Exact solutions for Fröhlich-Peierls Hamiltonian model via reduction method*

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Abstract

Fröhlich-Peierls Hamiltonian model for electrons which interact with phonons only with some infinite discrete modes is studied using group-theoretical methods. Group analysis of this model in nonequilibrium state in one dimensional case is carried out and some exact solutions had been found.

1. Introduction

Equations for equilibrium and nonequilibrium states of Fröhlich-Peierls Hamiltonian model for electrons which interact with phonons only with some infinite discrete modes are obtained in [7]. Nonequilibrium states in one dimensional case are described by the system of coupled equations

$$-\frac{\partial^2}{\partial t^2}W(t,x)+\omega_0^2\frac{\partial^2}{\partial x^2}W(t,x)=-4\alpha\frac{\partial^2}{\partial x^2}|\Psi(t,x)|^2, \tag{1}$$

$$i\frac{\partial}{\partial t}\Psi(t,x)=\left(-\frac{1}{2m}\frac{\partial^2}{\partial x^2}-\mu\right)\Psi(t,x)+W(t,x)\Psi(t,x), \tag{2}$$

where ω_0, m, μ, α are real parameters.

Wave function $\Psi(t,x)$ satisfies Schrödinger equation with a time depended potential. It is defined by a real function $W(t,x)$ which is a solution of the non-linear d'Alembert eq. with the right–hand side term $-4\alpha\frac{\partial^2}{\partial x^2}|\Psi(t,x)|^2$.

A particular solution of equations (1), (2) was obtained in [7]. It is a soliton solution defined by the following functions

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$$W(t,x) = \frac{-16\,\alpha\eta^2}{\omega_0^2 - V^2} \frac{1}{\cosh^2 \frac{u}{2}(x - vt - x_0)}\,, \tag{3}$$

$$\Psi(t,x) = 2\,i\eta\ \frac{\exp i\left(vmx - \left(\frac{v^2}{2}m - \frac{u^2}{8m} + \mu\right)t - \varphi_0\right)}{\cosh \frac{u}{2}(x - vt - x_0)}\,, \tag{4}$$

where

$$v = -4\,\xi\sqrt{\frac{\alpha}{(\omega_0^2 - V^2)m}}\,,\quad u = 4\,\eta\sqrt{\frac{4\,\alpha\,m}{\omega_0^2 - V^2}}\,,\quad \xi, \eta, x_0, \varphi_0 \text{ are the constants.}$$

The question, whether additional exact solutions for this model exist, continued to be opened. We will give a positive answer to this question and present an extended class of such solutions.

A regular way for construction of exact solutions of PDEs is given by group analysis which was developed by Sophus Lie almost 150 years ago.

The Lie algorithm for construction of exact solutions in particular case of equations (1), (2) can be formulated in the following way:

1. To find a group of continuous transformations which leave the equation invariant and find the corresponding Lie algebra.

2. To choose the optimal system of all one-dimensional subalgebras of this algebra.

3. There exist changes of variables (which corresponds to any of this subalgebra) that reduces the equations to a system of second order ODEs. Integrating the obtained system it is possible to reconstruct exact solution of the initial system (1), (2).

2. Group analysis of Fröhlich-Peierls Hamiltonian model. It is evident, that the system of equations (1), (2) can not have extended symmetries since the left–hand side of equation (1) is invariant with respect to the Lorentz transformation but the left–hand side of equation (2) is invariant with respect to the Galilean transformation.

To find symmetries of (1), (2) we start with the change of variables

$$W(t,x) = \widetilde{W}(t,\tilde{x}), \qquad \Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\, e^{i\mu t}\widetilde{\Psi}(t,\tilde{x}), \qquad \tilde{x} = \frac{x}{\omega_0}$$

which transform system (1), (2) to the following form:

$$-\frac{\partial^2}{\partial t^2}\,\widetilde{W}(t,\tilde{x}) + \frac{\partial^2}{\partial \tilde{x}^2}\,\widetilde{W}(t,\tilde{x}) = -\frac{\partial^2}{\partial \tilde{x}^2}\,|\widetilde{\Psi}(t,\tilde{x})|^2, \tag{5}$$

$$i\frac{\partial}{\partial t}\,\widetilde{\Psi}(t,\tilde{x}) = -\frac{1}{2\,m\,\omega_0^2}\,\frac{\partial^2}{\partial \tilde{x}^2}\,\widetilde{\Psi}(t,\tilde{x}) + \widetilde{W}(t,\tilde{x})\widetilde{\Psi}(t,\tilde{x}). \tag{6}$$

