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Entropy bound of horizons for charged and rotating black holes

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ABSTRACT

We revisit the entropy product, entropy sum and other thermodynamic relations of charged and rotating black holes. Based on these relations, we derive the entropy (area) bound for both event horizon and Cauchy horizon. We establish these results for variant class of 4-dimensional charged and rotating black holes in Einstein(–Maxwell) gravity and higher derivative gravity. We also generalize the discussion to black holes with NUT charge. The validity of this formula, which seems to be universal for black holes with two horizons, gives further clue on the crucial role that the thermodynamic relations of multi-horizons play in black hole thermodynamics and understanding the entropy at the microscopic level.

1. Introduction

Recently, the entropy product of multi-horizons black holes were studied widely in a lot literatures [1–24]. They are always independent on the mass of black holes. The universal character of this relation holds for many charged and rotating black holes [1–13], even all known five-dimensional asymptotically flat black rings, and black strings [14]. This relation is expected to be helpful of understanding the black hole entropy at the microscopic level. Actually, the entropy product, in conjunction with Cauchy horizon thermodynamics, can be used to determine whether the corresponding Bekenstein–Hawking entropy can be written as a Cardy formula, hence providing some evidence for a CFT description of the corresponding microstates [14,15]. This also makes it important to study the thermodynamics of Cauchy horizon.

On the other hand, the mass-independence of entropy product fails for some multi-horizons black holes [15–19]. Then the entropy sum [12,13,16,20,23] and other thermodynamic relations [16,17,20–22,24] are introduced, which also have mass-independence for some cases and seem to be universal as well. Especially for the relation $T_+S_+ = T_-S_-$, which was linked closely with the mass-independence of entropy product, was also understood well and physically by the holographic description, i.e. the thermodynamic method of black hole/CFT (BH/CFT) correspondence [7,30–35]. This relation $T_+S_+ = T_-S_-$ may be taken as the criterion whether there is a 2-dimensional CFT dual for the black holes in the Einstein gravity and other diffeomorphism invariant gravity theories [7,30–35]. Besides, It was found that the thermodynamic relation $T_+S_+ = T_-S_-$ is equivalent to the central charge being the same (i.e. $c_R = c_L$) for some two-horizons black holes. For the whole perspective of black hole thermodynamics, the first law and Smarr relation are also derived which are consistence with known results in previous literature.

However, it is still unclear how other thermodynamic relations of multi-horizons work for constructing the holographic description of black hole, in the general background spacetime. The aim of this paper is trying to give some clues. We link thermodynamic relations of multi-horizons with black hole thermodynamics, including the entropy (area) bound, the first law of thermodynamics and Smarr relation, for event horizon and Cauchy horizon of charged and rotating black holes. In this way, one can also observe that thermodynamics of Cauchy horizon is closely related to thermodynamics of event horizon. In this work, we first revisit the entropy product, entropy sum and other thermodynamic relations of charged and rotating black holes. Based on entropy product and entropy sum, we obtain entropy (area) bound for event horizon and Cauchy horizon. Especially for the upper area bound of event horizon, it is actually the Penrose-like inequality. Totally, it is found that the electric charge Q diminishes the physical bound of entropy (area) for event horizon, while it enlarges that for Cauchy horizon; the angular momentum J enlarges them for Cauchy horizon, while it does nothing with that for event horizon; the NUT charge n always enlarges them for both event horizon and Cauchy

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horizon. Besides, consider the first derivative of entropy product and entropy sum together, we derive the first law of thermodynamics for event horizon and Cauchy horizon. Moreover, using the scaling discussion and thermodynamic relations, we also obtain Smarr relation of horizons. We establish this result for a large class of charged and rotating black holes, in Einstein gravity and higher derivative gravity. We also generalize the discussion to black holes with NUT charge. The validity of this formula, which seems to be universal for two-horizons black holes, gives further clue on the crucial role that the thermodynamic relations of multi-horizons play in black hole thermodynamics and understanding the entropy at the microscopic level.

This paper is organized as follows. In next section, we firstly take Kerr black hole as a detailed example, to derive the entropy (area) bound. In Section 3, the discussion is generalized to the Kerr family, including the Kerr–Newman black hole, Kerr–Newman black hole in Gauss–Bonnet and Kerr–Taub–NUT black hole. Section 4 is devoted to the conclusions and discussions. In Appendix A, we also derive the first law of black hole thermodynamics and Smarr relation using the thermodynamic relations of multi-horizons.

