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Yang Mills short distance potential and perturbation theory

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We compute the coupling α_{qq} defined in terms of the static quark force by simulating the SU(3) Yang-Mills theory at lattice spacings down to 10^{-2} fm, keeping the volume large. In order to systematically improve the approach to the continuum, we subtract the leading cutoff effects in Symanzik's effective theory, resumming the leading $\log(a/r)$-term by renormalization group improvement. Subsequently we extrapolate with $\bar{g}^2(a^{-1})^{\hat{\gamma}_1}\,(a/r)^2$ corrections to the continuum limit. We finally investigate the applicability of continuum perturbation theory, extract the pure-gauge Λ-parameter at different values of α_{qq} and different orders of perturbation theory and compare to other methods.

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1. Introduction

Since a few years the most precise and reliable determinations of the strong coupling arise from low-energy experiments combined with lattice gauge theory. The determined value,

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1182(8) \quad \text{FLAG 19 (lattice) [1]}, \tag{1.1}$$

agrees well with the phenomenological/perturbative analysis of experimental results at larger energy scales evolved to M_Z,

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1174(16) \quad \text{PDG 16 (non-lattice) [2]}. \tag{1.2}$$

Bazavov et al. used the short-distance potential computed on the lattice and matched to perturbation theory [3]. The cited result,

$$\alpha^{(5)}_{\overline{\rm MS}}(M_Z) = 0.1166(10) \quad [3] \tag{1.3}$$

enters the average in (1.1), but is not in too good agreement with it. This motivates a precision study of the short-distance potential and its comparison to perturbation theory. In general, if we want to reduce the error of $\alpha_{\overline{\rm MS}}$ further, we need to understand perturbative errors better [4].

1.1 $\alpha_{\overline{\rm MS}}$ from the static potential

The potential is of particular interest because its perturbative expansion is known to a high order of perturbation theory (see [5] and references therein). Counting the leading one-gluon exchange potential as no loop, the accuracy is three loops.

Unfortunately there are also some caveats. First, the static potential is infrared divergent, starting at three loops [6]. It is known how to re-sum the divergent graphs, but the final expression involves an enhanced $\log(\alpha)\alpha^4$-term on top of the α^4 one; moreover there is a remaining infrared sensitivity hidden in the choice of the "ultra-soft" scale made in the resummation based on pNRQCD [5]. Second, the potential is defined from the large-t behavior of Wilson loops, $W(r,t)$, and the limit needs to be controlled. Third, since only the perturbative expansion of the large-volume potential is known, one needs very large lattices in order to reach small distances at a few lattice spacings. The third point can be avoided by finite-volume couplings and a step-scaling strategy [7].

To be more specific about the second point, we consider the coupling

$$\bar{g}^2_{\rm qq}(1/r) = 3\pi r^2 F(r), \quad F(r) = \frac{\rm d}{{\rm d}r}V(r). \tag{1.4}$$

and its β-function, $\beta_{\rm qq}(\bar{g}_{\rm qq}(\mu)) = \mu\frac{\rm d}{{\rm d}\mu}\bar{g}_{\rm qq}(\mu)$ in the form

$$\beta_{\rm qq}(\bar{g}_{\rm qq}) = -\bar{g}^3_{\rm qq}[b_0 + b_1\bar{g}^2_{\rm qq} + b_2\bar{g}^4_{\rm qq} + (b_3 + b_{3\rm L}l)\bar{g}^6_{\rm qq} + (b_4 + b_{4\rm L}l + b_{4\rm LL}l^2)\bar{g}^8_{\rm qq} + \ldots] \tag{1.5}$$

$$b_2(4\pi)^3 = 1.6524,\ b_3(4\pi)^4 = 4.9449,\ b_{3\rm L}(4\pi)^4 = 1.2538, \tag{1.6}$$

$$b_{4\rm L}(4\pi)^5 = 9.8088,\ b_{4\rm LL}(4\pi)^5 = -1.6463, \qquad l = \log(3\bar{g}^2_{\rm qq}/(8\pi)). \tag{1.7}$$

