KTH ROYAL INSTITUTE
OF TECHNOLOGY

Doctoral Thesis in Physics

Models of SO(10) Grand Unified Theories

Yukawa Sector and Gauge Coupling Unification

MARCUS PERNOW

Stockholm, Sweden 2021

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Academic Dissertation which, with due permission of the KTH Royal Institute of Technology, is submitted for public defence for the Degree of Doctor of Philosophy on Tuesday the 30th of November 2021, at 1:00 pm in FR4, Roslagstullsbacken 21, Stockholm

Doctoral Thesis in Physics
KTH Royal Institute of Technology
Stockholm, Sweden 2021

ISBN 978-91-8040-028-2
TRITA-SCI-FOU 2021:40

Printed by: Universitetsservice US-AB, Sweden 2021

Abstract

The Standard Model (SM) of particle physics is a highly successful description of observed phenomena exhibited by particles. Despite this, it suffers from several shortcomings. Among the most crucial are that the SM does not provide a mechanism to generate neutrino masses, nor does it include dark matter. Furthermore, several of its aspects seem *ad hoc*, such as the choice of gauge group, the structure of the fermionic sector, and the cancellation of gauge anomalies.

These shortcomings can be interpreted to suggest a theory beyond the SM. Many such models have been proposed and one of the most popular classes of models is grand unified theories based on the SO(10) gauge group. It predicts that the three gauge groups in the SM are embedded into one at a high energy, which contains the SM as a subgroup. The work that comprises this thesis investigates certain aspects of such SO(10)-based models. In particular, we investigate the constraints imposed by the requirements of unification of the three gauge couplings of the SM and the fact that the Yukawa sector should reproduce the measured parameter values.

The energy scale at which the gauge couplings unify is related to the rate of proton decay. Since protons have not been observed to decay, this results in an upper bound on the decay rate, which translates to a lower bound on the unification scale. By solving the renormalization group equations for the gauge couplings of the SM and comparing the predicted unification scale to the constraints from proton decay, we can analyze which models of grand unification are viable and which are ruled out. Those that are ruled out can be salvaged by higher-order threshold corrections at the unification scale.

The difference in energy between the electroweak scale and the unification scale spans around 14 orders of magnitude. To relate the parameters of the SO(10) model to those of the SM, one must solve the renormalization group equations. Using this and numerical fits, we can investigate the ability of the SO(10) models to accommodate the known fermion masses and mixing parameters, taking into account thresholds at which heavy intermediate scale particles are integrated out of the theory. Although the results depend on the details of the particular model under consideration, there are some general results that appear to hold true. The observables of the Yukawa sector can in general be accommodated into SO(10) models only if the neutrino masses are normally ordered. Furthermore, we find that numerical fits to the data favor type I seesaw over type II seesaw for the generation of neutrino masses. When both are included, the dominant contribution arises from the type I seesaw mechanism.

Key words: grand unified theories, renormalization group equations, neutrino masses, threshold effects, gauge coupling unification.

Sammanfattning

Partikelfysikens standardmodell (SM) är en mycket framgångsrik beskrivning av observerade fenomen. Trots detta så lider den av ett antal brister. Bland de mest framstående av dessa finner vi det faktum att SM inte innehåller en mekanism för att generera neutrinomassor, samt att den inte inkluderar mörk materia. Dessutom verkar ett antal av dess aspekter *ad hoc*, så som valet av gaugegrupp, strukturen i fermionsektorn och att gaugeanomalierna oväntat tar ut varandra.

Dessa brister kan tolkas som en antydan till en teori bortom SM. Ett flertal olika sådana modeller har föreslagits och en av de mest populära är storförenade teorier baserade på gaugegruppen SO(10). Den förutsäger att de tre gaugegrupperna i SM, vid hög energi, förenas i en gaugegrupp som innehåller SM som en delgrupp. Arbetet som omfattas av den här avhandlingen utforskar ett antal aspekter av modeller baserade på SO(10). Särskilt undersöker vi de olika begränsningar som härstammar från kravet på förening av gaugekopplingarna samt att Yukawasektorn bör återge uppmätta parametervärden.

Energiskalan vid vilken gaugekopplingarna förenas kan relateras till protonens sönderfallshastighet. Från det faktum att protonsönderfall inte har observerats kan man härleda en övre gräns för sönderfallshastigheten vilken motsvarar en nedre gräns för föreningsskalan. Genom att lösa renormeringsgruppsekvationerna för gaugekopplingarna i SM och jämföra den resulterande föreningsskalan med begränsningen från protonsönderfall kan man studera vilka storförenade modeller som är tillåtna och vilka som är uteslutna. De som är uteslutna kan tillåtas om man tar hänsyn till tröskeleffekter vid en högre ordning i störningsräkning vid föreningsskalan.

Skillnaden i energiskala mellan den elektrosvaga skalan och föreningsskalan sträcker sig över ungefär 14 storleksordningar. Därför relateras parametrarna i SO(10)-modellen till de i SM med hjälp av renormeringsgruppsekvationer. Genom dessa och numeriska anpassningar utforskas möjligheten för SO(10)-modeller att återge de kända fermionmassorna och blandningsparametrarna med beaktande av trösklar vid vilka tunga partiklar med intermediär massa utintegreras ur teorin. Även om resultaten beror på detaljer av den särkilda modellen som studeras så erhålls ett antal resultat som verkar gälla allmänt. Observablerna i Yukawasektorn kan allmänt sett rymmas i SO(10)-modeller endast om neutrinomassorna är normalt ordnade. Dessutom gynnar numeriska anpassningar till data typ I-gungbrädemekanismen över typ II-gungbrädemekanismen för generering av neutrinomassor. När båda inkluderas kommer det dominanta bidraget från typ I-mekanismen.

Nyckelord: storförenade teorier, renormeringsgruppsekvationer, neutrinomassor, tröskeleffekter, gaugekopplingsförening.

Preface

This thesis is the result of my research at the Department of Physics from August 2017 to November 2021. The first part of the thesis presents an introduction to the subjects relevant for the scientific work of this thesis. These include the Standard Model and its shortcomings, models of neutrino mass, grand unification and SO(10), as well as renormalization group equations, threshold corrections, and the numerical methods used. The second part contains the five papers that are included in this thesis.

List of papers included in this thesis

The scientific papers included in this thesis are:

Paper I [1]
T. Ohlsson and *M. Pernow*
Running of fermion observables in non-supersymmetric SO(10) models
J. High Energy Phys. **11**, 028 (2018)
`arXiv:1804.04560`

Paper II [2]
S. M. Boucenna, T. Ohlsson and *M. Pernow*
A minimal non-supersymmetric SO(10) model with Peccei–Quinn symmetry
Phys. Lett. B **792**, 251 (2019)
`arXiv:1812.10548`

Paper III [3]
T. Ohlsson and *M. Pernow*
Fits to non-supersymmetric SO(10) models with type I and II seesaw mechanisms using renormalization group evolution
J. High Energy Phys. **06**, 085 (2019)
`arXiv:1903.08241`

Paper IV [4]
D. Meloni, T. Ohlsson and *M. Pernow*
Threshold effects in SO(10) models with one intermediate breaking scale
Eur. Phys. J C **80**, 840 (2020)
`arXiv:1911.11411`

Paper V [5]
T. Ohlsson and *M. Pernow*
Flavor symmetries in the Yukawa sector of non-supersymmetric SO(10): numerical fits using renormalization group running
J. High Energy Phys. **09**, 111 (2021)
`arXiv:2107.08771`

List of papers not included in this thesis

Paper VI [6]
T. Ohlsson, *M. Pernow* and E. Sönnerlind
Realizing unification in two different SO(10) models with one intermediate breaking scale
Eur. Phys. J. C **80**, 1089 (2020)
`arXiv:2006.13936`

Paper VII [7]
M. Lindestam, T. Ohlsson and *M. Pernow*
Flavor Symmetries in an SU(5) Model of Grand Unification
`arXiv:2110.09533`

The thesis author's contribution to the papers

I participated in the scientific work as well as in the writing of all papers included in this thesis. I am also the corresponding author of all the papers.

I I modified large parts of the code and ran all numerical computations. All figures were produced by me and I wrote the majority of the paper.

II All calculations were performed by me and I wrote the code for the numerical computations. I produced all figures and wrote large parts of the paper.

III I performed all numerical computations, produced all figures, and wrote most of the paper.

IV I performed all computations, produced all the figures, and wrote most of the paper.

V I performed all numerical computations and wrote a large part of the paper.

Acknowledgements

First and foremost, I would like to thank Tommy Ohlsson for taking me on as a PhD student. I am grateful to him for the opportunity to do research in theoretical particle physics and for everything I have learnt during my time at KTH, both about physics and about research in general. He is also the most meticulous proof-reader I have encountered and his comments have improved my writing immensely. Nevertheles, I am fully responsible for any errors in this thesis that I have managed to sneak past him.

I am also grateful to all the teachers that have shown me the joy and beauty of physics and mathematics—from all those who answered my annoying questions as a child and provided me with puzzles to think about, to those that I have had the pleasure to learn from through research. In reversed roles, I have learnt a lot from the students who have attended my classes in Mathematical Methods, Special Relativity, and Theoretical Particle Physics and challenged me with their questions. I have probably learnt more from teaching the subjects than from attending classes myself. In particular, I would like to thank Erik Sönnerlind and Malte Lindestam, whose Master's theses I have co-supervised. I can only hope that the future will continue to bring such great opportunties to learn.

Thank you to all the current and former members of the Division for Particle and Astroparticle Physics and the former Department of Theoretical Physics. In particular, I am thankful to Sofiane for everything he taught me about particle physics and model building, to Mattias for all the hours spent in the gym together, and to my office mates Stefan, Mózsi, Nirmal, and Rakhee for making the office such a nice place to work. Thank you to everyone else for the lunchtime discussions, after-work beers, and most importantly the fika.

For financial support, I am grateful to Stiftelsen Olle Engkvist Byggmästare for funding the first two years of my PhD. I also thank Roland Gustafssons Stiftelse för teoretisk fysik and Signeuls Stiftelse for travel scholarships. The computations that make up a large part of this thesis were performed using resources provided by Swedish National Infrastructure for Computing (SNIC) at PDC Center for High Performance Computing (PDC-HPC) at KTH Royal Institute of Technology.

I owe a great deal to my family for their continual support during my studies. Thank you to mamma and pappa for their support and encouragement. Thank you to Jonathan for unapologetically questioning the ways of the world. Thank you to mormor and morfar for their encouragement, and especially to morfar for inspiring me to study physics in the first place. Last, but certainly not least, thank you Lara. Your relentless love and support has made all the difference. This thesis is dedicated to you.

Contents

Part I

Introduction and background material

Chapter 1

Introduction

Physics is the science through which we aim to understand the natural world. Specifically, it deals with the study of matter, energy, space, and time, as well as the relations between these concepts. To understand Nature means to observe physical phenomena and describe these in mathematical models, from which one can predict the future behavior of similar systems and verify or falsify the predictions. Furthermore, progress can be made by unifying several such models into one more fundamental model which describes several phenomena in one framework. Although it may seem as though this hints at an explanation rather than a description of Nature, it remains a description rather than an explanation since the models originate in observations.

A historically successful guiding principle in the search for more fundamental descriptions of Nature has been the concept of symmetries. In addition to the objective advantage of leading to more compact mathematical descriptions of several phenomena, they also have a subjective aesthetic appeal. This combination of objective advantage and subjective appeal makes it tempting to view symmetries as leading to fundamentally true models. However, the success of a theory always has to be judged by how well the predictions agree with reality. As Richard Feynman said, "It doesn't matter how beautiful your theory is, it doesn't matter how smart you are. If it disagrees with experiment, it's wrong."

Related to symmetries is the concept of unification of various phenomena. Sir Isaac Newton unified the celestial mechanics with the mechanics of everyday objects that we observe on Earth by suggesting that the same mathematical descriptions apply to both. James Clerk Maxwell unified the phenomena of electricity and magnetism into one theory of electromagnetism. A feature of Maxwell's theory is symmetry under relative motion, such that two observers moving relative to each other observe the same physics and, particularly, measure the same value of the speed of light. This symmetry is central in Albert Einstein's theory of relativity, which unifies space and time into spacetime.

Particle physics is the branch of physics that deals with the description of Nature at the shortest length scales and highest energy scales. It aims to understand the elementary particles that make up the world around us and the processes in which they partake. As such, it classifies the particles that have been observed and the ways in which they interact through the electromagnetic, the strong nuclear, and the weak nuclear forces. The electromagnetic force is responsible for electric and magnetic phenomena such as binding electrons and nuclei into atoms and atoms into molecules, the strong nuclear force is responsible for binding quarks into protons and neutrons as well as binding protons and neutrons into nuclei, and the weak nuclear force is responsible for β-decay. The fourth force, gravity, is typically not relevant for particle physics since it is much weaker than the other forces. For example, the electrostatic force between two electrons is about 43 orders of magnitude larger than the gravitational force between them.

The current framework for understanding the particles and their interactions is the Standard Model (SM). It classifies the quantum fields corresponding to the various particles based on how they interact with the three forces. These forces are mediated by virtual particles, which are also described in terms of quantum fields. The mathematical model that describes these interactions relies heavily on a type of symmetry called gauge symmetry, which is such that there exist local transformations of the quantum fields that leave the mathematical model unchanged. A necessary consequence of gauge symmetry is that the quantum fields interact with each other. In this way, the existence of gauge symmetries necessarily leads to interactions between the quantum fields, which manifests itself as interactions of matter with the fundamental forces.

In order for the gauge symmetries that give rise to the electromagnetic and weak nuclear forces to be fully consistent with observations, these two symmetries have to be unified into what is known as the electroweak symmetry. Through the Higgs mechanism, the electroweak symmetry is broken to give the two separate phenomena. In this way, the electromagnetic and weak nuclear forces are two different manifestations of the same gauge symmetry.

Evidence for the success of the SM and its underlying principles is provided by predictions that led to the discoveries of new particles. An example is the discovery of the charm quark from the J/ψ resonance in 1974 by both SLAC and BNL [8, 9]. The strong force was further supported by the discovery of the gluon at DESY in 1979 [10, 11]. In 1995, the top quark was finally discovered at Tevatron [12, 13], thus completing the list of quarks. In the electroweak sector, the W and Z bosons were discovered at CERN in 1983 [14–17]. The most recent discovery, which provided the final piece of the SM, was the Higgs boson, discovered by ATLAS and CMS at CERN in 2012 [18, 19].

Based on the success of symmetries and unification in the SM, it is tempting to take it further and unify the electroweak theory and the strong nuclear force into one model with a larger and more unified symmetry as its basis. This is called a Grand Unified Theory (GUT) and is the subject of this thesis. Unifying the three forces into one would imply that all forces but gravity are, in fact, different manifestations of the same force. This would extend the SM and provide us with phenomena beyond those predicted by the SM. The fact that gravity is not included, however, means that grand unification does not claim to provide a final theory, but is a step towards a theory of particle physics that appears more fundamental.

Except for the purely aesthetic motivations for unifying the SM into a GUT, there are several more pragmatic motivations for extending our theories beyond the SM. Despite its successes, it has several shortcomings and open questions, some of which can be addressed naturally within the framework of GUTs.

As always, the success of a model is decided by its agreement with experiments. Some of the data that are available are the known parameters of the SM, as well as constraints stemming from the fact that protons have not been observed to decay. Hence, the work that this thesis is based on has tested the viability of various types of GUT models using numerical fits to these known parameters as well as the constraints due to the non-observation of proton decay.

1.1 Outline

This thesis is divided into two parts. Part I contains a description of the theory related to the research work and an introduction to the papers. This is divided into six chapters. In Ch. 2, we introduce the main features of the SM as well as some of its shortcomings and open questions. Next, in Ch. 3, we describe in more detail the specifics of one of the most well-established shortcomings of the SM, namely neutrino mass. Then, in Ch. 4, we discuss the framework of GUTs, paying special attention to those based on the SO(10) gauge group, and motivate them as natural extensions of the SM. Ch. 5 contains a description of renormalization as well as some of the numerical methods used to investigate the viability of GUTs in the research papers. Finally, in Ch. 6, we conclude this thesis. Following this introduction to the subject, Part II contains the papers upon which this thesis is based.

Chapter 2

The Standard Model

The Standard Model is the current description of particle physics. It has so far been extraordinarily powerful in predicting experimental results at the LHC and other particle physics experiments. However, there are many open questions associated with it. This chapter will summarize the features of the SM relevant for the work in this thesis and outline some of the open problems that suggest physics beyond the SM.

2.1 Overview of the Standard Model

The SM is a quantum field theory (QFT) that describes the particles and interactions observed in terms of interactions of quantum fields. The mathematical formulation is in terms of a Lagrangian density $\mathcal{L}_{\mathrm{SM}}$ which is a function of the fields and their derivatives. It is invariant under certain symmetry transformations that are local non-Abelian gauge transformations, making the SM a Yang–Mills (YM) theory [20]. The Lie group that this symmetry belongs to is[1]

$$\mathcal{G}_{\mathrm{SM}} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{Y}. \tag{2.1}$$

The labels C ("color"), L ("left"), and Y (hypercharge) describe the quantum numbers relevant to each of these gauge groups, which are carried by each field. The $\mathrm{SU}(3)_{\mathrm{C}}$ factor is responsible for the strong nuclear force through the theory of quantum chromodynamics (QCD) [23–25]. The remaining $\mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{Y}$ factor is the electroweak theory [26–29], which is responsible for the electromagnetic and weak nuclear forces.

[1] The more general gauge group of the SM is $\mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{Y}/\mathcal{Z}$, where $\mathcal{Z}$ is one of the subgroups of $\mathbb{Z}_6$, namely $\mathbb{Z}_6$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, or $\mathbb{1}$ [21]. This does not influence results of typical particle interactions, which only rely on the local structure, but is relevant for topological defects such as monopoles, which depend on the global group structure [22].

To specify the model, one must determine the particle content and the transformation properties of each corresponding field under $\mathcal{G}_{\text{SM}}$. Then, the conditions of Lorentz invariance, invariance under $\mathcal{G}_{\text{SM}}$, and renormalizability determine the theory by dictating the allowed terms in the Lagrangian density.

2.1.1 Gauge bosons

The force carriers through which the SM gauge interactions occur correspond to gauge bosons, which are spin-1 Lorentz vectors. In a YM theory, these are defined from the gauge group of the model. Specifically, each generator of the gauge group corresponds to a gauge boson.

There is one vector field B_μ corresponding to $\text{U}(1)_Y$, since its generator acting on a field is simply the hypercharge of that field. There are three weak gauge bosons W^i_μ, with $i \in \{1, 2, 3\}$, corresponding to the three generators $t^i = \frac{1}{2}\sigma^i$ of $\text{SU}(2)_\text{L}$, where σ^i are the Pauli matrices. In QCD, there are eight gauge bosons, the gluons G^a_μ, with $a \in \{1, 2, \ldots, 8\}$, corresponding to the eight generators $t^a = \frac{1}{2}\lambda^a$ of $\text{SU}(3)_\text{C}$, where λ^a are the Gell-Mann matrices. The normalization of the generators for both SU(2) and SU(3) is such that

$$\text{Tr}\left(t^m t^n\right) = \frac{1}{2}\delta^{mn}. \tag{2.2}$$

From these vector fields, we form the gauge invariant field strength tensors[2]

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \tag{2.3}$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \epsilon^{ijk} W^j_\mu W^k_\nu, \tag{2.4}$$

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_3 f^{abc} G^b_\mu G^c_\nu, \tag{2.5}$$

where g_2 and g_3 are the $\text{SU}(2)_\text{L}$ and $\text{SU}(3)_\text{C}$ gauge coupling constants, respectively, and ϵ^{ijk} and f^{abc} are the structure constants of SU(2) and SU(3), respectively. The kinetic terms in the Lagrangian density for the gauge fields are

$$\mathcal{L}_\text{gauge} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i,\mu\nu} - \frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu}. \tag{2.6}$$

From this and the definitions of the field strength tensors in Eqs. (2.3)–(2.5), it is evident that, in contrast to the hypercharge gauge boson, the gluons and the weak gauge bosons have self-interactions via cubic and quartic couplings. Group theoretically, this is a consequence of the non-Abelian nature of these gauge groups, resulting in the gauge bosons carrying charges under the corresponding groups.

[2] We employ the slight abuse of notation that the rank-two tensor $A_{\mu\nu}$ is the field strength tensor corresponding to the vector field A_μ denoted by the same symbol. These two should not be confused.

2.1.2 Fermionic particle content

Matter is composed of spin-1/2 fermions, which are the quarks and leptons. They can be organized into three generations, which have identical properties under the SM gauge group and only differ in their masses. In the first generation, the leptons are the electron and the electron neutrino, and the quarks are the up quark and the down quark. Correspondingly, the second generation contains the muon and the muon neutrino as well as the charm quark and the strange quark. Similarly, the third generation contains the tau, the tau neutrino, the top quark and the bottom quark.

Mathematically, they are described by spinors ψ, which may be decomposed into left-handed and right-handed chiral components using the projection operators

$$P_{\mathrm{L}} = \frac{\mathbb{1} - \gamma^5}{2}, \quad P_{\mathrm{R}} = \frac{\mathbb{1} + \gamma^5}{2}, \tag{2.7}$$

where $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$, in which the γ^μ are the Dirac matrices satisfying the Clifford algebra defined by $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$, with $g^{\mu\nu}$ being the Minkowski metric. Using these, the left- and right-handed chiral components can then be calculated as $\psi_{\mathrm{L}} = P_{\mathrm{L}}\psi$ and $\psi_{\mathrm{R}} = P_{\mathrm{R}}\psi$. The SM is a chiral theory, meaning that the two chiral components of the fermions transform differently under $\mathcal{G}_{\mathrm{SM}}$.

The quarks carry quantum numbers under all three of the gauge groups of the SM. They come in three different color charges, called red, green, and blue, and transform as the **3** representation of $\mathrm{SU}(3)_{\mathrm{C}}$. Under $\mathrm{SU}(2)_{\mathrm{L}}$ transformations, the left-handed u_{L} and d_{L} quarks of the first generation form a doublet, $Q_{\mathrm{L}} = (u_{\mathrm{L}} \quad d_{\mathrm{L}})^T$, while the two right-handed u_{R} and d_{R} are singlets. Thus, the first generation of quarks are organized as

$$Q_{\mathrm{L}} = \begin{pmatrix} u_{\mathrm{L}}^r & u_{\mathrm{L}}^g & u_{\mathrm{L}}^b \\ d_{\mathrm{L}}^r & d_{\mathrm{L}}^g & d_{\mathrm{L}}^b \end{pmatrix} \sim (\mathbf{3}, \mathbf{2})_{1/6},$$
$$(u_{\mathrm{R}}^r \quad u_{\mathrm{R}}^g \quad u_{\mathrm{R}}^b) \sim (\mathbf{3}, \mathbf{1})_{2/3}, \quad (d_{\mathrm{R}}^r \quad d_{\mathrm{R}}^g \quad d_{\mathrm{R}}^b) \sim (\mathbf{3}, \mathbf{1})_{-1/3},$$

where the numbers within the parentheses give their representations under $\mathrm{SU}(3)_{\mathrm{C}}$ and $\mathrm{SU}(2)_{\mathrm{L}}$, and the hypercharge is given in the subscript. The next generation second and third generations follow a similar pattern but with u and d replaced with c and s or t and b.

The leptons have a similar structure, except that they are all neutral under $\mathrm{SU}(3)_{\mathrm{C}}$ and there are no right-handed neutrinos in the SM. In the first generation, the left-handed components of the electron e and its corresponding electron neutrino ν_e belong to an $\mathrm{SU}(2)_{\mathrm{L}}$ doublet, while the right-handed component of e is an $\mathrm{SU}(2)_{\mathrm{L}}$ singlet. Hence they are organized as

$$L_{\mathrm{L}} = \begin{pmatrix} \nu_{e\mathrm{L}} \\ e_{\mathrm{L}} \end{pmatrix} \sim (\mathbf{1}, \mathbf{2})_{-1/2},$$
$$e_{\mathrm{R}} \sim (\mathbf{1}, \mathbf{1})_{-1}.$$

The second and third generations again follow the same pattern, but with e and ν_e replaced by μ and ν_μ or τ and ν_τ.

Thus, each generation of fermions in the SM contains 15 Weyl spinors, with all right-handed fermions transforming as singlets under $\mathrm{SU}(2)_\mathrm{L}$. In other words, the weak interaction only couples to the left-handed components of the fermions. This explains the meaning of the subscript L for "left".

For any spinor field ψ, the kinetic term in the Lagrangian density is

$$\mathcal{L}_\mathrm{Dirac} = \mathrm{i}\overline{\psi}\gamma^\mu\partial_\mu\psi, \tag{2.8}$$

where ∂_μ is a spacetime derivative. To make gauge invariance explicit, we write the fermion fields as multiplets under $\mathcal{G}_\mathrm{SM}$ and denote them as Ψ. Then, to maintain gauge invariance, the term in the Lagrangian density becomes

$$\mathcal{L}_\mathrm{Dirac} = \mathrm{i}\overline{\Psi}\gamma^\mu D_\mu\Psi, \tag{2.9}$$

where D_μ is the covariant derivative, defined as

$$D_\mu \equiv \partial_\mu - \mathrm{i}g_3 G^a_\mu t^a - \mathrm{i}g_2 W^i_\mu t^i - \mathrm{i}g_1 Y B_\mu, \tag{2.10}$$

with the coefficients g_3, g_2, and g_1 being the coupling constants corresponding to their respective gauge fields. In this way, we see that the gauge interactions of the fermions are uniquely determined from the requirement of gauge invariance.

2.1.3 The Higgs mechanism

The final piece of the SM was confirmed in 2012 by the discovery of the Higgs boson by the ATLAS [18] and CMS [19] collaborations. To allow the fermions and electroweak gauge bosons to obtain their experimentally measured masses, the $\mathrm{SU}(2)_\mathrm{L}$ symmetry must be broken. Accomplishing this requires the Higgs field [30–33], which is a complex scalar $\mathrm{SU}(2)_\mathrm{L}$ doublet,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \sim (\mathbf{1},\mathbf{2})_{1/2}.$$

It has a kinetic and potential term in the Lagrangian density,

$$\mathcal{L}_\mathrm{Higgs} = (D_\mu\Phi)^\dagger(D^\mu\Phi) - V(\Phi), \tag{2.11}$$

where the Higgs potential is

$$V(\Phi) = -\mu^2(\Phi^\dagger\Phi) + \frac{\lambda}{4}(\Phi^\dagger\Phi)^2. \tag{2.12}$$

Minimizing this potential gives

$$(\Phi^\dagger\Phi)_\mathrm{min} = \frac{2\mu^2}{\lambda}. \tag{2.13}$$

Thus, the Higgs field acquires a vacuum expectation value (vev) of $\langle|\Phi|\rangle = \sqrt{2\mu^2/\lambda} \equiv v/\sqrt{2}$ with $v \approx 246\,\mathrm{GeV}$ [34]. Perturbation expansions must then be performed with this value taking the place of the vacuum. Expanding about the vev, the Higgs field may be written as

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi^+ \\ v + h + \mathrm{i}\sigma \end{pmatrix}, \tag{2.14}$$

up to $\mathrm{SU}(2)_\mathrm{L}$ transformations. Here, h and σ are real scalar fields, which are excitations around the (also real) vev v.

