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UNIQUENESS OF TOPOLOGICAL SOLUTIONS FOR THE GUDNASON MODEL

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Abstract. In this paper, we consider the Gudnason model of $\mathcal{N} = 2$ supersymmetric field theory, where the gauge field dynamics is governed by two Chern-Simons terms. Recently, it was shown by Han et al. that for a prescribed configuration of vortex points, there exist at least two distinct solutions for the Gudnason model in a flat two-torus, where a sufficient condition was obtained for the existence. Furthermore, one of these solutions has the asymptotic behavior of topological type. In this paper, we prove that such doubly periodic topological solutions are uniquely determined by the location of their vortex points in a weak-coupling regime.

1. Introduction

Chern-Simons gauge field theories have been developed in various physics models to study high temperature superconductivity [45,51], the Bose-Einstein condensates [40,44], the quantum Hall effect [61], optics [8], and superfluids [58] (see also [1, 3, 6, 7, 10–13, 19, 21, 23, 30, 33, 37, 38, 41–43, 47, 49, 52–54, 56, 57, 62–64, 66, 69–71] and references therein). For a supersymmetric gauge field theory, Gudnason in [26, 27] formulated a non-abelian Chern-Simons model (see also [2, 4, 5, 28, 35, 36, 50, 59, 60] and references therein for the backgrounds). Under a suitable physical ansatz for vortex solutions, the Gudnason model of $\mathcal{N} = 2$ supersymmetric field theory, where the dynamics of gauge fields is governed by two Chern-Simons terms, was reduced to the following nonlinear elliptic system

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(refer to [26, 27, 32] for the details):

$$
\begin{cases}
\Delta u_{\alpha,\beta} = \alpha^2(e^{u_{\alpha,\beta}+v_{\alpha,\beta}} + e^{u_{\alpha,\beta}-v_{\alpha,\beta}})(e^{u_{\alpha,\beta}+v_{\alpha,\beta}} + e^{u_{\alpha,\beta}-v_{\alpha,\beta}} - 2) \\
\qquad\qquad + \alpha\beta(e^{u_{\alpha,\beta}+v_{\alpha,\beta}} - e^{u_{\alpha,\beta}-v_{\alpha,\beta}})^2 + 4\pi \sum_{i=1}^{N} m_i \delta_{p_i}, \\
\Delta v_{\alpha,\beta} = \alpha\beta(e^{u_{\alpha,\beta}+v_{\alpha,\beta}} - e^{u_{\alpha,\beta}-v_{\alpha,\beta}})(e^{u_{\alpha,\beta}+v_{\alpha,\beta}} + e^{u_{\alpha,\beta}-v_{\alpha,\beta}} - 2) \\
\qquad\qquad + \beta^2(e^{2u_{\alpha,\beta}+2v_{\alpha,\beta}} - e^{2u_{\alpha,\beta}-2v_{\alpha,\beta}}) + 4\pi \sum_{i=1}^{N} m_i \delta_{p_i}.
\end{cases} \tag{1}
$$

Here, $\alpha > 0$ and $\beta > 0$ are positive parameters, δ_{p_i} stands for the Dirac measure concentrated at p_i, and $p_i \neq p_j$ if $i \neq j$. Each p_i is called a vortex point and $m_i \in \mathbb{N}$ is the multiplicity of p_i.

In this paper, we are concerned with solutions to the elliptic system (1) over a two-dimensional flat torus $\mathbb{T}$, a doubly periodic domain in $\mathbb{R}^2$, due to the lattice structures in a condensed matter system (for instance, see [1, 66]), and the theory suggested by 't Hooft in [68]. In a flat two-torus $\mathbb{T}$, Han, Lin, Tarantello, and Yang in [32] proved that there exist at least two distinct solutions to the system (1) for any prescribed distribution of vortices by applying a variational approach. Here, a necessary condition and some sufficient condition on the coupling parameters were derived for the existence results. In [32], they also established planar topological vortex solutions of a generalized m-coupled system ($m \geq 2$) in the whole plane $\mathbb{R}^2$. We refer to [14,17,18,22,25,31,33,34,39,48,55,67] for recent developments on the equation (1).

In [32], it was also shown that one of two constructed solutions to (1) over $\mathbb{T}$ satisfies that both components converge to 0 pointwise a.e. in $\mathbb{T}$ as coupling parameters tend to infinity. More precisely, we quote some of results as follows:

Theorem A ([32, Theorem 2.2]). *For any given constant $\eta > 1$, assume $1 < \frac{\beta}{\alpha} < \eta$. Then there exists a constant $M_\eta > 0$ such that if $\alpha > M_\eta$, then the system* (1) *admits at least two distinct solutions over $\mathbb{T}$. Furthermore, one of the solutions satisfies the following asymptotic behavior*:

$$
u_{\alpha,\beta} + v_{\alpha,\beta} \to 0, \quad u_{\alpha,\beta} - v_{\alpha,\beta} \to 0 \quad \textit{as} \quad \alpha \to \infty \quad \textit{pointwise a.e. in } \mathbb{T}. \tag{2}
$$

The aim of this paper is to prove that solutions to (1) satisfying (2) are uniquely determined by a given configuration of vortex points p_i ($1 \leq i \leq N$), provided that $\beta/\alpha > 1$ is sufficiently close to 1 and $\alpha > 0$ is large enough. As in Section 4 of [32], let us consider $(u_1, u_2) := (u_{\alpha,\beta} + v_{\alpha,\beta}, u_{\alpha,\beta} - v_{\alpha,\beta})$ for a solution $(u_{\alpha,\beta}, v_{\alpha,\beta})$ to (1). Then (1) is transformed into the following system in terms of (u_1, u_2):

$$\begin{cases} \Delta u_1 = (\alpha+\beta)^2 e^{u_1}(e^{u_1}-1) + (\alpha-\beta)^2 e^{u_2}(e^{u_1}-1) \\ \qquad\qquad - (\beta^2-\alpha^2)(e^{u_1}+e^{u_2})(e^{u_2}-1) + 8\pi \sum_{i=1}^{N} m_i \delta_{p_i}, \\ \Delta u_2 = (\alpha+\beta)^2 e^{u_2}(e^{u_2}-1) + (\alpha-\beta)^2 e^{u_1}(e^{u_2}-1) \\ \qquad\qquad - (\beta^2-\alpha^2)(e^{u_1}+e^{u_2})(e^{u_1}-1). \end{cases} \tag{3}$$

In the case when $\beta/\alpha > 1$ is sufficiently close to 1 and $\alpha+\beta$ is sufficiently large, the system (3) is almost decoupled, and each equation of (3) can be regarded as a perturbation of the following elliptic equation:

$$\Delta v_\varepsilon + \frac{1}{\varepsilon^2} e^{v_\varepsilon}(1-e^{v_\varepsilon}) = 4\pi \sum_{i=0}^{N_0} n_i \delta_{q_i} \tag{4}$$

with $\varepsilon := 1/(\alpha+\beta)$. The equation (4) arises in the study of high temperature superconductivity. In the relativistic Chern-Simons model suggested by Hong, Kim and Pac [37] and Jackiw and Weinberg [41], self-dual equations satisfied by energy minimizers can be reduced to the nonlinear elliptic equation (4) involving exponential nonlinearity (see [37, 41, 67, 71] for the details). During the last few decades, the equation (4) has been extensively studied, and we refer the readers to [13, 20, 21, 38, 47, 53, 64, 67]. Among them, in [20, 65], the uniqueness of topological solutions to (4) has been proved not only over $\mathbb{T}$ but also over $\mathbb{R}^2$. Here, a solution v_ε of (4) over $\mathbb{T}$ is called a topological solution if

$$v_\varepsilon \to 0 \quad \text{as} \quad \varepsilon \to 0 \quad \text{pointwise a.e. in } \mathbb{T}.$$

Similarly, we say that $(u_{\alpha,\beta}, v_{\alpha,\beta})$ is a topological solution to (1) over $\mathbb{T}$ if it satisfies (2). Note that $\beta \to \infty$ as $\alpha \to \infty$ assuming that $1 < \frac{\beta}{\alpha} < \eta$ in Theorem A.

