m, \tag{28}$$

$$\frac{k^2}{a^2}\Psi = 4\pi G\left(\frac{4 + M_1a^2/k^2}{3 + M_1a^2/k^2}\right)\rho_m\delta_m, \tag{29}$$

and the liner growth rate is given by

$$\widehat{\mathcal{L}}\delta_m = 0, \quad \widehat{\mathcal{L}} \equiv \frac{d^2}{dt^2} + 2H\frac{d}{dt} - \frac{\kappa^2}{2}\left(\frac{4 + M_1a^2/k^2}{3 + M_1a^2/k^2}\right)\rho_m. \tag{30}$$

In this paper, we consider a function $f(R)$ of the form [32]

$$f(R) \propto \frac{R^n}{AR^n + 1}, \tag{31}$$

where A is a constant with dimensions of length squared and n is an integer. In the following we take $n = 1$. In the limit $R \to 0$, $f(R) \to 0$ and there is no cosmological constant. For high curvature $AR \gg 1$, $f(R)$ can be expanded as

$$f(R) = -2\kappa^2\rho_\Lambda - f_{R0}\frac{\bar{R}_0}{R}, \tag{32}$$

where ρ_Λ is determined by A, $\bar{R}_0$ is the background curvature today and we defined f_{R0} as $f_{R0} = \bar{f}_R(R_0)$. As we mentioned before, we take $|f_{R0}| \ll 1$ and assume that the background expansion follows the ΛCDM history with the same ρ_Λ. The M_1 term determines the mass of the BD field $m_{\rm BD} = (M_1/3)^{1/2}$ as

$$m_{\rm BD}(t) \equiv \sqrt{\frac{R_f}{3}} = \left(\frac{R_0}{6|\bar{f}_R|}\sqrt{\frac{f_{R0}}{\bar{f}_R}}\right)^{1/2}. \tag{33}$$

Above the compton length $m_{\rm BD}^{-1}$, the BD scalar interaction decays exponentially and we recover GR. On small scales, we recover the BD theory with $\omega_{\rm BD} = 0$. Then the Newton constant is 4/3 times large than GR. Thus the linear power spectrum acquires a scale dependent enhancement on small scales (Fig. 2).

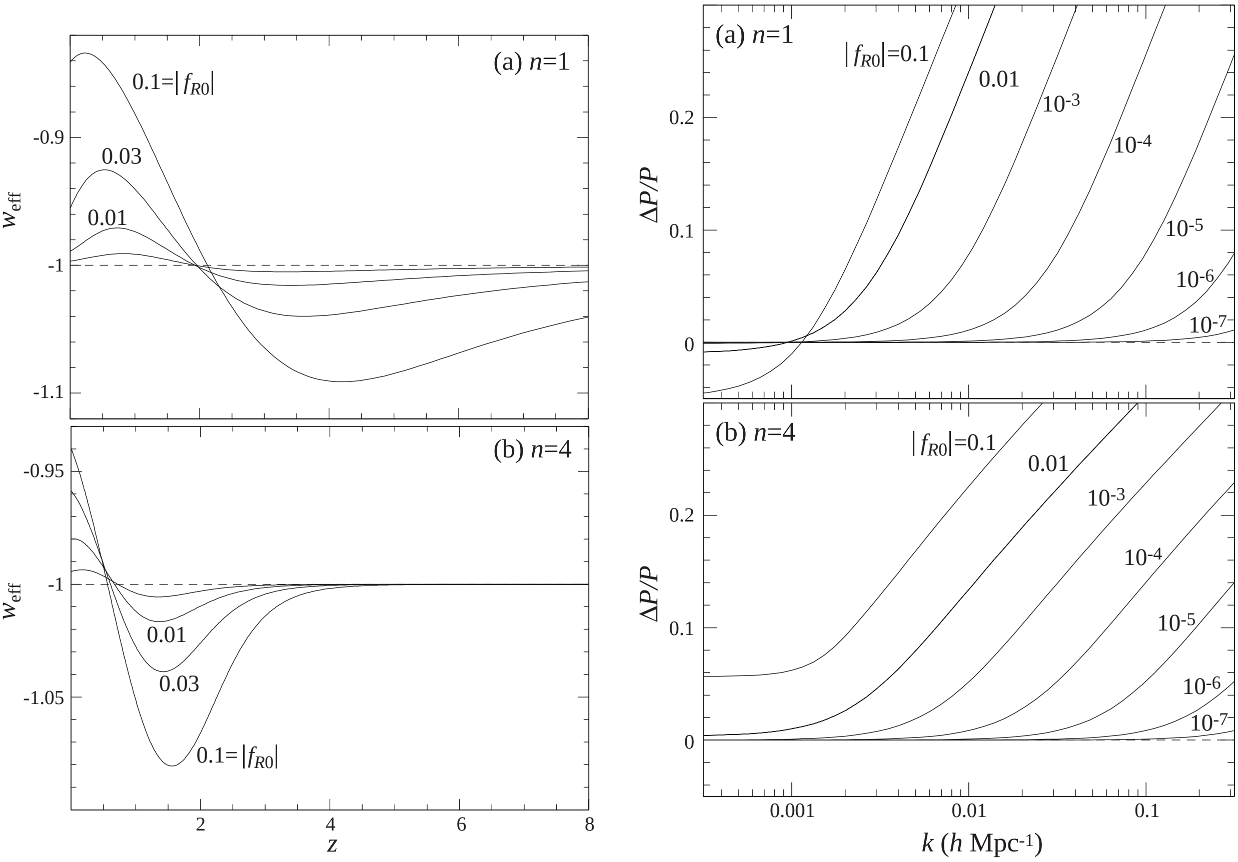

Figure 2: Left: Evolution of the effective equation of state w_{de} for $n = 1, 4$ for several values of the cosmological field amplitude today, f_{R0}. Right: Fractional change in the matter power spectrum $P(k)$ relative to ΛCDM for a series of the cosmological field amplitude today, f_{R0}, for $n = 1, 4$ models. For scales that are below the cosmological Compton wavelength during the acceleration epoch $k/a \gg M_1$ perturbation dynamics transition to the low-curvature regime where $\omega_{\rm BD} = 0$ and density growth is enhanced. From [32].

3.4 Distinguish between modified gravity and dark energy models

Fig.3 illustrates how one can detect the failure of GR from future observations. Suppose that our Universe is described by the DGP model. However, astronomers still try to fit the data by dark energy models in GR. For example, they use the parametrization of the equation of state of dark energy

$$w = w_0 + w_1 z. \tag{34}$$

Combining SN observations, CMB shift parameter and weak lensing, there appears an inconsistency. This is because weak lensing probes the growth of structure and the growth rate in the DGP model cannot be fitted by the growth rate in GR models given the same expansion history.