Then we represent function $\widetilde{\Psi}(t,\widetilde{x})$ through amplitude and phase

$$\widetilde{\Psi}(t,\widetilde{x}) = \rho(t,\widetilde{x}) e^{i\varphi(t,\widetilde{x})}$$

and reduce system (5), (6) to the following one:

$$\widetilde{W}_{tt} - \widetilde{W}_{\tilde{x}\tilde{x}} = 2\left(\rho_{\tilde{x}}^2 + \rho\rho_{\tilde{x}\tilde{x}}\right), \tag{7}$$

$$\rho_t = -\frac{1}{2\,m\,\omega_0^2}\left(2\,\rho_{\tilde{x}}\,\varphi_{\tilde{x}} + \rho\,\varphi_{\tilde{x}\tilde{x}}\right), \tag{8}$$

$$\rho\varphi_t = \frac{1}{2\,m\,\omega_0^2}\left(\rho_{\tilde{x}\tilde{x}} - \rho\,\varphi_{\tilde{x}}^2\right) - \widetilde{W}\rho. \tag{9}$$

Using standard Lie algorithm we find that the infinitesimal operator of invariance group of system (7) - (9) is a linear combination of the following generators:

$$e_1 = -\partial_\varphi,\ e_2 = -\partial_{\widetilde{W}} + t\partial_\varphi,\ e_3 = t\partial_{\widetilde{W}} - \frac{1}{2}t^2\partial_\varphi,\ e_4 = \partial_t,\ e_5 = \partial_{\tilde{x}}. \tag{10}$$

Returning in (10) to complex variables $\widetilde{\Psi}$ we can formulate the following statement.

Theorem 1 *The system of equations (5), (6) admits five-dimensional Lie algebra, whose basis elements can be chosen in the form*

$$e_1 = -i\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right), \qquad e_2 = -\partial_{\widetilde{W}} + it\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right),$$
$$e_3 = t\partial_{\widetilde{W}} - \frac{i}{2}t^2\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right), \qquad e_4 = \partial_t, \qquad e_5 = \partial_{\tilde{x}}.$$

Their nonzero commutators are $[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2.$

Therefore, derived operators form algebra $A_{4.1} \bigoplus A_1$ according to Mubarakzyanov classification [4].

Our next step is to find all one-dimensional subalgebras of found invariance algebra.

3. One-dimensional subalgebras of algebra $A_{4.1} \bigoplus A_1$. One-dimensional subalgebras of algebra $A_{4.1} \bigoplus A_1$ we find using the method proposed in [6]. This method consists in that we take generic element $e = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5$ of algebra $A_{4.1} \bigoplus A_1$ and subject it to various adjoint transformations so that simplify it as much as possible. In [1] we have found the optimal system of one-dimensional subalgebras, which can be described by theorem 2.

Theorem 2 *Up to the group of inner automorphisms there are five inequivalent one-dimensional subalgebras of algebra $A_{4.1} \bigoplus A_1$, whose basis elements can be chosen in the following form:*

$$pe_3 + e_4 + \varkappa e_5 = \partial_t + \varkappa\partial_{\tilde{x}} + pt\partial_{\widetilde{W}} - \frac{p}{2}t^2 i\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right),$$

$$qe_1 + e_3 + \varkappa e_5 = \varkappa\partial_{\tilde{x}} + t\partial_{\widetilde{W}} - \left(\frac{1}{2}t^2 + q\right) i\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right),$$

$$e_2 + \varkappa e_5 = \varkappa\partial_{\tilde{x}} - \partial_{\widetilde{W}} + ti\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right),$$

$$e_1 + \varkappa e_5 = \varkappa\partial_{\tilde{x}} - i\left(\widetilde{\Psi}\partial_{\widetilde{\Psi}} - \widetilde{\Psi}^*\partial_{\widetilde{\Psi}^*}\right), \qquad e_5 = \partial_{\tilde{x}}.$$

4. Exact solutions of Fröhlich-Peierls Hamiltonian model. Corresponding change of variables for each of derived one-dimensional subalgebras which reduces equations (7) - (9) to ODEs. As new variables we choose invariants of the related group of transformations. We can find corresponding anzatzes and reduced equations for each of the one-dimensional subalgebras are listed in the following table.