The coefficients b_i are evaluated for $N_{\rm f} = 0$ and b_4 is not known. $\beta_{\rm qq}$ is obtained from the perturbative expansion of the static force (including the so-called soft and ultra-soft contributions) and setting the ultra-soft scale to the recommended value $\mu_{\rm US} = \mu \times 3\bar{g}^2_{\rm qq}/(8\pi)$ [5]. The scale of the coupling is $\mu = 1/r$. In the above formulae, factors of 4π are arranged such that we see coefficients of $\alpha_{\rm qq} = \bar{g}^2_{\rm qq}/(4\pi)$ inside the bracket. The b_3 and $b_{4\rm L}$ terms give significant contributions for a typical range $0.2 \le \alpha_{\rm qq} \le 0.3$.

1.2 Yang-Mills theory

Understanding the above pros and cons may be possible in a simplified setting, namely the pure gauge theory, where we can afford to simulate very small lattice spacings. Furthermore, recent determinations of the Λ-parameter in the $N_f = 0$ theory,

$$\sqrt{8t_0}\Lambda^{(0)}_{\overline{\rm MS}} = 0.6227(98) \quad [8] \tag{1.8}$$

$$\sqrt{8t_0}\Lambda^{(0)}_{\overline{\rm MS}} = 0.5968(33) \;\leftarrow w_0\,\Lambda_{\overline{\rm MS}} = 0.2154(12) \quad [9]\,. \tag{1.9}$$

are not in agreement (we converted from w_0-units to t_0 units with the help of our simulations described below) [10]. While there is little doubt about the computation of [8], since $\alpha \approx 0.1$ was reached by non-perturbative step-scaling, a confirmation/improvement of (1.8) would be welcome, as the pure gauge theory result enters the decoupling strategy for renormalization problems [11]. For these reasons we study again the pure gauge theory and consider the extraction of α from the short-distance potential.

2. Simulations

We simulated lattices with $L \approx 4r_0 \approx 2$ fm and resolutions L/a ranging from 32 to 192. The smallest lattice spacing is $\approx 10^{-2}$ fm. In comparison to [3] we reach a factor $\approx 1/15$ further down in the relevant variable a^2. However, we use a standard Wilson action, in comparison to the tree-level Symanzik improved action of [3]. The smallest lattice spacing is new in comparison to the results presented in [12]. Topology freezing is avoided by open boundaries in time [13].

One reason why we opted against Symanzik improvement is that we need the limit

$$V(r) = \frac{1}{a}\lim_{t\to\infty} \log\left(W(r,t)/W(r,t+a)\right)\,. \tag{2.1}$$

Unfortunately, even in the pure gauge theory, where we employ the one-link integral [14] for the time-like links, the signal-to-noise ratio grows rapidly with t: thus, both the smearing of the spatial links of the Wilson loops and the GEVP [15, 16] are crucial in obtaining a reliable result. The latter method is based on correlation functions with a positive spectral representation. Since violations of positivity are quite strong for improved actions [17, 18], we avoid their use. In fig.1 we demonstrate a typical case for the standard Wilson action. Extracting the ground state potential is not trivial at large r/a, even when strict positivity gives a rigorous mathematical basis for the GEVP method.

3. Strategy

Our strategy is similar to [12]. We eliminate the unphysical self-energy of the static potential, consider the force and analyse it in terms of the physical (regularisation-independent) coupling $\bar{g}_{\rm qq}$, see (1.4). From this coupling we evaluate the Λ-parameter, by integrating the renormalization group equation exactly, after truncating the β-function at a given order. The perturbative uncertainty in Λ due to the missing b_4 is a relative error $\sim \alpha^3_{\rm qq}$. However, since we are dealing with an asymptotic series, the more reliable error estimate is the effect of the last known term, $\Delta\beta_{\rm qq} = -(b_3 + b_{3\rm L}l)\bar{g}^9_{\rm qq}$, i.e. the difference between three-loop and four-loop running.