2. Kerr black hole

The metric of Kerr black hole is

$$ds^2 = -\frac{\Delta}{\Sigma}[dt - \chi d\phi]^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - a dt\right]^2 + \Sigma\left[\frac{dr^2}{\Delta} + d\theta^2\right], \tag{2.1}$$

$$\Sigma = r^2 + a^2\cos^2\theta, \quad \chi = a\sin^2\theta, \quad a = \frac{J}{M},$$
$$\Delta = r^2 - 2Mr + a^2, \tag{2.2}$$

where M and J are the mass and angular momentum, respectively. The roots of metric function Δ correspond to event horizon r_E and Cauchy horizon r_C

$$r_E = M + \sqrt{M^2 - a^2}, \qquad r_C = M - \sqrt{M^2 - a^2}. \tag{2.3}$$

The Hawking temperature of event horizon is

$$T_E = \frac{r_E - r_C}{4\pi(r_E^2 + a^2)}, \tag{2.4}$$

while the Hawking temperature of Cauchy horizon meets the following relation [7]

$$T_C = -T_E|_{r_E \leftrightarrow r_C} = \frac{r_E - r_C}{4\pi(r_C^2 + a^2)}. \tag{2.5}$$

The entropy of horizons are

$$S_i = \frac{A_i}{4} = \pi(r_i^2 + a^2) \quad (i = E, C) \tag{2.6}$$

and the angular velocity of corresponding horizons are

$$\Omega_i = \frac{a}{r_i^2 + a^2} = \frac{\pi J}{M S_i} \quad (i = E, C). \tag{2.7}$$

Based on these thermodynamic quantities, some mass-independent and mass-dependent thermodynamic relations are introduced [24], including the entropy product

$$S_E S_C = 4\pi^2 J^2, \tag{2.8}$$

the entropy sum

$$S_E + S_C = 4\pi M^2, \tag{2.9}$$

the entropy minus

$$S_E - S_C = 8\pi M T_E S_E = 8\pi M T_C S_C, \tag{2.10}$$

and the sum of entropy inverse

$$\frac{1}{S_E} + \frac{1}{S_C} = \frac{M^2}{\pi J^2}. \tag{2.11}$$

Other useful thermodynamic relations, the sum of angular velocity and relation of temperature and entropy are

$$\Omega_E + \Omega_C = \frac{M}{J}, \tag{2.12}$$
$$T_E S_E = T_C S_C \tag{2.13}$$

respectively.

Actually, this relation (2.13) and entropy product are really useful of providing some evidence for a CFT description of the corresponding microstates. (See [14,15] and [7,30–35], respectively.) In what follows, we present the further application of other thermodynamic relations of multi-horizons in black hole thermodynamics.

We consider the entropy (area) bound. However, in order to avoid naked singularity, one must introduce the famous Kerr bound $M \geq a$, i.e. $M^2 \geq J$. As $r_E \geq r_C$, one can find $S_E \geq S_C \geq 0$. Then the entropy product (2.8) leads to

$$S_E \geq \sqrt{S_E S_C} = 2\pi J \geq S_C,$$

and the entropy sum (2.9) results in

$$4\pi M^2 = (S_E + S_C) \geq S_E \geq \frac{(S_E + S_C)}{2} = 2\pi M^2 \geq S_C.$$

Totally, we get the entropy bound for event horizon and Cauchy horizon

$$S_E \in [2\pi M^2, 4\pi M^2], \qquad S_C \in [0, 2\pi J], \tag{2.14}$$

where the Kerr bound is used. Besides, the area entropy (2.6) leads to the area bound

$$\sqrt{\frac{A_E}{16\pi}} \in \left[\frac{M}{\sqrt{2}}, M\right], \qquad \sqrt{\frac{A_C}{16\pi}} \in \left[0, \sqrt{\frac{J}{2}}\right], \tag{2.15}$$

where the upper bound of area for event horizon is actually the exact Penrose inequality which is the first geometrical inequality of black hole [36].

We have derived the entropy (area) bound for Kerr black hole. Straightforwardly this procedure could be extended to the Kerr-family black holes due to the similar solution structure.