However, there is a subtlety in testing modified gravity models using large scale structure of the Universe. In any successful modified gravity models, we should recover GR on small scales. Indeed, unless there is an additional mechanism to screen the scalar interaction which changes the growth rate of structure formation, the modification of gravity contradicts to the stringent constraints on the deviation from GR at solar system scales. This mechanism affects the non-linear clustering of dark matter. We expect that the power-spectrum of dark matter perturbations approaches the one in the GR dark energy model with the same expansion history of the Universe because the modification of gravity disappears on small scales. Then the difference between a modified gravity model and a dark energy model with the same expansion history becomes smaller on smaller scales. This recovery of GR has important implications

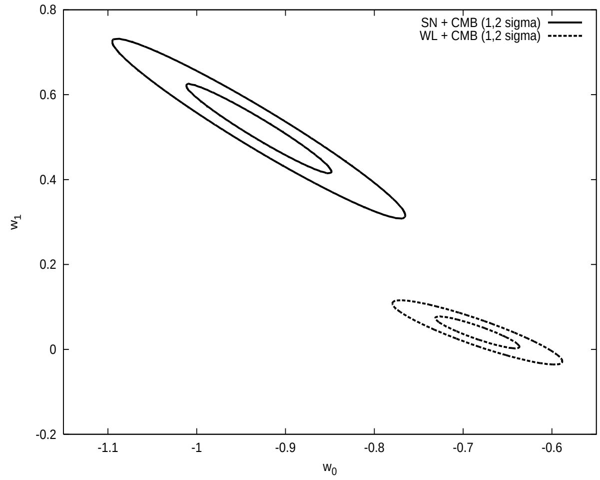

Figure 3: Equations of state found using two different combinations of data sets. Solid contours are for fits to SN Ia and CMB data, while dashed contours are for fits to weak lensing and CMB data. The significant difference (inconsistency) between the equations of state found using these two combinations is a signature of the DGP model. The inconsistency is an observational detection of the underlying modified gravity DGP model (assumed here to generate the data). From [8].

for weak lensing measurements because the strongest signals in weak lensing measurements come from non-linear scales.

In almost all of the literature including the result in Fig. 3, the non-linear power spectrum in modified gravity models was derived using the mapping formula between the linear power spectrum and the non-linear power spectrum. This is equivalent to assume that gravity is modified down to small scales in the same way as in the linear regime which contradicts to the solar system constraints. Thus this approach overestimates the difference between modified gravity models and dark energy models. This was explicitly shown by N-body simulations in the context of $f(R)$ gravity [33–35]. By tuning the function f, it is possible to make the Compton wavelength of the BD scalar short at solar system scales and screen the BD scalar interaction [32,36,37]. N-body simulations show that, due to this mechanism, the deviation of the non-linear power spectrum from GR is suppressed on small scales. It was shown that the mapping formula failed to describe this recovery of GR and it overestimated the deviation from GR.

4 Non-linear clustering in modified gravity models

In the section, we develop a formalism to treat the quasi non-linear evolution of the power spectrum in modified gravity models by properly taking into account the mechanism to recover GR on small scales [38]. Our formalism is based on the closure approximation which gives a closed set of evolution equations for the matter power spectrum [39]. These evolution equations reproduce the one-loop results of the standard perturbation theory (SPT) by replacing the quantities in the non-linear terms with linear-order ones. The SPT in GR is tested against N-body simulations extensively recently and it has been shown that, at the quasi non-linear regime, it can predict the power-spectrum with a sub-percent accuracy [40]. Although the validity regime of the perturbation theory is limited, it is the most relevant regime to distinguish between modified gravity models and dark energy models in GR because the difference in the two models is large in the linear and quasi-non-linear regime.

4.1 Evolution equations for perturbations

The Fourier transform of the fluid equations (8) and (9) become

$$H^{-1}\frac{\partial\delta(\boldsymbol{k})}{\partial t}+\theta(\boldsymbol{k})=-\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\delta(\boldsymbol{k}_2), \tag{35}$$

$$H^{-1}\frac{\partial\theta(\boldsymbol{k})}{\partial t}+\left(2+\frac{\dot{H}}{H^2}\right)\theta(\boldsymbol{k})-\left(\frac{k}{a\,H}\right)^2\psi(\boldsymbol{k})=-\frac{1}{2}\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2)\beta(\boldsymbol{k}_1,\boldsymbol{k}_2)\,\theta(\boldsymbol{k}_1)\theta(\boldsymbol{k}_2), \tag{36}$$

where the kernels in the Fourier integrals, α and β, are given by

$$\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2)=1+\frac{\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{|\boldsymbol{k}_1|^2},\qquad \beta(\boldsymbol{k}_1,\boldsymbol{k}_2)=\frac{(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\,|\boldsymbol{k}_1+\boldsymbol{k}_2|^2}{|\boldsymbol{k}_1|^2|\boldsymbol{k}_2|^2}. \tag{37}$$

We take into account the non-linear interaction terms $\mathcal{I}$ in the BD scalar equation. Due to the non-linear interactions, the potential ψ is couples to δ through the BD scalar φ in a fully non-linear way. To derive closed equations for δ and θ, we must employ the perturbative approach to Eq. (6). By solving Eq. (6) perturbatively assuming $\varphi<1$, ψ can be expressed in terms of δ as

$$-\left(\frac{k}{a}\right)^2\psi=\frac{1}{2}\kappa^2\,\overline{\rho}_{\rm m}\left[1+\frac{1}{3}\frac{(k/a)^2}{\Pi(k)}\right]\delta(\boldsymbol{k})+\frac{1}{2}\left(\frac{k}{a}\right)^2 S(\boldsymbol{k}), \tag{38}$$

where

$$\Pi(k)=\frac{1}{3}\left((3+2\omega_{BD})\frac{k^2}{a^2}+M_1\right), \tag{39}$$

and $\kappa^2=8\pi G$. The function $S(\boldsymbol{k})$ is the non-linear source term which is obtained perturbatively using (4) as

$$\begin{aligned}S(\boldsymbol{k})\;=\;&-\frac{1}{6\Pi(\boldsymbol{k})}\left(\frac{\kappa^2\,\overline{\rho}_{\rm m}}{3}\right)^2\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})M_2(\boldsymbol{k}_1,\boldsymbol{k}_2)\frac{\delta(\boldsymbol{k}_1)\,\delta(\boldsymbol{k}_2)}{\Pi(\boldsymbol{k}_1)\Pi(\boldsymbol{k}_2)}\\&-\frac{1}{18\,\Pi(\boldsymbol{k})}\left(\frac{\kappa^2\,\overline{\rho}_{\rm m}}{3}\right)^3\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2 d^3\boldsymbol{k}_3}{(2\pi)^6}\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{123})\left\{M_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)-\frac{M_2(\boldsymbol{k}_1,\boldsymbol{k}_2+\boldsymbol{k}_3)M_2(\boldsymbol{k}_2,\boldsymbol{k}_3)}{\Pi(\boldsymbol{k}_{23})}\right\}\\&\times\frac{\delta(\boldsymbol{k}_1)\,\delta(\boldsymbol{k}_2),\delta(\boldsymbol{k}_3)}{\Pi(\boldsymbol{k}_1)\Pi(\boldsymbol{k}_2)\Pi(\boldsymbol{k}_3)},\end{aligned} \tag{40}$$

The expression (40) is valid up to the third-order in δ.

