



Seiberg–Witten theory and monstrous moonshine

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 We study the relation between the instanton expansion of the Seiberg–Witten (SW) prepotential for $D = 4$, $\mathcal{N} = 2$ $SU(2)$ SUSY gauge theory for $N_f = 0$ and 1 and the monstrous moonshine. By utilizing a newly developed simple method to obtain the SW prepotential, it is shown that the coefficients of the expansion of $q = e^{2\pi i\tau}$ in terms of $A^2 = \frac{\Lambda^2}{16a^2}$ ($N_f = 0$) or $\frac{\Lambda^2}{32a^2}$ ($N_f = 1$) are all integer-coefficient polynomials of the moonshine coefficients of the modular j -function. A relationship between the Alday–Gaiotto–Tachikawa (AGT) $c = 25$ Liouville conformal field theory (CFT) and the $c = 24$ vertex operator algebra CFT of the moonshine module is also suggested.

Subject Index C32, H10, H16

Seiberg–Witten (SW) theory [1,2] is a method, or the entire framework thereof, for solving analytically (and/or geometrically) the low-energy strongly coupled dynamics of $D = 4$, $\mathcal{N} = 2$ (originally $SU(2)$) SUSY gauge theory, based on the idea that, after all, the complexified gauge coupling can be identified (for the $SU(2)$ case) as the complex structure modulus of an elliptic fibration (see Ref. [3] for a review). It succeeded in determining the exact low-energy effective prepotential including the full instanton contributions [4] in terms of some integrals on the Seiberg–Witten curve. It has developed in connection with various research areas of modern string theory and mathematical physics, such as embedding into M-theory/Gaiotto duality [5, 6], 7-brane system/F-theory [7–9], E-strings [10–13], Alday–Gaiotto–Tachikawa (AGT) 4D/2D correspondence [14], and matrix models (see, e.g., Refs. [15–19]).

Monstrous moonshine, on the other hand, refers to the curious fact, first noticed by John McKay in 1979, that the Fourier coefficients of the modular j -function can be written as a simple linear combination of dimensions of irreducible representations of the monster group [20] (see Ref. [21] for a review including interesting anecdotes about monstrous moonshine). The reasoning for this coincidence was given by constructing $c = 24$ vertex operator algebra conformal field theory (CFT) whose character is the j -function such that the monster group acts on this module as a symmetry [22, 23]. Later it was shown that this was a Z_2 asymmetric orbifold [24].

In this letter, we study the relation between the two. In fact, the fact that the two are related is not in itself surprising. This is because an SW curve is a rational elliptic surface over the u -plane and the j -function is a fundamental function in the theory of elliptic functions. However, the details of this specific relationship have not been known until now. This letter fills this gap.

We use a newly developed method for easily deriving the $SU(2)$ SW prepotential to show that the coefficients of the instanton expansion of the prepotential are related to the monstrous moonshine in a way specifically explained in the text. In particular, it is shown that, in the $N_f = 0$ case, the coefficients of the expansion of $q = e^{2\pi i\tau}$ in terms of $A^2 = \frac{\Lambda^2}{16a^2}$ are all integer-coefficient polynomials of the moonshine coefficients. A similar thing holds for $N_f = 1$.

The idea of the new method is very simple. The modular j -function has an expansion in terms of $q = e^{2\pi i\tau}$ as¹

$$j(\tau) = \frac{1}{q} + 744 + 196\,884q + 21\,493\,760q^2 + 864\,299\,970q^3 + 20\,245\,856\,256q^4 + O(q^5). \tag{1}$$