3. Generalize to Kerr black hole family

In this section, we generalize the discussion about entropy bound to Kerr black hole family, including the Kerr–Newman black hole, Kerr–Newman black hole in Gauss–Bonnet gravity and Kerr–Taub–NUT black hole. One can expect that this formula always works well for the black holes with two horizons. We also derive the first law of black hole thermodynamics and Smarr relation using the thermodynamic relations of multi-horizons in Appendix A, which is consistence with known results in previous literature.

3.1. Kerr–Newman black hole

Adding the electric charge to the discussion, we can consider the Kerr–Newman black hole, which has the metric (2.1) with different metric function

$$\Delta(r) = r^2 - 2Mr + a^2 + Q^2. \tag{3.1}$$

The event horizon r_E and Cauchy horizon r_C locate at the zeros of $\Delta(r)$

$$r_E = M + \sqrt{M^2 - a^2 - Q^2}, \qquad r_C = M - \sqrt{M^2 - a^2 - Q^2}. \tag{3.2}$$

Where the electric potential for event horizon and Cauchy horizon are simplified as

$$\Phi_E = \frac{Q r_E}{r_E^2 + a^2}, \quad \Phi_C = \frac{Q r_C}{r_C^2 + a^2}. \tag{3.3}$$

The Hawking temperature of event horizon and Cauchy horizon still behave as Eq. (2.4) and Eq. (2.5), while the corresponding entropy and angular velocity of horizons are also the same with Eq. (2.6) and Eq. (2.7), respectively. For this case, thermodynamic relations are modified by the electric charge [18,22]. For example, the mass-independent entropy product becomes $S_E S_C = \pi^2(4J^2 + Q^4)$; and the entropy sum turns to $S_E + S_C = 2\pi(2M^2 - Q^2)$. Other relations still keep the same form with that for Kerr black hole, including the entropy minus Eq. (2.10) and thermodynamic relation of temperature and entropy (2.13).

The existence of black hole horizons leads to the Kerr-like bound $M^2 - Q^2 - a^2 \geq 0$, or working in terms of J as it is better to rephrase this as $M^4 - M^2Q^2 - J^2 \geq 0$. This results in the relation $M^2 \geq \frac{Q^2 + \sqrt{Q^4 + 4J^2}}{2}$. Similarly, the entropy product leads to

$$S_E \geq \pi\sqrt{Q^4 + 4J^2} \geq S_C,$$

and the entropy sum gives

$$2\pi(2M^2 - Q^2) \geq S_E \geq \pi(2M^2 - Q^2) \geq S_C.$$

Consider the above bound and Kerr-like bound together, we get the entropy bound for event horizon and Cauchy horizon

$$S_E \in \left[2\pi M^2, 4\pi M^2\right] \times \left(1 - \frac{Q^2}{2M^2}\right),$$
$$S_C \in \left[0, 2\pi J\right] \times \sqrt{1 + \frac{Q^4}{4J^2}} \tag{3.4}$$

and the corresponding area bound

$$\sqrt{\frac{A_E}{16\pi}} \in \left[\frac{M}{\sqrt{2}}, M\right] \times \sqrt{1 - \frac{Q^2}{2M^2}},$$
$$\sqrt{\frac{A_C}{16\pi}} \in \left[0, \sqrt{\frac{J}{2}}\right] \times \left(1 + \frac{Q^4}{4J^2}\right)^{1/4}. \tag{3.5}$$

Part of which are consistent with that in [6]. Hence, for gravity with Maxwell source, one can find that the electric charge Q diminishes the physical bound of entropy (area) for event horizon, while it enlarges that for Cauchy horizon. When one consider the pure gravity, i.e. the electric charge is vanishing, the upper bound of area for event horizon degenerates to the exact Penrose inequality of black hole [36].

One can focus on the degenerated case $a = 0$ of the Kerr–Newman black hole, i.e. the Reissner–Nordström black hole. Thus we can find the entropy product reduces to $S_E S_C = \pi^2 Q^4$, and the entropy sum becomes $S_E + S_C = 2\pi(2M^2 - Q^2)$. The Kerr-like bound is $M \geq Q$. The entropy (area) bound for event horizon and Cauchy horizon are simplified as

$$S_E \in \left[2\pi M^2, 4\pi M^2\right] \times \left(1 - \frac{Q^2}{2M^2}\right), \qquad S_C \in \left[0, \pi Q^2\right], \tag{3.6}$$

and

$$\sqrt{\frac{A_E}{16\pi}} \in \left[\frac{M}{\sqrt{2}}, M\right] \times \sqrt{1 - \frac{Q^2}{2M^2}}, \qquad \sqrt{\frac{A_C}{16\pi}} \in \left[0, \frac{Q}{2}\right], \tag{3.7}$$

respectively. Comparing with that in Kerr–Newman black hole, one can find that the angular momentum J enlarges the physical bound of entropy (area) for Cauchy horizon, while it does nothing with that for event horizon. The upper bound of area for event horizon can be seen as Penrose-like inequality.