In the paper, we shall prove that a topological solution to (1) over $\mathbb{T}$ is unique under the assumption that $\beta/\alpha > 1$ is sufficiently close to 1 and $\alpha > 0$ is sufficiently large. For the purpose, we will study the asymptotic behavior of solutions (u_1, u_2) to the reduced system (3) which is a perturbed system of (4) as α tends to infinity. In the analysis of solutions to (3), a uniform boundedness of L^1 norm of nonlinear terms with respect to coupling parameters would play an important role. But it seems difficult to derive a uniform L^1 boundedness of the nonlinearities when $\beta^2 - \alpha^2$ is too big. In order to guarantee the uniform L^1 boundedness with respect to large $\alpha > 0$, a weak coupling effect is imposed throughout the paper, that is, positive parameters α and β satisfy

$$0 < \beta^2 - \alpha^2 = (\beta-\alpha)(\alpha+\beta) \le \mathfrak{N} \tag{5}$$

for some constant $\mathfrak{N} > 0$. Notice that the assumption (5) implies that

$$\frac{\beta}{\alpha} \to 1 \quad \text{as} \quad \alpha \to \infty.$$

We also mention that a classification result of Brezis-Merle type for solutions to (1) over $\mathbb{T}$ was obtained in [46] in terms of their asymptotic behavior as $\alpha \to \infty$ under the assumption (5). In fact, the asymptotic behavior (2) is one of the alternatives; see Proposition 2.5 for some preliminary results.

Now we state our main results in the paper. Throughout the paper, let us fix vortex points $p_i \in \mathbb{T}$ $(1 \le i \le N)$ with multiplicity $m_i \in \mathbb{N}$ which satisfy $p_i \ne p_j$ for $i \ne j$. Before proving the uniqueness of topological solutions to (1), we will first obtain a priori estimate of topological solutions comparing with the entire topological solutions on the whole plane $\mathbb{R}^2$. For each $1 \le j \le N$, let us consider the following elliptic problem:

$$\begin{cases} \Delta U_j + e^{U_j}(1 - e^{U_j}) = 8\pi m_j \delta_0 \quad \text{in } \mathbb{R}^2, \\ \lim_{|x| \to +\infty} U_j(x) = 0. \end{cases} \tag{6}$$

The problem (6) admits a unique solution U_j, and we call U_j the entire topological solution to (6). Indeed, U_j is radially symmetric about the origin and negative in $\mathbb{R}^2$ (see Section 2 for important properties of U_j). Then we have a priori estimate of topological solutions as follows:

Theorem 1.1. *Assume that $\beta > \alpha > 0$ satisfy the condition* (5). *Let $(u_{\alpha,\beta}, v_{\alpha,\beta})$ be a sequence of solutions to* (1) *satisfying* (2). *Then, up to subsequences, the following asymptotic behavior hold*:

$$\left\| u_{\alpha,\beta} + v_{\alpha,\beta} - \sum_{j=1}^{N} U_j\Big((\alpha+\beta)(x-p_j)\Big)\chi_d(|x-p_j|) \right\|_{L^\infty(\mathbb{T})} = O\left(\frac{1}{(\alpha+\beta)^2}\right)$$

and

$$\|u_{\alpha,\beta} - v_{\alpha,\beta}\|_{L^\infty(\mathbb{T})} = O\left(\frac{1}{(\alpha+\beta)^2}\right) \qquad \text{as } \alpha \to \infty.$$

Here, U_j solves the problem (6), *and a cut-off function χ_d satisfies that $\chi_d \equiv 1$ on $B_{\frac{d}{2}}(0)$, $\chi_d \equiv 0$ on $\mathbb{R}^2 \setminus B_d(0)$ and $0 \le \chi_d \le 1$ with a small fixed constant $d > 0$ depending only on p_j $(\le j \le N)$.*

In order to show a prior estimate in Theorem 1.1, we employ a blow up analysis around each vortex point, and uniform estimates of relevant linearized operators (see Theorem 2.7).

Based on a priori estimate in Theorem 1.1, we prove the uniqueness of topological solutions to (1).

Theorem 1.2. *For given $N \in \mathbb{N}$, let $p_i \in \mathbb{T}$ $(1 \le i \le N)$ be vortex points with multiplicity $m_i \in \mathbb{N}$ such that $p_i \ne p_j$ for $i \ne j$. Assume that $\beta > \alpha > 0$ satisfy the condition* (5). *Then a topological solution of* (1) *is unique for sufficiently large $\alpha > 0$.*

The rest of the paper is organized as follows. In Section 2, we review some preliminary results used in the paper. In Section 3, we study a priori estimate of topological solutions and prove Theorem 1.1. Also, the uniqueness of topological solutions in Theorem 1.2 is obtained.

2. Preliminaries

Throughout the paper, we will use the following notation:

$$\varepsilon := \frac{1}{\alpha+\beta} \quad \text{and} \quad \sigma_{\alpha,\beta} := \frac{\beta-\alpha}{\alpha+\beta} \tag{7}$$

for positive parameters $\beta > \alpha > 0$. For notational simplicity, we denote $\sigma_{\alpha,\beta}$ by σ_ε. Then the assumption of a weak coupling effect (5) is equivalent to

$$0 < \sigma_\varepsilon = \sigma_{\alpha,\beta} = \frac{\beta-\alpha}{\alpha+\beta} \le \mathfrak{N}\varepsilon^2, \tag{8}$$

and the elliptic system (3) can be rewritten as follows:

$$\begin{cases} \Delta u_{1,\varepsilon} = \dfrac{1}{\varepsilon^2}\left\{e^{u_{1,\varepsilon}}(e^{u_{1,\varepsilon}}-1) + \sigma_\varepsilon^2 e^{u_{2,\varepsilon}}(e^{u_{1,\varepsilon}}-1) - \sigma_\varepsilon(e^{u_{1,\varepsilon}}+e^{u_{2,\varepsilon}})(e^{u_{2,\varepsilon}}-1)\right\} \\ \qquad\qquad + 8\pi \displaystyle\sum_{i=1}^N m_i \delta_{p_i}, \\ \Delta u_{2,\varepsilon} = \dfrac{1}{\varepsilon^2}\left\{e^{u_{2,\varepsilon}}(e^{u_{2,\varepsilon}}-1) + \sigma_\varepsilon^2 e^{u_{1,\varepsilon}}(e^{u_{2,\varepsilon}}-1) - \sigma_\varepsilon(e^{u_{1,\varepsilon}}+e^{u_{2,\varepsilon}})(e^{u_{1,\varepsilon}}-1)\right\}. \end{cases} \tag{9}$$

As mentioned in the introduction, (9) can be considered as a perturbed system of (4). For the system above, $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is said to be a topological solution to (9) if

$$u_{i,\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \text{ pointwise a.e. in } \mathbb{T} \quad \text{for } i = 1, 2. \tag{10}$$

We first recall some results on the limiting equation of (4). Let w be a solution to the elliptic equation

$$\Delta w + e^w(1-e^w) = 4\pi m \delta_0 \quad \text{in } \mathbb{R}^2. \tag{11}$$

Lemma 2.1 ([21, Lemma 3.2], [9,16])**.** *Let m be a nonnegative integer, and w be a solution of* (11) *with $e^w(1-e^w) \in L^1(\mathbb{R}^2)$. Then, either*

(i) *$w(x) \to 0$ as $|x| \to \infty$, or*

(ii) *$w(x) = -\gamma_0 \ln|x| + O(1)$ as $|x| \to \infty$, where a constant $\gamma_0 > 0$ is given by*

$$\gamma_0 = -2m + \frac{1}{2\pi}\int_{\mathbb{R}^2} e^w(1-e^w)dx.$$

Under the assumption that w satisfies the boundary condition (ii)*, we have*

$$\int_{\mathbb{R}^2} e^{2w}dx = \pi(\gamma_0^2 - 4\gamma_0 - 4m^2 - 8m) \text{ and } \int_{\mathbb{R}^2} e^w dx = \pi(\gamma_0^2 - 2\gamma_0 - 4m^2 - 4m).$$

In particular, $\int_{\mathbb{R}^2} e^w(1-e^w)dx > 8\pi(1+m)$, and thus $\gamma_0 > 2(m+2)$.

When $m = 0$, it has been known that the integral $\int_{\mathbb{R}^2} e^w(1-e^w)dx$ depends on the maximum value of w, and has a lower bound as follows.

Lemma 2.2 ([13, Theorem 2.1], [15, Theorem 3.2], [63, Theorem 2.2])**.** *Let $m = 0$, and w be a solution of* (11) *with $e^w(1-e^w) \in L^1(\mathbb{R}^2)$. Then, $w(x)$ is smooth, radially symmetric with respect to some point x_0 in $\mathbb{R}^2$, and it is a nonpositive, decreasing function of $r = |x - x_0|$.*

Moreover, suppose that $w(r; s)$ is the radially symmetric solution with respect to the origin of (11) *satisfying*

$$\lim_{r\to 0} w(r; s) = s \quad \textit{and} \quad \lim_{r\to 0} w'(r; s) = 0,$$

where $w'(r; s)$ denotes $\frac{dw}{dr}(r; s)$. Then one has

(i) $w(\cdot; 0) \equiv 0$.