Thus q is conversely expanded in terms of $\frac{1}{j}$ as

$$q = \frac{1}{j} + a_0 \left(\frac{1}{j}\right)^2 + (a_0^2 + a_1) \left(\frac{1}{j}\right)^3 + (a_0^3 + 3a_1a_0 + a_2) \left(\frac{1}{j}\right)^4 + O\left(\left(\frac{1}{j}\right)^5\right), \tag{2}$$

where $a_0 = 744$, $a_1 = 196\,884$, $a_2 = 21\,493\,760, \dots$ are the coefficients of the expansion (1). Note that the coefficient of the $\left(\frac{1}{j}\right)^k$ term in Eq. (2) is an integer given by a $(k - 1)$ th-order homogeneous polynomial of a_i if the ‘‘degree’’ of a_i is counted as $i + 1$. Since $q = e^{2\pi i\tau}$, we can obtain $2\pi i$ times the prepotential by taking its logarithm and integrating it with respect to a twice. On the other hand, if we suppose that the SW curve is given in the Weierstrass form

$$Y^2 = X^3 + f(u)X + g(u), \tag{3}$$

then its complex structure modulus τ is found by inversely solving the equation

$$j(\tau) = \frac{12^3 \cdot 4f(u)^3}{4f(u)^3 + 27g(u)^2}. \tag{4}$$

We expand the rhs of Eq. (4) by u around $u = \infty$, thereby obtaining a $1/j$ -expansion of $1/u$. Furthermore, since the $1/a$ -expansion of $1/u$ is obtained by the period integral on the SW curve, we end up with the $1/a$ -expansion of q , from which the $1/a$ -expansion of the prepotential \mathcal{F} is obtained.

As a concrete example, let us consider the pure ($N_f = 0$) $SU(2)$ $\mathcal{N} = 2$ SYM theory. The SW curve is given in the quartic-polynomial representation as [25]

$$y^2 = C(x)^2 - G(x), \quad C(x) = x^2 - u, \quad G(x) = \Lambda^4. \tag{5}$$

This is equivalent to the Weierstrass form

$$\begin{aligned} Y^2 &= X^3 + f(u)X + g(u), \\ f(u) &= -\frac{16}{3}u^2 + 4\Lambda^4, \\ g(u) &= -\frac{128}{27}u^3 + \frac{16}{3}\Lambda^4u. \end{aligned} \tag{6}$$

The integral of the holomorphic differential along the cycle that collapses as $u \rightarrow \infty$ yields $\frac{\partial a}{\partial u}$, which can be obtained by, for instance, solving the Picard–Fuchs equation [26]. The result is

$$\frac{\partial a}{\partial u} = \frac{\sqrt{2}}{4\sqrt{u}} F\left(\frac{1}{4}, \frac{3}{4}, 1; \frac{\Lambda^4}{u^2}\right). \tag{7}$$

¹All the series expansions performed in this letter have been assisted by Mathematica.

Integrating this with respect to u , we find

$$a = -\frac{\sqrt{u}}{\sqrt{2}} \sum_{n=0}^{\infty} \frac{(4n-3)!!}{4^{2n}(n!)^2} \left(\frac{\Lambda^4}{u^2}\right)^n, \tag{8}$$

where we have used the infinite series representation of the hyperelliptic function.

Looking at this, it might appear that, except for its prefactor, a has a series expansion in $\frac{\Lambda^4}{u^2}$ with rational-number coefficients whose denominators are integers containing very many prime factors. This is not case, however, as all the factors of powers of odd prime integers in $(n!)^2$ are contained in $(4n-3)!!$ and hence cancel out, leaving only powers of 2 in the denominator. Moreover, these factors of powers of 2 turn out to be absorbed if we take the expansion parameter to be $\frac{\Lambda^4}{8u^2}$ instead of $\frac{\Lambda^4}{u^2}$. Therefore, defining

