RESEARCH ARTICLE | APRIL 14 2025

Kirkwood–Dirac quasiprobabilities for measurements related to mutually unbiased equiangular tight frames

Alexey E. Rastegin

APL Quantum 2, 026108 (2025)
https://doi.org/10.1063/5.0254873

APL Quantum ARTICLE pubs.aip.org/aip/apq

Kirkwood–Dirac quasiprobabilities for measurements related to mutually unbiased equiangular tight frames

Cite as: APL Quantum 2, 026108 (2025); doi: 10.1063/5.0254873
Submitted: 25 December 2024 • Accepted: 28 March 2025 •
Published Online: 14 April 2025

Alexey E. Rastegin[a)]

AFFILIATIONS

Irkutsk State University, K. Marx St. 1, Irkutsk 664003, Russia

[a)]Author to whom correspondence should be addressed: alexrastegin@mail.ru

ABSTRACT

Protocols of quantum information science often realize in terms of specially selected states. In particular, such states are used to perform measurements at the final stage of a protocol. This study aims to explore measurements assigned to a mutually unbiased-equiangular tight frame. The utilized method deals with Kirkwood–Dirac quasiprobabilities, which are increasingly used in contemporary research. These quasiprobabilities constitute a matrix that can be linked to unravelings of certain quantum channels. Using states of the given frame to build principal Kraus operators leads to quasiprobabilities that represent the measured state. The structure of a mutually unbiased-equiangular tight frame allows one to characterize entropies associated with a particular unraveling. To do this, we estimate some of the Schatten and Ky Fan norms of the matrix consisted of quasiprobabilities. New uncertainty relations in terms of Rényi and Tsallis entropies follow from the obtained inequalities. A utility of the presented inequalities is exemplified with mutually unbiased bases of a qubit and equiangular tight frames of a ququart.

I. INTRODUCTION

Quantum technologies of information processing are currently the subject of active research. Their role will only increase in recent future. Several physical platforms are recognized as a feasible way to build quantum processors.[1–4] Quantum measurements are a necessary step to complete any protocol. Specially constructed sets of quantum states are requisite for such purposes. Mutually unbiased bases[5] are seemingly the most known example of important discrete structures in Hilbert spaces.[6] They were first considered by Schwinger[7] and applied to quantum state determination in Refs. 8 and 9. In fact, two mutually unbiased bases are used in the BB84 scheme of quantum cryptography.[10] Recently, equiangular tight frames have been shown to be useful in quantum information processing. Such frames of finite-dimensional vectors were originally studied independently of applications.[11,12] The concept of mutually unbiased equiangular tight frames was also proposed rather as a way to produce new frames from existing ones.[13] Extending mutually unbiased bases, this concept deserves further development, including potential usage in quantum information science.

As a rule, quantum measurements of the considered type differ from the most familiar case of an orthonormal basis. It is important in both theory and practice that the number of different outcomes can exceed the dimensionality. The use of overcomplete sets of vectors is often significant, for example, as with mutually unbiased bases.[14] Properties of the measurements of interest can be described in terms of quasiprobabilities. The Wigner functions[15] are a popular example of quasiprobabilities used in various questions.[16–20] The Kirkwood–Dirac quasiprobabilities are another especially important direction of investigations. They are now exploited more widely[21] than it was intended initially.[22,23] In fact, Kirkwood gave a phase-space methodology to calculate partition functions and dealt with canonically conjugate variables.[22]

At present, the Kirkwood–Dirac quasiprobabilities have found use in quantum state tomography,[24–27] information scrambling,[28–31] postselected metrology,[32–34] quantum thermodynamics,[35–37] and conceptual questions.[38–41] Reference 42 used the Kirkwood–Dirac quasiprobabilities to characterize the unravelings of quantum channel assigned to an equiangular tight frame.

This study examines the Kirkwood–Dirac quasiprobabilities for measurements assigned to mutually unbiased equiangular tight frames. First, it generalizes to several measurements the consideration originally given in Ref. 42. Second, the introduced matrices of quasiprobabilities will be described in terms of the Schatten and Ky Fan norms. Relations between various characteristics of probability distributions are important because some of them are easier to obtain than others. The structure of a mutually unbiased-equiangular tight frame allows one to derive many useful relations. New entropic uncertainty relations will be given for unravelings of the induced quantum channels. It is also of interest, since entropic functions are hardly exposed to measure immediately. This paper is organized as follows: Sec. II reviews the preliminary facts and gives the notation. Section III aims to characterize the quasiprobabilities in terms of the Schatten and Ky Fan norms. Entropic uncertainty relations for unravelings of the corresponding quantum channels will also be examined. The considered examples include complementary finite-dimensional observables with mutually unbiased eigenbases. Section IV concludes this paper.

II. PRELIMINARIES

This section reviews the required material. First, one recalls some definitions concerning finite-dimensional operators and their norms. Second, the concept of mutually unbiased equiangular tight frames is briefly discussed. Furthermore, Kirkwood–Dirac quasiprobabilities in connection with unravelings of a quantum channel are discussed. Finally, we recall the Rényi and Tsallis entropies to characterize probability distributions of interest.

A. Required facts from linear algebra

Let $\mathcal{L}(\mathcal{H}_d)$ denote the space of linear operators on d-dimensional Hilbert space $\mathcal{H}_d$. By $\mathcal{L}_+(\mathcal{H}_d)$ and $\mathcal{L}_{sa}(\mathcal{H}_d)$, one, respectively, means the set of positive semi-definite operators and the real space of Hermitian ones. The quantum states are represented by density matrices such that $\boldsymbol{\rho}\in\mathcal{L}_+(\mathcal{H})$ and $\mathrm{tr}(\boldsymbol{\rho})=1$. The set of pure states contains density matrices of the form $\boldsymbol{\rho}=|\psi\rangle\langle\psi|$, where $|\psi\rangle\in\mathcal{H}_d$ and $\langle\psi|\psi\rangle=1$. The operators of interest are described by rectangular matrices with complex entries. Let $\mathbb{M}_{m\times n}(\mathbb{C})$ be the space of all $m\times n$ complex matrices. The space of $n\times n$ matrices will be denoted by $\mathbb{M}_n(\mathbb{C})$. By $\mathbb{M}_n^{(sa)}(\mathbb{C})$ and $\mathbb{M}_n^{+}(\mathbb{C})$, we, respectively, mean the space of Hermitian $n\times n$ matrices and the set of positive semi-definite ones. For every $\mathsf{G}\in\mathbb{M}_{m\times n}(\mathbb{C})$, the square matrices $\mathsf{G}^\dagger\mathsf{G}$ and $\mathsf{G}\mathsf{G}^\dagger$ have the same non-zero eigenvalues. The positive square roots of these eigenvalues are the singular values $\sigma_j(\mathsf{G})$ of G.[43] For real $\alpha>0$, we will use norm-like functional,

$$\|\mathsf{G}\|_\alpha=\left(\sum\nolimits_j\sigma_j(\mathsf{G})^\alpha\right)^{1/\alpha}, \tag{1}$$

where the sum is actually taken over non-zero singular values of G. The legitimate norm arises for $\alpha\geq1$, and it is the Schatten α-norm. In particular, this family includes the trace norm for $\alpha=1$, the Hilbert–Schmidt norm, or the Frobenius norm,

$$\|\mathsf{G}\|_2=\sqrt{\mathrm{tr}(\mathsf{G}^\dagger\mathsf{G})} \tag{2}$$

for $\alpha=2$, and the spectral norm $\|\mathsf{G}\|_\infty=\max\sigma_j(\mathsf{G})$.

The Ky Fan norms form another especially important family of unitarily invariant norms. These norms are used to formulate the Ky Fan maximum principle.[44] For $k=1,2,\ldots$, the Ky Fan k-norm is defined as[43]

$$\|\mathsf{G}\|_{(k)}=\sum_{j=1}^{k}\sigma_j(\mathsf{G})^{\downarrow}. \tag{3}$$

Here, the down arrow indicates that singular values should be put in non-increasing order. The above norms can also be applied to finite-dimensional vectors. For a probability distribution $P=\{p_j\}$ and $\alpha>0$, we will use the quantity

$$\|P\|_\alpha=\left(\sum\nolimits_j p_j^\alpha\right)^{1/\alpha}. \tag{4}$$

Furthermore, the Ky Fan k-norm reads as

$$\|P\|_{(k)}=\sum_{j=1}^{k}p_j^{\downarrow}. \tag{5}$$

For a distribution with N probabilities, the index of coincidence is defined as

$$I(P)=\sum_{j=1}^{N}p_j^2. \tag{6}$$

The above quantities will be applied to characterize probability distributions quantitatively.

