

# Supersymmetry and Darboux transformations

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## Abstract

We study supersymmetry and Darboux transformations for generalized Schrödinger equations with a position-dependent mass and with linearly energy-dependent potentials. The formally adjoint generators of supersymmetry and two superpartner Hamiltonians are constructed and they close a quadratic pseudo-superalgebra for our class of equations.

Keywords: Supersymmetry , Intertwining operators, Pseudo-superalgebra

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## 1 Introduction

Over the last years the study of quantum systems with position-dependent mass [1]-[5] and energy-dependent potentials [6, 7] has been of great interest due to the rapid development of opto- and nano-electronics, the basic elements of which are low-dimensional structures such as quantum wells, wires, dots and superlattices [8]-[12]. One of the most significant questions of quantum engineering is the construction of multi-quantum well structures possessing desirable spectral properties. In recent years supersymmetric quantum mechanics (SUSY QM) has been applied to research shape invariant potentials [13], quasi-exactly solved models [14] conditionally exactly solvable problems [15], and to receive new exactly solvable potentials [16], [17] searching for pairs of quantum Hamiltonians coupled supersymmetrically by an intertwining operator [5, 18, 19]. This supersymmetry procedure [20] is closely tied to factorization method [21, 22] and equivalent to the application of the Darboux transformations [7], [23]-[27]. In this paper we research the Darboux transformation operator technique and supersymmetry for the generalized Schrödinger equations with position-dependent effective mass and linearly energy-dependent potentials and demonstrate how to make the quantum well potentials with a given spectrum. Although supersymmetry relations are similar in form with the standard supersymmetry ones, the intertwining transformation operators are different. These operators are formally adjoint, have a more complicate form and have to factorize the pair of pseudo-Hermitian Hamiltonians. Here, we shall establish a correspondence between the spaces of solutions of the initial and transformed equations. The paper is organized as follows. Section 2 is devoted to the generalized Darboux transformation of the first-order. In Section 3 we elaborate supersymmetry formalism for generalized Schrödinger equations. In Section 4 we construct chains of Darboux transformations by iteration of first-order Darboux transformations. In Section 5 we illustrate our generalized transformations by concrete example.

## 2 The intertwining technique

As it is well known, Darboux transformations (or supersymmetry) in nonrelativistic quantum mechanics allows one to produce Hamiltonians whose spectra can differ in one bound state [24].

We will extend the approach to the one-dimensional Schrödinger-type equation with position-dependent mass and weighted energy

$$-\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right]\phi(x) + v(x)\phi(x) = q(x)\mathcal{E}\phi(x). \quad (1)$$

Here  $m(x)$  stands for the particle's effective mass,  $q(x)\mathcal{E}$  is a linearly energy-dependent potential and  $v(x)$  denote the potential,  $\phi(x)$  is the wave function and  $\mathcal{E}$  denotes real-valued energy and we use atomic units. The potential functions  $v$ ,  $q$  and  $m$  are assumed to be real and integrable on  $[a, b]$  on which  $m(x)$ ,  $q(x)$  and  $v(x)$  are defined, and  $v(x) \in L^1_1$  i.e.,  $\int_a^b dx |v(x)|(1+|x|) < \infty$ . We assume that  $m(x) \neq 0$  and  $q(x) \neq 0$  on the interval  $[a, b]$ . In principle the conditions can be more weak if it is necessary to consider the physical situation with an abrupt interface. It means that effective mass has discontinuities at some points of  $[a, b]$ . We assume that potential functions  $v(x)$  remain finite, but not necessarily continuous, the function  $\phi(x)$  must be continuous  $\phi_-(x) = \phi_+(x)$  and its derivative should satisfy the condition  $\left(\frac{1}{m(x)}\frac{d\phi(x)}{dx}\right)_- = \left(\frac{1}{m(x)}\frac{d\phi(x)}{dx}\right)_+$ , where sub-indices  $-$  and  $+$  denote the left- and right-hand sides of mass discontinuity [28]. Even if the potential function has points of singularities, one can use the Darboux transformation method, reconstructing potential excluding the points of singularities (see, e.g. [29]).

The presence of a linearly energy-dependent potential leads to the modification of the scalar product with the weight of  $q(x)$ :  $(f, \chi)_q = \int f^*(x)q(x)\chi(x)dx$ . As a consequence, in contrast with conventional Schrödinger equation the role of the self-adjoint operator plays the operator  $\mathcal{H}^\dagger = q^{-1}\mathcal{H}^+q$  instead of  $\mathcal{H}^+$ . It can be shown that the operator  $\mathcal{H}^\dagger = \mathcal{H}$ , where  $\mathcal{H} = -\frac{1}{q(x)}\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right] + \frac{v(x)}{q(x)}$  is the second-order differential operator of our wave equation (1). The corresponding eigenfunctions should satisfy the following conditions:  $\int_a^b dx q(x)|\phi(x)|^2 < \infty$ , which is an analog of the square integrability of the Schrödinger equation eigenfunctions. By the definition [30], operators  $D$  are said to be pseudo-Hermitian with respect to  $q$  if  $D^\dagger = q^{-1}D^+q = D$ . The operator  $\mathcal{H}$  of (1) satisfies this condition.