$$A \equiv \frac{\Lambda}{4a}, \quad U \equiv \frac{\Lambda^2}{8u}, \tag{9}$$

A^2 is expanded by U as

$$\begin{aligned} A^2 &= \frac{U}{\left(\sum_{n=0}^{\infty} \frac{(4n-3)!!}{(n!)^2} (4U^2)^n\right)^2} \\ &= U + 8U^3 + 168U^5 + 5056U^7 + 184\,040U^9 + 7525\,440U^{11} + 332\,612\,800U^{13} \\ &\quad + 15\,538\,219\,520U^{15} + 756\,483\,502\,440U^{17} + 38\,023\,703\,291\,200U^{19} \\ &\quad + 1960\,287\,432\,256\,832U^{21} + 103\,165\,644\,665\,826\,816U^{23} + O(U^{25}), \end{aligned} \tag{10}$$

where the coefficients are all positive integers. Note that the factor of 8 in the denominator of U is the smallest one that can absorb all the factors of powers of 2^{-1} in the expansion. This U is inversely expanded by A^2 as

$$\begin{aligned} U &= A^2 - 8A^6 + 24A^{10} - 448A^{14} - 4520A^{18} - 151\,872A^{22} - 4095\,296A^{26} \\ &\quad - 124\,070\,400A^{30} - 3886\,030\,632A^{34} - 126\,167\,064\,640A^{38} - 4206\,822\,732\,736A^{42} \\ &\quad - 143\,383\,813\,565\,952A^{46} + O(A^{50}). \end{aligned} \tag{11}$$

On the other hand, by plugging Eq. (6) into Eq. (4), we have

$$\begin{aligned} \frac{1}{j} &= \frac{U^4(1-64U^2)}{(1-48U^2)^3} \\ &= U^4 + 80U^6 + 4608U^8 + 221\,184U^{10} + 8847\,360U^{12} + 254\,803\,968U^{14} \\ &\quad - 782\,757\,789\,696U^{18} - 84\,537\,841\,287\,168U^{20} - 6763\,027\,302\,973\,440U^{22} \\ &\quad - 476\,117\,122\,129\,330\,176U^{24} + O(U^{25}). \end{aligned} \tag{12}$$

Thus, using this in Eq. (11), we obtain an A^2 -expansion of $\frac{1}{j}$. This expansion can be further used in Eq. (2) to finally obtain an A^2 -expansion of q . We can see from Eqs. (2), (12), and (11) that its expansion coefficients are all integer-coefficient polynomials of the a_i . Though we do not present an explicit expression for this expansion of q , we instead show the expansion of its

Table 1. The instanton expansion of the prepotential for $N_f = 0$ and the monstrous moonshine. If we use the actual values of the Fourier coefficients a_i of the j -function in the middle column, we re-derive the correct \mathcal{F}_k in the right column on the corresponding row. Since each a_i is an integer-coefficient linear combination of the dimensions of irreducible representations of the monster, \mathcal{F}_k is also a rational-coefficient polynomial of them.

k	Rational-coefficient polynomial of a_i	\mathcal{F}_k
1	$\frac{48}{4^4 \cdot 3 \cdot 2}$	$\frac{1}{2^5}$
2	$\frac{a_0 + 96}{4^8 \cdot 7 \cdot 6}$	$\frac{5}{2^{14}}$
3	$\frac{48(a_0 - 304)}{4^{12} \cdot 11 \cdot 10}$	$\frac{3}{2^{18}}$
4	$\frac{\frac{1}{2}(a_0^2 + 2496a_0 + 2a_1 - 1570368)}{4^{16} \cdot 15 \cdot 14}$	$\frac{1469}{2^{31}}$
5	$\frac{\frac{48}{5}(5a_0^2 + 880a_0 + 10a_1 - 3352464)}{4^{20} \cdot 19 \cdot 18}$	$\frac{4471}{2^{34} \cdot 5}$
6	$\frac{\frac{1}{3}(a_0^3 + 7200a_0^2 + 6a_1a_0 - 3447648a_0 + 14400a_1 + 3a_2 - 3648416256)}{4^{24} \cdot 23 \cdot 22}$	$\frac{40397}{2^{43}}$
7	$\frac{\frac{48}{7}(7a_0^3 + 9968a_0^2 + 42a_1a_0 - 12245520a_0 + 19936a_1 + 21a_2 - 6513833472)}{4^{28} \cdot 27 \cdot 26}$	$\frac{441325}{2^{47} \cdot 7}$
8	$\frac{\frac{1}{4}(a_0^4 + 14208a_0^3 + 12a_1a_0^2 + 140160a_0^2 + 85248a_1a_0 + 12a_2a_0 - 15707695104a_0 + 6a_1^2 + 280320a_1 + 42624a_2 + 4a_3 - 6519374734464)}{4^{32} \cdot 31 \cdot 30}$	$\frac{866589165}{2^{64}}$
\vdots	\vdots	\vdots