Quantum measurements are represented by the non-orthogonal resolutions of the identity, also known as positive operator-valued measures (POVMs).[45] The set $\mathcal{E}=\{\mathsf{E}_j\}_{j=1}^N$ of operators $\mathsf{E}_j\in\mathcal{L}_+(\mathcal{H}_d)$ is a POVM, when the completeness relation holds,

$$\sum_{j=1}^{N}\mathsf{E}_j=\mathbb{1}_d. \tag{7}$$

For the pre-measurement density matrix $\boldsymbol{\rho}$, the probability of the jth outcome reads as

$$p_j(\mathcal{E};\boldsymbol{\rho})=\mathrm{tr}(\mathsf{E}_j\boldsymbol{\rho}). \tag{8}$$

It is principally important for quantum information processing that the number of outcomes N can exceed the dimensionality d.

B. Mutually unbiased equiangular tight frames

A set $\{|\phi_j\rangle\}$ of $n\geq d$ unit vectors of $\mathcal{H}_d$ gives a tight frame, when[46]

$$\sum_{j=1}^{n}|\phi_j\rangle\langle\phi_j|=\frac{n}{d}\mathbb{1}_d. \tag{9}$$

If the formula

$$|\langle\phi_i|\phi_j\rangle|^2 = c \tag{10}$$

holds for all $i \neq j$, then we deal with an equiangular tight frame (ETF). It can be shown that $n \le d^2$ and

$$c = \frac{n-d}{(n-1)d}. \tag{11}$$

An equiangular tight frame of $n = d^2$ vectors, if it exists in $\mathcal{H}_d$, leads to a symmetric informationally complete measurement (SIC-POVM) with elements,

$$\frac{1}{d}|\phi_j\rangle\langle\phi_j|. \tag{12}$$

The existence of such sets for all d was conjectured by Zauner.[6] Reference 47 examined SIC-POVMs in more detail. Informationally complete measurements are interesting in various issues, including quantum tomography.[48]

The concept of a mutually unbiased-equiangular tight frame was first introduced in Ref. 13. Suppose that $1 \le M$ and $1 \le d \le n$. A sequence $\{|\phi_{\mu j}\rangle\}$ of unit vectors with $\mu = 1, \dots, M$ and $j = 1, \dots, n$ is a mutually unbiased-equiangular tight frame if[13]

$$|\langle\phi_{\mu i}|\phi_{\nu j}\rangle|^2 = \begin{cases} c, & \mu = \nu \text{ and } i \neq j, \\ d^{-1}, & \mu \neq \nu, \end{cases} \tag{13}$$

where c is given by (11). In other words, this mutually unbiased-equiangular tight frame consists of M usual ETFs. The special case with $n = d$ and $c = 0$ reduces to a set of M mutually unbiased bases (MUBs). Such bases provide an example of complementary observables in finite dimensions.[49] The authors of Ref. 13 also gave a recipe to produce new ETFs with the use of mutually unbiased ETFs. The authors of Ref. 50 proposed mutually unbiased measurements. These measurements are similar to MUBs, but rank-one elements are not required. The case of several ETFs with the fixed overlap between vectors of different frames can be treated as another modification of MUBs. We still use POVMs with rank-one elements, but the number of outcomes can exceed dimensionality.

Let us exemplify shortly a mutually unbiased-equiangular tight frame with $c \neq 0$. It deals with a ququart in dimension four. The vectors of mutually unbiased ETFs arise as columns of the matrices $\mathbf{\Psi}$, $\mathbf{\Delta\Psi}$, and $\mathbf{\Delta}^2\mathbf{\Psi}$, where[13]

$$\mathbf{\Delta} = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega^2 & 0 & 0 \\ 0 & 0 & \omega^8 & 0 \\ 0 & 0 & 0 & \omega^4 \end{pmatrix}, \qquad \mathbf{\Psi} = \frac{1}{2}\begin{pmatrix} 1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} \\ 1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 \\ 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 \\ 1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 \end{pmatrix}, \tag{14}$$

and $\omega = \exp(\mathrm{i}2\pi/15)$. In this example, we have $d = 4$, $n = 5$, and $M = 3$.

Each of M ETFs induces a non-orthogonal resolution $\mathcal{F}_\mu = \{\mathsf{F}_{\mu i}\}_{i=1}^{n}$ of the identity with rank-one elements,

$$\mathsf{F}_{\mu i} = \frac{d}{n}|\phi_{\mu i}\rangle\langle\phi_{\mu i}|, \tag{15}$$

since

$$\frac{d}{n}\sum_{i=1}^{n}|\phi_{\mu i}\rangle\langle\phi_{\mu i}| = \mathbb{1}_d. \tag{16}$$

For each μ, the probability of the ith outcome reads as

$$p_i(\mathcal{F}_\mu;\boldsymbol{\rho}) = \frac{d}{n}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{17}$$

Substituting these probabilities in (6) gives the μth index of coincidence $I(\mathcal{F}_\mu;\boldsymbol{\rho})$. To a mutually unbiased-equiangular tight frame, we also assign a single POVM with rank-one elements,

$$\frac{d}{Mn}|\phi_{\mu i}\rangle\langle\phi_{\mu i}|. \tag{18}$$

The probability of the μith outcome then appears as

$$p_{\mu i}(\mathcal{F};\boldsymbol{\rho}) = \frac{d}{Mn}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{19}$$

It follows from (17) and (19) that

$$I(\mathcal{F};\boldsymbol{\rho}) = \frac{1}{M^2}\sum_{\mu=1}^{M}I(\mathcal{F}_\mu;\boldsymbol{\rho}). \tag{20}$$

Here, the total index of coincidence $I(\mathcal{F};\boldsymbol{\rho})$ is obtained from (6) by substituting the probabilities (19). Formula (20) connects two different interpretations of measurement data.

C. Kirkwood–Dirac quasiprobabilities and channel unravelings

The concept of Kirkwood–Dirac quasiprobabilities was originally introduced for orthonormal bases and later extended to the case of two POVMs.[51] In this paper, we will use the following reformulation. To the given POVM $\mathcal{E} = \{\mathsf{E}_j\}_{j=1}^{N}$ and density matrix $\boldsymbol{\rho}$, one assigns N^2 quantities of the form $\mathrm{tr}(\mathsf{E}_i\mathsf{E}_j\boldsymbol{\rho})$. Following Ref. 42, these quantities will be referred to as generalized Kirkwood–Dirac quasiprobabilities. To each POVM, one can also assign trace-preserving completely positive map, or quantum channel,

$$\boldsymbol{\rho} \mapsto \Psi(\boldsymbol{\rho}) = \sum_{j=1}^{N}\mathsf{A}_j\boldsymbol{\rho}\mathsf{A}_j^{\dagger}, \tag{21}$$

where

$$\mathsf{A}_j = \sqrt{\mathsf{E}_j}. \tag{22}$$

The set $\{\mathsf{A}_j\}_{j=1}^{N}$ is a particular operator-sum representation of Ψ in terms of Kraus operators. Following Ref. 52, we will call it an unraveling of Ψ. The operators of form (22) are measurement operators according to Sec. 2.2.6 of Ref. 45. The operators A_j will be referred to as the principal Kraus operators.[53]

In the case of POVM elements from a mutually unbiased-equiangular tight frame, measurement statistics can be interpreted in two different ways. The first deals with the quasiprobabilities expressed as

$$\frac{d^2}{M^2n^2}\langle\phi_{\mu i}|\phi_{\nu j}\rangle\langle\phi_{\nu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{23}$$

By $\Pi(\mathcal{F};\boldsymbol{\rho})$, we denote the $Mn \times Mn$ matrix constituted by these quasiprobabilities. When $\mu \neq \nu$, quantity (23) involves states of the two different frames. This is consistent with the approach to a generalized Kirkwood–Dirac distribution proposed in Ref. 51. Extending Ref. 42, we define the quantum channel as follows:

$$\boldsymbol{\rho} \mapsto \Phi(\boldsymbol{\rho}) = \sum_{\mu=1}^{M}\sum_{i=1}^{n} p_{\mu i}(\mathcal{F};\boldsymbol{\rho})\,|\phi_{\mu i}\rangle\langle\phi_{\mu i}|, \tag{24}$$

$$\mathsf{A}_{\mu i} = \sqrt{\frac{d}{Mn}}|\phi_{\mu i}\rangle\langle\phi_{\mu i}|, \tag{25}$$

where $p_{\mu i}(\mathcal{F};\boldsymbol{\rho})$ reads as (19). In the following, we will also use the $Mn \times Mn$ matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with elements,

$$\mathrm{tr}\Big(\mathsf{A}_{\mu i}^{\dagger}\mathsf{A}_{\nu j}\boldsymbol{\rho}\Big) = \frac{d}{Mn}\langle\phi_{\mu i}|\phi_{\nu j}\rangle\langle\phi_{\nu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{26}$$

The latter differs from (23) only by a factor. Therefore, the matrix equation

$$\Pi(\mathcal{F};\boldsymbol{\rho}) = \frac{d}{Mn}\,\Lambda(\mathcal{A};\boldsymbol{\rho}) \tag{27}$$

is valid due to the chosen form of POVM elements. It is obvious that $\Lambda(\mathcal{A};\boldsymbol{\rho}) \in \mathbb{M}_{Mn}^{(sa)}(\mathbb{C})$ and $\Pi(\mathcal{F};\boldsymbol{\rho}) \in \mathbb{M}_{Mn}^{(sa)}(\mathbb{C})$. It will be proved in the following that these matrices are positive semi-definite.