Consider two generalized Schrödinger equations

$$\mathcal{H}\phi = \mathcal{E}\phi, \quad \mathcal{H} = -\frac{1}{q(x)}\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right] + \frac{v(x)}{q(x)}, \quad (2)$$

$$\tilde{\mathcal{H}}\tilde{\phi} = \mathcal{E}\tilde{\phi}, \quad \tilde{\mathcal{H}} = -\frac{1}{q(x)}\left[\frac{d}{dx}\left(\frac{1}{m(x)}\right)\frac{d}{dx}\right] + \frac{\tilde{v}(x)}{q(x)}, \quad (3)$$

where Hamiltonians  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  differ only in potentials  $v$  and  $\tilde{v}$ .

Suppose that solutions of the eigenvalue problem to the initial equation. are known and we want to solve a similar problem for another Hamiltonian  $\tilde{\mathcal{H}}$ , spectrum of which differs from the spectrum of  $\mathcal{H}$  by a single quantum state. In accordance with the strategy of intertwining operators, we solve this problem by finding an intertwiner (or Darboux transformation operator)  $\mathcal{L}$  from the intertwining relation

$$\mathcal{L}\mathcal{H} = \tilde{\mathcal{H}}\mathcal{L}. \quad (4)$$

We search for the intertwiner  $\mathcal{L}$  in a form of a linear, 1st-order differential operator  $\mathcal{L} = A(x) + B(x)d/dx$ , where the coefficients  $A$  and  $B$  are to be determined. To this end we insert  $\mathcal{L}$  and the explicit form of  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  into (4) and apply it to the solution  $\phi$  of (2). As a result, we get

$$B = \frac{\beta}{\sqrt{q}m}, \quad A = BK, \quad K = -\frac{\mathcal{U}'}{\mathcal{U}}, \quad A = -\frac{1}{\sqrt{q}m}\frac{\mathcal{U}'}{\mathcal{U}}.$$

The function  $\mathcal{U}$  satisfies the initial equation at  $\mathcal{E} = \lambda$ . Hence, it is known.

Finally, the operator  $\mathcal{L}$ , the new potential  $\tilde{v}(x)$  and corresponding solutions  $\tilde{\phi}(x)$  can be written as

$$\mathcal{L} = \frac{1}{\sqrt{q} m} \left( \frac{d}{dx} + K \right) = \frac{1}{\sqrt{q} m} \left[ \frac{d}{dx} - \frac{\mathcal{U}'}{\mathcal{U}} \right], \quad (5)$$

$$\tilde{v} = v - 2\sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{\sqrt{q} m} \frac{\mathcal{U}'}{\mathcal{U}} \right] - \sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{q} \frac{d}{dx} \left( \sqrt{\frac{q}{m}} \right) \right], \quad (6)$$

$$\tilde{\phi} = \mathcal{L}\phi = \frac{1}{\sqrt{q} m} \left[ \frac{d}{dx} - \frac{\mathcal{U}'}{\mathcal{U}} \right] \phi. \quad (7)$$

Note that relation (7) connects the solutions of two equations (2) and (3) at arbitrary energy except for solutions at energy of transformation,  $\mathcal{E} = \lambda$ . Evidently, at  $\mathcal{E} = \lambda$  the action of Darboux transformation (5) on the function  $\mathcal{U}$  and on functions  $\phi$  linearly dependent to  $\mathcal{U}$  gives us  $\mathcal{L}\mathcal{U} = 0$ . In order to obtain a solution of the transformed equation (3) at energy  $\lambda$ , we replace  $\mathcal{U}$  by a linearly independent solution  $\hat{\mathcal{U}}$

$$\hat{\mathcal{U}} = \mathcal{U} \int_{x_0}^x dx' \frac{m(x')}{|\mathcal{U}(x')|^2}. \quad (8)$$

The action of  $\mathcal{L}$  on the function  $\hat{\mathcal{U}}$  gives us a solution  $\eta$  of the transformed equation (3) at energy  $\lambda$

$$\eta = \mathcal{L}\hat{\mathcal{U}} = \sqrt{\frac{m}{q}} \frac{1}{\mathcal{U}}. \quad (9)$$

By using the generalized Liouville formula (8) once more, one can get a second solution  $\hat{\eta}$  of (3) at energy  $\lambda$ . For this  $\mathcal{U}$  is replaced by  $\eta$  in (8) and with (9) we get

$$\hat{\eta} = \eta \int_{x_0}^x dx' \frac{m(x')}{|\eta(x')|^2} = \sqrt{\frac{m}{q}} \frac{1}{\mathcal{U}} \int_{x_0}^x dx' q(x') |\mathcal{U}(x')|^2. \quad (10)$$

Thus, if we know all solutions of the initial equation (2) we can determine all solutions of the transformed equation (3), including the solutions at energy of transformation. Notice that if the transformation function  $\mathcal{U}$  replies to the bound state of  $\mathcal{H}$ , then the function  $\eta$  defined by (9) at the energy of transformation  $\lambda$  cannot be normalized. Such bound state is excluded from the spectrum of  $\mathcal{H}$ . Therefore, the Hamiltonians  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  are isospectral except for the bound state with energy  $\lambda$ , which is removed from the spectrum of  $\mathcal{H}$ .