logarithm:

$$\begin{aligned}
 2\pi i\tau &= \log A^8 + 48A^4 + (a_0 + 96)A^8 + 48(a_0 - 304)A^{12} \\
 &+ \frac{1}{2}(a_0^2 + 2496a_0 + 2a_1 - 1570368)A^{16} + \frac{48}{5}(5a_0^2 + 880a_0 + 10a_1 - 3352464)A^{20} \\
 &+ \frac{1}{3}(a_0^3 + 7200a_0^2 + 6a_1a_0 - 3447648a_0 + 14400a_1 + 3a_2 - 3648416256)A^{24} \\
 &+ \frac{48}{7}(7a_0^3 + 9968a_0^2 + 42a_1a_0 - 12245520a_0 + 19936a_1 + 21a_2 - 6513833472)A^{28} \\
 &+ \frac{1}{4}(a_0^4 + 14208a_0^3 + 12a_1a_0^2 + 140160a_0^2 + 85248a_1a_0 + 12a_2a_0 \\
 &- 15707695104a_0 + 6a_1^2 + 280320a_1 + 42624a_2 + 4a_3 \\
 &- 6519374734464)A^{32} + O(A^{34}). \tag{13}
 \end{aligned}$$

Thus we can find the expansion of $2\pi i$ times the prepotential \mathcal{F} by integrating Eq. (13) with respect to a twice. For example, if we integrate the second term $48A^4$ with respect to a twice, we get

$$48A^4 \xrightarrow{(\int da)^2} \frac{48}{4^4 \cdot 3 \cdot 2} \frac{\Lambda^4}{a^2} = \frac{1}{32} \frac{\Lambda^4}{a^2}, \tag{14}$$

which agrees with the $k = 1$ term of the known $N_f = 0$ prepotential

$$\mathcal{F} = \frac{ia^2}{2\pi} \left(4 \log \frac{a}{\Lambda} - 6 + 8 \log 2 - \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{a} \right)^{4k} \right) \tag{15}$$

(Table 1). From the third term $(a_0 + 96)A^8$ we can compute the $k = 2$ term. By using $a_0 = 744$ we find

$$(a_0 + 96)A^8 \xrightarrow{(\int da)^2} \frac{a_0 + 96}{4^8 \cdot 7 \cdot 6} \cdot \frac{\Lambda^8}{a^6} = \frac{5}{16384} \frac{\Lambda^8}{a^6} = \frac{5}{2^{14}} \frac{\Lambda^8}{a^6}, \tag{16}$$

which is also the correct result.

In fact, $a_0 = 744 (= 3 \times \dim E_8)$ has nothing to do with the monster; the first Fourier coefficient a_1 related to monster representations appears for the first time in the A^{16} term. From this, by using $a_0 = 744$, $a_1 = 196\,884$ and integrating with respect to a twice, we find

$$\begin{aligned} \frac{1}{2}(-1570\,368 + 2496a_0 + a_0^2 + 2a_1)A^{16} &\stackrel{(f da)^2}{\rightarrow} \frac{\frac{1}{2}(-1570\,368 + 2496a_0 + a_0^2 + 2a_1) \Lambda^{16}}{4^{16} \cdot 15 \cdot 14} \frac{1}{a^{14}} \\ &= \frac{1469 \Lambda^{16}}{2^{31} a^{14}}, \end{aligned} \tag{17}$$

which is a correct answer. In this way, all the known results can be correctly recovered. Incidentally, since this method directly determines τ , it also correctly produces the perturbative part of \mathcal{F} (15). Indeed, $\frac{1}{2\pi i} \times$

$$\int da \int da \log A^8 = 6a^2 + \frac{1}{2}a^2 \log \left(\frac{\Lambda^8}{2^{16}a^8} \right) \tag{18}$$

coincides with the perturbative part.

