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The physical interpretation of point interactions in one-dimensional relativistic quantum mechanics

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Abstract

We investigate point interactions (PIs) in one-dimensional relativistic quantum mechanics using a distributional approach based on Schwartz's theory of distributions. From the properties of the most general covariant distribution describing relativistic PIs (RPIs) we obtain the physical parameters associated with the point potentials that behave as a scalar, a pseudo-scalar and a vector under Lorentz transformations. Then, we establish a one-to-one relationship between these physical parameters and the well-known set of four parameters giving the boundary conditions at the singular point(s), which define a self-adjoint Hamiltonian. By considering the non-relativistic limit, we obtain the most general PI in the Schrödinger equation in terms of these four physical point potentials. Finally, we study the symmetries of the RPIs under space inversion, time reversal and charge conjugation, and investigate how requirements of invariance under these symmetry transformations can be used to restrict the set of physical parameters.

Keywords: Dirac equation with point interactions, delta interaction, gauge transformations

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1. Introduction

Contact interactions in one-dimensional relativistic quantum mechanics—also known as point interactions (PIs)—have long been used to model short-range potentials. For instance, PIs have been considered to describe the Casimir effect (e.g. [1, 2] and references therein), to investigate tunneling resonances [3–5], and in connection with confinement in quark models [6]. From the mathematical point of view, just like their non-relativistic counterparts, relativistic PIs (RPIs) are singular interactions whose treatment warrants the use of special mathematical methods in order to avoid ambiguities—see [7] for a recent collection of papers on methods and applications of contact interactions.

One method extensively employed in the treatment of PIs, both in the non-relativistic (see, e.g. [8–14]) and relativistic [11, 12, 15–19] regimes, is to consider sequences of regular potentials converging to the singular point potential. This method, while simple and intuitively pleasing, must be used with great care, since it is known that different regularizations may lead to ambiguous or even contradictory results [15, 16, 18], as it is often the case when dealing with δ'-converging sequences in non-relativistic quantum mechanics [20, 21]. There are, however, mathematically rigorous approaches to the regularization of PI addressing these issues, e.g. [22–26].

The well-established method of self-adjoint extensions (SAEs) has also been used extensively to investigate PI (see, e.g. [27–33]) and RPI [34–38]. An important result demonstrated in the SAE approach is that, similarly to the non-relativistic case, RPI form a four-parameter family of interactions completely characterized by the boundary conditions (b.c.) satisfied by the wave-function [36].

Typically, works on RPI either adopt the formalism of self-adjoint Dirac operators specified by the Λ-matrix parameters characterizing the b.c., see equation (35), or use a regularized Hamiltonian given in terms of regular physical potentials characterized by Lorentz transformations, see equation (2), and take the limit to the singular potential at the end in order to obtain the b.c. around the singular point. The latter approach is often restricted to the so-called vector/electrostatic (the *time component* of a Lorentz vector) and scalar interactions [18, 36, 39], and its results usually depend on the specific limiting procedure used or on the formal manipulations of ambiguously defined expressions [20, 21]. It should be noted, however, that the self-adjointness and spectral analysis of Dirac operators with singular vector and scalar Lorentz interactions of the δ-shell type (i.e. interactions supported on the boundary of compact and smooth domains in $\mathbb{R}^2$ or $\mathbb{R}^3$) have been recently considered in depth—see, e.g. [40–45] and references therein; in [46] this analysis is also performed for the one-dimensional case, using the general approach of boundary triples. Nevertheless, a systematic investigation of the general relationships between the two sets of parameters in one dimension (i.e. the Λ-matrix and the physical parameters characterizing the point potentials) is missing from the literature and its derivation is the main goal of this paper (see, however, [18, 47], which deal with some special cases in the relativistic and non-relativistic cases, respectively).

In this work we investigate RPI with the goal of obtaining mathematically rigorous results in terms of the physical parameters, not using any regularization procedure, thus eliminating the ambiguities mentioned above. We obtain explicit relationships between the set of the four physical parameters and the usual set of four Λ-matrix parameters giving the b.c. for the Dirac spinor at the singular point. We also investigate the behavior of such interactions under the transformations of space inversion, time reversal and charge conjugation, and show that restrictions on the physical parameters emerge as a consequence of requiring invariance under such symmetry transformations. In addition, by taking the non-relativistic limit, we discuss several common non-RPIs and establish an explicit relationship between these non-RPI and

the physical point potentials. In particular, our results shed light on the possibility of defining a PI associated with a potential given by the derivative of a Dirac delta distribution.