Since the vacuum is at a point in field space that is not invariant under $\mathrm{SU}(2)_\mathrm{L}$ transformations, the symmetry is spontaneously broken as

$$\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_Y \to \mathrm{SU}(3)_\mathrm{C} \times \mathrm{U}(1)_Q, \tag{2.15}$$

where Q is the electric charge given by the Gell-Mann–Nihshijima formula [35, 36]

$$Q = T_3 + Y, \tag{2.16}$$

with T_3 being the third generator of $\mathrm{SU}(2)_\mathrm{L}$.

Out of the four generators of the electroweak gauge group, three are broken and one combination remains unbroken, corresponding to electromagnetic $\mathrm{U}(1)_Q$ symmetry. Thus, there are three Nambu–Goldstone bosons [37–39], which are the two components of the complex ϕ^+ and σ in Eq. (2.14). They are absorbed ("eaten") by the gauge bosons corresponding to the broken generators and become their longitudinal polarization degrees of freedom [33]. This is the mechanism through which the gauge bosons corresponding to the broken generators acquire mass terms.

Since we expand the Higgs field about its vev as in Eq. (2.14), the covariant derivative term in the Higgs Lagrangian density Eq. (2.11) contains the terms

$$\mathcal{L}_\mathrm{Higgs} \supset \frac{v^2}{8} \left[g_2^2 W_\mu^1 W^{1,\mu} + g_2^2 W_\mu^2 W^{2,\mu} + \left(g_2 W_\mu^3 - g_1 B_\mu \right) \left(g_2 W^{3,\mu} - g_1 B^\mu \right) \right]. \tag{2.17}$$

These mass terms can be diagonalized to give the physical vector bosons and their masses. The three massive gauge bosons corresponding to the W_μ^+, W_μ^-, and Z_μ^0 (where the superscripts refer to their electric charge) are

$$W_\mu^\pm = \frac{1}{\sqrt{2}} \left(W_\mu^1 \mp \mathrm{i} W_\mu^2 \right), \quad Z_\mu^0 = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_2 W_\mu^3 - g_1 B_\mu \right), \tag{2.18}$$

with tree-level masses

$$M_W = \frac{g_2 v}{2}, \quad M_Z = \sqrt{g_1^2 + g_2^2}\, \frac{v}{2}. \tag{2.19}$$

The fourth gauge boson is the one that is orthogonal to Z^0, namely

$$A_\mu = \frac{1}{\sqrt{g_1^2 + g_2^2}} \left(g_1 W_\mu^3 + g_2 B_\mu \right). \tag{2.20}$$

It remains massless and corresponds to the gauge boson of $\mathrm{U}(1)_Q$, which is the photon.

It is customary to define the mixing angle between W_μ^3 and B_μ, called the Weinberg angle, as

$$\tan\theta_W = \frac{g_1}{g_2}. \tag{2.21}$$

Then, the physical electrically neutral states can be written as

$$Z_\mu^0 = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu, \tag{2.22}$$

$$A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu. \tag{2.23}$$

The ratio of masses between the heavy gauge bosons in terms of the Weinberg angle is given by

$$\frac{M_W}{M_Z} = \cos\theta_W. \tag{2.24}$$

2.1.4 Fermion masses

With the introduction of the Higgs boson, we now also have the Yukawa terms which couple the Higgs boson to the fermions. As the symmetry is spontaneously broken and the Higgs field acquires a vev, it will generate mass terms for the fermions. Without the Higgs field, this would not be present in the theory and the fermions would not have mass.

With a slight abuse of notation, let $i \in \{1, 2, 3\}$ label the generation, and let Q_L^i be the left-handed quark doublets, u_R^i the vector of right-handed up, charm, and top quarks, let d_R^i the vector of right-handed down, strange, and bottom quarks. Similarly for the leptons, let L_L^i be the lepton doublets and ℓ_R^i the vector of right-handed electrons, muons, and taus.

Then, the Lagrangian density for the Yukawa interactions is

$$\mathcal{L}_\mathrm{Yukawa} = -Y_d^{ij}\overline{Q_\mathrm{L}^i}\Phi d_\mathrm{R}^j - Y_u^{ij}\overline{Q_\mathrm{L}^i}\widetilde{\Phi} u_\mathrm{R}^j - Y_\ell^{ij}\overline{L_\mathrm{L}^i}\Phi \ell_\mathrm{R} + \text{h.c.}, \tag{2.25}$$

where Y_d^{ij}, Y_u^{ij}, and Y_ℓ^{ij} are dimensionless components of general complex Yukawa coupling matrices and $\widetilde{\Phi} = \mathrm{i}\sigma_2\Phi$.

As the Higgs field acquires a vev, the fermions are given masses proportional to the Higgs vev and the Yukawa couplings. The mass matrices are

$$M_d = Y_d\frac{v}{\sqrt{2}}, \quad M_u = Y_u\frac{v}{\sqrt{2}}, \quad M_\ell = Y_\ell\frac{v}{\sqrt{2}}. \tag{2.26}$$

In order to extract physical masses and mass states of the fermions, we diagonalize the mass matrices. Any complex matrix may be diagonalized by a bi-unitary transformation.

$$M_d = U_{d_\mathrm{L}}^\dagger \tilde{M}_d U_{d_\mathrm{R}}, \quad M_u = U_{u_\mathrm{L}}^\dagger \tilde{M}_u U_{u_\mathrm{R}}, \quad M_\ell = U_{\ell_\mathrm{L}}^\dagger \tilde{M}_\ell U_{\ell_\mathrm{R}}, \tag{2.27}$$

in which U_{d_L}, U_{d_R}, U_{u_L}, U_{u_R}, U_{ℓ_L}, and U_{ℓ_R} are unitary matrices in generation space and $\tilde{M}_d$, $\tilde{M}_u$, and $\tilde{M}_\ell$ are diagonal mass matrices. We correspondingly rotate the fermion fields as

$$(d'_\mathrm{L})^i = (U_{d_\mathrm{L}})^{ij}(d_\mathrm{L})^j, \quad (d'_\mathrm{R})^i = (U_{d_\mathrm{R}})^{ij}(d_\mathrm{R})^j, \tag{2.28}$$

$$(u'_\mathrm{L})^i = (U_{u_\mathrm{L}})^{ij}(u_\mathrm{L})^j, \quad (u'_\mathrm{R})^i = (U_{u_\mathrm{R}})^{ij}(u_\mathrm{R})^j, \tag{2.29}$$

$$(L'_\mathrm{L})^i = (U_{\ell_\mathrm{L}})^{ij}(L_\mathrm{L})^j, \quad (\ell'_\mathrm{R})^i = (U_{\ell_\mathrm{R}})^{ij}(\ell_\mathrm{R})^j. \tag{2.30}$$

Since the left-handed quark doublet Q_L appears in two of the terms of Eq. (2.25), we must rotate its two components u_L and d_L separately in generation space, while the lepton doublet L_L only appears once, so we can rotate the whole doublet together in generation space. Hence, the same rotation in generation space is applied to both the ν_L and the ℓ_L components of the L_L doublet.

These rotations in generation space have a physical effect on the interactions with the W boson. The interaction current transforms under these rotations as

$$\begin{aligned} j^\mu_W &= \frac{1}{\sqrt{2}}(\overline{\nu_\mathrm{L}}\gamma^\mu \ell_\mathrm{L} + \overline{u_\mathrm{L}}\gamma^\mu d_\mathrm{L}) + \text{h.c.} \\ &= \frac{1}{\sqrt{2}}(\overline{\nu'_\mathrm{L}}U_{\ell_\mathrm{L}}\gamma^\mu U^\dagger_{\ell_\mathrm{L}}\ell_\mathrm{L} + \overline{u_\mathrm{L}}U_{u_\mathrm{L}}\gamma^\mu U^\dagger_{d_\mathrm{L}}d_\mathrm{L}) + \text{h.c.} \\ &= \frac{1}{\sqrt{2}}(\overline{\nu'_\mathrm{L}}\gamma^\mu \ell_\mathrm{L} + \overline{u_\mathrm{L}}V_\mathrm{CKM}\gamma^\mu d_\mathrm{L}) + \text{h.c.}, \end{aligned} \tag{2.31}$$

where we have arrived at the Cabibbo–Kobayashi–Maskawa (CKM) matrix [40, 41], defined as

$$V_\mathrm{CKM} = U_{u_\mathrm{L}}U^\dagger_{d_\mathrm{L}}. \tag{2.32}$$

This matrix relates the mass eigenstates of the quarks to their eigenstates in the weak interaction basis and gives rise to the phenomenon of quark mixing in weak interactions. Since the left-handed neutrinos and charged leptons were rotated by the same unitary matrix, no such mixing matrix appeared in the leptonic interaction current.

The CKM matrix is a unitary 3×3 matrix. It therefore has nine real parameters, three of which are mixing angles and six of which are complex phases. However, we have the freedom of absorbing five of these phases into the six quark fields u^i_L and d^i_L. The last phase cannot be removed in this way, since it can be taken to be a global redefinition of the phases of the quark fields. We are thus left with one complex phase in the CKM matrix and three real mixing angles. There are different ways of parametrizing such a matrix, and we will use the standard parametrization [34]

$$\begin{aligned} V_\mathrm{CKM} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ 0 & 1 & 0 \\ -s_{13}\mathrm{e}^{\mathrm{i}\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix}, \end{aligned} \tag{2.33}$$

where $s_{ij} \equiv \sin\theta^q_{ij}$ and $c_{ij} \equiv \cos\theta^q_{ij}$ for the three mixing angles θ^q_{12}, θ^q_{13}, and θ^q_{23}, and δ is the complex phase. This phase gives rise to $\mathcal{CP}$ violation in hadronic processes, such as different decay rates for B^0 and $\overline{B^0}$ mesons or mixing in the $K^0 - \overline{K^0}$ system.

2.1.5 Parameters of the Standard Model

The complete Lagrangian density may now be constructed from the different parts discussed above,[3] *i.e.*

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{Higgs}} + \mathcal{L}_{\text{Yukawa}}, \tag{2.34}$$

where $\mathcal{L}_{\text{gauge}}$ can be found in Eq. (2.6), $\mathcal{L}_{\text{Dirac}}$ in Eq. (2.9), $\mathcal{L}_{\text{Higgs}}$ in Eq. (2.11), and $\mathcal{L}_{\text{Yukawa}}$ in Eq. (2.25).

There are 18 parameters in total in this Lagrangian.[4] These are: the three gauge couplings, the Higgs vev and quartic coupling λ, the nine Yukawa couplings of the quarks and leptons, the three CKM mixing angles, and the $\mathcal{CP}$-violating phase of the CKM matrix. Their values are given in Tab. 2.1. The particle content of the SM is summarized in Tab. 2.2.

2.2 Open questions and some proposed solutions

Although the SM is a highly successful theory of particles and their interactions, there are still a number of open questions and shortcomings of the SM. They can be divided into those that derive from disagreements with observations and those that have a more aesthetic nature, in which the SM appears to have an accidental property that suggests a more fundamental underlying model.

2.2.1 Observational problems

The main shortcomings of the SM are that neutrinos, while massless in the SM, are known to be massive, that the observed dark matter abundance is not accounted for in the SM, as well as the fact that the observed baryon asymmetry cannot be produced without extending the SM. Furthermore, there are some hints of new physics in the muon anomalous magnetic moment and in hadronic decay observables. In this section, we briefly review these and discuss some proposed solutions.

Neutrino masses

Due to the absence of right-handed neutrinos in the SM, there is no term in the Yukawa Lagrangian of Eq. (2.25) through which the neutrinos obtain mass in the

[3] There are also additional terms relating to gauge-fixing and ghosts, but these are not relevant for the rest of this thesis.

[4] Neglecting the QCD $\mathcal{CP}$-violating parameter θ.

Parameter	Value
g_1	0.461
g_2	0.652
g_3	1.22
v	246 GeV
λ	0.516
y_e	2.79×10^{-6}
y_μ	5.90×10^{-4}
y_τ	1.00×10^{-2}
y_u	7.80×10^{-6}
y_c	3.65×10^{-3}
y_t	0.990
y_d	1.66×10^{-5}
y_s	3.10×10^{-4}
y_b	1.65×10^{-2}
θ^q_{12}	0.227
θ^q_{13}	3.71×10^{-3}
θ^q_{23}	4.18×10^{-2}
δ	1.14

Table 2.1: Parameters of the SM, evaluated at the energy scale M_Z in the $\overline{\text{MS}}$ scheme. The coupling constants g_i and Higgs parameters v and λ are taken from Ref. [34] and the Yukawa couplings y_i are taken from Ref. [42]. The quark mixing parameters are taken from the 2019 update of Ref. [43]. All other parameters, *e.g.* θ_W or the gauge boson masses may be derived from these.

same way as the other fermions. However, the phenomenon of neutrino oscillations [44] between flavor eigenstates during their propagation requires that at least two of the neutrino mass eigenstates have non-zero mass. These oscillations are a result of the propagating mass eigenstates being linear combinations of the flavor eigenstates in which production and detection of neutrinos occur. More details on neutrino oscillations and mechanisms for generating neutrino masses can be found in Ch. 3.

	Field	Representation
Gauge bosons	B_μ	$(\mathbf{1},\mathbf{1})_0$
	W_μ	$(\mathbf{1},\mathbf{3})_0$
	G_μ	$(\mathbf{8},\mathbf{1})_0$
Fermions	L^i_{L}	$(\mathbf{1},\mathbf{2})_{-1/2}$
	ℓ^i_{R}	$(\mathbf{1},\mathbf{1})_{-1}$
	Q^i_{L}	$(\mathbf{3},\mathbf{2})_{1/6}$
	u^i_{R}	$(\mathbf{3},\mathbf{1})_{2/3}$
	d^i_{R}	$(\mathbf{3},\mathbf{1})_{-1/3}$
Scalar bosons	Φ	$(\mathbf{1},\mathbf{2})_{1/2}$

Table 2.2: Particle content of the SM. The index $i \in \{1,2,3\}$ labels the generation of the given fermion.

Dark matter

Another shortcoming of the SM is that it does not contain any candidate for dark matter (DM), while its existence has been verified through several astrophysical and cosmological observations. These include galactic rotation curves [45–48], gravitational lensing [49], and the cosmic microwave background radiation [50, 51]. The current best fit suggests a DM density of [51]

$$\Omega_{\mathrm{DM}} h^2 \approx 0.12, \tag{2.35}$$

where h is the Hubble constant in units of 100 km/s/Mpc. To explain this abundance requires postulating a new type of particle beyond the SM. There many such candidates, including sterile neutrinos [52], axions [53–55] (see also the strong $\mathcal{CP}$ problem below), and the lightest supersymmetric partner [56].

Baryon asymmetry

Also of a cosmological origin is the baryon asymmetry of the Universe (BAU). The asymmetry between the number density of baryons and antibaryons, normalized to the photon number density, has been measured to be [51]

$$\eta_{\mathrm{B}} \equiv \frac{n_{\mathrm{B}} - n_{\overline{\mathrm{B}}}}{n_\gamma} \approx 6.14 \times 10^{-10}. \tag{2.36}$$

To produce such an asymmetry, there needs to exist processes that satisfy the three Sakharov conditions [57], namely that there should be baryon number violation,

$\mathcal{C}$- and $\mathcal{CP}$-violation, as well as out-of-equilibrium interactions. In particular, the second of these is not sufficiently provided by the SM [58], making the BAU a phenomenon of physics beyond the SM. One proposed mechanism to produce the baryon asymmetry is through leptogenesis, in which the asymmetry is generated in the lepton sector and then transferred to the baryon sector [59].

Other hints

There is also a discrepancy between SM predictions and measurements of the muon anomalous magnetic moment at the level of 4.2σ [60]. If this tension persists, it would signal new physics which enters at some higher energy scale less than about 10^3 TeV [61], set by unitarity bounds. This may be in the form of new scalar leptoquark bosons [62], or an additional gauge boson from a gauged $\mathrm{U}(1)_{L_\mu - L_\tau}$ symmetry [63, 64] to name a few proposals. It should also be noted that this tension may decrease in the future with more precise theoretical calculations, especially from lattice QCD [65].

Furthermore, there are a number of deviations of observables related to leptonic B-meson decay compared to the predicted SM values. In particular the branching ratio of decays to muons compared to electrons exhibits a tension at the level of 3.1σ [66]. Similarly to the muon magnetic moment tension, it may be resolved by new physics at a higher energy scale, which should be below about 80 TeV [67]. Examples of such physics models are (vector or scalar) leptoquarks [68] or a gauge boson from an additional U(1) group together with additional colored fermions [69].

2.2.2 Aesthetic problems

Aside from the observational problems in the SM, there are certain features of the SM that appear somewhat *ad hoc* and suggest that there exist a more fundamental description of Nature in which these features appear naturally.

Structure of the SM

Firstly, we may ask why the gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ is the one that describes Nature. Secondly, there is the question of why there are three generations of fermions with identical quantum numbers and successively larger masses. Thirdly, it is unclear why the fermions have the charges that they do, and why the hypercharge (and hence electric charge) is quantized.

Anomaly cancellation

Since the SM is a gauge theory, the gauge anomalies [70, 71] need to vanish in order for it to be consistent, which puts constraints on the allowed charges. With this in mind, the hypercharges seem to conspire such that the anomalies cancel exactly. Thus, the hypercharges appear to be chosen in such a way that the anomalies cancel and the SM gauge group is allowed. Although anomaly cancellation in the

SM can be related to hypercharge quantization [72–77], that does not explain why the hypercharges take the values that they do.

The flavor puzzle

The masses of the three generations of fermions in the SM are strongly hierarchical and span five orders of magnitude. Furthermore, the CKM matrix is close to diagonal, while the elements of the leptonic mixing matrix are all of similar sizes. The flavor puzzle [78–82] is then the question of the origin of the three generations of fermions, their hierarchical masses, and the relations between elements of the CKM and leptonic mixing matrices.

There are many proposed models to address this puzzle. One is the Froggatt–Nielsen (FN) mechanism [83], which introduces a new symmetry, $\mathrm{U}(1)_{\mathrm{FN}}$ under which the different generations of the fermions have different charges, as well as an additional scalar field η, known as a flavon field. Let the ith generation of fermion ψ, the flavon, and the Higgs boson carry charges under $\mathrm{U}(1)_{\mathrm{FN}}$ as

$$\mathrm{FN}(\psi_{\mathrm{L},i}) = n_i q, \quad \mathrm{FN}(\psi_{\mathrm{R},i}) = -n_i q, \tag{2.37}$$

$$\mathrm{FN}(\eta) = 2q, \qquad \mathrm{FN}(\Phi) = 0, \tag{2.38}$$

for some real parameter q. A Yukawa term invariant under $\mathrm{U}(1)_{\mathrm{FN}}$ would then be

$$\mathcal{L}_{\mathrm{Yukawa}} \supset -c_i \left(\frac{\eta}{\Lambda}\right)^{n_i} \overline{\psi_{\mathrm{L},i}} \psi_{\mathrm{R},i} \Phi, \tag{2.39}$$

where Λ corresponds to some energy scale relevant for a high-energy completion of this effective operator. Assigning a vev v_η to the flavon, the Yukawa couplings are

$$y_i = c_i (v_\eta/\Lambda)^{n_i}. \tag{2.40}$$

For $v_\eta < \Lambda$, and with the parameters n_i satisfying $n_1 > n_2 > n_3$, we naturally generate the hierarchy $y_1 < y_2 < y_3$ without requiring any hierarchy in the coupling constants c_i. A high-energy completion of this involves the addition of a set of fermions of mass Λ that mediate this interaction.

Similar to the FN mechanism is the clockwork mechanism, which adds several global symmetries and additional fermions to generate the observed mass spectrum [84–86]. Many other models based on both discrete and continuous flavor symmetries have also been proposed [87–93].

Since the mass spectrum of the fermions is a result of the Yukawa matrices, imposing texture zeros (that is, setting certain elements to zero) is another way of addressing the flavor puzzle [94–96]. Many of these matrix textures can be derived from family symmetries [97]. It has also been suggested that a hierarchical spectrum can be generated from products of random matrices with all elements $\mathcal{O}(1)$ [98].

Naturalness

Another unsatisfactory feature of the SM is the apparent fine-tuning of the Higgs mass. The problem is that quantum corrections due to physics at higher scales in the theory, such as the Planck scale, should place the Higgs mass at that higher scale. However, the Higgs mass has been measured to be 125 GeV [18, 19]. The hierarchy problem [99–101] is then to understand why there is this hierarchy of scales between the Higgs mass and the Planck scale despite large quantum corrections.

One way in which this could occur is if there were very precise cancellation of the contributions to the quantum corrections. Such cancellation would have to occur to a precision of more than 30 significant figures, which is considered unnatural. Another proposed solution is supersymmetry (SUSY), in which each particle has a superpartner, such that fermions have bosonic partners and bosons have fermionic partners. (For an introduction, see for example Ref. [102].) The high-scale contributions to the Higgs mass are then canceled by opposite contributions from the supersymmetric partners. Hence, the precise cancellation is imposed by symmetry and no fine-tuning is necessary.

The issue of fine-tuning also occurs in the strong $\mathcal{CP}$ problem, which notes that the SM Lagrangian density in general contains the term [103, 104]

$$\mathcal{L}_{\text{SM}} \supset \theta \frac{g_3^2}{32\pi^2} \widetilde{G}^{a,\mu\nu} G^a_{\mu\nu}, \tag{2.41}$$

where $\widetilde{G}^{a,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} G^a_{\rho\sigma}$. This term violates $\mathcal{CP}$ and would induce an electric dipole moment for the neutron. However, the non-observation of the neutron electric dipole moment implies an upper bound of $|\theta| \lesssim 10^{-10}$ [105–107]. The problem is thus one of fine-tuning: Is there any reason for this parameter to be so small? One possible solution to the strong $\mathcal{CP}$ problem is to introduce a Peccei–Quinn symmetry, $\text{U}(1)_{\text{PQ}}$. The angle θ is then replaced by the axion field which is the pseudo-Nambu–Goldstone boson associated with the spontaneous symmetry breaking of $\text{U}(1)_{\text{PQ}}$ [108–111]. In that way, there is a dynamical explanation for the smallness of θ and fine-tuning is therefore not required.

Chapter 3

Neutrino masses

3.1 Evidence for neutrino mass

As mentioned in Ch. 2, neutrinos have been observed to have mass. In this chapter, we review the evidence for neutrino masses and the experimental status.

3.1.1 Neutrino oscillations

If neutrinos are massive, then they appear in the Lagrangian both in a mass term and in a gauge interaction term. In analogy with quarks, the mass and interaction bases may be different if the two terms are not simultaneously diagonalizable. The mismatch between them gives rise to a mixing matrix, analogous to the CKM matrix, and is observable through the phenomenon of neutrino oscillations.

Neutrinos are produced and detected through their weak interactions, which couple to the flavor basis states. However, they propagate as eigenstates of the Hamiltonian, which are the mass eigenstates. The fact that the interaction and propagation eigenbases are different causes the neutrinos to oscillate, in the sense that the detected neutrino flavor may be different from the one that was emitted [44, 112–116].

We can relate the flavor state $|\nu_\alpha\rangle$, with $\alpha \in \{e, \mu, \tau\}$, of a neutrino to its mass state $|\nu_i\rangle$, with $i \in \{1, 2, 3\}$, through

$$|\nu_\alpha\rangle = U^*_{\alpha i} |\nu_i\rangle , \tag{3.1}$$

$$|\nu_i\rangle = U_{\alpha i} |\nu_\alpha\rangle , \tag{3.2}$$

where $U_{\alpha i}$ are the elements of a unitary mixing matrix.

The mass states propagate as plane wave solutions, $|\nu_i(t)\rangle = \mathrm{e}^{-\mathrm{i}E_i t}\,|\nu_i(t=0)\rangle$, such that a flavor state evolves in time as

$$|\nu_\alpha(t)\rangle = U_{\alpha i}^* \mathrm{e}^{-\mathrm{i}E_i t}\,|\nu_i(t=0)\rangle = U_{\alpha i}^* \mathrm{e}^{-\mathrm{i}E_i t} U_{\beta i}\,|\nu_\beta(t=0)\rangle\,. \tag{3.3}$$

This results in an amplitude for transitions from flavor β to flavor α given by

$$A_{\beta\to\alpha}(t) = \langle\nu_\beta(t=0)|\nu_\alpha(t)\rangle = U_{\alpha i}^* \mathrm{e}^{-\mathrm{i}E_i t} U_{\beta i} \tag{3.4}$$

and a transition probability

$$P_{\beta\to\alpha}(t) = |A_{\beta\to\alpha}(t)|^2 = \sum_{i,j} U_{\alpha i}^* U_{\beta i} U_{\alpha j} U_{\beta j}^* \mathrm{e}^{-\mathrm{i}(E_i - E_j)t}. \tag{3.5}$$

Since the neutrinos are ultrarelativistic, we may write the energy as

$$E_i = \sqrt{|\mathbf{p}|^2 + m_i^2} \simeq E + \frac{m_i^2}{2E}, \tag{3.6}$$

with $E \simeq |\mathbf{p}|^2$. This gives the difference

$$E_i - E_j \simeq \frac{\Delta m_{ij}^2}{2E}, \tag{3.7}$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ is the mass-squared difference between two mass eigenvalues m_i and m_j. Furthermore, we can write the propgation time as $t \simeq L$ since the neutrinos travel at almost the speed of light.

With this, the transition probability can be written as

$$\begin{aligned} P_{\beta\to\alpha}(t) = &\sum_i |U_{\beta i}|^2 |U_{\alpha i}|^2 + 2\sum_{j<i} \mathrm{Re}(U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*) \cos\left(\frac{\Delta m_{ij}^2 L}{2E}\right) \\ &+ 2\sum_{j<i} \mathrm{Im}(U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*) \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right). \end{aligned} \tag{3.8}$$

In particular, we have the survival probability

$$P_{\alpha\to\alpha}(t) = 1 - 4\sum_{j<i} |U_{\alpha i}|^2 |U_{\alpha j}|^2 \sin^2\left(\frac{\Delta m_{ij}^2 L}{4E}\right), \tag{3.9}$$

which makes it evident that it is the existence of a non-zero mass-squared difference that enables neutrino oscillations.

Under a $\mathcal{CP}$ transformation, each state transforms to its antiparticle. Then, $P_{\beta\to\alpha}$ becomes $P_{\overline{\beta}\to\overline{\alpha}}$, which has the same expression except for a negative sign in front of the imaginary part of Eq. (3.8). With this, we define the $\mathcal{CP}$ asymmetry as

$$\Delta_{\beta\to\alpha}^{\mathcal{CP}} \equiv P_{\beta\to\alpha} - P_{\overline{\beta}\to\overline{\alpha}} = 4\sum_{j<i} \mathrm{Im}(U_{\beta i}^* U_{\alpha i} U_{\beta j} U_{\alpha j}^*) \sin\left(\frac{\Delta m_{ij}^2 L}{2E}\right), \tag{3.10}$$

showing that the $\mathcal{CP}$ asymmetry can only be observed in transitions, and not in measurements of the survival probability.

