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Unification by Generalising Proper Time Beyond the Extra Spatial Dimensions Paradigm

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Abstract

The main arguments supporting a novel proposal for the construction of a unified theory are presented. Theories with additional spatial dimensions have been investigated for around one hundred years, without clear empirical success or evidence for the existence of such a geometric extension to 4-dimensional spacetime. The new approach, dropping the central assumption of a geometric quadratic form in such models, is based upon a generalisation of proper time intervals to higher-order forms. The properties of the residual components over the original local 4-dimensional spacetime form are interpreted as the basis for the elementary structure of matter. We argue that the new approach is not only better motivated in terms of a conceptual foundation for unification, but also provides a much-improved framework for the empirical phenomena to be accounted for by a unified theory. In particular, we describe how generalised proper time accounts for features of the Standard Model particle multiplet structure and discuss the possible areas of new physics beyond. The manner in which the theory also provides a suitable setting for amalgamating gravity with quantum theory and incorporates a dark sector as appropriate for cosmological observations will also be described. Open questions for this novel unification scheme, in all of these areas, and the corresponding direction and opportunities for further development, will be outlined.

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1 Introduction: Motivation for Generalising Proper Time

Throughout the history of physics, dating back to the Newtonian worldview of classical mechanics, a key element has been the role of time as a continuous real parameter, which we can write simply as:

$$s \in \mathbb{R} \tag{1}$$

This remained the case through developments of the 19$^{\text{th}}$ century, as for Maxwell's equations of electrodynamics, and the 20$^{\text{th}}$ century, as for example in quantum theory in the equations of motion of Schrödinger and Dirac. The continuous nature of time is also prominent in Einstein's special and general theories of relativity, with 3-dimensional space no longer considered independent of time but rather collectively subsumed into a 4-dimensional spacetime continuum.

General relativity utilises the arena of a curved 4-dimensional spacetime with a geometry described by a symmetric metric tensor $g_{\mu\nu}(x)$, a ten-component function of the location x in spacetime, which can also be interpreted as the gravitational field. At any location an infinitesimal interval of 'proper time' δs satisfies the quadratic relation:

$$(\delta s)^2 \,=\, g_{\mu\nu}(x)\delta x^\mu \delta x^\nu \tag{2}$$

This defines 'proper' time in the sense that $\delta s \in \mathbb{R}$ is invariant under transformations between different general coordinate systems $\{x^\mu\}$ (with the convention of summing over repeated indices adopted in this paper, here for μ and ν over $0, 1, 2, 3$ as general spacetime coordinate indices).

Soon after the publication of Einstein's general relativistic theory of gravity [1] the first theories with an extra special dimension were proposed in the 1920s [2, 3]. These models of Kaluza and Klein augmented the 4-dimensional spacetime geometry by appending one extra dimension of space, with the metric tensor of general relativity and equation 2 correspondingly extended to a 5×5 symmetric matrix of components while maintaining a quadratic form for the proper time interval. On extracting the original 4-dimensional spacetime, four of the five residual metric components could be interpreted as a basis for the electromagnetic field, hence identified together with the gravitational field in a single framework.

Since the 1970s analogous models with a greater number of extra spatial dimensions have been proposed which, in addition to gravity and electromagnetism, are able to accommodate non-Abelian gauge theories (for example [4, 5]), with the ultimate ambition of accommodating the esoteric properties of the Standard Model of particle physics itself (for example [6, 7]). More generally, wide-ranging means of implementing extra spatial dimensions remain ubiquitous in 21$^{\text{st}}$ century model building (for a selection of approaches see [8, 9, 10, 11, 12] and references therein).

Circa the 1920s the initial aim was to unify gravity and electromagnetism as classical field theories, and hence it was appropriate to seek a *global* augmentation to the 4-dimensional spacetime geometry by extending the metric field $g_{\mu\nu}(x)$ of equation 2, in a manner such as proposed by Kaluza and Klein. A similar strategy is still predominantly followed in modern-day theories with further extra spatial dimensions, with a 'compactification' scheme then typically applied to the higher-dimensional spacetime manifold, similarly as had been originally proposed by Klein, to extract a

4-dimensional spacetime containing structures of matter associated with the compactified extra dimensions.

However, on noting that for contemporary particle physics we are particularly interested in *local* particle interactions and *local* gauge symmetry groups, rather than setting out with the global 4-dimensional spacetime metric of equation 2, we might also consider a more direct route by starting at the level of the *local* 4-dimensional spacetime structure. This then provides some motivation for *beginning* at the simpler local level of an infinitesimal inertial reference frame of general relativity in which an infinitesimal proper time interval can be expressed as (in local inertial coordinates $\{x^a\}$ with indices $a, b = 0, 1, 2, 3$):

$$(\delta s)^2 \;=\; \eta_{ab}\delta x^a \delta x^b \qquad (3)$$

Here $\eta_{ab} = \text{diag}(+1, -1, -1, -1)$ is the Lorentz metric of special relativity, as holds in a local inertial coordinate frame by the equivalence principle, and the proper time interval $\delta s \in \mathbb{R}$ is invariant under the local Lorentz transformations of the symmetry group $\text{SO}^+(1, 3)$ applied to the four local coordinates.

The central proposal of the new theory presented in this paper is the generalisation of this local expression for a proper time interval to the higher-order form:

$$(\delta s)^p \;=\; \alpha_{abc\ldots}\delta x^a \delta x^b \delta x^c \ldots \qquad (4)$$

with $p = 2, 3, 4, \ldots$ the homogeneous polynomial power, the constant coefficients $\alpha_{abc\ldots} \in \{-1, 0, +1\}$ generalising from the set of Lorentz metric coefficients η_{ab}, and a sum over each index $a, b, c, \ldots$ for the $n > 4$ components $\{\delta x^a\} \in \mathbb{R}^n$. A new full symmetry $\hat{G}$, as a generalisation from the Lorentz group with $\hat{G} \supset \text{SO}^+(1, 3)$, will act upon the $\{\delta x^a\} \in \mathbb{R}^n$ components leaving δs invariant, which is hence termed 'generalised proper time'.

By analogy with many models employing extra spatial dimensions, the residual components and broken symmetry structure, now in extracting the local 4-dimensional spacetime base and Lorentz symmetry of equation 3 from the general form of equation 4, will provide the basis for the elementary structure of matter. The main differences with the models utilising extra dimensions of space is then this initial focus upon the purely *local* level and the generalisation to forms for proper time beyond quadratic order in equation 4.

We also note that this *local* generalisation from equation 3 to equation 4 is *complementary* to the *global* generalisation from the flat Minkowski spacetime of special relativity, with the Lorentz metric of equation 3 applying for arbitrary finite intervals of proper time in global inertial coordinate frames, to the curved spacetime of general relativity itself as described by the metric tensor field $g_{\mu\nu}(x)$ of equation 2 in a general coordinate system. In this regard the new approach of generalised proper time is fully *compatible* with general relativity, with these complementary generalisations pictured in figure 1.

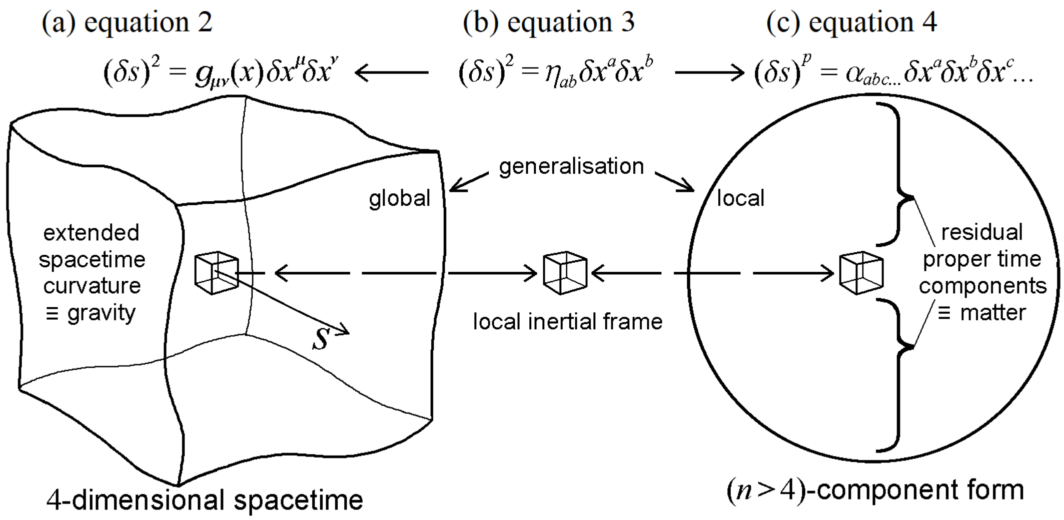

Figure 1: (a) The curved geometry of extended 4-dimensional spacetime described by equation 2, as the basis for the general relativistic theory of gravity, represents a global generalisation from Minkowski spacetime while retaining (b) the local inertial reference frames of equation 3 by the equivalence principle. This local form itself can be augmented to (c) the generalised local form for proper time beyond quadratic order in equation 4, with the residual components (represented by the large circle but having no geometric meaning), interpreted as the basis for matter, as described further in the text. In all three cases the flow of time is treated as the continuum $s \in \mathbb{R}$ of equation 1, as pictured in (a), and in all three cases a central property is the invariance of 'proper time', with respect to (a) global coordinate transformations, (b) local Lorentz transformations and (c) a local full $\hat{G}$ symmetry.

In general relativity the Einstein field equation directly relates the 4-dimensional spacetime geometry, described by the Einstein tensor $G^{\mu\nu}(x)$ as a function in up to the second derivatives of the metric field $g_{\mu\nu}(x)$, to the matter content, as represented by the energy-momentum tensor $T^{\mu\nu}(x)$, via a conventional normalisation constant $-\kappa$ as simply:

$$G^{\mu\nu} = -\kappa T^{\mu\nu} \quad (5)$$

Consistent with the ambition of the unified field theories of the 1920s and 1930s to account for the 'wood' of matter, alluding to our provisional phenomenological understanding of the nature of the energy-momentum on the right-hand side, by subsuming it into the 'marble' of geometry, building upon the precise mathematical expression of the Riemannian framework on the left-hand side of equation 5 ([13] page 370), the new theory proposes an augmentation from the local metric structure to provide such a basis for matter in 4-dimensional spacetime, as depicted in figure 1.

This figure also represents the conceptual progression from the purely linear one-dimensional flow of *time* $s \in \mathbb{R}$ in equation 1, as might parametrise a worldline as drawn in figure 1(a), to the 4-dimensional quadratic *space-time* form $(\delta s)^2$ of equation 2 or 3 as depicted in figures 1(a) and 1(b), and finally to the generalised unifying *space-time-matter* form $(\delta s)^p$ of equation 4 and figure 1(c). It is the observation that unifying extensions from the 4-dimensional spacetime structure are motivated by this ambition to accommodate a structure of *matter* and *not* more *space*, that permits the *quadratic assumption* of models with extra spatial dimensions to be *dropped* as we generalise from equation 3 to equation 4, allowing for both $n > 4$ components *and* $p > 2$ powers.

On the conceptual side, we emphasise that we are *not* here dealing with either extra 'spacelike' or 'timelike' dimensions, but rather with a non-trivial intrinsic substructure of *time itself*, with *all* components on the right-hand side of equation 4 being essentially subcomponents of time (as we shall elaborate further in particular in section 5). With the ideal of simplicity being a key quality typically desired of unification, rather than *appending anything* such as extra spatial dimensions, here the basic unifying entity is hence simply *time* alone in equation 1, entering on the left-hand side of equation 4. That is, given time as the real continuum of equation 1, infinitesimal intervals *have* the intrinsic arithmetic substructure of equation 4 which, considered as an augmentation from equation 3, can be written as (with $(\boldsymbol{\delta x}_{n-4})^{p-2}$ and $(\boldsymbol{\delta x}_n)^p$ denoting the appropriate polynomial expressions as consistent with equation 4):

$$(\delta s)^p = \overbrace{\alpha_{abc\ldots}\delta x^a \delta x^b \delta x^c \ldots}^{\substack{\text{generalised proper time} \\ a,b,c,\ldots\,=\,0,\ldots,n-1}} = \overbrace{(\eta_{ab}\delta x^a \delta x^b)}^{\substack{\text{4D spacetime} \\ a,b\,=\,0,1,2,3}} \times \overbrace{(\boldsymbol{\delta x}_{n-4})^{p-2} + (\boldsymbol{\delta x}_n)^p}^{\substack{\text{basis for matter} \\ \delta x^c;\, c\geq 4 \qquad \delta x^a;\, a \geq 0}} \tag{6}$$

$$\underbrace{\hat{G}}_{\text{full symmetry}} \supset \underbrace{\text{Lorentz}}_{\text{external}} \times \underbrace{G}_{\text{internal}} \tag{7}$$

These two equations make more explicit the residual components and broken symmetry which, while represented by the large circle in figure 1(c), have no spatial or geometric interpretation but rather directly provide the elementary local basis for matter. With the symmetry $\hat{G}$ of equation 4 broken on extracting a local quadratic substructure with $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\} \equiv \boldsymbol{\delta x}_4 \in \mathbb{R}^4$ identified as external 4-dimensional spacetime components, the residual components $\boldsymbol{\delta x}_{n-4} \in \mathbb{R}^{n-4}$ in equation 6, as a subset of the complete set $\boldsymbol{\delta x}_n \in \mathbb{R}^n$, make a central contribution to this foundation for matter. In addition to the residual internal symmetry G, the extracted external Lorentz symmetry of equation 7 will in general also act upon these matter components. The matter content will further include gauge fields associated with the internal symmetry G itself.

This symmetry breaking in extracting the four preferred components $\boldsymbol{\delta x}_4$ and the external Lorentz symmetry is *of necessity required* in order to identify 4-dimensional spacetime itself as well as all the physics it contains. The resulting product structure of equation 7, as will apply for this physics, is hence consistent for example with the Coleman-Mandula theorem for any relativistic particle scattering phenomenology that may arise [14].

Equations 4, 6 and 7 then indeed represent a *possible* generalisation of the local 4-dimensional spacetime structure as a means of identifying the source of matter, and with no clear immediate obstacles to developing such a unifying framework. In addition to making the case for the conceptual motivation, as emphasised in this section, a further main aim of this paper is to describe how unification with generalised proper time is also found to be very fitting from the practical perspective, given what is known empirically about the elementary properties of matter. We argue, in particular, that for generalised proper time *both* the conceptual motivation and the direct empirical successes offer a significant improvement in comparison with models deploying extra dimensions of space, while also highlighting the areas of the new theory where further progress is needed. These central arguments for the theory are also described in the presentation [15], upon which this paper was initially based.

As well as the motivation, emphasised before equation 3, for starting directly at the local level, as appropriate for the local properties of significance for contemporary particle physics, and further in comparison with the early unified field theories in general alluded to after equation 5, we now have the benefit of hindsight not only in the detailed properties of matter described by the Standard Model itself but also in the specific mathematical structures that can be employed to develop the theory of generalised proper time. In this context in the following section we first present a short argument explaining why the direct mathematical development and the symmetry breaking structure for forms of generalised proper time in equations 6 and 7 are of relevance for the Standard Model particle multiplet structure.

In particular we explain at the outset why this new approach involves symmetries $\hat{G}$ of equation 4 that include exceptional Lie groups with an octonionic construction. Since the 1970s unification models with such exceptional Lie groups [16, 17, 18] as well as the octonions [19, 20, 21] have been known to be of interest for the Standard Model. Here, significantly, rather than setting out in the pure mathematics of these structures, we begin with a firm conceptual foundation in generalised proper time which turns out to incorporate these unique mathematical forms in a manner fitting for the unique features of the Standard Model.

We then describe in more explicit detail through section 3 the features that can be directly identified, explaining how these mathematical structures are well-suited for accommodating the Standard Model in the context of the new theory, while also highlighting the empirically established features that remain to be accounted for. In section 4 we describe a way forward to uncover the complete Standard Model multiplet structure with this theory and outline the kinds of new physics that may be in turn accessible to empirical observation.

Given the suitability of generalised proper time in accounting for phenomena already observed in the particle physics laboratory, that this direct approach to unification has so far been overlooked may be due, not only to the familiarity of the assumed quadratic form for proper time as utilised by models with extra spatial dimensions, but in part also to the seeming implausibility of founding a unified theory essentially upon the continuum of time in equation 1 alone. In section 5 we address this concern by describing explicitly how a theory of matter in 4-dimensional spacetime, and indeed the physical universe itself, can be constructed solely from elements of generalised proper time, considered as an intrinsic substructure of temporal flow.

Returning to the empirical side, in sections 6 and 7 we describe how the construction of the theory is also well-suited for the other main questions to be addressed by a unified theory. In section 6 we explain how the approach of generalised proper time provides an appropriate setting for the amalgamation of quantum phenomena with general relativity, and motivate the overall strategy that can be adopted. Given the compatibility of the theory with general relativity, as discussed for figure 1, we make the case in general terms, counter to most approaches to 'quantum gravity', for a unifying framework in which the classical Einstein field equation 5 for a continuous spacetime geometry is fully preserved while quantum theory itself only strictly applies, for all non-gravitational fields, in the flat spacetime or non-relativistic limits to within an excellent approximation.

In section 7 we describe how the general form of proper time of equation 4, at the local level, in fact admits several mathematical branches, including the visible Standard Model sector with the exceptional Lie group symmetries described in sections 2–4. The additional branches have an *independent* internal gauge structure deriving from the symmetry breaking of equations 6 and 7 and are hence invisible, or *dark*, with respect to the Standard Model sector of matter. As a form of 'hidden QCD', distinct from the quantum chromodynamics (QCD) of the visible sector, we explain how both dark matter and dark energy candidates, of significance on the cosmological scale, can be identified in this manner.

In the concluding section 8 we first make further remarks about the conceptual foundations of the theory of generalised proper time. Through a head-to-head comparison with the most direct approaches employing extra spatial dimensions we then summarise how the new theory provides a much improved setting for the project of unification in particle physics and cosmology in general. The significant open questions will also be listed, highlighting the opportunities for the further development of this new approach based upon generalising proper time.

2 Explicit Mathematical Development of the Theory

We first present a short explanation regarding how the form for generalised proper time of equation 4, with the symmetry breaking extraction of a local 4-dimensional spacetime basis and the associated identification of an underlying foundation for matter as described for equations 6 and 7, leads very directly to mathematical structures of interest for the Standard Model of particle physics.

