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SOME FORMULAS FOR ESTIMATING TRACKING ERRORS*

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Abstract

Universal curves and approximate formulas for the elements of the track parameter error matrix are derived for homogeneous tracking detectors in the continuous measurement limit.

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1. Introduction

Formulas for estimating the track parameter error matrix resulting from diagonal uniformly weighted fits have been published in many places [1]. This note derives similar formulas for optimal fits. It also also demonstrates that in the limit of continuous measurement, the matrix elements are described by universal curves. The case treated is homogeneous systems of many layers. In general, heterogeneous systems require numerical calculation.

If the track is nearly straight, we can describe its projection on the plane perpendicular to the magnetic field as

$$y = a + bx + cx^2/2 \quad , \tag{1.1}$$

where a is the position at the beginning of the tracking system, b is the slope, c is the curvature, and x is the distance along the track. For the purposes of discussing errors, this formulation is good for very curved tracks also; we need only think of y as being the deviation from a reference track close to the actual track. We are interested in the entire variance matrix for a, b, and c.

If the magnetic field is uniform, then the errors in the bend and nonbend planes decouple, the errors can be expressed in a manner that is independent of the dip angle if appropriate projected quantities are used, and momentum errors are simply related to the curvature errors and the field strength.

2. Measurement Errors

Consider a detector with a total length L, N equally spaced layers, and an rms measurement error at each layer of σ. If we define the information density i as:

$$i \equiv \frac{(N+5)}{\sigma^2 L} \quad , \tag{2.1}$$

we may write the Gluckstern [2] formula for the curvature error when there is insignificant multiple scattering as:

$$V_{cc}^{r} \approx \frac{720}{iL^5} \quad . \tag{2.2}$$

In this case, the uniformly weighted diagonal fit is optimal. This illustrates clearly the $1/L^5$ dependence of a detector with fixed spacing between measurements.

3. Multiple Scattering Errors

Gluckstern's formula for V_{cc} when only multiple scattering errors are significant can be written as

$$V_{cc}^{s} = \frac{1.43s}{L} \quad , \tag{3.1}$$

where scattering density s is the mean square projected scattering angle per unit length.

The factor of 1.43 assumes an infinite number of equally spaced measurements and a uniformly weighted diagonal fit. According to Scott [3], if an optimal fit is done using all information about scattering errors at the measurement locations and their correlations, this factor is 1.00, i.e.,

$$V_{cc}^{s} \approx \frac{s}{L} \quad . \tag{3.2}$$

This gives us an upper limit on the possible improvement to be gained by doing an optimal fit.

4. Optimal Fits with Mixed Errors

Adding together (2.2) and (3.2), the optimal fit result is

$$V_{cc} \approx \frac{720}{iL^5} + \frac{s}{L} \quad . \tag{4.1}$$

This is all very fine, but you may have observed that the optimal fits which yielded the best values for V_{cc}^r and V_{cc}^s were different. Is adding the variances a good approximation for an optimal fit? To answer this and to find the other matrix elements, we must study the combined system. Note that $V_{ij} \not\approx V_{ij}^r + V_{ij}^s$ for V_{aa} , V_{bb} , or V_{ab} .

Billior [4] describes a recursive procedure which propagates the fit results from layer to layer, adding the information gain and loss at each layer. This method provides a straightforward means to calculate errors. He gives the following recursion relation for the information matrix $I \equiv V^{-1}$:

$$I_{n+1} = D^T \left(I_n^{-1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & sl & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)^{-1} D + \begin{bmatrix} il & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ,$$

$$D = \begin{bmatrix} 1 & l & l^2/2 \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} \quad , \tag{4.2}$$

$$l \equiv L/N \quad .$$

D is the matrix which transforms the parameters from those defined at one layer to those defined at the next layer.