3.1.2 Current results and bounds

Neutrino oscillations have been confirmed by a series of experiments [117–119]. Notably, the dependence of the oscillation probability on Δm^2_{ij} implies that all three neutrino mass states must be different, such that there exists two independent mass-squared differences. In effect, this means that at least two neutrino mass states must be massive, since one is still allowed to be massless. The absolute scale of the neutrino masses cannot be probed in oscillation experiments.

Also unknown is the ordering of the masses. The three mass states are defined by the linear combinations of the flavor states through the leptonic mixing matrix. Two of the mass-squared differences have been measured, with m_1 and m_2 being close together and m_3 being further away. However, it remains an open question whether the ordering is $m_1 < m_2 < m_3$ or $m_3 < m_1 < m_2$, *i.e.* whether the large mass-squared difference is between the two lightest or two heaviest neutrinos. The two mass orderings are respectively known as normal ordering (NO) and inverted ordering (IO).

The elements of the leptonic mixing matrix are parametrized by three mixing angles $\theta^\ell_{12}, \theta^\ell_{13}, \theta^\ell_{23}$ and one phase δ^ℓ in analogy to the CKM parametrization in Eq. (2.33). The values of these parameters, displayed in Tab. 3.1, depend on which mass ordering is realized. The latest results from global fits of the various neutrino experiments favor NO over IO at a significance of 2.7σ [120].

Parameter	Value
Δm^2_{21}	$7.50 \times 10^{-5}\,\mathrm{eV}^2$
Δm^2_{31} (NO)	$2.55 \times 10^{-3}\,\mathrm{eV}^2$
Δm^2_{31} (IO)	$-2.45 \times 10^{-3}\,\mathrm{eV}^2$
$\sin\theta^\ell_{12}$	0.318
$\sin\theta^\ell_{13}$ (NO)	0.0220
$\sin\theta^\ell_{13}$ (IO)	0.0223
$\sin\theta^\ell_{23}$ (NO)	0.574
$\sin\theta^\ell_{23}$ (IO)	0.578
δ^ℓ (NO)	3.39
δ^ℓ (IO)	4.96

Table 3.1: Central values of the neutrino parameters, derived from a global fit [120]. The abbreviations NO and IO stand for normal and inverted neutrino mass ordering, respectively.

Although one cannot probe the absolute scale in oscillation experiments, it is possible to derive an upper bound on the sum of the neutrino masses from cosmological observations. Both the cosmic microwave background radiation and the baryon acoustic oscillations are sensitive to the sum of the masses of the neutrinos, resulting in a combined 95 % confidence limit of [51]

$$\sum_{i=1}^{3} m_i < 0.12 \text{ eV}. \tag{3.11}$$

The absolute scale of neutrino masses can also be probed through β-decay experiments, in which a neutron is converted into a proton and emits an electron and an electron antineutrino. One observes an energy spectrum of the emitted electrons, the endpoint of which depends on the mass of the neutrino. Since it is the electron flavor state that is produced, one obtains information on the effective electron neutrino mass

$$m_{\nu_e}^2 = \sum_{i=1}^{3} |U_{ei}|^2 m_i^2. \tag{3.12}$$

Currently, the best limit is $m_{\nu_e} < 0.8\,\text{eV}$ at 90 % confidence, provided by the KATRIN experiment [121, 122].

3.2 Neutrino mass models

3.2.1 Dirac and Majorana mass terms

In order to incorporate neutrino masses, the SM needs to be extended to include a neutrino mass term. The simplest way to achieve that would be to add a set of right-handed neutrinos ν_R^i with a Yukawa term $-Y_\nu^{ij}\overline{L_\text{L}^i}\widetilde{\Phi}\nu_\text{R}^j$. This would produce a neutrino mass matrix $M_\nu = Y_\nu \frac{v}{\sqrt{2}}$ in analogy with the other fermions, as described in Sec. 2.1.4. Then, the neutrino field $\nu = (\nu_\text{L}, \nu_\text{R})$ would form a Dirac spinor with ν_L and ν_R being independent two-component spinors, in analogy with the charged fermions.

However, the bounds on the neutrino mass from cosmology and β-decay imply that the neutrino masses are six orders of magnitude smaller than the lightest charged fermion masses. As a result, the Yukawa coupling would have to be $\mathcal{O}(10^{-11})$. While this is not forbidden, it is unnaturally small and clearly sets the neutrinos apart from the other fermions.

Many models of neutrino mass instead consider neutrinos to be Majorana particles, in which case the two chiral components are related by charge conjugation. We then have $\nu_\text{R} = (\nu_\text{L})^c$, where the superscript c denotes the charge conjugation operator, which acts on a general spinor ψ as $\psi^c \equiv C\overline{\psi}^T$, with C being the charge conjugation matrix satisfying $C^{-1}\gamma_\mu C = -\gamma_\mu^T$. If neutrinos are Majorana particles, they are their own antiparticles.

In that case, one can write a Majorana mass terms for the neutrinos as

$$\mathcal{L}_{\text{Majorana}} = -\frac{1}{2} M_{\text{L}}^{ij} \overline{\nu_{\text{L}}^{i}} \nu_{\text{L}}^{c\,i} + \text{h.c.} \tag{3.13}$$

This is a different type of mass term compared to the other fermions and cannot be generated by the Higgs mechanism. Instead, one can look for other ways in which to generate such a term in a way that ensures that the masses are naturally small.

The Majorana nature of neutrinos can be tested through neutrinoless double β-decay, which is only allowed for Majorana neutrinos. This is the process in which two β-decays occur, and the two neutrinos annihilate each other [123–125]. Several experiments are currently searching for this process, including GERDA [126], NEMO-3 [127], CUORE [128], and KamLAND-Zen [129], but it has so far not been observed. As a result, there is an upper bound on the effective mass parameter [124]

$$\langle m_{\beta\beta} \rangle \equiv \left| \sum_i U_{ei}^2 m_i \right| \lesssim 0.2\,\text{eV}. \tag{3.14}$$

3.2.2 Weinberg operator

The Majorana mass matrix for the neutrinos can be described by higher-dimensional operators [130]. At dimension five, the only operator that exists is the Weinberg operator [131]

$$\mathcal{L}_{\text{Weinberg}} = -\frac{1}{4} \kappa_{ij} (\overline{L_{\text{L}}^{c}}^{i} \Phi)(\Phi^T L_{\text{L}}^{c\,j}) + \text{h.c.} \tag{3.15}$$

The corresponding Feynman diagram is shown in the upper left panel of Fig. 3.1. After electroweak symmetry breaking, this operator gives the neutrino mass matrix

$$m_\nu = -\frac{1}{4} \kappa v^2. \tag{3.16}$$

Being an effective operator of dimension five, the coefficient κ must be inversely proportional to some energy scale Λ corresponding to the energy scale of the new physics that realizes the Weinberg operator. Typically, this is the mass of some heavy particle involved in the neutrino mass mechanism. This suppression by a heavy mass scale is a common feature of effective field theories and means that the effect coming from the high-energy theory is small at low energies. In the case of neutrino masses, this suppression of the effective operator provides a natural explanation of their smallness.

3.2.3 Seesaw mechanisms

Seesaw mechanisms are tree-level realizations of the Weinberg operator involving a heavy field. The type I seesaw mechanism [132–136] introduces three heavy right-handed neutrinos N_{R} with a Majorana mass term, resulting in the Lagrangian density

$$\mathcal{L}_{\text{seesaw-I}} = -\frac{1}{2} M_{\text{R}}^{ij} \overline{N_{\text{R}}^{ci}} N_{\text{R}}^{j} - Y_{\nu}^{ij} \overline{L_{\text{L}}^{i}} \widetilde{\Phi} N_{\text{R}}^{j} + \text{h.c.} \tag{3.17}$$

In a basis formed by the column vectors $(\nu_{\text{L}}^{c}, N_{\text{R}})^{T}$, in which both ν_{L} and N_{R} are vectors containing the three generations, the mass matrix can be written as

$$\mathcal{M}_{\nu} = \begin{pmatrix} 0 & \frac{v}{\sqrt{2}} Y_{\nu} \\ \frac{v}{\sqrt{2}} Y_{\nu}^{T} & M_{\text{R}} \end{pmatrix}. \tag{3.18}$$

After a bi-unitary transformation to block-diagonal form, one obtains the light neutrino mass matrix

$$m_{\nu}^{\text{I}} = -\frac{1}{2} v^{2} Y_{\nu} M_{\text{R}}^{-1} Y_{\nu}^{T} \tag{3.19}$$

for the left-handed neutrinos. Comparing this to the Weinberg operator, one finds that

$$\kappa = 2 Y_{\nu} M_{\text{R}}^{-1} Y_{\nu}^{T}, \tag{3.20}$$

such that M_{R} is the large energy scale that suppresses the dimension-five operator. This explains the reason behind the name “seesaw”: the smallness of the left-handed neutrino mass is a direct result of the largeness of M_{R}. The Feynman diagram for the generation of neutrino mass through the type I seesaw mechanism is shown in the upper right panel of Fig. 3.1.

In the type II seesaw mechanism [137–139], a heavy scalar field Δ that transforms as a triplet under $\text{SU}(2)_{\text{L}}$ and carries hypercharge $Y = 1$ is introduced, which allows for a Yukawa coupling of the lepton doublet with itself through this triplet. The relevant parts of the Lagrangian are

$$\mathcal{L}_{\text{seesaw-II}} \supset -\frac{1}{\sqrt{2}} Y_{\Delta}^{ij} \overline{L_{\text{L}}^{ic}} \text{i}\sigma_{2} \Delta L_{\text{L}}^{j} - \frac{1}{2} M_{\Delta}^{2} \operatorname{Tr}\left(\Delta^{\dagger}\Delta\right) - \frac{\lambda_{\Delta}}{\sqrt{2}} \Phi^{T} \text{i}\sigma_{2} \Delta \Phi + \text{h.c.}, \tag{3.21}$$

where Y_{Δ} is the Yukawa coupling matrix, M_{Δ} is the mass of the triplet, and λ_{Δ} is a coupling constant for the triplet to the Higgs boson. Electroweak symmetry breaking induces a vev

$$v_{\Delta} = \frac{\lambda_{\Delta} v^{2}}{M_{\Delta}^{2}} \tag{3.22}$$

for the electrically neutral component of the triplet. This results in the left-handed neutrinos obtaining a Majorana mass term

$$m_{\nu}^{\text{II}} = \frac{1}{2} Y_{\Delta} v_{\Delta}. \tag{3.23}$$

Since v_{Δ} is inversely proportional to M_{Δ}, a large mass for the triplet ensures that the light neutrino masses are small. In terms of the Weinberg operator, the type II seesaw mechanism results in

$$\kappa = -2 Y_{\Delta} \frac{v_{\Delta}}{v^{2}}. \tag{3.24}$$

The Feynman diagram of this mechanism is shown in the lower left panel of Fig. 3.1.

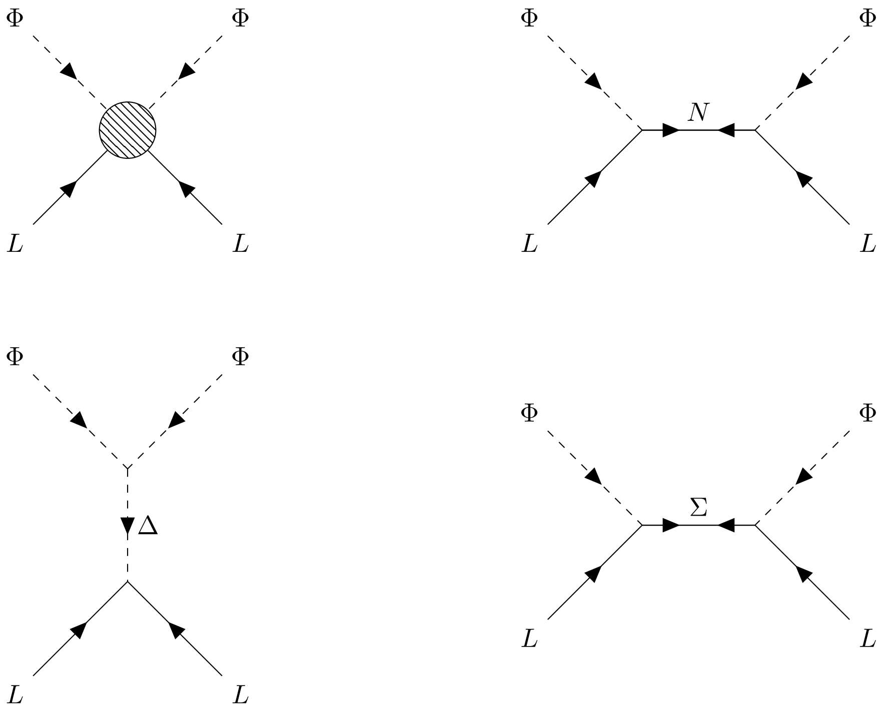


Figure 3.1: Feynman diagrams for the effective Weinberg operator (top left) as well as seesaw mechanisms of type I (top right), II (bottom left), and III (bottom right).

The type III seesaw mechanism [140] adds a number of heavy fermion fields Σ^i that transforms as a triplet under $\mathrm{SU}(2)_\mathrm{L}$ and has neutral hypercharge. Its mass term and coupling to the leptons is given by

$$\mathcal{L}_{\text{seesaw-III}} \supset -Y_\Sigma^{ij} \overline{L_\mathrm{L}^{ic}} \mathrm{i}\sigma_2 \Sigma^j \Phi - \frac{1}{2} M_\Sigma^{ij} \,\mathrm{Tr}\left(\overline{\Sigma}^i \Sigma^j\right) + \text{h.c.}, \tag{3.25}$$

where Y_Σ is the Yukawa coupling matrix and M_Σ the mass matrix of Σ. Similar to the type I mechanism, we find the light neutrino mass matrix to be given by

$$m_\nu^{\mathrm{III}} = -\frac{1}{2} v^2 Y_\Sigma M_\Sigma^{-1} Y_\Sigma^T, \tag{3.26}$$

showing that a heavy triplet Σ leads to small neutrino masses. In terms of the Weinberg operator, the type III seesaw mechanism leads to an effective neutrino mass matrix

$$\kappa = 2 Y_\Sigma M_\Sigma^{-1} Y_\Sigma^T. \tag{3.27}$$

The Feynman diagram for the type III seesaw mechanism is shown in the lower right panel of Fig. 3.1.

3.2.4 Radiative models

Another option for generating small neutrino masses is using a radiative mechanism, in which a non-zero neutrino mass matrix is generated only at loop level. The perturbation expansion suppresses loop-order contributions relative to tree-level, which naturally leads to the prediction that the neutrinos are lighter than the charged leptons. Some examples of these radiative mechanisms are the Zee model [141], the scotogenic model [142], and the Zee–Babu model [143, 144]. These and other radiative neutrino mass models are reviewed in Refs. [130, 145].

Chapter 4

Grand unified theories and SO(10)

One approach to address some of the problems and open questions of the SM discussed in Ch. 2 is to postulate a larger gauge group from which the SM gauge group arises through symmetry breaking. As such, the idea of unification exemplified by the electroweak theory is extended and the question of the choice of gauge group is directly addressed by embedding all three gauge group factors into one larger gauge group. This is the idea of a Grand Unified Theory (GUT).

The idea behind grand unification is that, at some higher energy, the gauge group factors of $\mathcal{G}_{\text{SM}} = \text{SU}(3)_{\text{C}} \times \text{SU}(2)_{\text{L}} \times \text{U}(1)_{Y}$ unify and we have only one gauge group $\mathcal{G}_{\text{GUT}}$. This unification corresponds to a unification of the three forces described by the SM. Since our observations are mostly in agreement with a model based on the gauge group $\mathcal{G}_{\text{SM}}$, we require that $\mathcal{G}_{\text{GUT}}$ has $\mathcal{G}_{\text{SM}}$ as a subgroup. Similarly to the Higgs mechanism, we assume that $\mathcal{G}_{\text{GUT}}$ spontaneously breaks to $\mathcal{G}_{\text{SM}}$ at some high energy scale. Cosmologically, higher energies correspond to earlier times, since the Universe was hotter earlier. Thus, a unified theory is more fundamental in the sense that the SM is a result of it.

Further, $\mathcal{G}_{\text{GUT}}$ should accommodate the fermion representations of the SM and its chiral structure. It is conventional to write the fermion representations as all left-handed spinors using the charge-conjugation matrix. In that notation, we have[1]

$$\begin{aligned} Q_{\text{L}} \sim (\mathbf{3}, \mathbf{2})_{1/6}, \quad (u_{\text{R}})^c \sim (\overline{\mathbf{3}}, \mathbf{1})_{-2/3}, \quad (d_{\text{R}})^c \sim (\overline{\mathbf{3}}, \mathbf{1})_{1/3}, \\ L_{\text{L}} \sim (\mathbf{1}, \mathbf{2})_{-1/2}, \quad (e_{\text{R}})^c \sim (\mathbf{1}, \mathbf{1})_{1}. \end{aligned} \tag{4.1}$$

Charge conjugation affects the chirality of the spinor such that $(\psi_{\text{R}})^c = (\psi^c)_{\text{L}}$, meaning that all spinors in Eq. (4.1) have left-handed chirality.

[1] We will in the future drop the “L” and “R” subscripts since all spinors are assumed to be left-handed.

4.1 Gauge coupling unification

In a gauge theory, each gauge group has an associated gauge coupling constant that parametrizes the strength of the interaction. The value of this constant depends on the energy scale involved in the interaction through the renormalization group equation (RGE).[2] They are thus said to "run" or "evolve" with the energy scale. For the three gauge group factors in $\mathcal{G}_{\text{SM}}$ to be unified into one, we require that the three gauge couplings of the SM meet at the energy scale at which $\mathcal{G}_{\text{GUT}}$ spontaneously breaks. Hence, the scale at which the coupling constants unify defines the scale at which the GUT is the appropriate theory [146].

Before computing the renormalization group running of the gauge couplings, we must first consider the normalization of the hypercharges of the SM fermions. When embedding $\text{U}(1)_Y$ into a larger group, the hypercharges must be normalized in a way that is consistent with the other groups in the embedding. For any representation, there is a diagonal generator—one of the Cartan generators—that acts on each state in a representation to give the hypercharge of that state. Since hypercharge is embedded together with $\text{SU}(3)_\text{C}$ and $\text{SU}(2)_\text{L}$, the normalization of the hypercharge generator must correspond to the normalization of the Cartan generators of $\text{SU}(3)_\text{C}$ and $\text{SU}(2)_\text{L}$. Of course, this will depend on how the fermions are embedded into the unified gauge group. Consider a model with all SM fermions embedded into one representation (which, as we will see, is the case for SO(10)-based models). Then the trace of the squared T_3 operator should give the same result as the trace of the squred properly normalized hypercharge operator. The trace of these two operators are

$$\text{Tr}\, T_3^2 = \left(\frac{1}{2}\right)^2 \cdot 3 + \left(\frac{-1}{2}\right)^2 \cdot 3 + \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 = 2, \tag{4.2}$$

$$\text{Tr}\, Y^2 = \left(\frac{1}{6}\right)^2 \cdot 6 + \left(\frac{-2}{3}\right)^2 \cdot 3 + \left(\frac{1}{3}\right)^2 \cdot 3 + \left(\frac{-1}{2}\right)^2 \cdot 2 + (1)^2 = \frac{10}{3}. \tag{4.3}$$

In order to have hypercharge normalized in the same way as $\text{SU}(2)_\text{L}$, we define the GUT-normalized hypercharge, related to the SM hypercharge through

$$Y_{\text{GUT}} = \sqrt{\frac{3}{5}} Y_{\text{SM}}. \tag{4.4}$$

Since the hypercharge always appears in the covariant derivative together with the gauge coupling in the combination $g_1 Y$, this product must retain its value when altering the normalization. Thus, we also redefine the gauge coupling as

$$g_{1_{\text{GUT}}} = \sqrt{\frac{5}{3}} g_{1_{\text{SM}}}. \tag{4.5}$$

In what follows, all references to the $\text{U}(1)_Y$ gauge coupling are to be understood as GUT-normalized. For clarity, we will continue to list the hypercharges themselves

[2] Details on renormalization can be found in Sec. 5.1.

with the familiar SM normalization and it is to be understood that they should be normalized appropriately before being used in calculations.

The RGE for the coupling constant g_i evaluated at one-loop order in perturbation theory is given by

$$\frac{\partial g_i}{\partial \ln \mu} = -\frac{g_i^3}{16\pi^2} a_i, \tag{4.6}$$

where a_i is a coefficient that depends on the particle content of the theory. The solution to this differential equation is conventionally given in terms of the parameter α_i, defined as

$$\alpha_i \equiv \frac{g_i^2}{4\pi}, \tag{4.7}$$

or its inverse, α_i^{-1}. With this parametrization, the solution to the RGE relating α_i^{-1} at the scale $\mu = M_2$ to the scale $\mu = M_1$ is

$$\alpha_i^{-1}(M_2) = \alpha_i^{-1}(M_1) - \frac{a_i}{2\pi} \ln\left(\frac{M_2}{M_1}\right). \tag{4.8}$$

We therefore need the values of the gauge couplings at some given energy scale. One convenient energy scale is the electroweak scale, $M_Z \simeq 91.1876\,\text{GeV}$, at which the values of the gauge couplings are [34]

$$g_1(M_Z) \simeq 0.461, \quad g_2(M_Z) \simeq 0.652, \quad g_3(M_Z) \simeq 1.22. \tag{4.9}$$

The general formula for calculating the coefficients a_i can be found in Sec. 5.1. In the SM, they are

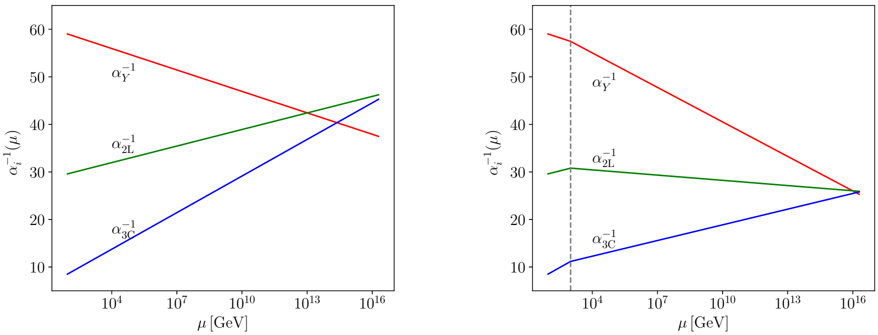
$$a_1 = \frac{41}{10}, \quad a_2 = -\frac{19}{6}, \quad a_3 = -7. \tag{4.10}$$

In the left panel of Fig. 4.1, the renormalization group running is shown in the SM. As can be seen, the gauge couplings are close to unifying, but do not quite do so in the SM. They are much closer to unifying in SUSY models, which is why these are often considered together with grand unification. In the Minimal Supersymmetric Standard Model (MSSM), the coefficients a_i are

$$a_1 = \frac{33}{5}, \quad a_2 = 1, \quad a_3 = -3. \tag{4.11}$$

The renormalization group (RG) running in the MSSM with a SUSY-breaking scale of 1 TeV is shown in the right panel of Fig. 4.1.

The near unification is a further reason to take the idea of unification seriously. We will focus on non-SUSY models in the rest of this thesis and consider other ways in which to achieve precise unification of the gauge couplings. One may imagine that in the vast so-called "desert" between M_Z and M_{GUT} spanning 14 orders of magnitude, some new physics enters. For example, this could be particles with masses somewhere in that interval which influence the RG running of the gauge couplings such that precision unification is achieved, as considered in Paper II.



(a) SM **(b)** MSSM

Figure 4.1: Renormalization group running of the gauge couplings to one-loop order in the SM (left) and the MSSM with a SUSY-breaking scale at 1 TeV (right).

There could also be a chain of symmetry breaking between the GUT and the SM which also modifies the RG running and produces precision unification, as considered in Paper I. This will be discussed in more details in Sec. 4.3.1. These two scenarios are shown in Fig. 4.2 with intermediate scale particles shown in the left panel and an intermediate symmetry breaking step shown in the right panel. In the left panel, the RG running is modified by scalars transforming as $(\mathbf{8},\mathbf{1})_1$ and $(\mathbf{8},\mathbf{3})_0$ at masses 3.1 TeV and 2.34×10^8 GeV, respectively. The right panel has an intermediate Pati–Salam (PS) breaking scale at 1.28×10^{11} GeV.

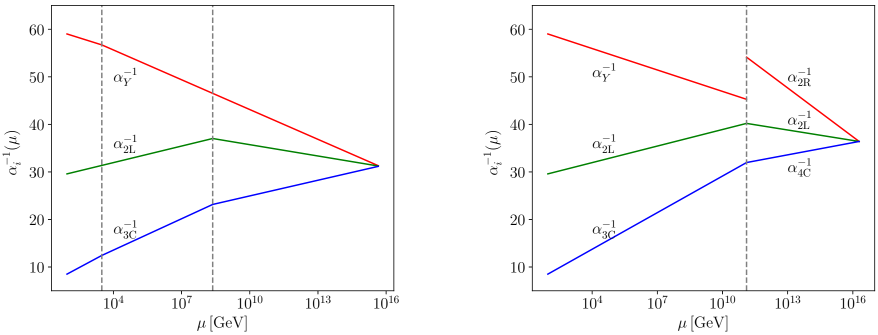
The solutions to the RGEs are in general altered by increasing the order in perturbation theory to which they are computed. That should be accompanied by the matching at some threshold being computed to a higher order, which entails the inclusion of threshold effects [147, 148]. These can considerably affect the scale at which the gauge couplings unify, and hence also the related phenomenology [149–153], as explored in Paper IV.

4.2 Candidate gauge groups

In order to be a viable candidate for a unified gauge group, there are several requirements that need to be fulfilled [154].

Rank

A unified gauge group needs to be large enough to contain $\mathcal{G}_{\rm SM}$ as a subgroup. Since $\mathcal{G}_{\rm SM}$ is of rank four, the rank of $\mathcal{G}_{\rm GUT}$ needs to be at least four.



(a) Intermediate particles

(b) PS intermediate symmetry

Figure 4.2: Gauge coupling unification with one-loop renormalization group running using two intermediate-scale fields (left) or an intermediate Pati–Salam gauge symmetry in the breaking chain (right).

SM as a subgroup

To reproduce the low-energy phenomenology of the SM that has been experimentally observed, the unified gauge group needs to contain $\mathcal{G}_{\text{SM}}$ as a subgroup.