Similar is true for $N_f = 1$. The SW curve for $N_f = 1$ is

$$y^2 = C(x)^2 - G(x), \quad C(x) = x^2 - u, \quad G(x) = \Lambda^3(x + m) \tag{19}$$

in the quartic-polynomial representation, whose equivalent Weierstrass form reads

$$\begin{aligned} Y^2 &= X^3 + f(u, m)X + g(u, m), \\ f(u, m) &= -\frac{16}{3}u^2 + 4\Lambda^3m, \\ g(u, m) &= -\frac{128}{27}u^3 + \frac{16}{3}\Lambda^3mu - \Lambda^6. \end{aligned} \tag{20}$$

From these data, $1/j$ can be computed as

$$\begin{aligned} \frac{1}{j} &= \frac{U^3 ((64\hat{m}^3 + 432) U^3 - 72\hat{m}U^2 - \hat{m}^2U + 1)}{(48\hat{m}U^2 - 1)^3} \\ &= -U^3 + \hat{m}^2U^4 - 72\hat{m}U^5 + (80\hat{m}^3 - 432) U^6 - 3456\hat{m}^2U^7 \\ &\quad + (13\,824\hat{m}^4 - 144\hat{m} (64\hat{m}^3 + 432)) U^8 - 110\,592\hat{m}^3U^9 \\ &\quad + (1105\,920\hat{m}^5 - 13\,824\hat{m}^2 (64\hat{m}^3 + 432)) U^{10} \\ &\quad + (79\,626\,240\hat{m}^6 - 1105\,920\hat{m}^3 (64\hat{m}^3 + 432)) U^{12} + 382\,205\,952\hat{m}^5U^{13} \\ &\quad + (5350\,883\,328\hat{m}^7 - 79\,626\,240\hat{m}^4 (64\hat{m}^3 + 432)) U^{14} + 42\,807\,066\,624\hat{m}^6U^{15} \\ &\quad + (342\,456\,532\,992\hat{m}^8 - 5350\,883\,328\hat{m}^5 (64\hat{m}^3 + 432)) U^{16} + O(U^{17}), \end{aligned} \tag{21}$$

where $U \equiv \frac{\Lambda^2}{16u}$, $\hat{m} \equiv \frac{4m}{\Lambda}$ in this $N_f = 1$ case.

On the other hand, $\frac{\partial a}{\partial u}$ is given by using the quadratic and cubic transformations of the hypergeometric functions [27]

$$\frac{\partial a}{\partial u} = \frac{F\left(\frac{1}{12}, \frac{5}{12}, 1; \frac{12^3}{j}\right)}{\sqrt{2}(-3f(u, m))^{\frac{1}{4}}}. \tag{22}$$