No	Subalgebras	Invariants	Reduction equations
1	$pe_3+e_4+\varkappa e_5$	$\widetilde{W} = f(\omega) + \frac{pt^2}{2}$, $\rho = h(\omega)$, $\varphi = g(\omega) - \frac{p}{6}t^3$, $\omega = \tilde{x} - \varkappa t$	$\ddot{f}(\varkappa^2-1)+p-2(\dot{h}^2+h\ddot{h})=0$, $\ddot{g} + \frac{2\dot{h}}{h}(\dot{g} - m\omega_0^2\varkappa) = 0$, $\ddot{h}-h\dot{g}^2+2m\omega_0^2 h\,(\varkappa\dot{g}-f)=0$
2	$qe_1+e_3+\varkappa e_5$	$\widetilde{W} = f(\omega) + \frac{t\tilde{x}}{\varkappa}$, $\rho = h(\omega)$, $\omega = t$, $\varphi=g(\omega)-\left(\frac{1}{2}t^2+q\right)\frac{\tilde{x}}{\varkappa}$	$\ddot{f} = 0$, $\dot{h} = 0$, $\dot{g}+\frac{1}{2m\omega_0^2\varkappa^2}\left(\frac{1}{2}\omega^2+q\right)^2+f=0$
3	$e_2 + \varkappa e_5$	$\widetilde{W} = f(\omega) - \frac{\tilde{x}}{\varkappa}$, $\rho = h(\omega)$, $\omega = t$, $\varphi = g(\omega) + \frac{t\tilde{x}}{\varkappa}$	$\ddot{f} = 0$, $\dot{h} = 0$, $\dot{g} + \frac{\omega^2}{2m\omega_0^2\varkappa^2} + f = 0$
4	$e_1 + \varkappa e_5$	$\widetilde{W} = f(\omega)$, $\rho = h(\omega)$, $\varphi = g(\omega) - \frac{\tilde{x}}{\varkappa}$, $\omega = t$	$\ddot{f} = 0$, $\dot{h} = 0$, $\dot{g} + \frac{1}{2m\omega_0^2\varkappa^2} + f = 0$
5	e_5	$\widetilde{W} = f(\omega)$, $\rho = h(\omega)$, $\varphi = g(\omega)$, $\omega = t$	$\ddot{f} = 0$, $\dot{h} = 0$, $\dot{g} + f = 0$

The reduced systems present in Items 2-5 give us next exact solutions:

1) $W(t,x) = C_1 t + \dfrac{tx}{\varkappa\omega_0}$,

$$\Psi(t,x)=\frac{C_3\omega_0}{2\sqrt{\alpha}}\exp i\!\left(\!\left(\mu-\frac{q^2}{2m\omega_0^2\varkappa^2}-C_2\right)\!t-\frac{3t^5+20qt^3}{120m\omega_0^2\varkappa^2}-\frac{C_1}{2}t^2-\left(\frac{1}{2}t^2+q\right)\frac{x}{\varkappa\omega_0}\right)\!.$$

$$\begin{aligned}
&2)\quad W(t,x) = C_1 t - \frac{x}{\varkappa\omega_0},\\
&\quad\ \Psi(t,x) = \frac{C_3\omega_0}{2\sqrt{\alpha}}\exp i\left(\frac{tx}{\varkappa\omega_0} - \frac{t^3}{6m\omega_0^2\varkappa^2} - \frac{C_1}{2}t^2 + (\mu - C_2)\,t\right).\\
&3)\quad W(t,x) = C_1 t, \qquad\qquad (11)\\
&\quad\ \Psi(t,x) = \frac{C_3\omega_0}{2\sqrt{\alpha}}\exp i\left(\left(\mu - \frac{1}{2m\omega_0^2\varkappa^2} - C_2\right)t - \frac{C_1}{2}t^2 - \frac{x}{\varkappa\omega_0}\right).\\
&4)\quad W(t,x) = C_1 t \quad \Psi(t,x) = \frac{C_3\omega_0}{2\sqrt{\alpha}}\exp i\left((\mu - C_2)\,t - \frac{C_1}{2}t^2\right).
\end{aligned}$$

In these solutions the amplitude is a constant and phase has a form of polynomial of order no greater than five.