The second interpretation uses M POVMs $\mathcal{F}_\mu$ and M quantum channels. For each $\mu = 1,\ldots,M$, we have the $n \times n$ matrix $\Pi(\mathcal{F}_\mu;\boldsymbol{\rho})$, which consisted of quasiprobabilities of the form

$$\frac{d^2}{n^2}\,\langle\phi_{\mu i}|\phi_{\mu j}\rangle\langle\phi_{\mu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{28}$$

The corresponding quantum channel and its principal Kraus operators read as

$$\boldsymbol{\rho} \mapsto \Phi_\mu(\boldsymbol{\rho}) = \sum_{i=1}^{n} p_i(\mathcal{F}_\mu;\boldsymbol{\rho})\,|\phi_{\mu i}\rangle\langle\phi_{\mu i}|, \tag{29}$$

$$\sqrt{M}\,\mathsf{A}_{\mu i} = \sqrt{\frac{d}{n}}|\phi_{\mu i}\rangle\langle\phi_{\mu i}|, \tag{30}$$

where $p_i(\mathcal{F}_\mu;\boldsymbol{\rho})$ is given by (17). The $n \times n$ matrix $\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})$ contains entries of the form

$$M\,\mathrm{tr}\Big(\mathsf{A}_{\mu i}^{\dagger}\mathsf{A}_{\mu j}\boldsymbol{\rho}\Big) = \frac{d}{n}\langle\phi_{\mu i}|\phi_{\mu j}\rangle\langle\phi_{\mu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle. \tag{31}$$

It is obvious that $\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho}) \in \mathbb{M}_{n}^{(sa)}(\mathbb{C})$ and $\Pi(\mathcal{F}_\mu;\boldsymbol{\rho}) \in \mathbb{M}_{n}^{(sa)}(\mathbb{C})$. These matrices are positive semi-definite and $\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_1 = 1.$[42]

D. Generalized entropies

To characterize probability distributions, we will use the Rényi[54] and Tsallis entropies.[55] It will be convenient to begin with the second case. Dealing with the functions of the Tsallis type, we utilize the α-logarithm of positive variable. It is defined as

$$\ln_\alpha(X) = \begin{cases} \dfrac{X^{1-\alpha}-1}{1-\alpha} & \text{if } 0 < \alpha \neq 1, \\ \ln X & \text{if } \alpha = 1. \end{cases} \tag{32}$$

For $\alpha > 0$, the Tsallis α-entropy reads as[55]

$$H_\alpha(P) = \frac{1}{1-\alpha}\left(\sum_j p_j^\alpha - 1\right) = \sum_j p_j \ln_\alpha\!\left(\frac{1}{p_j}\right). \tag{33}$$

The Rényi α-entropy is defined as[54]

$$R_\alpha(P) = \frac{1}{1-\alpha}\ln\left(\sum_j p_j^\alpha\right). \tag{34}$$

The limit $\alpha \to \infty$ gives the min-entropy expressed as

$$R_\infty(P) = -\ln(\max p_j). \tag{35}$$

In the limit $\alpha \to 1$, both entropies (33) and (34) reduce to the Shannon entropy,

$$H_1(P) = -\sum_j p_j \ln p_j. \tag{36}$$

The basic properties of the Rényi and Tsallis entropies are considered in Sec. 2.7 of Ref. 56. It follows from (33) and (34) that

$$R_\alpha(P) = \frac{1}{1-\alpha}\ln\left[1 + (1-\alpha)H_\alpha(P)\right]. \tag{37}$$

Due to this link, each of Tsallis-entropy inequalities potentially has a Rényi-entropy counterpart, and vice versa. Using quantity (4), we, respectively, write

$$H_\alpha(P) = \frac{\|P\|_\alpha^\alpha - 1}{1-\alpha}, \tag{38}$$

$$R_\alpha(P) = \frac{\alpha}{1-\alpha}\ln\|P\|_\alpha. \tag{39}$$

The above entropies have found use in various disciplines. Part II of Ref. 57 discussed concrete physical examples, which actualize the Tsallis entropies. For physical applications of the Rényi entropies, see Ref. 58 and references therein.

III. MAIN RESULTS

This section aims to report the main results. First, the Schatten norms of the corresponding matrices are characterized. Furthermore, some Ky Fan norms are estimated from above. Third, the obtained inequalities are used to formulate new uncertainty relations for unravelings of the considered quantum channels. Finally, we give examples of Kirkwood–Dirac quasiprobabilities defined in terms of mutually unbiased ETFs. The first example deals with a pair of complementary observables in dimension two.

A. Inequalities for estimating some Schatten norms

The inner structure of a mutually unbiased-equiangular tight frame allows us to evaluate the Hilbert–Schmidt norms of the matrices of interest. This result is posed as follows:

Proposition 1. Let the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with elements (26) be assigned to the density matrix $\boldsymbol{\rho}$ and mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ with $\mu=1,\dots,M$ and $j=1,\dots,n$; then, $\Lambda(\mathcal{A};\boldsymbol{\rho})\in\mathbb{M}_{Mn}^{+}(\mathbb{C})$, $\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_1=1$, and

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_2^2=(1-c)I(\mathcal{F};\boldsymbol{\rho})+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2). \tag{40}$$

Proof. The matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ is Hermitian and can be diagonalized as

$$\mathsf{V}^{\dagger}\Lambda(\mathcal{A};\boldsymbol{\rho})\mathsf{V}=\mathrm{diag}(\lambda_1,\dots,\lambda_{Mn}), \tag{41}$$

where V is a unitary $Mn\times Mn$ matrix. Due to a unitary freedom in the operator-sum representation,[45] map (24) can be rewritten with a new Kraus operator,

$$\mathsf{A}_{\mu i}^{(ex)}=\sum\nolimits_j \mathsf{A}_{\nu j}v_{\nu j,\mu i}. \tag{42}$$

It is immediate to check that $\lambda_{\mu i}=p_{\mu i}(\mathcal{A}^{(ex)};\boldsymbol{\rho})\geq 0$. Thus, the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ is positive semi-definite. Its trace is equal to 1 and also coincides with the norm $\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_1$.

The $(\mu i,\mu i)$-entry of $\Lambda(\mathcal{A};\boldsymbol{\rho})^2$ reads as

$$\frac{d^2}{M^2n^2}\sum_{j=1}^{n}|\langle\phi_{\mu i}|\phi_{\mu j}\rangle|^2\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\mu j}\rangle\langle\phi_{\mu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle + \frac{d}{M^2n^2}\sum_{\nu\neq\mu}\sum_{j=1}^{n}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\nu j}\rangle\langle\phi_{\nu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle, \tag{43}$$

where we used the second row of (13). Summing the second term in (43) over $\mu=1,\dots,M$ and $i=1,\dots,n$ results in

$$\frac{M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2) \tag{44}$$

due to (16). The first sum in (43) is rewritten as

$$\frac{d^2}{M^2n^2}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\mu i}\rangle^2+\frac{cd^2}{M^2n^2}\sum_{j\neq i}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\nu j}\rangle\langle\phi_{\nu j}|\boldsymbol{\rho}|\phi_{\mu i}\rangle = \frac{(1-c)d^2}{M^2n^2}\langle\phi_{\mu i}|\boldsymbol{\rho}|\phi_{\mu i}\rangle^2+\frac{cd}{M^2n}\langle\phi_{\mu i}|\boldsymbol{\rho}^2|\phi_{\mu i}\rangle. \tag{45}$$

Summing the latter over $\mu=1,\dots,M$ and $i=1,\dots,n$ gives

$$(1-c)I(\mathcal{F};\boldsymbol{\rho})+\frac{c}{M}\mathrm{tr}(\boldsymbol{\rho}^2). \tag{46}$$

Adding (44) to (46) provides the right-hand side of (40). ■

The result (40) will be used together with the following formula:

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2=\frac{1-c}{M}\sum_{\mu=1}^{M}I(\mathcal{F}_\mu;\boldsymbol{\rho})+c\,\mathrm{tr}(\boldsymbol{\rho}^2), \tag{47}$$

which follows from

$$\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2=(1-c)I(\mathcal{F}_\mu;\boldsymbol{\rho})+c\,\mathrm{tr}(\boldsymbol{\rho}^2). \tag{48}$$

The latter was first proved in Ref. 42. It can also be obtained from (40) with $M=1$. Let us proceed to the estimation of the Hilbert–Schmidt norms from above.