### 3 Supersymmetry

It is known, the SUSY algebra provides a relation between superpartner Hermitian Hamiltonians, which can be presented in a factorized form in terms of Darboux tr. operators  $\mathcal{L}$  and its adjoint  $\mathcal{L}^+$ . But we have pseudo-Hermitian Hamiltonians with respect to  $q(x) \neq 0$

$$\mathcal{H}^\dagger = q^{-1} \mathcal{H}^+ q = \mathcal{H}.$$

In order to construct superalgebra we need in formally adjoint operators. Hence, instead of  $\mathcal{L}^+$  it is necessary to consider the operator  $\mathcal{L}^\dagger = q^{-1} \mathcal{L}^+ q$ . The operator  $\mathcal{L}^\dagger$  adjoint to  $\mathcal{L} = \frac{1}{\sqrt{q} m} \left( \frac{d}{dx} + K \right)$  can be written as

$$\mathcal{L}^\dagger = \frac{1}{\sqrt{q} m} \left( -\frac{d}{dx} + K \right) - \frac{1}{q} \frac{d}{dx} \sqrt{\frac{q}{m}}. \quad (11)$$

Since the operators  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$  are pseudo-Hermitian,  $\mathcal{H}^\dagger = \mathcal{H}$  and  $\tilde{\mathcal{H}}^\dagger = \tilde{\mathcal{H}}$  and with account  $\mathcal{L}^\dagger$  the intertwining relation adjoint to  $\mathcal{L}\mathcal{H} = \tilde{\mathcal{H}}\mathcal{L}$  reads

$$\mathcal{H}\mathcal{L}^\dagger = \mathcal{L}^\dagger\tilde{\mathcal{H}}. \quad (12)$$

The generalized Schrödinger equations (2) and (3) can then be written as one single matrix equation in the form

$$\begin{pmatrix} \mathcal{H} - \lambda & 0 \\ 0 & \tilde{\mathcal{H}} - \lambda \end{pmatrix} \begin{pmatrix} \phi \\ \tilde{\phi} \end{pmatrix} = 0. \quad (13)$$

On defining  $H_s = \text{diag}(\mathcal{H}, \tilde{\mathcal{H}})$  and  $\Phi = (\phi, \tilde{\phi})^T$ , the above matrix equation can be written as

$$[H_s - \lambda I] \Phi = 0, \quad (14)$$

where  $I$  is the  $2 \times 2$  unity matrix. As for the standard Schrödinger equation, we define two supercharge operators  $Q, Q^\dagger$

$$Q = \begin{pmatrix} 0 & 0 \\ \mathcal{L} & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & \mathcal{L}^\dagger \\ 0 & 0 \end{pmatrix}. \quad (15)$$

Since  $Q^\dagger$  can be determined as  $Q^\dagger = q^{-1}Q^\dagger q$  one can conclude, the operator  $Q^\dagger$  is pseudo-adjoint to  $Q$ . The matrix operators  $Q, Q^\dagger$  and  $H_s$  satisfy the following anti-commutation  $\{\cdot, \cdot\}$  and commutation  $[\cdot, \cdot]$  relations

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad (16)$$

$$[Q, H_s] = [H_s, Q^\dagger] = 0. \quad (17)$$

Equations (17) are equivalent to the intertwining ones.

Consider the complementing relations of SUSY algebra, namely, anti-commutators  $\{Q, Q^\dagger\}$  and  $\{Q^\dagger, Q\}$ . For this, we calculate  $\mathcal{L}^\dagger\mathcal{L}$  and  $\mathcal{L}\mathcal{L}^\dagger$ , and consider the connections of them with our Hamiltonians  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$ . After some algebraic transformations we arrive at

$$\mathcal{L}^\dagger\mathcal{L} = -\frac{1}{q} \frac{\partial_{xx}}{m} - \frac{1}{q} \left(\frac{1}{m}\right)' \partial_x + \frac{1}{q} \frac{1}{m} (|K|^2 - K') - \frac{1}{q} \left(\frac{1}{m}\right)' K,$$

$$\begin{aligned} \mathcal{L}\mathcal{L}^\dagger = & -\frac{1}{q} \frac{\partial_{xx}}{m} - \frac{1}{q} \left(\frac{1}{m}\right)' \partial_x + \frac{1}{q} \frac{1}{m} (|K|^2 + K') + \frac{1}{m} \left(\frac{1}{q}\right)' K - \\ & - \frac{1}{\sqrt{q} m} \left[ \frac{1}{q} \left(\sqrt{\frac{q}{m}}\right)' \right]'. \end{aligned}$$

We express the potential  $v$  from the Riccati equation

$$\frac{1}{q} \frac{1}{m} (-K' + K^2) - \frac{v}{q} - \frac{1}{q} \left(\frac{1}{m}\right)' K = -\lambda \quad (18)$$

in the form

$$v = \frac{1}{m} (-K' + K^2) - \left(\frac{1}{m}\right)' K + q\lambda. \quad (19)$$

