

## 2.13 Head-Tail and Transverse Mode-Coupling Instabilities (TMCI)

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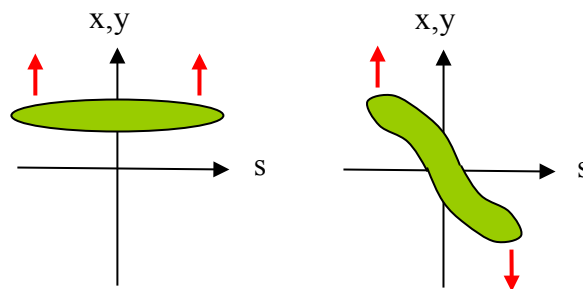
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### 2.13.1 Introduction

In this article, we first describe the head-tail instability, as much as visually, so that readers can get fresh and vivid ideas about this classical phenomenon. We then review Sacherer's head-tail instability theory and Transverse Mode-Coupling Instability (TMCI) theory to get more quantitative understanding. The article is concluded with some historical records on how the concept of "mode-coupling" was developed, not only in transverse instabilities (TMCI), but also in longitudinal ones (the bunch lengthening) in early 1980s.

### 2.13.2 Head-Tail Instability

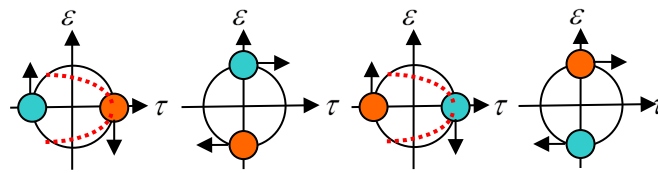
In a long proton bunch, every parts along the longitudinal direction do not have to oscillate together transversely. For example, it is possible that the head and the tail of the bunch oscillate 180 degrees out of phase, as illustrated in Fig. 1 (right).



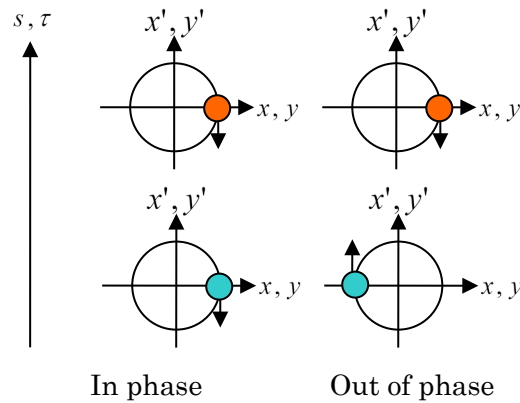
**Figure 1:** Snapshots of beam oscillations when the head and the tail of a bunch oscillate in phase (left) and out of phase (right).

In order for the head and the tail of the bunch to oscillate 180 degrees out of phase, we need a fast oscillating wake field which changes the phase by 180 degrees during the passage of the bunch. Such rapid phase oscillation is possible only by high frequency impedance. For synchronizing with the high frequency impedance, the intra-bunch motion needs to be more complex than a simple dipole oscillation. That introduces the synchrotron motion.

Let us explain the head-tail effect using the two-particle model (see Fig. 2). Only the particle behind can feel the wake fields created by the particle ahead. Since the degree of freedom is two (two particles), we have two modes in which the head and the tail of the bunch oscillate either in phase or out of phase, as illustrated in Fig. 3. Their motion can be unstable depending on the strength of the bunch intensity.

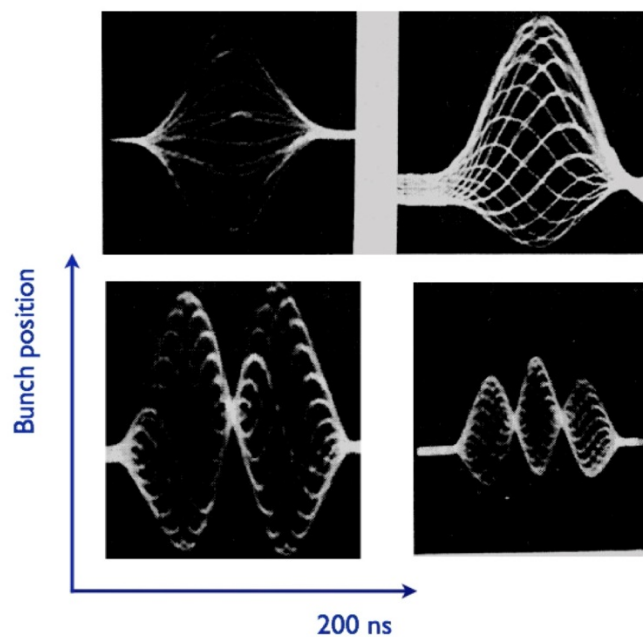


**Figure 2:** Position of the two particles in the synchrotron phase space and wake fields at every quarter of the synchrotron oscillation. The wake fields are denoted by the red dashed lines.