Proposition 2. Let the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with elements (26) be assigned to the density matrix $\boldsymbol{\rho}$ and mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ with $\mu=1,\dots,M$ and $j=1,\dots,n$; then,

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_2^2\leq\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{M^2nS}+\frac{1-c}{Mn}+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2), \tag{49}$$

where $S=n/d$. It also holds that

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2\leq\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MnS}+\frac{1-c}{n}+c\,\mathrm{tr}(\boldsymbol{\rho}^2). \tag{50}$$

Proof. It was shown in the Appendix of Ref. 59 that

$$\frac{1}{M}\sum_{\mu=1}^{M}I(\mathcal{F}_\mu;\boldsymbol{\rho})\leq\frac{(1-c)\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MnS}+\frac{1}{n}. \tag{51}$$

This result follows from the properties of generalized equiangular measurements given in Refs. 60 and 61. Combining (20) with (51) leads to

$$I(\mathcal{F};\boldsymbol{\rho})\leq\frac{(1-c)\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{M^2nS}+\frac{1}{Mn}. \tag{52}$$

Thus, the index of coincidence is estimated from above in terms of the purity. Claim (49) immediately follows from (40) and (52). Combining (47) with (51) leads to (50). ■

Inequality (49) estimates the square of the Hilbert–Schmidt norm of $\Lambda(\mathcal{A};\boldsymbol{\rho})$ from above. Due to (50), we can estimate the averaged Hilbert–Schmidt norm of the matrices $\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})$. These relations further lead to inequalities for certain Schatten norms. Let the piecewise smooth function $X\mapsto L_\alpha(X)$ be defined as

$$\begin{aligned}L_\alpha(X)&=(q+1)\ln_\alpha(q+1)-q\ln_\alpha(q)\\&\quad-q(q+1)[\ln_\alpha(q+1)-\ln_\alpha(q)]X,\\&\qquad X\in\left[\frac{1}{q+1},\frac{1}{q}\right],\end{aligned} \tag{53}$$

where integer $q\geq 1$. This function will be used to formulate desired estimates.

Proposition 3. Let the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with elements (26) be assigned to the density matrix $\boldsymbol{\rho}$ and mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ with $\mu=1,\dots,M$ and $j=1,\dots,n$. For $\alpha\in[1,2]$, it holds that

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_\alpha^\alpha\leq 1-(\alpha-1)L_\alpha\left(\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{M^2nS}+\frac{1-c}{Mn}+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2)\right), \tag{54}$$

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_\alpha^\alpha \le 1-(\alpha-1)L_\alpha\Bigg(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MnS} + \frac{1-c}{n}+c\,\mathrm{tr}(\boldsymbol{\rho}^2)\Bigg). \tag{55}$$

Proof. Reference 62 is devoted to estimating the Tsallis α-entropy at the given index of coincidence. It was shown therein that, for $\alpha\in(0,2]$,

$$\frac{1}{1-\alpha}\Big(\sum\nolimits_j\lambda_j^\alpha-1\Big)\ge L_\alpha\Big(\sum\nolimits_j\lambda_j^2\Big), \tag{56}$$

where positive numbers λ_j obey $\sum_j\lambda_j=1$. Let λ_j with $j=1,\dots,Mn$ denote the positive eigenvalues of $\Lambda(\mathcal{A};\boldsymbol{\rho})$. For $1<\alpha\le 2$, we directly obtain from (56) that

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_\alpha^\alpha\le 1-(\alpha-1)L_\alpha\big(\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_2^2\big). \tag{57}$$

The latter remains valid for $\alpha=1$ due to $\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_1=1$. By construction, the function $X\mapsto L_\alpha(X)$ is decreasing and convex. Using these properties, we combine (49) with (57) to prove (54).

In addition, the function $X\mapsto 1-(\alpha-1)L_\alpha(X)$ is increasing and concave for $\alpha\in(1,2]$. By concavity and (57), we have

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_\alpha^\alpha\le 1-(\alpha-1)L_\alpha\Bigg(\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2\Bigg).$$

The latter proves (55) due to (50). ■

For the prescribed values of α, we have estimated the Schatten norms of $\Lambda(\mathcal{A};\boldsymbol{\rho})$ from above. In a similar manner, inequality (55) deals with the averaged norms. Let us proceed to inequalities for some of the Ky Fan norms.

B. Inequalities for estimating some Ky Fan norms

For positive semi-definite matrices, the Ky Fan k-norm reduces to the sum of k largest eigenvalues. Such sums can be characterized in terms of the Hilbert–Schmidt norm. The following statement holds.

Proposition 4. Let the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with elements (26) be assigned to the density matrix $\boldsymbol{\rho}$ and mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ with $\mu=1,\dots,M$ and $j=1,\dots,n$; then,

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(1)}\le\frac{1}{Mn}\Bigg\{1+\sqrt{Mn-1}\,\Bigg(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MS} + S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}, \tag{58}$$

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(2)}\le\frac{1}{Mn}\Bigg\{2+\sqrt{2Mn-4}\,\Bigg(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MS} + S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}. \tag{59}$$

In addition, it holds that

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(1)}\le\frac{1}{n}\Bigg\{1+\sqrt{n-1}\,\Bigg(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MS} + nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}, \tag{60}$$

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(2)}\le\frac{1}{n}\Bigg\{2+\sqrt{2n-4}\,\Bigg(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MS} + nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}. \tag{61}$$

Proof. Due to $\Lambda(\mathcal{A};\boldsymbol{\rho})\in\mathbb{M}_{Mn}^{+}(\mathbb{C})$ and $\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_1=1$, the eigenvalues of $\Lambda(\mathcal{A};\boldsymbol{\rho})$ are positive and sum to 1. The left-hand side of (58) is equal to the maximal eigenvalue. To bind it from above, we recall that the maximal probability can be estimated in terms of the index of coincidence as[62]

$$\max_{1\le j\le N}p_j\le\frac{1}{N}\left(1+\sqrt{N-1}\,\sqrt{NI(P)-1}\right), \tag{62}$$

where N is the number of outcomes. Note also that the right-hand side of (62) increases with $I(P)$. Applying these facts to the eigenvalues of $\Lambda(\mathcal{A};\boldsymbol{\rho})$ with $N=Mn$ and (49) completes the proof of (58). The right-hand side of (62) is a concave function of the index of coincidence. This fact allows one to write

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(1)}\le\frac{1}{n}\times\Bigg\{1+\sqrt{n-1}\,\Bigg(\frac{n}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2-1\Bigg)^{1/2}\Bigg\}. \tag{63}$$

Combining the latter with (50) completes the proof of (60).

It was formulated in Theorem 2 of Ref. 63 that

$$\max_{j\ne k}\{p_j+p_k\}\le\frac{1}{N}\left(2+\sqrt{2N-4}\,\sqrt{NI(P)-1}\right). \tag{64}$$

The right-hand side of (64) increases with $I(P)$ so that the inequalities (49) and (64) together provide (59). Since the right-hand side of (64) is a concave function of $I(P)$, we also have

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(2)}\le\frac{1}{n}\times\Bigg\{2+\sqrt{2n-4}\,\Bigg(\frac{n}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2-1\Bigg)^{1/2}\Bigg\}, \tag{65}$$

whence the result (61) follows. ■

The statement of Proposition 4 allows one to estimate the first Ky Fan norms of $\Lambda(\mathcal{A};\boldsymbol{\rho})$ from above. For M matrices $\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})$, the averaged Ky Fan norms were estimated. In all these cases, the inequalities are expressed in terms of purity and M, n, and d. If the measurement statistics is sufficient to evaluate the corresponding indices of coincidence, we also have other formulations. For $\Lambda(\mathcal{A};\boldsymbol{\rho})$, it holds that

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(1)} \leq \frac{1}{Mn}\left\{1+\sqrt{Mn-1}\,\big(Mn(1-c)I(\mathcal{F};\boldsymbol{\rho}) + S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big)^{1/2}\right\}, \tag{66}$$

$$\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(2)} \leq \frac{1}{Mn}\left\{2+\sqrt{2Mn-4}\,\big(Mn(1-c)I(\mathcal{F};\boldsymbol{\rho}) + S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big)^{1/2}\right\}. \tag{67}$$

In addition, the averaged Ky Fan norms satisfy

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(1)} \leq \frac{1}{n} \times\left\{1+\sqrt{n-1}\left(\frac{n-nc}{M}\sum_{\mu=1}^{M}I(\mathcal{F}_\mu;\boldsymbol{\rho})+nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right)^{1/2}\right\}, \tag{68}$$

$$\frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(2)} \leq \frac{1}{n} \times\left\{2+\sqrt{2n-4}\left(\frac{n-nc}{M}\sum_{\mu=1}^{M}I(\mathcal{F}_\mu;\boldsymbol{\rho})+nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right)^{1/2}\right\}. \tag{69}$$

The inequalities for $k = 1$ are of particular importance since they lead to uncertainty relations in terms of the min-entropy. The first Ky Fan norm is estimated from below as follows: Let $X \mapsto \varLambda_p(X)$ be a piecewise smooth function such that[62]

$$\varLambda_p(X) = \frac{1}{q}\left(1+\sqrt{\frac{qX-1}{q-1}}\right), \qquad X \in \left[\frac{1}{q}, \frac{1}{q-1}\right], \tag{70}$$

where integer $q \geq 2$. For any distribution, the maximal probability is not less than $\varLambda_p(I(P))$,[62] whence

$$\varLambda_p\left((1-c)I(\mathcal{F};\boldsymbol{\rho})+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2)\right) = \varLambda_p\big(\|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_2^2\big) \leq \|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(1)}, \tag{71}$$

$$\varLambda_p\big((1-c)I(\mathcal{F}_\mu;\boldsymbol{\rho})+c\,\mathrm{tr}(\boldsymbol{\rho}^2)\big) = \varLambda_p\big(\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2\big) \leq \|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(1)}. \tag{72}$$

At this point, the measurement statistics allows us to evaluate the squared Hilbert–Schmidt norms due to relations (40) and (48). Inequalities (49) and (50) are not suitable here since the function $X \mapsto \varLambda_p(X)$ increases.