3.2. Kerr–Newman black hole in Gauss–Bonnet theory

Consider Kerr–Newman black hole in Gauss–Bonnet theory [25–29], which the Gauss–Bonnet term appears in the Lagrangian, i.e.

$$\mathcal{L} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big(R + \alpha(R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2)\Big) - \frac{1}{4}\int d^4x\sqrt{-g}F^{\mu\nu}F_{\mu\nu},$$

where α is the Gauss–Bonnet constant. The metric (2.1) and metric function (3.1) do not change and the ADM charges are not modified in asymptotical flat spacetime [18]. The event horizon and Cauchy horizon locate as the same as Eq. (3.2). The angular momentum and electric potential behave as Eq. (2.7) and Eq. (3.3). However, there is a shift for the entropy of horizons

$$S_E = \pi(r_E^2 + a^2 + 4\alpha) = S_E^0 + \alpha',$$
$$S_C = \pi(r_C^2 + a^2 + 4\alpha) = S_C^0 + \alpha'. \tag{3.8}$$

Where S_E^0 and S_C^0 are the entropies of Kerr–Newman black hole respectively and $\alpha' = 4\pi\alpha$. Straightforward, we can get the entropy product and entropy sum accordingly

$$S_E S_C = S_E^0 S_C^0 + \alpha'(S_E^0 + S_C^0) + \alpha'^2$$
$$= \pi^2(4J^2 + Q^4) + 2\pi\alpha'(2M^2 - Q^2) + \alpha'^2, \tag{3.9}$$

$$S_E + S_C = (S_E^0 + S_C^0) + 2\alpha'$$
$$= 2\pi(2M^2 - Q^2) + 2\alpha'. \tag{3.10}$$

Kerr-like bound is also valid here for the existence of the black hole and gives $\sqrt{4J^2 + Q^4} \leq 2M^2 - Q^2$. Then we find

$$\pi^2(4J^2 + Q^4) + 2\pi\alpha'(2M^2 - Q^2) + \alpha'^2 \leq (\pi(2M^2 - Q^2) + \alpha')^2,$$

which is useful for considering entropy bound as follow. We have

$$S_C \leq \sqrt{S_E S_C} = \sqrt{\pi^2(4J^2 + Q^4) + 2\pi\alpha'(2M^2 - Q^2) + \alpha'^2} \leq S_E,$$
$$S_C \leq \frac{S_E + S_C}{2} = \pi(2M^2 - Q^2) + \alpha' \leq S_E \leq S_C + S_E = 2(\pi(2M^2 - Q^2) + \alpha').$$

Thus, the entropy bound of event horizon and Cauchy horizon are also modified further by the parameter α following the same procedure

$$S_C \in \left[0,\ \sqrt{\pi^2(4J^2+Q^4)+2\pi\alpha'(2M^2-Q^2)+\alpha'^2}\right],$$
$$S_E \in \left[\pi(2M^2-Q^2)+\alpha',\ 2(\pi(2M^2-Q^2)+\alpha')\right]. \tag{3.11}$$

Since the Gauss–Bonnet term in Lagrange does not effect the metric in 4-dimensional spacetime [18], the area of horizons is similarly to Kerr–Newman black hole.