Using this for the transformed potential  $\tilde{v}$  (6) we get

$$\tilde{v} = \frac{1}{m}(K^2 + K') + \frac{q}{m} \left(\frac{1}{q}\right)' K - \frac{\sqrt{q}}{\sqrt{m}} \frac{d}{dx} \left[ \frac{1}{q} \left(\sqrt{\frac{q}{m}}\right)' \right] + q\lambda. \quad (20)$$

One can easily see that the potential difference is determined as

$$\tilde{v} - v = 2\sqrt{\frac{q}{m}} \frac{d}{dx} \frac{K}{\sqrt{q m}} - \sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{q} \frac{d}{dx} \left(\sqrt{\frac{q}{m}}\right) \right]. \quad (21)$$

After this the factorization formulae can be rewritten as

$$\mathcal{L}^\dagger \mathcal{L} = -\frac{1}{q} \left[ \frac{\partial}{\partial x} \left(\frac{1}{m}\right) \frac{\partial}{\partial x} \right] + \frac{v}{q} - \lambda = \mathcal{H} - \lambda; \quad (22)$$

$$\mathcal{L} \mathcal{L}^\dagger = -\frac{1}{q} \left[ \frac{\partial}{\partial x} \left(\frac{1}{m}\right) \frac{\partial}{\partial x} \right] + \frac{\tilde{v}}{q} - \lambda = \tilde{\mathcal{H}} - \lambda \quad (23)$$

and the anti-commutation relation reads

$$\{Q, Q^\dagger\} = \begin{pmatrix} \mathcal{L}^\dagger \mathcal{L} & 0 \\ 0 & \mathcal{L} \mathcal{L}^\dagger \end{pmatrix} = \begin{pmatrix} \mathcal{H} - \lambda & 0 \\ 0 & \tilde{\mathcal{H}} - \lambda \end{pmatrix} = \mathcal{H}_s - \lambda I. \quad (24)$$

Now, from (24) and with an account  $\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0$ ,  $[Q, H_s] = [H_s, Q^\dagger] = 0$  one can conclude, the operators  $\mathcal{H}_s$ ,  $Q$  and  $Q^\dagger$  close a pseudo-superalgebra. In components, the latter equality reads

$$\mathcal{H} = \mathcal{L}^\dagger \mathcal{L} + \lambda, \quad (25)$$

$$\tilde{\mathcal{H}} = \mathcal{L} \mathcal{L}^\dagger + \lambda. \quad (26)$$

One can conclude that the operators  $\mathcal{H}_s$ ,  $Q$  and  $Q^\dagger$  close a superalgebra and one can associate a pseudo-supersymmetry with a quantum system described by the Hamiltonian  $\mathcal{H}_s$ .

Note that as soon as the initial Hamiltonian  $\mathcal{H}$  is presented in the factorized form (25), one can get its supersymmetric partner in a factorized form (26), too. Indeed, multiplying equation (25) from the left by  $\mathcal{L}$  and taking into account the intertwining relation (12) we get

$$\mathcal{L} \mathcal{H} \phi = \mathcal{L} (\mathcal{L}^\dagger \mathcal{L} + \lambda) \phi = (\mathcal{L} \mathcal{L}^\dagger + \lambda) \mathcal{L} \phi = \tilde{\mathcal{H}} \mathcal{L} \phi. \quad (27)$$

It means, that  $\tilde{\mathcal{H}} = \mathcal{L} \mathcal{L}^\dagger + \lambda$ .

In summary, we obtained the explicit forms of the supersymmetric partner Hamiltonians  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$ . The Hamiltonians (25) and (26) are compatible with their definitions (2) and (3), respectively, if the transformed potentials  $v$  and  $\tilde{v}$  are given by (19) and (20). Finally, taking the difference of the factorized Hamiltonians (25) and (26) gives the potential difference (6) that we obtained for our Darboux transformation. Hence, the Darboux transformation is equivalent to the supersymmetry formalism.

The intertwining relation (12) means that the operator  $\mathcal{L}^\dagger$  is also the transformation operator and realizes the transformation from the solutions of (3) to the solutions of (2). Evidently, one can interchange the role of the initial generalized Schrödinger equation (2) and its transformed counterpart (3). To this end, let us express the operators  $\mathcal{L}$  and  $\mathcal{L}^\dagger$  in terms of functions  $\eta$  given in (9), which are solutions to the equation (3) at energy of transformation  $\lambda$ . First, we rewrite  $K$  by using  $\mathcal{U}$  from the relation (9)

$$K = -\frac{\mathcal{U}'}{\mathcal{U}} = \frac{1}{2} \frac{q'}{q} - \frac{1}{2} \frac{m'}{m} + \frac{\eta'}{\eta}.$$

Using this in (5) and (11), we obtain after simplifications

$$\mathcal{L}^\dagger = \frac{1}{\sqrt{q} m} \left( -\frac{d}{dx} + \frac{\eta'}{\eta} \right), \quad \mathcal{L} = \frac{1}{\sqrt{q} m} \left( \frac{d}{dx} + \frac{\eta'}{\eta} \right) + \frac{1}{q} \frac{d}{dx} \sqrt{\frac{q}{m}} \quad (28)$$