There is a standard way of rewriting the quadratic form of equation 3 over the four components $\boldsymbol{\delta x}_4 \equiv (\delta x^0, \delta x^1, \delta x^2, \delta x^3)$ as the determinant of a 2×2 complex Hermitian matrix over the components $\boldsymbol{\delta x}_4 \in \mathrm{h}_2\mathbb{C}$, that is with:

$$(\delta s)^2 = \det(\boldsymbol{\delta x}_4) = \det \begin{pmatrix} \delta x^0 + \delta x^3 & \delta x^1 - \delta x^2 i \\ \delta x^1 + \delta x^2 i & \delta x^0 - \delta x^3 \end{pmatrix} \tag{8}$$

The $\mathrm{SO}^+(1,3)$ Lorentz symmetry group is in turn replaced by its double cover $\mathrm{SL}(2,\mathbb{C})$, with the elements $S \in \mathrm{SL}(2,\mathbb{C})$ acting as Lorentz transformations $\boldsymbol{\delta x}_4 \to S\boldsymbol{\delta x}_4 S^\dagger$ leaving the proper time interval δs in equation 8 invariant.

The purpose here of writing equation 3 in the determinant form of equation 8 is to permit the direct augmentation from the 2×2 to the 3×3 matrix case, with the determinant now representing a possible *cubic* form for proper time as consistent with the generalisation of equation 4. That is, we can extend the form for proper time to a power $p = 3$, now over the $n = 9$ components of $\boldsymbol{\delta x}_9 \in \mathrm{h}_3\mathbb{C}$ and with a full $\hat{G} = \mathrm{SL}(3,\mathbb{C})$ symmetry, as can be written:

$$(\delta s)^3 = \det(\boldsymbol{\delta x}_9) = \det \left(\begin{array}{cc|c} \delta x^0 + \delta x^3 & \delta x^1 - \delta x^2 i & \delta x^4 + \delta x^5 i \\ \delta x^1 + \delta x^2 i & \delta x^0 - \delta x^3 & \delta x^6 + \delta x^7 i \\ \hline \delta x^4 - \delta x^5 i & \delta x^6 - \delta x^7 i & \delta x^8 \end{array} \right) \tag{9}$$

$$= \det \begin{pmatrix} \delta x^0 + \delta x^3 & \delta x^1 - \delta x^2 i \\ \delta x^1 + \delta x^2 i & \delta x^0 - \delta x^3 \end{pmatrix} \times \underbrace{\delta x_8 \quad + \quad (\boldsymbol{\delta x}_9)^3}_{\text{basis for matter}} \tag{10}$$

The straight line partitions in equation 9 highlight the embedding in the upper-left corner of the original 2×2 matrix of 4-dimensional spacetime components of equation 8, while equation 10 shows explicitly how this cubic expression can be written in the form of equation 6, here with $(\boldsymbol{\delta x}_9)^3$ denoting the residual terms given the extraction of the local form for 4-dimensional spacetime in the first term.

The corresponding symmetry breaking for this case of equation 7 is to the subgroup product structure $\hat{G} = \mathrm{SL}(3, \mathbb{C}) \supset \mathrm{SL}(2, \mathbb{C}) \times \mathrm{U}(1)$, where in addition to the external Lorentz $\mathrm{SL}(2, \mathbb{C})$ symmetry an internal $G = \mathrm{U}(1)$ symmetry, leaving the original four external $\boldsymbol{\delta x}_4$ components invariant, is identified. The residual 'matter' components of equations 9 and 10 include the external $\mathrm{SL}(2, \mathbb{C})$ Weyl spinor $\binom{\delta x^4 + \delta x^5 i}{\delta x^6 + \delta x^7 i} \in \mathbb{C}^2$ transforming as a charged state under the internal U(1) symmetry, together with the scalar component $\delta x_8 \in \mathbb{R}$ that is charge neutral. We note that the charged spin state already demonstrates how this approach of generalised proper time might be preferred to that of extra dimensions of space, given the known empirical properties of matter such as that of an elementary electron state.

The cubic form for proper time in equation 9 can also be augmented by generalising from the complex numbers $\mathbb{C}$ through the larger division algebras, via the 4-component quaternions $\mathbb{H}$ to the 8-component octonions $\mathbb{O}$. This leads to the 27-component 3×3 octonion Hermitian matrices $\mathrm{h}_3\mathbb{O}$ upon which, taking care for the non-associativity of the octonion algebra, a cubic determinant form can again be defined, now with a $\hat{G} = \mathrm{SL}(3, \mathbb{O}) \equiv \mathrm{E}_6$ symmetry [22, 23, 24]. These $\mathrm{h}_3\mathbb{O}$ elements can in turn be embedded within the 56-component space of the Freudenthal Triple System $F(\mathrm{h}_3\mathbb{O})$, upon which a (non-determinant) *quartic* form can be defined with a $\hat{G} = \mathrm{E}_7$ exceptional Lie group symmetry [25, 26, 27, 28]. These latter two augmentations to equation 3 consistent with the general form for proper time of equation 4, via equations 8 and 9, can be written respectively as:

$$(\delta s)^3 = \det(\boldsymbol{\delta x}_{27}) \quad \text{with} \quad \boldsymbol{\delta x}_{27} \in \mathrm{h}_3\mathbb{O} \tag{11}$$

$$(\delta s)^4 = q(\boldsymbol{\delta x}_{56}) \quad \text{with} \quad \boldsymbol{\delta x}_{56} \in F(\mathrm{h}_3\mathbb{O}) \tag{12}$$

Above we have listed these extensions with minimal detail in order to emphasise how *directly* the theory of generalised proper time leads to exceptional Lie group symmetries, specifically the real forms $\mathrm{E}_{6(-26)}$ and $\mathrm{E}_{7(-25)}$ for equations 11 and 12 respectively. It should also be noted that these constructions explicitly incorporate an octonionic description of the symmetry actions and the space of generalised proper time acted upon.

Connections between symmetry breaking properties of the exceptional Lie groups, particularly as the basis for an internal gauge symmetry in the context of a 'Grand Unified Theory' (GUT), with the Standard Model have been well-established since the 1970s [16, 17, 18], with parallel investigations also carried out involving properties of the division algebras [19, 20, 21], as noted in the previous section. Ongoing further developments continue to be of significant interest through to recent years (see for example [29, 30, 31, 32, 33, 34] for the exceptional Lie groups and [35, 36, 37, 38, 39, 40] for the octonions).

The direct access to such mathematical structures through equation 4, as described above for equations 11 and 12, suggests that the theory of generalised proper time, via the symmetry breaking mechanism of equations 6 and 7, might indeed be well-equipped to account for the esoteric features of the Standard Model. After describing this mathematical structure of the theory in more detail in the remainder of this section, in the following section we make this connection with particle physics more explicit.

For the analysis of the symmetry breaking structures it is convenient to avoid expressions with infinitesimal elements by introducing the generally finite components $v^a := \frac{\delta x^a}{\delta s}\big|_{\delta s \to 0}$ of an n-vector $\boldsymbol{v}_n \in \mathbb{R}^n$ and rewriting equation 4 for generalised proper time in the equivalent form:

$$L_p(\boldsymbol{v}_n)_{\hat{G}} \;:=\; \alpha_{abc\ldots}\frac{\delta x^a}{\delta s}\frac{\delta x^b}{\delta s}\frac{\delta x^c}{\delta s}\ldots\Big|_{\delta s\to 0} \;=\; \alpha_{abc\ldots}v^a v^b v^c\ldots \;=\; 1 \tag{13}$$

Here p is the homogeneous polynomial power of the general form for proper time and $\hat{G}$ is the full symmetry group acting upon the full set of n components $\{v^a\} \in \mathbb{R}^n$, with the latter to be associated with matter fields as identified in the symmetry breaking extraction of the local 4-dimensional spacetime subset of components. For example, equation 9 can then be written as:

$$L_3(\boldsymbol{v}_9)_{\mathrm{SL}(3,\mathbb{C})} = \det(\boldsymbol{v}_9) = \det\left(\begin{array}{cc|c} v^0+v^3 & v^1-v^2i & v^4+v^5i \\ v^1+v^2i & v^0-v^3 & v^6+v^7i \\ \hline v^4-v^5i & v^6-v^7i & v^8 \end{array}\right) = \det\left(\begin{array}{c|c} \boldsymbol{v}_4 & \psi \\ \hline \psi^\dagger & v^8 \end{array}\right) = 1 \tag{14}$$

with $\boldsymbol{v}_9 \in \mathrm{h}_3\mathbb{C}$, $\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$, $\psi \in \mathbb{C}^2$ and $v^8 \in \mathbb{R}$. The symmetry breaking decomposition described following equation 10, under the extraction of the external spacetime components $\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$, can in turn be written out as:

$$\hat{G} \;=\; \mathrm{SL}(3,\mathbb{C}) \;\;\to\;\; \mathrm{SL}(2,\mathbb{C}) \;\times\; \mathrm{U}(1) \tag{15}$$

$$\boldsymbol{v}_9 \to \left\{\begin{array}{lclcl} \boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C} & : & \text{vector} & 0 & : \text{tangent vector} \\ \psi \in \mathbb{C}^2 & : & L\text{-spinor} & 1 & : \text{matter field} \\ v^8 \in \mathbb{R} & : & \text{scalar} & 0 & : \text{matter field} \end{array}\right. \tag{16}$$

where the four components $(v^0, v^1, v^2, v^3) \equiv \boldsymbol{v}_4$ form a tangent vector in the local 4-dimensional spacetime. The 2-component Weyl spinor $\psi = \binom{v^4+v^5i}{v^6+v^7i}$ is here taken to be left-handed 'L' and assigned the normalised unit charge '1' by convention, while v^8 is the neutral scalar as also indicated in equation 16.

For the case of the 27-component cubic form for proper time of equation 11 with a full $\hat{G} = \mathrm{E}_6 \equiv \mathrm{SL}(3,\mathbb{O})$ symmetry we follow the explicit construction, as well as notation conventions, of [24] (chapters 3 and 4) and [41, 42, 43], with an element of the set of octonion Hermitian matrices $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ written as:

$$\mathcal{X} = \begin{pmatrix} p & \bar{a} & c \\ a & m & \bar{b} \\ \bar{c} & b & n \end{pmatrix} \equiv \left(\begin{array}{c|c} X & \theta \\ \hline \theta^\dagger & n \end{array}\right) \in \mathrm{h}_3\mathbb{O} \tag{17}$$

with components $a, b, c \in \mathbb{O}$ and $p, m, n \in \mathbb{R}$ in the first expression, corresponding to a total of 27 real components, and $X \in \mathrm{h}_2\mathbb{O}$, $\theta \in \mathbb{O}^2$ and again $n \in \mathbb{R}$ in the second. Octonion conjugation $a \to \bar{a}$ reverses the sign of all seven imaginary components for any $a \in \mathbb{O}$.

Utilising the determinant form that can be defined on $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$, taking into account the non-associativity of the octonion algebra ([23] section 3.4, [24] section 3.3), writing equation 11 in the form of equation 13, and identifying the corresponding components as $\boldsymbol{v}_{27} \equiv \mathcal{X} \in \mathrm{h}_3\mathbb{O}$ in accordance with equation 17, we then express this cubic form for proper time as:

$$L_3(\boldsymbol{v}_{27})_{\mathrm{E}_6} = \det(\boldsymbol{v}_{27}) = pmn - p|b|^2 - m|c|^2 - n|a|^2 + 2\mathrm{Re}(\bar{a}\bar{b}\bar{c}) = 1 \tag{18}$$

$$= \det(\mathcal{X}) = \det\left(\begin{array}{c|c} X & \theta \\ \hline \theta^\dagger & n \end{array}\right) = 1 \tag{19}$$

The local 4-dimensional spacetime components, corresponding to $\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$ in equation 14, are now embedded as four subcomponents of the 10-component element $X \in \mathrm{h}_2\mathbb{O}$ of equation 19, on selecting one preferred octonion imaginary unit as part of this external spacetime. While the original Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry was considered a subgroup of the full $\mathrm{SL}(3,\mathbb{C})$ action on the $\mathrm{h}_3\mathbb{C}$ matrices in equation 14, the $\mathrm{SL}(3,\mathbb{C})$ symmetry can in turn be augmented to $\mathrm{E}_6 \equiv \mathrm{SL}(3,\mathbb{O})$, for which we again follow the construction in [24, 41, 42, 43].

In fact this construction of the $\mathrm{SL}(3,\mathbb{O})$ symmetry builds upon the Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry via an initial augmentation to $\mathrm{SL}(2,\mathbb{O})$, with the 2×2 matrices $M \in \mathrm{SL}(2,\mathbb{O}) \equiv \mathrm{Spin}^+(1,9)$ acting as higher-dimensional Lorentz transformations on the '10-dimensional spacetime' described by $X \in \mathrm{h}_2\mathbb{O}$ [44]. With $X \in \mathrm{h}_2\mathbb{O}$ embedded within $\mathrm{h}_3\mathbb{O}$ in equations 17 and 19, this $\mathrm{SL}(2,\mathbb{O})$ action is in turn embedded in the upper-left components of 3×3 matrices $\mathcal{M}$ to obtain a corresponding action for the 3×3 case, as can be written $R(\alpha) : \mathcal{X} \to \mathcal{M}\mathcal{X}\mathcal{M}^\dagger$ where $\alpha \in \mathbb{R}$ is associated with any one of the 45 one-parameter continuous subgroup actions with ([24] section 4.1.1):

$$\mathcal{M}\mathcal{X}\mathcal{M}^\dagger = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} X & \theta \\ \hline \theta^\dagger & n \end{array}\right)\left(\begin{array}{c|c} M & 0 \\ \hline 0 & 1 \end{array}\right)^\dagger = \left(\begin{array}{c|c} MXM^\dagger & M\theta \\ \hline \theta^\dagger M^\dagger & n \end{array}\right) \tag{20}$$

As well as the 10-vector transformation $R(\alpha) : X \to MXM^\dagger$ the above expression also describes the spinor $R(\alpha) : \theta \to M\theta$ and scalar $R(\alpha) : n \to 1n$ representations of $\mathrm{SL}(2, \mathbb{O})$. The full set of 45 transformations composing $\mathrm{SL}(2, \mathbb{O})$ respect the 3×3 block structure, leave the determinant of equations 18 and 19 invariant, and include, again taking into account the non-associativity of the octonions, nested compositions of the form:

$$\mathcal{X} \;\to\; \mathcal{M}_n(\ldots(\mathcal{M}_1(\mathcal{X})\mathcal{M}_1^\dagger)\ldots)\mathcal{M}_n^\dagger \tag{21}$$

which act for example on the spinor components as $\theta \to M_n(\ldots(M_1(\theta)))$.

Under the action of the Lorentz subgroup $\mathrm{SL}(2, \mathbb{C}) \subset \mathrm{SL}(2, \mathbb{O})$, as embedded in equation 20, the components of $\theta = \binom{c}{b} \in \mathbb{O}^2$ decompose into a set of *four* Weyl spinors. This is a non-standard representation of the Lorentz group, given the octonionic construction, with the four spinors transforming in an identical manner with the *same* handedness and taken to compose a set of four L-spinors as a generalisation from equation 16.

The 45 elements of the Lie algebra for $\mathrm{SL}(2, \mathbb{O})$, as the generators each associated with a given group action $R(\alpha) \in \mathrm{SL}(2, \mathbb{O})$ as parametrised by $\alpha \in \mathbb{R}$ and described for equation 20, can be identified with vector fields on the tangent space to $\mathrm{h}_3\mathbb{O}$ as defined at any $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ by ([43] figure 1):

$$\dot{R} = \frac{\partial\, R(\alpha)(\mathcal{X})}{\partial \alpha}\Big|_{\alpha=0} \;\in\; T\mathrm{h}_3\mathbb{O} \tag{22}$$

With a permutation of three ways in which to embed a 2×2 matrix inside a 3×3 matrix action, there are in fact three natural ways in which to embed the $\mathrm{SL}(2, \mathbb{O})$ action, with equation 20 only depicted the first type of embedding. This means that a total of $45 \times 3 = 135$ determinant preserving actions can be constructed on the space $\mathrm{h}_3\mathbb{O}$, each associated with a generator through equation 22. Using the linear dependences between the corresponding 135 vector fields in $T\mathrm{h}_3\mathbb{O}$ a linearly independent basis of 78 actions can be identified which provides a full description of the exceptional Lie group $\mathrm{E}_6 \equiv \mathrm{SL}(3, \mathbb{O})$ ([43] section 3.2).

Employing a preferred basis for these E_6 group actions, and the associated Lie algebra elements ([24] table A.1 page 177), in the context of the theory of generalised proper time in augmenting the form for proper time from the $\hat{G} = \mathrm{SL}(3, \mathbb{C})$ case of equation 14 to the octonionic case of equation 19 a corresponding augmentation from the broken symmetry of equation 15 can be identified with the subgroup product structure $\hat{G} = \mathrm{E}_6 \equiv \mathrm{SL}(3, \mathbb{O}) \to \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SU}(3) \times \mathrm{U}(1)$. That is, through the symmetry breaking extraction of the local 4-dimensional spacetime components from equation 19, as described for equations 6 and 7, alongside the external Lorentz $\mathrm{SL}(2, \mathbb{C})$ symmetry we now obtain a larger internal symmetry $G = \mathrm{SU}(3) \times \mathrm{U}(1)$. Before describing the decomposition of the full set of 27 components of the space $h_3\mathbb{O}$ under the broken symmetry, as an extension from equation 16, we first describe the nature of the further augmentation to the 56-component quartic form for proper time of equation 12.

An element of the Freudenthal Triple System $x \in F(h_3\mathbb{O})$, incorporating two copies of the space $h_3\mathbb{O}$, is typically written in the form of a '2×2 matrix' as ([26] equation 9.31):

$$x = \begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix}, \qquad \text{with } \mathcal{X}, \mathcal{Y} \in h_3\mathbb{O}, \quad \alpha, \beta \in \mathbb{R} \tag{23}$$

A quartic norm $q(x)$ with an E_7 symmetry can be defined on these components which, considered as the homogeneous quartic form for generalised proper time of equation 12 as expressed in the form of equation 13, can be written as ([26] equation 9.34, [27] equation 47, [28] equation 18):

$$L_4(\boldsymbol{v}_{56})_{E_7} \;=\; q(x) \;=\; -2[\alpha\beta - (\mathcal{X},\mathcal{Y})]^2 \;-\; 8[\alpha\det(\mathcal{X}) + \beta\det(\mathcal{Y}) - (\mathcal{X}^\sharp, \mathcal{Y}^\sharp)] = 1 \tag{24}$$

Here the bilinear form $(\mathcal{X},\mathcal{Y})$ is the trace of the Jordan product of $\mathcal{X},\mathcal{Y} \in h_3\mathbb{O}$, while the definition of the quadratic adjoint map $\mathcal{X}^\sharp$ can also be found in [26, 27]. Similarly as for equation 18 the above quartic form can also be expanded and written out explicitly in terms of the octonion and real subcomponents ([26] equation 9.51).