3.3. Kerr–Taub–NUT black hole

Black holes with NUT charge [40,41] is a vacuum solution of Einstein equation with the NUT parameter n. This NUT charge or dual mass has an intrinsic feature in General Relativity, which is the gravitational analogue to a magnetic monopole in Maxwell's electrodynamics [42]. For this case, the metric is slightly modified as

$$\mathrm{d}s^2 = -\frac{\Delta}{\Sigma}[\mathrm{d}t-\chi\,\mathrm{d}\phi]^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2+n^2)\mathrm{d}\phi - a\mathrm{d}t\right]^2 + \Sigma\left[\frac{\mathrm{d}r^2}{\Delta}+\mathrm{d}\theta^2\right],$$
$$\Sigma = r^2+(n+a\cos\theta)^2,\quad \chi = a\sin^2\theta - 2n\cos\theta,\quad a=\frac{J}{M},$$
$$\Delta = r^2-2Mr+a^2-n^2,$$

where the horizons locate at $r_E = M+\sqrt{M^2+n^2-a^2}$, $r_C = M-\sqrt{M^2+n^2-a^2}$. The area entropy behaves as $S_i = \pi(r_i^2+a^2+n^2)$ $(i=E,C)$. The temperatures of horizons are $T_E = \frac{r_E-r_C}{4\pi(r_E^2+a^2+n^2)}$ and $T_C = \frac{r_E-r_C}{4\pi(r_C^2+a^2+n^2)}$. Here we also choose the positive temperature for horizons. Thus the entropy product and entropy sum [21] can be derived as $S_E S_C = 4\pi^2(J^2+n^2(M^2+n^2))$ and $S_E+S_C = 4\pi(M^2+n^2)$, respectively.

For Kerr–Taub–NUT black hole, the Kerr-like bound is $M^2+n^2-a^2 \geq 0$, or in the form of angular momentum $M^4+n^2M^2-J^2\geq 0$, or equivalently $M^2 \geq \frac{-n^2+\sqrt{n^4+4J^2}}{2}$. Using this bound together with entropy product and sum, we obtain the entropy bound of event horizon and Cauchy horizon for Kerr–Taub–NUT black hole

$$S_E \in \left[2\pi M^2, 4\pi M^2\right]\times\left(1+\frac{n^2}{M^2}\right),$$
$$S_C \in \left[0, 2\pi J\right]\times\sqrt{1+\frac{n^2}{J^2}(M^2+n^2)}, \tag{3.12}$$

and the area bound

$$\sqrt{\frac{A_E}{16\pi}} \in \left[\frac{M}{\sqrt{2}}, M\right]\times\sqrt{1+\frac{n^2}{M^2}},$$
$$\sqrt{\frac{A_C}{16\pi}} \in \left[0, \sqrt{\frac{J}{2}}\right]\times\left(1+\frac{n^2}{J^2}(M^2+n^2)\right)^{1/4}. \tag{3.13}$$

Here the NUT charge always enlarges the physical bound of entropy (area) for both event horizon and Cauchy horizon, comparing with that of Kerr black hole.

4. Conclusions

In this paper, we have revisited the entropy product, entropy sum and other thermodynamic relations of multi-horizons of charged and rotating black holes. Based on the entropy product and entropy sum, we find entropy (area) bound of event horizon and Cauchy horizon. We have established this result for a large class of charged and rotating black holes, in Einstein gravity and higher derivative gravity. We also generalize the discussion to black holes with NUT charge. The validity of this formula, which seems to be universal for two-horizons black holes, gives further clue on the crucial role that the thermodynamic relations of multi-horizons play in black hole thermodynamics and understanding the entropy at the microscopic level. Especially for the upper bound of area for event horizon, it is actually the Penrose-like inequality of black holes. Totally, it is found that the electric charge Q diminishes the physical bound of entropy (area) for event horizon, while it enlarges that for Cauchy horizon; the angular momentum J enlarges that for Cauchy horizon, while it does nothing with that for event horizon; the NUT charge always enlarges the physical region of both event horizon and Cauchy horizon.

It is also interesting to generalize this discussion to black holes with different topology of the horizons, e.g. black ring. One can also focus on black holes in (A)dS spacetime, for which the first law of black hole thermodynamics and Smarr relation are still open questions. An interesting idea is treating the cosmological constant as a dynamic variable (see, e.g. [39,43–49]). This may be possibly checked following the similar procedure in this paper. Besides, as the entropy sum of (A)dS black holes are always solely dependent on the cosmological constant [12,13,16,20,23], one can expect that the area bound will be solely related to the cosmological radius. Hence the area bound may be the geometrical inequality of black holes. These are all left to be the future tasks.

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Appendix A. The first law of black hole thermodynamics and Smarr relation

The thermodynamics laws are fundamental in black hole physics. Using the thermodynamical relations, here we consider the first derivative of entropy product and entropy sum together, we can get the first law of thermodynamics for event horizon and Cauchy horizon. Smarr relation of horizons could be also obtained by the scaling discussion and using thermodynamic relations.