(ii) *If $s < 0$, then $w(r; s)$ is strictly decreasing in r, and $w(r; s) \to -\infty$ as $r \to \infty$.*

(iii) *Let $\gamma_0 : (-\infty, 0) \to (0, +\infty)$ be a function defined by*

$$\gamma_0(s) \equiv \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{w(r;s)}(1 - e^{w(r;s)})dx = \int_0^{\infty} e^{w(r;s)}(1 - e^{w(r;s)})rdr.$$

Then, $\gamma_0 : (-\infty, 0) \to (4, +\infty)$ is strictly increasing and bijective, and satisfies

$$\lim_{s\to -\infty} \gamma_0(s) = 4 \quad \textit{and} \quad \lim_{s\to 0^-} \gamma_0(s) = +\infty.$$

Now we are concerned with properties of topological solutions to (11) satisfying the boundary condition (i) in Lemma 2.1. For any positive integer m, consider the following problem

$$\begin{cases} \Delta U + e^U(1 - e^U) = 4\pi m\delta_0 \quad \text{in } \mathbb{R}^2, \\ \lim\limits_{|x|\to +\infty} U(x) = 0. \end{cases} \tag{12}$$

The existence and uniqueness of a radially symmetric solution to (12) was established in [15]. In [29, Theorem 1.3], Han showed that a solution to (12) should be radially symmetric about the origin. Then the uniqueness of the solution to (12) follows from [15]. Here, the unique solution U of (12) is called an entire topological solution of the Chern-Simons-Higgs equation. For $m > 0$, the entire topological solution $U(x) = U(|x|)$ is negative in $\mathbb{R}^2$ and strictly increasing with respect to $|x|$. Moreover, it was obtained in [65, Lemma 4.13] that for any $\delta \in (0, 1)$, there exists a constant $C_\delta > 0$ satisfying

$$(1 - e^{U(x)}) + |\nabla U(x)| + |U(x)| \le C_\delta e^{-(1-\delta)|x|} \quad \text{for any } |x| \ge 1. \tag{13}$$

Next, we present some results on a priori estimates of solutions $(u_{1,\varepsilon}, u_{2,\varepsilon})$ to (9).

Lemma 2.3 ([32, Proposition 4.1])**.** *Let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be solutions of* (9) *over* $\mathbb{T}$. *Then, it holds that*

$$u_{1,\varepsilon}(x) < 0 \quad \text{and} \quad u_{2,\varepsilon}(x) < 0 \quad \text{for any} \quad x \in \mathbb{T}.$$

Using the assumption (8), we have the uniform $L^1(\mathbb{T})$ boundedness of the main nonlinear terms in (9) with respect to $\varepsilon > 0$.

Lemma 2.4 ([46, Lemma 2.4])**.** *Assume that the condition* (8) *holds. Let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be solutions of* (9) *over* $\mathbb{T}$. *Then, we have*

$$\int_{\mathbb{T}} \frac{1}{\varepsilon^2} e^{u_{1,\varepsilon}} (1 - e^{u_{1,\varepsilon}})\, dx = \int_{\mathbb{T}} \frac{1}{\varepsilon^2} e^{u_{1,\varepsilon}} |1 - e^{u_{1,\varepsilon}}|\, dx \le 8\pi\mathfrak{M} + 2\mathfrak{N}|\mathbb{T}| =: 8\pi l_0,$$

where $\mathfrak{M} = \sum_{i=1}^N m_i$, *and*

$$\int_{\mathbb{T}} \frac{1}{\varepsilon^2} e^{u_{2,\varepsilon}} (1 - e^{u_{2,\varepsilon}})\, dx = \int_{\mathbb{T}} \frac{1}{\varepsilon^2} e^{u_{2,\varepsilon}} |1 - e^{u_{2,\varepsilon}}|\, dx \le 2\mathfrak{N}|\mathbb{T}|.$$

We also quote the following result on the asymptotic behavior of solutions to (9).

Proposition 2.5 ([46, Proposition 3.1])**.** *Assume that the condition* (8) *holds. Let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be a sequence of solutions to* (9) *over* $\mathbb{T}$, *and let* $Z := \{p_1, \ldots, p_N\}$. *Then the following properties hold.*

(i) *Up to subsequences, $u_{1,\varepsilon}$ satisfies one of the following*:
  (1a) *$u_{1,\varepsilon} \to 0$ uniformly on any compact subset of $\mathbb{T} \setminus Z$ as $\varepsilon \to 0$*;
  (1b) *there exists a constant $\nu_0 > 0$ such that*

$$\sup_{\varepsilon \to 0} \left(\sup_{\mathbb{T}} u_{1,\varepsilon} \right) \le -\nu_0.$$

(ii) *Up to subsequences, $u_{2,\varepsilon}$ satisfies one of the following*:
  (2a) *there is a constant $c_0 > 0$, independent of $\varepsilon > 0$, satisfying $\|u_{2,\varepsilon}\|_{L^\infty(\mathbb{T})} \le c_0 \varepsilon^2$*;
  (2b) *there exists a constant $\nu_0 > 0$ such that*

$$\sup_{\varepsilon \to 0} \left(\sup_{\mathbb{T}} u_{2,\varepsilon} \right) \le -\nu_0.$$

In order to remove singularities, let us introduce the Green function $G(x, y)$ which satisfies

$$-\Delta_x G(x, y) = \delta_y - \frac{1}{|\mathbb{T}|} \quad \forall x, y \in \mathbb{T} \qquad \text{with} \quad \int_{\mathbb{T}} G(x, y) dy = 0,$$

where $|\mathbb{T}|$ is the measure of $\mathbb{T}$. We denote the regular part of $G(x, y)$ by

$$\gamma(x, y) = G(x, y) + \frac{1}{2\pi} \ln |x - y|.$$

Let

$$u_0(x) := -8\pi \sum_{i=1}^N m_i G(x, p_i) \qquad \text{and} \qquad v_{1,\varepsilon} := u_{1,\varepsilon} - u_0.$$

Assuming that $|\mathbb{T}|=1$, (9) can be written in terms of $(v_{1,\varepsilon},u_{2,\varepsilon})$ as follows:

$$\begin{cases} \Delta v_{1,\varepsilon} = \dfrac{1}{\varepsilon^2}\left\{e^{v_{1,\varepsilon}+u_0}(e^{v_{1,\varepsilon}+u_0}-1)+\sigma_\varepsilon^2 e^{u_{2,\varepsilon}}(e^{v_{1,\varepsilon}+u_0}-1)\right. \\ \qquad\qquad \left. -\sigma_\varepsilon(e^{v_{1,\varepsilon}+u_0}+e^{u_{2,\varepsilon}})(e^{u_{2,\varepsilon}}-1)\right\} + 8\pi\mathfrak{M}, \\ \Delta u_{2,\varepsilon} = \dfrac{1}{\varepsilon^2}\left\{e^{u_{2,\varepsilon}}(e^{u_{2,\varepsilon}}-1)+\sigma_\varepsilon^2 e^{v_{1,\varepsilon}+u_0}(e^{u_{2,\varepsilon}}-1)\right. \\ \qquad\qquad \left. -\sigma_\varepsilon(e^{v_{1,\varepsilon}+u_0}+e^{u_{2,\varepsilon}})(e^{v_{1,\varepsilon}+u_0}-1)\right\}. \end{cases}$$

Then a priori gradient estimate of $(v_{1,\varepsilon},u_{2,\varepsilon})$ was obtained by using the Green representation.

Lemma 2.6 ([46, Lemma 2.6])**.** *Assume that the condition* (8) *holds. Let* $(u_{1,\varepsilon},u_{2,\varepsilon})$ *be a sequence of solutions to* (9) *over* $\mathbb{T}$. *Then there exists a uniform constant* $C>0$, *independent of* $\varepsilon>0$, *such that*

$$\|\nabla v_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}+\|\nabla u_{2,\varepsilon}\|_{L^\infty(\mathbb{T})}\le \frac{C}{\varepsilon}.$$

Lastly, let us define the following linearized operators:

$$\begin{cases} L_{1,\varepsilon}(\phi) := \Delta\phi + \dfrac{1}{\varepsilon^2}\left[\displaystyle\sum_{j=1}^N g'\left(U_j\left(\frac{x-p_j}{\varepsilon}\right)\right)\chi_d(|x-p_j|)\right. \\ \qquad\qquad\qquad \left. +g'(0)\Big(1-\displaystyle\sum_{j=1}^N \chi_d(|x-p_j|)\Big)\right]\phi, \\ L_{2,\varepsilon}(\phi) := \Delta\phi + \dfrac{g'(0)}{\varepsilon^2}\phi = \Delta\phi - \dfrac{1}{\varepsilon^2}\phi, \end{cases}$$

where

$$g(t) := e^t(1-e^t).$$

Employing the linearized operators enables us to deduce a refined asymptotic behavior of $(u_{1,\varepsilon},u_{2,\varepsilon})$. Following the proof of [24, Theorem 2.4], we have the the solvability results of the operators $L_{1,\varepsilon}$ and $L_{2,\varepsilon}$ together with uniform L^∞ estimates.