C. Uncertainty relations for unravelings of the considered quantum channels

The above inequalities lead to uncertainty relations for unravelings of channel (24). Information entropies provide a flexible way to address various measurement scenarios, including the cases of quantum memory[64–67] and multipartite systems.[68–70] For more applications of entropic uncertainty relations in quantum information, see Ref. 71 and references therein. For special types of measurements, uncertainty relations often follow from the estimation of the corresponding indices of coincidence. Using MUBs, this approach was given in Ref. 72. It was later applied to a SIC-POVM,[73] a general SIC-POVM,[74] and a single ETF.[75] We can similarly treat the case of mutually unbiased ETFs. Proposition 2 allows one to evaluate the Hilbert–Schmidt norm of all matrices of the form $\Lambda(\mathcal{B};\boldsymbol{\rho})$. These positive semi-definite matrices have the same non-zero eigenvalues, whence

$$\|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_\alpha = \|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_\alpha, \tag{73}$$

$$\|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_{(k)} = \|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(k)}. \tag{74}$$

For each of M quantum channels of form (29), we also write

$$\|\Lambda(\mathcal{B}_\mu;\boldsymbol{\rho})\|_\alpha = \|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_\alpha, \tag{75}$$

$$\|\Lambda(\mathcal{B}_\mu;\boldsymbol{\rho})\|_{(k)} = \|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(k)}. \tag{76}$$

Using the above connection, we have arrived at a conclusion.

Proposition 5. Let principal Kraus operators (24) be built of the states of mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ with $\mu = 1,\ldots,M$ and $j = 1,\ldots,n$. For arbitrary unraveling $\mathcal{B}$ of channel (24), each density matrix $\boldsymbol{\rho}$, and $\alpha \in (0,2]$, it holds that

$$H_\alpha(\mathcal{B};\boldsymbol{\rho}) \geq L_\alpha\left(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{M^2nS}+\frac{1-c}{Mn}+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2)\right). \tag{77}$$

Let $\mathcal{B}_\mu$ be an arbitrary unraveling of the quantum channel (29). For each density matrix $\boldsymbol{\rho}$ and $\alpha \in (0,2]$, we also have

$$\frac{1}{M}\sum_{\mu=1}^{M}H_\alpha(\mathcal{B}_\mu;\boldsymbol{\rho}) \geq L_\alpha\left(\frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{MnS}+\frac{1-c}{n}+c\,\mathrm{tr}(\boldsymbol{\rho}^2)\right). \tag{78}$$

Proof. It immediately follows from (56) that, for $\alpha \in (0,2]$,

$$H_\alpha(\mathcal{B};\boldsymbol{\rho}) \geq L_\alpha(I(\mathcal{B};\boldsymbol{\rho})). \tag{79}$$

To estimate $I(\mathcal{B};\boldsymbol{\rho})$ for an arbitrary unraveling $\mathcal{B}$, we use (49) and (73) to write

$$I(\mathcal{B};\boldsymbol{\rho}) \leq \|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_2^2 = \|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_2^2 \leq \frac{(1-c)^2\big[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\big]}{M^2nS} + \frac{1-c}{Mn}+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2). \tag{80}$$

Since the function $X \mapsto L_\alpha(X)$ decreases, combining (79) with (80) completes the proof of (77).

As the function $X \mapsto L_\alpha(X)$ is convex, one gets

$$\frac{1}{M}\sum_{\mu=1}^{M} H_\alpha(\mathcal{B}_\mu;\boldsymbol{\rho}) \geq L_\alpha\left(\frac{1}{M}\sum_{\mu=1}^{M} I(\mathcal{B}_\mu;\boldsymbol{\rho})\right). \tag{81}$$

According to (50), we also write

$$\begin{aligned}\frac{1}{M}\sum_{\mu=1}^{M} I(\mathcal{B}_\mu;\boldsymbol{\rho}) &\leq \frac{1}{M}\sum_{\mu=1}^{M} \|\Lambda(\mathcal{B}_\mu;\boldsymbol{\rho})\|_2^2 \\ &= \frac{1}{M}\sum_{\mu=1}^{M} \|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_2^2 \leq \frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MnS} \\ &+ \frac{1-c}{n} + c\,\mathrm{tr}(\boldsymbol{\rho}^2).\end{aligned} \tag{82}$$

Combining the latter with (81) leads to (78) due to the decreasing of the function $X \mapsto L_\alpha(X)$. ∎

The statement of Proposition 5 gives an estimate of $H_\alpha(\mathcal{B};\boldsymbol{\rho})$ for arbitrary unraveling of map (24). Using (37) allows us to obtain

$$\begin{aligned}R_\alpha(\mathcal{B};\boldsymbol{\rho}) \geq \frac{1}{1-\alpha}\ln\Bigg\{1+(1-\alpha)L_\alpha\Bigg(&\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{M^2nS} \\ &+\frac{1-c}{Mn}+\frac{cd+M-1}{Md}\mathrm{tr}(\boldsymbol{\rho}^2)\Bigg)\Bigg\},\end{aligned} \tag{83}$$

where $\alpha \in (0,2]$. Indeed, the function $X \mapsto (1-\alpha)^{-1}\ln\left[1+(1-\alpha)X\right]$ is increasing. For $\alpha \in [1,2]$, one also has

$$\begin{aligned}\frac{1}{M}\sum_{\mu=1}^{M} R_\alpha(\mathcal{B}_\mu;\boldsymbol{\rho}) \geq \frac{1}{1-\alpha}\ln\Bigg\{1+(1-\alpha)L_\alpha\Bigg(&\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MnS} \\ &+\frac{1-c}{n}+c\,\mathrm{tr}(\boldsymbol{\rho}^2)\Bigg)\Bigg\}.\end{aligned} \tag{84}$$

To derive the latter, we recall the following fact. If the function $X \mapsto f(X)$ is convex and the function $Y \mapsto g(Y)$ is increasing and convex, then their composition $X \mapsto g(f(X))$ is convex too. Combining this with the properties of $X \mapsto L_\alpha(X)$ and $Y \mapsto (1-\alpha)^{-1}\ln\{1+(1-\alpha)Y\}$ gives convexity of the function

$$X \mapsto \frac{1}{1-\alpha}\ln\{1+(1-\alpha)L_\alpha(X)\}$$

for $\alpha \in [1,2]$. In addition, this function decreases. Then, result (84) follows from (82) and

$$\frac{1}{M}\sum_{\mu=1}^{M} R_\alpha(\mathcal{B}_\mu;\boldsymbol{\rho}) \geq \frac{1}{M}\sum_{\mu=1}^{M} \frac{\ln\{1+(1-\alpha)L_\alpha(I(\mathcal{B}_\mu;\boldsymbol{\rho}))\}}{1-\alpha}. \tag{85}$$

Thus, we have formulated uncertainty relations in terms of the Tsallis and Rényi entropies for arbitrary unravelings of the quantum channels (24) and (29).

Uncertainty relations in terms of the min-entropy follow from the estimates of the maximal probabilities from above. The following statement is based on results (58) and (60).