A.1. Kerr–Newman black hole

We demonstrate a detail derivation of the first law and Smarr relation of Kerr black hole. Consider the first derivative of entropy product and entropy sum together again, we obtain

$$S_C\mathrm{d}S_E + S_E\mathrm{d}S_C = 4\pi^2(2J\mathrm{d}J + Q^3\mathrm{d}Q),$$
$$\mathrm{d}S_E + \mathrm{d}S_C = \pi(8M\mathrm{d}M - 4Q\,\mathrm{d}Q),$$

which lead to

$$\mathrm{d}M = \frac{S_E-S_C}{8\pi M S_E}\mathrm{d}S_E + \frac{\pi J}{M S_E}\mathrm{d}J + \left(\frac{Q}{2M}+\frac{\pi Q^3}{2MS_E}\right)\mathrm{d}Q,$$
$$\mathrm{d}M = -\frac{S_E-S_C}{8\pi M S_C}\mathrm{d}S_C + \frac{\pi J}{M S_C}\mathrm{d}J + \left(\frac{Q}{2M}+\frac{\pi Q^3}{2MS_C}\right)\mathrm{d}Q.$$

Using the entropy minus relation and angular velocity, we get the first law of thermodynamics for event horizon and Cauchy horizon of Kerr–Newman black hole [18]

$$\mathrm{d}M = +T_E \mathrm{d}S_E + \Omega_E \mathrm{d}J + \Phi_E \mathrm{d}Q, \tag{A.1}$$

$$\mathrm{d}M = -T_C \mathrm{d}S_C + \Omega_C \mathrm{d}J + \Phi_C \mathrm{d}Q. \tag{A.2}$$

We can introduce the sum of angular velocity and electric potential

$$\Omega_E + \Omega_C = \frac{2J}{M}\frac{(2M^2 - Q^2)}{(4J^2 + Q^4)},$$

$$\Phi_E + \Phi_C = \frac{Q}{M} + \frac{Q^3}{M}\frac{(2M^2 - Q^2)}{(4J^2 + Q^4)}. \tag{A.3}$$

For Smarr relation of event horizon, we consider the scaling argument as well. As M scales as $[\text{length}]^1$, S_E scales as $[\text{length}]^2$, J scales as $[\text{length}]^2$ and Q scales as $[\text{length}]^1$, we find [18]

$$M = 2(T_E S_E + \Omega_E J_E) + \Phi_E Q. \tag{A.4}$$

Again assuming that

$$M = a T_C S_C + b\Omega_C J + c\Phi_C Q, \tag{A.5}$$

where a, b and c are undetermined coefficients. Then inserting the relations (2.13), (A.3) successively, we can find

$$\begin{aligned} 2M &= 2(+T_E S_E + \Omega_E J) + \Phi_E Q + (a T_C S_C + b\Omega_C J + c\Phi_C Q) \\ &= (2+a) T_C S_C + (b-2)\Omega_C J + (c-1)\Phi_C Q + 2M \end{aligned}$$

implying $a = -2$, $b = 2$, $c = 1$. Finally, we get Smarr relation for the Cauchy horizon of Kerr–Newman black hole [18]

$$M = 2(-T_C S_C + \Omega_C J) + \Phi_C Q. \tag{A.6}$$

When the angular momentum parameter $a = 0$ the Kerr–Newman solution degenerate to Reissner–Nordström solution. The first law of black hole thermodynamics becomes

$$\mathrm{d}M = +T_E \mathrm{d}S_E + \Phi_E \mathrm{d}Q, \qquad \mathrm{d}M = -T_C \mathrm{d}S_C + \Phi_C \mathrm{d}Q, \tag{A.7}$$

where the electric potential are Coulomb potential: $\Phi_E = \frac{Q}{r_E}$, $\Phi_C = \frac{Q}{r_C}$. Furthermore, Smarr relation of Reissner–Nordström black hole behave as

$$M = 2T_E S_E + \Phi_E Q, \qquad M = -2T_C S_C + \Phi_C Q. \tag{A.8}$$

A.2. Kerr–Newman black hole in Gauss–Bonnet theory

Note that α should be considered as a thermodynamic variable here. Hence, the similar first derivative of entropy product and entropy sum lead to the first law of thermodynamics of event horizon and Cauchy horizon