As can be seen in equation 24, this quartic expression explicitly incorporates the cubic determinant form of equation 19. The cubic form for proper time of equations 18 and 19 is correspondingly subsumed into equation 24, with the $E_6 \equiv SL(3,\mathbb{O})$ action forming a subgroup of E_7 also leaving the proper time interval as expressed in the quartic form of equation 24 invariant. The actions $s \in E_6$ of this subgroup transform the elements of equation 23 with $x \to s(x) \in F(h_3\mathbb{O})$, leaving α and β invariant, in the manner (compare for example [26] equation 9.36c, [27] equation 55):

$$\begin{pmatrix} \alpha & \mathcal{X} \\ \mathcal{Y} & \beta \end{pmatrix} \to \begin{pmatrix} \alpha & s(\mathcal{X}) \\ s^{*-1}(\mathcal{Y}) & \beta \end{pmatrix} \tag{25}$$

This set of 78 actions for $E_{6(-26)}$ can be augmented by further transformations ([26] equations 9.36a–c, [27] equations 53–56) to construct the full set of 133 symmetry actions composing the non-compact real form $E_{7(-25)}$ of the exceptional Lie group E_7. The action $s^{*-1}(\mathcal{Y})$ in equation 25, also defined in [26, 27], is equivalent to the complex conjugate of the representation defined by the E_6 action $s(\mathcal{X})$ on the space $h_3\mathbb{O}$. Under the subgroup $E_6 \subset E_7$ the space $F(h_3\mathbb{O})$ correspondingly decomposes as the reducible representation ([26] equations 9.45 and 9.46):

$$\mathbf{56}_{E_7} \;\to\; (\mathbf{27} + \overline{\mathbf{27}} + \mathbf{1} + \mathbf{1})_{E_6} \tag{26}$$

Since complex conjugation interchanges left and right-handed representations of the Lorentz group, under the external Lorentz $SL(2,\mathbb{C}) \subset SL(2,\mathbb{O}) \subset E_6 \subset E_7$ action the set of four left-handed Weyl spinors identified for the $\theta \in \mathbb{O}^2$ components within $\mathcal{X}$ (corresponding to the '**27**') in equation 17, via equation 20, are hence accompanied by a set of four *right*-handed spinors from the $\theta' \in \mathbb{O}^2$ components within $\mathcal{Y}$ (the '$\overline{\mathbf{27}}$') in equation 23 under the symmetry breaking extraction of the local external spacetime components. Collectively these spin components hence form a set of four Dirac spinors $\psi = \binom{\psi_L}{\psi_R}$. The $\mathcal{X}$ and $\mathcal{Y}$ subspaces of equation 23 are then interpreted respectively as the 'left-handed' and 'right-handed' sectors in the context of the theory of generalised proper time.

The E_6 symmetry breaking described above, with an internal $G = \mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry, can be carried forward now for the augmented $\mathrm{E}_7 \supset \mathrm{E}_6$ case under the symmetry breaking extraction of the external spacetime 4-vector and Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry of equations 6 and 7. Applied for the 56-component quartic form for proper time of equation 24 the following symmetry breaking pattern, via the (X, θ, n) subcomponents of $\mathcal{X}$ in equation 17 and corresponding (Y, θ', n') subcomponents of $\mathcal{Y}$, is then identified:

$$\hat{G} = \mathrm{E}_7 \quad \Longrightarrow \quad \Longrightarrow \quad \Longrightarrow \quad \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SU}(3) \times \mathrm{U}(1) \tag{27}$$

$$\boldsymbol{v}_{56} \in F(\mathrm{h}_3\mathbb{O}) \to \left\{ \begin{array}{ll} \mathcal{X} \in \mathrm{h}_3\mathbb{O} : & \left\{ \begin{array}{ll} X \in \mathrm{h}_2\mathbb{O} & \left\{ \begin{array}{lcc} \text{vector} & \mathbf{1} & 0 \\ \text{scalar} & \mathbf{3} & \frac{2}{3} \end{array} \right. \\ \theta \in \mathbb{O}^2 & \left\{ \begin{array}{lcc} L\text{-spinor} & \mathbf{1} & 1 \\ L\text{-spinor} & \mathbf{3} & \frac{1}{3} \end{array} \right. \\ n \in \mathbb{R} & \left\{ \begin{array}{lcc} \text{scalar} & \mathbf{1} & 0 \end{array} \right. \end{array} \right. \\ \mathcal{Y} \in \mathrm{h}_3\mathbb{O} : & \left\{ \begin{array}{ll} Y \in \mathrm{h}_2\mathbb{O} & \left\{ \begin{array}{lcc} \text{vector} & \mathbf{1} & 0 \\ \text{scalar} & \mathbf{3} & \frac{2}{3} \end{array} \right. \\ \theta' \in \mathbb{O}^2 & \left\{ \begin{array}{lcc} R\text{-spinor} & \mathbf{1} & 1 \\ R\text{-spinor} & \mathbf{3} & \frac{1}{3} \end{array} \right. \\ n' \in \mathbb{R} & \left\{ \begin{array}{lcc} \text{scalar} & \mathbf{1} & 0 \end{array} \right. \end{array} \right. \\ \alpha, \beta \in \mathbb{R} : & \qquad\qquad\quad \begin{array}{lcc} \text{scalar} & \mathbf{1} & 0 \end{array} \end{array} \right. \tag{28}$$

The above symmetry breaking pattern has been derived explicitly as a mathematical structure. In addition to the vector, spinor and scalar representations under the external Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry, under the internal symmetry there are $\mathrm{SU}(3)$ singlets **1** and triplets **3** as well as a $\mathrm{U}(1)$ fractional charge structure, as listed by relative magnitude in being read off from the corresponding tangent vector components in $T\mathrm{h}_3\mathbb{O}$ as constructed in equation 22. The direct identification of these features, as an extension from the $\hat{G} = \mathrm{SL}(3,\mathbb{C})$ symmetry breaking case described after equation 10 and in equations 15–16, is an encouraging further development. In the following section we fill out the physical interpretation in the context of the Standard Model of particle physics.

3 Interpretation of Matter States for Particle Physics

A basis for the elementary structure of matter will be identified and physically interpreted in this section for the $\hat{G} = \mathrm{E}_6$ and $\hat{G} = \mathrm{E}_7$ levels of generalised proper time through the symmetry breaking of equations 6 and 7 as initially described for equations 27–28. The symmetry breaking pattern for the intermediate $\mathrm{E}_6 \subset \mathrm{E}_7$ cubic form of equation 19 is accommodated in the $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ components of equations 23 and 24. In turn, the extraction of the external 4-dimensional spacetime Lorentz 4-vector components from the $X \in \mathrm{h}_2\mathbb{O}$ subcomponents of $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ in equations 17 and 19 for the E_6 case could now be embedded in the top line of equation 28 for the E_7 case.

However, for the full E_7 case a further Lorentz 4-vector is identified as embedded within the corresponding $Y \in \mathrm{h}_2\mathbb{O}$ subcomponents of $\mathcal{Y} \in \mathrm{h}_3\mathbb{O}$ as also listed in equation 28. Given that there is only *one* external 4-dimensional spacetime manifold, only *one* of these sets of four real subcomponents is extracted to identify the local spacetime components. We then adopt the arbitrary choice with the four external spacetime tangent components $\boldsymbol{v}_4 (\equiv \boldsymbol{\delta x}_4/\delta s) \in \mathrm{h}_2\mathbb{C}$ taken from the $\mathcal{Y} \in \mathrm{h}_3\mathbb{O}$ subspace, that is with:

$$\begin{array}{rccccccc} \text{external } \boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C} & \subset & Y \in \mathrm{h}_2\mathbb{O} & \subset & \mathcal{Y} \in \mathrm{h}_3\mathbb{O} & \subset & \boldsymbol{v}_{56} \in F(\mathrm{h}_3\mathbb{O}) \\ \text{and Lorentz } \mathrm{SL}(2,\mathbb{C}) & \subset & \mathrm{SL}(2,\mathbb{O}) & \subset & \mathrm{SL}(3,\mathbb{O}) & \subset & \hat{G} = \mathrm{E}_7 \end{array} \quad (29)$$

Given that the $\mathcal{X}$ and $\mathcal{Y}$ components have been identified respectively with the left-handed and right-handed sectors of the theory for this E_7 symmetry breaking, as explained after equation 26, this necessary extraction of the four external spacetime components from one side alone in equation 29 *breaks* the left-right symmetry for the resulting physics. Such a left-right asymmetry is indeed a significant empirical property of particle physics observed in nature, as is generally put in by hand in theoretical descriptions as for the Standard Model itself.

As the basis for a Grand Unified Theory, the unification groups E_7 and E_8 are typically disfavoured since, unlike the case for E_6, they do not have complex representations and cannot directly accommodate such a left-right asymmetry in the interactions of the spinor states (that is, the chiral fermions of the Standard Model). However, the theory of generalised proper time is quite different to the GUT approach in that the external Lorentz symmetry, alongside the internal symmetry G, is *also* extracted from the unifying group $\hat{G}$ in equation 7 (as consistent with the Coleman-Mandula theorem as explained after equation 7). Here it is the left-right asymmetric embedding of the corresponding local external spacetime basis itself, extracted from equation 6 as the central symmetry breaking mechanism, in equation 29 that implies different properties for the left and right-handed sectors as an intrinsic property of the theory.

It can be seen from equations 27–28 that this direct symmetry breaking pattern in the decomposition of the 56-component form is rich in Standard Model features more generally. As already noted the L-spinors and R-spinors under the Lorentz symmetry collectively describe a set of four Dirac spinors $\psi = \binom{\psi_L}{\psi_R}$. Unlike the external symmetry, the compact subgroups of the internal $\mathrm{SU}(3) \times \mathrm{U}(1)$ symmetry in equation 27 act in exactly the same way on the $\mathcal{X}$ and $\mathcal{Y}$ subcomponents in equation 28. The singlet **1** and triplet **3** representations under the non-Abelian gauge symmetry are reminiscent of the properties of leptons and quarks under the colour gauge group $\mathrm{SU}(3)_c$ in the Standard Model. The fractional charges in the ratio $1 : \frac{2}{3} : \frac{1}{3} : 0$, written in terms of their relative magnitude in being read off from the corresponding generator element in the Lie algebra defined on the tangent space $T\mathrm{h}_3\mathbb{O}$, in turn match the electromagnetic $\mathrm{U}(1)_Q$ charges for lepton and quark states.

The above properties, together with elements of electroweak symmetry breaking including the above left-right asymmetry and as discussed further below, are sufficient to make the assignments in connection with a generation of Standard Model states listed in table 1. Similarly as for equations 27–28 this table again summarises the results of the symmetry breaking extraction of the local 4-dimensional spacetime basis in equations 6 and 7 for the specific quartic form of equation 24 with an E_7 symmetry, now with this provisional interpretation of the elementary matter states.

$56 \diagdown \mathrm{E}_7 \supset$	Lorentz	$\times$ $\mathrm{SU}(3)_c$	$\times$ $\mathrm{U}(1)_Q$	matter
$\begin{pmatrix} 4 \\ 4 \end{pmatrix}$ (upper)	<u>vector</u>	$\mathbf{1}$	0	'ν_L'
$\rightarrow$ (lower)	vector	$\mathbf{1}$	0	$\boldsymbol{v}_4$ 'Higgs'
$\rightarrow \begin{pmatrix} 4 \\ 4 \end{pmatrix}$	Dirac	$\mathbf{1}$	1	$\begin{pmatrix} e_L \\ e_R \end{pmatrix}$
$\rightarrow \begin{pmatrix} 6 \\ 6 \end{pmatrix}$	<u>scalar</u>	$\mathbf{3}$	$\frac{2}{3}$	$\begin{pmatrix} `u_L\text{'} \\ `u_R\text{'} \end{pmatrix}$
$\rightarrow \begin{pmatrix} 12 \\ 12 \end{pmatrix}$	Dirac	$\mathbf{3}$	$\frac{1}{3}$	$\begin{pmatrix} d_L \\ d_R \end{pmatrix}$
$\rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	scalar	$\mathbf{1}$	0	Yukawa coupling
2	scalar	$\mathbf{1}$	0	

Table 1: Symmetry breaking structure for the 56 components of the E_7 quartic form for generalised proper time of equation 24, under the product of the external Lorentz $\mathrm{SL}(2,\mathbb{C})$ symmetry and the internal gauge symmetry $G = \mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ for this case of equations 6 and 7. The arrows indicate the substructure fragmentation for the 27 components of the intermediate E_6 cubic form of equation 19, now associated with the R-handed $\mathcal{Y} \in \mathrm{h}_3\mathbb{O}$ sector in equations 27–28 incorporating the external spacetime 4-vector $\boldsymbol{v}_4$ components of equation 29, rather than the L-handed $\mathcal{X} \in \mathrm{h}_3\mathbb{O}$ sector which in turn is free to accommodate the 'ν_L' components. The provisional connection between the full set of representations and the multiplet structure of matter states in the Standard Model is indicated in the final column.

In table 1 identification of the four Dirac spinors with the electron $e = \binom{e_L}{e_R}$ and down-quark $d = \binom{d_L}{d_R}$ states is direct, given their transformation properties under the internal $\mathrm{SU}(3)_c \times \mathrm{U}(1)_Q$ symmetry. This same internal symmetry provisionally identifies the neutrino 'ν_L' and up-quark $u =$ '$\binom{u_L}{u_R}$' states, which are, however, listed in quote marks due to the incorrect non-spinor representations under the Lorentz symmetry, as also underlined in the table. We note that while such a 'ν_L' state can be accommodated in the left-handed sector of the theory, *there is no room* for a 'ν_R' state in the right-handed sector since the corresponding components are identified with the 4-vector $\boldsymbol{v}_4$ projected onto the local external 4-dimensional spacetime in equation 29. The above lepton and quark assignments are further suggested by the identification of elements of electroweak theory as outlined in the following (and detailed in [45, 46]).

The broken symmetry of equation 27 and the top of table 1 contains an $\mathrm{SU}(2) \times \mathrm{U}(1)_Q$ subgroup, where the $\mathrm{SU}(2) \equiv \mathrm{SO}(3)$ corresponds to the compact rotation subgroup of the external Lorentz symmetry. This broken symmetry and $\mathrm{SU}(2) \times \mathrm{U}(1)_Q$ subgroup is aligned with the original $\mathrm{SL}(2, \mathbb{O})$ embedding in the upper-left components of equation 20. As we noted after equation 22 there are two other possible similar embeddings, each giving rise to a corresponding permuted copy of this subgroup symmetry, each here denoted $\mathrm{SU}(2)' \times \mathrm{U}(1)'$, found to exhibit several features in common with the electroweak $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ symmetry of the Standard Model (with 'L' for left-handed and 'Y' for hypercharge).

Reminiscent of the Standard Model $\mathrm{SU}(2)_L$ symmetry, the above $\mathrm{SU}(2)'$ symmetries mix components within both the $\binom{\nu}{e}$ leptonic and $\binom{u}{d}$ quark sectors of table 1. Through impingement upon the external 4-vector $\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$ components in table 1 this 'mock electroweak symmetry' is broken with only the original $\mathrm{U}(1)_Q \subset \mathrm{SU}(2)' \times \mathrm{U}(1)'$ leaving the external spacetime components invariant. In fact such an $\mathrm{SU}(2)' \times \mathrm{U}(1)'$ is not independent of the external Lorentz $\mathrm{SL}(2, \mathbb{C})$ symmetry itself in table 1, but it *does* commute with the internal $\mathrm{SU}(3)_c$ symmetry, and indeed such a subgroup decomposition $\mathrm{SU}(3)_c \times \mathrm{SU}(2)' \times \mathrm{U}(1)' \subset \mathrm{E}_6 \subset \mathrm{E}_7$ is not unlike the identification of the full Standard Model gauge group in GUT models.

The local external 4-dimensional spacetime 4-vector $\boldsymbol{v}_4$ in table 1 is associated with a non-standard 'Higgs' since these components are not only central to the overall symmetry breaking mechanism of equations 6 and 7 but also to the provisional 'mock electroweak symmetry breaking' described above. The four real components $(v^0, v^1, v^2, v^3) \equiv \boldsymbol{v}_4$, here with $\boldsymbol{v}_4 \in \mathrm{h}_2\mathbb{C}$ in building upon equations 14–16, can be combined in a single Lorentz invariant scalar norm $|\boldsymbol{v}_4|$ defining:

$$h^2 := |\boldsymbol{v}_4|^2 = \det(\boldsymbol{v}_4) = \eta_{ab} v^a v^b = \frac{\eta_{ab} \delta x^a \delta x^b}{(\delta s)^2} \in \mathbb{R} \tag{30}$$

where, as a function of the external spacetime location, $h(x)$ is provisionally associated with the scalar Higgs field of the Standard Model. While taking a background 'vacuum value', this relative magnitude $h(x)$ of the projected 4-dimensional spacetime fragment of generalised proper time can locally fluctuate under the overall unit normalisation of the full form of equation 13, as for example for equations 14, 19 or 24.

Under the symmetry breaking of equations 6 and 7 the corresponding fragmentation of equation 13 can be expanded into a series of Lorentz and gauge invariant terms. For higher-order forms for generalised proper time terms containing 'matter' components, such as the leptons and quarks of table 1, as well as the $\boldsymbol{v}_4$ 'Higgs' components resemble the 'mass terms' of the Standard Model Lagrangian. Additional scalar components in such terms in turn resemble the 'Yukawa coupling' factors that determine the relative mass of the states, as also indicated in table 1.

Further, the symmetry breaking impingement of the above 'mock electroweak symmetry' $SU(2)' \times U(1)'$ upon the external $\boldsymbol{v}_4$ 'Higgs' components provides a prototype for the origin of the mass for the $W^{\pm}$ and Z^0 gauge bosons of the Standard Model, while the photon associated with the $U(1)_Q$ subgroup leaving the $\boldsymbol{v}_4$ components invariant remains massless. These connections with the role of the Standard Model Lagrangian, now played by the constraints of equation 13, in part motivating the notation 'L' in that central expression for generalised proper time, will be discussed further for equation 38b in section 6.