The perturbation equations (35), (36) and (40) can be further reduced to a compact form by introducing the following quantity:

$$\Phi_a(\boldsymbol{k})=\begin{pmatrix}\delta(\boldsymbol{k})\\-\theta(\boldsymbol{k})\end{pmatrix}. \tag{41}$$

We can write down the basic equations in a single form as

$$\begin{aligned}\frac{\partial\Phi_a(\boldsymbol{k};\eta)}{\partial\eta}+\Omega_{ab}(k;\eta)\,\Phi_b(\boldsymbol{k};\eta)=&\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2}{(2\pi)^3}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{12})\,\gamma_{abc}(\boldsymbol{k}_1,\boldsymbol{k}_2;\eta)\,\Phi_b(\boldsymbol{k}_1;\eta)\Phi_c(\boldsymbol{k}_2;\eta)\\&+\int\frac{d^3\boldsymbol{k}_1 d^3\boldsymbol{k}_2 d^3\boldsymbol{k}_3}{(2\pi)^6}\,\delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}_{123})\,\sigma_{abcd}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta)\,\Phi_b(\boldsymbol{k}_1;\eta)\Phi_c(\boldsymbol{k}_2\eta)\Phi_d(\boldsymbol{k}_3;\eta),\end{aligned} \tag{42}$$

where the time variable η is defined by $\eta=\ln a(t)$. The matrix Ω_{ab} is given by

$$\Omega_{ab}(k;\eta)=\begin{pmatrix}0 & -1\\ -\dfrac{\kappa^2}{2}\dfrac{\overline{\rho}_{\rm m}}{H^2}\left[1+\dfrac{1}{3}\dfrac{(k/a)^2}{\Pi(k)}\right] & 2+\dfrac{\dot{H}}{H^2}\end{pmatrix}. \tag{43}$$

From the $(2,1)$ component of Ω_{ab}, we can define the effective Newton constant as

$$G_{\rm eff} = G\left[1+\frac{1}{3}\frac{(k/a)^2}{\Pi(k)}\right]. \tag{44}$$

If M_1=0, the effective Newton constant is given by

$$G_{\rm eff} = \frac{2(2+\omega_{\rm BD})}{3+2\omega_{\rm BD}}G. \tag{45}$$

For a positive $\omega_{\rm BD} > 0$, the effective gravitational constant is larger than GR and the gravitational force is enhanced. On the other hand, if $M_1 \gg k^2/a^2$, $G_{\rm eff}$ becomes G. The quantity γ_{abc} is the vertex function as in the GR case, but new non-vanishing components arise in the case of modified gravity:

$$\gamma_{abc}(\boldsymbol{k}_1,\boldsymbol{k}_2;\eta) = \begin{cases} \dfrac{1}{2}\,\alpha(\boldsymbol{k}_2,\boldsymbol{k}_1) & ; \quad (a,b,c)=(1,1,2), \\[2ex] \dfrac{1}{2}\,\alpha(\boldsymbol{k}_1,\boldsymbol{k}_2) & ; \quad (a,b,c)=(1,2,1), \\[2ex] -\dfrac{1}{12H^2}\left(\dfrac{\kappa\,\bar{\rho}_{\rm m}}{3}\right)^2\left(\dfrac{k_{12}^2}{a^2}\right)\dfrac{M_2(\boldsymbol{k}_1,\boldsymbol{k}_2)}{\Pi(\boldsymbol{k}_{12})\Pi(\boldsymbol{k}_1)\Pi(\boldsymbol{k}_2)} & ; \quad (a,b,c)=(2,1,1), \\[2ex] \dfrac{1}{2}\,\beta(\boldsymbol{k}_1,\boldsymbol{k}_2) & ; \quad (a,b,c)=(2,2,2), \\[2ex] 0 & ; \quad \text{otherwise}. \end{cases} \tag{46}$$

Here γ_{211} is absent in GR. Note that the symmetric properties of the vertex function, $\gamma_{abc}(\boldsymbol{k}_1,\boldsymbol{k}_2) = \gamma_{acb}(\boldsymbol{k}_2,\boldsymbol{k}_1)$, still hold in the modified theory of gravity. In Eq. (42), there appears another vertex function coming from the non-linearity of Poisson equation. The explicit form of the higher-order vertex function σ_{abcd} is given by

$$\sigma_{abcd}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta) = \begin{cases} -\dfrac{1}{36H^2}\left(\dfrac{\kappa\bar{\rho}_{\rm m}}{3}\right)^3\left(\dfrac{k_{123}^2}{a^2}\right)\dfrac{M_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)}{\Pi(\boldsymbol{k}_{123})\Pi(\boldsymbol{k}_1)\Pi(\boldsymbol{k}_2)\Pi(\boldsymbol{k}_3)} & \\ \times\left[1-\dfrac{1}{3}\dfrac{1}{M_3(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)}\left\{\dfrac{M_2(\boldsymbol{k}_1,\boldsymbol{k}_2+\boldsymbol{k}_3)M_2(\boldsymbol{k}_2,\boldsymbol{k}_3)}{\Pi(\boldsymbol{k}_{23})}+\text{perm.}\right\}\right] & ; \quad (a,b,c,d)=(2,1,1,1), \\[2ex] 0 & ; \quad \text{otherwise}. \end{cases}$$

Again this term is absent in GR. The vertex function $\sigma_{abcd}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3;\eta)$ defined above is invariant under the permutation of $b \leftrightarrow c \leftrightarrow d$ or $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2 \leftrightarrow \boldsymbol{k}_3$.

4.2 Evolution equations for power spectrum

In this paper, we are especially concerned with the evolution of the matter power spectrum, defined by

$$\Big\langle \Phi_a(\boldsymbol{k};\eta)\Phi_b(\boldsymbol{k}';\eta)\Big\rangle = (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\,P_{ab}(|\boldsymbol{k}|;\eta). \tag{47}$$

Here the bracket $\langle\cdot\rangle$ stands for the ensemble average. Note that we obtain the three different power spectra: $P_{\delta\delta}$ from $(a,b)=(1,1)$, $-P_{\delta\theta}$ from $(a,b)=(1,2)$ and $(2,1)$, and $P_{\theta\theta}$ from $(a,b)=(2,2)$.

Let us consider how to compute the power spectrum. In the standard treatment of perturbation theory, we first solve Eq.(42) by expanding the quantity Φ_a as $\Phi_a = \Phi_a^{(1)} + \Phi_a^{(2)} + \cdots$. Substituting the perturbative solutions into the definition (47), we obtain the weakly non-linear corrections to the power spectrum. This treatment is straightforward, but it is not suited for numerical calculations. Furthermore, successive higher-order corrections generally converge poorly and SPT will be soon inapplicable at late-time stage of the non-linear evolution. Here in order to deal with modified gravity models in which analytical calculations are intractable in many cases, we take another approach. Our approach is based on the closure approximation proposed by Ref. [39], in which the evolution of power spectrum is obtained numerically by solving a closed set of evolution equations.