**Figure 3:** Two modes in which the head and the tail of the bunch oscillate in phase (left) and out of phase (right).

Unstable coherent oscillations with different betatron phase advances inside a bunch have been observed in many machines such as CERN PS. This phenomenon was called the head-tail instability [1-3]. Figure 4 shows the measured signals of the dipole moment at CERN PS Booster [2].



**Figure 4:** The measured signals of the dipole moment at CERN PS Booster [2].

### 2.13.3 Sacherer's Head-Tail Instability Theory

The important element of Sacherer's head-tail instability theory is the chromaticity. It is defined by

$$\xi = \frac{d\omega_\beta}{\omega_\beta} / \delta \quad (1)$$

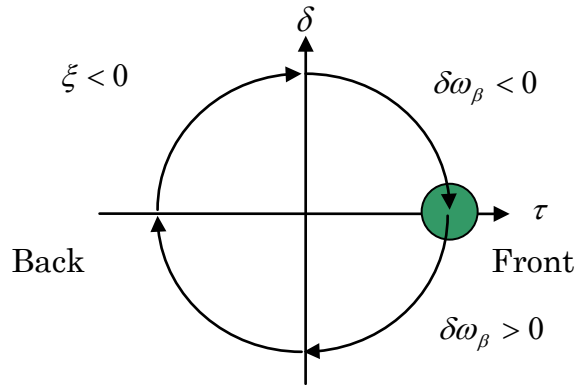
where  $\omega_\beta$  is the angular betatron frequency and  $\delta = \Delta p/p$  is the relative momentum deviation. In other words, the betatron frequency depends on the momentum deviation as

$$\omega_\beta(\delta) = \omega_{\beta 0}(1 + \xi\delta). \quad (2)$$

When a particle moves along the ring, the accumulated betatron phase advance is developed as

$$\begin{aligned} \phi_\beta(s) &= \int \omega_\beta(\delta) \frac{ds}{\beta c} = \omega_{\beta 0} \left( \frac{s}{\beta c} + \xi \int \delta \frac{ds}{\beta c} \right) \\ &= \omega_{\beta 0} \left( \frac{s}{\beta c} - \frac{\xi}{\eta} \tau \right) \equiv \omega_{\beta 0} \frac{s}{\beta c} - \omega_\xi \tau \end{aligned} \quad (3)$$

where  $\tau$  is the arrival time difference of the particle at the position  $s$  (positive toward the head of bunch),  $\beta c$  is the velocity of the particle, and  $\eta$  is the slippage factor. Thus, the betatron phase advance varies linearly along the bunch and attains its minimum (maximum) at the head (tail) of the bunch as illustrated in Fig. 5.



**Figure 5:** Change of the betatron oscillation below transition energy ( $\eta < 0$ ).

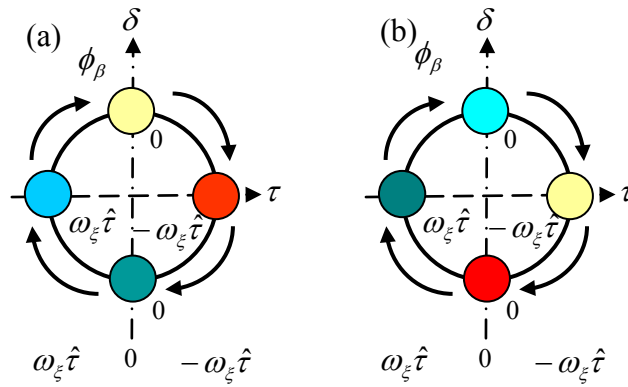
To investigate how the difference of phase advance between the head and the tail of the bunch varies over one synchrotron oscillation period, let us look at the four-particle model in the synchrotron phase space, as illustrated in Fig. 6 [4]. We assume that the arrival time is oscillating as

$$\tau = \hat{\tau} \cos(2\pi\nu_s k), \quad (4)$$

where  $\nu_s$  is the synchrotron oscillation tune and  $k$  is the revolution turn. We also assume that the chromaticity is negative and the ring is operated below the transition energy ( $\eta < 0$ ). Thus particles move clockwise in synchrotron phase space.