Theorem 2.7. *The operator*

$$L_{1,\varepsilon}: W^{2,2}(\mathbb{T})\to L^2(\mathbb{T})$$

is an isomorphism. Moreover, there exists a uniform constant $C>0$ *such that if* $(\phi,h)\in W^{2,2}(\mathbb{T})\times L^2(\mathbb{T})$ *satisfies* $L_{1,\varepsilon}(\phi)=h$ *on* $\mathbb{T}$ *with* $\phi,h\in L^\infty(\mathbb{T})$, *then*

$$\|\phi\|_{L^\infty(\mathbb{T})}\le C\varepsilon^2\|h\|_{L^\infty(\mathbb{T})}.$$

Also, $L_{2,\varepsilon}: W^{2,2}(\mathbb{T})\to L^2(\mathbb{T})$ *is an isomorphism, and if* $(\phi,h)\in W^{2,2}(\mathbb{T})\times L^2(\mathbb{T})$ *satisfies* $L_{2,\varepsilon}(\phi)=h$ *on* $\mathbb{T}$ *with* $\phi,h\in L^\infty(\mathbb{T})$, *then*

$$\|\phi\|_{L^\infty(\mathbb{T})}\le C\varepsilon^2\|h\|_{L^\infty(\mathbb{T})}.$$

Here, $C > 0$ is a uniform constant which is independent of $\varepsilon > 0$.

3. A priori estimate and uniqueness

3.1. Proof of Theorem 1.1

In the remaining part of the paper, assume that the condition (8) holds, and let $(u_{1,\varepsilon}, u_{2,\varepsilon})$ be a solution of (9) over $\mathbb{T}$ satisfying (10), that is,

$$u_{i,\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0 \text{ pointwise a.e. in } \mathbb{T} \quad \text{for } i = 1, 2.$$

Here, we recall that $(u_{1,\varepsilon}, u_{2,\varepsilon})$ is called a topological solution if it solves (9) over $\mathbb{T}$ satisfying (10).

Remark 3.1. We note that a topological solution $(u_{1,\varepsilon}, u_{2,\varepsilon})$ of (9) satisfies (1a) and (2a) of Proposition 2.5, i.e.,

- (1a) $u_{1,\varepsilon} \to 0$ uniformly on any compact subset of $\mathbb{T} \setminus Z$ as $\varepsilon \to 0$;
- (2a) $\|u_{2,\varepsilon}\|_{L^\infty(\mathbb{T})} \le c_0\varepsilon^2$ for some uniform constant $c_0 > 0$ (independent of $\varepsilon > 0$).

To analyze the asymptotic behavior of topological solutions, we compare $u_{1,\varepsilon}$ with a scaled function of the entire topological solution of (6) near singularities.

Lemma 3.2. *Assume that the condition* (8) *holds. Let* $(u_{1,\varepsilon}, u_{2,\varepsilon})$ *be a topological solution of* (9)*. Then we have*

$$\lim_{\varepsilon\to 0}\left\| u_{1,\varepsilon} - \sum_{j=1}^{N} U_j\Big(\frac{x-p_j}{\varepsilon}\Big)\chi_d(|x-p_j|)\right\|_{L^\infty(\mathbb{T})} = 0. \tag{14}$$

Here, U_j is the entire topological solution of (6)*, and a cut-off function χ_d satisfies that $\chi_d \equiv 1$ on $B_{\frac{d}{2}}(0)$, $\chi_d \equiv 0$ on $\mathbb{R}^2 \setminus B_d(0)$ and $0 \le \chi_d \le 1$ with a small fixed constant $d > 0$ depending only on p_j $(1 \le j \le N)$.*

Proof. (i) Fix $1 \le j \le N$ and let $\bar{u}_{i,\varepsilon}(x) := u_{i,\varepsilon}(\varepsilon x + p_j)$ for $i = 1, 2$. Then $\bar{u}_{1,\varepsilon}$ satisfies

$$\begin{cases} \Delta\bar{u}_{1,\varepsilon} + e^{\bar{u}_{1,\varepsilon}}(1-e^{\bar{u}_{1,\varepsilon}}) \\ = \sigma_\varepsilon^2 e^{\bar{u}_{2,\varepsilon}}(e^{\bar{u}_{1,\varepsilon}}-1) - \sigma_\varepsilon(e^{\bar{u}_{1,\varepsilon}} + e^{\bar{u}_{2,\varepsilon}})(e^{\bar{u}_{2,\varepsilon}}-1) + 8\pi m_j\delta_0 \quad \text{in } B_{\frac{d}{\varepsilon}}(0), \\ \displaystyle\int_{B_{\frac{d}{\varepsilon}}(0)} e^{\bar{u}_{1,\varepsilon}}|(1-e^{\bar{u}_{1,\varepsilon}})|dx \le 8\pi l_0. \end{cases}$$

Here we used Lemma 2.4 and $d > 0$ is chosen so that $B_{2d}(p_i) \cap B_{2d}(p_j) = \emptyset$ for $i \ne j$. Since $|\nabla(u_{1,\varepsilon} - u_0)| = O(\varepsilon^{-1})$ in $\mathbb{T}$ by Lemma 2.6, it follows that

$$\left|\nabla\bar{u}_{1,\varepsilon}(x) - \frac{4m_j x}{|x|^2}\right| = O(1) \quad \text{in } B_{\frac{d}{\varepsilon}}(0). \tag{15}$$

Firstly, we claim that there exists a constant $C_0 > 0$, independent of $\varepsilon > 0$, such that

$$\|\bar{u}_{1,\varepsilon}\|_{L^\infty(B_{\frac{d}{\varepsilon}}(0)\setminus B_1(0))} \le C_0 \quad \text{for small } \varepsilon > 0. \tag{16}$$

To prove the claim above, suppose to the contrary that up to a subsequence, there exists $x_\varepsilon \in B_{\frac{d}{\varepsilon}}(0) \setminus B_1(0)$ such that

$$\lim_{\varepsilon \to 0} |\bar{u}_{1,\varepsilon}(x_\varepsilon)| = +\infty.$$

Since $\max_{\mathbb{T}} u_{1,\varepsilon} \le 0$ by Lemma 2.3 , we see that $\lim_{\varepsilon \to 0} \bar{u}_{1,\varepsilon}(x_\varepsilon) = -\infty$. We shall now distinguish two cases.

Case 1: $\lim_{\varepsilon \to 0} |x_\varepsilon|$ is bounded.

We first observe that $\lim_{\varepsilon \to 0} (\sup_{|x| = \frac{r}{\varepsilon}} |\bar{u}_{1,\varepsilon}(x)|) = 0$ for any fixed constant $0 < r < d$ from Remark 3.1. Recalling the function γ_0 given in Lemma 2.2, let us fix a constant $s_0 < 0$ such that $\gamma_0(s_0) \ge 8 l_0$. Since $\lim_{\varepsilon \to 0} \bar{u}_{1,\varepsilon}(x_\varepsilon) = -\infty$, the intermediate value theorem implies that there exists $y_\varepsilon \in B_{\frac{d}{\varepsilon}}(0) \setminus B_1(0)$ satisfying

$$\bar{u}_{1,\varepsilon}(y_\varepsilon) = s_0 < 0.$$

Since $x_\varepsilon,\ y_\varepsilon \in B_{\frac{d}{\varepsilon}}(0) \setminus B_1(0)$, it holds from the gradient estimation (15) that

$$\begin{aligned} s_0 = \bar{u}_{1,\varepsilon}(y_\varepsilon) &= \bar{u}_{1,\varepsilon}(x_\varepsilon) + (y_\varepsilon - x_\varepsilon) \cdot \int_0^1 \nabla \bar{u}_{1,\varepsilon}(x_\varepsilon + t(y_\varepsilon - x_\varepsilon)) dt \\ &= \bar{u}_{1,\varepsilon}(x_\varepsilon) + O(|y_\varepsilon - x_\varepsilon|). \end{aligned} \tag{17}$$

Since $\lim_{\varepsilon \to 0} \bar{u}_{1,\varepsilon}(x_\varepsilon) = -\infty$ and $\lim_{\varepsilon \to 0} |x_\varepsilon|$ is bounded in this case, (17) implies that $\lim_{\varepsilon \to 0} |y_\varepsilon| = +\infty$. Then utilizing the fact that $\bar{u}_{1,\varepsilon}(y_\varepsilon) = s_0$ and the gradient estimation (15), we deduce from Ascoli's theorem that $\bar{u}_{1,\varepsilon}(\cdot + y_\varepsilon)$ converges to a function $\bar{u}$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$ up to a subsequence. By (8) and Lemma 2.3, we have that

$$\sigma_\varepsilon^2 e^{u_{2,\varepsilon}}(e^{u_{1,\varepsilon}} - 1) - \sigma_\varepsilon (e^{u_{1,\varepsilon}} + e^{u_{2,\varepsilon}})(e^{u_{2,\varepsilon}} - 1) \to 0 \quad \text{as } \varepsilon \to 0 \quad \text{uniformly in } \mathbb{T}. \tag{18}$$