Proposition 6. Let principal Kraus operators (24) be built of the states of mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ *with* $\mu = 1,\dots,M$ *and* $j = 1,\dots,n$. *For arbitrary unraveling* $\mathcal{B}$ *of channel (24) and each density matrix* $\boldsymbol{\rho}$, *it holds that*

$$\begin{aligned}R_\infty(\mathcal{B};\boldsymbol{\rho}) \geq \ln(Mn) - \ln\Bigg\{1+\sqrt{Mn-1}\Bigg(&\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} \\ &+S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}.\end{aligned} \tag{86}$$

Let $\mathcal{B}_\mu$ *be an arbitrary unraveling of the quantum channel (29). For each density matrix* $\boldsymbol{\rho}$, *we also have*

$$\begin{aligned}\frac{1}{M}\sum_{\mu=1}^{M} R_\infty(\mathcal{B}_\mu;\boldsymbol{\rho}) \geq \ln n - \ln\Bigg\{1+\sqrt{n-1}\Bigg(&\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} \\ &+nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}.\end{aligned} \tag{87}$$

Proof. To estimate the maximal probability for an arbitrary unraveling $\mathcal{B}$, we use (58) and (74) to get

$$\begin{aligned}\max_{1\leq j\leq Mn} p_j(\mathcal{B};\boldsymbol{\rho}) &\leq \|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_{(1)} = \|\Lambda(\mathcal{A};\boldsymbol{\rho})\|_{(1)} \\ &\leq \frac{1}{Mn}\Bigg\{1+\sqrt{Mn-1}\Bigg(\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} \\ &+S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}.\end{aligned} \tag{88}$$

Combining this with (35) immediately gives (86). In a similar manner, one uses (60) to write

$$\begin{aligned}\frac{1}{M}\sum_{\mu=1}^{M}\max_{1\leq j\leq n} p_j(\mathcal{B}_\mu;\boldsymbol{\rho}) &\leq \frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{B}_\mu;\boldsymbol{\rho})\|_{(1)} = \frac{1}{M}\sum_{\mu=1}^{M}\|\Lambda(\mathcal{A}_\mu;\boldsymbol{\rho})\|_{(1)} \\ &\leq \frac{1}{n}\Bigg\{1+\sqrt{n-1}\Bigg(\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} \\ &+nc\,\mathrm{tr}(\boldsymbol{\rho}^2)-c\Bigg)^{1/2}\Bigg\}.\end{aligned} \tag{89}$$

To obtain (87), we combine (35) with (89) since the function $X \mapsto -\ln X$ is convex and decreasing. ∎

Formulas (59) and (61) lead to uncertainty relations of the Landau–Pollak type. Inequalities of this kind are formulated in terms of the two maximal probabilities. Applications to uncertainty relations were mentioned in Ref. 76, whereas the original formulation[77] was focused on signal analysis. Here, we have arrived at a conclusion.

Proposition 7. Let principal Kraus operators (24) be built of the states of mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ *with*

$\mu = 1, \ldots, M$ and $j = 1, \ldots, n$. For arbitrary unraveling $\mathcal{B}$ of channel (24) and each density matrix $\boldsymbol{\rho}$, it holds that

$$\max_{j \neq k} \{p_j(\mathcal{B};\boldsymbol{\rho}) + p_k(\mathcal{B};\boldsymbol{\rho})\} \le \frac{1}{Mn} \times \left\{2 + \sqrt{2Mn-4}\left(\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} + S(cd+M-1)\,\mathrm{tr}(\boldsymbol{\rho}^2) - c\right)^{1/2}\right\}. \tag{90}$$

Let $\mathcal{B}_\mu$ be an arbitrary unraveling of the quantum channel (29). For each density matrix $\boldsymbol{\rho}$, we also have

$$\frac{1}{M}\sum_{\mu=1}^{M} \max_{j \neq k} \{p_j(\mathcal{B}_\mu;\boldsymbol{\rho}) + p_k(\mathcal{B}_\mu;\boldsymbol{\rho})\} \le \frac{1}{n} \times \left\{2 + \sqrt{2n-4}\left(\frac{(1-c)^2\left[d\,\mathrm{tr}(\boldsymbol{\rho}^2)-1\right]}{MS} + nc\,\mathrm{tr}(\boldsymbol{\rho}^2) - c\right)^{1/2}\right\}. \tag{91}$$

Proof. Recall that the matrix $\Lambda(\mathcal{B}_\mu, \boldsymbol{\rho})$ is positive semi-definite, and its diagonal elements represent probabilities $p_j(\mathcal{B};\boldsymbol{\rho})$. It also follows from Theorem 1 of Ky Fan[44] that

$$p_j(\mathcal{B}_\mu;\boldsymbol{\rho}) + p_k(\mathcal{B}_\mu;\boldsymbol{\rho}) \le \|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_{(2)}$$

for all $j \neq k$. Then, inequality (90) follows by combining (59) with (74) for $k = 2$. In a similar manner, inequality (91) is based on (61) and (76). ■

D. Examples of Kirkwood–Dirac quasiprobabilities for mutually unbiased ETFs

Let us discuss concrete examples of mutually unbiased equiangular tight frames. A traditional example of MUBs is provided by three eigenbases of the Pauli matrices X, Y, and Z. The corresponding six states are represented on the Bloch sphere by vertices of an octahedron as shown in Fig. 1. The normalized eigenstates are, respectively, denoted by $|x_\pm\rangle$, $|y_\pm\rangle$, and $|z_\pm\rangle$. We begin with the case $M = 2$ since a diagonal matrix appears for $M = 1$. The choice $M = 2$ gives a pair of complementary observables. The obtained quasiprobabilities can be interpreted as a finite-dimensional counterpart of the original formulation.[22] For definiteness, the four principal Kraus operators of the unraveling $\mathcal{A}$ read as $2^{-1}|x_\pm\rangle\langle x_\pm|$ and $2^{-1}|z_\pm\rangle\langle z_\pm|$. We begin with illustrating inequality (54). Figure 2 shows the Schatten 1.5-norm of the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ and its estimate from above as a function of square of the Bloch vector. The three orientations of the Bloch vector are used here. According to (73), the presented curves also characterize $\|\Lambda(\mathcal{B};\boldsymbol{\rho})\|_{1.5}$ for any unraveling $\mathcal{B}$. Errors of the estimate due to (54) are larger when the Bloch vector is directed along the y-axis. For this direction, the estimation error is maximal for pure states. However, it is less than seven percents in a relative scale. This example shows a utility of the obtained estimates.

The second example deals with a ququart in dimension four. The fifteen frame vectors arise as columns of the matrices $\boldsymbol{\Psi}$, $\boldsymbol{\Delta\Psi}$,

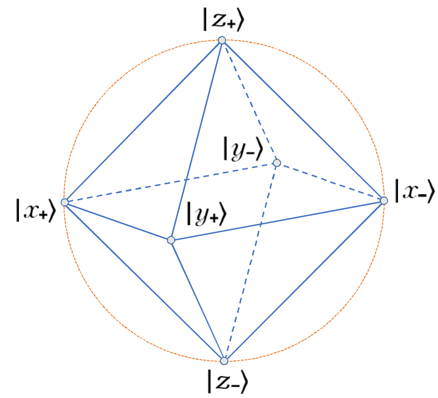

FIG. 1. Octahedron vertices corresponding to the six states of three mutually unbiased bases in dimension two.

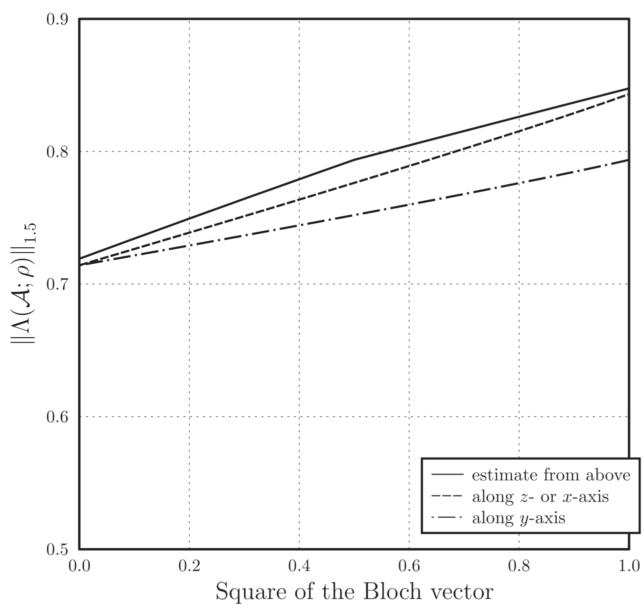

FIG. 2. Schatten 1.5-norm of the matrix $\Lambda(\mathcal{A};\boldsymbol{\rho})$ for few orientations of the Bloch vector together with the estimate from above obtained due to (54).

and $\boldsymbol{\Delta}^2\boldsymbol{\Psi}$, of which $\boldsymbol{\Psi}$ and $\boldsymbol{\Delta}$ are given in (14). Let us consider density matrices of the form

$$\boldsymbol{\rho} = v\,|\Phi'_{00}\rangle\langle\Phi'_{00}| + \frac{1-v}{4}\,\mathbb{1}_2 \otimes \mathbb{1}_2, \tag{92}$$

where $v \in [0,1]$ and $|\Phi'_{00}\rangle$ is one of the rotated Bell states. Such density matrices are similar to isotropic states typically used in testing criteria to detect non-classical correlations. For unitary qubit rotation

$$\mathsf{R}_{\hat{\ell}}(\theta) = \cos\left(\frac{\theta}{2}\right)\mathbb{1}_2 - \mathrm{i}\,\sin\left(\frac{\theta}{2}\right)(\ell_x\mathsf{X} + \ell_y\mathsf{Y} + \ell_z\mathsf{Z}) \tag{93}$$