However, there is a further significant reason for associating the external spacetime components $\boldsymbol{v}_4$ with the Higgs and the origin of mass in table 1. The above equation 30 can be trivially rearranged as:

$$(\delta s)^2 = \frac{\eta_{ab}\delta x^a \delta x^b}{h^2(x)} \tag{31}$$

Interpreting the local coordinates $\{x^a\}$ as a particular patch of general coordinates, by comparison with equation 2 this in turn implies that in the 4-dimensional spacetime $h(x)$ acts as a local conformal scaling factor with an effective local metric function:

$$g_{ab}(x) = \frac{\eta_{ab}}{h^2(x)} \tag{32}$$

Hence any interaction between matter or gauge fields and the external $\boldsymbol{v}_4$ 'Higgs' components, implying variation in $h(x)$, will generate local time dilation effects via equation 31 and an associated warping of the local 4-dimensional spacetime geometry described by equation 32. Through the Einstein equation 5 of general relativity, with the Einstein tensor $G^{\mu\nu}(x)$ in turn a function of the metric $g_{\mu\nu}(x)$, this warping of spacetime will then intrinsically accompany the apparent energy-momentum $T^{\mu\nu}(x)$ of particle states associated with the matter or gauge fields.

Significantly, these observations then directly connect the 'origin of mass' as attributed to the Higgs in the Standard Model with the 'presence of mass' as manifested by the curvature of the spacetime geometry in general relativity. A typical interpretation of general relativity involves the reading of the Einstein equation 5 from right-to-left as meaning 'matter curves spacetime'. However, here states that interact with the $\boldsymbol{v}_4$ 'Higgs' components in fact have mass *since* they perturb the spacetime geometry, providing a source of Ricci curvature as an *intrinsic property of matter*, reading equation 5 more from left-to-right as will be further explained for equation 39 in section 6.

While for the theory of generalised proper time it is then possible to step beyond the Standard Model, through this connection with the general relativistic theory of gravity, there remain clear significant discrepancies with a generation of leptons and quarks of the Standard Model itself in table 1 to be addressed. We first note, however, that in *nature* there is no such thing as a pristine 'single generation of the Standard Model' in isolation from the states of the other two generations. In particular there is a significant mismatch between weak interaction and mass eigenstates, in both the lepton and quark sectors, as associated with mixing between the full three generations of the Standard Model, which is correspondingly 'one thing'. If there does exist a larger, more complete, form for generalised proper time that can accommodate the full Standard Model structure, then there is no reason to expect the extraction of an intermediate partial structure, such as for table 1, to be precisely aligned with a single generation, and indeed as we have just noted that should not even be expected to be possible.

In seeking an augmentation from the E_6 cubic and E_7 quartic forms for proper time underlying table 1 the main further properties of Standard Model required are the correct spin representations for the ν-lepton and u-quark states, a full electroweak theory and indeed a full three generations. *All* of these features are correlated in particular by the need to identify a realistic weak $SU(2)_L$ symmetry, acting upon doublets of lepton and quark L-handed spin states and across the three generations. The interrelation between these properties also suggests that only one further level of augmentation may be needed.

Above we identified both an $SL(2,\mathbb{C}) \times SU(3)_c \times U(1)_Q \subset E_6 \subset E_7$ symmetry in equation 27 and table 1, in the form of equation 7 incorporating the external Lorentz and partial Standard Model gauge symmetry, and also a subgroup decomposition $SU(3)_c \times SU(2)' \times U(1)' \subset E_6 \subset E_7$ before equation 30 as a provisional GUT-like symmetry purely for the full Standard Model gauge group. However, the ambition of course is to obtain the external *and* full internal Standard Model groups *at the same time* in the context of a symmetry breaking for a generalised form for proper time with a larger symmetry. With the known subgroup sequence $E_6 \subset E_7 \subset E_8$ pointing uniquely to the largest exceptional Lie group E_8, this suggests seeking a form for generalised proper time with such a $\hat{G} = E_8$ symmetry and with a symmetry breaking pattern subsuming table 1 able to complete the Standard Model picture. We explore this possibility in the following section.

In the meantime, in this section we have demonstrated, through table 1 and the above discussion, how a series of significant elements of the Standard Model *can* be recovered via the approach of generalised proper time, far more directly than via the employment of extra dimensions of space, and with very little in the way of redundant features.

4 Towards the full Standard Model and Beyond

The incompleteness of table 1, with respect to the structure of the Standard Model of particle physics, points to the need for a yet further augmentation to the general form for proper time of equation 4 beyond the E_6 and E_7 symmetry levels and hence likely to involve a $\hat{G} = E_8$ symmetry as the largest exceptional Lie group. In particular, as a further augmentation beyond equations 11 and 12, with their respective $E_{6(-26)}$ and $E_{7(-25)}$ symmetries, we might seek an action of the real form $E_{8(-24)}$ upon an octonionic construction of the ultimate form for generalised proper time that can be provisionally written as:

$$(\delta s)^8 = Q(\boldsymbol{\delta x}_{248}) \quad \text{with} \quad \boldsymbol{\delta x}_{248} \in \mathcal{T}(\mathbb{O}) \tag{33}$$

Here, while the notation 'Q' is to suggest a form of at least *quintic* order, in extending from the quartic form 'q' of equation 12, the existence of on E_8 octic invariant [47, 48] hints at the possibility of a $p = 8$ form for equation 4. The space $\mathcal{T}(\mathbb{O})$

generically represents an octonionic construction of this ultimate form for generalised proper time, supplanting the space $F(\mathrm{h}_3\mathbb{O})$ of equation 12 in some manner. While the smallest non-trivial representations for E_6 and E_7 are 27 and 56-dimensional respectively, corresponding to the size of the spaces in equations 11 and 12, the space $\mathcal{T}(\mathbb{O})$ is here presumed to be 248-dimensional over the reals, corresponding to the smallest non-trivial representation for E_8, as also indicated in equation 33.

In fact around 200 components would be sufficient to accommodate three generations of Standard Model leptons and quarks together with the Higgs (the exact number depending on the nature of any right-handed neutrino sector as well as the Higgs itself). For the theory of generalised proper time the component degrees of freedom of the gauge boson states are *not* to be found in the $\boldsymbol{\delta x}_{248} \in \mathcal{T}(\mathbb{O})$ components of equation 33, through equation 6, but rather they will be associated with the internal symmetry G resulting from the breaking of the E_8 symmetry in equation 7. The ambition would then be to determine the corresponding symmetry breaking pattern, in projecting the local 4-dimensional spacetime components $\boldsymbol{\delta x}_4$ of equation 6 out of the full expression for generalised proper time of equation 33 with:

$$\mathrm{E}_8 \;\supset\; \mathrm{Lorentz} \times \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \tag{34}$$

yielding the internal symmetry G of the Standard Model gauge group structure in equation 7. In addition to this internal $G = \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ the theory should account in turn for the electroweak symmetry breaking $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$, over the external $\boldsymbol{v}_4 = \boldsymbol{\delta x}_4/\delta s$ 'Higgs' components while supplanting the 'mock electroweak theory' described after table 1 in the previous section, as consistent with the corresponding Standard Model phenomenology.

While there is a well-known branching for the 56-dimensional representation of E_7 under an E_6 subgroup as described in equation 26, E_8 also exhibits a branching pattern with an E_6 subgroup component of the form (see also [49] figure 5 and equation 2.26):

$$\mathrm{E}_8 \supset \mathrm{E}_6 \times \mathrm{SU}(3): \quad \mathbf{248} \;\to\; (\mathbf{27},\mathbf{3}) + (\overline{\mathbf{27}},\overline{\mathbf{3}}) + (\mathbf{78},\mathbf{1}) + (\mathbf{1},\mathbf{8}) \tag{35}$$

Hence in the sense that the E_7 level for generalised proper time of table 1 approximately accommodates one generation of Standard Model states with left and right-handed components, the three-fold nature of the 27-dimensional representations in equation 35, in comparison with equation 26, suggests that E_8 might indeed come close to accommodating a full three-generation picture, albeit not exactly in the form of the above decomposition. (See also [50, 51] for alternative embeddings of E_6 and E_7 in E_8 explicitly involving the octonions).

A number of E_8 models for unification are already under investigation, including for example [31, 32, 33, 34] as cited in section 2. We note, however, that direct analyses of E_8 symmetry breaking, in the form of equation 34 applied for the 248-dimensional representation, employing standard Lie group representation theory, does *not* yield the full set of Standard Model states, although in some respects a close connection can be identified. This is demonstrated for example in [52] in which an $\mathrm{E}_8 \supset \mathrm{F}_4 \times \mathrm{G}_2$ and $\mathrm{F}_4 \supset \mathrm{SO}(8)$ subgroup structure is analysed with respect to the 'complexified' Lie algebra. The SO(8) contains the external Lorentz $\mathrm{SU}(2) \times \mathrm{SU}(2)$

and internal $\mathrm{SU}(2)_L$ symmetries with a 3-way permutation in the representation structure across three generations related by SO(8) triality maps, in a manner that is only closely in accord with the Standard Model for the first generation. In general terms it can be shown that this problem cannot be fixed in standard Lie algebra representation theory since no real form of E_8 contains a sufficient number of non-compact generators to accommodate three generations of Standard Model spin states [53], although the assumptions of the proof do not prohibit all E_8 models [31, 33].

For our present considerations it is important to note that here we consider the *notion of symmetry* to be of a more elementary and fundamental significance than the *axioms of group theory* and the corresponding standard mathematical theory of Lie groups and their representations. In particular, group theory employs a standard axiom of *associativity* in the composition of group elements, as typically described by matrices with complex $\mathbb{C}$ entries, which serves a practical purpose in facilitating the classification of groups and their representation structure. However, octonion $\mathbb{O}$ composition forms a *non-associative* algebra, which nevertheless still expresses various *symmetry* transformations.

This is the case for the construction of the symmetries acting upon homogeneous polynomial forms for generalised proper time as seen for the E_6 action on the space $\mathrm{h}_3\mathbb{O}$ in equation 11, the E_7 action on the space $F(\mathrm{h}_3\mathbb{O})$ in equation 12, and as sought for the E_8 action on the space of elements $\boldsymbol{\delta x}_{248} \in \mathcal{T}(\mathbb{O})$ leaving $Q(\boldsymbol{\delta x}_{248})$ invariant in equation 33. Indeed, for the $\mathrm{E}_6 \equiv \mathrm{SL}(3,\mathbb{O})$ symmetry of equation 11, rewritten in the form of equation 19 and using the transformations of equation 20, the non-associativity of the octonions and the bracket structure of equation 21 is actually key to obtaining a complete description of the symmetry.

As discussed for equations 20–22 such an octonionic construction can then result in the description of a full Lie group, recovering the full Lie algebra commutation relations as isomorphic with the standard structure, as is the case for the $\mathrm{E}_6 \equiv \mathrm{SL}(3,\mathbb{O})$ construction (see the 'Multiplication Table' in [24]). However, as also noted after equation 21 in relation to the Weyl spinor structure, the use of an octonionic representation space can yield a *non-standard* representation structure under a symmetry action, and as already features significantly for the symmetry breaking pattern in table 1. While a standard Lie algebra analysis can indeed serve as an invaluable guide, for symmetry constructions incorporating the octonions with this possibility for non-standard representations, an *explicit* detailed construction of the symmetry breaking and transformation properties of the decomposed representation space components is required. This was the case for the analysis of the E_6 and E_7 levels of table 1 and will also be needed for the further extension to an E_8 symmetry.

A potentially particularly relevant property of the octonions is a 3-way triality that permutes left-handed spinor, right-handed spinor and vector representations, as a new feature not present in the standard analysis with complex components and as closely related to the property of SO(8) triality. In the octonionic construction of the space $\mathcal{T}(\mathbb{O})$ in equation 33 such an octonion triality could play a role in untangling the three SO(8)-triality permuted generations in [52], discussed after equation 35 above, and itself go some way to addressing the accommodation of the full Standard Model under an E_8-type symmetry.

Moreover, further input, not only concerning octonion triality and spinors [23, 36, 42, 54], but also for example the 'magic square of Lie algebras' ([55] for example section 3 tables, [56, 57]), the 'extended Freudenthal Triple System' ([27] section 3.5, [58]), as well as existing schemes for Standard Model unification utilising the exceptional Lie groups and/or octonions such as cited in this paper, may also assist in deducing an explicit construction for equation 33. In addition to the mathematical constraint of this homogeneous polynomial form, the key input here, absent in most other unification schemes employing E_8 and/or $\mathbb{O}$ structures, is a firm *conceptual basis* in the theory of generalised proper time.

Since it is not obvious that such a form for generalised proper time at this E_8 level should exist, with the various mathematical pieces coming together to accommodate the full set of Standard Model states under the symmetry breaking pattern of equation 34, this amounts to a *non-trivial mathematical prediction* of the theory. In the meantime it is already possible to anticipate the nature of the empirical consequences and predictions that might in turn be obtained in extrapolating from table 1, as we consider in the remainder of this section.

For example, while the construction and analysis of the full E_8 level may be needed for explicit predictions of new physics beyond the Standard Model, as described for table 1 at the E_7 level there is already a basis for a left-right asymmetry in matter states, as particularly marked in the neutrino sector. As associated with the corresponding vector components in equations 27–28, in table 1 a single 'ν_L' state in the left-handed sector is correlated with the $\boldsymbol{v}_4$ 'Higgs' components in the opposing right-handed sector of the theory. This suggests that, extrapolating from table 1, in a complete three-generation picture under E_8 the Higgs may effectively be composed from right-handed neutrino spin state degrees of freedom.

That is, under the external $SL(2, \mathbb{C}) \subset E_8$ symmetry these would-be ν_R spin state components might combine together in one way to form the local external spacetime 4-vector $\boldsymbol{v}_4$, if such a 4-vector cannot be identified directly as it is for the E_7 case in table 1 with $\boldsymbol{v}_4 \in h_2\mathbb{C}$, and at the same time in a further way to obtain a scalar Higgs field, similarly as proposed for equation 30. Rather than being held together by for example a new gauge interaction, it is the mathematical structure of the theory and *the necessity of* identifying the external 4-vector $\boldsymbol{v}_4$ in the local 4-dimensional spacetime that intrinsically fuses these spin components together. However, similarly as for existing models with a composite Higgs built from ν_R states [59, 60], this can have empirical implications for both Higgs and neutrino physics.

A non-standard composite Higgs can be probed through its couplings with fermions, that can differ from the Standard Model expectation by of order 10%, at the Large Hadron Collider [61] or at a future electron-positron collider [62]. The asymmetric embedding of the Higgs in the neutrino sector also suggests there may remain no more than two physical ν_R states, opposite the three physical ν_L states in the full three-generation picture, with a significant mass imbalance as reminiscent of neutrino 'seesaw' models ([63] section 2 and references therein). Existing models with two heavy ν_R states, evading direct detection owing to their large mass and minimal interactions, can also account for the observed solar and atmospheric left-handed neutrino oscillation phenomena ([64], [65] section 14) and play a pivotal role in the origin of the matter-antimatter asymmetry through the decay of heavy ν_R states in the early universe [66, 67].

In such models the lightest ν_L state is typically massless. This can be probed in the laboratory directly via tritium β-decay, with a current limit of $m_\nu < 0.45\,\mathrm{eV}$ [68], or in neutrinoless double β-decay experiments ([63] section 3.3, [65] section 14.9). A limit can also be set indirectly through cosmological observations, with a tight constraint of $\sum m_\nu < 0.0642\,\mathrm{eV}$ determined for the three ν_L states assuming the ΛCDM model [69, 70] (with Λ the cosmological constant and CDM cold dark matter). This cosmological limit strongly disfavours the neutrino inverted mass hierarchy scenario, while for the normal mass ordering it implies a lightest neutrino mass limit of $m_l < 0.023\,\mathrm{eV}$ albeit at the expense of a moderate tension with the neutrino oscillation data. This tension can be resolved by revising the ΛCDM model for example with evolving dark energy, which is itself indeed suggested by the cosmological observations as will be discussed towards the end of section 7.

In augmenting beyond the E_7 level of table 1 there is the possibility of not only recovering the full Standard Model internal gauge group but also identifying new gauge interactions between matter states, with potential consequence for particle physics experiments. Given the rank-8 size of the exceptional Lie group E_8, there is in fact room for an additional $\mathrm{SU}(2) \times \mathrm{U}(1)$ factor over and above the Standard Model gauge symmetry in equation 34, but the details will depend on the explicit nature of the symmetry breaking structure of equations 6 and 7 for the ultimate form of generalised proper time in equation 33.

We also noted in the discussion following equation 33 that only around 200 of the 248 components are actually needed to accommodate the Standard Model leptons and quarks together with a non-standard Higgs. While, as an extension from table 1, it is anticipated that combinations of the remaining 50 or so components may play a role analogous to Yukawa couplings in the Standard Model, as suggested after equation 30, these extra components could also potentially underlie new matter states beyond the Standard Model, with new particle types that might have implications in the laboratory or for cosmology. While this could include a contribution to 'dark matter', the theory of generalised proper time in fact accommodates a whole 'dark sector' as we shall explain in section 7.

The above possible areas for new physics beyond the Standard Model are both well-defined and relatively restricted. There is for example no prediction of 'mirror' particles accompanying all the Standard Model states or an extended array of 'supersymmetric' partners. Here the new physics is not posited in the form of a provisional phenomenological model, but rather is tightly constrained in deriving directly from the unifying theory of generalised proper time with the robust, simple and unique conceptual foundation described in section 1. Further, the theory has already achieved success in terms of the non-trivial inroads made into accounting for features of the Standard Model at the E_6 and E_7 levels of the theory described for table 1 in the previous section.

The *first prediction* of theory has more of a technical mathematical rather than empirical flavour, in the construction of an ultimate E_8 level consistent with the generalised form proper time of equation 4, based upon an octonionic construction and as provisionally introduced in equation 33 in this section, capable of accommodating the full Standard Model multiplet structure through the symmetry breaking extraction of the local 4-dimensional spacetime components in equations 6 and 7. This initial technical prediction will likely need to be verified before empirical predictions for physics beyond the Standard Model, for example in the areas outlined in this section, can be made in detail. (For further discussion and background on these questions see also [71, 72, 73] and references therein).

The simplicity of the theory, as emphasised for example in the discussion of figure 1, together with its general suitability for explaining properties of the Standard Model of particle physics, raises the question concerning why generalised proper time is not already widely employed as an alternative to the paradigm of extra spatial dimensions. We address some of the possible factors behind this, and in particular the apparent implausibility of making 'time' the sole fundamental entity underlying unification as it is for the theory of generalised proper time, in the following section.

5 Constructing a World from Elements of Time

With the theory of generalised proper time hinging upon a simple idea closely related to the approach of models with extra dimensions of space, with their century-long history, and given that the theory is found to be well-suited to the Standard Model, which was established fifty years ago, in this section we consider some of the factors that may account for why this new approach has so far been overlooked. A possible concern for the theory may be the starting point at the very local level of structure as depicted in figure 1 with, moreover, an apparent basis in 'time' alone as the basic entity of the theory perhaps seeming implausibly simple as the foundation for a complete unification scheme.