Obviously, the function  $\eta$  is also a transformation function. Notice,  $\mathcal{L}^\dagger \eta = 0$ , meaning that  $\eta$  belongs to the kernel of the operator  $\mathcal{L}^\dagger$ . As one can see from (28) and (10), the application of the operator  $\mathcal{L}^\dagger$  to the second linearly independent solution  $\hat{\eta}$  of equation (3) gives back the solutions  $\mathcal{U}$  of the initial problem at energy of transformation. Indeed,

$$\mathcal{L}^\dagger \hat{\eta} = \frac{1}{\sqrt{q} m} \left( -\frac{d}{dx} + \frac{\eta'}{\eta} \right) \eta \int_{x_0}^x dx' \frac{m(x')}{|\eta(x')|^2} = -\sqrt{\frac{m}{q}} \frac{1}{\eta} = -\mathcal{U}.$$

Hence, the solution at energy of transformation  $\lambda$  takes the form

$$\mathcal{U} = \sqrt{\frac{m}{q}} \frac{1}{\eta} \quad (29)$$

and, in principle, can reply to the new bound state. The second linearly independent solution  $\hat{\mathcal{U}}$  of (2) at energy  $\lambda$  can be written in terms of  $\eta$  as follows:

$$\hat{\mathcal{U}} = \sqrt{\frac{m}{q}} \frac{1}{\eta} \int^x dx' q |\eta|^2. \quad (30)$$

Introducing the function  $\tilde{K} = \tilde{K}(x)$  by  $\tilde{K} = \eta'/\eta$  and taking into account  $\frac{1}{\sqrt{q} m} \left( \frac{1}{2} \frac{q'}{q} - \frac{1}{2} \frac{m'}{m} \right) = \frac{1}{q} \frac{d}{dx} \sqrt{\frac{q}{m}}$ , the expressions (28) for operators  $\mathcal{L}^\dagger$  and  $\mathcal{L}$  can be rewritten as

$$\mathcal{L}^\dagger = \frac{1}{\sqrt{q} m} \left( -\frac{d}{dx} + \tilde{K} \right), \quad \mathcal{L} = \frac{1}{\sqrt{q} m} \left( \frac{d}{dx} + \tilde{K} \right) + \frac{1}{q} \frac{d}{dx} \sqrt{\frac{q}{m}}. \quad (31)$$

Using (21) the potential  $v$  can be expressed in terms of  $\tilde{v}$  as follows:

$$v = \tilde{v} - \sqrt{\frac{q}{m}} \left[ 2 \frac{d}{dx} \left( \frac{\tilde{K}}{\sqrt{q} m} \right) + \frac{d}{dx} \left( \frac{1}{q} \frac{d}{dx} \left( \sqrt{\frac{q}{m}} \right) \right) \right], \quad \tilde{K} = \frac{\eta'}{\eta} \quad (32)$$

and corresponding solution  $\phi$  are given by

$$\phi = \mathcal{L}^\dagger \tilde{\phi} = \frac{1}{\sqrt{q} m} \left[ -\frac{d}{dx} + \frac{\eta'}{\eta} \right] \tilde{\phi}. \quad (33)$$

Thus, the function  $\eta$  becomes a transformation function for the operator  $\mathcal{L}^\dagger$ , which performs the transformation from the potential  $\tilde{v}$  to the potential  $v$  and from the solutions of (3) to the solutions of (2). If within the first procedure (5)–(7) we constructed the potential  $\tilde{v}$  with one bound state removed, now we can construct the potential  $v$  with an additional bound state. Note that we have established a one-to-one correspondence between the spaces of solutions of equations (2) and (3). The operators  $\mathcal{L}$  and  $\mathcal{L}^\dagger$  realize this correspondence for any  $\mathcal{E} \neq \lambda$ . If  $\mathcal{E} = \lambda$ , the correspondence is ensured by mapping  $\eta \longleftrightarrow \hat{\mathcal{U}}$  and  $\hat{\eta} \longleftrightarrow \mathcal{U}$ .

In particular cases our generalized Darboux transformations are reduced correctly to the known expressions. In the case with a constant weighted energy potential, e.g.  $q(x) = 1$ , from our supersymmetry approach we get the supersymmetry for the effective mass Schrödinger equation [2, 5]. In the case with constant mass  $m(x) = m_0$  from our approach we obtain the supersymmetry for Schrödinger equation with weighted energy [7]. Finally, if  $m(x) = m_0$  and

$q(x) = 1$ , our expressions of supersymmetric algebra are correctly reduced to the conventional ones for the standard Schrödinger equation (see, e.g. [24]).

In summary, we obtained the explicit forms of the supersymmetric partner Hamiltonians  $\mathcal{H}$  and  $\tilde{\mathcal{H}}$ . The obtained Darboux transformations are equivalent to the supersymmetry formalism. Note, the intertwining relation

$$\mathcal{H}\mathcal{L}^\dagger = \mathcal{L}^\dagger\tilde{\mathcal{H}}$$

means that the operator  $\mathcal{L}^\dagger$  is also the transformation operator and realizes the transition from solutions of the transformed equation to solutions of the initial equation. Evidently, one can interchange the role of the initial equation and its transformed counterpart.