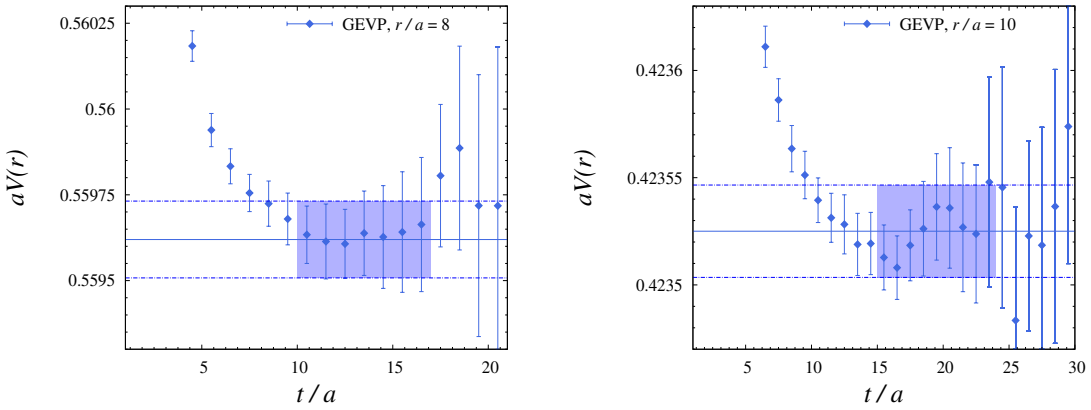

Figure 1: Extraction of $V(r)$. On the left for $a = 0.03$ fm and $r = 0.24$ fm (3x3 GEVP). On the right for $a = 0.01$ fm and $r = 0.1$ fm (2x2 GEVP). The smaller time needed in the GEVP is fixed to $4a$ on the left and to $5a$ on the right.

We set the scale by our own determination of t_0 [19] and evaluate $\bar{g}_{\rm qq}$ at $r/\sqrt{8t_0} = 0.4,\ldots,0.25$. At $r_{\rm switch} = 0.25\sqrt{8t_0}$ we switch to using the rather precise large-volume step scaling [12] and compute

$$\alpha_{\rm qq}((s^n r_{\rm switch})^{-1}) = \sigma(\alpha_{\rm qq}((s^{n-1} r_{\rm switch})^{-1}, s), \quad s = 3/4. \tag{3.1}$$

Apart from the plateau determinations as in fig. 1, the critical points of the computation are the continuum extrapolations $\sigma = \lim_{a\to 0}\Sigma$ of the lattice step-scaling function

$$\Sigma(\eta, s, a/r) = \alpha_{\rm qq}(1/(sr), a/(sr))\big|_{\alpha_{\rm qq}(1/r, a/r)=\eta}\,. \tag{3.2}$$

and the truncation of the perturbative series.

4. Continuum extrapolations

The standard definition of the force $F(r)$ from the on-axis lattice potential $V(r)$ is

$$F_{\rm n}(r) = \frac{1}{a}[V(r+a/2) - V(r-a/2)], \tag{4.1}$$

for $r = (n_{\rm r} + \frac{1}{2})a$, $n_{\rm r} = 2,3,\ldots$ In [12] and in many other works, tree-level improvement is implemented before continuum extrapolations. This, however, does not account for the logarithmic corrections of the form $[\bar{g}^2(1/a)]^{\hat{\gamma}} a^2/r^2$ to the static potential, which are due to the anomalous dimensions of the dimension-six fields in Symanzik's effective theory [20].