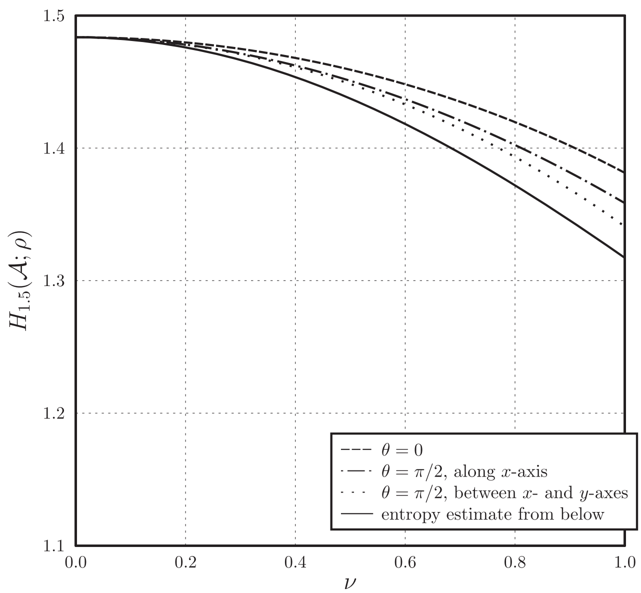

FIG. 3. Tsallis entropy $H_{1.5}(\mathcal{A};\rho)$ for few orientations of the vector $\hat{\ell}$ with the estimate from below due to (95).

and $a, b = 0, 1$, the rotated Bell states read as $|\Phi'_{ab}\rangle = \left(\mathsf{R}_{\hat{\ell}}(\theta)^{\dagger} \otimes \mathbb{1}_2\right)|\Phi_{ab}\rangle$. By $\hat{\ell} = (\ell_x, \ell_y, \ell_z)$, we mean here a real vector of unit length. The rotated Bell states are used in measurement based quantum computation.[78] The first Bell state reads as

$$|\Phi_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{94}$$

Let Kraus operators be built of the states of a mutually unbiased-equiangular tight frame $\{|\phi_{\mu j}\rangle\}$ according to (24). It follows from (52) and (56) that, for $\alpha \in (0, 2]$,

$$H_\alpha(\mathcal{A};\rho) \geq L_\alpha\left(\frac{(1-c)\left[d\,\mathrm{tr}(\rho^2) - 1\right]}{M^2 nS} + \frac{1}{Mn}\right). \tag{95}$$

Figure 3 shows the entropy $H_{1.5}(\mathcal{A};\rho)$ and its estimate from below as a function of $\nu \in [0, 1]$. The choice $\theta = 0$ implies the use of the original state (94). In addition, the rotation with $\theta = \pi$ was used with two orientations of the vector $\hat{\ell}$ along the x-axis and the middle between the x- and y-axes. All the curves are similar and decrease with the growth of ν. Similarly to Fig. 2, estimation errors become maximal for pure states. In a relative scale, they do not exceed five percents. A good quality of the given estimates is also observed.

IV. CONCLUSIONS

This paper considered Kirkwood–Dirac quasiprobabilities for measurements assigned to a mutually unbiased-equiangular tight frame. Mutually unbiased bases appeared as an important particular case. The contribution of this paper is characterized as threefold. First, the approach to quasiprobabilities given in Ref. 42 was extended to several measurements. It is consistent with the previous definition of generalized Kirkwood–Dirac quasiprobabilities.[51] Second, the matrices of quasiprobabilities were characterized in terms of unitarily invariant norms, such as the Schatten and Ky Fan ones. Third, improved entropic uncertainty relations for unravelings of the corresponding quantum channels were derived.

Finite tight frames have found use in various disciplines, including quantum information science. The considered Kirkwood–Dirac quasiprobabilities are easy to analyze in terms of unravelings of quantum channels whose principal Kraus operators are defined via states of the frame. The structure of a mutually unbiased-equiangular tight frame allows one to estimate from above some Schatten and Ky Fan norms of matrices formed by Kirkwood–Dirac quasiprobabilities. Such matrices are immediately connected with the matrices assigned to different unravelings of certain quantum channels. Here, many results are naturally formulated in terms of different unravelings of quantum channels of interest.

Since measurement statistics can be treated in two different ways, the following interpretations were considered. The first deals with a single POVM and only one quantum channel. The second treatment uses a collection of similar quantum channels defined in terms of mutually unbiased ETFs. Quantitative characteristics of the corresponding matrices were described due to the properties of a mutually unbiased-equiangular tight frame. For a set of several quantum channels, averaged Schatten and Ky Fan norms were estimated from above. The derived inequalities have led to uncertainty relations that hold for arbitrary unravelings of the considered channels. A utility of the derived inequalities was illustrated with qubit MUBs and ququart isotropic states defined in terms of rotated Bell states.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Alexey E. Rastegin: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

REFERENCES

[1] G. Wendin, "Quantum information processing with superconducting circuits: A review," Rep. Prog. Phys. **80**, 106001 (2017).
[2] F. Flamini, N. Spagnolo, and F. Sciarrino, "Photonic quantum information processing: A review," Rep. Prog. Phys. **82**, 016001 (2019).
[3] C. D. Bruzewicz, J. Chiaverini, R. McConnell, and J. M. Sage, "Trapped-ion quantum computing: Progress and challenges," Appl. Phys. Rev. **6**, 021314 (2019).
[4] L. Henriet, L. Beguin, A. Signoles, T. Lahaye, A. Browaeys, G.-O. Reymond, and C. Jurczak, "Quantum computing with neutral atoms," Quantum **4**, 327 (2020).