This combined with Lemma 2.4 yields that $\bar{u}$ satisfies

$$\begin{cases} \Delta \bar{u} + e^{\bar{u}}(1 - e^{\bar{u}}) = 0 \quad \text{in } \mathbb{R}^2, \\ \bar{u}(0) = s_0, \quad \displaystyle\int_{\mathbb{R}^2} e^{\bar{u}} |(1 - e^{\bar{u}})| dx \le 8\pi l_0. \end{cases} \tag{19}$$

In view of Lemma 2.2, $\bar{u}$ is a radially symmetric function with respect to some point $y_0 \in \mathbb{R}^2$. Since $\gamma_0(s)$ is increasing in s from Lemma 2.2 and $\bar{u}(x)$ is decreasing with respect to $|x - y_0|$, we have that

$$8\pi l_0 \ge \int_{\mathbb{R}^2} e^{\bar{u}}(1 - e^{\bar{u}}) dx \ge 2\pi \gamma_0(\bar{u}(y_0)) \ge 2\pi \gamma_0(s_0) \ge 16\pi l_0,$$

which implies a contradiction.

Case 2: $\lim_{\varepsilon \to 0} |x_\varepsilon| = +\infty$.

Since $\lim_{\varepsilon \to 0} \bar{u}_{1,\varepsilon}(x_\varepsilon) = -\infty$, it holds that $\lim_{\varepsilon \to 0} \varepsilon |x_\varepsilon| = 0$. Here we used the fact that $\lim_{\varepsilon\to 0}\sup_{\frac{r}{\varepsilon}\le|x|\le\frac{d}{\varepsilon}}|\bar{u}_{1,\varepsilon}(x)|=0$ for any fixed constant $0<r<d$ from Remark 3.1. Then by the intermediate value theorem, there exists $y_\varepsilon\in B_{\frac{d}{\varepsilon}}(0)\setminus B_{|x_\varepsilon|}(0)$ such that

$$\bar{u}_{1,\varepsilon}(y_\varepsilon)=s_0<0,$$

where a constant $s_0<0$ is chosen so that $\gamma_0(s_0)\ge 8l_0$. Since $\lim_{\varepsilon\to 0}|x_\varepsilon|=+\infty$ in this case, we see that $\lim_{\varepsilon\to 0}|y_\varepsilon|=+\infty$. Thus by a similar argument as for the case 1, using the fact that $\bar{u}_{1,\varepsilon}(y_\varepsilon)=s_0$, (15) and (18) implies that $\bar{u}_{1,\varepsilon}(\cdot+y_\varepsilon)$ converges to a function $\bar{u}$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$, where $\bar{u}$ satisfies (19). Then $\bar{u}$ is a radially symmetric with respect to some point $y_0\in\mathbb{R}^2$ by Lemma 2.2. Since γ_0 is an increasing function by Lemma 2.2, it follows that

$$8\pi l_0\ge\int_{\mathbb{R}^2}e^{\bar{u}}(1-e^{\bar{u}})dx\ge 2\pi\gamma_0(\bar{u}(y_0))\ge 2\pi\gamma_0(s_0)\ge 16\pi l_0,$$

which is a contradiction. Thus we have proved the claim (16) in both cases.

(ii) Next, we will prove that for any $1\le j\le N$,

$$u_{1,\varepsilon}(\varepsilon x+p_j)-U_j(x)\to 0\quad\text{in}\quad C^0_{\mathrm{loc}}(\mathbb{R}^2)\quad\text{as}\quad\varepsilon\to 0. \tag{20}$$

Fix $1\le j\le N$ and let $\bar{v}_{1,\varepsilon}(x):=\bar{u}_{1,\varepsilon}(x)-4m_j\ln|x|=u_{1,\varepsilon}(\varepsilon x+p_j)-4m_j\ln|x|$. Then, $\bar{v}_{1,\varepsilon}$ satisfies the following:

$$\begin{cases}\Delta\bar{v}_{1,\varepsilon}(x)+|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)}(1-|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)})\\ =\sigma_\varepsilon^2e^{\bar{u}_{2,\varepsilon}(x)}(|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)}-1)\\ -\sigma_\varepsilon(|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)}+e^{\bar{u}_{2,\varepsilon}(x)})(e^{\bar{u}_{2,\varepsilon}(x)}-1)\quad\text{in}\quad B_{\frac{d}{\varepsilon}}(0),\\ \displaystyle\int_{B_{\frac{d}{\varepsilon}}(0)}|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)}|(1-|x|^{4m_j}e^{\bar{v}_{1,\varepsilon}(x)})|dx\le 8\pi l_0.\end{cases}$$

From (16), it holds that

$$\|\bar{v}_{1,\varepsilon}\|_{L^\infty(\partial B_1(0))}\le C_0\quad\text{for small}\quad\varepsilon>0, \tag{21}$$

and by the gradient estimation (15), there exists a constant $c_0>0$ such that

$$\lim_{\varepsilon\to 0}\|\nabla\bar{v}_{1,\varepsilon}\|_{L^\infty(B_{\frac{d}{\varepsilon}}(0))}\le c_0. \tag{22}$$

Utilizing (21)-(22) and a similar argument as for (18), there is a function V_j such that $\bar{v}_{1,\varepsilon}(x)\to V_j(x)$ in $C^0_{\mathrm{loc}}(\mathbb{R}^2)$, up to subsequences, as $\varepsilon\to 0$, where V_j satisfies

$$\begin{cases}\Delta V_j+|x|^{4m_j}e^{V_j(x)}(1-|x|^{4m_j}e^{V_j(x)})=0\quad\text{in}\quad\mathbb{R}^2,\\ \displaystyle\int_{\mathbb{R}^2}|x|^{4m_j}e^{V_j(x)}|(1-|x|^{4m_j}e^{V_j(x)})|dx\le 8\pi l_0.\end{cases}$$

Let $\bar{U}_j := V_j + 4m_j \ln|x|$. Then $\bar{U}_j$ satisfies

$$\begin{cases} \Delta \bar{U}_j + e^{\bar{U}_j}(1 - e^{\bar{U}_j}) = 8\pi m_j \delta_0 \quad \text{in} \ \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{\bar{U}_j} |(1 - e^{\bar{U}_j})| dx \leq 8\pi l_0. \end{cases} \tag{23}$$

Since $\bar{v}_{1,\varepsilon}(x) = \bar{u}_{1,\varepsilon}(x) - 4m_j \ln|x| \to V_j(x) = \bar{U}_j(x) - 4m_j \ln|x|$ in $C^0_{\text{loc}}(\mathbb{R}^2)$, we obtain that $\bar{u}_{1,\varepsilon} - \bar{U}_j \to 0$ in $C^0_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$.

Now we will show that $\bar{U}_j$ is the topological solution of (6), i.e., $\bar{U}_j \equiv U_j$. In light of Lemma 2.1, suppose to the contrary that $\lim_{|x| \to +\infty} \bar{U}_j(x) = -\infty$. Since $\bar{u}_{1,\varepsilon} - \bar{U}_j \to 0$ in $C^0_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$, using the estimate (16) implies that

$$\|\bar{U}_j\|_{L^\infty(B_R(0) \setminus B_1(0))} \leq C_0 \quad \text{for any constant } R > 0,$$

where a constant $C_0 > 0$ is independent of $R > 0$. This contradicts the fact that $\lim_{|x| \to +\infty} \bar{U}_j(x) = -\infty$. Therefore, we deduce that $\bar{U}_j$ is the unique topological solution of (6) satisfying (23), namely, $\bar{U}_j \equiv U_j$. Since $\bar{u}_{1,\varepsilon}(x) - \bar{U}_j(x) = u_{1,\varepsilon}(\varepsilon x + p_j) - U_j(x) \to 0$ in $C^0_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$, we obtain the convergence in (20) for any $1 \leq j \leq N$.