Letting $l \to 0$ while holding i and s constant is the "continuous" detector limit. Let's assume for the moment that in this limit there is universal curve, i.e., that the same curve (up to scale factors in length and magnitude) which describes a matrix element's dependence on length is independent of i, and s. (I will demonstrate this later.) We can use (4.2) to calculate this curve by setting $i = s = 1$ and making l small. The successive I_n^{-1}s give the results we seek. Figure 3 shows this curve for V_{cc} and compares it to the approximate equation (4.1). We get the expected limits for thin and thick chambers. The transition between thick and thin is at $L \approx 7$. The numerical result is always larger than the approximation but never by more than approximately 25%. Figures 1 through 6 show the universal curves for all the matrix elements.

In one sense, our problem is solved once we determine the scale factors, but perhaps we can derive formulas for these curves. Define the characteristic scattering length of the the detector $\lambda \equiv 1/\sqrt[4]{is}$. For an infinitely long detector, Billoir gives the following power series in l/λ (translated into our notation):

$$
\begin{aligned}
V_{aa}^{\infty} &= \frac{\sqrt{2}}{i\lambda}\left(1 - \frac{1}{\sqrt{2}}\frac{l}{\lambda} - \frac{5}{8}\left(\frac{l}{\lambda}\right)^2 + \cdots\right) \\
V_{bb}^{\infty} &= \frac{\sqrt{2}}{i\lambda^3}\left(1 - \frac{1}{\sqrt{2}}\frac{l}{\lambda} + \frac{1}{8}\left(\frac{l}{\lambda}\right)^2 + \cdots\right) \quad .
\end{aligned}
\tag{4.3}
$$

We already know from the numerical solution that $V_{cc}^{\infty} = 0$. Taking the $l \rightarrow 0$ limit of (4.3) gives points at one end of our universal curves. The complete curves we seek are the elements of V as a function L in this limit.

In the continuous detector limit, equation (4.2) becomes a system of six coupled differential equations in the matrix elements of I. This system is first order, autonomous (L does not appear explicitly), and nonlinear. We know that all elements are zero at $L = 0$ (i.e., there is no information when there are no measurements.) We know that the solutions are well behaved everywhere except at $L = \infty$ (where

$I_{cc} \to \infty$). While closed form solutions are not easily obtained, we can find the a set of power series which satisfies the system of equations.

According to differential equation theory, these series converge for all L and are a unique solution. This matrix of power series can be inverted to yield a set of power series for the elements of V:

$$
\begin{aligned}
V_{aa} &= \frac{9}{i\lambda u}\left(1 + \frac{1}{5670}u^4 - \frac{139}{4,086,482,400}u^8 + - \cdots\right) \\
V_{bb} &= \frac{192}{i\lambda^3 u^3}\left(1 + \frac{1}{840}u^4 - \frac{47}{403,603,200}u^8 + - \cdots\right) \\
V_{cc} &= \frac{720}{i\lambda^5 u^5}\left(1 + \frac{1}{504}u^4 - \frac{19}{726,485,760}u^8 + - \cdots\right) \\
V_{ab} &= \frac{-36}{i\lambda^2 u^2}\left(1 + \frac{5}{2160}u^4 - \frac{37}{518,918,400}u^8 + - \cdots\right) \\
V_{bc} &= \frac{-360}{i\lambda^4 u^4}\left(1 + \frac{1}{1680}u^4 - \frac{19}{726,485,760}u^8 + - \cdots\right) \\
V_{ac} &= \frac{60}{i\lambda^3 u^3}\left(1 + \frac{1}{5040}u^4 - \frac{43}{3,632,428,800}u^8 + - \cdots\right)
\end{aligned}
\tag{4.4}
$$

where $u \equiv L/\lambda$. The first term is the usual expression for measurement error alone. We recognize the sum of the first two terms as the error one would get from a uniformly weighted diagonal fit. Thus the third term gives, to leading order in L/λ, the improvement due to optimal fitting. The above series are useful for $u < 7$, after which the number of terms required to get an accurate result grows rapidly. The form of this solution demonstrates the universal curve hypothesis.

Equation (4.2) can be rewritten as a recursion rule for V instead of I. From the resulting rule comes a system of differential equations in the elements of V.