Provided the basic equation (42), the evolution equation for power spectrum can be derived by truncating the infinite chain of the moment equations with a help of perturbative calculation, called closure approximation. We skip the details of the derivation and present the final results. Readers interested in the derivation can refer to Ref. [39]. The resultant evolution equations are the coupled equations characterized by the three statistical quantities including the power spectrum. We define

$$\begin{aligned}
\Big\langle \Phi_a(\boldsymbol{k};\eta)\Phi_b(\boldsymbol{k}';\eta')\Big\rangle &= (2\pi)^3\,\delta_{\rm D}(\boldsymbol{k}+\boldsymbol{k}')\,R_{ab}(|\boldsymbol{k}|;\eta,\eta') \quad ; \qquad \eta>\eta', \\
\Big\langle \frac{\delta\Phi_a(\boldsymbol{k};\eta)}{\delta\Phi_b(\boldsymbol{k}';\eta')}\Big\rangle &= \delta_{\rm D}(\boldsymbol{k}-\boldsymbol{k}')\,G_{ab}(|\boldsymbol{k}|;\eta,\eta') \quad ; \qquad \eta\geq\eta'.
\end{aligned} \tag{48}$$

The quantities R_{ab} and G_{ab} respectively denote the cross spectra between different times and the non-linear propagator. Note that $R_{ab}\neq R_{ba}$, in general. Then, the closure equations become

$$\begin{aligned}
\widehat{\Sigma}_{abcd}(k;\eta)\,P_{cd}(k;\eta) &= \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\Big[\gamma_{apq}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};\eta)\,F_{bpq}(-\boldsymbol{k},\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};\eta)+\gamma_{bpq}(\boldsymbol{q},-\boldsymbol{k}-\boldsymbol{q};\eta)\,F_{apq}(\boldsymbol{k},\boldsymbol{q},-\boldsymbol{k}-\boldsymbol{q};\eta)\Big] \\
&\quad +3\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\Big[\sigma_{apqr}(\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k};\eta)P_{pq}(q;\eta)P_{rb}(k;\eta)+\sigma_{bpqr}(\boldsymbol{q},-\boldsymbol{q},-\boldsymbol{k};\eta)\,P_{pq}(q;\eta)P_{ra}(k;\eta)\Big],
\end{aligned} \tag{49}$$

$$\begin{aligned}
\widehat{\Lambda}_{ab}(k;\eta)\,R_{bc}(k;\eta,\eta') &= \int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\gamma_{apq}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};\eta)\,K_{cpq}(-\boldsymbol{k},\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};\eta,\eta') \\
&\quad +3\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\sigma_{apqr}(\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k};\eta)P_{pq}(q;\eta)R_{rc}(k;\eta,\eta'),
\end{aligned} \tag{50}$$

$$\begin{aligned}
\widehat{\Lambda}_{ab}(k;\eta)\,G_{bc}(k|\eta,\eta') &= 4\int_{\eta'}^{\eta}d\eta''\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\gamma_{apq}(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q};\eta)\,\gamma_{lrs}(-\boldsymbol{q},\boldsymbol{k};\eta'')G_{ql}(|\boldsymbol{k}-\boldsymbol{q}||\eta,\eta'')R_{pr}(q;\eta,\eta'')G_{sc}(k|\eta'',\eta'), \\
&\quad +3\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\sigma_{apqr}(\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k};\eta)P_{pq}(q;\eta)G_{rc}(k|\eta,\eta'),
\end{aligned} \tag{51}$$

where the operators $\widehat{\Sigma}_{abcd}$ and $\widehat{\Lambda}_{ab}$ are defined as

$$\widehat{\Sigma}_{abcd}(k;\eta)=\delta_{ac}\delta_{bd}\frac{\partial}{\partial\eta}+\delta_{ac}\Omega_{bd}(k;\eta)+\delta_{bd}\Omega_{ac}(k;\eta),\quad \widehat{\Lambda}_{ab}(k;\eta)=\delta_{ab}\frac{\partial}{\partial\eta}+\Omega_{ab}(k;\eta), \tag{52}$$

The explicit expressions for the kernels F_{apq} and K_{cpq} are summarized as

$$\begin{aligned}
F_{apq}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q};\eta) &= 2\int_{\eta_0}^{\eta}d\eta''\,\Big[2\;G_{ql}(q|\eta,\eta'')\,\gamma_{lrs}(\boldsymbol{k},\boldsymbol{p};\eta'')\,R_{ar}(k;\eta,\eta'')R_{ps}(p;\eta,\eta'') \\
&\qquad +\;G_{al}(k|\eta,\eta'')\,\gamma_{lrs}(\boldsymbol{p},\boldsymbol{q};\eta'')\,R_{pr}(p;\eta,\eta'')\,R_{qs}(q;\eta,\eta'')\Big],
\end{aligned} \tag{53}$$

$$\begin{aligned}
&K_{cpq}(\boldsymbol{k}',\boldsymbol{p},\boldsymbol{q};\eta,\eta') \\
&\quad=\;4\int_{\eta_0}^{\eta}d\eta''\;G_{ql}(q|\eta,\eta'')\,\gamma_{lrs}(\boldsymbol{k}',\boldsymbol{p};\eta'')\,R_{ps}(p;\eta,\eta'') \\
&\qquad\qquad\times\Big\{R_{cr}(k';\eta',\eta'')\Theta(\eta'-\eta'')+R_{rc}(k';\eta'',\eta')\Theta(\eta''-\eta')\Big\} \\
&\quad\;+\;2\int_{\eta_0}^{\eta'}d\eta''\,G_{cl}(k'|\eta',\eta'')\,\gamma_{lrs}(\boldsymbol{p},\boldsymbol{q};\eta'')\,R_{pr}(p;\eta,\eta'')\,R_{qs}(q;\eta,\eta'').
\end{aligned} \tag{54}$$

The closure equations (50)–(51) are the integro-differential equations involving several non-linear terms, in which the information of the higher-order corrections in SPT is encoded. Thus, replacing all statistical quantities in these non-linear terms with linear-order ones, the solutions of closure equations automatically reproduce the leading-order results of SPT, called one-loop power spectra. Further, fully non-linear treatment of the closure equations is a non-perturbative description of the power spectra, and have an ability to predict the matter power spectra accurately, beyond one-loop SPT. Strictly speaking, the non-linear terms in the right-hand side of Eqs. (50)–(51) have only the information of the one-loop corrections. However, it has been shown in Ref. [39] that the present formulation is equivalent to the one-loop level of renormalized perturbation theory [41], and even the leading-order approximation still contain some non-perturbative effects. The application of the closure approximation, together with the detailed comparison with N-body simulations, is presented in Refs. [40, 42].