That is, whereas model with *extra* spatial dimensions set out by *adding* something to the 4-dimensional base manifold, as might conceivably relate to the matter content of spacetime, here we begin with something *less* than 4-dimensional spacetime. Not only matter, but also the extended external spacetime 'out there' is required to come from time alone as the continuous progression of equation 1. While in the previous sections we took the general form for a *local* proper time interval of equation 4, and considered the local symmetry breaking structure of equations 6 and 7 culminating in table 1, here we then address the question of the construction of an extended *global* 4-dimensional spacetime world for the theory.

A possible barrier to the adoption of the new perspective is the common representation of the flow of 'time' in equation 1 as a one-dimensional line, either drawn *in* space or depicted as a worldline *in* a spacetime diagram. However, such a representation does not capture what time *is*, in that *nothing* we know of in the physical world corresponds to such a vanishingly thin 'line' in space or spacetime. Instead our analysis should begin with a more abstract *mathematical*, rather than pictorially *geometric*, notion of time taken as the real continuum $s \in \mathbb{R}$ of equation 1. This equation of itself, purely representing the continuum of time, says *nothing* explicit about *any* relation to spatial extension or any other spatial properties. In particular, equation 1 clearly does *not* imply a 'line in space'.

At the most elementary level the real numbers $\mathbb{R}$ have the basic arithmetic properties not only of *additive* $\pm$-type but also *multiplicative* $\times\div$-type operations. Given that we are identifying time with the real continuum in equation 1 the crucial observation here is that this in turn implies that *time itself* will have these intrinsic arithmetic properties. Usually we only think of the additive $\pm$-type property applying to time, in which case time *could* be represented by a simple line in space, elements of which share these additive properties. That is, an interval of time $\delta s \in \mathbb{R}$, here infinitesimal or otherwise, could be expressed as:

$$\delta s = \delta x^1 + \delta x^2 + \delta x^3 + \ldots \tag{36}$$

with the intervals $\{\delta x^i \in \mathbb{R}; i = 1, 2, 3, \ldots\}$ 'fitting together' to compose a larger interval of the 'real line' $\mathbb{R}$ as might indeed be graphically drawn as a line in space.

However, on acknowledging the inherent further multiplicative $\times\div$-type arithmetic properties of the real continuum, we *can* also express an infinitesimal interval *of the continuum of time* as for example:

$$\begin{aligned}(\delta s)^2 &= (\delta x^0)^2 - (\delta x^1)^2 - (\delta x^2)^2 - (\delta x^3)^2 \\ &= (\delta x^0)^2 - \left[(\delta x^1)^2 + (\delta x^2)^2 + (\delta x^3)^2\right]\end{aligned} \tag{37}$$

as a possible *quadratic* composition in the intervals $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\} \in \mathbb{R}^4$.

Arithmetically adding intervals quadratically is isomorphic to adding lengths geometrically mutually positioned at right-angles in a Euclidean space. A structure of space *itself* can then be identified as a geometric *interpretation* of such a mathematical structure. Here from the intrinsic *arithmetic* substructure of an interval of time the properties of an element of 3-dimensional space *can then be identified* with the base components $\{\delta x^1, \delta x^2, \delta x^3\}$ via the quadratic form in the square brackets in equation 37. That is, as consistent with the basic rules of Euclidean geometry and the Pythagorean theorem, this arithmetic element *has* a geometric interpretation in which $\{\delta x^1, \delta x^2, \delta x^3\}$ are mutually positioned at right-angles forming a basis element for a 3-dimensional space. Further, given the minus signs in equation 37, the full 4-component quadratic form mathematically represents a local 4-dimensional geometric *spacetime* element with a causal light cone structure.

When a mathematical equation takes on a physical significance the humble 'equals sign' in that expression can in fact carry a lot of weight. This is the case in equation 37, which here should be read decisively *from-left-to-right* as an *expression for* an interval $\delta s \in \mathbb{R}$ of time. The invariance of this interval on the left-hand side of equation 37 under Lorentz symmetry transformations applied to the components on the right-hand side justifies naming δs 'proper time'. Only once equation 37 has been recognised as such an arithmetic expression *for* an interval of time can we make this interpretation and *begin* to speak of spacetime as a possible geometric form now identified implicitly *within* the substructure of the temporal continuum itself.

There is a long history of the two-way association between algebraic and geometric structures. This relation is typically *from* a geometric object *to* an algebraic expression, as for the Pythagorean theorem for right-angled triangles, dating from before 500 BC, or the coordinate geometry developed by Descartes and Fermat, in the mid-17$^{\text{th}}$ century, which allowed a more general transfer of all geometric relations in space to corresponding abstract number relations. Here we read this relation *the other way*, and with the mathematical structure of equation 37 not just underlying geometric objects in space but rather having a geometric interpretation as the basis for *space itself*. Further, perhaps reminiscent of the Erlangen Program initiated by Felix Klein in 1872, with symmetry groups of transformations and their invariants classifying geometric spaces, here it is the Lorentz symmetry group of equation 37, as an arithmetic form for proper time invariant under these transformations, that is central to the construction of a local Lorentzian spacetime.

Similarly as each δx^i in equation 36 is considered an interval of the real line $\mathbb{R}$, each δx^a, for $a = 0, 1, 2, 3$, in equation 37 is implicitly an interval of a corresponding local coordinate $x^a \in \mathbb{R}$. By contrast with the representation of the $\{\delta x^i\}$ intervals of equation 36 'fitting together' to form a one-dimensional line, as may be represented by a line in space, given the quadratic form of equation 37 the elements $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\}$, as a set of four intervals of independent local coordinates $\{x^0, x^1, x^2, x^3\}$, 'fit together' as local basis elements composing an extended region of the 4-dimensional space $\mathbb{R}^4$, with a geometric spacetime structure.

This construction of $\mathbb{R}^4$ can be taken to the continuum limit, with no residual discreteness, carrying the local causal light cone structure defined by equation 37 at every location. All local δs intervals, with $\delta s > 0$ and of course $(\delta s)^2 > 0$, fall within a future-directed light cone. With an extended continuous 4-dimensional spacetime manifold being constructed in this manner, through the innate arithmetic substructure of equation 37, the flow of time *can itself accommodate* the extended geometric form *through which* it propagates.

This construction of an extended spacetime manifold *from* the form for proper time in equation 37 can be compared and contrasted with the construction of the 4-dimensional spacetime world from 'atoms of spacetime' in causal set theory [74]. The key differences are that here the 'elements of time' identified with the local elements of equation 37 carry their own causal structure, as defined by the signature of the implicit Lorentz metric η_{ab} in this equation, and the construction is taken to the continuum, rather than 'atomic', limit (see also [75] section 3). A further significant difference is that the arithmetic expression for proper time in equation 37, identical to that in equation 3, further generalises to the form of equation 4 which, as we have described in detail in the previous sections, provides the source for *matter* in the 4-dimensional spacetime deriving from the residual components over and above the local spacetime elements. Following the mathematical structure implied by the theory of generalised proper time direct connections are identified between this matter content and the Standard Model of particle physics, as described for table 1 in section 3.

If the arithmetic form for proper time were to be limited to the 4-dimensional quadratic form of equation 37 the construction described above would lead to a global *flat* Minkowski spacetime. The source of curvature derives from the fact that equation 37, for an infinitesimal element $\delta s \in \mathbb{R}$ of equation 1, indeed immediately generalises further to equation 4. This general arithmetic form for proper time, with a full symmetry $\hat{G}$, can in turn be expressed as written in equation 6, from which the local geometric form of 4-dimensional spacetime in equation 37 can be extracted, breaking the full symmetry as described in equation 7. Piecing together *these* local geometric elements of $\mathbb{R}^4$ from equation 6 an extended 4-dimensional spacetime manifold can again be constructed, now in general with a curvature correlated with the matter content, which derives at the local level as described for equations 6 and 7 and now fills the extended spacetime at every location.

Via equation 37, written explicitly with the local metric η_{ab} in equation 3, as incorporated into equation 4 as expressed in equation 6, a local inertial reference frame and light cone structure can again be identified at any location, as now consistent with the equivalence principle in the curved spacetime setting of general relativity. The local basis elements $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\}$ again define local coordinate systems, with a family of such systems related by local Lorentz transformations. In the curved spacetime the *relative scale* of such a 4-dimensional spacetime element will depend upon the conformal scaling $h(x)$ of equations 30 and 31, and in turn the associated coupling with matter components in the projection out of the full form of proper time expressed in equation 6, which then directly relates to the local variation in metric scaling of equation 32.

More generally we note that '...the Hawking-King-McCarthy and Malament (HKMM) theorem [76, 77] ...implies that (M, g) is equivalent to $(M, \prec)$ plus a local volume element...' ([78] section 1). In fact a manifold 'M' with a causal structure '$\prec$' alone, without a volume element, would be sufficient to define a manifold with a metric (M, g) up to a local conformal factor. Hence the manifold M that we have constructed from the elements of equation 37 as subsumed into equation 6, together with the implicit local light cone causal structure and local conformal scale set by equations 30–32, is sufficient to fully define a global manifold with metric structure (M, g). On this manifold an invariant metric volume element could be defined in the usual way as $\sqrt{|g|}d^4x$ in general coordinates.

This metric tensor $g_{\mu\nu}(x)$, as originally here expressed in equation 2, more generally describes the global spacetime curvature in general coordinates, again as consistent with the arena of general relativity. In particular, the associated geometric constructions of differential geometry then follow, including the Levi-Civita linear connection and the Riemann curvature tensor, components of which define the Einstein tensor $G^{\mu\nu}(x)$ in the Einstein equation 5. The construction here of a curved 4-dimensional spacetime, together with a matter content, from elements of generalised proper time is represented in figure 2.

This construction of the world depicted in figure 2 convolves the left and right sides of figure 1, with the full 4-dimensional spacetime of figure 1(a) now constructed, together with the implied matter content, from the elements of generalised proper time represented in figure 1(c). At the same time the left-hand geometric 'marble' side and the right-hand phenomenological 'wood' side of the Einstein field equation 5 are intrinsically married together in this unifying picture, as we alluded to following figure 1.

The basis for this construction hinges upon the interpretation of equation 37, and the 'equals sign' therein, described earlier in this section. Rather than starting out with time and space coordinate components $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\}$ on the right-hand side and then *defining* proper time, here we set out from the left-hand side and read equation 37 as an intrinsic arithmetic substructure innate *within* a real time interval $\delta s \in \mathbb{R}$ itself. At this level the basis for this novel perspective is not something that can be *proven by* the mathematics, but rather entails a different way of *looking at* the mathematics. However, in addition to the conceptual motivation on the grounds of simplicity, the consequent generalisation for proper time in equation 4 is found to be very fruitful for known empirical physics.

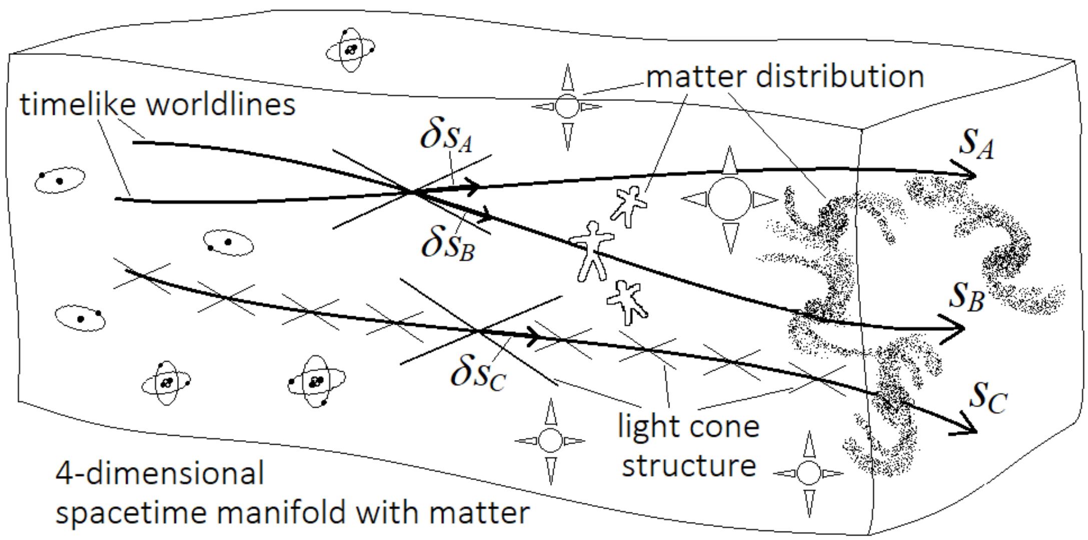

Figure 2: Depiction of an extended 4-dimensional spacetime manifold with a matter content constructed from elements of generalised proper time as described in this section. While timelike worldlines such as s_A, s_B and s_C can be drawn passing through any location, as might be associated with the idealised trajectory of for example an observer or a galaxy, they provide a very much incomplete representation of the nature of time itself. Rather, such worldlines are elements *within* the overall construction which is itself a *wholly temporal* entity in being built from 'elements of time' alone as expressed by the intrinsic local form of generalised proper time.

This idea may sound implausible, in beginning simply with the notion of time as the real continuum $s \in \mathbb{R}$ of equation 1 and interpreting the intrinsic arithmetic form for infinitesimal intervals in equation 4 as the source of both the 4-dimensional spacetime geometry and its matter content, as based upon equations 6 and 7, that extends to the entire physical universe. However, the ambition of *deriving as much as possible* from *as little as possible* is precisely the ideal of unification, here perhaps taken to an ultimate limit, and as we shall further discuss in section 8. (This conceptual change in perspective is also elaborated in [75, 79]).

The complete 4-dimensional spacetime 'block' world in figure 2, identified with the entire temporal and spatial expanse of our physical universe, contains everywhere a *flow of time* from which it is constructed, and which underlies the local 'dynamical' evolution of matter states. In this way the 'block' and 'dynamic' perspectives on the world are amalgamated in a single unifying picture. In turn, with the 4-dimensional-spacetime 'block' view associated with general relativity, and the full spacetime solutions of the Einstein field equation 5, while the one-temporal-parameter 'dynamic' view is associated more with quantum theory, for example with the evolution of states described by the Schrödinger equation, this also potentially provides a suitable arena for amalgamating gravity with quantum phenomena. This prospect of constructing a 'quantum gravity' framework for the theory of generalised proper time will be further elucidated in the following section.

6 Strategy for Uniting Gravity with Quantum Theory

Having presented the motivation for generalising proper time as the basis for unification in section 1, the direct connections established with the Standard Model in sections 2–4, and the means of constructing an extended physical world from this new perspective in section 5 as culminating in figure 2, we next consider the suitability of this framework for the main further questions for unification, firstly here for quantum gravity and secondly in the following section for large scale cosmology.

The compatibility between the theories of general relativity and generalised proper time, as emphasised for figure 1 as well as figure 2, suggests adopting an approach to 'quantum gravity' in which the classical gravity of the Einstein field equation 5 is fully retained while the standard calculational tools of quantum theory, being incompatible with general relativity, apply only as an excellent limiting approximation. After motivating this perspective of prioritising 'gravity' over 'quantum' in general terms we shall then describe the strategy for constructing such a unifying framework for the theory of generalised proper time, with the aim of amalgamating classical general relativity with a reformulated quantum physics while fully preserving the quantitative pragmatic empirical successes of standard quantum theory.

It is important to emphasise at the outset that classical general relativity itself is *not* inconsistent with *any known empirical phenomena*, including apparent indeterministic events as well as more generally, that have ever been recorded, either from individual experiments and observations or taken collectively, and in particular in the terrestrial laboratory as well as more broadly. General relativity is, however, incompatible with the *technical means* by which quantum theory calculates the likelihood of the events observed, in particular where indeterministic phenomena are involved. Hence in constructing a unifying 'quantum gravity' framework clearly something has to give way, and this is usually assumed to be on the 'gravity' side, leading to a range of attempts to 'quantise' either the gravitational field or 4-dimensional spacetime in some manner (see for example [80, 81]).

Again, we emphasise that classical general relativity has *not* been ruled out as a universal theory by *any* empirical observations or collection of observations and without exception. More specifically, there has been *no* demonstration that the foundations of quantum theory *cannot* be reformulated in a manner that is both fully consistent with classical general relativity and with all the empirical successes of existing quantum theory. Further, there are significantly more technical and conceptual problems for quantum theory in itself than there are for general relativity in itself, further motivating seeking a reformulated basis for the former.

While the main problematic issue for general relativity is the possibility of solutions for equation 5 implying 'singularities' in the spacetime geometry, which begs the question of the need for an amalgamation with quantum phenomena for a fuller understanding of physics on the smallest scales, for quantum theory there is a wide range of further significant issues. These include the pervasive infinities arising in amplitude calculations and the corresponding need for renormalisation, the extreme value in the calculation of the vacuum energy, and also the 'measurement problem' and the associated question of energy-momentum conservation throughout quantum processes. There is also the major issue of a lack of a firm conceptual foundation for quantum theory, the place of which is taken by a range of interpretations of the mathematical structure (none of which is entirely satisfactory), while general relativity is founded on the direct and simple conception of a geometrically curved spacetime as the origin of gravity.

Even from the early years of the theory Einstein in particular expressed concern over the nature of quantum mechanics, as based upon principles fundamentally different from the programmes of Newton and Maxwell: 'For the quantities that appear in its laws make no claim to describe Physical Reality *itself*, but only the *probabilities* for the appearances of a particular physical reality on which our attention is fixed' ([82] pages 72–73). He then suggested that dissatisfaction with 'such an indirect description of Reality' would remain even if Quantum Mechanics could be 'fitted successfully to the General Relativity postulates'.

Given the conceptual issues and seeming incompleteness of quantum theory in itself there may be a danger of overreaching in attempting to achieve such a fit with general relativity by also quantising gravity. Further, with general relativity the theory of the *external spacetime geometry* itself, there is no compelling reason why it should be quantised, given this quite different nature to the *matter fields in spacetime* to which quantum postulates have been successfully applied. However, through to the present day this is usually the perspective adopted, with quantum theory given the upper hand over gravity, perhaps in a large part due to the extensive laboratory testing and pragmatic success of quantum physics as applied to matter fields. Nevertheless, given that these successes of quantum theory may be in a large part down to consistency constraints, there may rather be an opportunity to redress some or all of the above conceptual and technical issues through a reformulation of quantum theory.