Plugging Eqs. (20), (21) into this equation, we find

$$\begin{aligned} \frac{\partial a}{\partial u} = & \frac{1}{2\sqrt{2u}}(1 + 12\hat{m}U^2 - 60U^3 + 420\hat{m}^2U^4 - 5040\hat{m}U^5 + (18\,480\hat{m}^3 + 13\,860)U^6 \\ & - 360\,360\hat{m}^2U^7 + (900\,900\hat{m}^4 + 2162\,160\hat{m})U^8 + (-24\,504\,480\hat{m}^3 - 4084\,080)U^9 \\ & + (46\,558\,512\hat{m}^5 + 232\,792\,560\hat{m}^2)U^{10} + (-1629\,547\,920\hat{m}^4 - 931\,170\,240\hat{m})U^{11} \\ & + (2498\,640\,144\hat{m}^6 + 21\,416\,915\,520\hat{m}^3 + 1338\,557\,220)U^{12} \\ & + (-107\,084\,577\,600\hat{m}^5 - 133\,855\,722\,000\hat{m}^2)U^{13} \\ & + (137\,680\,171\,200\hat{m}^7 + 1807\,052\,247\,000\hat{m}^4 + 401\,567\,166\,000\hat{m})U^{14} + O(U^{15}) \end{aligned} \quad (23)$$

Integrating this with respect to u , we derive a u -expansion of a , from which the expansion of $U \equiv \frac{\Lambda^2}{16u}$ in terms of $A \equiv \frac{\Lambda}{4\sqrt{2a}}$ is found as follows:

$$\begin{aligned} U = & A^2 - 8A^6\hat{m} + 24A^8 + 24A^{10}\hat{m}^2 + 64A^{12}\hat{m} - 8A^{14}(56\hat{m}^3 + 81) + 6960A^{16}\hat{m}^2 \\ & - 40A^{18}(\hat{m}(113\hat{m}^3 + 1128)) + 384A^{20}(449\hat{m}^3 + 268) - 1344A^{22}(\hat{m}^2(113\hat{m}^3 + 1517)) \\ & + 128A^{24}\hat{m}(46\,735\hat{m}^3 + 77\,976) - 8A^{26}(511\,912\hat{m}^6 + 11\,530\,560\hat{m}^3 + 2197\,485) \\ & + O(A^{27}). \end{aligned} \quad (24)$$

Thus $\frac{1}{j}$ is expanded by A^2 , whose coefficients are this time integer-coefficient polynomials of \hat{m} . This yields

$$\begin{aligned} q = & -A^6 + A^8\hat{m}^2 - 48A^{10}\hat{m} + A^{12}(a_0 + 48\hat{m}^3 - 504) - 2A^{14}((a_0 + 372)\hat{m}^2) \\ & + A^{16}((a_0 + 1248)\hat{m}^4 + 96(a_0 - 512)\hat{m}) \\ & + A^{18}(-192(a_0 - 212)\hat{m}^3 - a_0^2 + 1008a_0 - a_1 - 61\,992) \\ & + 3A^{20}\hat{m}^2(32(a_0 + 88)\hat{m}^3 + a_0^2 + 928a_0 + a_1 - 939\,920) \\ & - 3A^{22}(\hat{m}((a_0^2 + 2864a_0 + a_1 - 1343\,656)\hat{m}^3 + 48a_0^2 - 48\,896a_0 + 48(a_1 + 67\,320))) \\ & + A^{24}((a_0^2 + 4800a_0 + a_1 - 1149\,216)\hat{m}^6 + 48(9a_0^2 - 3264a_0 + 9a_1 - 2473\,280)\hat{m}^3 \\ & + a_0^3 - 1512a_0^2 - 1512a_1 + 3a_0(a_1 + 126\,000) + a_2 - 2282\,112) \\ & - 4A^{26}(\hat{m}^2(12(9a_0^2 + 2640a_0 + 9a_1 - 4424\,600)\hat{m}^3 + a_0^3 + 1530a_0^2 \\ & + 3(a_1 - 915\,340)a_0 + 1530a_1 + a_2 + 215\,271\,600)) \\ & + 6A^{28}(8(3a_0^2 + 2848a_0 + 3(a_1 - 583\,120))\hat{m}^7 + (a_0^3 + 4572a_0^2 + 3(a_1 - 1279\,560)a_0 \\ & + 4572a_1 + a_2 - 632\,801\,280)\hat{m}^4 + 32(a_0^3 - 1524a_0^2 + 3(a_1 + 130\,008)a_0 \\ & - 1524a_1 + a_2 - 2995\,968)\hat{m}) \\ & + A^{30}(-4(a_0^3 + 7614a_0^2 + 3(a_1 - 966\,660)a_0 + 7614a_1 + a_2 - 2146\,774\,992)\hat{m}^6 \\ & - 768(a_0^3 - 498a_0^2 + (3a_1 - 670\,444)a_0 - 498a_1 + a_2 + 73\,856\,560)\hat{m}^3 \\ & - a_0^4 + 2016a_0^3 - 2a_1^2 - 948\,024a_1 - 6a_0^2(a_1 + 158\,004) \\ & + a_0(6048a_1 - 4a_2 + 67\,052\,160) + 2016a_2 - a_3 - 146\,853\,000) \\ & + O(A^{32}), \end{aligned} \quad (25)$$