(iii) Now, we will improve the convergence result of (20) to show (14). More precisely, we claim that for each $1 \leq j \leq N$,

$$\lim_{\varepsilon \to 0} \|u_{1,\varepsilon}(\varepsilon x + p_j) - U_j(x)\chi_d(|\varepsilon x|)\|_{L^\infty(B_{\frac{d}{\varepsilon}}(0))} = 0. \tag{24}$$

To prove (24), we argue by contradiction and suppose that there exist $j \in \{1, \ldots, N\}$ and $\rho > 0$ satisfying

$$\lim_{\varepsilon \to 0} \|u_{1,\varepsilon}(\varepsilon x + p_j) - U_j(x)\chi_d(|\varepsilon x|)\|_{L^\infty(B_{\frac{d}{\varepsilon}}(0))} > \rho.$$

Fix a constant $s_0 \in (-\rho, 0)$ such that $\gamma_0(s_0) \geq 8l_0$ in light of Lemma 2.2. Then using (20) and the intermediate value theorem, there exists $y_\varepsilon \in B_{\frac{d}{\varepsilon}}(0)$ satisfying

$$|u_{1,\varepsilon}(\varepsilon y_\varepsilon + p_j) - U_j(y_\varepsilon)\chi_d(|\varepsilon y_\varepsilon|)| = |s_0| \neq 0. \tag{25}$$

In view of (20), we see that $\lim_{\varepsilon \to 0} |y_\varepsilon| = +\infty$. Then it follows that $\lim_{\varepsilon \to 0} U_j(y_\varepsilon) = 0$ since U_j is the topological solution satisfying (23). Thus we deduce that

$$\lim_{\varepsilon \to 0} u_{1,\varepsilon}(\varepsilon y_\varepsilon + p_j) = s_0 < 0$$

by using (25), Lemma 2.3 and the fact that $s_0 < 0$.

Now let $\bar{\bar{u}}_{i,\varepsilon}(x) := u_{i,\varepsilon}(\varepsilon(x+y_\varepsilon)+p_j)$ for $i=1,2$. Then $\bar{\bar{u}}_{1,\varepsilon}$ satisfies

$$\begin{cases} \Delta\bar{\bar{u}}_{1,\varepsilon} + e^{\bar{\bar{u}}_{1,\varepsilon}}(1-e^{\bar{\bar{u}}_{1,\varepsilon}}) \\ = \sigma_\varepsilon^2 e^{\bar{\bar{u}}_{2,\varepsilon}}\left(e^{\bar{\bar{u}}_{1,\varepsilon}}-1\right) - \sigma_\varepsilon(e^{\bar{\bar{u}}_{1,\varepsilon}}+e^{\bar{\bar{u}}_{2,\varepsilon}})(e^{\bar{\bar{u}}_{2,\varepsilon}}-1) \quad \text{in } B_{\frac{|y_\varepsilon|}{2}}(0), \\ \displaystyle\int_{B_{\frac{|y_\varepsilon|}{2}}(0)} e^{\bar{\bar{u}}_{1,\varepsilon}(x)}|(1-e^{\bar{\bar{u}}_{1,\varepsilon}(x)})|dx \le 8\pi l_0, \\ \bar{\bar{u}}_{1,\varepsilon}(0) = s_0 + o(1) \quad \text{as } \varepsilon\to 0. \end{cases}$$

Here, we notice that $\lim_{\varepsilon\to0}|y_\varepsilon| = +\infty$, and $|x+y_\varepsilon| \ge |y_\varepsilon|/2$ for $x\in B_{\frac{|y_\varepsilon|}{2}}(0)$. Note that $\lim_{\varepsilon\to0}\bar{\bar{u}}_{1,\varepsilon}(0)=s_0$ and

$$\left|\nabla\bar{\bar{u}}_{1,\varepsilon}(x) - \frac{4m_j(x+y_\varepsilon)}{|x+y_\varepsilon|^2}\right| = O(1) \quad \text{in } B_{\frac{|y_\varepsilon|}{2}}(0)$$

in light of (15). Together with a similar argument for (18), we deduce that $\bar{\bar{u}}_{1,\varepsilon}(x)$ converges to a function $\bar{\bar{u}}$ in $C^0_{\text{loc}}(\mathbb{R}^2)$, and $\bar{\bar{u}}$ satisfies

$$\begin{cases} \Delta\bar{\bar{u}} + e^{\bar{\bar{u}}}(1-e^{\bar{\bar{u}}}) = 0 \quad \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} e^{\bar{\bar{u}}(x)}|(1-e^{\bar{\bar{u}}(x)})|dx \le 8\pi l_0, \qquad \bar{\bar{u}}(0)=s_0. \end{cases}$$

Lemma 2.2 implies that $\bar{\bar{u}}$ is radially symmetric with respect to some point $y_0\in\mathbb{R}^2$. Since γ_0 is an increasing function by Lemma 2.2, we have that

$$8\pi l_0 \ge \int_{\mathbb{R}^2} e^{\bar{\bar{u}}}|1-e^{\bar{\bar{u}}}|dx \ge 2\pi\gamma_0(\bar{\bar{u}}(y_0)) \ge 2\pi\gamma_0(s_0) \ge 16\pi l_0.$$

This is a contradiction. So we have proved (24).

Since

$$\sup_{\mathbb{T}\setminus\cup_{i=1}^N B_d(p_i)} |u_{1,\varepsilon}| \to 0 \quad \text{as } \varepsilon\to 0$$

by Remark 3.1, and $\chi_d(|x|)\equiv 0$ if $|x|\ge d$, we complete the proof of Lemma 3.2 by using (24). □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. To prove Theorem 1.1, it suffices to show the following refined asymptotic behavior for the first component $u_{1,\varepsilon}$:

$$\left\| u_{1,\varepsilon} - \sum_{j=1}^N U_j\Big(\frac{x-p_j}{\varepsilon}\Big)\chi_d(|x-p_j|)\right\|_{L^\infty(\mathbb{T})} = O(\varepsilon^2). \tag{26}$$

Here, we use the notation $u_{1,\varepsilon}=u_{\alpha,\beta}+v_{\alpha,\beta}$ and $u_{2,\varepsilon}=u_{\alpha,\beta}-v_{\alpha,\beta}$ as seen in the introduction and Section 2, and recall Remark 3.1. Let $U_{j,\varepsilon}(x) := U_j\Big(\frac{x-p_j}{\varepsilon}\Big)$

for $1 \le j \le N$, and let

$$\phi_{1,\varepsilon} := u_{1,\varepsilon} - \sum_{j=1}^{N} U_{j,\varepsilon}(x)\chi_d(|x-p_j|).$$

Since $\phi_{1,\varepsilon}$ belongs to $W^{2,2}(\mathbb{T}) \cap L^\infty(\mathbb{T})$ in light of the proof of Lemma 3.2, Theorem 2.7 implies that there exists a constant $C > 0$ satisfying

$$\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})} \le C\varepsilon^2 \|L_{1,\varepsilon}(\phi_{1,\varepsilon})\|_{L^\infty(\mathbb{T})}, \tag{27}$$

provided that $L_{1,\varepsilon}(\phi_{1,\varepsilon}) \in L^\infty(\mathbb{T})$. Since U_j $(1 \le j \le N)$ is the topological solution and has exponential decay to zero at infinity (see (13)), it holds that

$$\sup_{\frac{d}{2} \le |x-p_j| \le d} \Big\{(1-e^{U_{j,\varepsilon}(x)}) + \varepsilon|\nabla U_{j,\varepsilon}(x)| + |U_{j,\varepsilon}(x)|\Big\} \le Ce^{-\frac{c}{\varepsilon}}, \tag{28}$$

for small $\varepsilon > 0$ with some uniform constants $C > c > 0$, and hence we deduce that

$$\begin{aligned} &\Delta\Big[\sum_{j=1}^{N} U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big] \\ &= -\frac{1}{\varepsilon^2}\sum_{j=1}^{N} e^{U_{j,\varepsilon}(x)}\left(1-e^{U_{j,\varepsilon}(x)}\right)\chi_d(|x-p_j|) + 8\pi\sum_{j=1}^{N} m_j\delta_{p_j} + o(1) \end{aligned} \tag{29}$$

as $\varepsilon \to 0$. Here we remind that a constant $d > 0$ is independent of $\varepsilon > 0$, and χ_d is a smooth function such that $0 \le \chi_d \le 1$, $\chi_d \equiv 1$ in $B_{\frac{d}{2}}(0)$ and $\chi_d \equiv 0$ in $\mathbb{R}^2 \setminus B_d(0)$. By (8) and Lemma 2.3, we have

$$\sigma_\varepsilon^2 e^{u_{2,\varepsilon}}(e^{u_{1,\varepsilon}}-1) - \sigma_\varepsilon(e^{u_{1,\varepsilon}}+e^{u_{2,\varepsilon}})(e^{u_{2,\varepsilon}}-1) \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly on } \mathbb{T}.$$

Then it follows from (29) that

$$\begin{aligned} &L_{1,\varepsilon}(\phi_{1,\varepsilon}) \\ &= \Delta\phi_{1,\varepsilon} + \frac{1}{\varepsilon^2}\left[\sum_{j=1}^{N} g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|) - \Big(1-\sum_{j=1}^{N}\chi_d(|x-p_j|)\Big)\right]\phi_{1,\varepsilon} \\ &= \frac{1}{\varepsilon^2}\left\{\sum_{j=1}^{N} e^{U_{j,\varepsilon}(x)}\left(1-e^{U_{j,\varepsilon}(x))}\right)\chi_d(|x-p_j|)\right\} - \frac{1}{\varepsilon^2}e^{u_{1,\varepsilon}}(1-e^{u_{1,\varepsilon}}) \\ &\quad + \frac{1}{\varepsilon^2}\left[\sum_{j=1}^{N} g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|) - \Big(1-\sum_{j=1}^{N}\chi_d(|x-p_j|)\Big)\right]\phi_{1,\varepsilon} + O(1) \end{aligned}$$

as $\varepsilon \to 0$. Letting

$$\Phi_{1,\varepsilon}(x) := \sum_{j=1}^{N} g\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|) - g\left(u_{1,\varepsilon}\right) + \left[\sum_{j=1}^{N} g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|) - \Big(1 - \sum_{j=1}^{N}\chi_d(|x-p_j|)\Big)\right]\phi_{1,\varepsilon},$$

we have that

$$\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})} \le O(\varepsilon^2) + C\|\Phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})} \tag{30}$$

in view of (27), provided that $\Phi_{1,\varepsilon} \in L^\infty(\mathbb{T})$.