Unfortunately, this system of equations is very badly behaved and complete series solutions are not readily obtained. This system does have the solution:

$$
\begin{aligned}
V_{aa} &= \sqrt{2}s\lambda^3\left(1+\frac{1}{\sqrt{2}u}\right) \\
V_{bb} &= \sqrt{2}s\lambda\left(1+\sqrt{2}\frac{1}{u}\right) \\
V_{cc} &= \frac{s}{\lambda u} \\
V_{ab} &= -s\lambda^2\left(1+\sqrt{2}\frac{1}{u}\right) \\
V_{bc} &= -\sqrt{2}s\frac{1}{u} \\
V_{ac} &= \frac{s\lambda}{u} \quad .
\end{aligned}
\tag{4.5}
$$

This is a complete solution but it is not unique. It does approach the desired solution as $L/\lambda \to \infty$. V_{aa}, V_{bb}, V_{ab} go to constant values in the large L limit. This is because the back of a detector contributes no information about the particles location and slope at the detector entrance once there has been sufficient scattering. V_{cc}, V_{bc}, V_{ac} continue to decrease as λ/L because essentially independent measurements of the curvature are made along the entire length of the track. The effective lever arm of the curvature measurements is limited by the multiple scattering.

5. Approximate Formulas for Optimal Fits

We get good simple approximations for V_{cc}, V_{bc}, and V_{ac} by adding (4.5) and the first terms from (4.4). In the case of V_{cc}, this is the result we obtained previously in (4.1). These formulas are represented by the dotted curves on the figures. V_{aa}, V_{bb}, V_{ab} are not so simple. Equation (4.4) truncated at two terms is shown on the figures as a dotted curve. Equation (4.5) is displayed as a dashed curve. A useful approximation can be had by using (4.4) for $L/\lambda < 7$ and (4.5) above. These recommendations are summarized below.

One caution: Remember that these formulas are for the continuous chamber approximation in the large N limit, i.e., $L/N \ll \lambda$ and $N > 10$. The formulas for V_{aa}, V_{bb}, and V_{ab}, in particular, will be overestimates if the former condition is violated. Since this effect is worst at $L \to \infty$, equation (4.3) may be used to get an upper limit on the deviation.

6. Summary of Approximate Formulas

Here is a summary of the recommended approximate formulas for the optimal fit variance matrix elements in the continuous detector large N limit.

$$
\begin{aligned}
V_{aa} &\approx \frac{9}{i\lambda}\left(\frac{1}{u}+\frac{1}{5760}u^3\right) \qquad \text{if } u<7 \\
V_{aa} &\approx \frac{1}{i\lambda}\left(\frac{1}{u}+\sqrt{2}\right) \qquad \text{if } u>7 \\
V_{bb} &\approx \frac{192}{i\lambda^3}\left(\frac{1}{u^3}+\frac{1}{840}u\right) \qquad \text{if } u<7 \\
V_{bb} &\approx \frac{1}{i\lambda^3}\left(\frac{2}{u}+\sqrt{2}\right) \qquad \text{if } u>7 \\
V_{cc} &\approx \frac{1}{i\lambda^5}\left(\frac{720}{u^5}+\frac{1}{u}\right) \\
V_{ab} &\approx \frac{-36}{i\lambda^2}\left(\frac{1}{u^2}+\frac{5}{2160}u^2\right) \qquad \text{if } u<7 \\
V_{ab} &\approx \frac{-1}{i\lambda^2}\left(\frac{\sqrt{2}}{u}+1\right) \qquad \text{if } u>7 \\
V_{bc} &\approx \frac{-1}{i\lambda^4}\left(\frac{360}{u^4}+\frac{\sqrt{2}}{u}\right) \\
V_{ac} &\approx \frac{1}{i\lambda^3}\left(\frac{60}{u^3}+\frac{1}{u}\right) \quad .
\end{aligned}
\tag{6.1}
$$

where for a detector with N layers, length L, and rms measurement error per layer of σ: $i \equiv (N+5)/(\sigma^2 L)$, $s \equiv$ mean square projected scattering angle per unit length, $\lambda \equiv 1/\sqrt[4]{is}$, and $u \equiv L/\lambda$.

References

[1] Particle Data Group, *Phys. Lett.* **B239** (1990) III.1.