In this paper, we mainly use the closure equations for the purpose of computing the one-loop power spectra. The results for fully non-linear treatment of the closure equations will be presented elsewhere. The numerical scheme to solve the closure equations is basically the same as described in Ref. [39]. In Appendix A, we briefly review the numerical scheme and summarize several modifications.

Before closing this section, we note here that the resultant equations (50)–(51) contain the additional non-linear terms originating from the modification of the Poisson equation (see Eq.(38)). In particular, for the terms containing the higher-order vertex function σ_{abcd} can be effectively absorbed into the matrix Ω_{ab}. Since the non-vanishing contribution of the higher-order vertex function only comes from σ_{2111}, this means that the effective Newton constant $G_{\rm eff}$ defined by (44) is renormalized as $G_{\rm eff} \to G_{\rm eff} + \delta G_{\rm eff}$, with $\delta G_{\rm eff}$ given by

$$\delta G_{\rm eff} = \frac{3H^2}{4\pi\,\rho_m} \int \frac{d^3\boldsymbol{q}}{(2\pi)^3}\, \sigma_{2111}(\boldsymbol{q}, -\boldsymbol{q}, \boldsymbol{k}; \eta)\, P_{11}(q; \eta). \tag{55}$$

This is a clear manifestation of the mechanism that non-linear clustering does generically alter the growth rate of the structure formation through the renormalisation of the Newton constant and, due to this mechanism, successful modified gravity models are expected to recover GR on small scales.

5 Non-linear power spectrum in modified gravity models

In this section, we show solutions for the non-lienar power spectrum in two examples, DGP models and $f(R)$ gravity models. We also extend these results to fully non-linear scales using the so-called Parametrized Post Friedmann (PPF) framework.

5.1 PPF formalism

Sawicki and Hu proposed a fitting formula for the non-linear power spectrum in modified gravity models [5]. The fitting formula based on the observation that the non-linear power spectrum should approach the one in the GR model that follows the same expansion history of the Universe due to the recovery of GR on small scales. They postulate that the full non-linear power spectrum in a modified gravity model is given by the formula

$$P(k, z) = \frac{P_{\rm non-GR}(k, z) + c_{\rm nl}\Sigma^2(k, z) P_{\rm GR}(k, z)}{1 + c_{\rm nl}\Sigma^2(k, z)}, \tag{56}$$

where z is a red-shift. Here $P_{\rm non-GR}(k, z)$ is the non-linear power spectrum which is obtained without the non-linear interactions that are responsible for the recovery of GR. This is equivalent to assume that gravity is modified down to the small scales in the same way as in the linear regime. $P_{\rm GR}(k, z)$ is the non-linear power spectrum obtained in the GR dark energy model that follows the same expansion history of the Universe as in the modified gravity model. The function $\Sigma^2(k, z)$ determines the degree of non-linearity at a relevant wavenumber k. They propose to take $\Sigma^2(k) = k^3 P_{\rm lin}(k, z)/2\pi^2$, where $P_{\rm lin}(k, z)$ is the linear power spectrum in the modified gravity model. Finally, $c_{\rm nl}$ is a parameter in this framework which controls the scale at which the theory approaches GR.

Once we obtain the quasi non-linear power spectrum, we can check whether the PPF framework works and determines $c_{\rm nl}$ in the quasi non-linear regime. In our formalism, $P_{\rm non-GR}(k, z)$ is obtained by neglecting the non-linear interaction $\mathcal{I}$. $P_{\rm GR}(k, z)$ can be obtained by taking $\omega_{BD} \to \infty$ limit and also neglecting $\mathcal{I}$. We again consider two explicit examples.

5.2 DGP models

In DGP models, the BD scalar acquires a large second order interaction given by

$$\mathcal{I}(\varphi) = \frac{r_c^2}{a^4}\Big[(\nabla^2\varphi)^2 - (\nabla_i\nabla_j\varphi)^2\Big]. \tag{57}$$

Note that r_c is tuned to be the present-day horizon scale. Thus this second order term has a large effect. The higher order terms than the second order are suppressed by the additional powers of the 4D Planck scale and, in the Newtonian limit, we can safely ignore them. Therefore, in this model, we have

$$M_1 = 0, \quad M_2(\boldsymbol{k}_1, \boldsymbol{k}_2) = 2\frac{r_c^2}{a^4}\Big[k_1^2 k_2^2 - (\boldsymbol{k}_1\cdot\boldsymbol{k}_2)^2\Big], \quad M_3 = 0, \tag{58}$$

$$\Pi(k,\eta) = \beta(\eta)\frac{k^2}{a^2}. \tag{59}$$

In this paper, we only consider the self-accelerating branch solutions and use the best-fit cosmological parameters for the flat self-accelerating universe: $\Omega_m = 0.257, \Omega_b = 0.0544, h = 0.66, n_s = 0.998$ [28]. In the left panel of Fig. 4, the solid (black) line shows the fractional difference between the power spectrum in DGP models and dark energy models that follow the same expansion history obtained by solving the linearized closure equation numerically. The dashed (red) line shows the result obtained by neglecting the non-linear term M_2, $P_{\mathrm{non-GR}}$. we can see that the non-linear interaction enhances the non-linear power spectrum. This is natural because in the self-accelerating branch, the linear growth rate is suppressed compared to the GR model that follows the same expansion history due to the negative BD parameter $\omega_{\mathrm{BD}} < 0$ which makes the Newton constant smaller than GR. This is closely related to the fact that the BD scalar becomes a ghost in this model. Classically, the ghost mediates a repulsive force and suppresses the gravitational collapse. The non-linear interaction makes the theory approach GR. Thus it effectively increases the Newton constant by screening the BD scalar. Then the power-spectrum receives an enhancement compared with the case without the non-linear interaction.

For DGP models, we find that the PPF formalism with

$$\Sigma^2(k,z) = \frac{k^3}{2\pi^2}P_{\mathrm{lin}}(k,z) \tag{60}$$

as proposed by Sawicki and Hu gives a nice fit to the results obtained in the perturbation theory. We find that by allowing the time dependence in c_{nl}, it is possible to recover the solutions for t he non-linear power spectrum very well within the validity regime of the perturbation theory. At $z = 0$, c_{nl} is given by 0.3 and there is a slight redshift dependence (the dotted (blue) line in the left panel of Fig. 4).