The most interesting solutions of Fröhlich-Peierls Hamiltonian model can be obtained, using the first of the reduced systems, namely, the system

$$\ddot{f}(\varkappa^2 - 1) + p - 2(\dot{h}^2 + h\ddot{h}) = 0, \qquad (12)$$

$$\ddot{g} + \frac{2\dot{h}}{h}(\dot{g} - m\omega_0^2\varkappa) = 0, \qquad (13)$$

$$\ddot{h} - h\dot{g}^2 + 2m\omega_0^2 h\,(\varkappa\dot{g} - f) = 0. \qquad (14)$$

We were unable to find the general solution of this system but we can present some particular solutions.

From equations (12) and (13) we find:

$$(\varkappa^2 - 1)f + \frac{p}{2}\omega^2 + C_1\omega + C_2 = h^2, \qquad (15)$$

$$g = m\omega_0^2\varkappa\omega + C_3\int\frac{d\omega}{h^2} + C_4, \qquad (16)$$

where $C_1, \ldots, C_4$ are the integration constants.

Substituting the derived expressions (15) - (16) to the equation (14) we obtain the ODE:

$$\begin{aligned}
0 = (\varkappa^2 - 1)\ddot{h} &+ m\omega_0^2 h\left(p\omega^2 + 2C_1\omega + 2C_2 + m\omega_0^2\varkappa^2(\varkappa^2 - 1)\right)\\
&- 2m\omega_0^2 h^3 - \frac{C_3^2(\varkappa^2 - 1)}{h^3}\,. \qquad (17)
\end{aligned}$$

Particular solutions of this equation can be obtained for the following values of parameters:

$$1)\quad \varkappa^2 - 1 = 0; \qquad 2)\quad \varkappa^2 - 1 \neq 0, \quad C_1 = p = 0;$$

$$3)\quad \varkappa^2 - 1 \neq 0, \quad C_3 = p = 0, \quad C_2 = \frac{m\omega_0^2\varkappa^2(1 - \varkappa^2)}{2}\,.$$

1) Let us examine the case, where $\varkappa = 1$. Further integration depends on the value of the determinant $C_1^2 - 2p\,C_2$.

1.1. If $C_1^2 - 2p\,C_2 > 0$

$$W(t,x) = \frac{\left(2pC_2 - C_1^2 - 4C_3^2\right)\omega_0^2}{2m\left(p(x-t\omega_0)^2 + 2C_1\omega_0(x-t\omega_0) + 2C_2\omega_0^2\right)^2} + \frac{m\omega_0^2 + pt^2}{2}\ ,$$

$$\Psi(t,x) = \frac{1}{2\sqrt{\alpha}}\sqrt{\frac{p}{2}(x-t\omega_0)^2 + C_1\omega_0(x-t\omega_0) + C_2\omega_0^2}\,\exp i\Bigg(m\omega_0 x + \\ + t(\mu - m\omega_0^2) - \frac{pt^3}{6} + \frac{C_3}{\sqrt{C_1^2 - 2pC_2}}\ln\left|\frac{p(x-t\omega_0) + C_1\omega_0 - \omega_0\sqrt{C_1^2 - 2pC_2}}{p(x-t\omega_0) + C_1\omega_0 + \omega_0\sqrt{C_1^2 - 2pC_2}}\right|\Bigg).$$

1.2. If $C_1^2 - 2p\,C_2 = 0$

$$W(t,x) = -\frac{2C_3^2 p^2 \omega_0^2}{m\left(p(x-t\omega_0) + C_1\omega_0\right)^4} + \frac{m\omega_0^2 + pt^2}{2},$$

$$\Psi(t,x) = \frac{|p(x-t\omega_0) + C_1\omega_0|}{2\sqrt{2p\alpha}}\exp i\Bigg(m\omega_0 x + t(\mu - m\omega_0^2) - \frac{pt^3}{6} - \\ - \frac{2C_3\omega_0}{p(x-t\omega_0) + C_1\omega_0}\Bigg).$$

1.3. If $C_1^2 - 2p\,C_2 < 0$

$$W(t,x) = \frac{\left(2pC_2 - C_1^2 - 4C_3^2\right)\omega_0^2}{2m\left(p(x-t\omega_0)^2 + 2C_1\omega_0(x-t\omega_0) + 2C_2\omega_0^2\right)^2} + \frac{m\omega_0^2 + pt^2}{2}\ ,$$

$$\Psi(t,x) = \frac{1}{2\sqrt{\alpha}}\sqrt{\frac{p}{2}(x-t\omega_0)^2 + C_1\omega_0(x-t\omega_0) + C_2\omega_0^2}\,\exp i\Bigg(m\omega_0 x - \frac{pt^3}{6} + \\ + t(\mu - m\omega_0^2) + \frac{2C_3}{\sqrt{2pC_2 - C_1^2}}\arctan\frac{p(x-t\omega_0) + C_1\omega_0}{\omega_0\sqrt{2pC_2 - C_1^2}}\Bigg).$$

The case, where $\varkappa = -1$ can be examined in the same way.