[2] R. L. Gluckstern, *Nucl. Inst. Meth.* **24** (1963) 381.

[3] W. T. Scott, *Phys. Rev.* **76** (1949) 212.

[4] P. Billoir, *Nucl. Inst. Meth.* **225** (1984) 352.

Figure Captions

1. V_{aa}. The solid curve is the numerical calculation. The dotted curve is result of the diagonal unweighted fit. The dashed curve is the large u solution.
2. V_{bb}. The solid curve is the numerical calculation. The dotted curve is result of the diagonal unweighted fit. The dashed curve is the large u solution.
3. V_{cc}. The solid curve is the numerical calculation. The dotted curve is result of the approximation (4.1).
4. V_{ab}. The solid curve is the numerical calculation. The dotted curve is result of the diagonal unweighted fit. The dashed curve is the large u solution.
5. V_{bc}. The solid curve is the numerical calculation. The dotted curve is result of the approximation.
6. V_{ac}. The solid curve is the numerical calculation. The dotted curve is result of the approximation.

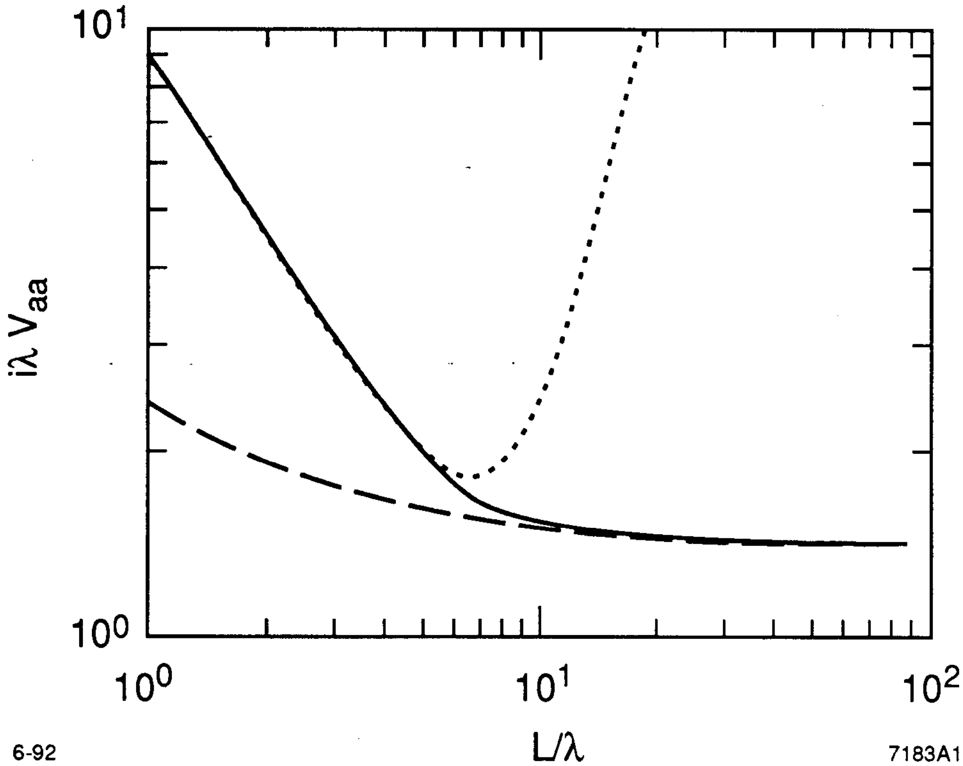


Fig. 1

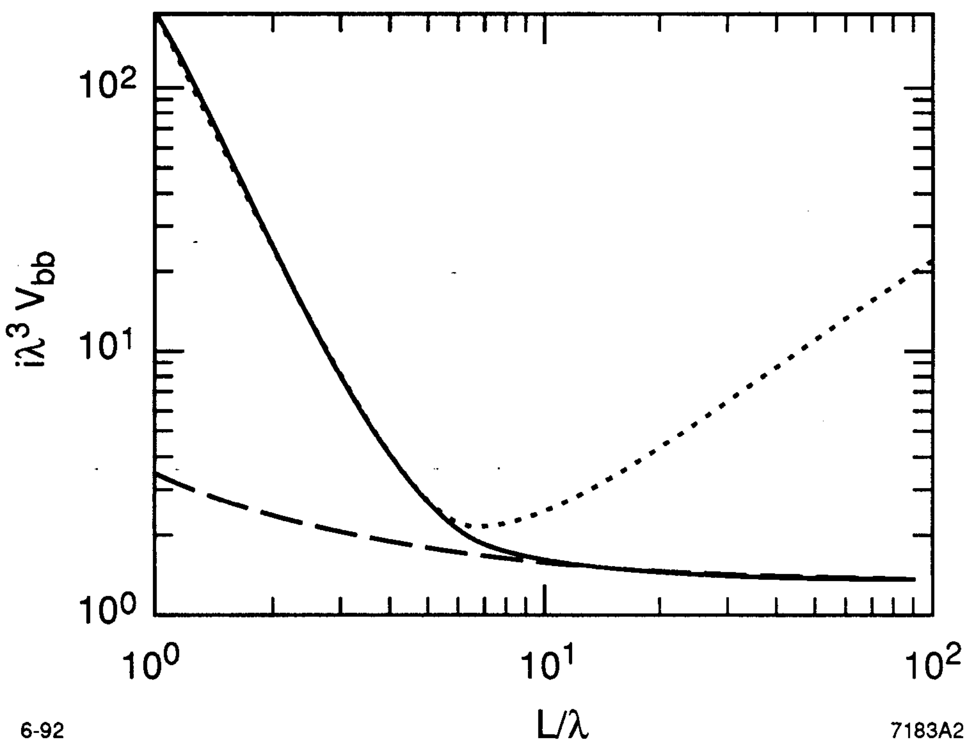


Fig. 2

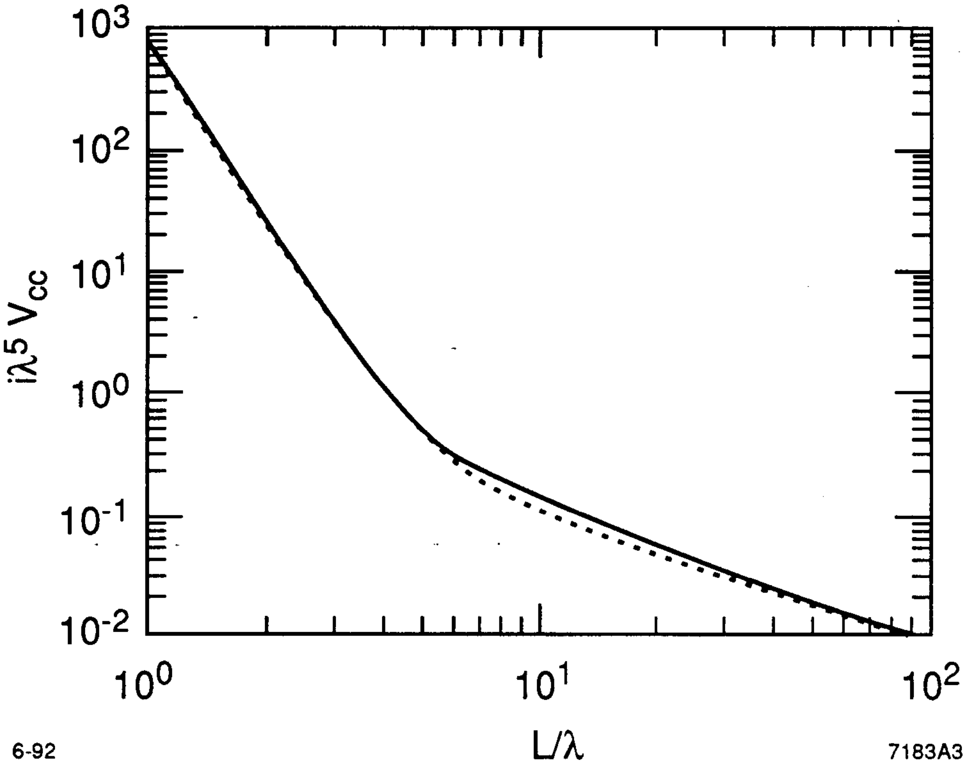

10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
$i\lambda^5 V_{cc}$
10^0
10^1
10^2
L/λ
6-92
7183A3