In principle, the betatron phase advance slows down (quickens up) by  $\omega_\xi \hat{\tau}$  for every quarter period of synchrotron oscillation as the particle moves forward (backward) along the synchrotron orbit, respectively. As can be seen in Fig. 6, the initial phase relationship along the bunch is preserved after the quarter period of synchrotron oscillation. One can quickly check that the phase pattern remains stationary over the full period of the synchrotron oscillation. In other words, the difference of the phase advance between the head and the tail of the bunch is constant, and we denote this constant as  $\chi$ :

$$\chi = \phi_{\beta Tail} - \phi_{\beta Head} = 2\omega_\xi \hat{\tau} = \text{constant} \quad (5)$$



**Figure 6:** (a) The initial phase advance setting of the four particles. (b) The phase advance after a quarter period of the full synchrotron oscillation. The phase difference between the head and the tail of the bunch remains the same:  $\chi = \phi_{\beta Tail} - \phi_{\beta Head} = 2\omega_\xi \hat{\tau}$  after a quarter period of the full synchrotron oscillation.

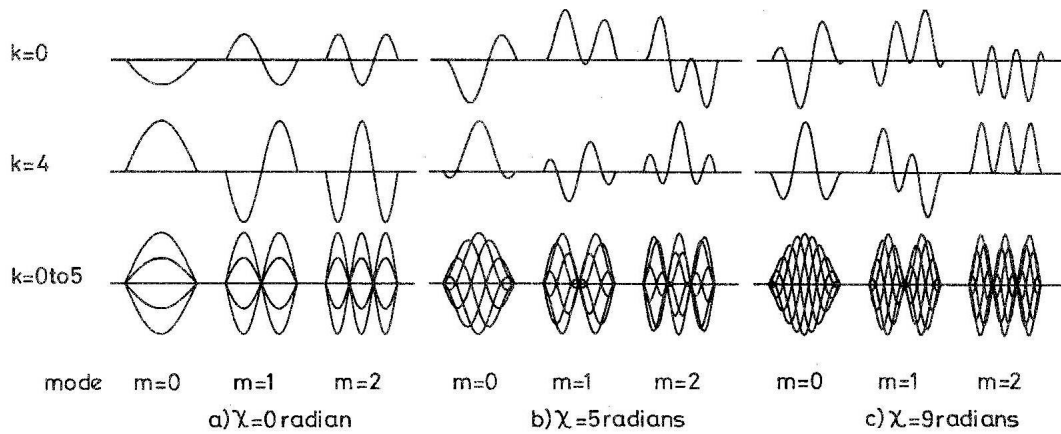
Let us assume that the dipole moment observed at a single point in the ring has the following standing wave pattern [1]:

$$p_m(t) = \begin{cases} \cos(m+1)\pi \frac{t}{\tau_L} \propto \frac{1}{2} \left( \exp(i(m+1)\pi \frac{t}{\tau_L}) + \exp(-i(m+1)\pi \frac{t}{\tau_L}) \right) & m = 0, 2, 4, \dots \\ \sin(m+1)\pi \frac{t}{\tau_L} \propto \frac{1}{2i} \left( \exp(i(m+1)\pi \frac{t}{\tau_L}) - \exp(-i(m+1)\pi \frac{t}{\tau_L}) \right) & m = 1, 3, 5, \dots \end{cases} \quad (6)$$

where  $\tau_L$  is the total bunch length. The transverse pick-up signal observed at that point on the  $k$ -th revolution turn is given by

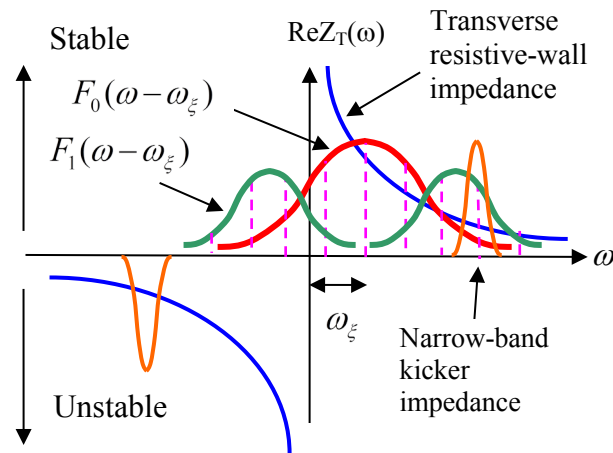
$$I_m(t) \propto p_m(t) \exp(i\phi_\beta(s)) = p_m(t) \exp(i\omega_\xi t + i2\pi k \nu_{\beta 0}), \quad (7)$$

where  $\nu_{\beta 0}$  is the betatron tune. Typical time evolutions of the transverse oscillation signals are shown in Fig. 7.