## 4 Chain

Darboux transformations of the second-order small Now define the 2nd-order Darboux transformation as a sequence of two 1st-order Darboux transformations  $\mathcal{L}_1 = \frac{1}{\sqrt{q}m} \left( \frac{d}{dx} + K_1 \right)$  and  $\mathcal{L}_2$

$$\mathcal{L} = \mathcal{L}_2\mathcal{L}_1, \quad (34)$$

where  $\mathcal{L}_1$  is actually  $\mathcal{L}$  given in (5) and  $\mathcal{L}_2$  is determined as follows

$$\mathcal{L}_2 = \frac{1}{\sqrt{q}m} \left( \frac{d}{dx} + K_2 \right), \quad K_2 = -\frac{\chi_1'}{\chi_1}. \quad (35)$$

The function  $\chi_1$  is obtained by means of the 1st-order Darboux transformation, applied to an auxiliary solution  $\mathcal{U}_2$  of the initial equation taken at energy  $\lambda_2$

$$\chi_1 = \mathcal{L}_1\mathcal{U}_2 = \frac{1}{\sqrt{q}m} \left( \frac{d}{dx} + K_1 \right) \mathcal{U}_2. \quad (36)$$

Clearly,  $\chi_1$  is the solution of the transformed equation with the potential  $v_1 = \tilde{v}$ .

Now  $\chi_1$  can be taken as a new transformation function for the Hamiltonian  $\mathcal{H}_1 = \tilde{\mathcal{H}}$  to generate a new potential

$$v_2 = v_1 + 2\sqrt{\frac{q}{m}} \frac{d}{dx} \frac{K_2}{\sqrt{q}m} - \sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{q} \frac{d}{dx} \left( \sqrt{\frac{q}{m}} \right) \right], \quad (37)$$

and corresponding solutions

$$\phi_2 = \mathcal{L}_2\phi_1 = \frac{1}{\sqrt{q}m} \left( \frac{d}{dx} + K_2 \right) \phi_1, \quad \phi_1 = \mathcal{L}_1\phi. \quad (38)$$

Hence,

$$\phi_2 = \mathcal{L} \phi = \mathcal{L}_2 \mathcal{L}_1 \phi. \quad (39)$$

The function  $\phi_1 = \tilde{\phi}$  is the solution of equation with Hamiltonian  $\mathcal{H}_1 = \tilde{\mathcal{H}}$ . In summary, the action of the 2nd-order operator  $\mathcal{L} = \mathcal{L}_2\mathcal{L}_1$  on solutions  $\phi$  of the initial equation leads to solutions  $\phi_2$  of the following equation

$$\mathcal{H}_2 \phi_2(x) = \mathcal{E}\phi_2(x), \quad \mathcal{H}_2 = -\frac{1}{q} \left[ \frac{d}{dx} \left( \frac{1}{m} \right) \frac{d}{dx} \right] + \frac{v_2}{q}. \quad (40)$$

Iterating the procedure  $n$  times in regard to the given operator  $\mathcal{H}$  leads to  $\mathcal{H}_n$  with the transformed potentials  $v_n$ , which satisfy the following recursion relation

$$v_n = v_{n-1} + 2\sqrt{\frac{q}{m}} \frac{d}{dx} \frac{K_n}{\sqrt{q} m} - \sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{q} \frac{d}{dx} \left( \sqrt{\frac{q}{m}} \right) \right]. \quad (41)$$

The corresponding solutions are

$$\phi_n = \mathcal{L}\phi = \mathcal{L}_n\phi_{n-1} = \mathcal{L}_n\mathcal{L}_{n-1}\dots\mathcal{L}_1\phi, \quad (42)$$

where  $\mathcal{L}$  is the  $n$ th-order operator:

$$\mathcal{L} = \mathcal{L}_n\mathcal{L}_{n-1}\dots\mathcal{L}_1, \quad \mathcal{L}_n = \frac{1}{\sqrt{q} m} \left( \frac{d}{dx} + K_n \right), \quad K_n = -\frac{\chi'_{n-1}}{\chi_{n-1}}. \quad (43)$$

Thus, the chain of  $n$  first-order Darboux transformations results in a chain of exactly solvable Hamiltonians  $\mathcal{H} \rightarrow \mathcal{H}_1 \rightarrow \dots \rightarrow \mathcal{H}_n$ .