Chirality

Furthermore, to reproduce the chiral nature of the SM, the GUT group needs to allow for complex representations. The reason for this is that, using the charge conjugation matrix, the left- and right-handed fermions may be related as

$$\psi_{\text{R}} = C\overline{\psi_{\text{L}}^{c}}^{T}. \tag{4.12}$$

Hence, if the left-handed fermions transform according to a (possibly reducible) representation $\mathbf{F}_{\text{L}}$, then the right-handed counterparts will transform according to a representation $\mathbf{F}_{\text{R}} = \overline{\mathbf{F}_{\text{L}}}$. Our knowledge of the SM implies that we must have $\overline{\mathbf{F}_{\text{L}}} \neq \mathbf{F}_{\text{L}}$. To be concrete, in the SM we have the left-handed fermions in Eq. (4.1) in a reducible representation

$$\mathbf{F}_{\text{L}} = (\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{1})_{1}. \tag{4.13}$$

Thus, the right-handed counterparts transform according to $\mathbf{F}_{\text{R}} = \overline{\mathbf{F}_{\text{L}}} \neq \mathbf{F}_{\text{L}}$, and hence, the SM is chiral. This must also hold for any grand unified model.

4.2.1 Viable gauge groups

The above list of requirements narrows down the list of possible simple gauge groups considerably [133, 155–157]. At rank four, the only possibility satisfying the constraints is SU(5), which was the original GUT proposed by Georgi and Glashow [158]. Going to rank five, the only possibility satisfying all conditions is SO(10), which was first proposed by Fritzsch and Minkowski [159] and independently by Georgi [160, 161]. Turning to rank six, we find the exceptional group $E(6)$ [162–164] as the only possibility.

There are also some semisimple gauge groups that allow for partial unification of $\mathcal{G}_{\rm SM}$ into a product of simple gauge groups. To be able to reprodcue the SM hypercharge, the rank has to be at least five, which presents several possibilities. One possibility is to enlarge the SU(5) group to SU(5) × U(1), which allows for a so-called "flipped" embedding of hypercharge [165–169]. There are also the left-right symmetric models $\mathrm{SU}(3)_{\rm C} \times \mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R} \times \mathrm{U}(1)_{B-L}$ [170–173], in which the left-handed weak interaction is mirrored by a right-handed one, and the Pati–Salam group $\mathcal{G}_{\rm PS} = \mathrm{SU}(4)_{\rm C} \times \mathrm{SU}(2)_{\rm L} \times \mathrm{SU}(2)_{\rm R}$ [174], in which the leptons are unified with the quarks as a fourth color.

Below follow some comments regarding the three most important gauge groups for unified model building, before we focus on models based on the SO(10) gauge group.

4.2.2 The SU(5) group

Being the unique embedding of the SM at rank four, the SU(5) group is the simplest possible candidate for a GUT [158]. The fermions of each generation are embedded in $\overline{\mathbf{5}}$ and $\mathbf{10}$, since under decomposition to $\mathcal{G}_{\rm SM}$, they become

$$\overline{\mathbf{5}}_F \to (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2} = d^c \oplus L, \tag{4.14}$$

$$\mathbf{10}_F \to (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{1})_{1} = u^c \oplus Q \oplus e^c. \tag{4.15}$$

Since the adjoint representation is **24**, there are 24 gauge bosons in the model. The structure of the gauge sector of the Lagrangian follows in the same manner as for SU(3) or SU(2) discussed in Ch. 2. The 24 gauge bosons may be written in matrix form using the generators t^a, where $a \in \{1, 2, \ldots, 24\}$, of SU(5) as

$$A_\mu = A_\mu^a t^a. \tag{4.16}$$

After symmetry breaking, eight of the 24 gauge bosons are identified with the gluons G_μ^a, three are identified with the SU(2) gauge bosons W_μ^i, and one is identified with the hypercharge gauge boson B_μ. There remain twelve gauge bosons which acquire masses around the symmetry breaking scale.

To break the SU(5) symmetry down to $\mathcal{G}_{\rm SM}$, we employ a mechanism based on the same principle as the Higgs mechanism in the SM. This is achieved by introducing scalar fields transforming as the adjoint, **24**. Using the adjoint to spontaneously break a group conserves its rank, since the vev may be diagonalized by the group generators, and hence commutes with the Cartan generators. Thus, the number of unbroken Cartan generators remains the same [175].

We therefore have 24 scalar fields ϕ^a, which can be written in matrix form

$$\mathbf{24}_H = \phi^a t^a \tag{4.17}$$

and the scalar potential is

$$V_{24}(\mathbf{24}_H) = -\mu^2 \,\mathrm{Tr}(\mathbf{24}_H^2) + \lambda_1 \,\mathrm{Tr}(\mathbf{24}_H^2)^2 + \lambda_2 \,\mathrm{Tr}(\mathbf{24}_H^4), \tag{4.18}$$

neglecting the cubic term by imposing a discrete symmetry $\mathbf{24}_H \to -\mathbf{24}_H$. Minimizing this potential under the assumptions $\lambda_2 > 0$ and $\lambda_1 > -7/30\lambda_2$, one finds that $\mathbf{24}_H$ takes the vev

$$\langle \mathbf{24}_H \rangle = v\,\mathrm{diag}(2,2,2,-3,-3), \tag{4.19}$$

with v given by

$$v = \frac{\mu^2}{2(30\lambda_1 + 7\lambda_2)}. \tag{4.20}$$

This achieves the desired breaking to the SM.

The scalar sector also needs to include an SU(2) doublet that will provide the Higgs mechanism of the electroweak theory. The simplest choice is $\mathbf{5}_H$, which decomposes as

$$\mathbf{5} \to (\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}. \tag{4.21}$$

The latter of these can be identified with the SM Higgs, while the first one is an extra SU(3) triplet that should have a large mass in order not to affect low-energy phenomenology. This in general requires fine-tuning of the parameters in the scalar potential $V(\mathbf{24}_H, \mathbf{5}_H) = V_{24}(\mathbf{24}_H) + V_5(\mathbf{5}_H) + V_{\mathrm{mix}}(\mathbf{24}_H, \mathbf{5}_H)$ [176] and is known as the "doublet-triplet splitting" problem. See Ref. [177] for a discussion regarding the amount of fine-tuning necessary for this type of splitting in various models.

Fermion masses are a result of Yukawa couplings. In particular, we require the fermions to couple to $\mathbf{5}_H$ and not $\mathbf{24}_H$, since the SM Higgs resides in $\mathbf{5}_H$. It happens that Yukawa couplings with $\mathbf{24}_H$ are forbidden by SU(5) symmetry, as required. In a compact notation (suppressing Lorentz, family, and SU(5) indices), the Yukawa terms of the Lagrangian density is given by

$$\mathcal{L}_{\mathrm{Yukawa}} = -Y_5 \overline{\mathbf{5}}_F \mathbf{10}_F \mathbf{5}_H^* - \frac{1}{8} Y_{10} \epsilon_5 \mathbf{10}_F \mathbf{10}_F \mathbf{5}_H + \mathrm{h.c.}, \tag{4.22}$$

where Y_5 and Y_{10} are the Yukawa matrices in family space and ϵ_5 is the 5-index completely antisymmetric tensor in SU(5)-space. Writing out the charge-conjugation matrix and spinor transposition explicitly, as well as the family indices $(i,j) \in \{1,2,3\}$ and SU(5) indices $(\alpha,\beta,\gamma,\delta,\epsilon) \in \{1,2,\ldots,5\}$, we obtain

$$\mathcal{L}_{\text{Yukawa}} = -(Y_5)^{ij}(\overline{\mathbf{5}}_F^T)^i_\alpha C(\mathbf{10}_F)^j_{\alpha\beta}(\mathbf{5}_H^*)_\beta - \frac{1}{8}(Y_{10})^{ij}\epsilon_{\alpha\beta\gamma\delta\epsilon}(\mathbf{10}_F^T)^i_{\alpha\beta}C(\mathbf{10}_F)^j_{\gamma\delta}(\mathbf{5}_H)_\epsilon + \text{h.c.} \quad (4.23)$$

Expanding the SU(5) multiplication to write it in terms of the SM fields, the first term gives

$$-Y_5(d^c)^T CQH^* - Y_5 L^T Ce^c H^* \quad (4.24)$$

and the second term gives

$$-\frac{1}{2}(Y_{10} - Y_{10}^T)(u^c)^T CQH. \quad (4.25)$$

Therefore, the SU(5) symmetry predicts the mass relations

$$M_\ell = M_d^T, \quad M_u = M_u^T, \quad (4.26)$$

the first of which is clearly false [158] and is not entirely corrected by renormalization effects [176]. They can, however, be corrected by adding an additional Higgs field in the **45** representation [178] or taking into account Planck-scale suppressed non-renormalizable operators [179].

4.2.3 The Pati–Salam group

The Pati–Salam model [174] of partial unification is based on the gauge group $\mathcal{G}_{\text{PS}} = \text{SU}(4)_\text{C} \times \text{SU}(2)_\text{L} \times \text{SU}(2)_\text{R}$ and unifies the quarks and leptons by treating the leptons as a fourth color. Each family of fermions is embedded as

$$F_\text{L} = \begin{pmatrix} u^r & u^g & u^b & \nu \\ d^r & d^g & d^b & e \end{pmatrix} \sim (\mathbf{4},\mathbf{2},\mathbf{1}), \quad (4.27)$$

$$F_\text{R} = \begin{pmatrix} d^{r\,c} & d^{g\,c} & d^{b\,c} & e^c \\ -u^{r\,c} & -u^{g\,c} & -u^{b\,c} & -\nu^c \end{pmatrix} \sim (\overline{\mathbf{4}},\mathbf{1},\mathbf{2}). \quad (4.28)$$

This fermion embedding makes evident a symmetry between the left-handed and right-handed fermions which the SM lacks. A result of this symmetry is the introduction of a right-handed neutrino ν^c.

It is possible to have a discrete symmetry called D-parity in this model [170, 172, 180–182]. It acts on the fermions as a left-right transformation such that $D(\mathbf{4},\mathbf{2},\mathbf{1}) = (\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$, similar to a regular parity transformation. The scalar representations, however, transform in a more complicated way such that one may have two scalar representations with the same transformation properties under $\text{SU}(4) \times \text{SU}(2) \times \text{SU}(2)$ that have opposite D-parity. Phenomenologically, a model with D-parity has identical $\mathrm{SU}(2)_\mathrm{L}$ and $\mathrm{SU}(2)_\mathrm{R}$ sectors, which implies that the corresponding gauge couplings for the two groups are equal.

The gauge bosons of $\mathrm{SU}(4)_\mathrm{C}$ reside in the adjoint representation $(\mathbf{15},\mathbf{1},\mathbf{1})$, which contains the eight gluons after breaking to the SM. The three gauge bosons of $\mathrm{SU}(2)_\mathrm{L}$ in $(\mathbf{1},\mathbf{3},\mathbf{1})$ are the same as those in the SM. Finally, $\mathrm{SU}(2)_\mathrm{R}$ has its gauge bosons in $(\mathbf{1},\mathbf{1},\mathbf{3})$. The hypercharge gauge boson is a linear combination of the two $(\mathbf{1},\mathbf{1})_0$ components in $(\mathbf{15},\mathbf{1},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{3})$ such that the hypercharge is embedded as

$$Y = \frac{B-L}{2} - T_{R,3}, \tag{4.29}$$

where $B-L$ is the (suggestively named) unbroken generator of $\mathrm{SU}(4)_\mathrm{C}$ that does not belong to $\mathrm{SU}(3)_\mathrm{C}$ and $T_{R,3}$ is the third generator of $\mathrm{SU}(2)_\mathrm{R}$.

The Pati–Salam symmetry can be broken directly to the SM by including scalar bosons transforming as $(\overline{\mathbf{10}},\mathbf{1},\mathbf{3})$ and arranging the scalar potential such that it takes a vev in the direction that is a singlet under the SM gauge group. However, there are other breaking chains possible that break the symmetry to the SM in several steps. A popular such route is via the left-right symmetric group $\mathrm{SU}(3)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{SU}(2)_\mathrm{R} \times \mathrm{U}(1)_{B-L}$ [170–173], which is a subgroup of $\mathcal{G}_\mathrm{PS}$ and in turn contains $\mathcal{G}_\mathrm{SM}$ as a subgroup. This can be achieved by including scalar bosons in $(\mathbf{15},\mathbf{1},\mathbf{1})$ and assigning a vev in the appropriate direction of it.

Finally, we need to include a representation of scalar bosons that contains the SM Higgs in order to achieve the Higgs mechanism and give fermions mass. This can be done with $\Phi \sim (\mathbf{1},\mathbf{2},\mathbf{2})$, which contains the SM Higgs doublet. However, this would enable only one Yukawa term, *i.e.*

$$\mathcal{L}_\mathrm{Yukawa} = -Y\overline{F_\mathrm{L}}\Phi F_\mathrm{R} + \mathrm{h.c.}, \tag{4.30}$$

which clearly gives the wrong mass relations, just like in the $\mathrm{SU}(5)$ model. Again, this may be remedied by introducing multiple scalars that each take vevs and couple to the fermions [174], for example more copies of $(\mathbf{1},\mathbf{2},\mathbf{2})$ representations or $(\mathbf{15},\mathbf{2},\mathbf{2})$ representations.

4.2.4 The SO(10) group

The SO(10) group [159–161] is a popular candidate for unification for several reasons. Firstly, it contains both the Pati–Salam group and $\mathrm{SU}(5) \times \mathrm{U}(1)$ (and hence also $\mathrm{SU}(5)$) as subgroups and is therefore more unified in a sense. It also embeds all SM fermions of each generation, plus a right-handed neutrino, into one single representation, namely $\mathbf{16}_F$.

Orthogonal groups $\mathrm{SO}(N)$ would *a priori* be thought to contain only real representations, and hence be unsuitable for embedding the SM. However, they also contain spinorial representations which are complex for even values of N and can therefore accommodate the SM fermions [133, 156, 157, 183, 184].

Under decomposition to SU(5), **16** becomes

$$\mathbf{16} \to \mathbf{10} \oplus \overline{\mathbf{5}} \oplus \mathbf{1}, \tag{4.31}$$

while in the PS model, it becomes

$$\mathbf{16} \to (\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2}). \tag{4.32}$$

Since it reproduces the fermion sector of the SU(5) model and of $\mathcal{G}_{\text{PS}}$, it also reproduces the SM fermion sector with an additional singlet.

The adjoint representation of SO(10) is **45**, so there are 45 gauge bosons in SO(10) models. As with the previous versions of GUTs discussed, when the symmetry is spontaneously broken, only the gauge bosons corresponding to the generators that are left unbroken remain massless. The rest of the gauge bosons acquire masses of the order of the symmetry breaking scale M_{GUT}.

The scalar sector of SO(10) models is very rich due to the number of possible ways of breaking it to $\mathcal{G}_{\text{SM}}$. It may be broken either through the $\mathcal{G}_{\text{PS}}$ route or through the SU(5) route with many possibilities for intermediate gauge groups before reaching $\mathcal{G}_{\text{SM}}$. There are also three different possibilities for embedding the Higgs doublet into SO(10), namely $\mathbf{10}_H$, $\overline{\mathbf{126}}_H$, and $\mathbf{120}_H$, which produce different mass relations. This will be discussed in more detail in Sec. 4.3.

4.3 Aspects of SO(10) model building

4.3.1 Symmetry breaking

Since SO(10) has rank five, which is one more than $\mathcal{G}_{\text{SM}}$, there are several possibilities for symmetry breaking. On the one hand, this produces more rich features, but on the other hand, it introduces arbitrariness into the model in terms of the choice of scalar sector and potential. Furthermore, since the scalar masses and the intermediate symmetry breaking steps affect the RG running, the breaking chain chosen can have an effect on the unification scale and hence the related phenomenology [185–188].

To analyze the symmetry breaking chains, we enumerate the decompositions of different multiplets to be used in the breaking under the intermediate gauge groups. The symmetry is broken by assigning a vev to a component of a multiplet and the resulting symmetry group is the largest group under which that component is a singlet, in analogy with the Higgs mechanism in the SM. Such decompositions can be found in Tabs. A.1 and A.2 in App. A. A schematic diagram of the subgroups and breaking chains is shown in Fig. 4.3. The different breaking chains have been analyzed in *e.g.* Refs. [181, 189–199].

One can first look for patterns of symmetry breaking that occur in one step. It turns out that this can be achieved with scalars transforming under the **144** representation [201], since it contains a singlet under $\mathcal{G}_{\text{SM}}$ but not under any of the other intermediate subgroups of SO(10).

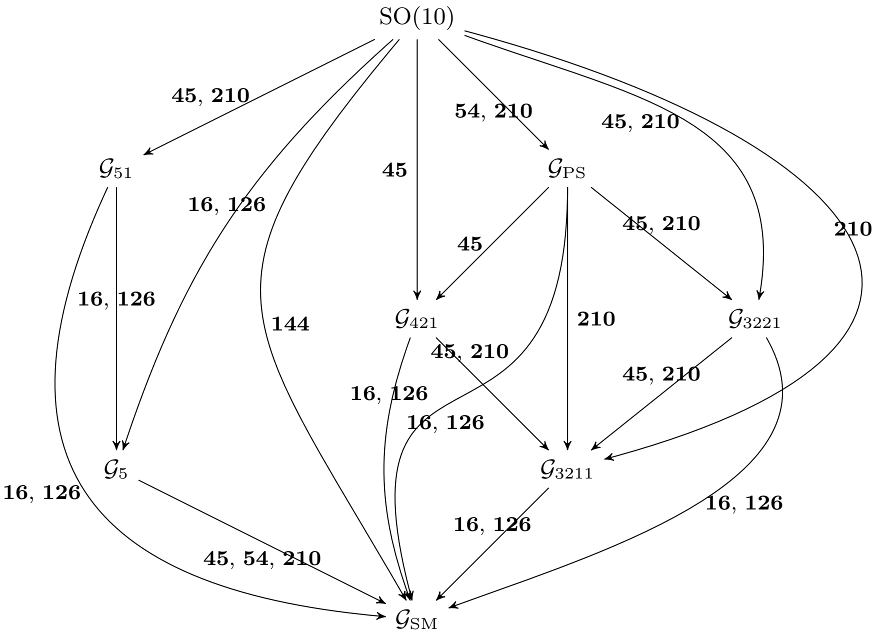


Figure 4.3: Possible breaking chains of SO(10) to $\mathcal{G}_{\mathrm{SM}}$. The representations written next to each arrow are the representation of SO(10) which contain multiplets that can achieve that particular breaking. We introduce the shorthand notation for the different groups, such that $\mathcal{G}_{51} = \mathrm{SU}(5) \times \mathrm{U}(1)$, $\mathcal{G}_5 = \mathrm{SU}(5)$, $\mathcal{G}_{\mathrm{PS}} = \mathrm{SU}(4)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}}$, $\mathcal{G}_{421} = \mathrm{SU}(4)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{B-L}$, $\mathcal{G}_{3221} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{B-L}$, and $\mathcal{G}_{3211} = \mathrm{SU}(3)_{\mathrm{C}} \times \mathrm{SU}(2)_{\mathrm{L}} \times \mathrm{U}(1)_{\mathrm{R}} \times \mathrm{U}(1)_{B-L}$. This figure is based on Refs. [181, 200].

As can be seen in Fig. 4.3, the spindle of breaking chains separates into two separate sectors, namely the SU(5) path and the PS path. Starting with the SU(5) path, we see that we have singlets under SU(5) × U(1) in **45** and **210**. If we have the flipped embedding, we can then break the symmetry directly to $\mathcal{G}_{\mathrm{SM}}$ using **16** or **126**. With the standard embedding, we have to go via SU(5) using the same and then using **45**, **54**, or **210** to break to $\mathcal{G}_{\mathrm{SM}}$. Alternatively, we can bypass SU(5) × U(1) and go directly to SU(5) using **16** or **126**, since the SU(5) singlets therein have a U(1) charge, and then break it to $\mathcal{G}_{\mathrm{SM}}$ as before.

Note that if the motivation for using an intermediate symmetry is to allow for successful gauge coupling unification, then it does not make sense to have SU(5) as an intermediate gauge group. The reason is that it would require the three gauge couplings to unify at the intermediate breaking scale, and therefore does not solve the problem. However, this is not required for flipped SU(5) × U(1), so having it as an intermediate gauge group can help achieve gauge coupling unification.

For the Pati–Salam breaking chain, we can break SO(10) to $\mathcal{G}_{\mathrm{PS}}$ by assigning a vev to the appropriate submultiplet of **54** or **210**. To break $\mathcal{G}_{\mathrm{PS}}$, or to break SO(10) directly to any of the subgroups of $\mathcal{G}_{\mathrm{PS}}$, we can assign a vev to the appropriate directions of **45** or **210**. At the end of the breaking chain to $\mathcal{G}_{\mathrm{SM}}$, either **16** or **126** needs to be assigned a vev. The reason is that those multiplets contain multiplets that break the $\mathrm{SU}(2)_{\mathrm{R}}$ and $\mathrm{U}(1)_{\mathrm{R}}$ symmetry.

The $\mathcal{G}_{\mathrm{PS}}$ and $\mathcal{G}_{3221}$ subgroups may also have D-parity, as discussed in Sec. 4.2.3. This is a transformation that belongs to the SO(10) algebra and hence its appearance after spontaneous symmetry breaking depends on which representation is used in the breaking [202–204]. For example, breaking SO(10) to $\mathcal{G}_{\mathrm{PS}}$ with a vev in the **54** representation leaves D-parity invariant since the direction that takes vev is invariant under D-transformation, and we get $\mathcal{G}_{\mathrm{PS}} \times D$ as a subgroup. On the other hand, using **210** breaks D-parity. If we instead want to break the symmetry down to $\mathcal{G}_{3221}$, the relevant vev in **210** is invariant under D-transformations [205]. Therefore, we can achieve the breaking chain $\mathrm{SO}(10) \to \mathcal{G}_{3221} \times D$ using **210**, or $\mathrm{SO}(10) \to \mathcal{G}_{\mathrm{PS}} \times D \to \mathcal{G}_{3221} \times D$ using **54** in the first step and **210** in the second step. Had we instead used **45** in the second step, D-parity would not be conserved. For more details on models based on the $\mathcal{G}_{3221}$ group, see Refs. [206, 207].

Any of the symmetry breaking chains may be achieved by assigning a vev to the relevant component of the relevant multiplet. In order to do so, the scalar potential needs to be constructed in such a way as to create a vev in the appropriate direction. This may, in general, include some level of fine-tuning of the parameters in the potential.

As an example of the vevs that cause spontaneous symmetry breaking, we can consider breaking SO(10) to SU(5) using **16**. Assigning a vev to the SU(5) singlet component such that [189]

$$\langle \mathbf{16} \rangle = (0, 0, \ldots, 0, \chi), \tag{4.33}$$

with $\chi \sim M_{\mathrm{GUT}}$, breaks the $\mathrm{SU}(2)_{\mathrm{R}}$ subgroup of SO(10) and leaves the SU(5) transformations invariant. If we instead wish to break SO(10) to $\mathcal{G}_{\mathrm{PS}}$, we can use the symmetric rank-two tensorial **54** representation and assign a vev [193, 196]

$$\langle \mathbf{54} \rangle = \omega \, \mathrm{diag} \left(-\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, -\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5} \right), \tag{4.34}$$

with $\omega \sim M_{\mathrm{GUT}}$. Here, it is evident how the vev causes the breaking to $\mathcal{G}_{\mathrm{PS}}$, as it partitions the 10×10 matrix into two blocks of 6×6 and 4×4, representing the breaking to $\mathrm{SO}(6) \times \mathrm{SO}(4)$. Since SO(6) is isomorphic to SU(4) and SO(4) is isomorphic to $\mathrm{SU}(2) \times \mathrm{SU}(2)$, we see that the breaking indeed gives $\mathcal{G}_{\mathrm{PS}}$.

It is in general possible to break the symmetry at one scale by letting several breaking steps to occur at the same scale. For example, in a model with a two-step breaking $\mathrm{SO}(10) \to \mathcal{G}_{\mathrm{PS}} \to \mathcal{G}_{\mathrm{SM}}$ using vevs in **54** and **126**, if the two vevs coincide in energy scale, the breaking chain effectively collapses to a one-step breaking directly to $\mathcal{G}_{\mathrm{SM}}$. The viability of such scenarios depends on detailed considerations of the scalar potential, which may restrict ratios of vevs [193].

4.3.2 Yukawa sector

For the Higgs mechanism of the SM, we require not only that we have an $\mathrm{SU}(2)_{\mathrm{L}}$ doublet embedded in a representation of SO(10), but also that it couples to the fermions, which are in $\mathbf{16}_F$. Hence, we need one or more representations $\mathbf{R}$ such that the coupling $\mathbf{16} \cdot \mathbf{R} \cdot \mathbf{16}$ is invariant under SO(10) transformations. There are only three possibilities for $\mathbf{R}$, namely $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$ [208]. Therefore, the Yukawa sector will contain scalars in these three representations. In a compact notation, the Yukawa sector of the Lagrangian density can then be written as

$$\mathcal{L}_{\mathrm{Yukawa}} = -\mathbf{16}_F (Y_{10} \mathbf{10}_H + Y_{120} \mathbf{120}_H + Y_{126} \overline{\mathbf{126}}_H) \mathbf{16}_F + \mathrm{h.c.}, \tag{4.35}$$

where Y_{10}, Y_{120}, and Y_{126} are 3×3 Yukawa matrices in family space. The SO(10) structure of this coupling dictates that Y_{10} and Y_{126} are symmetric while Y_{120} is antisymmetric. See App. A.3 for more details.