Contrary to our ability to observe states of matter, owing to the weakness of gravity we have *no* empirical information about the gravitational field accompanying particle and quantum phenomena in the laboratory. This does, however, provide significant freedom to assess the possibility of a non-trivial role for gravity, and the associated spacetime curvature, in this environment as guided by theoretical consistency. Retaining classical general relativity, as consistent with the theory of generalised proper time, will here itself provide constraints on the reconstruction of quantum physics.

There is in fact an important precedent regarding the consistent fusing together of quantum theory with relativity, dating from around the 1940s, in the amalgamation of quantum mechanics with *special* relativity. In that case the foundations of classical special relativity have been *fully* preserved, including properties such as Lorentz invariance and local causality, while quantum mechanics was wholly recast in the framework of quantum field theory (QFT). This already demonstrates a significant degree of malleability on the 'quantum' side, within the bounds of various consistency constraints as we further describe shortly below.

While writing three detailed volumes on the technical calculational tools of the quantum theory of fields [83] Weinberg also acknowledged the provisional nature of the theory. It may be, he wrote, that 'the quantum field theories of which we are so proud are mere "effective field theories" ', with QFT being 'the way it is because (with certain qualifications) this is the only way to reconcile quantum mechanics with special relativity' and such theories valid at energies accessible to us as 'low-energy approximations to a more fundamental theory that may not be a field theory at all' ([83]

volume I, chapters 1, 2 and 12 openings). We here then suggest a further, perhaps more significant, reformulation for quantum theory in amalgamation with *general* relativity, with an altogether different and more fundamental conceptual basis as constructed within the context of the theory of generalised proper time.

That a reasonable set of constraints can go a long way as the basis for predicting, or at least heavily restricting, the possible outcomes of an experiment involving quantum phenomena has been demonstrated by the employment of 'bootstrap' techniques. The 'S-matrix bootstrap' dating from the 1960s and applied for hadronic physics [84, 85] employed constraints of mathematical consistency such as unitarity (the quantum representation of probability conservation) as well as locality, causality and symmetries such as Lorentz invariance (as collectively demanded by special relativity), while achieving some degree of success.

The main difference between bootstrap techniques and a full quantum field theory (such as for Quantum Chromodynamics (QCD) or the full Standard Model itself), is that *in addition to* the above 'bootstrap' constraints specific details of the microphysical contents, in terms of the elementary fields and their interactions as described by the Lagrangian for the theory, are employed together with explicit perturbative calculations. Even then, if the coupling of the fields is large and prevents an effective application of perturbation theory, bootstrap constraints such as listed above can be employed to tightly restrict the higher-order contributions, as for example with the 'QCD bootstrap' in more recent years [86].

These observations suggest that, at the level of the most elementary microscopic processes, having an appropriate collection of bootstrap-type constraints as well as something like an explicit Lagrangian may provide a complete account of the information necessary to 'churn out' empirically reliable calculations. That is, from a pragmatic perspective the 'right answer' might be obtained albeit the 'wrong way' in the sense that these constraints do *not originate* from a realistic underlying conceptual foundation in accord with how nature actually works. That successful QFT calculations can be performed with completely unlocalised plane wave field states, of a very different nature to the localised particle states empirically detected, and the need for renormalisation, with an element of calibration against the measured mass and charge of the states, are amongst the symptoms that this may indeed be the case.

Further, if a new framework with a new formulation for quantum physics, now based upon a realistic conceptual foundation, implied a similar or equivalent set of constraints collectively composing the complete information necessary, and in particular with something to take the place of an explicit Lagrangian at the microscopic level, a quite *different means* of calculation might still yield *equivalent end* results, fully reproducing the successes of quantum theory for these most elementary processes.

One obvious difficulty then concerns the question of where to start in constructing such a deep-rooted reformulation for quantum theory. With this in mind we note that given the compatibility of the theory of generalised proper time with general relativity, the former may provide a possible way forward for constructing a unifying framework with classical gravity having the upper hand over quantum theory. As discussed earlier in this section, this remains at least a logical possibility. Below we first list, more generally, three mutually-related reasons why the theory of generalised proper time may provide a suitable setting for the 'quantum gravity' problem.

- As described for equation 4 and figure 1, for the theory of generalised proper time we begin at the very *local* level of infinitesimal intervals of proper time and their general arithmetic form. This provides the basis for both a local element of 4-dimensional spacetime and its elementary matter content, as described for equations 6 and 7. The extended 4-dimensional spacetime is *then* constructed, with the associated matter content, as described leading to figure 2. There is hence not an issue here of having a structure of classical fields describing matter in spacetime that should *then* be 'quantised' in some manner, rather the basis for quantum and gravitational phenomena can be mutually formulated from the *outset* through the apparent 'matter' composition of the extended spacetime, as built up from this local construction.

- For many approaches to quantum gravity there is a 'problem of time' associated with the very different, and incompatible, treatment of time in quantum theory, as the main independent parameter underlying the 'dynamic' evolution of states, and in general relativity, as a non-unique coordinate parameter of a spacetime 'block' with its own dynamic properties [87, 88]. For the 'theory of generalised proper time' this problem is effectively turned inside out since the whole framework can be considered, in brief, a 'theory of time'. In constructing the extended physical world 'around time', as the basic entity of equation 1 underlying the theory, these one-parameter 'dynamic' and 4-dimensional spacetime 'block' perspectives can be amalgamated as described for figure 2.

- A central feature of a quantum field theory, and the main additional feature over and above bootstrap constraints as discussed above, is the employment of a specific Lagrangian function of fields in the context of a 'principle of stationary action'. Both historically and as employed today there is a close connection between the action principle and the notion of *paths over time*, with the action now defined by an integral of the Lagrangian function over time. Further, in general relativity a geodesic trajectory, as followed by a free body in the gravitational field, is described by the stationarity of the integral of proper time itself, as defined in equation 2, between two points in the 4-dimensional spacetime geometry. Here it is the form for *generalised* proper time of equation 4, in the symmetry breaking projection over a 4-dimensional spacetime base as locally described for equations 6 and 7, that will be critical in providing the constraints to replace an explicit Lagrangian and action principle.

Further to the third point above, there is in fact a notable historical precedent in the first paper with an extra spatial dimension dating from 1921. There Kaluza showed how the geodesic equation in a 5-dimensional spacetime could, under certain assumptions, be interpreted as the motion of a charged body in the gravitational and electromagnetic fields in 4-dimensional spacetime ([2] equations 11, 11a and 12). Here we note that the local symmetry breaking structure for generalised proper time, for the full $\hat{G} = E_6$ and $\hat{G} = E_7$ levels of equations 11 and 12, directly resembles features of the Standard Model multiplet structure, as described for table 1, and with the ambition to recover the full Standard Model at an ultimate $\hat{G} = E_8$ level as outlined in section 4. Hence constraints deriving from this breaking of generalised proper time over the extended 4-dimensional spacetime base will inevitably in turn have some relation to the Standard Model Lagrangian, albeit with the potential for new physics beyond as also considered in section 4.

We note that all three factors above would also be relevant for approaches of ‘quantising gravity’. However, in addition to the consistency of the new theory with classical general relativity on the global scale, the ability of the theory of generalised proper time to directly account for Standard Model particle multiplet features at the local level itself provides motivation to also seek the *derivation* of a mathematical foundation, and indeed explanation, for quantum phenomena themselves from first principles, without resorting to a standard set of postulates.

The central concept that replaces the ‘principle of stationary action’ postulate is the integral of generalised proper time now over a finite duration S_n, where n denotes the number of components in the generalised form of equation 4:

$$S_n \;=\; \int \delta s \;=\; \int \left(\alpha_{abc\ldots} \frac{\delta x^a}{\delta s} \frac{\delta x^b}{\delta s} \frac{\delta x^c}{\delta s} \ldots \right)^{\frac{1}{p}} \delta s \qquad \text{with} \qquad \Delta S_n = 0 \tag{38a}$$

This expression is essentially simply equation 1 with $S_n = s \in \mathbb{R}$, now recognising the n-component arithmetic substructure of equation 4. Since here ‘time’ is the fundamental given entity we automatically have the stationarity condition in the vanishing of $\Delta S_n = 0$ as a symmetry under any variation in the ‘path’ taken through the $\{\delta x^a\} \in \mathbb{R}^n$ subcomponents. This symmetry can be considered an extension for finite time intervals from the full symmetry $\hat{G}$ for the infinitesimal intervals of equation 4.

Constraints on physical structures are realised when this invariance of equation 38a is applied in the context of the local symmetry breaking projection of equations 6 and 7, through the selection of $\{\delta x^0, \delta x^1, \delta x^2, \delta x^3\}$ as four preferred components describing the local external spacetime geometry. In building the theory ‘around time’ the intrinsic invariance $\Delta S_n = 0$ of equation 38a is then interpreted as replacing, while also explaining the pragmatic utility of, a principle of stationary action; with both approaches directly implying constraints on the equations of motion in spacetime. Here these constraints represent the global implications of constructing the extended 4-dimensional spacetime from elements of equation 4 via the local substructure of equation 6.

The function under the integral in the centre of equation 38a is essentially equation 4 written in the form of equation 13, with the notation ‘L’ in $L_p(\boldsymbol{v}_n)_{\hat{G}} = 1$ chosen in part through the pivotal role of this expression in replacing a Lagrangian function ‘$\mathcal{L}$’ as anticipated shortly before equation 31. It is the broken form of this relation that features in this replacement and which, as equivalent to equation 6 with the broken symmetry of equation 7, can be written as:

$$\not{L}_p(\boldsymbol{v}_n)_{\text{Lorentz}\times G} \;=\; 1 \tag{38b}$$

It should be noted that this is *not* an exact substitution for a Lagrangian function (with $\not{L} \equiv \mathcal{L}$), since equation 38b is *not* an equation for fields in a pre-existing spacetime, but rather a constraint on the building of spacetime itself. It was, nevertheless, described in the discussion of equations 30–32 how terms in the expansion of equation 38b can exhibit the properties of Lagrangian mass terms in featuring Yukawa, Higgs and matter component factors. Further, the constant constraint of unity on the right-hand side of equation 38b also implies the vanishing of the gauge covariant derivative $D_\mu \not{L}_p(\boldsymbol{v}_n)_{\mathrm{Lorentz}\times G} = 0$, terms in the expansion of which will describe an 'interaction coupling' between gauge and matter field components.

These observations indicate how field interactions and equations of motion under the constraints represented by equation 38ab, as applied for the symmetry breaking structures building upon table 1, can parallel those implied by the constraints of the Standard Model Lagrangian. (With the integrand of equation 38a being essentially equation 38b we refer to these expressions collectively as equation 38ab). More generally, while the 4-dimensional spacetime integral over proper time in general relativity translates into an action integral over the kinetic minus potential energy in the non-relativistic gravitational limit, here the aim is to establish an effective correspondence between the n-component time integral of equation 38ab, as employed in the construction of the extended spacetime itself, and the combined Einstein-Hilbert and Standard Model action integral in an appropriate limit.

This construction of the extended 4-dimensional spacetime from local elements of generalised proper time is compatible with general relativity as noted for figure 1. Through this construction the geometric curvature of the extended spacetime, as represented in figure 2 and described by the Einstein tensor $G^{\mu\nu}(x)$ as a function of the metric $g_{\mu\nu}(x)$, will correlate with the residual matter field $\psi(x)$ and gauge field $A(x)$ components deriving from equations 6 and 7. We write this relation generically as:

$$G^{\mu\nu} \;=\; f^{\mu\nu}(A,\psi) \;=:\; -\kappa T^{\mu\nu} \tag{39}$$

A relation between the external geometry $G^{\mu\nu}(x)$ and the gauge fields $A(x)$, associated with the internal symmetry G, can be provisionally identified by analogy with the geometric constructions of non-Abelian Kaluza-Klein theories in the setting of a principal fibre bundle space $P = M \times G$, where M is the base spacetime manifold and G the group fibres [5, 89, 90, 91, 92]. In this manner self-interactions arise for the non-Abelian gauge fields through terms quadratic in the field strength tensor [93]. The relation between the spacetime geometry and the matter fields $\psi(x)$ will be determined in part by the couplings to the 'Higgs' as described for equations 31–32 through the terms of equation 38b discussed above.

Through equation 39 the energy-momentum tensor $T^{\mu\nu}(x)$ is in turn directly *defined* in terms of the composition $f^{\mu\nu}(A,\psi)$ of the extended spacetime construction, as consistent with the Einstein field equation 5 of general relativity. The geometric property of the Bianchi identity implies that $\nabla_\mu G^{\mu\nu} = 0$, on contracting the covariant derivative ∇_μ defined in the spacetime with the Einstein tensor it is acting upon. This is here taken to *motivate the choice* of the Einstein tensor $G^{\mu\nu}(x)$ on the left-hand side of equation 39 since this in turn implies that $\nabla_\mu T^{\mu\nu} = 0$, and hence the energy-momentum as defined on the right-hand side of equation 39 will be locally conserved in the flat spacetime approximation.

Since equation 39 incorporates the single smooth geometry of classical general relativity there is no quantisation of gravity, and hence none of the associated technical difficulties, and there are no 'graviton' states. On the other hand empirically observed particle states, such as the discrete particle quanta of the Standard Model, can be provisionally described by propagating wave-packets in both the external gravitational field $g_{\mu\nu}(x)$, as undulations in $G^{\mu\nu}(x)$ and the associated Ricci curvature, and in the corresponding matter components, here including the gauge fields $A(x)$ as well as $\psi(x)$, through equation 39. Given the extremely small gravitational fields involved, and corresponding near-flat spacetime, these wave-packets and their Fourier components can be described by linear wave equations using a linearised gravitation approximation. It then remains to account for the empirical quantum phenomena of the non-gravitational fields $A(x)$ and $\psi(x)$, and the associated particle states, and indeed in a manner that will amount to the basis for a reconceptualisation of the quantum world itself.

For these quantum elements the central goal here is to bypass the employment of complex amplitudes in the calculation of an event likelihood. Rather, whereas a *classical* probability might be considered to be proportional to the 'number of ways' something can 'happen in spacetime', a *quantum* probability will relate to the 'number of ways' it is possible to 'build spacetime itself' through equation 39. This provides a unifying notion of likelihood for classical and quantum events, as a relative frequency, while the respective phenomenology can still in general be quite different. In particular, while classical probabilities typically reflect our ignorance of the precise details of a deterministic event, such as the tossing of a coin, quantum probabilities identified through equation 39 can be intrinsically indeterministic, while still being consistent with the Einstein field equation and local energy-momentum conservation. This enveloping of quantum processes within the general relativistic environment can be signified by writing that above equation as follows:

$$\overbrace{G^{\mu\nu} \;=\; \underbrace{f^{\mu\nu}(A,\psi)}_{\text{Quantum Theory}} \;=:\; -\kappa T^{\mu\nu}}^{\text{General Relativity}} \tag{39$'$}$$

While particle states are described by the real functions of propagating wave-packets they can be decomposed into $e^{+ik\cdot x}$ and $e^{-ik\cdot x}$ Fourier modes for the $A(x)$ and $\psi(x)$ field components in the flat spacetime approximation, with k the wave 4-vector for the wave components. In building the extended 4-dimensional spacetime in equation $39'$ from the local level, the likelihood of a particle interaction will then be proportional to the *mathematical degeneracy* in these matter field contributions underlying the actual local external geometry. This degeneracy can be expressed as the sum over possible field $e^{+ik\cdot x}$ mode exchanges D_+, multiplied by the independent sum of $e^{-ik\cdot x}$ mode exchanges D_-, with all such exchanges preserving the local function $f^{\mu\nu}(A,\psi)$ as a real hybrid of complex field components in equation $39'$. The $e^{\pm ik\cdot x}$ mode field exchanges are permitted if they also leave terms of the constraint equations, associated with equation 38ab, invariant *by analogy with* the field exchanges permitted in a quantum field theory through the terms of an interaction Lagrangian when expanded in field creation and annihilation operators.

Following through these parallel means of calculating the probability P_{fi} for a process from an initial state i to a final state f, via the local field contribution degeneracy underlying equation 39′ or via a standard quantum field theory formalism, a provisional correspondence can be expressed as follows:

$$\underbrace{P_{fi} \;\propto\; D_+ D_-}_{\text{prob. as degeneracy}} \;\overbrace{\sim}^{\text{formal similarity } (D_+D_- \sim \mathrm{Im}(\mathcal{M}_{ii}))}\; \underbrace{\mathrm{Im}(\mathcal{M}_{ii}) \;\propto\; \textstyle\sum_f |\mathcal{M}_{fi}|^2}_{\text{optical theorem}} \;\propto\; \sigma_{\text{tot}} \qquad \overbrace{\textstyle\sum_f |\mathcal{M}_{fi}|^2 \propto \sigma_{\text{tot}}}^{|\text{amplitude}|^2 \text{ for prob.}} \tag{40}$$

The pivotal connection is between the mathematical degeneracy D_+D_- for a particular process and *contributions to*, as represented by the '$\sim$' link, the imaginary part of the forward scattering amplitude $\mathrm{Im}(\mathcal{M}_{ii})$, which is also a real number. In particular this 'formal similarity' involves the correspondence, at each order, between an iterative expansion in field component exchanges describing the local mathematical degeneracy D_+D_- underlying solutions for equation 39′ and terms of the perturbative expansion for $\mathrm{Im}(\mathcal{M}_{ii})$ as calculated by applying 'cutting rules' to Feynman diagrams (see for example [94, 95] and references therein. We note that such rules also play a central role in the bootstrap program).

In this manner we arrive at a provisional connection between a 'classical' calculation of probability, as the local degeneracy in the matter field composition underlying the *construction of the extended spacetime itself*, and the 'quantum' calculation, pragmatically employing amplitudes and unitarity to model probability conservation and consistently relate observations, and indeed as connects $\mathrm{Im}(\mathcal{M}_{ii})$ with the total cross-section σ_{tot} via the optical theorem as indicated in equation 40. While this connection through equation 40 is indeed provisional, crucially we observe that, with reference to the earlier discussion of bootstrap techniques in relation to full QFT calculations, here we have an *equivalent full set* of constraints on these elementary processes as we describe in the following.

From the compatibility of the new theory with general relativity, and solutions for equation 39′ for a complete 4-dimensional spacetime geometry, these constraints include locality, causality and Lorentz invariance. The local causal structure of the spacetime is defined by the array of local light cones as described for figure 2 and as associated with the local spacetime elements projected from equation 6. These elements also underlie the local Lorentz symmetry as consistent with the equivalence principle, holding exactly at the infinitesimal level and to a very good approximation in the near-flat spacetime environment of the particle interaction processes. As discussed for equation 38b, the role of a quantum field theory Lagrangian, describing details of the microphysics, is replaced by the constraints provided by an explicit form for generalised proper time in the local symmetry breaking projection over the 4-dimensional spacetime.