Armed with this result, it is tempting to extend our analysis to the fully non-linear regime. In GR, there are several fitting formulae which provide the mapping between the linear power spectrum and non-linear power spectrum. It is impossible to apply these mapping formulae to modified gravity models as the mapping does not take into account the non-linear interaction terms in the Poisson equation $\mathcal{I}$. If we apply the GR mapping formula to the linear power spectrum, we would get the non-linear power spectrum without $\mathcal{I}$, $P_{\mathrm{non-GR}}(k)$. In fact, there exist N-body simulations in DGP models performed by neglecting the non-linear interaction terms [43]. It was shown that the power spectrum obtained in these N-body simulations can be fitted well by the mapping formulae in GR. The mapping formulae should be valid in GR models, so we can also predict $P_{\mathrm{GR}}(k)$. Then using the PPF formalism (56), we can predict the non-linear power spectrum if c_{nl} is known. In the right panel of Fig. 4, we plotted the fractional difference between the power spectrum in the DGP model and the GR model with the same expansion history. We used the fitting formula developed by Smith et.al. [44]. If we could extrapolate the result in the quasi non-linear regime, we would have $c_{\mathrm{nl}} = 0.3$ at $z = 0$. If this is correct, we find that even at $k = 10\mathrm{Mpc\ h}^{-1}$, the difference between the power spectrum in DGP and that in the equivalent GR model remains at 7% level. This is crucial to distinguish between the two models using weak lensing as the signal to noise ratio is larger on smaller scales. Of course, we should emphasize that there is no guarantee that c_{nl} measured in the quasi non-linear regime is valid down to the fully non-linear scales and this should be tested using N-body simulations. Still there is no N-body simulation available in DGP models and it would be important to test the prediction against N-body simulations.

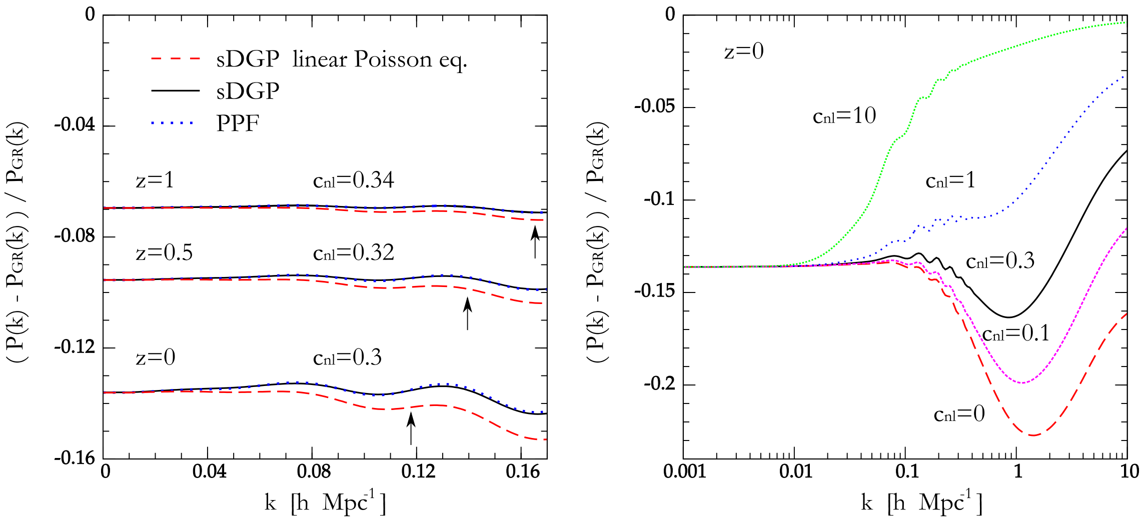

Figure 4: The fractional difference between the power spectrum in the DGP model and the one in the GR model which has the same expansion history as the DGP. The solid (black) line shows the perturbation theory solution and the dashed (red) line shows the perturbation theory solution without the non-linear interaction terms in the Poisson equation. The dotted (blue) line shows the PPF fitting. By allowing the redshift dependence of $c_{\rm nl}$, we can fit the power spectrum very well within the validity regime of the perturbation theory indicated by arrows. The right panel shows the results at $z = 0$ obtained from the fitting formula by Smith et.al. for $P_{\rm non-GR}$ and $P_{\rm GR}$. If $c_{\rm nl} = 0.3$ obtained by the perturbation theory is applicable, the the solid (black) line is the prediction on non-linear scales. The cosmological parameters are the same as in Fig 1.

5.3 $f(R)$ gravity models

Next, we consider $f(R)$ gravity models. In these models the potential gives the non-linear interaction term

$$\mathcal{I}(\varphi) = \delta R(\varphi). \tag{61}$$

Then we find

$$M_1 = \overline{R}_f(\eta) \equiv \frac{d\overline{R}(f_R)}{df_R}, \quad M_2 = \overline{R}_{ff}(\eta) \equiv \frac{d^2\overline{R}(f_R)}{df_R^2}, \quad M_3 = \overline{R}_{fff}(\eta) \equiv \frac{d^3\overline{R}(f_R)}{df_R^3},$$
$$\Pi(k,\eta) = \left(\frac{k}{a}\right)^2 + \frac{\overline{R}_f(\eta)}{3}.$$

In this paper, we consider the model described in section 3.3 where $f(R)$ is given by Eq. (31). We adopt the cosmological parameters given by $f_{R0} = 10^{-4}, n_s = 0.958, \Omega_m = 0.24, \Omega_b = 0.046, \Omega_\Lambda = 0.76, h = 0.73$. In this model, the solar system constraints are satisfied but it has been pointed that the chameleon mechanism does not work for strong gravity and Neutron stars cannot exist [45, 46]. A fine-tuned higher curvature corrections to $f(R)$ are needed to cure the problem [47]. In this paper, we perturbatively take into account the chameleon mechanism in the cosmological background and the quasi non-linear power spectrum would be insensitive to the high curvature corrections.

The left panel of Fig. 5, the solid (black) line shows the fractional difference between the power spectrum in $f(R)$ models and ΛCDM by solving the linearized closure equation numerically. The dashed (red) line shows the result obtained by neglecting the non-linear terms, $P_{\rm non-GR}$. The linear power spectrum acquires a scale dependent enhancement on small scales due to the larger effective Newton constant $G_{\rm eff} = 4G/3$. The higher order terms $M_i (i > 1)$ are responsible for the suppression of this modification of gravity on small scales. Thus the non-linear interaction terms $\mathcal{I}$ suppress the non-linear power spectrum.