2) Let us examine the case, where $\varkappa^2 - 1 \neq 0$ and $C_1 = p = 0$. Considering this case we obtain the following solutions of the system (1), (2):

2.1.

$$W(t,x) = \frac{4\nu^2}{\varkappa^2-1}\left(\tan^2(2\nu\xi(x-\varkappa\omega_0 t)) + \frac{2}{3}\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)}\ ,$$

$$\Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\sqrt{4\nu^2\left(\tan^2(2\nu\xi(x-\varkappa\omega_0 t)) + \frac{2}{3}\right) + \frac{2}{3}\eta}\, \exp i\Bigg(\mu t + m\omega_0\varkappa$$

$$\times(x-\varkappa\omega_0 t) + \frac{3C_3\,(x-\varkappa\omega_0 t)}{2\omega_0\,(\eta-2\nu^2)} - \frac{3\sqrt{3}C_3 \arctan\frac{2\sqrt{3}\nu\tan(2\nu\xi(x-\varkappa\omega_0 t))}{\sqrt{8\nu^2+2\eta}}}{4\omega_0\xi\,(\eta-2\nu^2)\sqrt{8\nu^2+2\eta}}\Bigg),$$

where $\xi = \sqrt{\dfrac{m}{\varkappa^2-1}}$, $\eta = C_2 + \dfrac{1}{2}m\omega_0^2\varkappa^2(\varkappa^2-1)$.

2.2.

$$W(t,x) = \frac{2\nu^2}{\varkappa^2-1}\left(\tanh^2\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right) - \frac{2}{3}\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)},$$

$$\Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\sqrt{2\nu^2\left(\tanh^2\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right) - \frac{2}{3}\right) + \frac{2}{3}\eta}\, \exp i\Bigg(\mu t + m\omega_0\varkappa$$

$$\times(x-\varkappa\omega_0 t) + \frac{3C_3\,(x-\varkappa\omega_0 t)}{2\omega_0(\nu^2+\eta)} + \frac{3C_3\sqrt{3}\arctan\frac{\nu\sqrt{3}\tanh\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right)}{\sqrt{\eta-2\nu^2}}}{2\omega_0\xi(\nu^2+\eta)\sqrt{2\eta-4\nu^2}}\Bigg).$$

2.3.

$$W(t,x) = \frac{2\nu^2}{\varkappa^2-1}\left(\coth^2\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right) - \frac{2}{3}\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)},$$

$$\Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\sqrt{2\nu^2\left(\coth^2\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right) - \frac{2}{3}\right) + \frac{2}{3}\eta}\, \exp i\Bigg(\mu t + m\omega_0\varkappa$$

$$\times(x-\varkappa\omega_0 t) + \frac{3C_3\,(x-\varkappa\omega_0 t)}{2\omega_0(\nu^2+\eta)} + \frac{3C_3\sqrt{3}\arctan\frac{\nu\sqrt{3}\coth\left(\nu\xi\sqrt{2}(x-\varkappa\omega_0 t)\right)}{\sqrt{\eta-2\nu^2}}}{2\omega_0\xi(\nu^2+\eta)\sqrt{2\eta-4\nu^2}}\Bigg).$$

2.4.

$$W(t,x) = \frac{4\nu^2}{\varkappa^2-1}\left(\coth^2(2\nu\xi(x-\varkappa\omega_0 t)) - \frac{2}{3}\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)},$$

$$\Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\sqrt{4\nu^2\left(\coth^2(2\nu\xi(x-\varkappa\omega_0 t)) - \frac{2}{3}\right) + \frac{2}{3}\eta}\, \exp i\Bigg(\mu t + m\omega_0\varkappa$$

$$\times (x - \varkappa\omega_0 t) + \frac{3C_3\,(x - \varkappa\omega_0 t)}{2\omega_0(2\nu^2+\eta)}\frac{3C_3\sqrt{3}\arctan\frac{2\nu\sqrt{3}\coth(2\nu\xi(x-\varkappa\omega_0 t))}{\sqrt{2\eta-8\nu^2}}}{2\omega_0\xi(2\nu^2+\eta)\sqrt{2\eta-8\nu^2}}\Bigg).$$