Finally, the unitarity of quantum theory, utilised on the right-hand side of equation 40, is replaced by *explicit* probability conservation, in terms of a relative degeneracy count as determined on the left-hand side. The links through equation 40 are identified by constructing an expression for the degeneracy count from the left-hand side and taking apart a standard QFT calculation from the right-hand side, meeting midway as described for the '$\sim$' connection. Since then essentially *all the same* information is input from both ends of equation 40 as applied through a parallel iterative expansion, albeit presented in a quite different conceptual form, in principle equivalent quantitative results might be obtained for these most elementary systems and processes studied in the laboratory.

The large degree of local degeneracy in the $A(x)$ and $\psi(x)$ field components and exchanges underlying D_+D_- in equation 40 will also be reflected in the non-unique nature of the final state of an interaction, in which the ultimate $e^{\pm ik\cdot x}$ components combine to make the distinct real physical imprint of specific propagating final states in the solution for equation 39′. The different possible outcomes then correspond to different 4-dimensional spacetime geometries, with 'our world' being one of 'many solutions', each consistent with the equation 39′ and local energy-momentum conservation throughout, and each with an intrinsic prior indeterminacy in the outcome of such interaction events.

To sum up, classical general relativity and quantum phenomena are here proposed to be unified through the construction of equation 39 as elaborated for equation 39′. This equation is bracketed by the Einstein field equation 5 while the central function $f^{\mu\nu}(A,\psi)$ accommodates the degenerate matter field composition of the spacetime geometry associated with indeterministic quantum events, as permitted by the constraints described for equation 38ab and with the relative likelihood of equation 40, as proposed above. It remains to be seen whether this strategy outlined for equations 38ab–40 can be mathematically completed and consistently maintained for rigorous quantitative calculations for the broad range of quantum systems that have been empirically investigated.

It is possible that the construction of the extended physical world in the framework of generalised proper time, for the same reasons listed in the three bullet points before equation 38a, might also provide the basis for an approach to quantum gravity in which standard quantum theory takes the upper hand and with classical gravity emerging in a limiting approximation. However, given the natural compatibility of generalised proper time with classical general relativity as described for the construction in figure 2, the severe technical difficulties encountered by attempts to 'quantise gravity', and also the conceptual issues with standard quantum theory itself, it is very much worthwhile exploring this opposite strategy of giving gravity the upper hand.

While the focus has been upon this amalgamation of general relativity with quantum theory there is then indeed also the challenge of accounting for the significant technical and conceptual difficulties for quantum theory in itself as we have noted. As an example of a possible means of meeting such a challenge, with regard to the apparent 'spooky action at a distance' associated with measurements made in quantum mechanics, here a significant role is played by the 4-dimensional spacetime solutions for equation 39′, which are in some sense 'outside causality' in *defining* the causal structure of spacetime itself.

While for the new approach complex amplitudes are bypassed in the calculations for relativistic particle interaction events, similarly there are no complex wavefunctions intrinsic to the theory in the non-relativistic limit. There is in turn no physical notion of 'wavefunction collapse' for any quantum process and no associated 'measurement problem' or 'spooky action'. Rather there is a single continuous smooth

4-dimensional spacetime geometry, as a metric $g_{\mu\nu}(x)$ solution for equation 39′, that envelopes all systems with a continuous interface between the 'quantum' states and the 'classical' apparatus. Since we cannot detect the minute variations in this *physical* metric gravitational field, our best knowledge of a state before a measurement might nevertheless be *represented* by a wavefunction as a purely pragmatic means of calculating event likelihoods. It is simply this knowledge that is abruptly updated in the apparent 'collapse of the wavefunction'.

Moreover, in developing this new mathematical formulation the aim of providing a coherent framework keeping in mind and quantitatively matching what we *actually see* in experiments should be maintained throughout. An explicit correspondence between the left and right-hand sides of equation 40 that matches in every detail is not required, since indeed we wish to avoid reproducing the 'infinities' and the need for 'renormalisation' associated with calculations on the right-hand side. Rather it is an overall quantitative agreement between finite calculations from the left-hand side and the pragmatic results of amplitude calculations that is sought. More generally, a direct match is *not* required with the explicit mathematical constructions of any existing quantum theory formalism, including the wavefunctions and superpositions of states that we never see.

While we also cannot detect the metric field $g_{\mu\nu}(x)$ in particle experiments, we can seek a consistent description of its properties and behaviour. With particles themselves only observed by way of their web of interactions, here the explanation of what physically happens *between*, as well as through, such interactions is intimately connected to the external field $g_{\mu\nu}(x)$ itself through solutions for equation 39′ for particle phenomena. In many cases this will involve single propagating wave-packet particle states, but for others such as the double-slit experiment the solutions may be less straightforward ([96] section 7.4, figure 5).

Hence here classical general relativity, playing a holistic role, very much takes priority for this amalgamation of gravity with quantum phenomena, in a unified framework which also provides an account of 'what is actually physically happening' in the microscopic quantum world. In place of a set of general 'bootstrap' consistency principles, here an exhaustive set of constraints derive from the simple basic entity of time in equation 1, as expressed through the theory of generalised proper time and explicit calculation aiming to realistically replicate particle phenomena. In turn, there is no need for an 'interpretation', as there is for quantum mechanics, rather the theory provides an explicit microphysical description of these processes and is built upon a well-defined conceptual foundation.

For this new approach, consistent with the overall strategy outlined in this section, equations 38ab–40 currently amount to three holding expressions, in lieu of a more complete quantum gravity theory. While these three equations can be developed to some extent independently, what is really needed is further mathematical development in which they are more intimately interrelated and employed together. This will involve likelihoods expressed through equation 40 as consistent with the constraints described for equation 38ab under a single 4-dimensional spacetime solution for equation 39. While further details are described elsewhere ([45] chapter 11, [96], [97] section 3.2), further work is clearly still needed.

If such a project were to prove intractable, that itself might provide evidence of the need to quantise gravity in some manner, although this would forfeit the potential significant benefits of tackling the issues with quantum theory itself. Here these standard formalisms for quantum field theory and quantum mechanics are rather seen as limiting pragmatic approximations. These approximations work extremely well, from a practical quantitative calculational perspective, due to the tight consistency constraints that can be imposed for analysis of the most elementary physical systems, but are conceptually lacking in clarity and in the provision of a realistic representation of the world.

Given the distinction from such a standard quantum formalism, and in here describing the detailed internal anatomy of quantum and particle systems studied in the laboratory, one area in which the new theory might ultimately be tested through predictions will be at the interface between the 'quantum' and 'classical' elements. Such tests would then be in addition to the potential new physics 'beyond the Standard Model' described in section 4. Deviations in the phenomena associated with the elementary forms of matter deriving from generalised proper time, in comparison with existing physical theory, are not, however, limited to observations in the terrestrial quantum and particle physics laboratory. In the following section we describe how further *sectors* of matter can be identified for the new theory, of potentially great significance on the cosmological scale.

7 Basis for an Extensive Dark Sector in Cosmology

The symmetry breaking pattern, over the local 4-dimensional spacetime base components, for the forms of generalised proper time in equations 11 and 12 lead directly to unique features of the Standard Model of particle physics as described for table 1. The predicted further augmentation described for the form of equation 33 may not only complete the Standard Model picture but also imply new physics accessible to laboratory experiments as also provisionally assessed in section 4. However, this series of generalisations for proper time, through the exceptional Lie group symmetries $\hat{G} = \mathrm{E}_6$, $\hat{G} = \mathrm{E}_7$ and potentially as far as $\hat{G} = \mathrm{E}_8$, is *not* the unique and only means of augmenting the form for proper time in equation 3 as consistent with the general expression of equation 4.

By contrast, while models with extra dimensions of space share the main goal of deriving the elementary structure of matter through an augmentation from the 4-dimensional spacetime form, at the local level the most literal extra spatial dimensions generalisation from equation 3 is *limited* simply to the $(n>4)$-dimensional quadratic form:

$$(\delta s)^2 = \hat{\eta}_{ab}\delta x^a \delta x^b \qquad (41)$$

with $\hat{\eta}_{ab} = \mathrm{diag}(+1, -1, \ldots, -1)$ the augmented Lorentz signature metric, indices $a, b = 0, \ldots, n-1$ for the $n > 4$ spacetime dimensions, and with an $\mathrm{SO}^+(1, n-1)$ higher-dimensional Lorentz symmetry. In practice, models with extra spatial dimensions are far more diverse, in allowing non-Euclidean variations on equation 41 for the additional dimensions and in starting with an extended global n-dimensional bulk spacetime with a wide range of possible geometries and means of compactifying down to, or otherwise extracting, a 4-dimensional spacetime world [8, 9, 10, 11, 12].

While, within the choice of $n>4$, this local expression is *unique* for the simplest models with extra spatial dimensions, equation 41 also represents a *special case* for the theory of generalised proper time with a quadratic power $p=2$ in equation 4. Again following the local symmetry breaking over the external 4-dimensional spacetime base described for equations 6 and 7, for the full $\hat{G} = \mathrm{SO}^+(1, n-1)$ symmetry of this particular case in equation 41 expressed in the form of equation 13 as $L_2(\boldsymbol{v}_n)_{\mathrm{SO}^+(1,n-1)} = 1$, in contrast to table 1 we now find the structure listed in table 2.

n \ $\mathrm{SO}^+(1, n-1) \supset$	Lorentz	$\times$ $\mathrm{SO}(m)_D$	matter
4	4-vector	invariant	$\boldsymbol{v}_4$ external
m $(= n-4)$	scalar	m-vector	$\boldsymbol{v}_m$ dark

Table 2: Symmetry breaking pattern for the 'extra spatial dimensions' special case of generalised proper time in equation 41, as can be taken to an $n \to \infty$ limit. By comparison with table 1 this simplest and most direct extraction of a source of matter from extra dimensions of space over and above the 4-dimensional spacetime base, with a local external Lorentz $\mathrm{SO}^+(1,3)$ symmetry, looks nothing like the structures of the Standard Model. Moreover, while the external $\boldsymbol{v}_4$ local 4-dimensional spacetime components as extracted in equation 6 are in common with table 1, as is the associated extended gravitational arena, here the internal gauge symmetry $G = \mathrm{SO}(m)_D$ identified from equation 7 is entirely *independent* of the visible Standard Model sector. The above matter content, based upon the scalar $\boldsymbol{v}_m$ components transforming under the local $\mathrm{SO}(m)_D$ gauge symmetry, hence underlies a suitable dark matter candidate as described in the text.

The shared external local 4-dimensional spacetime components $\boldsymbol{v}_4$ in tables 1 and 2 imply that the alternative sector of matter of the latter is both part of *our* universe and gravitationally connected to us through the classical general relativity of equation 39. That is, the extended 4-dimensional spacetime manifold and geometric curvature pictured in figure 2 is a common arena for both the 'exceptional Lie group' branch of table 1 as well as the 'extra spatial dimensions' branch of generalised proper time in table 2.

By contrast, the internal symmetry in table 2 is parallel to but 'hidden from' the Standard Model as an independent gauge symmetry sector, identified as $G = \mathrm{SO}(m)_D$ with the subscript 'D' denoting its 'dark' nature. This internal $\mathrm{SO}(m)_D$ acts upon the scalar matter components $\boldsymbol{v}_m$ which hence can be interpreted as the basis for 'scalar dark quarks', by analogy with the familiar fermionic quark states of QCD in the Standard Model. Such a 'hidden QCD' sector, with compact non-Abelian gauge group $\mathrm{SO}(m)_D$, could provide a source of massive 'dark glueball, hadron or pion'-type states as a candidate for self-interacting dark matter (SIDM) [98, 99]. As an alternative to cold dark matter (CDM), which is assumed to be collisionless, studies show how SIDM models may help address 'small scale' galactic issue relating to 'core-cusp', 'rotation discrepancy' and 'satellite subhalo' problems for the CDM models [100].

There is then a question of whether *any* kind of non-gravitational interaction with visible matter may be possible. Given that the external $\boldsymbol{v}_4$ components in table 1 are also associated with a non-standard 'Higgs', for the reasons described in section 3, this also hints at the possibility of a 'Higgs-portal' type interaction between the visible sector and the states of table 2, analogous to that proposed by a range of models [101, 102]. However, this would require a more thorough understanding of Higgs physics in the context of the new theory, and in particular under the full E_8 level for generalised proper time proposed in section 4. There it was also suggested that in the full picture for this Standard Model branch of the theory the 'Higgs' might be effectively composed of underlying right-handed neutrino spin degrees of freedom, rather than directly in terms of the 4-vector components of table 1 that are in common with table 2. This itself raises some doubt over the possibility of a physical scalar Higgs-like interaction bridging the two sectors.

While equation 41 and table 2 represent a special case for generalised proper time, in addition to the branch of equations 11, 12 and 33 and table 1, there is also the question of whether there are further mathematically permitted branches consistent with equation 4, over and above the local form of equation 3, and their possible physical implications. As a further example, the augmentation from the 2×2 matrix determinant form for the local 4-dimensional spacetime in equation 8 to the cubic 3×3 matrix determinant form for proper time in equation 9 could itself also be extended to the $p \times p$ matrix determinant form:

$$(\delta s)^p = \det(\boldsymbol{\delta x}_{p^2}) \quad \text{with} \quad \boldsymbol{\delta x}_{p^2} \in \mathrm{h}_p\mathbb{C} \tag{42}$$

with a full symmetry $\hat{G} = \mathrm{SL}(p, \mathbb{C})$ acting upon the $n = p^2$ real components of the $p \times p$ complex Hermitian matrices $\boldsymbol{\delta x}_{p^2} \in \mathrm{h}_p\mathbb{C}$ for any integer $p > 3$. With this branch for generalised proper time rewritten in the form of equation 13 as $L_p(\boldsymbol{v}_{p^2})_{\mathrm{SL}(p,\mathbb{C})} = 1$, as an extension from equation 14, the symmetry breaking pattern in the extraction of the external local 4-dimensional spacetime components $\boldsymbol{v}_4$, correspondingly extending from equations 15–16, is listed for this new branch in table 3.

Here an internal $G = \mathrm{SL}(k, \mathbb{C})_D \times \mathrm{U}(1)_D$, resulting from the symmetry breaking of equations 6 and 7 applied to the branch for generalised proper time in equation 42, acts upon the fragmented $\boldsymbol{v}_{p^2}$ matter components. While again sharing the common external $\boldsymbol{v}_4$ components and 4-dimensional spacetime gravitational arena, the independence of this gauge sector from that of the branch of table 1, and indeed also that of table 2, suggests a further contribution to the dark sector. There is, however, a new feature in table 3 in the *non-compact* nature of the non-Abelian gauge symmetry group $\mathrm{SL}(k, \mathbb{C})_D$ for $k = p - 2 \geq 2$.

This means that for the $\mathrm{SL}(k, \mathbb{C})_D$ gauge symmetry as well as the associated positive kinetic energy gauge bosons, denoted A_+, there will also be *negative kinetic energy* gauge bosons A_-, implying a potential instability of the vacuum state. By the symmetry between the A_+ and A_- states in being perpetually and profusely created out of the vacuum while also mutually annihilating, through the self-interactions of this non-Abelian gauge group component, the net positive and negative contributions to both the energy density ρ and pressure p may nevertheless be perfectly balanced in this vacuum sector, other than on the microscopic scale of fluctuations in the presence

$p^2 \diagdown \mathrm{SL}(p,\mathbb{C}) \supset$	Lorentz $\times$	$\mathrm{SL}(k,\mathbb{C})_D$	$\times$ $\mathrm{U}(1)_D$	matter
4	4-vector	invariant	0	$\boldsymbol{v}_4$ external
$4k$ $(k = p-2)$	Weyl	$\boldsymbol{v}_{4k} \to S_k \boldsymbol{v}_{4k}$	1	$\boldsymbol{v}_{4k}$ dark
k^2	scalar	$\boldsymbol{v}_{k^2} \to S_k \boldsymbol{v}_{k^2} S_k^\dagger$	0	$\boldsymbol{v}_{k^2}$ dark

Table 3: Symmetry breaking pattern over the external $\boldsymbol{v}_4$ components for the $p \times p$ matrix branch of generalised proper time in equation 42, as can be taken to a $p \to \infty$ limit. The original full $\hat{G} = \mathrm{SL}(p,\mathbb{C})$ symmetry is broken to the product of the external Lorentz $\mathrm{SL}(2,\mathbb{C})$ and an internal $\mathrm{SL}(k,\mathbb{C})_D \times \mathrm{U}(1)_D$ symmetry. The basis for matter states identified includes a set of k 2-component complex Weyl spinors in the $k \times 2$ complex matrix $\boldsymbol{v}_{4k}$, transforming in the standard representation under the internal $S_k \in \mathrm{SL}(k,\mathbb{C})_D$ actions and charged under the internal dark $\mathrm{U}(1)_D$, as well as a set of k^2 scalars transforming as the matrix $\boldsymbol{v}_{k^2} \in \mathrm{h}_k\mathbb{C}$ under determinant-preserving $\mathrm{SL}(k,\mathbb{C})_D$ actions and neutral under $\mathrm{U}(1)_D$. The non-compact property of the gauge group $\mathrm{SL}(k,\mathbb{C})_D$ leads to a possible source of dark energy as explained in the text.

of individual A_+ and A_- states. With the only link with the visible Standard Model branch of generalised proper time again being through the common $\boldsymbol{v}_4$ components of tables 1 and 3 and the classical general relativity of equation 39 this dark vacuum could hence be entirely gravitationally benign on macroscopic scales, acting as a 'perfect fluid' with net $\rho = 0$ and $p = 0$ and hence vanishing energy-momentum, and not only stable from our perspective but also completely undetectable.

This does not, however, take into account the $\boldsymbol{v}_{4k}$ and $\boldsymbol{v}_{k^2}$ matter states in table 3. If these matter states, denoted M_+, in interaction with the A_+ and A_- states, introduce a small contribution of positive kinetic energy and pressure and a corresponding small *asymmetric* perturbation in the equation of state this can in principle generate a gravitational net dark energy effect. That is, a raging sea of $\{M_+, A_+, A_-\}$ states, with energy and pressure contributions nearly cancelling in a stable equilibrium of mutual production and annihilation, could compose a vacuum state with both a very small net positive energy density $\rho > 0$ and a very small net negative pressure $p < 0$, with the latter being a key characteristic requirement for dark energy. In being ever replenished out of the vacuum as the universe expands, such a state would also be largely uniform in time and space as a further attribute of dark energy broadly consistent with the observed cosmological expansion history.