Fig. 3

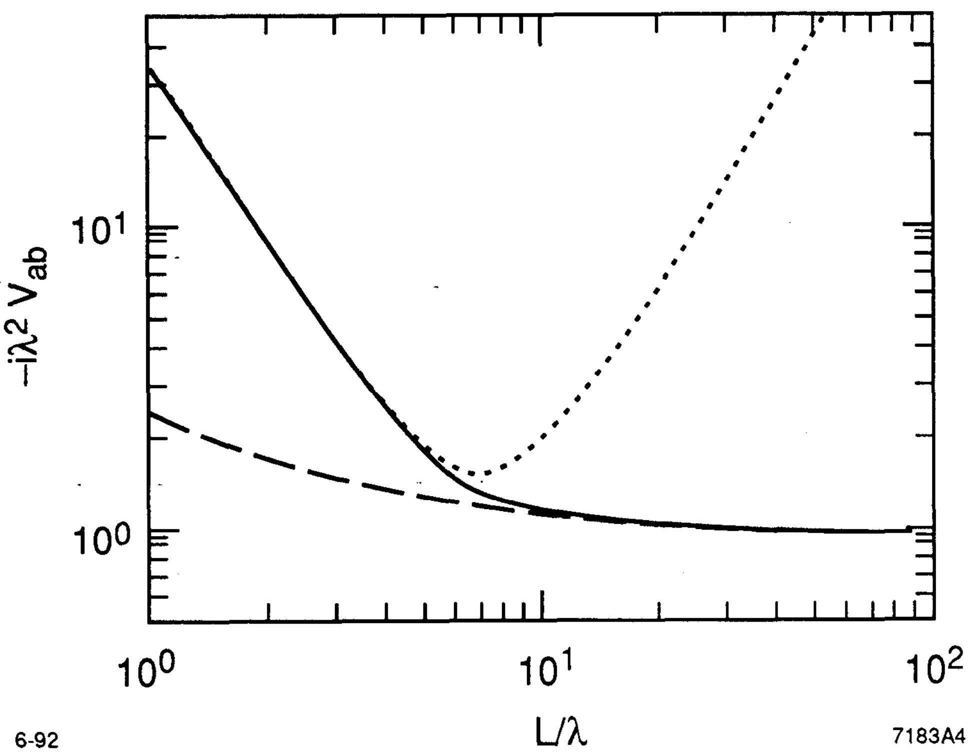
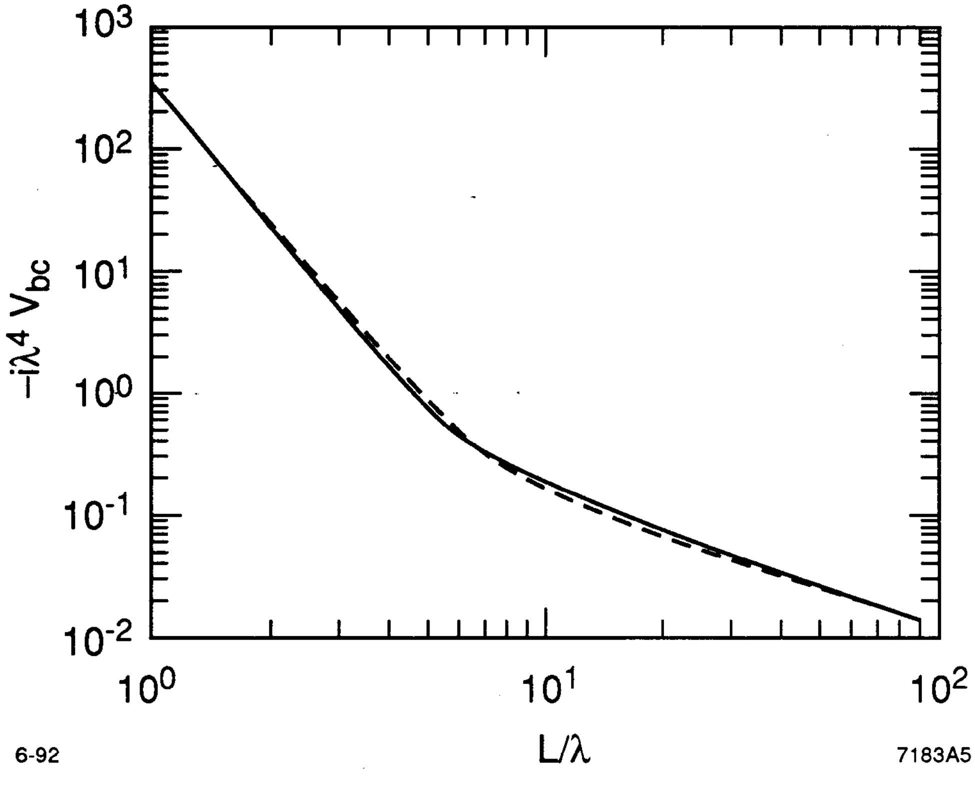
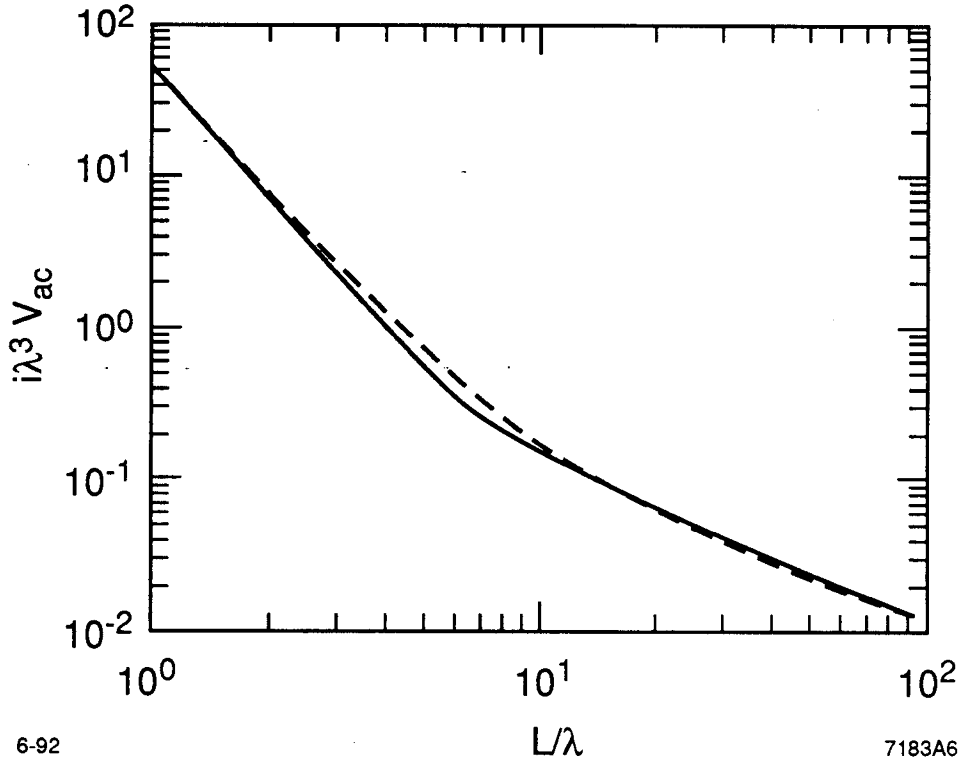


Fig. 4



10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
$-i\lambda^4 V_{bc}$
10^0
10^1
10^2
L/λ
6-92
7183A5

Fig. 5



10^2
10^1
10^0
10^{-1}
10^{-2}
$i\lambda^3 V_{ac}$
10^0
10^1
10^2
L/λ
6-92
7183A6

Fig. 6