2.5.

$$W(t,x) = \frac{4\nu^2}{\varkappa^2-1}\left(\tanh^2(2\nu\xi(x-\varkappa\omega_0 t)) - \frac{2}{3}\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)},$$

$$\Psi(t,x) = \sqrt{4\nu^2\left(\tanh^2(2\nu\xi(x-\varkappa\omega_0 t)) - \frac{2}{3}\right) + \frac{2}{3}\eta}\exp i\Bigg(\frac{3C_3\,(x-\varkappa\omega_0 t)}{2\omega_0(2\nu^2+\eta)} + m\omega_0\varkappa$$

$$\times (x-\varkappa\omega_0 t) + \mu t + \frac{3C_3\sqrt{3}\arctan\frac{2\nu\sqrt{3}\tanh(2\nu\xi(x-\varkappa\omega_0 t))}{\sqrt{2\eta-8\nu^2}}}{2\omega_0\xi(2\nu^2+\eta)\sqrt{2\eta-8\nu^2}}\Bigg).$$

2.6.

$$W(t,x) = \frac{1}{\xi^2\,(\varkappa^2-1)\,(x-\varkappa\omega_0 t)^2} + \frac{\eta-2C_2}{3(\varkappa^2-1)},$$

$$\Psi(t,x) = \frac{\omega_0}{2\sqrt{\alpha}}\sqrt{\frac{1}{\xi^2\,(x-\varkappa\omega_0 t)^2} + \frac{2}{3}\eta}\exp i\Bigg(m\omega_0\varkappa(x-\varkappa\omega_0 t) + \mu t +$$

$$+ \frac{3C_3(x-\varkappa\omega_0 t)}{2\omega_0\eta} - \frac{3\sqrt{3}C_3\arctan\frac{1}{3}\xi\sqrt{6\eta}\,(x-\varkappa\omega_0 t)}{2\omega_0\xi\eta\sqrt{2\eta}}\Bigg).$$

2.7. The potential which is a part of one more solution has the following form:

$$W(t,x) = \frac{2}{\varkappa^2-1}\wp\left(\xi\sqrt{2}(x-\varkappa\omega_0 t)\right) + \frac{\eta-2C_2}{3(\varkappa^2-1)} -$$

$$- \frac{p}{2(\varkappa^2-1)}\left(\frac{x}{\omega_0} - \varkappa t\right)^2 - \frac{C_1}{\varkappa^2-1}\left(\frac{x}{\omega_0} - \varkappa t\right) + \frac{pt^2}{2},$$

where $\wp$ is a two-periodic Weierstrass function (refer, e.g., [8]), which is meromorphic on the whole complex plane range. The realated function $\Psi(t,x)$ can be found from equation (2).

3) Let us examine the case, where $\varkappa^2-1 \neq 0$, $C_3 = p = 0$ $C_2 = \frac{m\omega_0^2\varkappa^2(1-\varkappa^2)}{2}$. Then the reduced equation (17) has the form given by relation:

$$\ddot{h} = \frac{2m\omega_0^2}{\varkappa^2-1}(h^3 - C_1\omega h). \qquad (18)$$

Equation (18) is the second Painlevé equation, and function h is the Painlevé transcendental. Substituting this function in (15) and (16), we obtain a solution of reduced system, then we find the explicit forms of the functions $W(t, x)$ and $\Psi(t, x)$.

Using translations with respect to t and x we can find families of exact solutions, which depend on more parameters.

5. Conclusion. Thus, we have found all possible Lie reductions for the system of equations, which describes nonequilibrium states of Fröhlich-Peierls Hamiltonian model. Among the found exact solutions we can indicate periodic solutions and soliton like ones. A rather nontrivial class of solution is given by potential which includes Weierstrass function and also solutions which can be expressed via the second Painlevé transcendental.

In this paper we restrict ourself to construction of the exact solutions for this model. A physical interpretation of the presented results will be a subject of future publication.

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