$$\mathrm{d}M = +T_E \mathrm{d}S_E + \Omega_E \mathrm{d}J + \Phi_E \mathrm{d}Q + \Theta_E \mathrm{d}\alpha, \tag{A.9}$$

$$\mathrm{d}M = -T_C \mathrm{d}S_C + \Omega_C \mathrm{d}J + \Phi_C \mathrm{d}Q + \Theta_C \mathrm{d}\alpha, \tag{A.10}$$

where the thermodynamic potential conjugate to α is defined to be $\Theta_E \equiv \left(\frac{\partial M}{\partial \alpha}\right)_{S_E,J,Q} = -4\pi T_E$, $\Theta_C \equiv \left(\frac{\partial M}{\partial \alpha}\right)_{S_C,J,Q} = +4\pi T_C$.

Then consider Smarr relation of event horizon, we use the scaling argument as well. Here M scales as $[\text{length}]^1$, S_E scales as $[\text{length}]^2$, J scales as $[\text{length}]^2$, Q scales as $[\text{length}]^1$ and α scales as $[\text{length}]^2$, we find

$$M = 2(T_E S_E + \Omega_E J_E) + \Phi_E Q + 2\Theta_E \alpha. \tag{A.11}$$

For the one of Cauchy horizon, we assume that

$$M = a T_C S_C + b\Omega_C J + c\Phi_C Q + d\Theta_C \alpha,$$

where a, b, c and d are undetermined coefficients. Considering the relation of entropy and temperature $T_E S_E = T_C S_C$, the sum of angular velocity, the sum of electric potential and the sum of thermodynamic potential successively, we can find

$$\begin{aligned} 2M &= 2(+T_E S_E + \Omega_E J) + \Phi_E Q + 2\Theta_E \alpha \\ &\quad + (a T_C S_C + b\Omega_C J + c\Phi_C Q) + d\Theta_C \alpha \\ &= (2+a) T_C S_C + (b-2)\Omega_C J + (c-1)\Phi_C Q + (d-2)\Theta_C \alpha \\ &\quad + 2M, \end{aligned}$$

implying $a = -2$, $b = 2$, $c = 1$, $d = 2$. Finally, we get Smarr relation for the Cauchy horizon of Kerr–Newman black hole in Gauss–Bonnet theory

$$M = 2(-T_C S_C + \Omega_C J) + \Phi_C Q + 2\Theta_C \alpha. \tag{A.12}$$

The above first law of thermodynamics (A.9), (A.10) and Smarr relation (A.11), (A.12) for the event horizon and the Cauchy horizon of Kerr–Newman black hole in Gauss–Bonnet theory are consistent with that in [18,37–39].

A.3. Kerr–Taub–NUT black hole

Using the first derivative of entropy product and entropy sum together, we obtain the first law of thermodynamics of event horizon and Cauchy horizon [21]

$$\mathrm{d}M = +T_E \mathrm{d}S_E + \Omega_E \mathrm{d}J + \Phi^n_E \mathrm{d}n, \tag{A.13}$$

$$\mathrm{d}M = -T_C \mathrm{d}S_C + \Omega_C \mathrm{d}J + \Phi^n_C \mathrm{d}Q, \tag{A.14}$$

where the angular velocity is $\Omega_E = \frac{a}{2Mr_E + 2n^2}$, $\Omega_C = \frac{a}{2Mr_C + 2n^2}$, and the Taub–NUT potential conjugate to NUT charge n is defined to be $\Phi^n_E \equiv \left(\frac{\partial M}{\partial n}\right)_{S_E,J}$, $\Phi^n_C \equiv \left(\frac{\partial M}{\partial n}\right)_{S_C,J}$.

Besides, the scaling argument is as follows: M scales as $[\text{length}]^1$, S_E scales as $[\text{length}]^2$, J scales as $[\text{length}]^2$ and n scales as $[\text{length}]^1$, which leads to

$$M = 2(T_E S_E + \Omega_E J_E) + \Phi^n_E n. \tag{A.15}$$

For the Cauchy horizon, one can get

$$M = 2(-T_C S_C + \Omega_C J_C) + \Phi^n_C n. \tag{A.16}$$

Note the discussion can be easily generalized to the case of Kerr–Newman–Taub–NUT black hole and reduced to that for Taub–NUT black hole.

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