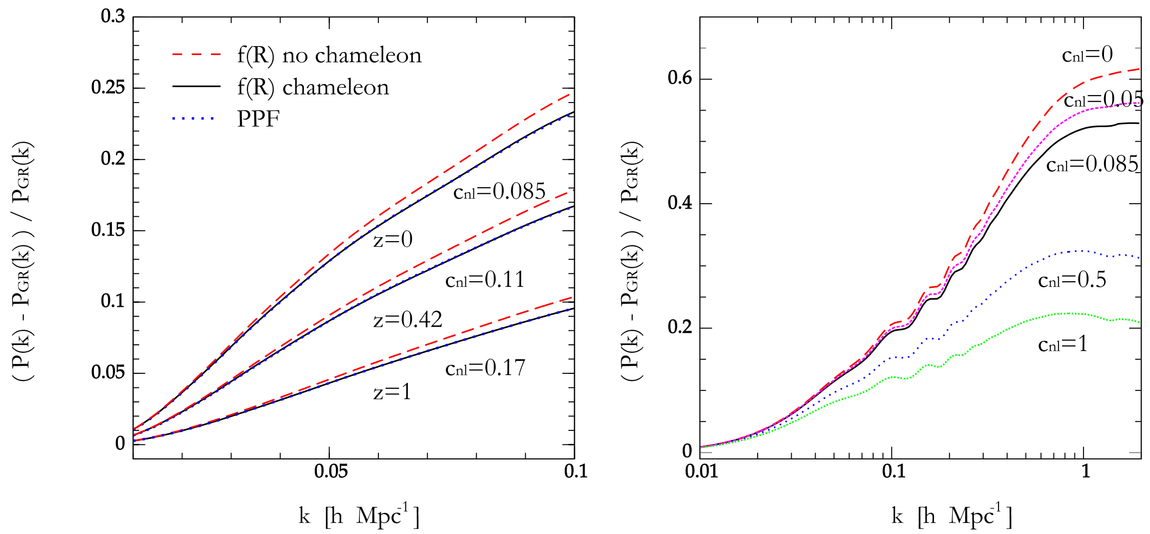

Figure 5: The same as Fig. 4 in $f(R)$ gravity models

As in the DGP model, we first check if we can reproduce the perturbation theory results by the PPF fitting. We find that the fitting is not very good if we adopt $\Sigma(k,z)$ as is proposed by Hu and Sawicki. Instead, if we choose $\Sigma(k,z)$ as

$$\Sigma^2(k,z) = \left(\frac{k^3}{2\pi^2} P_{\rm lin}(k,z)\right)^{1/3}, \tag{62}$$

the solutions in the perturbation theory are fitted by the PPF formula very well by allowing the redshift dependence in $c_{\rm nl}$. At $z=0$, $c_{\rm nl}=0.085$ gives an excellent fitting to the power spectrum within the validity regime of the perturbation theory. In Fig. 5, we also show the prediction for the fractional difference between the power spectrum in $f(R)$ theory and that in ΛCDM model in fully non-linear regime for several $c_{\rm nl}$.

In $f(R)$ gravity models, it is possible to check our predictions against the No-body simulations. Fig. 6 shows the comparison between the PPF prediction and N-body simulations. The dashed line corresponds to non-Chameleon case with $c_{\rm nl}=0$. The corresponding N-body results are shown by triangles. We again used the fitting formula by Smith et.al. to derive the non-linear power spectrum from the linear power spectrum. Compared with the N-body results, the formula by Smith et.al. slightly underestimates the power spectrum around $0.03h{\rm Mpc}^{-1} < k < 0.5h{\rm Mpc}^{-1}$ and overestimates the power at $k > 0.5h{\rm Mpc}^{-1}$ though N-body simulations have large errors in this regime. The solid line shows the case with the chameleon mechanism. Again the PPF formula underestimates the power spectrum in the same region as the non-chameleon case. If we take the ratio between the non-chameleon case and chameleon case, the PPF formalism nicely recovers the N-body results up to $k \sim 0.5h{\rm Mpc}^{-1}$.Beyond that, N-body simulations have large errors. We should emphasize that the perturbation theory is valid only up to $k = 0.08h{\rm Mpc}^{-1}$ at $z=0$. Thus the PPF formula using $c_{\rm nl}$ derived by the perturbation theory describe the effect of the chameleon mechanism on non-linear scales beyond the validity regime of the perturbation theory.

This observation suggests that an improvement of the PPF formalism can be made by getting a more accurate power spectrum without the chameleon mechanism because the PPF formalism describes the effect of the chameleon mechanism very well. In order to demonstrate this fact, we derive the power spectrum without the chameleon mechanism $P_{\rm non-GR}$ by interpolating the N-body results. Using this power spectrum as the non-chameleon power spectrum in the PPF formula, we find that the power spectrum with the chameleon mechanism can be very well described by the PPF formula where $c_{\rm nl}$ is derived by the perturbation theory. We should emphasize that the ratio between the power spectra with and without the chameleon mechanism is insensitive to the non-chameleon power spectrum. This also indicates that the PPF formalism with $c_{\rm nl}$ determined by the perturbation theory describes the effect of the chameleon mechanism very well at least up to $k \sim 0.5h{\rm Mpc}^{-1}$. For larger k, N-body simulations also do not have enough resolutions and it is difficult to tell whether this extrapolation is good or not. More

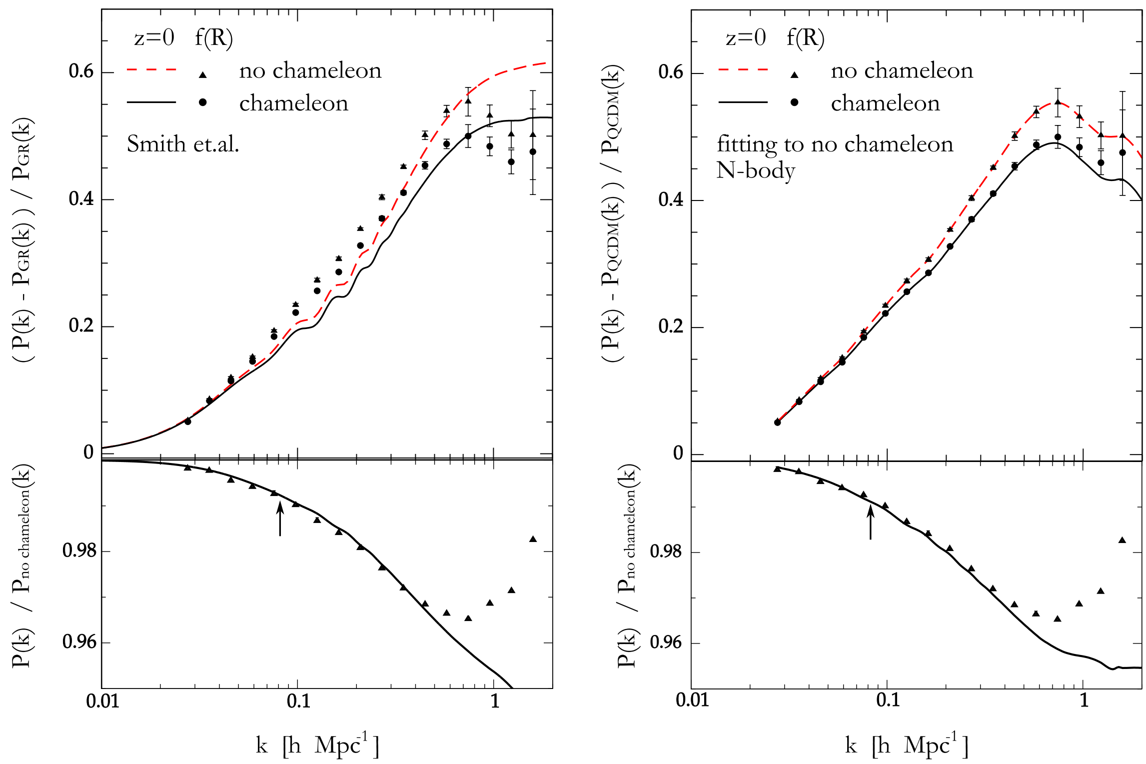

Figure 6: Comparison between the PPF prediction and N-body simulations. In the left panel, Smith et.al. fitting formula is used to predict $P_{\rm non}$ and $P_{\rm GR}$. We used $c_{\rm nl}$ determined by the perturbation theory $c_{\rm nl} = 0.085$ at $z = 0$. In the left panel, we fitted N-body results without the chameleon mechanism to derive $P_{\rm non}$.

detailed study is needed to address the power spectrum at larger k, but the PPF formalism is likely to give a promising way to develop a fitting formula for the non-linear power spectrum in modified gravity models.