To write out the SO(10) index structure of this multiplication explicitly, we need to keep in mind some facts about the spinor structure of the $\mathbf{16}_F$ representation. These can be found in App. A.3. Since the elements of $\mathbf{16}_F$ are Lorentz spinors, we need to have the usual Lorentz spinor structure with the transposition and charge conjugation matrix C acting in Lorentz spinor space. We also have the 16-dimensional space corresponding to the spinor representations of SO(10). This will be labeled by indices $(a, b, c, d, e, f, g) \in \{1, 2, \ldots, 16\}$. The 10-dimensional space of the tensorial representations of SO(10) will be labeled by indices $(\alpha, \beta, \gamma, \delta, \epsilon) \in \{1, 2, \ldots, 10\}$. Family indices are labeled by $(i, j) \in \{1, 2, 3\}$. Finally, we need the equivalent of the charge conjugation matrix and the Dirac γ matrices acting in the space of the spinorial SO(10) representation. These will be denoted by B and Γ_α, respectively, and are 16×16 matrices. Then, the SO(10) structure of the Yukawa coupling is

$$\begin{aligned}\mathcal{L}_{\rm Yukawa} = -(\mathbf{16}_F^T)^{ia} C B^{ab}\Big[&(Y_{10})^{ij}(\Gamma_\alpha)^{bg}(\mathbf{10}_H)_\alpha \\ &+ (Y_{120})^{ij}(\Gamma_\alpha)^{bc}(\Gamma_\beta)^{cd}(\Gamma_\gamma)^{dg}(\mathbf{120}_H)_{\alpha\beta\gamma} \\ + (Y_{126})^{ij}(\Gamma_\alpha)^{bc}(\Gamma_\beta)^{cd}(\Gamma_\gamma)^{de}&(\Gamma_\delta)^{ef}(\Gamma_\epsilon)^{fg}(\overline{\mathbf{126}}_H)_{\alpha\beta\gamma\delta\epsilon}\Big](\mathbf{16}_F)^{jg} + \text{h.c.}\end{aligned} \tag{4.36}$$

Since we have three different representations with Higgs doublets, one may ask which one is the SM Higgs doublet. The components of the three representations of scalars in the Yukawa sector under the $\mathcal{G}_{\rm SM}$ decomposition that are relevant for fermion masses are

$$\mathbf{10}_H \supset (\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \equiv \Phi_{10}^d \oplus \Phi_{10}^u, \tag{4.37}$$

$$\begin{aligned}\mathbf{120}_H &\supset (\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \\ &\equiv \Phi_{120}^d \oplus \Phi_{120}^u \oplus \Sigma_{120}^d \oplus \Sigma_{120}^u,\end{aligned} \tag{4.38}$$

$$\begin{aligned}\overline{\mathbf{126}}_H &\supset (\mathbf{1},\mathbf{2})_{1/2} \oplus (\mathbf{1},\mathbf{2})_{-1/2} \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_1 \\ &\equiv \Sigma_{126}^d \oplus \Sigma_{126}^u \oplus \sigma_{\rm R} \oplus \Delta_{\rm L}.\end{aligned} \tag{4.39}$$

We thus have eight different $\mathrm{SU(2)_L}$ doublets that play the roles of Φ and $\widetilde{\Phi}$ in the SM Yukawa Lagrangian Eq. (2.25). They can all develop vevs, depending on the scalar potential of the model, and thereby provide mass to the fermions. The light Higgs boson, which has been observed at the LHC, is in general a linear combination of the $\mathrm{SU(2)_L}$ doublets above. This may require some fine-tuning of the parameters of the scalar potential.

When we decompose the Yukawa Lagrangian Eq. (4.36), we find terms that produce the fermion masses as well as some additional terms involving the additional fields found in $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$. These mediate exotic processes such as proton decay, which will be discussed in Sec. 4.5. The terms that provide fermion masses are identical to those of the SM Yukawa Lagrangian Eq. (2.25), except that the mass matrices are now given in terms of the Yukawa matrices Y_{10}, Y_{120}, and Y_{126} as well as the vevs of the fields in Eqs. (4.37)–(4.39). The matching conditions between the SO(10) parameters and the SM mass matrices are [152, 209–211]

$$M_u = v_{10}^u Y_{10} + v_{126}^u Y_{126} + (v_{120_1}^u + v_{120_2}^u) Y_{120}, \tag{4.40}$$

$$M_d = v_{10}^d Y_{10} + v_{126}^d Y_{126} + (v_{120_1}^d + v_{120_2}^d) Y_{120}, \tag{4.41}$$

$$M_\nu = v_{10}^u Y_{10} - 3v_{126}^u Y_{126} + (v_{120_1}^u - 3v_{120_2}^u) Y_{120}, \tag{4.42}$$

$$M_\ell = v_{10}^d Y_{10} - 3v_{126}^d Y_{126} + (v_{120_1}^d - 3v_{120_2}^d) Y_{120}, \tag{4.43}$$

$$M_{\rm R} = v_{\rm R} Y_{126}, \tag{4.44}$$

$$M_{\rm L} = v_{\rm L} Y_{126}, \tag{4.45}$$

where the vevs are defined as

$$v_{10}^{u,d} = \langle \Phi_{10}^{u,d} \rangle, \tag{4.46}$$

$$v_{120_1}^{u,d} = \langle \Phi_{120}^{u,d} \rangle, \quad v_{120_2}^{u,d} = \langle \Sigma_{120}^{u,d} \rangle, \tag{4.47}$$

$$v_{126}^{u,d} = \langle \Sigma_{126}^{u,d} \rangle, \quad v_{\mathrm{R}} = \langle \sigma_{\mathrm{R}} \rangle, \quad v_{\mathrm{L}} = \langle \Delta_{\mathrm{L}} \rangle. \tag{4.48}$$

The factors of 3 and relative signs of the terms are due to Clebsch–Gordan coefficients. The matrices M_{R} and M_{L} are the mass matrices of the heavy right-handed neutrinos and the contribution to the light neutrino masses from type II seesaw, respectively, as explained in Sec. 2.2.1.

Since all fields that are charged under $\mathrm{SU(2)_L}$ couple to the $\mathrm{SU(2)_L}$ gauge bosons, their vevs contribute to the gauge boson masses. These have been measured to a high degree of accuracy, meaning that there is a constraint on the sum of the vevs, *i.e.*

$$|v_{10}^u|^2 + |v_{10}^d|^2 + |v_{120_1}^u|^2 + |v_{120_1}^d|^2 + |v_{120_2}^u|^2 + |v_{120_2}^d|^2 + |v_{126}^u|^2 + |v_{126}^d|^2 + 2|v_{\mathrm{L}}|^2 = v_{\mathrm{SM}}^2. \tag{4.49}$$

The factor of 2 in front of $|v_{\mathrm{L}}|^2$ is due to a Clebsch–Gordan coefficient from the coupling of the triplet to the gauge bosons. This constraint is dependent on the model in the sense that it depends on which $\mathrm{SU(2)_L}$ multiplets are included. For example, if one considers a model without $\mathbf{120}_H$, then the terms $|v_{120_{1,2}}^{u,d}|$ should be removed from Eq. (4.49). Likewise, if the model contains further multiplets charged under $\mathrm{SU(2)_L}$, they should be added to Eq. (4.49).

The constraint Eq. (4.49) allows for most of the vevs to be many orders of magnitude smaller than v_{SM} and one being of the order of v_{SM}. However, since the vevs of the $\mathrm{SU(2)_L}$ doublets are versions of the SM Higgs doublet, one would expect them to be of a similar order of magnitude. The vev v_{L} is expected to be very small, since it is the induced vev involved in type II seesaw. Finally, the vev v_{R} is expected to be large, since it is involved in the breaking chain of SO(10) to $\mathcal{G}_{\mathrm{SM}}$. It is thus of the order of either M_{GUT} or a relevant intermediate breaking scale.

In specifying a model, one must determine which of the three possible representations of scalars to include for fermion masses. A common guiding principle in model building is minimality, which means choosing the smallest possible number of fields required to construct a viable model. At least two different multiplets are needed, since otherwise the mass matrices in Eqs. (4.40)–(4.45) will all be proportional, which contradicts the observation of mixing. Minimality would therefore favor a Yukawa sector consisting of only two out of $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$. One of these should be $\overline{\mathbf{126}}_H$ in order to produce the seesaw mechanisms. For the second one, minimality would dictate that we choose $\mathbf{10}_H$ since it contains the fewest number of states. This is the model considered in Papers II and III.

Since $\mathbf{10}_H$ is a real representation of SO(10), $\overline{\mathbf{10}}_H = \mathbf{10}_H$ implies that $v_{10}^u = v_{10}^d$, which results in wrong predictions for the mass relations [212, 213]. It can, however, be complexified by adding a second $\mathbf{10}_H$ and forming the combination $\mathbf{10}_H = \mathbf{10}_{H,1} + \mathrm{i}\mathbf{10}_{H,2}$, so that we can use $\mathbf{10}_H$ and $\mathbf{10}_H^*$. Although this solves the problem of wrong mass relations, it decreases the minimality and predictivity of the model, since $\mathbf{10}_H$ and $\mathbf{10}_H^*$ in general have two independent Yukawa coupling matrices. This can, however, be solved by introducing a global Peccei–Quinn symmetry $\mathrm{U}(1)_{\mathrm{PQ}}$ [108, 109] with charge assignment

$$\mathbf{16}_F \to \mathrm{e}^{\mathrm{i}\alpha}\mathbf{16}_F, \quad \mathbf{10}_H \to \mathrm{e}^{-2\mathrm{i}\alpha}\mathbf{10}_H, \quad \overline{\mathbf{126}}_H \to \mathrm{e}^{-2\mathrm{i}\alpha}\overline{\mathbf{126}}_H, \tag{4.50}$$

for some real parameter α. The PQ symmetry forbids $\mathbf{10}_H^*$ from coupling to the fermion bilinear, meaning that we can have two different vevs v_{10}^u and v_{10}^d, while only having one Yukawa matrix for the coupling to $\mathbf{10}_H$. Additionally, the $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry can solve the strong $\mathcal{CP}$ problem and provide axion dark matter, as discussed in Sec. 2.2. For more details on axions in SO(10) models, see Ref. [214].

One can also consider more extended models which include $\mathbf{120}_H$, as in Papers I and V. Since each representation couples to a subset of the other representations in the model, a model with more representations also contains more parameters. This can allow for better fits to the measured parameter values of the Yukawa sector, but it decreases the minimality of the model. Other Yukawa sectors are of course also possible, such as $\overline{\mathbf{126}}_H \oplus \mathbf{120}_H$, or several copies of each of the three representations. Models without $\overline{\mathbf{126}}_H$ are also possible if one adds scalars in a $\overline{\mathbf{16}}_H$ representation. Then, the neutrino mass can be generated by a non-renormalizable interaction $\mathbf{16}_F \cdot \overline{\mathbf{16}}_H \cdot \overline{\mathbf{16}}_H \cdot \mathbf{16}_F$ via the Witten mechanism [215], since the product of two $\overline{\mathbf{16}}$s contains $\overline{\mathbf{126}}$.

4.4 Flavor models in SO(10)

The problem of flavor as discussed in Sec. 2.2.2 persists in SO(10). The existence of three generations of fermions means that there must be three copies of $\mathbf{16}_F$. This does not have any natural origin in standard SO(10) models. Furthermore, although the fermion masses and mixing parameters are related to each other to a higher degree than in the SM through Eqs. (4.40)–(4.45), there is no explanation for the observed hierarchy of masses or values of the mixing parameters. In order to address this, one can construct models based on flavor symmetries within the SO(10) framework.

One approach to explain the existence of three generations of fermions is to assume a larger group with representations such that we do not need to manually add three copies of the fermions, but instead use representations that are large enough such that three copies of the same fermion particle content are not necessary. Early proposed examples of such models are based on the groups SU(11) [154] and SU(7) [216] (see also Refs. [217, 218]), with the objective of having as few copies as possible of each representation. Embedding all fermions in one representation requires a larger group such as SU(19) [219], with all fermions in a $\mathbf{171}$ representation, which decomposes into $3 \times \mathbf{16} \oplus \mathbf{120} \oplus 3 \times \mathbf{1}$ under SO(10). Other examples are SO(18) [220–225], with the fermions in **256**, which decomposes into $(\mathbf{16}, \mathbf{8}') \oplus (\overline{\mathbf{16}}, \mathbf{8}'')$ under SO(10) × SO(8) and O(14) [226–228], with the fermions in **128**, which decomposes into to $4 \times \mathbf{16} \oplus 4 \times \overline{\mathbf{16}}$ under SO(10). The drawback of these models is that there are many additional fermions, which need to have large masses in order to avoid tension with current measurements. As a result, more fine-tuning is required.[3] For the models based on SU(N) groups, we also need to ensure that they are anomaly-free, which can be achieved by adding conjugate representations.

There are also a number of models that, starting from SO(10), impose a flavor symmetry, sometimes also called a horizontal or family symmetry [229, 230]. If this group, which we can denote $\mathcal{G}_f$ contains a three-dimensional representation, then the fermions are typically assigned to $(\mathbf{16}, \mathbf{3})$ under SO(10) × $\mathcal{G}_f$, which explains the existence of three generations. Examples of $\mathcal{G}_f$ that have been studied are SU(3) [231, 232], SU(2) [233], U(2) [234], D_3 [235–237], $\Delta(27)$ [238], $\Delta(75)$ [239], A_4 [240, 241], and S_4 [242–245].

Since the SM does not contain $\mathcal{G}_f$, the flavor symmetry needs to be broken. This can be done by introducing scalar fields with non-trivial transformations under $\mathcal{G}_f$, called flavons, which take vevs. The flavons ϕ couple to fermions ψ and the Higgs doublet Φ as

$$\mathcal{L} \supset \frac{y}{\Lambda} \phi \overline{\psi} \Phi \psi, \tag{4.51}$$

where y is a coupling constant and Λ some energy scale relevant for the theory. If there are several flavons whose vevs are hierarchical and have a non-trivial flavor structure, then coupling the flavons to the fermions can induce a mass hierarchy between the fermion generations. Furthermore, it can produce relations among elements of the mass matrices and set some elements to zero, leading to predictions of the mixing parameters.

A more direct approach to explain the parameters related to masses and mixing parameters is to impose textures on the Yukawa matrices, which is to set some of the elements to zero [246–249]. This is useful since it limits the number of free parameters in the Yukawa sector, that can be quite large in SO(10) models. These zeros can originate in flavor symmetries under which the different generations of fermions transform differently. For example, if the first generation of fermions and $\mathbf{10}_H$ transform as $\mathbf{16}_{F1} \to \mathrm{e}^{\mathrm{i}\pi/2} \mathbf{16}_{F1}$ and $\mathbf{10}_H \to \mathrm{e}^{\mathrm{i}\pi/2} \mathbf{10}_H$, then a coupling $\mathbf{16}_{F1} \mathbf{10}_H \mathbf{16}_{F1}$ would be forbidden, since it acquires a non-zero phase under the given transformation. If both the second and third generations transform as $\mathbf{16}_{F2,3} \to \mathrm{e}^{3\mathrm{i}\pi/4} \mathbf{16}_{F2,3}$, then these are allowed to couple to $\mathbf{10}_H$. In this example, the corresponding Yukawa matrix would have a texture given by

$$Y_{10} \sim \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \tag{4.52}$$

[3] This problem also occurs with the additional scalar and vector bosons in standard SO(10) or SU(5) models, but now the problem occurs also for fermions. Additionally, larger groups have larger representations, meaning that there are now many more fields at the high-energy scale.

where $\times$ denotes a general non-zero coupling. In general, if each generation of fermions $\mathbf{16}_{Fi}$ transforms with a phase α_i and the scalar representations Φ_a picks up a phase ϕ_a (with the label a enumerating the possible choices $\mathbf{10}_H$, $\overline{\mathbf{126}}_H$, and $\mathbf{120}_H$), a coupling $\mathbf{16}_{Fi}\Phi_a\mathbf{16}_{Fj}$ will result in a factor $\exp(\mathrm{i}(\alpha_i + \alpha_j + \phi_a))$. If $\alpha_i + \alpha_j + \phi_a = 2n\pi$ for some integer n, then this coupling is allowed. Otherwise, it imposes a texture zero in the corresponding Yukawa matrix.

In Ref. [250], all possible textures and corresponding transformations were enumerated by starting from all possible Yukawa matrix textures and imposing that the non-zero elements be allowed. This procedure is dependent on the choice of Yukawa sector, with the typical one being $\mathbf{10}_H \oplus \overline{\mathbf{126}}_H \oplus \mathbf{210}_H$ such that there are three independent Yukawa matrices, two of which are symmetric and one of which is antisymmetric. Some of these sets of matrices will result in models that are equivalent. In particular, if one set of matrices is related to another by a permutation of generations, then these produce the same model. This is equivalent to saying that there is no preferred flavor basis and that a global relabeling of the generation of all fermions does not have a physical effect.

For a given model of Yukawa matrix textures, each non-zero element leads to a constraint on the phases. For example, a non-zero element in the $(2,3)$ element of Y_{10} leads to the constraint $\alpha_2 + \alpha_3 + \phi_{10} = 2n\pi$. If there are N such constraints, they can be written in an $N \times 6$ matrix multiplying the 6-element vector of phases $(\alpha_1, \alpha_2, \alpha_3, \phi_{10}, \phi_{126}, \phi_{120})^T$. The solution to these equations can be obtained by transforming the matrix of coefficients to Smith normal form [250–252], which is a diagonal form for non-square matrices of integers. The result is a set of restrictions on the phases of the form

$$m_i \tilde{\alpha}_i = 2\pi \tilde{n}_i, \tag{4.53}$$

where the $\tilde{\alpha}_i$ are linear combinations of the six original phases and m_i and $\tilde{n}_i$ are integer coefficients. This allows us to read off the symmetry associated with each phase: $m_i = 1$ corresponds to no symmetry since the phase is restricted to be a multiple of 2π, $m_i = p > 1$ corresponds to a $\mathbb{Z}_p$ symmetry, and $m_i = 0$ corresponds to a U(1) symmetry since the phase is unconstrained. Finally, one U(1) symmetry must be discarded since it corresponds to a global rephasing of the fermions by δ and the scalars by -2δ. One can extend this to include non-Abelian symmetries that can give rise to permutations of the generations or the scalar representations.

It is also possible to combine models such that several separate sets of transformations are applied to the fields. This can in principle be done in all possible combinations, but will in most cases yield symmetries that have already been found. Furthermore, there will be several models with the same symmetry. If the set of non-zero elements of one model is a subset of the set of non-zero elements of another model with the same symmetry, then that model is not considered unique and is thus discarded. Furthermore, there is the phenomenological constraint that no generation decouples in the mixing matrices. This is equivalent to saying that no element can be the only non-zero one in both its row and column in all Yukawa matrices simultaneously. Finally, Y_{126} cannot be singular, since its inverse is required for the type I seesaw mechanism.[4]

With a scalar sector consisting of one of each of $\mathbf{10}_H$, $\overline{\mathbf{126}}_H$, and $\mathbf{120}_H$, there are 14 inequivalent models with rephasing symmetries of the Yukawa sector as described above [249]. The symmetries of these are U(1), $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. There are no non-Abelian symmetries, since all possibilities either cause one generation to decouple or require more than one copy of each scalar representation [250]. The complete set of 14 models were investigated in Ref. [249] in a SUSY setting and in Paper V in a non-SUSY setting. Furthermore, Paper V takes into account the renormalization group running of the parameters, which allows for a consistent integration out of intermediate scale particles. Both with and without SUSY, the result is that a fit is only possible in two of the models, which have $\mathbb{Z}_2$ symmetry. This is not surprising since $\mathbb{Z}_2$ is the smallest possible symmetry, and thus gives the least restrictions on the Yukawa matrices. As a result, models with this symmetry have more free parameters than those with larger symmetries.

There are also other approaches to flavor in SO(10) models. One such example is the model presented in Ref. [253], in which only $\mathbf{10}_H$ was used in the Yukawa sector, but a pair of vector-like fermions interacting with SM fermions were added. This improves the predictivity of the model and allows it to reproduce measured masses and mixing parameters. Another model is the clockwork mechanism [254], in which the hierarchy in fermion masses is generated by each generation being connected to a sequence of additional vector-like fermions.

4.5 Proton decay

Grand unified theories generically predict exotic interactions via their additional gauge bosons and scalars, which can mediate proton decay [156, 255, 256]. The gauge boson-mediated proton decay can be seen in the covariant derivative of the fermions, in which gauge bosons in **45** of SO(10) couple to both quarks and leptons. They are thus called "leptoquark" gauge bosons and can convert leptons to quarks. For example, there are scalars in $\mathbf{10}_H$, $\mathbf{120}_H$, and $\overline{\mathbf{126}}_H$ that do not contribute to the fermion masses and are leptoquark scalars that can mediate proton decay. One possible channel of proton decay with a gauge boson is illustrated in Fig. 4.4.

These leptoquark gauge and scalar bosons violate baryon number. They must have masses around the unification scale in order to suppress the proton decay rate. Hence, the non-observation of proton decay places a lower bound on the scale M_{GUT}. For low-energy phenomenology, one can integrate out the leptoquark bosons and find effective operators of dimension six that describe proton decay [131, 257–259].

Although proton decay is a generic prediction of GUTs, there is some model dependence in the allowed effective operators, depending on which couplings are

[4]In a model without the type I seesaw mechanism, this restriction may be relaxed.

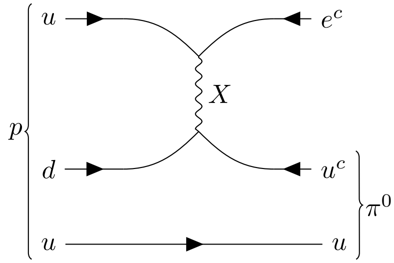


Figure 4.4: Example of a process leading to proton decay through a leptoquark gauge boson X.

present in the full theory [165]. Furthermore, there is a difference in models with or without SUSY, since SUSY allows also for dimension-four and -five effective operators which may lead to faster proton decay [260, 261].

An order of magnitude estimate of the proton decay width with a GUT-scale gauge mediator gives

$$\Gamma \sim \alpha_{\mathrm{GUT}}^2 \frac{m_p^5}{M_{\mathrm{GUT}}^4}, \tag{4.54}$$

where $\alpha_{\mathrm{GUT}} = g_{\mathrm{GUT}}^2/4\pi$, with g_{GUT} being the gauge coupling at the scale M_{GUT}, and m_p is the proton mass. The decay lifetime is the inverse of this, and is therefore proportional to $M_{\mathrm{GUT}}^4/\alpha_{\mathrm{GUT}}^2$. As a result, a long lifetime requires a combination of a small coupling constant α_{GUT}^2 and a large scale M_{GUT}^4. Since the dependence on M_{GUT} is stronger than that on α_{GUT} and since it is easier to modify M_{GUT} than α_{GUT}, the non-observation of proton decay in practice implies a lower bound on M_{GUT}.

In order to derive a more precise prediction of the proton lifetime, we need to consider in more detail the various allowed channels. Since the most constraining decay channel is through gauge bosons [255], we only consider the gauge interactions. The adjoint representation **45** contains, among others, $X \sim (\mathbf{3},\mathbf{2})_{-5/6}$, $X' \sim (\mathbf{3},\mathbf{2})_{1/6}$, $Y \sim (\overline{\mathbf{3}},\mathbf{2})_{5/6}$, and $Y' \sim (\overline{\mathbf{3}},\mathbf{2})_{-1/6}$, where the transformation properties are given under $\mathcal{G}_{\mathrm{SM}}$. To see the interactions between the fermions and gauge bosons that lead to proton decay, we write down the Dirac term

$$\mathcal{L}_{\mathrm{Dirac}} = \mathrm{i}\overline{\mathbf{16}}_F \gamma^\mu D_\mu \mathbf{16}_F, \tag{4.55}$$

where SO(10) indices have been suppressed and the covariant derivative contains the adjoint representation **45** in which the gauge bosons reside. By computing this product of representations in terms of the SM representations, we find the terms invloving the components X, X', Y, and Y' to be [262]

$$\begin{aligned}\mathcal{L}_{\text{Dirac}} \supset\ & \frac{g_{\text{GUT}}}{2} X_\mu \left(\overline{u^c}\gamma^\mu Q + \overline{Q}\gamma^\mu e^c - \overline{L}\gamma^\mu d^c\right) \\ & + \frac{g_{\text{GUT}}}{2} X'_\mu \left(\overline{d^c}\gamma^\mu Q + \overline{L}\gamma^\mu u^c - \overline{Q}\gamma^\mu \nu^c\right) \\ & + \frac{g_{\text{GUT}}}{2} Y_\mu \left(\overline{Q}\gamma^\mu u^c + \overline{d^c}\gamma^\mu L - \overline{e^c}\gamma^\mu Q\right) \\ & + \frac{g_{\text{GUT}}}{2} Y'_\mu \left(\overline{Q}\gamma^\mu d^c + \overline{u^c}\gamma^\mu L - \overline{\nu^c}\gamma^\mu Q\right), \end{aligned} \tag{4.56}$$

where the $\text{SU}(3)_\text{C}$, $\text{SU}(2)_\text{L}$, and family indices have been suppressed. Integrating out the heavy gauge bosons, either by a functional approach or by replacing them by the solutions to their equations of motion, we find four independent dimension-six effective operators that are relevant for proton decay [131, 255, 257–259]:

$$\mathcal{O}_\text{I} = \overline{u^c}_i \gamma^\mu Q_i \overline{e^c}_j \gamma_\mu Q_j, \tag{4.57}$$

$$\mathcal{O}_\text{II} = \overline{u^c}_i \gamma^\mu Q_i \overline{d^c}_j \gamma_\mu L_j, \tag{4.58}$$

$$\mathcal{O}_\text{III} = \overline{d^c}_i \gamma^\mu Q_i \overline{u^c}_j \gamma_\mu L_j, \tag{4.59}$$

$$\mathcal{O}_\text{IV} = \overline{d^c}_i \gamma^\mu Q_i \overline{\nu^c}_j \gamma_\mu Q_j, \tag{4.60}$$

where $i, j \in \{1, 2, 3\}$ are family indices. The operators in Eqs. (4.57)–(4.60) are diagonal in family space, since they are written in the interaction basis. The coefficients of these operators will be proportional to $g_\text{GUT}^2/M_\text{GUT}^2$, as can be seen from the fact that they are dimension-six operators and from explicitly considering their tree-level realizations.

To derive the decay rate from these operator, several considerations need to be taken into account. Firstly, since the proton contains physical quark mass states, a mixing factor coming from the Yukawa sector needs to be included. Secondly, a renormalization factor is needed to account for the fact that the relevant energy scale for the proton is $\mu \simeq 1\,\text{GeV}$ while the operators were formed by integrating out the gauge bosons at a scale $\mu \simeq M_\text{GUT}$. Thirdly, a hadronic matrix element is needed for the projection of the initial proton state of the constituent quarks onto the final meson state. Putting this together, we have for the decay to a pion and a positron, which is the most restrictive one, an approximate decay width given by [255, 263]

$$\Gamma(p \to e^+\pi^0) \simeq \frac{m_p}{64\pi f_\pi^2} \frac{g_\text{GUT}^4}{M_\text{GUT}^4} A_\text{L}^2 \alpha_H^2 F_q, \tag{4.61}$$

where $f_\pi \simeq 139\,\text{MeV}$ is the pion decay constant, $A_\text{L} \simeq 2.726$ is a renormalization factor, $\alpha_H \simeq 0.012\,\text{GeV}^3$ is the hadronic matrix element, and $F_q \simeq 7.6$ is a quark-mixing factor. This gives an estimate for the proton lifetime

$$\tau(p \to e^+\pi^0) \simeq (7.47 \times 10^{35}\,\text{yr}) \left(\frac{M_\text{GUT}}{10^{16}\,\text{GeV}}\right)^4 \left(\frac{0.03}{\alpha_\text{GUT}}\right)^2. \tag{4.62}$$

If proton decay were observed in different channels, then it would be useful to consider the decay rates to the different channels in order to determine details of the underlying GUT model. However, the non-observation of proton decay to date puts a lower bound on the proton lifetime. The current best lower bound is from Super-Kamiokande [264–266], which results the bounds $\tau(p \to e^+\pi^0) > 1.67 \times 10^{34}$ yr, $\tau(p \to \mu^+\pi^0) > 7.78 \times 10^{33}$ yr, and $\tau(p \to \nu K^+) > 6.61 \times 10^{33}$ yr at 90 % confidence level. The projected Hyper-Kamiokande is expected to increase these bounds to $\tau(p \to e^+\pi^0) > 5.5 \times 10^{34}$ yr and $\tau(p \to \nu K^+) > 1.8 \times 10^{34}$ yr after five years of collecting data [267]. With a coupling $\alpha_{\mathrm{GUT}} \approx 0.03$, the current bound on the proton lifetime implies $M_{\mathrm{GUT}} \gtrsim 4 \times 10^{15}$ GeV.