Since the continuum extrapolation is crucial, we want to treat the leading $\mathrm{O}(a^2)$ scaling violations as accurately as possible. We thus modify naive tree-level improvement by renormalization group improvement, resumming the $\log(a/r)$ terms [20]. We start by improving the discrete derivative (4.1) in the form

$$F_{\rm impr}(r) = F_{\rm n}(r) - \frac{1}{24}[F_{\rm n}(r+a) - 2F_{\rm n}(r) + F_{\rm n}(r-a)]. \tag{4.2}$$

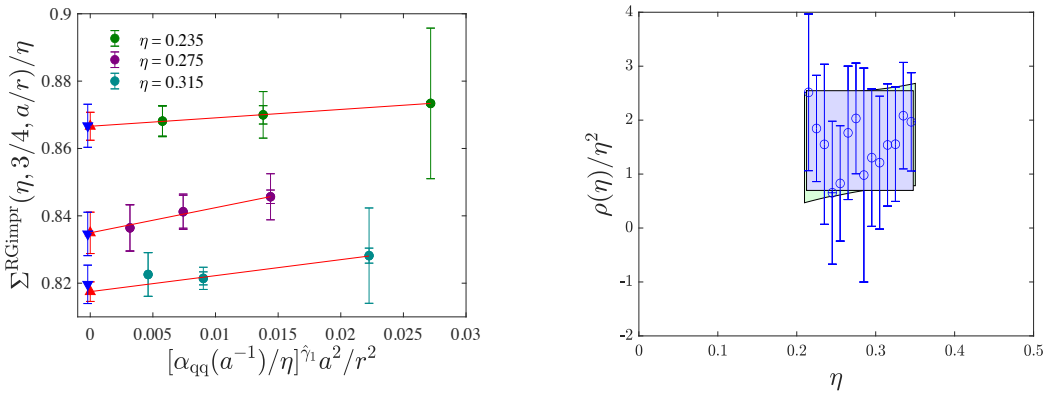

Figure 2: Left: Continuum extrapolations (4.8) at different η_i: upward pointing triangles and smaller errorbars have the Ba^4/r^4 term (see text) turned off. Right: resulting slopes ρ_i and the fits (4.9) with one (blue band) and three free parameters (green band).

The improved $F_{\mathrm{impr}}(r)$ is accurate up to $\mathrm{O}(a^4)$ except for $\mathrm{O}(a^2)$ effects which originate from $V(r)$ itself. The latter are now known in the form [20]

$$\alpha_{\mathrm{qq}}(1/r, a/r) - \alpha_{\mathrm{qq}}(1/r, 0) = \alpha_s(r^{-1})\frac{a^2}{r^2} \times \left\{ \left[\frac{\alpha_s(a^{-1})}{\alpha_s(r^{-1})}\right]^{\hat{\gamma}_1} A_1(r) + \left[\frac{\alpha_s(a^{-1})}{\alpha_s(r^{-1})}\right]^{\hat{\gamma}_2} A_2(r) \right\} \quad (4.3)$$
$$\times(1 + \mathrm{O}(\alpha_s(a^{-1})), \quad \hat{\gamma}_1 = \frac{7}{11} \approx 0.636, \quad \hat{\gamma}_2 = \frac{63}{55} \approx 1.145.$$

Here we have written the expansion in terms of the renormalised coupling α_s in a scheme s which is irrelevant at the considered order. The functions $A_i(r)$ are (RGI) matrix elements of the $d = 6$ operators in Symanzik's effective action. At short distances they can be expanded

$$A_i(r) = A_{i,0} + A_{i,1}\alpha_s(1/r) + A_{i,2}\alpha_s(1/r)^2 + \ldots \quad (4.4)$$

and for our Wilson action we have

$$A_{1,0} = 0, \quad A_{2,0} = \frac{3}{4}. \quad (4.5)$$

At small lattice spacing, the first term in the curly bracket in (4.3) dominates, but the second one is suppressed by one power less of $\alpha(r^{-1})$ at small r.