[5]T. Durt, B.-G. Englert, I. Bengtsson, and K. Życzkowski," On mutually unbiased bases," Int. J. Quantum Inf. **8**, 535–640 (2010).
[6]G. Zauner, "Quantum designs: Foundations of a noncommutative design theory," Int. J. Quantum Inf. **9**, 445–507 (2011).
[7]J. Schwinger, "Unitary operator bases," Proc. Natl. Acad. Sci. U. S. A. **46**, 570–579 (1960).
[8]I. D. Ivanović, "Geometrical description of quantal state determination," J. Phys. A: Math. Gen. **14**, 3241–3245 (1981).
[9]W. K. Wootters and B. D. Fields, "Optimal state-determination by mutually unbiased measurements," Ann. Phys. **191**, 363–381 (1989).
[10]C. H. Bennett and G. Brassard, "Quantum cryptography: Public key distribution and coin tossing," Theor. Comput. Sci. **560**, 7–11 (2014).
[11]J. H. v. Lint and J. J. Seidel, "Equilateral point sets in elliptic geometry," Indagationes Mathematicae **69**, 335–348 (1966).
[12]P. W. H. Lemmens and J. J. Seidel, "Equiangular lines," J. Algebra **24**, 494–512 (1973).
[13]M. Fickus and B. R. Mayo, "Mutually unbiased equiangular tight frames," IEEE Trans. Inf. Theory **67**, 1656–1667 (2021).
[14]M. A. Ballester and S. Wehner, "Entropic uncertainty relations and locking: Tight bounds for mutually unbiased bases," Phys. Rev. A **75**, 022319 (2007).
[15]E. Wigner, "On the quantum correction for thermodynamic equilibrium," Phys. Rev. **40**, 749–759 (1932).
[16]A. Casado, S. Guerra, and J. Plácido, "Wigner representation for experiments on quantum cryptography using two-photon polarization entanglement produced in parametric downconversion," J. Phys. B: At., Mol. Opt. Phys. **41**, 045501 (2008).
[17]D. Sels, F. Brosens, and W. Magnus, "Wigner distribution functions for complex dynamical systems: A path integral approach," Physica A **392**, 326–335 (2013).
[18]R. A. Earnshaw, C. Lei, J. Li, S. Mugassabi, and A. Vourdas, "Large-scale data analysis using the Wigner function," Physica A **391**, 2401–2407 (2012).
[19]F. Siyouri, M. El Baz, and Y. Hassouni, "The negativity of Wigner function as a measure of quantum correlations," Quantum Inf. Process. **15**, 4237–4252 (2016).
[20]G. S. Thekkadath, F. Hufnagel, and J. S. Lundeen, "Determining complementary properties using weak-measurement: Uncertainty, predictability, and disturbance," New J. Phys. **20**, 113034 (2018).
[21]D. R. M. Arvidsson-Shukur *et al.*, "Properties and applications of the Kirkwood–Dirac distribution," New J. Phys. **26**, 121201 (2024).
[22]J. G. Kirkwood, "Quantum statistics of almost classical assemblies," Phys. Rev. **44**, 31–37 (1933).
[23]P. A. M. Dirac, "On the analogy between classical and quantum mechanics," Rev. Mod. Phys. **17**, 195–199 (1945).
[24]L. M. Johansen, "Quantum theory of successive projective measurements," Phys. Rev. A **76**, 012119 (2007).
[25]J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, "Direct measurement of the quantum wavefunction," Nature **474**, 188–191 (2011).
[26]J. S. Lundeen and C. Bamber, "Procedure for direct measurement of general quantum states using weak measurement," Phys. Rev. Lett. **108**, 070402 (2012).
[27]C. Bamber and J. S. Lundeen, "Observing Dirac's classical phase space analog to the quantum state," Phys. Rev. Lett. **112**, 070405 (2014).
[28]N. Yunger Halpern, "Jarzynski-like equality for the out-of-time-ordered correlator," Phys. Rev. A **95**, 012120 (2017).
[29]N. Yunger Halpern, B. Swingle, and J. Dressel, "Quasiprobability behind the out-of-time-ordered correlator," Phys. Rev. A **97**, 042105 (2018).
[30]J. R. González Alonso, N. Yunger Halpern, and J. Dressel, "Out-of-time-ordered-correlator quasiprobabilities robustly witness scrambling," Phys. Rev. Lett. **122**, 040404 (2019).
[31]N. Yunger Halpern, A. Bartolotta, and J. Pollack, "Entropic uncertainty relations for quantum information scrambling," Commun. Phys. **2**, 92 (2019).
[32]J. Dressel, "Weak values as interference phenomena," Phys. Rev. A **91**, 032116 (2015).
[33]R. Kunjwal, M. Lostaglio, and M. F. Pusey, "Anomalous weak values and contextuality: Robustness, tightness, and imaginary parts," Phys. Rev. A **100**, 042116 (2019).
[34]D. R. M. Arvidsson-Shukur, N. Yunger Halpern, H. V. Lepage, A. A. Lasek, C. H. W. Barnes, and S. Lloyd, "Quantum advantage in postselected metrology," Nat. Commun. **11**, 3775 (2020).
[35]A. E. Allahverdyan, "Nonequilibrium quantum fluctuations of work," Phys. Rev. E **90**, 032137 (2014).
[36]H. J. D. Miller and J. Anders, "Time-reversal symmetric work distributions for closed quantum dynamics in the histories framework," New J. Phys. **19**, 062001 (2017).
[37]A. Levy and M. Lostaglio, "Quasiprobability distribution for heat fluctuations in the quantum regime," PRX Quantum **1**, 010309 (2020).
[38]H. F. Hofmann, "On the role of complex phases in the quantum statistics of weak measurements," New J. Phys. **13**, 103009 (2011).
[39]J. J. Halliwell, "Leggett–Garg inequalities and no-signaling in time: A quasiprobability approach," Phys. Rev. A **93**, 022123 (2016).
[40]D. R. M. Arvidsson-Shukur, J. C. Drori, and N. Y. Halpern, "Conditions tighter than noncommutation needed for nonclassicality," J. Phys. A: Math. Theor. **54**, 284001 (2021).
[41]S. De Bièvre, "Complete incompatibility, support uncertainty, and Kirkwood–Dirac nonclassicality," Phys. Rev. Lett. **127**, 190404 (2021).
[42]A. E. Rastegin, "On Kirkwood–Dirac quasiprobabilities and unravelings of quantum channel assigned to a tight frame," Physica A **629**, 129208 (2023).
[43]R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1990).
[44]K. Fan, "On a theorem of Weyl concerning eigenvalues of linear transformations I," Proc. Natl. Acad. Sci. U. S. A. **35**, 652–655 (1949).
[45]M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2010).
[46]F. Caro Pérez, V. Gonzalez Avella, and D. Goyeneche, "Mutually unbiased frames," Quantum **6**, 851 (2022).
[47]J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, "Symmetric informationally complete quantum measurements," J. Math. Phys. **45**, 2171–2180 (2004).
[48]A. Czerwinski, "Quantum state tomography with informationally complete POVMs generated in the time domain," Quantum Inf. Process. **20**, 105 (2021).
[49]K. Kraus, "Complementary observables and uncertainty relations," Phys. Rev. D **35**, 3070–3075 (1987).
[50]A. Kalev and G. Gour, "Mutually unbiased measurements in finite dimensions," New J. Phys. **16**, 053038 (2014).
[51]N. Lupu-Gladstein *et al.*, "Negative quasiprobabilities enhance phase estimation in quantum-optics experiment," Phys. Rev. Lett. **128**, 220504 (2022).
[52]H. J. Carmichael, *An Open Systems Approach to Quantum Optics*, *Lecture Notes in Physics* (Springer, Berlin, 1993), Vol. m18.
[53]J. B. DeBrota and B. C. Stacey, "Lüders channels and the existence of symmetric-informationally-complete measurements," Phys. Rev. A **100**, 062327 (2019).
[54]A. Rényi, "On measures of entropy and information," in *Proceedings of 4th Berkeley Symposium on Mathematical Statistics and Probability*, edited by J. Neyman (University of California Press, Berkeley, 1961), Vol. I, pp. 547–561.
[55]C. Tsallis, "Possible generalization of Boltzmann–Gibbs statistics," J. Stat. Phys. **52**, 479–487 (1988).
[56]I. Bengtsson and K. Życzkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge University Press, Cambridge, 2017).
[57]C. Tsallis, *Introduction to Nonextensive Statistical Mechanics: Approaching a Complex World* (Springer, New York, 2023).
[58]P. Jizba and T. Arimitsu, "The world according to Rényi: Thermodynamics of multifractal systems," Ann. Phys. **312**, 17–59 (2004).
[59]A. E. Rastegin, "Uncertainty relations for quantum coherence with respect to mutually unbiased equiangular tight frames," Phys. Scr. **99**, 115109 (2024).
[60]K. Siudzińska, "Informationally overcomplete measurements from generalized equiangular tight frames," J. Phys. A: Math. Theor. **57**, 335302 (2024).
[61]A. E. Rastegin, "Uncertainty relations for quantum measurements from generalized equiangular tight frames," Ann. Phys. **470**, 169785 (2024).
[62]A. E. Rastegin, "Uncertainty relations in terms of generalized entropies derived from information diagrams," arXiv:2305.18005 [quant-ph] (2023).
[63]A. E. Rastegin, "Entropic uncertainty relations for measurements assigned to a projective two-design," APL Quantum **1**, 026111 (2024).
[64]M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, "The uncertainty principle in the presence of quantum memory," Nat. Phys. **6**, 659 (2010).

[65] F. Ming, D. Wang, X.-G. Fan, W.-N. Shi, L. Ye, and J.-L. Chen, "Improved tripartite uncertainty relation with quantum memory," Phys. Rev. A **102**, 012206 (2020).
[66] B.-F. Xie, F. Ming, D. Wang, L. Ye, and J.-L. Chen, "Optimized entropic uncertainty relations for multiple measurements," Phys. Rev. A **104**, 062204 (2021).
[67] T.-Y. Wang and D. Wang, "Stronger entropic uncertainty relations with multiple quantum memories," Phys. Lett. A **499**, 129364 (2024).
[68] L. Wu, L. Ye, and D. Wang, "Tighter generalized entropic uncertainty relations in multipartite systems," Phys. Rev. A **106**, 062219 (2022).
[69] Z.-A. Wang, B.-F. Xie, F. Ming, Y.-T. Wang, D. Wang, Y. Meng, Z.-H. Liu, K. Xu, J.-S. Tang, L. Ye, C.-F. Li, G.-C. Guo, and S. Kais, "Experimental test of generalized multipartite entropic uncertainty relations," Phys. Rev. A **110**, 062220 (2024).
[70] T.-Y. Wang and D. Wang, "Entropic uncertainty relations in Schwarzschild space-time," Phys. Lett. B **855**, 138876 (2024).
[71] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, "Entropic uncertainty relations and their applications," Rev. Mod. Phys. **89**, 015002 (2017).
[72] S. Wu, S. Yu, and K. Mølmer, "Entropic uncertainty relation for mutually unbiased bases," Phys. Rev. A **79**, 022104 (2009).
[73] A. E. Rastegin, "Uncertainty relations for MUBs and SIC-POVMs in terms of generalized entropies," Eur. Phys. J. D **67**, 269 (2013).
[74] A. E. Rastegin, "Notes on general SIC-POVMs," Phys. Scr. **89**, 085101 (2014).
[75] A. E. Rastegin, "Entropic uncertainty relations from equiangular tight frames and their applications," Proc. R. Soc. A **479**, 20220546 (2023).
[76] H. Maassen and J. B. M. Uffink, "Generalized entropic uncertainty relations," Phys. Rev. Lett. **60**, 1103–1106 (1988).
[77] H. J. Landau and H. O. Pollak, "Prolate spheroidal wave functions, Fourier analysis and uncertainty - II," Bell Syst. Tech. J. **40**, 65–84 (1961).
[78] R. Jozsa, "An introduction to measurement based quantum computation," arXiv:quant-ph/0508124 (2005).