6 Conclusion

Modified gravity models are still under developments and still no fully consistent models are developed. Although DGP models and $f(R)$ models provide us interesting indications of the nature of modified gravity models, these models face severe problems. However, the general feature of these models that large scale structure of the Universe is affected by the modification of gravity would be valid in many modified gravity models. This feature can be used to distinguish modified gravity models from dark energy models in GR. In the study of large scale structure in modified gravity models, an emphasis should be made on the recovery of GR on small scales which is essential to evade the stringent constraints on the deviation from GR at solar system scales. Any successful modified gravity models should have a mechanism to recover GR at solar system scales and this mechanism affects the non-linear power spectrum of dark matter perturbations. The study of non-linear clustering of dark matter has just started but the perturbations theory and N-body simulations begin to reveal the nature of non-linear clustering in modified gravity models. Cosmology is now experiencing a golden age of discovery driven by on-going and future massive new surveys of the sky. Rapid progress in cosmological observations will enable us to distinguish modified gravity models from dark energy models in general relativity, and to provide a test of general relativity on cosmological scales

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References

[1] S. Nojiri and S. D. Odintsov, ECONF **C0602061**, 06 (2006), hep-th/0601213.

[2] K. Koyama, Gen. Rel. Grav. **40**, 421 (2008), 0706.1557.

[3] R. Durrer and R. Maartens, (2008), 0811.4132.

[4] R. Durrer and R. Maartens, Gen. Rel. Grav. **40**, 301 (2008), 0711.0077.

[5] W. Hu and I. Sawicki, (2007), arXiv:0708.1190 [astro-ph].

[6] J.-P. Uzan and F. Bernardeau, Phys. Rev. **D64**, 083004 (2001), hep-ph/0012011.

[7] A. Lue, R. Scoccimarro, and G. Starkman, Phys. Rev. **D69**, 044005 (2004), astro-ph/0307034.

[8] M. Ishak, A. Upadhye, and D. N. Spergel, Phys. Rev. **D74**, 043513 (2006), astro-ph/0507184.

[9] L. Knox, Y.-S. Song, and J. A. Tyson, Phys. Rev. **D74**, 023512 (2006).

[10] K. Koyama, JCAP **0603**, 017 (2006), astro-ph/0601220.

[11] T. Chiba and R. Takahashi, Phys. Rev. **D75**, 101301 (2007), astro-ph/0703347.

[12] L. Amendola, M. Kunz, and D. Sapone, JCAP **0804**, 013 (2008), 0704.2421.

[13] K. Yamamoto, D. Parkinson, T. Hamana, R. C. Nichol, and Y. Suto, Phys. Rev. **D76**, 023504 (2007), 0704.2949.

[14] K. Yamamoto, T. Sato, and G. Huetsi, Prog. Theor. Phys. **120**, 609 (2008), 0805.4789.

[15] Y.-S. Song and W. J. Percival, (2008), 0807.0810.

[16] M. Kunz and D. Sapone, Phys. Rev. Lett. **98**, 121301 (2007), astro-ph/0612452.

[17] B. Jain and P. Zhang, (2007), 0709.2375.

[18] Y.-S. Song and K. Koyama, JCAP **01**, 048 (2009), 0802.3897.

[19] Y.-S. Song and O. Dore, (2008), 0812.0002.

[20] J. Khoury and A. Weltman, Phys. Rev. **D69**, 044026 (2004), astro-ph/0309411.

[21] G. R. Dvali, G. Gabadadze, and M. Porrati, Phys. Lett. **B485**, 208 (2000), hep-th/0005016.

[22] M. A. Luty, M. Porrati, and R. Rattazzi, JHEP **09**, 029 (2003), hep-th/0303116.

[23] T. Tanaka, Phys. Rev. **D69**, 024001 (2004), gr-qc/0305031.

[24] K. Koyama and F. P. Silva, Phys. Rev. **D75**, 084040 (2007), hep-th/0702169.

[25] C. Deffayet, G. R. Dvali, G. Gabadadze, and A. I. Vainshtein, Phys. Rev. **D65**, 044026 (2002), hep-th/0106001.

[26] C. Deffayet, Phys. Lett. **B502**, 199 (2001), hep-th/0010186.

[27] K. Koyama, Class. Quant. Grav. **24**, R231 (2007), 0709.2399.

[28] W. Fang *et al.*, Phys. Rev. **D78**, 103509 (2008), 0808.2208.

[29] A. Lue, R. Scoccimarro, and G. D. Starkman, Phys. Rev. **D69**, 124015 (2004), astro-ph/0401515.

[30] K. Koyama and R. Maartens, JCAP **0601**, 016 (2006), astro-ph/0511634.

[31] T. Chiba, Phys. Lett. **B575**, 1 (2003), astro-ph/0307338.

[32] W. Hu and I. Sawicki, Phys. Rev. **D76**, 064004 (2007), 0705.1158.

[33] H. Oyaizu, Phys. Rev. **D78**, 123523 (2008), 0807.2449.

[34] H. Oyaizu, M. Lima, and W. Hu, Phys. Rev. **D78**, 123524 (2008), 0807.2462.

[35] F. Schmidt, M. V. Lima, H. Oyaizu, and W. Hu, (2008), 0812.0545.

[36] A. A. Starobinsky, JETP Lett. **86**, 157 (2007), 0706.2041.

[37] S. A. Appleby and R. A. Battye, Phys. Lett. **B654**, 7 (2007), 0705.3199.

[38] K. Koyama, A. Taruya, and T. Hiramatsu, (2009), in preparation.

[39] A. Taruya and T. Hiramatsu, (2007), 0708.1367.

[40] T. Nishimichi *et al.*, (2008), 0810.0813.

[41] M. Crocce and R. Scoccimarro, Phys. Rev. **D73**, 063519 (2006), astro-ph/0509418.

[42] A. Taruya, T. Nishimichi, S. Saito, and T. Hiramatsu, (2009), in preparation.

[43] I. Laszlo and R. Bean, Phys. Rev. **D77**, 024048 (2008), 0709.0307.

[44] The Virgo Consortium, R. E. Smith *et al.*, Mon. Not. Roy. Astron. Soc. **341**, 1311 (2003), astro-ph/0207664.

[45] A. V. Frolov, Phys. Rev. Lett. **101**, 061103 (2008), 0803.2500.

[46] T. Kobayashi and K.-i. Maeda, (2008), 0807.2503.

[47] T. Kobayashi and K.-i. Maeda, Phys. Rev. **D79**, 024009 (2009), 0810.5664.