Now we will estimate $\|\Phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}$ in terms of $\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}$. Firstly, we see that in $B_{\frac{d}{2}}(p_j)$ for each $1 \le j \le N$,

$$\begin{aligned}\Phi_{1,\varepsilon}(x) &= g\Big(U_{j,\varepsilon}(x)\Big) - g(u_{1,\varepsilon}) + g'\Big(U_{j,\varepsilon}(x)\Big)\Big(u_{1,\varepsilon} - U_{j,\varepsilon}(x)\Big)\\ &= O\Big(|u_{1,\varepsilon} - U_{j,\varepsilon}|^2\Big) = O(|\phi_{1,\varepsilon}|^2).\end{aligned} \tag{31}$$

Here we recall that $u_{1,\varepsilon} < 0$ and $U_{j,\varepsilon} < 0$. Since $g'(0) = -1$, we get that in $B_d(p_j) \setminus B_{\frac{d}{2}}(p_j)$ for each $1 \le j \le N$,

$$\begin{aligned}\Phi_{1,\varepsilon}(x) = {}& g\Big(U_{j,\varepsilon}(x)\Big)\chi_d(|x-p_j|) - g\Big(U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big)\\ &+ g\Big(U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big) - g(u_{1,\varepsilon})\\ &+ g'\Big(U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big)\Big(u_{1,\varepsilon} - U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big)\\ &- \Big[g'\Big(U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big) - g'(0)\Big]\Big(u_{1,\varepsilon} - U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big)\\ &+ \Big[g'\Big(U_{j,\varepsilon}(x)\Big) - g'(0)\Big]\chi_d(|x-p_j|)\Big(u_{1,\varepsilon} - U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\Big).\end{aligned}$$

Then using (28), we deduce that in $B_d(p_j) \setminus B_{\frac{d}{2}}(p_j)$ for each $1 \le j \le N$,

$$\Phi_{1,\varepsilon}(x) = O(|\phi_{1,\varepsilon}|^2) + o(|\phi_{1,\varepsilon}|) + O\Big(e^{-\frac{c}{\varepsilon}}\Big). \tag{32}$$

In $\mathbb{T} \setminus \cup_{j=1}^N B_d(p_j)$, we have that $\chi_d(|x-p_j|) = 0$ and $\phi_{1,\varepsilon} = u_{1,\varepsilon}$. Since $g(0) = 0$ and $g'(0) = -1$, it holds that

$$\begin{aligned}\Phi_{1,\varepsilon}(x) &= -g(u_{1,\varepsilon}) - \phi_{1,\varepsilon} = -g(0) - g'(0)u_{1,\varepsilon} + O(|u_{1,\varepsilon}|^2) - \phi_{1,\varepsilon}\\ &= O(|\phi_{1,\varepsilon}|^2).\end{aligned} \tag{33}$$

In view of (30)-(33), we deduce that

$$\begin{aligned}\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})} &\le O(\varepsilon^2) + C\|\Phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}\\ &= O(\varepsilon^2) + O(\|\phi_{1,\varepsilon}\|^2_{L^\infty(\mathbb{T})}) + o(\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}).\end{aligned} \tag{34}$$

Since

$$\lim_{\varepsilon\to 0}\|\phi_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}=\lim_{\varepsilon\to 0}\left\|u_{1,\varepsilon}-\sum_{j=1}^N U_j\Big(\frac{x-p_j}{\varepsilon}\Big)\chi_d(|x-p_j|)\right\|_{L^\infty(\mathbb{T})}=0$$

from Lemma 3.2, this combined with (34) yields (26), which completes the proof of Theorem 1.1. □

3.2. Uniqueness of topological solutions

Lastly, we will prove the uniqueness of topological solutions to (1) as stated in Theorem 1.2.

Proof of Theorem 1.2. To prove Theorem 1.2, suppose that there exist two distinct solutions $(u_{1,\varepsilon},u_{2,\varepsilon})$ and $(\tilde u_{1,\varepsilon},\tilde u_{2,\varepsilon})$ of (9) such that

$$u_{i,\varepsilon},\ \tilde u_{i,\varepsilon}\to 0 \quad \text{as } \varepsilon\to 0 \ \text{ pointwise a.e. in } \mathbb{T} \quad \text{for } i=1,2.$$

As in the proof of Theorem 1.1, let $U_{j,\varepsilon}(x):=U_j\Big(\frac{x-p_j}{\varepsilon}\Big)$ for $1\le j\le N$. Then the difference $u_{1,\varepsilon}-\tilde u_{1,\varepsilon}$ of the first components solves

$$\begin{aligned}
&L_{1,\varepsilon}(u_{1,\varepsilon}-\tilde u_{1,\varepsilon})\\
&=\Delta(u_{1,\varepsilon}-\tilde u_{1,\varepsilon})\\
&\quad+\frac{1}{\varepsilon^2}\Big[\sum_{j=1}^N g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|)+g'\left(0\right)\Big(1-\sum_{j=1}^N\chi_d(|x-p_j|)\Big)\Big](u_{1,\varepsilon}-\tilde u_{1,\varepsilon})\\
&=\frac{1}{\varepsilon^2}\Big[\sum_{j=1}^N g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|)+g'\left(0\right)\Big(1-\sum_{j=1}^N\chi_d(|x-p_j|)\Big)\Big](u_{1,\varepsilon}-\tilde u_{1,\varepsilon})\\
&\quad+H_{1,\varepsilon}(x),
\end{aligned}$$

where

$$\begin{aligned}
H_{1,\varepsilon}(x):=&-\frac{1}{\varepsilon^2}\left(g(u_{1,\varepsilon})-g(\tilde u_{1,\varepsilon})\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}\left(g(u_{2,\varepsilon})-g(\tilde u_{2,\varepsilon})\right)\\
&-\frac{\sigma_\varepsilon^2}{\varepsilon^2}\left(e^{u_{2,\varepsilon}}-e^{\tilde u_{2,\varepsilon}}\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}\left(e^{u_{1,\varepsilon}}-e^{\tilde u_{1,\varepsilon}}\right)\\
&+\left(\frac{\sigma_\varepsilon^2}{\varepsilon^2}-\frac{\sigma_\varepsilon}{\varepsilon^2}\right)\left(e^{u_{1,\varepsilon}+u_{2,\varepsilon}}-e^{\tilde u_{1,\varepsilon}+\tilde u_{2,\varepsilon}}\right).
\end{aligned}$$