4.6 Phenomenology in SO(10) models

A large range of phenomenology can be embedded in SO(10) models, with the scale of interest being around M_{GUT} or some intermediate scale in the case that symmetry breaking occurs in multiple steps.

For example, axions are naturally embedded due to the $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry, as mentioned in Sec. 4.3.2, and can also provide a solution to dark matter. This is an appealing option since the $\mathrm{U}(1)_{\mathrm{PQ}}$ symmetry is motivated from considerations of the Yukawa sector. Being the psuedo-Nambu–Goldstone boson resulting from spontaneous symmetry breaking of $\mathrm{U}(1)_{\mathrm{PQ}}$, the decay constant of the axion will be of the order of the energy scale at which the symmetry breaking occurs. The mass of the axion is inversely proportional to its decay constant and is therefore naturally very small if the symmetry breaking occurs at M_{GUT} [152, 210, 214, 268–270].

Dark matter may also be in the form of a scalar or fermion field, which is stabilized due to an inherent parity symmetry inside the SO(10) group structure [200, 271–280]. Another option is to have the dark matter candidate be a pseudo-Nambu–Goldstone boson, since that naturally evades all current observational bounds [281, 282].

Since SO(10) naturally embeds a heavy right-handed neutrino, it provides the necessary ingredients for generating the baryon asymmetry through the mechanism of leptogenesis [138, 210, 277, 283–290]. Furthermore, due to the additional scalar and vector leptoquarks, both SO(10) and its PS subgroup are interesting from the point of view of the B-physics anomalies [291–294].

From the symmetry breaking of SO(10) down to the SM, the scalar fields can form particular configurations, known as topological defects. These may be domain walls, cosmic strings, or magnetic monopoles, depending on the symmetry breaking chain. Phenomenologically, these defects can have an effect on cosmology [153, 295, 296]. In particular, domain walls and magnetic monopoles are problematic in the sense that they overclose the Universe and it is therefore assumed that inflation occurs after their production such that their abundance becomes exponentially suppressed [297]. Cosmic strings do not share this problem and can therefore be allowed to persist [295, 298]. Furthermore, cosmic strings are produced by the symmetry breaking of U(1) while monopoles are produced by the symmetry breaking of non-Abelian groups. Therefore, if inflation occurs between the symmetry breaking of SO(10) and an intermediate symmetry involving U(1), then it is possible to dilute the monopole abundance while retaining substantial cosmic strings. These can interact with each other, and decay away with their energy transformed into gravitational waves [299]. With recent searches for the stochastic gravitational wave background [300, 301], it has been possible to place bounds on the generation of cosmic strings and hence a new window to constrain models of grand unification has opened up [302–306].

4.7 Current status of SO(10) models

Models based on the SO(10) gauge group are still viable models for physics beyond the SM. There are essentially only two ways in which such models could be ruled out. The first is if they predict wrong mass relations. This is the case for the most minimal models in which only one Higgs representation is used. However, it is easy to generalize the model to include another Higgs representation in order to save the mass relations. Thus, only very specific model details may be ruled out on this ground. In fact, SO(10) models with Higgs fields in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations are not only able to accommodate the correct mass relations, but also the neutrino masses through the seesaw mechanism.

The second way in which SO(10) models and GUT models in general may be ruled out is by constraints on the unification scale, coming from proton decay or gravitational waves from cosmic strings. At the time of writing this thesis, proton decay has not yet been observed. If this non-observation continues, then it will keep increasing the lower bound on the proton lifetime. This, in effect, increases the minimum value of M_{GUT} allowed. The constraints that can be derived from gravitational waves are typically related to intermediate symmetry breaking scales and depend on the details of the symmetry breaking chain. Constraints on M_{GUT} can rule out the simplest models, but more freedom can always be introduced by adding intermediate symmetry breaking scales or extra fields that alter the RG running. Due to the freedom in model building, grand unification is still a viable source for physics beyond the SM.

Chapter 5

Renormalization group running and numerical methods

In order to compare models of grand unification to observations, we need to relate the physics at the unification scale to the physics at energy scales accessible in experiments, which corresponds to a difference in energy of around 14 orders of magnitude. This leads to significant renormalization group effects between the two scales, as demonstrated for gauge couplings in Sec. 4.1. Further, there may be intermediate scale physics to take into account when solving the renormalization group running, such as fields with masses between the two scales of interest or intermediate symmetry breaking steps. This needs to be appropriately taken into account in renormalization group running.

An important viability test of GUT models is if they can accommodate the measured observables of the SM, namely masses and mixing parameters. There are two components required to be able to test this. The first is a way to relate the parameters at M_{GUT} to the observables measured in experiments, which is provided by renormalization group running. The second is a method to explore the parameter space of the model being tested, for which a numerical optimization algorithm is required.

This chapter deals with the methods involved in relating the high energy phenomenology of SO(10) models to the measured values of the parameters. This includes the masses and mixing parameters in the Yukawa sector as well as the gauge couplings. We start with a general description of renormalization and in particular the renormalization group running of gauge couplings. Then, we describe the process of integrating out intermediate particles such as the heavy right-handed neutrinos (RHNs) between M_{GUT} and M_Z. Finally, we describe the algorithm used in performing the numerical fits.

5.1 Renormalization group running

A key feature of QFTs is renormalization. Calculations performed in perturbation theory suffer from infinities that arise from divergent momentum integrals when computing Feynman diagrams. In order to extract meaningful finite results, one has to regularize these divergences, which can be performed in a number of ways and typically introduces a dependence on some energy scale. Since the physics of the original theory must be invariant under changes of that energy scale, the parameters of the Lagrangian must change accordingly to counter the effect of changes in the scale. This gives the renormalized parameters of the theory, which depend on the energy scale through the RGEs. More details on this process may be found in numerous resources, for example Refs. [307–309].

5.1.1 Regularization

There are several ways to regularize the integrals that arise from loops in Feynman diagrams. As an example, consider the integral over a scalar propagator in a loop, *i.e.*

$$\int \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{5.1}$$

where the integral runs over the whole range of four-momentum k. By dimensional analysis, one observes that the contributions from arbitrarily large four-momenta will cause the integral to diverge. Regularization is the procedure in which we extract a meaningful answer from integrals such as Eq. (5.1).

An intuitively simple procedure is to introduce a momentum cutoff Λ as the upper limit of the integral, such that

$$\int_0^\infty \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon} \to \int_0^\Lambda \frac{\mathrm{d}^4 k}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{5.2}$$

which is finite. The parameter Λ parametrizes the divergence and the original integral is recovered in the limit $\Lambda \to \infty$.

Another procedure of regularization is the Pauli–Villars procedure [310], which involves modifying the integrand as

$$\frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} \to \frac{1}{k^2 - m^2 + \mathrm{i}\epsilon} - \frac{1}{k^2 - \Lambda^2 + \mathrm{i}\epsilon}. \tag{5.3}$$

Here, Λ is the parametrization of the divergence and the original expression is again recovered in the limit $\Lambda \to \infty$. The second term may be interpreted as the contribution of a fictitious particle with the wrong sign of the propagator.

The most widely used procedure of regularization is dimensional regularization [311], in which the number of spacetime dimensions is altered from $d = 4$ to non-integer $d = 4 - \epsilon$, such that the original divergent expression is recovered in the limit $\epsilon \to 0$. We must similarly modify all expressions that depend on the number of spacetime dimensions, such as spinor algebra and surface integrals, to their equivalents in non-integer spacetime dimensions.

Requiring that the action

$$S = \int \mathrm{d}^d x \, \mathcal{L} \tag{5.4}$$

be dimensionless and that the mass parameters have dimensions of energy, we find that the dimensions of scalar, vector, and spinorial fields are

$$[\phi] = [A_\mu] = \frac{d-2}{2}, \quad [\psi] = \frac{d-1}{2}. \tag{5.5}$$

In order to keep the coupling constants dimensionless, we redefine them by extracting the relevant mass dimension. For the gauge, Yukawa, and scalar quartic couplings, we find

$$g \to \mu^{\epsilon/2} g, \quad Y \to \mu^{\epsilon/2} Y, \quad \lambda \to \mu^{\epsilon} \lambda, \tag{5.6}$$

where μ is some arbitrary parameter of mass dimension 1.

Within this scheme, the loop corrections to quantities may be calculated and the divergences are encapsulated in poles as $\epsilon \to 0$. For example, consider the integral

$$I = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{g^2}{k^2 - m^2 + \mathrm{i}\epsilon} \to \mu^{\epsilon} \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{g^2}{k^2 - m^2 + \mathrm{i}\epsilon}, \tag{5.7}$$

which evaluates to [307]

$$I = \frac{-\mathrm{i}g^2}{16\pi^2} m^2 \frac{\mu^{\epsilon}(4\pi)^{\epsilon/2}}{(m^2)^{\epsilon/2}} \Gamma\left(\frac{\epsilon}{2} - 1\right), \tag{5.8}$$

where Γ is the gamma function with $\Gamma(n) = (n-1)!$ for integers and in general it holds that $\Gamma(x+1) = x\Gamma(x)$. By expanding in powers of ϵ, the result is

$$I = \frac{\mathrm{i}g^2 m^2}{16\pi^2} \left[\frac{2}{\epsilon} + 1 - \gamma_{\mathrm{E}} + \ln(4\pi) + \ln\left(\frac{\mu^2}{m^2}\right) + \mathcal{O}(\epsilon) \right], \tag{5.9}$$

where $\gamma_{\mathrm{E}} \simeq 0.577$ is the Euler–Mascheroni constant. As shown in this result, dimensional regularization separates the divergence as $\epsilon \to 0$ from the finite part of the integral. Furthermore, the dependence on the arbitrary energy scale μ is made clear. These calculations are implemented in several software packages such as `FeynCalc` [312–314] and `Package-X` [315].

5.1.2 Renormalization

After regularizing the loop integrals, the next step is renormalization, which corresponds to removing the divergent parts of the result so that we are left with a meaningful physical quantity. This is done by adding to the original (called "bare")

Lagrangian counterterms that are constructed so as to cancel the divergences in calculated physical quantities, *i.e.*

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{ct}}, \tag{5.10}$$

where $\mathcal{L}_0$ is the bare Lagrangian, $\mathcal{L}_{\text{ct}}$ contains the counterterms, and $\mathcal{L}$ is known as the renormalized Lagrangian, from which finite physical quantities may be derived. The counterterm Lagrangian adds a number of additional Feynman rules, which give rise to divergent contributions of the same magnitude and opposite sign to the original divergences. In this way, the final result is finite.

Since there are many ways to divide a divergent integral into a convergent part and divergent part, there is freedom in choosing a convenient method. One convenient renormalization scheme is the "modified minimal subtraction" scheme, $\overline{\text{MS}}$ [316], which subtracts the $1/\epsilon$ pole as well as extra constants $\ln 4\pi - \gamma_{\text{E}}$ that often arise. This is in contrast to the minimal subtraction scheme, MS [317, 318], which subtracts only the $1/\epsilon$ pole. Both of these schemes require the regularization to be performed using dimensional regularization.

In the process of dimensional regularization, the energy scale μ was introduced. Any quantity computed in renormalized perturbation theory will, in general, depend on this scale. However, the original bare quantities are independent of it. Thus, we can relate the bare and renormalized quantities to each other and impose the condition that the bare quantity is independent of μ. As a result, the dependence of the renormalized quantity on μ is absorbed into the renormalized coupling constants.

Consider an $(n+m)$-point Green's function involving n fermion fields ψ and m scalar fields ϕ with a coupling y. The bare and renormalized Green's functions will be related via a field rescaling, namely

$$G_0^{(n,m)}(\{x_i\}, y_0) = Z_\psi^{n/2} Z_\phi^{m/2} G^{(n,m)}(\{x_i\}, y, \mu), \tag{5.11}$$

where y_0 is the bare coupling. The bare Green's function must be independent of μ, since the scale μ is an artifact of the renormalization process and not a part of the bare theory. In other words, we find that

$$\begin{aligned} 0 &= \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_0^{(n,m)}(\{x_i\}, y_0, \mu) \\ &= Z_\psi^{n/2} Z_\phi^{m/2} \left(\mu \frac{\partial}{\partial \mu} + \frac{n}{2} \frac{\mu}{Z_\psi} \frac{\partial Z_\psi}{\partial \mu} + \frac{m}{2} \frac{\mu}{Z_\phi} \frac{\partial Z_\phi}{\partial \mu} + \mu \frac{\partial y}{\partial \mu} \frac{\partial}{\partial y} \right) G^{(n,m)}(\{x_i\}, y, \mu). \end{aligned} \tag{5.12}$$

This is the Callan–Symanzik equation [319–321], from which we define the β function for the coupling y as

$$\beta_y \equiv \mu \frac{\partial y}{\partial \mu}. \tag{5.13}$$

If the Green's function depends on several couplings c_i, then there are more terms like the last one in Eq. (5.12) with y replaced by c_i and each one has a corresponding β function.

By computing the Green's functions and counterterms perturbatively, one can solve the Callan–Symanzik equation for the β function. This gives the RGE for the coupling y, which determines its running behavior, meaning that its value depends on the center-of-mass energy of the interaction. There are standard formulas for the β functions of the gauge couplings in non-Abelian gauge theories as well as the Yukawa couplings and the scalar quartic couplings [322–328].

The gauge coupling g_i of a non-Abelian theory with group $\mathcal{G}_i$ has β function to two-loop order given by [326, 329, 330]

$$\beta(g_i) = -\frac{g_i^3}{(4\pi)^2}a_i - \frac{g_i^3}{(4\pi)^4}\left[\sum_j g_j^2 b_{ij} + 2\kappa_F Y_4(\mathbf{F}_i)\right], \tag{5.14}$$

where

$$a_i = \frac{11}{3}C_2(\mathbf{G}_i) - \frac{4}{3}\kappa_F S_2(\mathbf{F}_i) - \frac{1}{3}\kappa_S S_2(\mathbf{S}_i), \tag{5.15}$$

$$\begin{aligned} b_{ij} = \frac{34}{3}\left[C_2(\mathbf{G}_j)\right]^2\delta_{ij} - \kappa_F\left[4C_2(\mathbf{F}_j) + \frac{20}{3}C_2(\mathbf{G}_j)\delta_{ij}\right]S_2(\mathbf{F}_i) \\ - \kappa_S\left[4C_2(\mathbf{S}_j) + \frac{2}{3}C_2(\mathbf{G}_j)\delta_{ij}\right]S_2(\mathbf{S}_i). \end{aligned} \tag{5.16}$$

Here, $C_2(\mathbf{r})$ and $S_2(\mathbf{r})$ are the quadratic Casimir and Dynkin indices, respectively, of the representation $\mathbf{r}$. The representations $\mathbf{G}_i$, $\mathbf{F}_i$, and $\mathbf{S}_i$ denote the adjoint, fermion, and scalar representations, respectively, under the gauge group $\mathcal{G}_i$, summed over all degrees of freedom in the model. The coefficient κ_F is 1 ($\frac{1}{2}$) for Dirac (Weyl) fermions and κ_S is 1 ($\frac{1}{2}$) for complex (real) scalars. The sum over j runs over all gauge group factors in the semisimple group $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_n$. Finally, we have

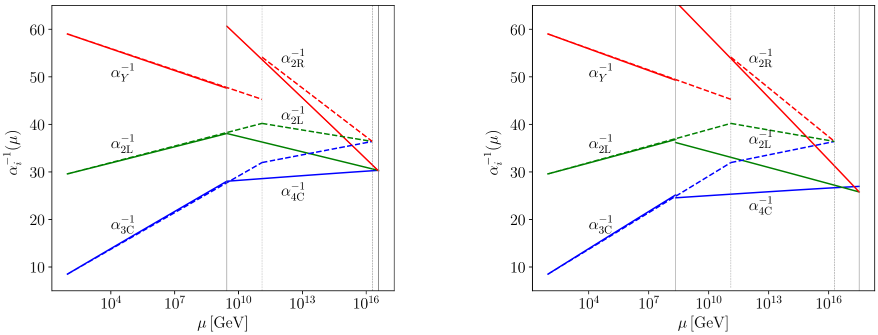
$$Y_4(\mathbf{F}_i) = \frac{1}{d(\mathbf{G}_i)}\sum_a C_2(\mathbf{f}_i^a)\,\mathrm{Tr}\left(Y^a Y^{a\dagger}\right), \tag{5.17}$$

where the sum over a runs over the various Yukawa coupling matrices in the theory and $\mathbf{f}_i^a$ denotes the particular fermion representations involved in the Yukawa coupling Y^a. In practice, the term with the Yukawa couplings is often neglected when analyzing the RG running of gauge couplings, since it is highly model dependent and is usually small compared to the rest of the two-loop contribution.

Inserting the SM field content into Eqs. (5.15)–(5.16), we find the one-loop coefficients a_i and the two-loop coefficients b_{ij} given by

$$a_i = \left(-7, -\frac{19}{6}, \frac{41}{10}\right), \quad b_{ij} = \begin{pmatrix} -26 & \frac{9}{2} & \frac{11}{10} \\ 12 & \frac{35}{6} & \frac{9}{10} \\ \frac{44}{5} & \frac{27}{10} & \frac{199}{50} \end{pmatrix}. \tag{5.18}$$

In the left panel of Fig. 5.1, we show the two-loop RG running in the Pati–Salam model with the one-loop result shown by the dashed lines. In particular, we



(a) Two-loop RG running **(b)** One-loop threshold corrections

Figure 5.1: Gauge coupling unification wiht Pati–Salam intermediate symmetry using two-loop RG running (left) and two-loop RG running with one-loop threshold corrections (right). The dashed lines in both panels show the result for one-loop RG running without threshold corrections. Vertical lines denote the intermediate energy scale and M_{GUT}

see that the difference between one- and two-loop order is larger in the PS model than in the SM. The intermediate energy scale is lowered when including two-loop RGEs, while M_{GUT} is increased by only a small amount.

The general formulas for the β functions of the Yukawa and scalar quartic couplings are somewhat more complicated, and we refer the reader to Refs. [326–328, 331, 332]. They are also implemented in several software packages, such as `PyR@TE` [331] and `SARAH` [333].

The total set of RGEs that are needed to perform the RG running from M_{GUT} to M_Z are given in App. B. These are the β functions for the gauge couplings, the Yukawa coupling matrices, the right-handed neutrino mass matrix, and the Higgs quartic self-coupling.

5.2 Effective field theory and thresholds

In general, there are several scales involved in a theory, for example M_Z and M_{GUT}. These scales are highly separated and one would expect the physics at M_Z to be largely independent of the physics at M_{GUT}. This separation of scales is described by the Appelquist–Carazzone theorem [334], which states that a high-energy theory may be described by an effective field theory that coincides with the high-energy theory in the limit of small momenta. The effect of a particle in the high-energy theory with mass M contributes to the amplitudes with a suppression factor p^2/M^2. Hence, heavy particles decouple in the limit $p^2 \ll M^2$. The theorem was proven in an on-shell renormalization scheme, but also holds in mass-independent schemes such as $\overline{\mathrm{MS}}$ [335–337]. In such schemes, the particles need to be removed from the theory (integrated out) by hand, and the suppressed low-energy corrections from the high-energy theory enter as higher-dimensional operators and finite shifts of the parameters of the theory. Below the scale in question, one considers an effective field theory in which the heavy particles do not enter [338].

A familiar example is Fermi's four-fermion interaction, which is an effective theory valid at energies well below the electroweak scale. After integrating out the electroweak gauge bosons, one ends up with an effective four-fermion interaction. In the effective theory, this is a non-renormalizable dimension-six operator, with coefficient proportional to $1/M_Z^2$.

During the renormalization group running over some energy range, one may encounter several such mass thresholds that separate two theories from each other. This can be due to symmetry breaking, which gives rise to massive gauge bosons, or due to other massive particles in the theory. One example is the spontaneous symmetry breaking of a GUT to its subgroup at the scale M_{GUT}, at which scale some gauge and scalar bosons receive masses at the breaking scale. Another example is the mass scale of heavy right-handed neutrinos, which results in the effective neutrino mass operator described in Sec. 3.2.2

5.2.1 Symmetry breaking and matching of gauge couplings

At an energy scale where spontaneous symmetry breaking occurs, the gauge couplings of the theory below that scale need to be matched to those of the theory above it. It is typically the case that a number of fields have masses around the energy scale at which the gauge couplings are to be matched. Those heavy fields then need to be integrated out of the theory.

The integration out of the heavy fields affects the matching of the gauge couplings, which can be derived by comparing the gauge boson two-point function in the full and the effective theories [147, 148, 336, 337]. By considering this for one of the gauge bosons, one finds loop corrections in the full theory that do not exist in the effective theory. As an effect, one obtains a redefinition of the fields, which also leads to an effective coupling constant that differs from the direct tree-level matching.

At an energy scale $M_{m\to n}$ at which a group $\mathcal{G}_m$ breaks to $\mathcal{G}_n$, the matching of the inverse squared gauge couplings to one-loop order may be written as [147, 148]

$$\alpha_n^{-1}(M_{m\to n}) = \alpha_m^{-1}(M_{m\to n}) - \frac{1}{12\pi}\left\{\sum_{i\in\text{vectors}} k_{\mathbf{V}_i} S_2(\mathbf{V}_i)\left[1 - 21\ln\left(\frac{M_{\mathbf{V}_i}}{M_{m\to n}}\right)\right]\right.$$
$$\left. + \sum_{i\in\text{scalars}} \kappa_{\mathbf{S}_i} k_{\mathbf{S}_i} S_2(\mathbf{S}_i)\ln\left(\frac{M_{\mathbf{S}_i}}{M_{m\to n}}\right) + 8\sum_{i\in\text{fermions}} k_{\mathbf{F}_i}\ln\left(\frac{M_{\mathbf{F}_i}}{M_{m\to n}}\right)\right\}, \quad (5.19)$$

where each sum runs over the fields that are integrated out around the scale $M_{m\to n}$, S_2 is the Dynkin index, k_i is the multiplicity of the field i taking into account the dimensions of the representation under the other gauge groups, and $\kappa_{\mathbf{S}_i}$ is 1 (2) for real (complex) scalar fields.

As such, the matching at loop order receives threshold corrections that depend on the particle content that lies around the matching scale. Typically, the gauge bosons have masses around the matching scale, so the last term on the first line of Eq. (5.19) can be ignored. Furthermore, most standard SO(10) models do not have any fermions around the matching scale, in which case the only dependence on the masses comes from the spectrum of the scalar fields through the first term in the second line.

As is typical for SO(10) models, if there is a large number of fields around the matching scale, this can lighten the restrictions of gauge coupling unification. Additionally, threshold corrections can modify the scale of unification and hence allow models that were previously ruled out by constraints from proton decay. As an example, the right panel of Fig. 5.1 shows the RG running of the gauge couplings in the Pati–Salam model at two-loop order with one-loop threshold corrections. The one-loop RG running is shown in dashed lines for comparison. Due to the threshold corrections, the gauge couplings do not meet at a single point at M_{GUT}, but they are still unified. A similar effect can be seen in the matching of $\alpha_{2\mathrm{L}}^{-1}$ between the two theories, as well as the matching of $\alpha_{3\mathrm{C}}^{-1}$ to $\alpha_{4\mathrm{C}}^{-1}$. As a result, M_{GUT} is increased from 3.85×10^{16} GeV to 3.33×10^{17} GeV.

5.2.2 Neutrino effective field theory

Another important application of effective field theories in the RG running between M_Z and M_{GUT} is the integrating out of the heavy right-handed neutrinos, which are relevant for generating neutrino masses through the type I seesaw mechanism. When integrating them out, they are no longer part of the theory, but contribute to the effective dimension-five operator [258], which generates neutrino masses. Therefore, as they are integrated out, their couplings to the light neutrinos are removed and their contribution to the neutrino masses is encoded by altering the coefficient of the effective operator, as mentioned in Ch. 3.

The relevant quantities for this procedure are the Dirac neutrino Yukawa matrix Y_ν, the right-handed Majorana mass matrix M_{R}, and the effective neutrino mass matrix κ. In the full theory, they are all 3×3 matrices with κ being zero. At the threshold corresponding to the heaviest RHN N_3, the relevant coefficients are removed from Y_ν and M_{R} and added to κ, following the process outlined in Refs. [339, 340]. Thus, the last row of Y_ν is removed and it becomes a 2×3 matrix. The last row and column of M_{R} are removed and it becomes a 2×2 matrix. The effective neutrino mass matrix κ is updated to

$$\kappa \to \kappa + \frac{2}{M_3} \left(Y_\nu^{(3)}\right)^T \left(Y_\nu^{(3)}\right), \tag{5.20}$$

where $Y_\nu^{(3)}$ is the last row of Y_ν, which was removed, and M_3 is the mass of N_3. On the right-hand side of Eq. (5.20), κ is typically zero, but it may be non-zero if it received some other contribution at an energy above this. The procedure of removing rows or columns from matrices is clearly basis-dependent. It is therefore important to transform to a basis in which M_R is diagonal and apply the corresponding basis transformation to Y_ν before applying this procedure.

The matching procedure at the second heaviest RHN N_2 follows the same method. The last row is removed from Y_ν such that it becomes a 1×3 matrix and the last row and last column of M_R are removed such that it becomes a 1×1 matrix. The effective neutrino mass matrix is updated according to

$$\kappa \to \kappa + \frac{2}{M_2} \left(Y_\nu^{(2)}\right)^T \left(Y_\nu^{(2)}\right). \tag{5.21}$$

At the last threshold, the lightest RHN N_1 is integrated out and the matching is

$$\kappa \to \kappa + \frac{2}{M_1} \left(Y_\nu^{(1)}\right)^T \left(Y_\nu^{(1)}\right). \tag{5.22}$$

The difference now is that all entries in the matrices Y_ν and M_R have been removed and these quantities are no longer present in the theory.

If there is also a scalar triplet causing the type II seesaw mechanism, it needs to be integrated out at its mass scale. This is considerably more straightforward than integrating out the RHNs, since there is only one mass threshold. After solving the RGEs down to the mass scale of the scalar triplet, it is integrated out from the theory and the effective neutrino mass matrix is updated to [341]

$$\kappa \to \kappa - 4\frac{v_\mathrm{L}}{v_\mathrm{SM}^2} Y_\mathrm{L}, \tag{5.23}$$

where Y_L is the Yukawa matrix of the coupling between the neutrinos and the scalar triplet, which in the case of an SO(10) model is Y_{126}.

The above matching conditions to κ are at tree level. At loop level, there will be an additional contribution to the matching condition, which is generated from loop-level effects due to the heavy fields that have been integrated out, in the same way that the gauge couplings receive threshold corrections as described in Sec. 5.2.1. This has been computed both in SUSY [342] and non-SUSY [343] models.