By contrast with 'phantom dark energy' models, which posit a scalar field of negative kinetic energy as associated with a more dramatic cosmic acceleration [103, 104, 105], the parallel analysis incorporating the negative kinetic energy gauge bosons from table 3 may be more consistent with a cosmological constant Λ effect for this dark vacuum state, with a corresponding equation of state parameter $w = p/\rho = -1$. If these A_- states are indeed physical and not prohibited by some consistency constraint then any particle-like interaction between this dark vacuum and the Standard Model could make the vacuum unstable from our visible matter perspective, with potentially catastrophic consequences. In the case of phantom model such a mediation via graviton states is typically presumed ([106] figure 1), however, here the dark energy sector of table 3 only has a classical gravitational connection with the Standard Model through equation 39 as noted above. These considerations also suggest that a 'Higgs portal'-like interaction between the sectors may be unlikely.

Indeed, the special case of generalised proper time in equation 41 could itself be augmented to a quadratic form with a non-Lorentzian signature, as might be associated with extra 'timelike' as well as 'spacelike' dimensions in other models [107]. Again we emphasise that here, however, there is no *interpretation* as either timelike or spacelike extra dimensions, and hence no issues for causality in the former case. Rather it is *time itself* in equation 1 that has the non-trivial internal structure of equation 4, with the extra components over the 4-dimensional spacetime base interpreted directly as the basis for matter in equation 6, as discussed before that equation in section 1. However, such a generalisation would also imply a *non-compact* non-Abelian gauge group $G = \mathrm{SO}(m_+, m_-)_D$, containing the compact subgroup $\mathrm{SO}(m)_D \subset \mathrm{SO}(m_+, m_-)_D$ (with $m = m_+$) of table 2, implying in turn a further source of negative kinetic energy gauge bosons also in that branch. We also note that the $p \times p$ matrix branch of table 3 contains a compact subgroup $\mathrm{SU}(k)_D \subset \mathrm{SL}(k, \mathbb{C})_D$ gauge symmetry, furthering the similarity of the two dark branches.

With 'hidden QCD' compact gauge groups here associated with dark matter candidates and non-compact gauge groups associated with dark energy candidates, this suggests the possibility that sectors with interacting dark matter *and* dark energy might be identified for *both* the branch of generalised proper time of equation 41/table 2 and the branch of equation 42/table 3. While comparisons can be made with models of self-interacting dark matter and phantom dark energy as well as with models of interacting dark matter and dark energy [108, 109], crucially here, given also the difficulties of direct observation, these structures and the implied details arise from a fundamental theory based upon generalised proper time.

Further, as can be seen for example for the specific form for generalised proper time in equations 9, 10 and 14, within the invariance of δs the relative magnitude of the projected local 4-dimensional spacetime part $\det(\boldsymbol{v}_4)$, as mutually related to the $v^8 \in \mathbb{R}$ component of equation 14 by dilation transformations, could be very different in the initial state of the universe, potentially implying very different physical conditions before stability is achieved (corresponding to the present 'Higgs' vacuum value for $h(x)$ discussed for equation 30). Whether this could generate a much more rapid, or even 'inflationary', accelerating expansion in the very early universe is another open question.

In any case, fluctuations in the vacuum creation of A_+ and A_- gauge bosons and M_+ matter states at that earliest epoch, followed by a transition phase from most of that dark energy to dark matter, could act as the source of cosmic structure formation as observable today, with for example the primordial production of A_- states being ultimately correlated with the size and distribution of cosmic voids. Calculations of this process, perhaps incorporating techniques similar to the 'cosmological bootstrap' [110, 111], could then potentially have testable consequences for observations of the cosmic microwave background and large-scale galactic distribution.

This mechanism for interaction between dark energy and dark matter, and in particular the potential for dark energy to decay into dark matter, also suggests the possibility of a dark energy equation of state parameter $w = p/\rho$ that evolves more generally with cosmic time. Such cosmic history is indicated by the latest cosmological observations, which favour a long-term evolution from a protracted phantom regime with $w < -1$ at an earlier epoch, via a phantom crossing around five billion years ago, to a weakening dark energy era with $w > -1$ through to the present ([69], [112] figure 7). These observations disfavour the standard ΛCDM cosmological model, with $w = -1$ throughout, at up to the 4.2σ level.

The aim of this section has been to describe the broad picture regarding how the theory of generalised proper time, via the identification of several mathematical branches, may generate a rich dark sector of cosmological significance. As well as the need for further development of the physics, there also remains the mathematical question of identifying all possible branches for generalised proper time. Of particular interest would be the identification of a dark sector branch that only contains *compact* non-Abelian gauge groups, as the basis for a hidden QCD dark matter candidate without any dark energy component, as well as the investigation of any potential means for non-gravitational interaction with the visible Standard Model branch.

Further details regarding the initial developments of these cosmological aspects of the theory and further background references are described in [113, 114, 115] (see also [116, 117]). In the following concluding section, after reviewing the conceptual foundations, we shall indicate the main open questions concerning all elements of the theory presented in this paper more generally and the principal areas where further development is required.

8 Conclusions: Status of the Theory and Open Questions

In the opening section of this paper we argued that an inherent assumption of models with extra dimensions of *space*, namely that of a quadratic form, can be dropped to motivate the theory of generalised proper *time*, given the aim of both approaches to construct a unifying framework accounting for the elementary structure of *matter*. In fact, in terms of the relations between the basic entities of *space*, *time* and *matter*, there is of course a further, and indeed the original, means of constructing a theory, in the *phenomenology* of *matter* itself as the most prominent entity taking centre stage in a *background* of either Newtonian or relativistic space and time, and as indeed still provides the basis for almost all of science. It is the ambition of *explaining* the properties of matter from a more fundamental level that drives the promotion of space or time, and their generalisations, to the foreground.

As we described again for equation 41 in the previous section, the main aim of the models appending extra dimensions of space is to derive the elementary structures of matter from an augmentation of *spacetime*, with the emphasis on the right-hand side of that equation. By contrast for the theory of generalised proper time we shift the emphasis to the left-hand side and *generalise the whole expression for proper time*, allowing not only for $n > 4$ components *but also* powers $p > 2$ as expressed in equation 4. For the new theory the elementary structure of both matter *and* space derive *from time alone* as the fundamental entity, as described for equations 6 and 7 in section 1. The key observation is that a power '$p = 2$' is *not* needed either for time on the left-hand side of equation 6 nor for the matter components identified on the right-hand side, but *only* for the extracted external space components. As a possible alternative to models with extra spatial dimensions for deriving structures of matter, albeit with a related mathematical basis, we have argued in this paper how the approach of generalised proper time is both better motivated conceptually and better suited empirically.

At the basic *mathematical* level the two approaches are not so different, in that progressing from extra spatial dimensions to generalised proper time corresponds locally to replacing the power '2' on the left-hand side in equation 41 with the power 'p' in equation 4, as suitably balanced by the right-hand side. The new theory is not, however, motivated in terms of such a 'mathematical ansatz', but rather in terms of a much more significant *conceptual* shift away from a basis in 'space-time' to a basis in 'time' alone. In terms of unifying relations between space, time and matter, this approach can be considered a further step in the progression from Newtonian physics, through special and general relativity and via extra spatial dimensions, to generalised proper time as potentially the ultimate basis for unification (see also [75] table 1).

In progressing from special to general relativity the *global* spacetime flatness assumption was dropped, with the curved geometry of the left-hand side of figure 1 and of the Einstein field equation 5 providing the basis for *gravity*. By contrast, in the progression from extra spatial dimensions to generalised proper time it is the *local* quadratic form assumption that is being dropped, with a suitable foundation for *matter* identified in the right-hand side of figure 1, via equations 6 and 7, for the right-hand side of equation 5, as further explained for equation 39. Moreover, the ambition of the theory is ultimately to bypass phenomenological 'models' altogether, in particular as apply for the 'wood' of $T^{\mu\nu}(x)$ as described after equation 5, by providing an account of how the world is actually constituted at the most elementary level.

This supplanting of extra dimensions of space by the generalisation of proper time defines a theory that is more unique, in essentially starting with the continuum $s \in \mathbb{R}$ of equation 1 at the local level, and more unifying, with all forms of matter and even space itself deriving from time. Further, whereas we have never observed or experienced *anything* in extra dimensions of space, by stark contrast we do observe and experience *everything* in time. The new theory hence also provides a more conservative, in the sense of this familiarity of time, as well as more direct, via the intrinsic substructure of time in equation 6, basis for the elementary structure of matter. The theory of generalised proper time also arguably represents the *simplest possible* unifying relation between space, time and matter. All of these features are consistent with the ideal of unification.

Perhaps the main obstacle to adopting the new approach is this sheer simplicity and the associated seeming implausibility. That is, it might at first seem *obvious* that a unifying theory accounting for all the vast variety of material phenomena in a complex physical world could not possibly be constructed simply from the conception of time alone as the continuous progression of equation 1, as we acknowledged in the opening of section 5. However, on overcoming this barrier, through what essentially amounts to an all-embracing 'Gestalt shift' in worldview, it might also seem *obvious* that an appropriate understanding of the nature of 'time' might in fact provide a natural unifying basis for a comprehensive physical theory since, in an almost prosaic sense, indeed 'everything happens in time'. This notion is then encapsulated by the theory of generalised proper time as an intuitively natural basic unifying principle.

To paraphrase the much-quoted words of Einstein, in establishing a physical theory 'everything should be made as simple as possible, but not simpler' [118]. The *trick* is then to see how it is indeed possible to construct a unifying theory based simply on the notion of the continuum of time in equation 1. As was the focus of section 5, this is achieved by utilising the full arithmetic substructure of the real continuum $\mathbb{R}$ in representing time, infinitesimal intervals of which can then be intrinsically expressed in the form of equation 4, providing all the elements needed, as made explicit for equations 6 and 7, to construct an entire physical world *from* these local elements of time as described culminating in figure 2. The above Gestalt shift is then away from a perspective in which 'time is something *in* the world' towards rather the view that 'the world is something *in* time'.

This unifying picture also has the often desired feature of being based upon 'one simple equation', that is equation 4 for generalised proper time, which, moreover, is conceptually firmly grounded as we have summarised above. Here we are not *adding* anything to time, rather the continuum of time itself *has* a non-trivial arithmetic substructure as expressed in equation 6, which is here utilised as the basis for elements of 4-dimensional spacetime and a matter content in constructing the full physical world. On recognising the possibility of constructing the universe from these elements of time, as founded upon such a simple principle, a compelling case can be made that this may indeed be the way the world works, particularly if the development of the resulting theory is met with empirical success.

Prior to the pertinent question of whether this approach can provide the basis for *the ultimate* unification scheme, there is the initial question of whether the present status of theory merits further investigation. An affirmative answer is suggested based in part on the conceptual motivation, summarised above, and in part on the suitability of theory to account for known properties of matter as also described in this paper. Both the conceptual foundations *and* the empirical effectiveness, in essentially all of the key areas, arguably show a marked improvement over models with extra spatial dimensions and also, indeed, qualify the theory of generalised proper time as a candidate for unification in its own right. We summarise the main differences with the approach of extra spatial dimensions in making this case for generalised proper time in table 4.

The case presented for the theory of generalised proper time in this paper and as summarised in table 4 is not intended to be of an 'all-or-nothing' nature. Rather, while adopting a fundamental basis in the generalisation of proper time there may be other ways of developing the theory in some areas. For example, the theory could be employed to account for the basic properties of the Standard Model and predict new physics beyond, together with elements of the dark sector, while still applying standard tools of quantum theory to the associated matter fields, albeit with the usual questions remaining for the foundations of quantum theory itself and the nature of 'quantum gravity'. The overall strategy has, however, been to address as many issues as possible, conceptual and empirical, with minimal assumptions (see also [97] table 2 on page 71 and references therein).

Extra Spatial Dimensions	Generalised Proper Time
Locally $(\delta s)^2 = \underline{\hat{\eta}_{ab}\delta x^a \delta x^b}$	Locally $\underline{(\delta s)^p} = \alpha_{abc\ldots}\delta x^a \delta x^b \delta x^c \ldots$
In practice augment the *global* metric $g_{\mu\nu}$ of equation 2 and figure 1(a) in an extended higher-dimensional spacetime	Basis for matter directly from the *local* level in figure 1(b,c), from which build up extended spacetime of figure 2
No direct connection to the Standard Model without adding further structure and assumptions	Symmetry breaking for exceptional Lie group branch makes direct connections with the Standard Model (table 1)
In general, much additional structure needed to identify a 'quantum gravity' framework, in which usually aim to quantise gravity	Constructing the theory 'around time' provides a natural framework for 'quantum gravity', in particular with classical gravity prevailing (section 6)
Typically require 'compactification' from the global structure down to 4-dimensional spacetime, resulting in much ambiguity in matter content, as for the 'landscape problem' in string theory [119] (a 'bad' form of non-uniqueness). All technical and conceptual issues with quantum theory itself remain	Vast degeneracy in local matter composition in equation 39′, underlying this construction, as basis for quantum indeterminacy and phenomena generally (a 'good' form of non-uniqueness). With standard quantum theory seen as a limiting approximation, opportunity to address issues including 'the measurement problem'
Lightest Kaluza-Klein particle mode can be incorporated as a potential dark matter candidate [120], but limited direct dark sector physics	Directly identify further possible local branches of generalised proper time with suitable features for dark matter and dark energy (section 7)
More familiar, in being extensively employed since the 1970s, and may sound more plausible, since *adding* something to 4-dimensional spacetime	Novel approach, may sound counterintuitive, and yet more conservative, in starting from *time alone* as something itself very familiar
Wide range of related models proposed, without any empirical evidence, direct or indirect, for extra dimensions of space established	More unique and with suitability for known empirical structures demonstrated; constraints of the theory imply explanatory and predictive power

Table 4: Head-to-head comparison between the approaches of appending extra dimensions of space and generalising proper time as the basis for a unified theory. In the local expressions for a proper time interval in the first line the right-hand side of equation 41 has been underlined to emphasise this augmentation of *spacetime* as the origin of matter, while it is the left-hand side of equation 4 that is underlined to emphasise the generalisation of the whole expression for *proper time* in the new theory.

The significant and broad explanatory power identified for the new theory in this manner is further summarised for the three main areas below. While further progress may in addition be possible on the conceptual foundations, the main questions and opportunities for further development on the empirical side are also listed.

Standard Model: We described in section 2 how directly the conceptual basis for the theory, via the extensions from equation 8 to equations 9, 11 and 12, leads to the exceptional Lie groups and the octonions in expressing the symmetries for $p > 2$ forms of generalised proper time in equation 4. These mathematical structures are well-known to be of interest for the Standard Model of particle physics. Indeed, the symmetry breaking of equations 6 and 7 for the E_7 level of equations 12 and 24 leads directly to features resembling a generation of leptons and quarks in the Standard Model, as described for table 1 in section 3, and with very little redundancy. While still incomplete and with some discrepancies, these connections are much closer than can be achieved with a similarly direct approach using extra spatial dimensions, as noted for table 2 in section 7.

This incompleteness at the E_6 and E_7 levels motivates the mathematical prediction of an octonion-based E_8-type symmetry of an ultimate form for generalised proper time, as provisionally proposed in equation 33, to recover the full three-generation Standard Model particle multiplet picture. As the 'first prediction' of the new theory, this marks an accessible and tractable means of testing the theory through an explicit algebraic construction of this proposed E_8 level consistent with both the constraints of generalised proper time and the known empirical properties of the Standard Model. Likely areas for subsequent laboratory predictions and testing beyond the Standard Model, for example in the closely connected Higgs and neutrino sectors, were also described in section 4.

Quantum Gravity: While the above Standard Model multiplet structure is to be recovered at the *local* symmetry breaking level, in constructing the *global* 4-dimensional spacetime from elements of generalised proper time, as described for figure 2 in section 5, an appropriate framework for 'quantum gravity' can be identified, motivating a strategy for amalgamating classical general relativity with quantum phenomena, as described in section 6. With the calculational tools of quantum field theory and quantum mechanics now to be identified as excellent limiting approximations, the development of the new framework is closely guided by the actual laboratory observations to be accounted for. The new theory provides an equivalent set of constraints upon the associated elementary physical phenomena, including a replacement for an action principle and Lagrangian as explained for equations 38a and 38b, through which the pragmatic successes of quantum theory might in principle be reproduced.

A key question concerns the full understanding of precisely how equation 38ab for the theory of generalised proper time, in the local projection over 4-dimensional spacetime, replaces the Lagrangian constraints, in particular for the Standard Model. This expression needs to work in unison with equation 39, describing the geometric construction of the extended spacetime itself as further elaborated for equation 39′, and equation 40, for the calculation of particle interaction probabilities based on local matter field composition degeneracies. These three equations 38ab, 39 and 40 require collective further development. Further conceptual understanding may also be needed together with the mathematical development in this area of theory. Deviations from standard quantum theory, in terms of observable quantities, may in turn lead to testable laboratory predictions.

Cosmology: While connections with the Standard Model of particle physics follow from the unique sequence of mathematical structures described in sections 2–4, this in not *the* unique means of generalising proper time. In section 7 we described how the further two branches of equations 41 and 42 yield quite different sets of matter properties, as explained for tables 2 and 3 respectively. In exhibiting only a classical general relativistic interaction with the visible matter branch, through 4-dimensional spacetime solutions for equation 39, these further branches make natural 'hidden QCD' candidates for both dark matter and dark energy. Fluctuations in the production of positive and negative energy states of the dark energy in the very early universe, together with the possibility of interaction between dark energy and dark matter within each new branch, imply a possible source of cosmological structure formation.

An open question here is again a purely mathematical one, concerning the classification of all possible forms consistent with the homogeneous polynomial expression of equation 4 in generalising proper time from equation 3. Beyond that, the study of the phenomenology of compact and non-compact gauge groups in the hidden sectors, as associated with dark matter and dark energy and their potential interaction, will require significant work. Bootstrap techniques, under simplifying assumptions and symmetries applied for the very early universe, may assist in the calculations for structure formation, which may result in characteristic observable signatures. More generally in developing the theory a basis for something approximating the standard ΛCDM cosmological model, while accounting for any tensions between ΛCDM and observations, might be sought.

To conclude, the focus of this paper has been on a new conceptual foundation for the construction of the world and a corresponding basis for building a unified theory. While based upon the simple idea that the basic unifying entity is the continuum of time alone in equation 1, via the intrinsic infinitesimal arithmetic substructure of equation 4, the initial development of theory already directly makes non-trivial inroads into the empirical world: in providing a basis for the Standard Model of particle physics, a framework for quantum gravity and a source for the dark sector in cosmology. As for the first steps taken with any new theory there are open questions remaining. However, the firm grounding provided by the simple conceptual motivation together with the explanatory power achieved and direction forward in all these pertinent areas, as summarised above, are all factors making the case for further investigation into this generalised proper time approach to unification.

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