We therefore also use the known $A_{2,0}$ and define

$$\alpha_{\mathrm{qq}}^{\mathrm{RGimpr}} = \frac{\alpha_{\mathrm{qq}}}{1 + A_{2,0}\left[\frac{\alpha_{\mathrm{qq}}(a^{-1})}{\alpha_{\mathrm{qq}}(r^{-1})}\right]^{\hat{\gamma}_2}\frac{a^2}{r^2}}, \quad (4.6)$$

where the coupling at the cutoff $\alpha_{\mathrm{qq}}(a^{-1})$ is obtained from the measured $\alpha_{\mathrm{qq}}(r^{-1}) = \frac{3}{4}F_{\mathrm{n}}(r)r^2$ at $r = 2.5a$ by four-loop evolution (here we use the unimproved force F_{n}). Discretisation errors of $\alpha_{\mathrm{qq}}^{\mathrm{RGimpr}}$ are then modelled as $\frac{a^2}{r^2}\left[\frac{\alpha_s(a^{-1})}{\alpha_s(r^{-1})}\right]^{\hat{\gamma}_1} \alpha_{\mathrm{qq}}(r^{-1})A_1(r)$, with $A_1(r) = \mathrm{O}(\alpha_{\mathrm{qq}}(r^{-1}))$. The evaluation of Σ requires α_{qq} as a continuum function of r, which is easily obtained by a *local* interpolation. The

continuum limit $\sigma(\eta, s)$ is reached, asymptotically, as

$$\Sigma^{\mathrm{RGimpr}}(\eta, s, a/r) = \sigma(\eta, s) + \rho(\eta) \left[\frac{\alpha_{\mathrm{qq}}(a^{-1})}{\eta}\right]^{\hat{\gamma}_1} \frac{a^2}{r^2}, \quad \rho(\eta) = \mathrm{O}(\eta^2). \tag{4.7}$$

We assume that form for the continuum extrapolations, which we split into a three-step procedure.

1. We choose values η_i covering the accessible range of η with rather small separation; in practice we have $0.215 \le \eta_i \le 0.345$, separated by $\eta_{i+1} - \eta_i = 0.01$. At these values we fit (fig. 2)

$$\Sigma^{\mathrm{RGimpr}}(\eta_i, s, a/r) = \sigma_i + \rho_i \left[\frac{\alpha_{\mathrm{qq}}(a^{-1})}{\eta_i}\right]^{\hat{\gamma}_1} \frac{a^2}{r^2}. \tag{4.8}$$

Of course, we should restrict the fits to data with reasonably small lattice spacings. In practice, an exclusion of the potential at the cutoff $V(a)$ implies $r/a \ge 3.5$, which is enough for our extrapolations (see also below).

2. We then test that the slopes ρ_i are compatible with the expected form,

$$\rho_i = \eta_i^2 \times (\rho^{(1)} + \rho^{(2)}\eta_i + \ldots), \tag{4.9}$$

fitting to this form with parameters $\rho^{(j)}$, see fig. 2. At this point the discretization errors are known in the form

$$\Sigma(\eta, s, a/r) = \sigma(\eta, s) + \left[\frac{\alpha_{\mathrm{qq}}(a^{-1})}{\eta}\right]^{\hat{\gamma}_1} \frac{a^2}{r^2} (\rho^{(1)}\eta^2 + \rho^{(2)}\eta^3 + \ldots). \tag{4.10}$$

3. This formula, with the determined parameters $\rho^{(1)}$, $\rho^{(2)}$, is now used to extract the step-scaling functions at the desired points $\alpha((s^k r_{\mathrm{start}})^{-1})$, $k = 1, 2, \ldots$.