5.3 Numerical fitting procedure

To numerically fit the parameters of the SO(10) model to the measured observables of the SM and the neutrino sector, we have to solve the RGEs, taking into account the RHN mass thresholds. Since the matching conditions Eqs. (5.20)–(5.22) contain more parameters on the right-hand side than the left-hand side, the matching can only be performed from high energy to low energy and one must solve the system of RGEs from $M_{\rm GUT}$ down to M_Z. The opposite approach involves extrapolating the observables up to $M_{\rm GUT}$, at which scale the fitting can be performed. Since the matching at the RHN thresholds cannot be performed upwards in a unique way, this procedure must necessarily rely on an approximate RG running for the neutrinos. Particularly, the predicted values for the neutrino masses and leptonic mixing parameters are substantially affected by the details of the RG running and matching [339]. Nevertheless, it provides an indication of whether an SO(10) model is viable and has therefore been used extensively in numerical fits [152, 209, 210, 246, 247, 253, 344–347].

In the work that comprises Papers I, II, III, and V, we perform the RG running from the high-energy theory down to the experimentally accessible energy at M_Z, as this enables a consistent analysis of the effects of RG running and matching at RHN and scalar triplet thresholds and symmetry breaking scales. The solution of the RGEs in numerical fits has been done previously in some models, for example [42, 211, 348, 349].

5.3.1 Parametrization

For simplicity, consider a model with a Higgs sector consisting of a complexified $\mathbf{10}_H$ and a $\overline{\mathbf{126}}_H$, and with a $\mathrm{U}(1)_{\rm PQ}$ symmetry as discussed in Sec. 4.3.2. Including $\mathbf{120}_H$ as in Papers I and V leads to the same logic, but only adds more parameters. Assume that the neutrino masses are generated purely by the type I seesaw mechanism such that we can ignore any contributions from the scalar triplet that leads to the type II seesaw mechanism.

The first step is to sample the parameters of the SO(10) model, which are the elements of the Yukawa coupling matrices Y_{10} and Y_{126}, as well as the vevs $v^{u,d}_{10,126}$ and $v_{\rm R}$. It is convenient to rescale the parameters such that [211, 246]

$$\begin{aligned} H &\equiv \frac{v^d_{10}}{v_{\rm SM}} Y_{10}, \quad F \equiv \frac{v^d_{126}}{v_{\rm SM}} Y_{126}, \quad r \equiv \frac{v^u_{10}}{v^d_{10}}, \\ s &\equiv \frac{1}{r}\frac{v^u_{126}}{v^d_{126}} = \frac{v^d_{10}}{v^u_{10}}\frac{v^u_{126}}{v^d_{126}}, \quad r_{\rm R} \equiv v_{\rm R}\frac{v_{\rm SM}}{v^d_{126}}, \end{aligned} \tag{5.24}$$

and sample these instead. Here, $v_{\rm SM} = v/\sqrt{2}$, with v being the vev as given in Sec. 2.1.3. The matching conditions for this model to the SM Yukawa matrices Eqs. (4.40)–(4.44) are then more simply written as

$$Y_u = r(H + sF), \tag{5.25}$$
$$Y_d = H + F, \tag{5.26}$$
$$Y_\nu = r(H - 3sF), \tag{5.27}$$
$$Y_\ell = H - 3F, \tag{5.28}$$
$$M_{\rm R} = r_{\rm R} F. \tag{5.29}$$

The Yukawa matrices Y_{10} and Y_{126} are in general complex symmetric 3×3 matrices. One may, however, choose a basis in which Y_{10} (and hence H) is real and diagonal. This means that there are three parameters in H and twelve parameters in F. Since r and r_{R} are overall multiplicative factors, their complex phases will have no effect on the fermion observables, so they can be taken to be real. Finally, the complex phase of s will have an effect and thus it remains complex. The total number of parameters in this model is thus $3(H)+12(F)+1(r)+2(s)+1(r_{\mathrm{R}}) = 19$.

The fitting procedure is performed by first sampling these 19 parameters according to some priors that reflect the expected orders of magnitude of these parameters. For example, we expect the elements of the Yukawa couplings to be between 10^{-6} and 1, while we expect r_{R} to lie around M_{GUT}. For parameters that can vary over several orders of magnitude, it is reasonable to assume a logarithmic prior such that we sample the base-10 logarithm uniformly.

We can rewrite the constraint in Eq. (4.49) in terms of the newly introduced parameters. Neglecting v_{L} and the vevs from $\mathbf{120}_H$, this gives

$$\left(\frac{v_{10}^d}{v_{\mathrm{SM}}}\right)^2 (1+r^2) + \left(\frac{v_{126}^d}{v_{\mathrm{SM}}}\right)^2 (1+r^2 s^2) = 1. \tag{5.30}$$

Since the sampled parameters are r and s, we have some freedom in choosing v_{10}^d and v_{126}^d such that the constraint in Eq. (5.30) is satisfied. The only lower bound on the vevs is that the Yukawa couplings are perturbative. Taking $Y_{10} = v_{\mathrm{SM}} H / v_{10}^d \leq 4\pi$ and $Y_{126} = v_{\mathrm{SM}} F / v_{126}^d \leq 4\pi$ as the perturbativity bound, the constraint in Eq. (5.30) reads

$$\max |H|^2 (1+r^2) + \max |F|^2 (1+r^2 s^2) \leq 16\pi^2, \tag{5.31}$$

where $\max |M|^2$ denotes the largest squared element of the matrix M. Although typical parameter values satisfy this inequality, it should be checked explicitly after each fit.

5.3.2 RG running

After the parameter values have been sampled, they are transformed to the Yukawa couplings of the SM using the matching conditions Eqs. (5.25)–(5.29). Then, they are evolved down from M_{GUT} to M_3 by solving the RGEs. There, N_3 is integrated out following the procedure outlined in Sec. 5.2.2. This modifies Y_ν and M_{R} and introduces κ into the theory. Thereafter, the RGEs are solved from M_3 down to M_2, where N_2 is integrated out. From there, the RGEs are solved from M_2 to M_1 where N_1 is integrated out. Then, Y_ν and M_{R} are no longer part of the theory, so they have no corresponding RGEs. Finally, the RGEs are solved from M_1 down to M_Z. This procedure is depicted in Fig. 5.2.

If we wish to include the type II seesaw mechanism, we simply have to include an extra step in the procedure, by integrating out the scalar triplet at the appropriate energy scale. In that case, we must first identify the mass of the scalar triplet in

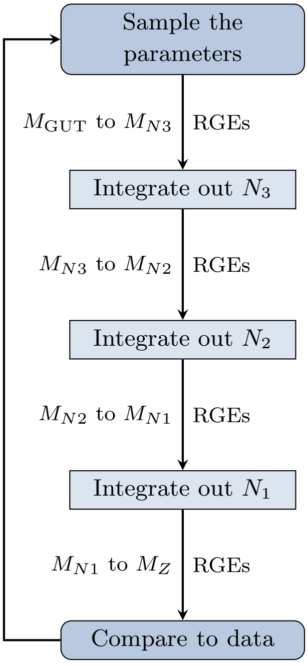


Figure 5.2: Flowchart showing the procedure used in performing the numerical fits for a model with the type I seesaw mechanism. The type II seesaw mechanism may be added by introducing another threshold at which the scalar triplet is integrated out. The procedure is coupled to a sampling algorithm, which samples new sets of parameter values based on the output χ^2.

relation to M_1, M_2, and M_3 in order to integrate it out at the correct step. Similarly, the procedure has to be appropriately modified if there are other intermediate particles or an intermediate symmetry breaking step.

5.3.3 Fitting

Once we have the SM parameters at M_Z, they are transformed to fermion masses and mixing parameters. These are then compared to the data through a χ^2 goodness-of-fit function, defined as

$$\chi^2 = \sum_{i=1}^{N} \frac{(x_i - X_i)^2}{\sigma_i^2}, \tag{5.32}$$

where x_i denotes the predicted value of the ith observable obtained from the RG running and X_i denotes the actual value with corresponding error σ_i. Note that we do not assign any statistical meaning to the χ^2 function and merely use it as a measure of the goodness of fit in order to compare models. As noted in Refs. [42, 245, 350], it is unclear how to define the number of degrees of freedom for a model in which the output parameters are non-linear in the input parameters, and especially if the number of input parameters is larger than the number of output parameters.

The program that takes as input the set of parameter values, performs the RG running and outputs the χ^2 value is linked to a sampling algorithm which performs the minimization of the χ^2 function and returns an optimal set of parameter values. There are many possible optimization algorithms and packages available to choose from. One is `MultiNest` [351–353], which is a nested sampling algorithm with capabilities to perform Bayesian inference. This was used in Paper I. Another one is `Diver` [354], which is part of the ScannerBit package from the GAMBIT collaboration. It is a differential evolution algorithm and has been shown to outperform `MultiNest` in high-dimensional parameter spaces such as the one in our problem. This was used in Papers II, III, and V.

The sampling algorithm was run on a computational cluster, utilizing up to 240 CPU cores. This was repeated several times in order to increase confidence that the minimum was a global one. After reasonable convergence, another local minimization procedure was used, such as the basin-hopping algorithm [355] from the `Scipy` library [356], which perturbs the point around the starting point in parameter space, and a Nelder–Mead simplex algorithm [357], which further improves the best-fit point by traversing the χ^2 manifold downhill. Despite this, it is not, in general, possible to guarantee that the set of parameter values found corresponds to the true optimum. This is a general feature of optimization of non-linear functions with a large number of parameters, as in our problem. Nevertheless, repeating the computation several times with similar results increases the confidence that no better optimum can be found.

Chapter 6

Summary and conclusions

In Part I of this thesis, we introduced the background material relevant to the papers included in Part II. We started in Ch. 2 with the SM of particle physics and the reasons for investigating theories beyond it. These include shortcomings such as the massless nature of neutrinos according to the SM, the lack of a dark matter candidate, and the unknown mechanism for producing a matter-antimatter asymmetry of the Universe. Certain aesthetic shortcomings are also suggestive of a theory beyond the SM, including the origin of the gauge group of the SM, the underlying reasons for anomaly cancellation and charge quantization, as well as problems related to naturalness.

In Ch. 4, we introduced GUTs as an extension of the SM and a possible solution to some of the issues discussed in the preceding chapter. Apart from addressing some of the aesthetic shortcomings of the SM, there is the tantalizing clue of approximate gauge coupling unification at a high scale. We discussed some general features of GUTs and the requirements that a gauge group has to fulfill in order to be a viable GUT. Some of the popular candidates are SU(5), the Pati–Salam group, and SO(10), the features of which we outlined. Being central to this thesis, we focused on the properties of models based on SO(10), including the generation of fermion masses, the many paths through which to recover the SM from SO(10) through spontaneous symmetry breaking, and the constraints due to proton decay. Finally, we commented on some of the phenomenology that SO(10) models may exhibit, including axions, dark matter, and gravitational waves due to cosmological defects.

Finally, in Ch. 5, we discussed how to relate the physics at M_{GUT} to that accessible to us through observations and experiments. This involves solving the renormalization group equations for the parameters of the model. We also discussed how to match the parameters of theories above and below some energy scale. For the gauge couplings, integrating out heavy degrees of freedom can, at the loop level, give rise to threshold corrections, which lighten the constraints of gauge coupling unification. For neutrinos, the seesaw mechanisms imply that there exist heavy fields such as right-handed neutrinos or a scalar triplet. Interating those out gives rise to the effective neutrino mass operator. In order to properly treat the thresholds due to heavy fields, one must perform the RG running from M_{GUT} down to M_Z rather than the other way around. Since this is comparatively more complex, most previous works have used the approximation of extrapolating the measured observables up to M_{GUT} and fitting at that scale. This is a significant approximation and has a large effect on the parameters of the neutrino sector.

In Paper I, we investigated the RG running of fermion observables in an SO(10) model with intermediate symmetry breaking via the PS group. In this paper, we made the assumption that the heavy RHNs all have masses around the intermediate scale such that they are all integrated out at that scale. We compared two different models, namely the minimal model which has a Yukawa sector consisting of scalars in only the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations, and the extended model which additionally has scalars in the $\mathbf{120}_H$ representation. It was found that it is difficult to find a good fit with the minimal model, while the extended model is able to accommodate the fermion observables. Fits to neutrino masses were performed both for normal and inverted ordering and it was found that it is only possible to accommodate normal neutrino mass ordering. The difficulty in finding a good fit comes predominantly from the value of the leptonic mixing angle θ_{23}^{ℓ}, since the model consistently favored values smaller than the actual value.

In Paper II, we considered a model with symmetry breaking of SO(10) to the SM occurring in one step. Gauge coupling unification is achieved by including extra scalars originating in $\mathbf{210}_H$ with masses between M_Z and M_{GUT}, as shown in the left panel of Fig. 4.2. An analysis of gauge coupling unification resulted in a correlation between the masses of these intermediate scalars and the proton lifetime. If proton decay is not observed at Hyper-Kamiokande within five years of the start of operations, the resulting bound on the proton lifetime would, together with LHC bounds on the mass of the extra scalars, rule out this model. The Yukawa sector contained scalars in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations only. It was found that the model can accommodate the fermion observables well, and also that the presence of the intermediate-scale scalars help to stabilize the vacuum. Again, the largest source of difficulty in fitting the fermion observables is a too low predicted value of θ_{23}^{ℓ}.

In Paper III, we considered a generic set of SO(10) models with one-step symmetry breaking. In order to remain agnostic about the mechanism for achieving gauge couplings unification, we did not impose gauge coupling unification, but instead tested the sensitivity of the results to changes in M_{GUT}. The Yukawa sector contained scalars in the $\mathbf{10}_H$ and $\overline{\mathbf{126}}_H$ representations and neutrino masses were generated by the type I or type II seesaw mechanisms, or a combination of both. The results showed that a pure type II seesaw mechanism is disfavored by the fits and that when both were combined, the type I mechanism is the dominant contributor to neutrino masses. As in Paper I, neutrino masses can only be accommodated with normal ordering and not with inverted ordering. It was seen that the results of the fit are fairly insensitive to changes in M_{GUT} within an order of magnitude, showing that the problem of determining the unification scale is, to some extent, decoupled from the parameters in the fermion sector. Once again, it was consistently observed that the largest difficulty in finding a good fit comes from the favored value of θ_{23}^{ℓ} being too low compared to the actual value.

Paper IV, dealt in detail with gauge coupling unification and the constraints placed on models by the proton lifetime. We considered RG running of the gauge couplings at the two-loop level and included the threshold corrections at the symmetry breaking scale, as discussed in Sec. 5.2.1. For all possible models with either direct symmetry breaking to the SM or one intermediate symmetry, we computed the threshold corrections assuming minimal particle content. We then investigated the required sizes of threshold corrections, parametrized by the largest deviation of a heavy field from the energy scale of symmetry breaking, in order to satisfy the proton lifetime constraint. The results showed that only $\mathcal{G}_{\mathrm{PS}}$ and $\mathcal{G}_{3221}$ are allowed if all heavy fields have masses at precisely the symmetry breaking scale. Allowing the logarithmic deviation of the masses to be within $[-1, +1]$, we also found that $\mathcal{G}_{422\mathrm{D}}$ and $\mathcal{G}_{3221\mathrm{D}}$ are allowed. Increasing this to $[-2, +2]$ allows also the model with direct symmetry breaking to the SM.

Finally, in Paper V, we considered models of flavor in SO(10) based on phase transformations of the fermion and scalar fields that result in Yukawa matrix textures, as discussed in Sec. 4.4. These models had been derived in previous works and investigated in a SUSY setting. We performed fits to their non-SUSY counterparts, properly taking into account the RG running and integrating out heavy degrees of freedom, as done in Papers II and III. Out of the 14 models, only two, which have a $\mathbb{Z}_2$ symmetry, resulted in acceptable fits. These were the same as in the SUSY case. The neutrino masses were generated by a combination of type I and type II seesaw mechanisms and we found that the dominant contribution is from the type I seesaw mechanism, as in Paper III. In contrast with the previous fits, the largest tension comes from the fermion masses, rather than the mixing parameters.

The work in this thesis has focused on the Yukawa sector of non-SUSY SO(10)-based GUT models. The question that has led this research is: Is SO(10) a viable model for the fermion masses and mixing parameters, taking into account the RG running between M_{GUT} and M_Z? The answer is: Yes, depending on the details of the model. Since a proper treatment of the RGEs and thresholds of heavy RHNs have a large impact on the neutrino observables, it is important to deal with these correctly.

SO(10) in particular and grand unification in general continue to provide a promising framework for physics beyond the SM. In addition to providing more structure behind some of the arbitrary aspects of the SM, it can incorporate a wide range of phenomenology that addresses some of the current open questions in particle physics. The criticism against grand unification is primarily that the unification scale is much higher than will be reachable in the near future and that it is thus not testable. While it is true that the unification scale will most likely not be reachable in colliders in the foreseeable future, there are still some signatures of grand unification that can be observed. The most notable of these is proton decay, the detection of which may still be possible in future experiments. We can also test particular models of grand unification through fits to the parameters of the Yukawa sector. As the simple models become ruled out, we can construct increasingly more complicated ones that are able to accommodate the measured parameters.

The arguments in favor of grand unification are based on simplicity and mathematical elegance. This is now in tension with the complexity required of specific models in order to accommodate the known parameters.

Appendix A

Group theory

This appendix briefly summarizes some aspects of group and representation theory that are relevant to the study of GUTs. Further details about group theory can be found in textbooks such as Ref. [358] or the review Ref. [157], which contains numerous tables that are useful for model building. There are several software packages that are capable of performing group theoretic calculations, such as `Susyno` [359] or `LieART` [360].

A.1 Basics of Lie groups, Lie algebras, and representations

A Lie group is a group in which the elements depend on a set of continuous parameters. Elements can be written in terms of the generators t^a of the group via the exponential map, namely

$$g(\alpha) = \exp(\mathrm{i}\alpha^a t^a). \tag{A.1}$$

The generators form a Lie algebra with the Lie bracket

$$[t^a, t^b] = \mathrm{i} f^{abc} t^c, \tag{A.2}$$

where f^{abc} are the structure constants of the Lie algebra.

We may define a representation of the group as a map from the group elements to square matrices such that the elements of the group representation act on elements of a vector space. The representation of the group is also a representation of the algebra, in the sense that the representation matrices of the algebra generate the representation matrices of the group via the exponential map. The dimension of the representation is the number of dimensions of the vector space on which the representation matrices act. Thus, a representation with matrices of size 3×3 is 3-dimensional.

We are usually interested in irreducible representations, which are representations with no invariant subspaces. In contrast, reducible representations contain invariant subspaces and may therefore be decomposed into direct products of irreducible representations.

For each representation, there exists a set of mutually commuting Hermitian generators. The matrices that form the largest such set are called the Cartan generators. They are useful because they can be simultaneously diagonalized. Hence, they can be used to assign quantum numbers to states within a representation, such as the T_3 generator of $\mathrm{SU}(2)_\mathrm{L}$ in the Gell-Mann–Nishijima formula in Eq. (2.16). This becomes particularly relevant in symmetry breaking, since the Abelian charges will be combinations of these quantum numbers. The number of Cartan generators of an algebra is known as its rank.

Since the SM contains an SU(2) and an SU(3) group, SU(N) is highly relevant for particle physics. This is the group defined by $N \times N$ special unitary matrices. That is, its elements in the defining representation are $N \times N$ unitary matrices with determinant 1. There are $N^2 - 1$ such matrices, meaning that there are $N^2 - 1$ generators of SU(N). These are $N \times N$ traceless Hermitian matrices.

Relevant for GUT model building are also the SO(N) groups. The defining representation of these are the set of $N \times N$ orthogonal matrices with determinant 1, of which there are $N(N-1)/2$. Hence the generators in this representation are antisymmetric traceless $N \times N$ matrices.

Two useful constants for calculations of β-functions are the quadratic Casimir and the Dynkin index. The quadratic Casimir $C_2(\mathbf{r})$ is defined for a representation $\mathbf{r}$ of a Lie algebra as

$$t^a_\mathbf{r} t^a_\mathbf{r} = C_2(\mathbf{r})\mathbb{1}, \tag{A.3}$$

where the index a is summed over the generators of that representation. The Dynkin index $S_2(\mathbf{r})$ is defined as

$$\mathrm{Tr}(t^a_\mathbf{r} t^b_\mathbf{r}) = S_2(\mathbf{r})\delta^{ab}. \tag{A.4}$$

The two are related by the relation

$$S_2(\mathbf{r}) = \frac{\dim(\mathbf{r})}{\dim(\mathbf{G})} C_2(\mathbf{r}), \tag{A.5}$$

where $\dim(\mathbf{r})$ is the dimension of representation $\mathbf{r}$ and $\mathbf{G}$ denotes the adjoint representation.

A.2 Decompositions of some SO(10) representations

To construct a model beyond the SM, we need to make sure that it can reproduce the SM. Thus, the GUT symmetry needs to be broken down to $\mathcal{G}_\mathrm{SM}$. Therefore, it is useful to know how the different representations of SO(10) decompose under its different subgroups. This is necessary both for the study of how to produce each breaking chain by looking for the singlets under the different subgroups and also for tracking how a given representation traverses the breaking chain down to $\mathcal{G}_{\rm SM}$. Thus, we give the decompositions of the representations up to dimension 210 in the SU(5) breaking chain in Tab. A.1 and the PS breaking chain in Tab. A.2, computed using `Susyno` [359].

SO(10)	$\mathcal{G}_{51}$	$\mathcal{G}_{\rm SM}$
$\mathbf{10}$	$\mathbf{5}_{-2}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{2}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
$\mathbf{16}$	$\mathbf{10}_{1}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{5}}_{-3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{1}_{5}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{45}$	$\mathbf{24}_{0}$	$(\mathbf{8},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{1},\mathbf{1})_{0}$
	$\mathbf{10}_{-4}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{10}}_{4}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{1}_{0}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{54}$	$\mathbf{24}_{0}$	$(\mathbf{8},\mathbf{1})_{0} \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_{0} \oplus (\mathbf{1},\mathbf{1})_{0}$
	$\mathbf{15}_{-4}$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{3})_{1}$
	$\overline{\mathbf{15}}_{4}$	$(\mathbf{6},\mathbf{1})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{1},\mathbf{3})_{-1}$
$\mathbf{120}$	$\mathbf{45}_{-2}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6}$ $\oplus (\mathbf{3},\mathbf{1})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{45}}_{2}$	$(\mathbf{8},\mathbf{2})_{-1/2} \oplus (\overline{\mathbf{6}},\mathbf{1})_{1/3} \oplus (\overline{\mathbf{3}},\mathbf{3})_{1/3} \oplus (\mathbf{3},\mathbf{2})_{7/6}$ $\oplus (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{3},\mathbf{1})_{-4/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{10}_{6}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_{1}$
	$\overline{\mathbf{10}}_{-6}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{5}_{-2}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{2}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$

$\overline{\mathbf{126}}$	$\overline{\mathbf{50}}_2$	$(\mathbf{8},\mathbf{2})_{-1/2} \oplus (\overline{\mathbf{6}},\mathbf{3})_{1/3} \oplus (\mathbf{6},\mathbf{1})_{-4/3} \oplus (\mathbf{3},\mathbf{2})_{7/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{1})_2$
	$\mathbf{45}_{-2}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6} \oplus (\mathbf{3},\mathbf{1})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{4/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{15}}_{-6}$	$(\mathbf{6},\mathbf{1})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{1},\mathbf{3})_{-1}$
	$\mathbf{10}_6$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\overline{\mathbf{5}}_2$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{1}_{10}$	$(\mathbf{1},\mathbf{1})_0$
$\mathbf{144}$	$\overline{\mathbf{45}}_{-3}$	$(\mathbf{8},\mathbf{2})_{1/2} \oplus (\mathbf{6},\mathbf{1})_{-1/3} \oplus (\mathbf{3},\mathbf{3})_{-1/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
	$\mathbf{40}_1$	$(\mathbf{6},\mathbf{2})_{1/6} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\overline{\mathbf{3}},\mathbf{3})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{2})_{-3/2}$
	$\mathbf{24}_5$	$(\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{15}_1$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6} \oplus (\mathbf{1},\mathbf{3})_1$
	$\mathbf{10}_1$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\mathbf{5}_{-7}$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{-3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
$\mathbf{210}$	$\mathbf{75}_0$	$(\mathbf{8},\mathbf{3})_0 \oplus (\mathbf{6},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{6}},\mathbf{2})_{5/6} \oplus (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{3},\mathbf{1})_{5/3} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-5/3} \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{40}_{-4}$	$(\mathbf{6},\mathbf{2})_{1/6} \oplus (\mathbf{8},\mathbf{1})_1 \oplus (\overline{\mathbf{3}},\mathbf{3})_{-2/3} \oplus (\mathbf{3},\mathbf{2})_{1/6}$
	$\overline{\mathbf{40}}_4$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6} \oplus (\mathbf{8},\mathbf{1})_{-1} \oplus (\mathbf{3},\mathbf{3})_{2/3} \oplus (\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
	$\mathbf{24}_0$	$(\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-5/6} \oplus (\overline{\mathbf{3}},\mathbf{2})_{5/6} \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0$
	$\mathbf{10}_{-4}$	$(\mathbf{3},\mathbf{2})_{1/6} \oplus (\overline{\mathbf{3}},\mathbf{1})_{-2/3} \oplus (\mathbf{1},\mathbf{1})_1$
	$\overline{\mathbf{10}}_4$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6} \oplus (\mathbf{3},\mathbf{1})_{2/3} \oplus (\mathbf{1},\mathbf{1})_{-1}$
	$\mathbf{5}_8$	$(\mathbf{3},\mathbf{1})_{-1/3} \oplus (\mathbf{1},\mathbf{2})_{1/2}$
	$\overline{\mathbf{5}}_{-8}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3} \oplus (\mathbf{1},\mathbf{2})_{-1/2}$
	$\mathbf{1}_0$	$(\mathbf{1},\mathbf{1})_0$

Table A.1: Decompositions of representations of SO(10) up to 210 dimensions under the SU(5) breaking chain. Here, $\mathcal{G}_{51} = \mathrm{SU}(5) \times \mathrm{U}(1)$. We assume the standard hypercharge embedding in $\mathrm{SU}(5) \times \mathrm{U}(1)$. For the flipped hypercharge embedding, the hypercharge should be $Y = \frac{1}{5}(X - Y')$, where Y' is the Abelian charge from within the SU(5) group and X is the charge of the external Abelian group. For the decomposition under $\mathcal{G}_5$, simply remove the U(1) charge from the $\mathcal{G}_{51}$ representations.