## 5 Application

Consider the following simple example. We start with generalized equation with the repulsive coulomb potential  $v(x) = 1/(4x)$

$$-\left[ \frac{d}{dx} \left( \frac{1}{m(x)} \right) \frac{d}{dx} \right] \phi(x) + \frac{1}{4x} \phi(x) = q(x) \mathcal{E} \phi(x), \quad (44)$$

where we choose effective mass as  $m = 1/x$  and  $q = x$ . The general solution of this equation can be written as

$$\phi(x) = \frac{C_1 \sin(kx)}{k\sqrt{x}} + \frac{C_2 \cos(kx)}{k\sqrt{x}}. \quad (45)$$

Now we obtain potentials with one bound state at energy  $\mathcal{E}_1 = -\kappa_1^2$  and corresponding solutions by applying first-order Darboux transformations to a special case  $\mathcal{U}$  of the general solution (45)

$$\mathcal{U} = \frac{C \cosh(\kappa_1 x)}{\kappa_1 \sqrt{x}}.$$

We obtain the transformation operator  $K = \mathcal{U}'/\mathcal{U}$  in the form

$$K = -\frac{1}{2x} + \kappa_1 \tanh \kappa_1 x,$$

the potential  $\tilde{v}$  and corresponding solutions  $\tilde{\phi}$  at  $\mathcal{E} \neq \mathcal{E}_1$

$$\tilde{v}(x) = \frac{1}{4x} - 2x\kappa_1^2 \left( 1 - \tanh^2(\kappa_1 x) \right) = \frac{1}{4x} - \frac{2x\kappa_1^2}{\cosh^2(\kappa_1 x)}, \quad (46)$$

$$\tilde{\phi} = \left[ -\frac{d}{dx} + K \right] \phi = \left[ -\frac{d}{dx} - \frac{1}{2x} + \kappa_1 \tanh \kappa_1 x \right] \phi. \quad (47)$$

On Fig.1a potentials  $v_1(x) \equiv \tilde{v}(x)$  corresponding to one bound state created at different energies are given.

Now employing Darboux transformations of the second-order, we shall construct potentials and solutions of the generalized equation with two bound states. We define the auxiliary functions  $\eta_1$  and  $\eta_2$  as follows

$$\eta_1 = \frac{\cosh(\kappa_1 x)}{\sqrt{\kappa_1 x}}, \quad \eta_2 = \frac{\sinh(\kappa_2 x)}{\sqrt{\kappa_2 x}}.$$

For the second step we have  $\widetilde{K} = \widetilde{K}_1 + \widetilde{K}_2$ , where

$$\widetilde{K}_1 = \frac{\eta'_1}{\eta_1}, \quad \widetilde{K}_2 = \frac{\chi'_1}{\chi_1}, \quad \chi_1 = \frac{1}{\sqrt{q} m} \left( -\eta'_2 + \widetilde{K}_1 \eta_2 \right), \quad \widetilde{K} = -\frac{d}{dx} \left( \ln \frac{W_{12}}{\sqrt{q} m} \right).$$

The transformed potential  $v_2$  having two bound states at energies  $\lambda_1 = -\kappa_1^2$  and  $\lambda_2 = -\kappa_2^2$  can be written as

$$v_2 = v - \frac{2\sqrt{q}}{\sqrt{m}} \frac{d}{dx} \left( \frac{\widetilde{K}}{\sqrt{q} m} \right) - 2 \sqrt{\frac{q}{m}} \frac{d}{dx} \left[ \frac{1}{q} \frac{d}{dx} \left( \sqrt{\frac{q}{m}} \right) \right]. \quad (48)$$

Finally, for our choice of  $v$ ,  $m$  and  $q$  we obtain

$$v_2 = \frac{9}{4x} - 2x \frac{d^2}{dx^2} \ln W_{1,2}, \quad (49)$$

where  $W_{1,2} = W(\eta_1, \eta_2) = \frac{1}{x\sqrt{\kappa_2\kappa_1}} \left( \kappa_2 \cosh(\kappa_1 x) \cosh(\kappa_2 x) - \kappa_1 \sinh(\kappa_2 x) \sinh(\kappa_1 x) \right)$  and the corresponding solutions are

$$\phi_2 = \frac{\sqrt{\kappa_1} x}{\cosh(\kappa_1 x)} \left( \frac{d}{dx} W_{1,\varepsilon} - \frac{d(\ln W_{1,2})}{dx} W_{1,\varepsilon} \right),$$

where  $W_{1,\varepsilon} = \frac{C}{x\sqrt{\kappa_1}} \left( k \cosh(\kappa_1 x) \cos(kx) - \kappa_1 \sinh(\kappa_1 x) \sin(kx) \right)$  if  $\phi(x)$  is chosen as  $\phi(x) = \frac{C \sin(kx)}{k\sqrt{x}}$ . We put  $\kappa_1 < \kappa_2$ . The potential having two bound states are depicted in Fig.1b.