Systematic errors

The above procedure is based on a number of assumptions. We assume that after the renormalization group improved removal of the leading (in the expansion (4.6)) a^2 errors, the left over ones are reasonably modeled by the leading term in (4.3). Deviations are higher orders in $\alpha_{\mathrm{qq}}(a^{-1})$ which do not vary so much with a. They are thus expected to be *effectively* taken into account in the fit (4.8). However, we also discard a^4 terms in that fit. We try to estimate their effect in our errors, by adding a term Ba^4/r^4 to the force with a coefficient $B = 0 \pm 3.3$, where the ± 3.3 error is the size of the a^4-term found at tree-level. The effect of B is propagated into all quantities by standard, quadratic, error propagation. Note that it also reduces the weight of points with large a/r in the extrapolations (4.8) . It becomes unnecessary to apply a cut beyond the mentioned $r/a \ge 3.5$. The effect of B is included in the above figures as well as the following results.

5. Results and conclusions

The step-scaling function at the mentioned points η_i is shown in the left panel of fig. 3. It is somewhat below the perturbative estimates at different orders. Despite the additional lattice spacing, error bars are larger than in the previous analysis [12], due to the additional systematic error coming from the Ba^4/r^4 term. A semi-quantitative agreement with perturbation theory is

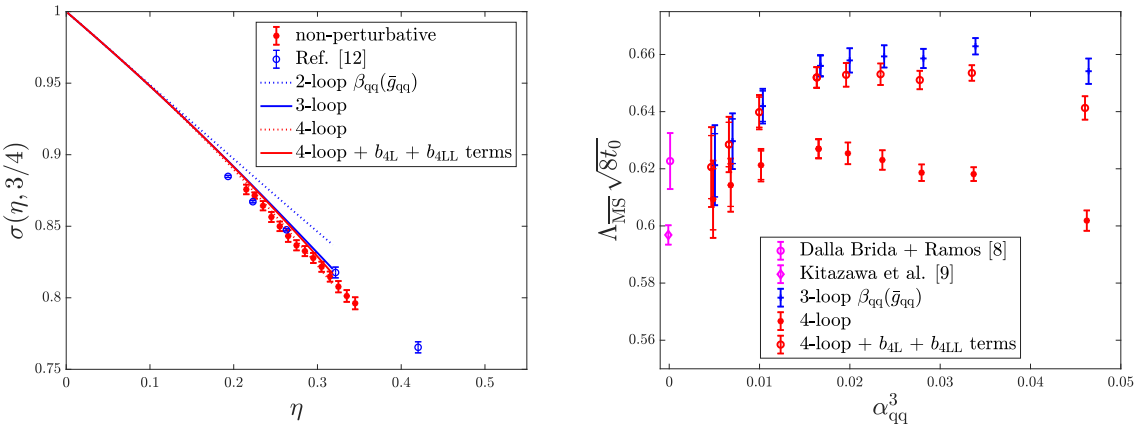

Figure 3: Left: Step scaling function compared with PT. Right: Λ-parameters determined at various values of α_{qq} together with different orders of PT.

found over quite a large range of α at the level of 5-10% in Λ, as seen on the right panel of fig. 3. For central values we should concentrate on the results using the 4-loop β-function. An extrapolation of the last few points in α^3 appears to agree better with the result of [9] than with [8]. However, the precision achieved is not good enough to make a real distinction.

The fact that the b_{4L} and b_{4LL} terms are as big as the 4-loop contribution to the β-function in the relevant range of α_{qq} suggests that the limitation of the asymptotic perturbative series is reached with the 4-loop β-function in most of our range, while 5-loop accuracy may help at the edge of $\alpha \approx 0.21$. With data at $\alpha \approx 0.25$ and above, the difference of numbers with 3-loop and 4-loop (or equivalently 4-loop and 4-loop $+b_{4L}, b_{4LL}$ terms puts a bound on the precision.

On the other hand smaller values of α_{qq} are afflicted with too large errors at present. The main reason is that our estimate of the a^4/r^4 uncertainty is too large compared to the very high precision required to obtain a value of Λ below the 5% level.

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