**Figure 7:** Typical time evolutions of the transverse oscillation signals for different  $\chi$  [2].

The effect of the travelling-wave component  $\exp(i\omega_\xi t + i2\pi k v_{\beta 0})$  over the standing-wave dipole moment  $p_m(t)$  is to shift the bunch spectrum by  $\omega_\xi$ . Figure 8 shows an example where the transverse resistive-wall impedance and the narrow-band kicker impedance are shown as impedance examples. The spectra are drawn for the head-tail modes 0 and 1 with positive phase difference  $\chi$ . In this example, the mode 0 is stabilized, while the mode 1 becomes unstable by the transverse resistive-wall impedance at low frequency.



**Figure 8:** An example of the bunch spectrum shift by non-zero chromaticity.

The physical reason for the spectrum shift is as follows. When  $\chi$  is large, the head and the tail of the bunch oscillate with large betatron phase difference. In order to excite such an intra-bunch oscillation, the wake field has to change its phase rapidly during the passage of the bunch. In other words, the impedance around the frequency  $\omega_\xi$  can now best synchronize with the fast intra-bunch motion.

The shift of the left peak of the bunch spectrum of the mode 1 means that a part of the mode now oscillates slowly. The head and the tail of the bunch will move almost in phase,

not out of phase, to synchronize with the low frequency impedance. On the other hand, the shift of the right peak of the mode 1 bunch spectrum implies that this part of the mode now oscillates faster. In summary, the mode  $m=1$  is degenerated at zero chromaticity, but are now split to slower and faster oscillating parts by non-zero chromaticity. These split modes are equally excited due to the standing-wave condition of head-tail modes, and thus they always have a node at the center just like the  $m=1$  mode at zero chromaticity.

The coherent tune shift of head-tail mode can be calculated by solving the Vlasov equation. Here, we only show the analytical result with the airbag model [3]:

$$\begin{aligned} \nu - \nu_{\beta 0} - m \nu_{s0} &\cong -i \frac{I_b R}{4\pi \nu_{\beta 0} E_0 / e} \sum_{p=-\infty}^{\infty} Z_T(\omega'_p) \\ &\times J_m^2((\omega'_p - \omega_\xi) \hat{\tau}) \end{aligned} \quad (8)$$

Here  $Z_T(\omega)$  is the transverse impedance and

$$\omega'_p = (p + \nu_{\beta 0} + m \nu_{s0}) \omega_0. \quad (9)$$

The other parameters are:  $I_b$  is the bunch current,  $R$  is the ring radius,  $E_0$  is the beam energy and  $J_m(x)$  is the Bessel function.

#### 2.13.4 TMCI Theory

When the beam intensity is increased, the interaction between different head-tail modes becomes non-negligible. We now have to solve the entire matrix including the interaction between various head-tail modes [5,6]. The eigen-solution  $\nu$ , the coherent tune, can be obtained by solving the following equation (for the airbag model [3]):

$$\det((\nu - \nu_{\beta 0}) \mathbf{I} - \mathbf{A}) = 0. \quad (10)$$

Here  $\mathbf{I}$  is the unit matrix and the matrix element of  $\mathbf{A}$  is given by

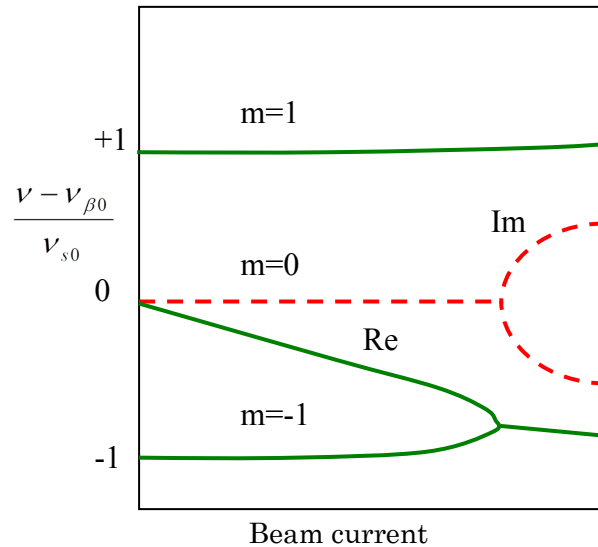
$$A_{mn} = m \nu_{s0} \delta_{mn} + M_{mn}, \quad (11)$$

where

$$M_{mn} = -i \frac{I_b R}{4\pi \nu_{\beta 0} E_0 / e} i^{m-n} \sum_p Z_T(\omega'_p) \cdot J_m((\omega'_p - \omega_\xi) \hat{\tau}) \cdot J_n((\omega'_p - \omega_\xi) \hat{\tau}) \quad (12)$$