By using (41) one can construct the potential  $v_3$  for the generalized Schrödinger equation (3) having three bound states

$$v_3 = \frac{13}{4x} - 2x \frac{d^2}{dx^2} \ln W_{1,2,3}, \quad (50)$$

where the auxiliary functions  $\eta_1$ ,  $\eta_2$  and  $\eta_3$  determined as

$$\eta_1 = \frac{\cosh(\kappa_1 x)}{\sqrt{\kappa_1} x}, \quad \eta_2 = \frac{\sinh(\kappa_2 x)}{\sqrt{\kappa_2} x}, \quad \eta_3 = \frac{\cosh(\kappa_3 x)}{\sqrt{\kappa_3} x}$$

give us the Wronskian  $W_{1,2,3}$

$$\begin{aligned} W_{1,2,3} = & \frac{1}{x^{3/2} \sqrt{\kappa_1 \kappa_2 \kappa_3}} \left[ \cosh(\kappa_1 x) \cosh(\kappa_2 x) \kappa_2 \sinh(\kappa_3 x) \kappa_3^2 - \cosh(\kappa_1 x) \sinh(\kappa_2 x) \kappa_2^2 \cosh(\kappa_3 x) \kappa_3 - \right. \\ & - \sinh(\kappa_1 x) \kappa_1 \sinh(\kappa_2 x) \sinh(\kappa_3 x) \kappa_3^2 + \sinh(\kappa_1 x) \kappa_1 \sinh(\kappa_3 x) \sinh(\kappa_2 x) \kappa_2^2 + \\ & \left. + \cosh(\kappa_1 x) \sinh(\kappa_2 x) \kappa_1^2 \cosh(\kappa_3 x) \kappa_3 - \cosh(\kappa_1 x) \kappa_1^2 \cosh(\kappa_2 x) \kappa_2 \sinh(\kappa_3 x) \right]. \quad (51) \end{aligned}$$

As an illustrative example we present the potentials  $v_1$ , obtained at energy of transformation  $\mathcal{E}_1 = -4$ ,  $v_2$  obtained at the energies  $\mathcal{E}_1 = -4$ ,  $\mathcal{E}_2 = -4.84$  and  $v_3$  calculated at the energies of transformation  $\mathcal{E}_1 = -4$ ,  $\mathcal{E}_2 = -4.84$ ,  $\mathcal{E}_3 = -16$ . They are depicted in Fig.1b.

## 6 Conclusion

A supersymmetric approach (SUSY) for generalized Schrödinger equations is developed. The formally adjoint generators of supersymmetry and two formally self-adjoint superpartner Hamiltonians are constructed and they close a quadratic pseudo-superalgebra. Hamiltonians with a different number of levels are produced.

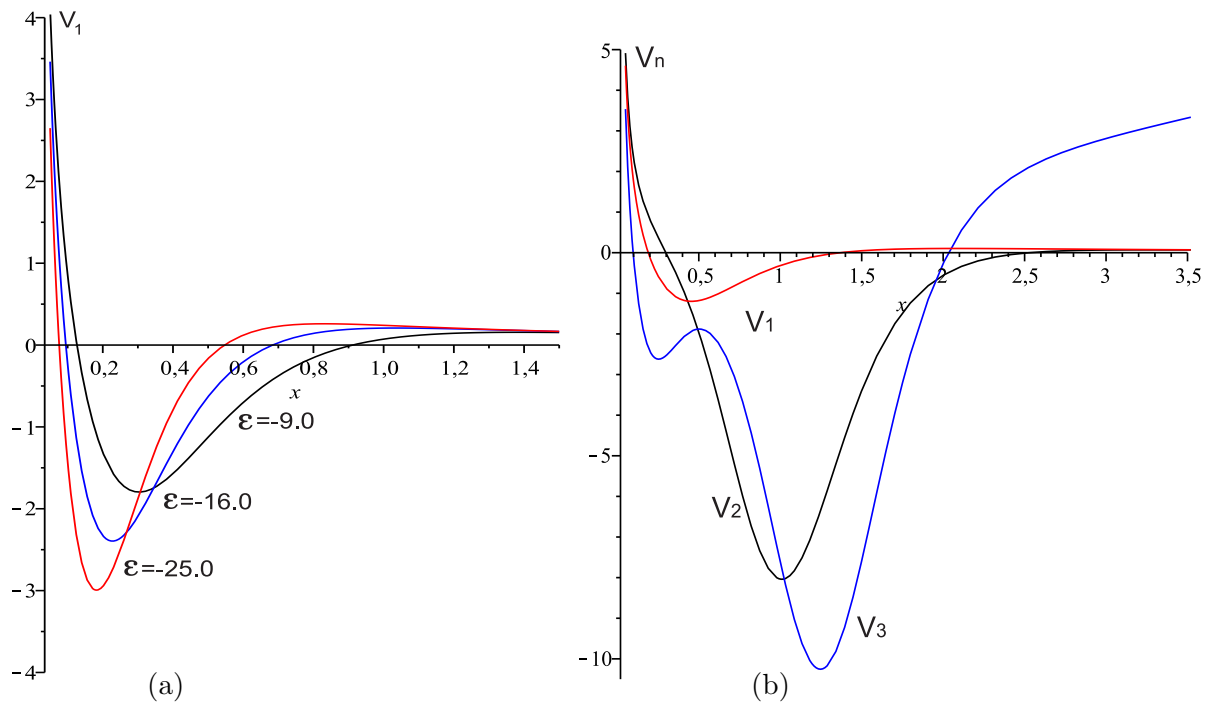


Figure 1: (a) Potentials  $v_1(x)$  corresponding to one bound state created at different energies. (b) Potentials  $v_n$ ,  $n = 1, 2, 3$  having one, two and three bound states, respectively.

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