SO(10)	$\mathcal{G}_{\rm PS}$	$\mathcal{G}_{3221}$	$\mathcal{G}_{3211}$	$\mathcal{G}_{\rm SM}$
$\mathbf{10}$	$(\mathbf{6},\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{2},\mathbf{2})_{0}$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
$\mathbf{16}$	$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(\mathbf{3},\mathbf{2},\mathbf{1})_{1/3}$	$(\mathbf{3},\mathbf{2})_{0,1/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{1})_{-1}$	$(\mathbf{1},\mathbf{2})_{0,-1}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
	$(\overline{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-1/2,-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{1/2,-1/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{2})_{1}$	$(\mathbf{1},\mathbf{1})_{1/2,1}$	$(\mathbf{1},\mathbf{1})_{1}$
			$(\mathbf{1},\mathbf{1})_{-1/2,1}$	$(\mathbf{1},\mathbf{1})_{0}$
$\mathbf{45}$	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
	$(\mathbf{1},\mathbf{3},\mathbf{1})$	$(\mathbf{1},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\mathbf{1},\mathbf{1},\mathbf{3})$	$(\mathbf{1},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{1})_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{-1,0}$	$(\mathbf{1},\mathbf{1})_{1}$
$\mathbf{54}$	$(\mathbf{20'},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{6},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{6},\mathbf{1})_{0,4/3}$	$(\mathbf{6},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{6}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{-2/3}$
	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{1},\mathbf{3},\mathbf{3})$	$(\mathbf{1},\mathbf{3},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{3})_{1,0}$	$(\mathbf{1},\mathbf{3})_{-1}$
			$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
			$(\mathbf{1},\mathbf{3})_{-1,0}$	$(\mathbf{1},\mathbf{3})_{1}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$

$\mathbf{120}$	$(\mathbf{15},\mathbf{2},\mathbf{2})$	$(\mathbf{8},\mathbf{2},\mathbf{2})_0$	$(\mathbf{8},\mathbf{2})_{1/2,0}$	$(\mathbf{8},\mathbf{2})_{-1/2}$
			$(\mathbf{8},\mathbf{2})_{-1/2,0}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{4/3}$	$(\mathbf{3},\mathbf{2})_{1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{7/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
	$(\mathbf{6},\mathbf{3},\mathbf{1})$	$(\mathbf{3},\mathbf{3},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{3})_{0,-2/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{1/3}$
	$(\mathbf{6},\mathbf{1},\mathbf{3})$	$(\mathbf{3},\mathbf{1},\mathbf{3})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{1,-2/3}$	$(\mathbf{3},\mathbf{1})_{-4/3}$
			$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1,-2/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
	$(\mathbf{10},\mathbf{1},\mathbf{1})$	$(\overline{\mathbf{6}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{6}},\mathbf{1})_{1/3}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{-2}$	$(\mathbf{1},\mathbf{1})_{0,-2}$	$(\mathbf{1},\mathbf{1})_{-1}$
	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{1})$	$(\mathbf{6},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{6},\mathbf{1})_{0,-2/3}$	$(\mathbf{6},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{2}$	$(\mathbf{1},\mathbf{1})_{0,2}$	$(\mathbf{1},\mathbf{1})_{1}$
	$(\mathbf{1},\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
$\overline{\mathbf{126}}$	$(\mathbf{15},\mathbf{2},\mathbf{2})$	$(\mathbf{8},\mathbf{2},\mathbf{2})_0$	$(\mathbf{8},\mathbf{2})_{1/2,0}$	$(\mathbf{8},\mathbf{2})_{-1/2}$
			$(\mathbf{8},\mathbf{2})_{-1/2,0}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{4/3}$	$(\mathbf{3},\mathbf{2})_{1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,4/3}$	$(\mathbf{3},\mathbf{2})_{7/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,-4/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_0$	$(\mathbf{1},\mathbf{2})_{1/2,0}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,0}$	$(\mathbf{1},\mathbf{2})_{1/2}$

	$(\overline{\mathbf{10}},\mathbf{1},\mathbf{3})$	$(\mathbf{6},\mathbf{1},\mathbf{3})_{-2/3}$	$(\mathbf{6},\mathbf{1})_{1,-2/3}$	$(\mathbf{6},\mathbf{1})_{-4/3}$
			$(\mathbf{6},\mathbf{1})_{0,-2/3}$	$(\mathbf{6},\mathbf{1})_{-1/3}$
			$(\mathbf{6},\mathbf{1})_{-1,-2/3}$	$(\mathbf{6},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{3})_{2}$	$(\mathbf{1},\mathbf{1})_{1,2}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{0,2}$	$(\mathbf{1},\mathbf{1})_{1}$
			$(\mathbf{1},\mathbf{1})_{-1,2}$	$(\mathbf{1},\mathbf{1})_{2}$
	$(\mathbf{10},\mathbf{3},\mathbf{1})$	$(\overline{\mathbf{6}},\mathbf{3},\mathbf{1})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{3})_{0,2/3}$	$(\overline{\mathbf{6}},\mathbf{3})_{1/3}$
		$(\mathbf{3},\mathbf{3},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{3})_{0,-2/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{1})_{-2}$	$(\mathbf{1},\mathbf{3})_{0,-2}$	$(\mathbf{1},\mathbf{3})_{-1}$
	$(\mathbf{6},\mathbf{1},\mathbf{1})$	$(\mathbf{3},\mathbf{1},\mathbf{1})_{-2/3}$	$(\mathbf{3},\mathbf{1})_{0,-2/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,2/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
144	$(\mathbf{20},\mathbf{1},\mathbf{2})$	$(\mathbf{8},\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{8},\mathbf{1})_{1/2,-1}$	$(\mathbf{8},\mathbf{1})_{-1}$
			$(\mathbf{8},\mathbf{1})_{-1/2,-1}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{6},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{6},\mathbf{1})_{1/2,1/3}$	$(\mathbf{6},\mathbf{1})_{-1/3}$
			$(\mathbf{6},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{6},\mathbf{1})_{2/3}$
		$(\mathbf{3},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{1})_{1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{2})_{5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/2,5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1/2,5/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{4/3}$
	$(\overline{\mathbf{20}},\mathbf{2},\mathbf{1})$	$(\mathbf{8},\mathbf{2},\mathbf{1})_{1}$	$(\mathbf{8},\mathbf{2})_{0,1}$	$(\mathbf{8},\mathbf{2})_{1/2}$
		$(\overline{\mathbf{6}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6}$
		$(\mathbf{3},\mathbf{2},\mathbf{1})_{-5/3}$	$(\mathbf{3},\mathbf{2})_{0,-5/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
	$(\mathbf{4},\mathbf{3},\mathbf{2})$	$(\mathbf{3},\mathbf{3},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{3})_{1/2,1/3}$	$(\mathbf{3},\mathbf{3})_{-1/3}$
			$(\mathbf{3},\mathbf{3})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{3})_{2/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{3})_{1/2,-1}$	$(\mathbf{1},\mathbf{3})_{-1}$
			$(\mathbf{1},\mathbf{3})_{-1/2,-1}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\overline{\mathbf{4}},\mathbf{2},\mathbf{3})$	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{3})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-7/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{3})_{1}$	$(\mathbf{1},\mathbf{2})_{1,1}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
			$(\mathbf{1},\mathbf{2})_{0,1}$	$(\mathbf{1},\mathbf{2})_{1/2}$
			$(\mathbf{1},\mathbf{2})_{-1,1}$	$(\mathbf{1},\mathbf{2})_{3/2}$
	$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(\mathbf{3},\mathbf{1},\mathbf{2})_{1/3}$	$(\mathbf{3},\mathbf{1})_{1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{-1/2,1/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{2})_{-1}$	$(\mathbf{1},\mathbf{1})_{1/2,-1}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{-1/2,-1}$	$(\mathbf{1},\mathbf{1})_{0}$

	$(\overline{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(\overline{\mathbf{3}},\mathbf{2},\mathbf{1})_{-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{0,-1/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{1})_{1}$	$(\mathbf{1},\mathbf{2})_{0,1}$	$(\mathbf{1},\mathbf{2})_{1/2}$
210	$(\mathbf{15},\mathbf{3},\mathbf{1})$	$(\mathbf{8},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{3})_{0,0}$	$(\mathbf{8},\mathbf{3})_{0}$
		$(\mathbf{3},\mathbf{3},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{3})_{0,4/3}$	$(\mathbf{3},\mathbf{3})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{3},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{3})_{-2/3}$
		$(\mathbf{1},\mathbf{3},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{3})_{0,0}$	$(\mathbf{1},\mathbf{3})_{0}$
	$(\mathbf{15},\mathbf{1},\mathbf{3})$	$(\mathbf{8},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{8},\mathbf{1})_{1,0}$	$(\mathbf{8},\mathbf{1})_{-1}$
			$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
			$(\mathbf{8},\mathbf{1})_{-1,0}$	$(\mathbf{8},\mathbf{1})_{1}$
		$(\mathbf{3},\mathbf{1},\mathbf{3})_{4/3}$	$(\mathbf{3},\mathbf{1})_{1,4/3}$	$(\mathbf{3},\mathbf{1})_{-1/3}$
			$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
			$(\mathbf{3},\mathbf{1})_{-1,4/3}$	$(\mathbf{3},\mathbf{1})_{5/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{3})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-5/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
			$(\overline{\mathbf{3}},\mathbf{1})_{-1,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{1/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{3})_{0}$	$(\mathbf{1},\mathbf{1})_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$
			$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
			$(\mathbf{1},\mathbf{1})_{-1,0}$	$(\mathbf{1},\mathbf{1})_{1}$
	$(\mathbf{10},\mathbf{2},\mathbf{2})$	$(\overline{\mathbf{6}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{6}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{6}},\mathbf{2})_{5/6}$
		$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_{-2}$	$(\mathbf{1},\mathbf{2})_{1/2,-2}$	$(\mathbf{1},\mathbf{2})_{-3/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,-2}$	$(\mathbf{1},\mathbf{2})_{-1/2}$
	$(\overline{\mathbf{10}},\mathbf{2},\mathbf{2})$	$(\mathbf{6},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{6},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{6},\mathbf{2})_{1/6}$
			$(\mathbf{6},\mathbf{2})_{-(1/2),-2/3}$	$(\mathbf{6},\mathbf{2})_{-5/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
		$(\mathbf{1},\mathbf{2},\mathbf{2})_{2}$	$(\mathbf{1},\mathbf{2})_{1/2,2}$	$(\mathbf{1},\mathbf{2})_{1/2}$
			$(\mathbf{1},\mathbf{2})_{-1/2,2}$	$(\mathbf{1},\mathbf{2})_{3/2}$
	$(\mathbf{6},\mathbf{2},\mathbf{2})$	$(\mathbf{3},\mathbf{2},\mathbf{2})_{-2/3}$	$(\mathbf{3},\mathbf{2})_{1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{-5/6}$
			$(\mathbf{3},\mathbf{2})_{-1/2,-2/3}$	$(\mathbf{3},\mathbf{2})_{1/6}$
		$(\overline{\mathbf{3}},\mathbf{2},\mathbf{2})_{2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{-1/6}$
			$(\overline{\mathbf{3}},\mathbf{2})_{-1/2,2/3}$	$(\overline{\mathbf{3}},\mathbf{2})_{5/6}$
	$(\mathbf{15},\mathbf{1},\mathbf{1})$	$(\mathbf{8},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{8},\mathbf{1})_{0,0}$	$(\mathbf{8},\mathbf{1})_{0}$
		$(\mathbf{3},\mathbf{1},\mathbf{1})_{4/3}$	$(\mathbf{3},\mathbf{1})_{0,4/3}$	$(\mathbf{3},\mathbf{1})_{2/3}$
		$(\overline{\mathbf{3}},\mathbf{1},\mathbf{1})_{-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{0,-4/3}$	$(\overline{\mathbf{3}},\mathbf{1})_{-2/3}$
		$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$
	$(\mathbf{1},\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1},\mathbf{1})_{0}$	$(\mathbf{1},\mathbf{1})_{0,0}$	$(\mathbf{1},\mathbf{1})_{0}$

Table A.2: Decompositions of the representations of SO(10) up to 210 dimensions under the PS breaking chain. Note that the decomposition under $\mathcal{G}_{421}$ is not shown since it is easy to obtain from $\mathcal{G}_{\mathrm{PS}}$. The hypercharge is related to $B-L$ and the third $\mathrm{SU}(2)_{\mathrm{R}}$ generator by $Y = \frac{B-L}{2} - T_{R,3}$.

A.3 Spinorial representations

A feature of SO(N) groups is that, in addition to the tensorial representations, they also contain spinorial representations. These differ for even and odd values of N. We will therefore first discuss the case for $N = 2n$ and then comment on how this construction is modified for the case of $N = 2n - 1$.

In particular, there exists a representation that is generated by

$$\Sigma_{ij} = \frac{\mathrm{i}}{2}[\Gamma_i, \Gamma_j], \tag{A.6}$$

where the N matrices Γ_i are Hermitian and satisfy the Clifford algebra

$$\{\Gamma_i, \Gamma_j\} = \frac{1}{2}\delta_{ij}\mathbb{1}. \tag{A.7}$$

The proof of their existence follows most easily from their explicit construction, starting with two of the Pauli matrices for $n = 1$ and iteratively extending the matrices for n. Concretely, we can take [184]

$$\Gamma_1^{(n=1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_2^{(n=1)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{A.8}$$

and build up larger representations according to

$$\begin{aligned} \Gamma_i^{(n+1)} &= \begin{pmatrix} \Gamma_i^{(n)} & 0 \\ 0 & -\Gamma_i^{(n)} \end{pmatrix} \text{ for } i \in \{1, \dots, 2n\}, \\ \Gamma_{2n+1}^{(n+1)} &= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \Gamma_{2n+2}^{(n+1)} = \begin{pmatrix} 0 & -\mathrm{i}\mathbb{1} \\ \mathrm{i}\mathbb{1} & 0 \end{pmatrix}. \end{aligned} \tag{A.9}$$

These can be verified to satisfy the Clifford algebra Eq. (A.7). Hence, we have constructed the $2n + 2$ matrices with dimension 2^{n+1} for $\mathrm{SO}(2n + 2)$. We thus see that the spinorial representation that is generated by Σ_{ij} is 2^n-dimensional. However, it is not an irreducible 2^n-dimensional representation, but splits into two irreducible 2^{n-1}-dimensional representations, in analogy with the chirality of Lorentz spinors. This can be done by defining an additional Γ-matrix by

$$\Gamma_0 = (-\mathrm{i})^n(\Gamma_1\Gamma_2\cdots\Gamma_{2n}), \tag{A.10}$$

in analogy to the Dirac γ^5 matrix. The matrix Γ_0 satisfies

$$\Gamma_0^2 = \mathbb{1}, \quad [\Gamma_0, \Sigma_{ij}] = 0, \quad \{\Gamma_0, \Gamma_i\} = 0. \tag{A.11}$$

The existence of this matrix which commutes with all the generators shows that the representation is reducible. Defining the projection operators

$$P_\pm = \frac{1}{2}(\mathbb{1} \pm \Gamma_0), \tag{A.12}$$

we can split the 2^n-dimensional spinor representation into two 2^{n-1}-dimensional irreducible representations.

Specifying to SO(10), *i.e.* $n = 5$, we find that there are two irreducible $2^{n-1} = 16$-dimensional spinorial representations with opposite eigenvalues of Γ_0. This is the representation in which all fermions of one generation of the SM fit.

In order to write down fermion mass terms, we need a bilinear in $\mathbf{16}_F$ which we can use to couple the fermions to the scalars. The simplest version would be

$$\mathbf{16}_F^T\mathbf{16}_F, \tag{A.13}$$

but this is not invariant since $\mathbf{16}_F^T$ does not transform like a conjugate spinor [183]. To solve this, we introduce a matrix B, which is analogous to the charge conjugation matrix for Lorentz spinors. If this matrix satisfies

$$B^{-1}\Sigma_{ij}^T B = -\Sigma_{ij}, \tag{A.14}$$

then the bilinear $\mathbf{16}_F^T B\mathbf{16}_F$ is invariant under SO(10) transformations. An explicit construction is to start with

$$B^{(n=1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A.15}$$

and iteratively build up the matrix for larger n by

$$B^{(n+1)} = \begin{pmatrix} 0 & B^{(n)} \\ (-1)^{n+1}B^{(n)} & 0 \end{pmatrix}, \tag{A.16}$$

from which one can also deduce the properties

$$B^{-1}\Gamma_i^T B = (-1)^n\Gamma_i, \quad B^{-1}\Gamma_0 B = (-1)^n\Gamma_0. \tag{A.17}$$

Using this, we can build the fermion bilinears. One can show that the bilinear

$$\mathbf{16}_F^T B\Gamma_{i_1}\cdots\Gamma_{i_k}\mathbf{16}_F \tag{A.18}$$

transforms as a k-index tensor under SO(10) transformations. Using the properties of the charge conjugation matrix and the fact that $\mathbf{16}_F$ is an eigenstate of Γ_0, one can show that Eq. (A.18) is zero unless k is odd. This is the case for **10**, **120**, and $\overline{\mathbf{126}}$, which are the ones that have an invariant coupling to two **16**s.

To find whether the Yukawa couplings should be symmetric or antisymmetric, we write down the coupling

$$\mathbf{16}_F^T CB\Gamma_{i_1}\cdots\Gamma_{i_k}\mathbf{16}_F, \tag{A.19}$$

where C is the Lorentz charge conjugation matrix. Taking the transpose of this and using the facts that the fermion fields are anticommuting, $C^T = -C$ and $B^T = -B$, we find

$$-\mathbf{16}_F^T\Gamma_{i_k}^T\cdots\Gamma_{i_1}^T BC\mathbf{16}_F = -\mathbf{16}_F^T BB^{-1}\Gamma_{i_k}^T B\cdots B^{-1}\Gamma_{i_1}^T BC\mathbf{16}_F, \tag{A.20}$$

where in the last equality we have inserted factors of $BB^{-1} = \mathbb{1}$ between each Γ-matrix. Using Eq. (A.17), we can write the sequence of Γ-matrices as

$$BB^{-1}\Gamma_{i_k}^T B\cdots B^{-1}\Gamma_{i_1}^T B = (-1)^k B\Gamma_{i_k}\cdots\Gamma_{i_1}. \tag{A.21}$$

The scalar representations that couple to the bilinear all have antisymmetric indices or just one index in the case of the **10**, meaning that the sequence of Γ-matrices must all be different. Thus, we can use the Clifford algebra to permute the Γ-matrices as

$$\Gamma_{i_k}\cdots\Gamma_{i_1} = (-1)^{k(k-1)/2}\Gamma_{i_1}\cdots\Gamma_{i_k}. \tag{A.22}$$

Overall, we have, with the Yukawa coupling matrix Y,

$$Y\mathbf{16}_F^T CB\Gamma_{i_1}\cdots\Gamma_{i_k}\mathbf{16}_F = (-1)^{k+k(k-1)/2+1}Y^T\mathbf{16}_F^T CB\Gamma_{i_1}\cdots\Gamma_{i_k}\mathbf{16}_F. \tag{A.23}$$

In order for this equality to hold, we must have $Y^T = \pm Y$, with the sign being the same as the factor $(-1)^{k+k(k-1)/2+1}$. For $k = 1$, *i.e.* **10**, we find $Y_{10} = Y_{10}^T$, for $k = 3$, *i.e.* **120**, $Y_{120} = -Y_{120}^T$, and for $k = 5$, *i.e.* $\overline{\mathbf{126}}$, $Y_{126} = Y_{126}^T$.

If we are instead interested in odd $N = 2n - 1$, we find that the irreducible spinor representation still has dimension 2^{n-1}, just like $N = 2n$ [184]. The same procedure for constructing the representation applies as for $N = 2n$, but we omit the matrix Γ_{2n}. The matrix Γ_0 can be taken to be the same as for $N = 2n$.

A.4 Anomalies

Computation of the triangle anomalies [70, 71], mentioned in Sec. 2.2.2, involves a group-theoretic factor

$$\mathcal{A}^{abc} = \mathrm{Tr}(\{t^a, t^b\}t^c), \tag{A.24}$$

where t^a are the generators of the representation to which the fields in the triangle diagram belong and $\{\cdot,\cdot\}$ denotes anti-commutator. The question of anomalies of a model therefore lends itself to a group-theoretic consideration [361–363].

For SO(N), we may label the $N(N-1)/2$ generators by two indices $i, j \in \{1, 2, \ldots, N\}$, such that $t^{ij} = -t^{ji}$ are $N \times N$ matrices. That is, the indices i and j label the generator and are not matrix indices. Then, the anomaly factor

$$\mathcal{A}^{ijklmn} = \mathrm{Tr}(\{t^{ij}, t^{kl}\} t^{mn}) \tag{A.25}$$

is a six-index object. The SO(N) algebra implies that it must be antisymmetric under the exchange of indices $i \leftrightarrow j$, $k \leftrightarrow l$, and $m \leftrightarrow n$, while the structure of the anomaly factor implies that it must be symmetric under the exchange of pairs $ij \leftrightarrow kl$, $kl \leftrightarrow mn$, and $ij \leftrightarrow mn$. However, these two requirements are incompatible, since the most general object consistent with the antisymmetry requirement is

$$\begin{aligned}\mathcal{A}^{ijklmn} \propto (&\delta^{ik}\delta^{lm}\delta^{jn} - \delta^{jk}\delta^{lm}\delta^{in} - \delta^{il}\delta^{km}\delta^{jn} + \delta^{jl}\delta^{km}\delta^{in} \\ &- \delta^{ik}\delta^{ln}\delta^{jm} + \delta^{jk}\delta^{ln}\delta^{im} + \delta^{il}\delta^{kn}\delta^{jm} - \delta^{jl}\delta^{kn}\delta^{im}).\end{aligned} \tag{A.26}$$

However, under the interchange $ij \leftrightarrow kl$, this is antisymmetric. Hence, it must vanish and these algebras are identically anomaly-free.

For SO(6), this is not the most general six-index object due to the existence of the antisymmetric tensor ϵ^{ijklmn}, which is invariant under the exchange $ij \leftrightarrow kl$. The anomaly factor may be proportional to this, which means that SO(6) is not automatically anomaly-free. The same applies to SO(N) algebras with $N < 6$, since their N-index totally antisymmetric tensors ruin this proof. However, SO(5) is anomaly-free since one cannot construct a six-index object from its five-index antisymmetric tensor and Kronecker deltas.

Appendix B

Renormalization group equations

In this appendix, we list the RGEs that are required to perform the RG running from M_{GUT} to M_Z at the one-loop order. The relevant parameters that run are: the gauge couplings g_i, the Higgs quartic self-coupling λ, the Yukawa coupling matrices for the up-type quarks Y_u, down-type quarks Y_d, neutrinos Y_ν, charged leptons Y_ℓ, scalar triplet Y_Δ (for type II seesaw), the right-handed neutrino Majorana mass matrix M_R, and the effective neutrino mass matrix κ. Note that the β-functions for the gauge couplings g_1 and g_2 include contributions from the scalar triplet. If the scalar triplet is not in the model, or below its mass threshold, one should use only the first terms of Eqs. (B.1) and (B.2), as well as removing any contribution from Y_Δ. For details on the numerical procedure used in solving the RGEs and integrating out heavy right-handed neutrinos and the scalar triplet, see Ch. 5. To one-loop order, the complete set of RGEs used are [211, 326–328, 339–341, 364]

$$16\pi^2\beta_{g_1} = \frac{41}{10}g_1^3 + \frac{3}{5}g_1^3 = \frac{47}{10}g_1^3, \tag{B.1}$$

$$16\pi^2\beta_{g_2} = -\frac{19}{6}g_2^3 + \frac{2}{3}g_2^3 = -\frac{5}{2}g_2^3, \tag{B.2}$$

$$16\pi^2\beta_{g_3} = -7g_3^3, \tag{B.3}$$

$$\begin{aligned} 16\pi^2\beta_\lambda = {} & 6\lambda^2 - 3\lambda\left(3g_2^2 + \frac{3}{5}g_1^2\right) + 3g_2^4 + \frac{3}{2}\left(\frac{3}{5}g_1^2 + g_2^2\right)^2 \\ & + 4\lambda\mathrm{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right) \\ & - 8\mathrm{Tr}\left(Y_\ell^\dagger Y_\ell Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d Y_d^\dagger Y_d + 3Y_u^\dagger Y_u Y_u^\dagger Y_u\right), \end{aligned} \tag{B.4}$$

$$16\pi^2\beta_{Y_u} = Y_u\Bigg[\frac{3}{2}Y_u^\dagger Y_u - \frac{3}{2}Y_d^\dagger Y_d - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \tag{B.5}$$

$$+ \operatorname{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right)\Bigg], \tag{B.6}$$

$$16\pi^2\beta_{Y_d} = Y_d\Bigg[\frac{3}{2}Y_d^\dagger Y_d - \frac{3}{2}Y_u^\dagger Y_u - \frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2 \tag{B.7}$$

$$+ \operatorname{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right)\Bigg], \tag{B.8}$$

$$16\pi^2\beta_{Y_\nu} = Y_\nu\Bigg[\frac{3}{2}Y_\nu^\dagger Y_\nu - \frac{3}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta - \frac{9}{20}g_1^2 - \frac{9}{4}g_2^2 \tag{B.9}$$

$$+ \operatorname{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right)\Bigg], \tag{B.10}$$

$$16\pi^2\beta_{Y_\ell} = Y_\ell\Bigg[\frac{3}{2}Y_\ell^\dagger Y_\ell - \frac{3}{2}Y_\nu^\dagger Y_\nu + \frac{3}{2}Y_\Delta^\dagger Y_\Delta - \frac{9}{4}g_1^2 - \frac{9}{4}g_2^2 \tag{B.11}$$

$$+ \operatorname{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_d^\dagger Y_d + 3Y_u^\dagger Y_u\right)\Bigg], \tag{B.12}$$

$$\begin{aligned} 16\pi^2\beta_{Y_\Delta} &= \left(\frac{1}{2}Y_\nu^\dagger Y_\nu + \frac{1}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta\right)^T Y_\Delta + Y_\Delta\left(\frac{1}{2}Y_\nu^\dagger Y_\nu + \frac{1}{2}Y_\ell^\dagger Y_\ell + \frac{3}{2}Y_\Delta^\dagger Y_\Delta\right) \\ &+ \left[-\frac{3}{2}\left(\frac{3}{5}g_1^2 + 3g_2^2\right) + \operatorname{Tr}\left(Y_\Delta^\dagger Y_\Delta\right)\right]Y_\Delta, \end{aligned} \tag{B.13}$$

$$16\pi^2\beta_{M_R} = (Y_\nu Y_\nu^\dagger)M_R + M_R(Y_\nu Y_\nu^\dagger)^T, \tag{B.14}$$

$$\begin{aligned} 16\pi^2\beta_\kappa &= \frac{1}{2}(Y_\nu^\dagger Y_\nu - 3Y_\ell^\dagger Y_\ell + 3Y_\Delta^\dagger Y_\Delta)^T\kappa + \frac{1}{2}\kappa(Y_\nu^\dagger Y_\nu - 3Y_\ell^\dagger Y_\ell + 3Y_\Delta^\dagger Y_\Delta) \\ &+ 2\operatorname{Tr}\left(Y_\ell^\dagger Y_\ell + Y_\nu^\dagger Y_\nu + 3Y_u^\dagger Y_u + 3Y_d^\dagger Y_d\right)\kappa - 3g_2^2\kappa + \lambda\kappa. \end{aligned} \tag{B.15}$$

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Part II

Scientific papers