Let us examine how the interaction between head-tail modes changes the behavior of the instability. The real part of the coherent tune (the coherent tune shift from the unperturbed tune  $\nu_{\beta 0} + \nu_{s0}$ ) is determined by the sum of the imaginary part (multiplied by the bunch spectrum) of the transverse impedance over frequency. Therefore, broadband impedance, rather than narrow one, tends to excite a larger tune shift. In conventional head-tail instability, the broadband impedance has a little effect on its excitation. But, when the beam intensity is increased and exceeds a certain threshold value, the tunes of two modes can merge and then suddenly, a strong instability occurs, as illustrated in Fig. 9. The growth

time of the instability can be as fast as the synchrotron oscillation period. This instability is called the Transverse Mode-Coupling Instability (TMCI) or the strong head-tail instability. TMCI has been observed mostly in electron machines such as PETRA, PEP [5] and LEP [7] where a bunch is short, but it has been observed also in proton machines such as SPS [8] where a bunch is relatively short compared to most of other proton machines.



**Figure 9:** Typical behavior of coherent tune shifts as a function of the beam current. The real and the imaginary parts of the tune shifts are denoted by the solid and dashed lines. When the two tune shifts merge, the imaginary part of the tune shift (the growth rate) starts to increase rapidly.

### 2.13.5 Historical Record

By 1983, accelerator theorists faced two challenging instability problems to solve. One was the bunch lengthening phenomenon (longitudinal) and other was a transverse one, which is now called TMCI. The both phenomena have clear thresholds in the bunch current, and thus they are distinct from conventional Robinson-type instabilities.

By that time, there was common understanding among the accelerator theorists that the both phenomena are caused by mode-couplings. At the first glance, the bunch lengthening seemed to be easier to solve, since it involves only the longitudinal motion and the construction of the theory was simpler. However, such attempt encountered a big problem: the calculated threshold is much higher than the measured one. It is because the bunch lengthening is accompanied with the potential-well distortion, and thus the non-linear effect should have a significant role in the theory, which a simple linear theory cannot cope with. The breakthrough of this dilemma came out when Oide and Yokoya introduced the action-angle variable technique to solve the linearized Vlasov equation with an arbitrary potential well [9].

The TMCI theory was more complicated, but straightforward, since it involves no non-linearity. It is interesting to point out that the period of time when the TMCI theory was developed coincided with time when the independent variable in the instability theory was changed from “time  $t$ ” to “position  $s$  or  $\theta$ ” in a ring, in order to be more precise and straightforward. Up to then, instability theories were written with time as the independent variable, as seen in Sacherer’s model Eqs. (6) and (8). This introduces annoying correction terms to be consistent with measurements, and there was strong demand to straighten the

theoretical foundation. Nowadays, it sounds too natural to describe beam instabilities developing as a beam moves along a ring, but the snapshot type understanding used to be dominant those days.

The picture like Fig. 9, now familiar with TMCI, first appeared in Chin and Satoh's theoretical paper [5]. As seen from Fig. 9, the first mode-coupling happens usually between  $m=0$  and  $m=-1$  modes. There is a clear asymmetrical behavior between positive and negative  $m$  modes. However, in Sacherer's head-tail model (Eqs. (6) and (8)), positive and negative  $m$  modes are supposed to be symmetrical. The only difference between them is that they rotate in the opposite direction in the synchrotron phase space. In this way, they together create a standing wave pattern and Sacherer's head-tail modes are described only by positive  $m$  numbers. Notice that the bunch spectrums in Fig. 8 are only for positive (no negative)  $m$ . Measured dipole signals such as shown in Fig. 4 have clear nodes, which demonstrates the validity of the standing wave model. On the other hands, measured tune shifts show the behavior, as illustrated in Fig. 9, and positive and negative  $m$  modes are clearly distinct, especially close to the TMCI threshold. This thinking exercise reveals that Sacherer's standing wave model and the bunch spectrum shift by non-zero chromaticity, as illustrated in Fig. 8, are valid only at low beam intensity. It may be interesting to investigate how the degeneracy of positive and negative  $m$  modes starts to break down as a function of the beam intensity and how head-tail modes look like in that regime, in particular, close to the TMCI threshold, to bridge Sacherer's model and the TMCI theory.

### 2.13.6 Acknowledgements

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### 2.13.7 References

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