Using the mean value theorem, we have that

$$\begin{aligned}
H_{1,\varepsilon}(x)=&-\frac{1}{\varepsilon^2}g'(\xi_{1,\varepsilon})\left(u_{1,\varepsilon}-\tilde u_{1,\varepsilon}\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}g'(\xi_{2,\varepsilon})\left(u_{2,\varepsilon}-\tilde u_{2,\varepsilon}\right)\\
&-\frac{\sigma_\varepsilon^2}{\varepsilon^2}e^{\eta_{2,\varepsilon}}\left(u_{2,\varepsilon}-\tilde u_{2,\varepsilon}\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}e^{\eta_{1,\varepsilon}}\left(u_{1,\varepsilon}-\tilde u_{1,\varepsilon}\right)\\
&+\left(\frac{\sigma_\varepsilon^2}{\varepsilon^2}-\frac{\sigma_\varepsilon}{\varepsilon^2}\right)e^{\zeta_{1,\varepsilon}+\zeta_{2,\varepsilon}}\left(u_{1,\varepsilon}+u_{2,\varepsilon}-\tilde u_{1,\varepsilon}-\tilde u_{2,\varepsilon}\right),
\end{aligned}$$

where $\xi_{i,\varepsilon}$ and $\eta_{i,\varepsilon}$ are numbers between $u_{i,\varepsilon}$ and $\tilde{u}_{i,\varepsilon}$ for $i=1,2$, and $\sum_{i=1}^{2}\zeta_{,\varepsilon}$ lies between $\sum_{i=1}^{2}u_{i,\varepsilon}$ and $\sum_{i=1}^{2}\tilde{u}_{i,\varepsilon}$. In view of Theorem 1.1, we have

$$(35)\quad \|\xi_{1,\varepsilon}-\sum_{j=1}^{N}U_{j,\varepsilon}(x)\chi_d(|x-p_j|)\|_{L^\infty(\mathbb{T})}=O(\varepsilon^2) \text{ and } \|\xi_{2,\varepsilon}\|_{L^\infty(\mathbb{T})}=O(\varepsilon^2)$$

as $\varepsilon\to 0$. From our assumption (8) and Theorem 2.7, we get that

$$\begin{aligned}(36)\quad &\|u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}\\ &\le C\Big\|\Big\{-g'(\xi_{1,\varepsilon})+\sum_{j=1}^{N}g'\left(U_{j,\varepsilon}(x)\right)\chi_d(|x-p_j|)\\ &\qquad +g'(0)\Big(1-\sum_{j=1}^{N}\chi_d(|x-p_j|)\Big)\Big\}(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon})\Big\|_{L^\infty(\mathbb{T})}\\ &\quad +C\Big\|\sigma_\varepsilon g'(\xi_{2,\varepsilon})\left(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)-\sigma_\varepsilon^2 e^{\eta_{2,\varepsilon}}\left(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)+\sigma_\varepsilon e^{\eta_{1,\varepsilon}}\left(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\right)\Big\|_{L^\infty(\mathbb{T})}\\ &\quad +C\Big\|\left(\sigma_\varepsilon^2-\sigma_\varepsilon\right)e^{\zeta_{1,\varepsilon}+\zeta_{2,\varepsilon}}\left(u_{1,\varepsilon}+u_{2,\varepsilon}-\tilde{u}_{1,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)\Big\|_{L^\infty(\mathbb{T})},\end{aligned}$$

where a uniform constant $C>0$ is independent of $\varepsilon>0$. Since the first term on the right-hand side of (36) is given by

$$\|\{-g'(\xi_{1,\varepsilon})+g'(0)+\sum_{j=1}^{N}[g'(U_{j,\varepsilon}(x))-g'(0)]\chi_d(|x-p_j|)\}(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon})\|_{L^\infty(\mathbb{T})}=:I_1,$$

by utilizing (35) and the exponential decay estimate (28) of $U_{j,\varepsilon}$, it holds that

$$I_1=O(\varepsilon^2)\|(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon})\|_{L^\infty(\mathbb{T})} \quad \text{as} \quad \varepsilon\to 0.$$

Here, we also note that $u_{i,\varepsilon}<0$, $\tilde{u}_{i,\varepsilon}<0$ and $U_{j,\varepsilon}<0$. Hence by the assumption (8), we have

$$\begin{aligned}&\|u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}\\ &\le o(1)\|(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon})\|_{L^\infty(\mathbb{T})}+O(\varepsilon^2)\|u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\|_{L^\infty(\mathbb{T})} \quad \text{as} \quad \varepsilon\to 0.\end{aligned}$$

This implies that

$$(37)\qquad \|u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}\le O(\varepsilon^2)\|u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\|_{L^\infty(\mathbb{T})} \quad \text{as} \quad \varepsilon\to 0.$$

Next, the difference $u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}$ satisfies

$$\begin{aligned}L_{2,\varepsilon}(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})&=\Delta(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})+\frac{g'(0)}{\varepsilon^2}(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})\\ &=\frac{g'(0)}{\varepsilon^2}(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})+H_{2,\varepsilon}(x),\end{aligned}$$

where

$$\begin{aligned}
&H_{2,\varepsilon}(x)\\
= &-\frac{1}{\varepsilon^2}\left(g(u_{2,\varepsilon})-g(\tilde{u}_{2,\varepsilon})\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}\left(g(u_{1,\varepsilon})-g(\tilde{u}_{1,\varepsilon})\right)-\frac{\sigma_\varepsilon^2}{\varepsilon^2}\left(e^{u_{1,\varepsilon}}-e^{\tilde{u}_{1,\varepsilon}}\right)\\
&+\frac{\sigma_\varepsilon}{\varepsilon^2}\left(e^{u_{2,\varepsilon}}-e^{\tilde{u}_{2,\varepsilon}}\right)+\left(\frac{\sigma_\varepsilon^2}{\varepsilon^2}-\frac{\sigma_\varepsilon}{\varepsilon^2}\right)\left(e^{u_{1,\varepsilon}+u_{2,\varepsilon}}-e^{\tilde{u}_{1,\varepsilon}+\tilde{u}_{2,\varepsilon}}\right).
\end{aligned}$$

By the mean value theorem, we have that

$$\begin{aligned}
&H_{2,\varepsilon}(x)\\
= &-\frac{1}{\varepsilon^2}g'(\xi_{2,\varepsilon})\left(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)+\frac{\sigma_\varepsilon}{\varepsilon^2}g'(\xi_{1,\varepsilon})\left(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\right)-\frac{\sigma_\varepsilon^2}{\varepsilon^2}e^{\eta_{1,\varepsilon}}\left(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\right)\\
&+\frac{\sigma_\varepsilon}{\varepsilon^2}e^{\eta_{2,\varepsilon}}\left(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)+\left(\frac{\sigma_\varepsilon^2}{\varepsilon^2}-\frac{\sigma_\varepsilon}{\varepsilon^2}\right)e^{\zeta_{1,\varepsilon}+\zeta_{2,\varepsilon}}\left(u_{1,\varepsilon}+u_{2,\varepsilon}-\tilde{u}_{1,\varepsilon}-\tilde{u}_{2,\varepsilon}\right),
\end{aligned}$$

where $\xi_{i,\varepsilon}$ and $\eta_{i,\varepsilon}$ are numbers between $u_{i,\varepsilon}$ and $\tilde{u}_{i,\varepsilon}$ for $i=1,2$, and $\sum_{i=1}^{2}\zeta_{,\varepsilon}$ lies between $\sum_{i=1}^{2}u_{i,\varepsilon}$ and $\sum_{i=1}^{2}\tilde{u}_{i,\varepsilon}$. Here, we also note that $u_{i,\varepsilon}<0$ and $\tilde{u}_{i,\varepsilon}<0$. Then it holds from Theorem 2.7 that

$$\begin{aligned}
&\|u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\|_{L^\infty(\mathbb{T})}\\
\le\ & C\Big\|\Big(g'(0)-g'(\xi_{2,\varepsilon})\Big)(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})\Big\|_{L^\infty(\mathbb{T})}\\
&+C\Big\|\sigma_\varepsilon g'(\xi_{1,\varepsilon})\left(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\right)-\sigma_\varepsilon^2 e^{\eta_{1,\varepsilon}}\left(u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\right)+\sigma_\varepsilon e^{\eta_{2,\varepsilon}}\left(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)\Big\|_{L^\infty(\mathbb{T})}\\
&+C\Big\|\left(\sigma_\varepsilon^2-\sigma_\varepsilon\right)e^{\zeta_{1,\varepsilon}+\zeta_{2,\varepsilon}}\left(u_{1,\varepsilon}+u_{2,\varepsilon}-\tilde{u}_{1,\varepsilon}-\tilde{u}_{2,\varepsilon}\right)\Big\|_{L^\infty(\mathbb{T})}.
\end{aligned}$$

Here a uniform constant $C>0$ is independent of $\varepsilon>0$. Thus using (35) and the assumption (8), we deduce that

$$\|u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\|_{L^\infty(\mathbb{T})}\le o(1)\|(u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon})\|_{L^\infty(\mathbb{T})}+O(\varepsilon^2)\|u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\|_{L^\infty(\mathbb{T})},$$

as $\varepsilon\to 0$, which implies that

$$\|u_{2,\varepsilon}-\tilde{u}_{2,\varepsilon}\|_{L^\infty(\mathbb{T})}\le O(\varepsilon^2)\|u_{1,\varepsilon}-\tilde{u}_{1,\varepsilon}\|_{L^\infty(\mathbb{T})}\quad\text{as}\quad\varepsilon\to 0. \tag{38}$$

Therefore the estimations (37) and (38) yield that $(u_{1,\varepsilon},u_{2,\varepsilon})\equiv(\tilde{u}_{1,\varepsilon},\tilde{u}_{2,\varepsilon})$ in $\mathbb{T}$ for small $\varepsilon>0$. This completes the proof of the uniqueness of topological solutions to (1) for large $\alpha>0$. □

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