Table 2. The instanton expansion of the prepotential for $N_f = 1$ and monstrous moonshine. We have defined $\tilde{m} \equiv \frac{m}{\Lambda}$ and $\hat{m} \equiv \frac{4m}{\Lambda}$. Note that $\mathcal{F}_k^{N_f=1}$ in this table is related to \mathcal{F}_k^1 in Appendix D of Ref. [28] as $\mathcal{F}_k^1 = -\frac{1}{2}\Lambda^{2k}\mathcal{F}_k^{N_f=1}$ (here), $\Lambda_1 = \Lambda$ (here).

k	Rational-coefficient polynomial of a_i	$\mathcal{F}_k^{N_f=1}$
2	$\frac{\frac{1}{2}(96\hat{m}-\hat{m}^4)}{(4\sqrt{2})^4 \cdot 3 \cdot 2}$	$\frac{\hat{m}}{32} - \frac{\hat{m}^4}{48}$
3	$\frac{\frac{1}{3}(1512-3a_0-\hat{m}^6)}{(4\sqrt{2})^6 \cdot 5 \cdot 4}$	$-\frac{3}{8192} - \frac{\hat{m}^6}{480}$
4	$\frac{\frac{1}{4}((4a_0+384)\hat{m}^2-\hat{m}^8)}{(4\sqrt{2})^8 \cdot 7 \cdot 6}$	$\frac{5\hat{m}^2}{16384} - \frac{\hat{m}^8}{2688}$
5	$\frac{\frac{1}{5}((-240a_0+124800)\hat{m}-\hat{m}^{10})}{(4\sqrt{2})^{10} \cdot 9 \cdot 8}$	$-\frac{7\hat{m}}{393216} - \frac{\hat{m}^{10}}{11520}$
6	$\frac{\frac{1}{6}(-390096-3024a_0+3a_0^2+6a_1+(-87552+288a_0)\hat{m}^3-\hat{m}^{12})}{(4\sqrt{2})^{12} \cdot 11 \cdot 10}$	$\frac{153}{536870912} + \frac{3\hat{m}^3}{262144} - \frac{\hat{m}^{12}}{42240}$
7	$\frac{\frac{1}{7}((7382760-5208a_0-7a_0^2-14a_1)\hat{m}^2-\hat{m}^{14})}{(4\sqrt{2})^{14} \cdot 13 \cdot 12}$	$-\frac{715\hat{m}^2}{536870912} - \frac{\hat{m}^{14}}{139776}$
8	$\frac{\frac{1}{8}((-46891008-393216a_0+384a_0^2+768a_1)\hat{m}+(-6281472+9984a_0+4a_0^2+8a_1)\hat{m}^4-\hat{m}^{16})}{(4\sqrt{2})^{16} \cdot 15 \cdot 14}$	$\frac{1131\hat{m}}{21474836480} + \frac{1469\hat{m}^4}{2147483648} - \frac{\hat{m}^{16}}{430080}$
9	$\frac{\frac{1}{9}(-864a_0^2\hat{m}^3 + 366336a_0\hat{m}^3 - 1728a_1\hat{m}^3 - 3a_0^3 + 4536a_0^2 - 557928a_0 - 18a_0a_1 + 9072a_1 - 9a_2 - \hat{m}^{18} + 381411072\hat{m}^3 + 123415488)}{(4\sqrt{2})^{18} \cdot 17 \cdot 16}$	$-\frac{385}{549755813888} - \frac{525\hat{m}^3}{4294967296} - \frac{\hat{m}^{18}}{1253376}$
\vdots	\vdots	\vdots

whose coefficients are again integer-coefficient polynomials of a_i and \hat{m} . The a_1 , a_2 , and a_3 related to the monster group representation appear for the first time in the coefficients of A^{18} , A^{26} , and A^{30} , respectively. Taking the logarithm and integrating with respect to a twice, we can derive $2\pi i$ times the $N_f = 1$ prepotential

$$\mathcal{F}^{N_f=1} = \frac{ia^2}{2\pi} \left(3 \log \frac{a}{\Lambda} + \frac{1}{2} (-9 + 15 \log 2 + \pi i) - \frac{m^2 \log \frac{a}{\Lambda}}{2a^2} - \sum_{k=2}^{\infty} \mathcal{F}_k^{N_f=1} \left(\frac{\Lambda}{a} \right)^{2k} \right) \quad (26)$$

with $\mathcal{F}_k^{N_f=1}$ shown in Table 2.

In this letter, we have shown that, in both the $N_f = 0$ and 1 cases, q is expanded by A^2 whose coefficients are integer-coefficient polynomials of a_i (as well as \hat{m} for $N_f = 1$) if $A \propto a^{-1}$ is appropriately defined. As a result, the coefficients of the instanton expansion of the prepotential are expressed as rational-coefficient polynomials of the ‘‘moonshine coefficients’’ a_i (as well as \hat{m} for $N_f = 1$) as shown in Tables 1 and 2. We expect a similar relationship to hold for $N_f = 2$ and 3, and perhaps even for E-string theory.

From these tables, we can see that in both cases the polynomials contain ‘‘inhomogeneous terms’’; i.e., the terms that remain after all a_i are set to 0. Of course, these numbers are not arbitrarily determined, but are determined by the given SW curves. In fact, setting all a_i to 0 in Eq. (1) amounts to approximating $j(\tau)$ by $\frac{1}{q}$. Since

$$j(\tau) = \left(\frac{\vartheta_3^8 + \vartheta_4^8 + \vartheta_2^8}{2\eta^8} \right)^3, \quad (27)$$

this implies that the whole affine E_8 character is replaced solely by the contribution of the ground state. Thus we may conclude that these inhomogeneous terms represent information on the ground state of the moonshine module.

We note that something similar occurs with the asymptotic behavior of the Atiyah–Hitchin (AH) metric, where the similar replacements $\vartheta_3 \rightarrow 1$, $\vartheta_4 \rightarrow 1$, $\vartheta_2 \rightarrow 0$, and $\eta \rightarrow q^{\frac{1}{24}}$ lead to the Taub-NUT metric with a negative NUT charge [29–32]. The AH space is known to be the moduli space of $D = 3$ $\mathcal{N} = 2$ $SU(2)$ SUSY gauge theory [33], and the instanton corrections

resolve the singularity of the negative-charge Taub-NUT. Since the Taub-NUT space is the transverse space of the M-theory lift of a D6-brane and the AH space corresponds to an O6-plane, our results in the present analysis may be considered to show the corresponding facts in the D7/O7 system.

The results in Tables 1 and 2 also show that the coefficients of the polynomials of a_i contain large integers (unlike the moonshine). Since each term of the instanton expansion of the prepotential is known to be expressed as a $c = 25$ Liouville correlation function [14], while the moonshine module is a Z_2 orbifold of $c = 24$ vertex operator algebra CFT [24], we suspect that these mysterious large numbers may be interpreted as a contribution coming from the missing $c = 1$ CFT.

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