To conduct this analysis of RPI, we adopt the distributional approach to singular interactions in one dimension introduced in [48, 49]. The use of distribution theory to deal with singularities is well-known in quantum field theory, where the causal approach, introduced by Epstein and Glaser in [50] and further developed by Scharf [51] and collaborators, has been applied to many physical systems/problems (see, e.g. [52–55] and references therein). In one-dimensional quantum mechanics, the distributional approach provides a mathematically rigorous method that allows us to write the interaction explicitly in terms of the Dirac spinor, $\psi(x)$, and the parameters characterizing the b.c. at the singular point and, thus, it is especially suitable for analyzing the symmetries of the theory. In this approach, the time-independent Dirac equation with an external and static singular (point) potential $V(x)$ is treated from the distributional point of view, that is, both the interaction and the spinor are given in terms of well-defined Schwartz distributions. According to this method, the ill-defined product between $V(x)$ and $\psi(x)$ must be substituted by an interaction distribution, $D[\psi](x)$, to be determined from simple mathematical requirements (see section 2 and [49, 56, 57]). We note that another approach to PI using Schwartz distributions in the context of one-dimensional Schrödinger operators can be found in [58]; in that work, the authors use an extension of the Hörmander product of distributions to define (pseudo) potential operators associated to the b.c. describing PIs. In the present approach, we never need to deal with the issue of defining products of distributions. Instead, we replace the ill-defined product $V(x)\psi(x)$ by a well-defined distribution that is obtained from a set of very basic mathematical requirements.

This work is organized as follows. In section 2 we revisit the distributional treatment of the Dirac equation with PIs by treating it from first principles (in [49] some of us considered this problem by reducing it to the case of the Schrödinger equation in terms of the spinor components) and, along the way, we refine the distributional method by reducing the number of formal mathematical requirements necessary. Furthermore, we rewrite the interaction distribution, as compared to [49], in order to obtain an expression that is more appropriated for the covariant formulation and symmetry analyses. In section 3 we obtain the relationship between the set of physical (Lorentz) parameters and the usual set of four parameters of the b.c. Λ matrix. At the end of this section we discuss our results in the non-relativistic limit. The constraints imposed on RPI by invariance of the distributional Dirac equation under the symmetries of parity, time reversal and charge conjugation transformations are considered in section 4. Section 5 presents the final comments.

2. The distributional approach for RPIs

In this work we will say that a potential is *regular* if it is described by a function $V(x)$ that is locally integrable in the Lebesgue sense and is of *slow growth*, i.e. $\lim_{x\to\infty}|x|^{-N}V(x)=0$, for some non-negative integer N [59]. The Dirac equation for a particle of mass m in the presence of an external regular potential $V(x^\mu)\equiv\gamma^0\mathcal{I}(x)$, is given by [we adopt natural units, $\hbar=c=1$, the metric is $g^{\mu\nu}=\mathrm{diag}(1,-1)$, and $x=(x^0,x^1)$]

$$(i\gamma^\mu\partial_\mu - m\mathbb{1})\,\Psi(x)=\mathcal{I}(x)\,\Psi(x)\,, \tag{1}$$

where $\mu\in\{0,1\}$, $\mathbb{1}$ is the 2×2 identity matrix, $\mathcal{I}(x)$ is a 2×2 matrix that behaves as a Lorentz scalar, $\Psi(x)$ is a two-component Lorentz spinor and γ^μ are the 2×2 Dirac matrices, which can be represented using the Pauli matrices as $\gamma^0=\sigma_3$ and $\gamma^1=i\sigma_2$. Since the set of all three

Pauli matrices plus the identity is a basis for the space of complex, 2×2 matrices, the most general regular interaction in $(1+1)$-dimensions is such that $\mathcal{I}(x)$ can be written as [39, 60]

$$\mathcal{I}(x) = B(x)\,\mathbb{1} + A_\mu(x)\,\gamma^\mu + i\,W(x)\,\gamma^5, \tag{2}$$

where $\gamma^5 = \gamma^0\gamma^1 = \sigma_1$. In (2), the physical potentials $B(x), A_0(x), A_1(x)$, and $W(x)$ are regular real functions that behave as a Lorentz scalar, the two components of a Lorentz vector, and a Lorentz pseudo scalar, respectively. The $B(x)$ term is physically associated with a mass-like interaction (or mass-jump) [61, 62], $A_0(x)$ gives the electrostatic interaction and $A_1(x)$ would be associated with a 'magnetic' interaction (but see below for the role of $A_1(x)$ in one dimension).

In what follows we will be concerned with potentials that are time-independent in some particular Lorentz frame, referred to as the *laboratory* reference frame (later, in section 3.1, we will write the time-dependent Dirac equation for PIs in a Lorentz covariant way, in order to obtain the physical interpretation to the parameters characterizing the PI). Thus, we begin by restricting our analysis to the time-independent Dirac equation for a particle with energy E in the laboratory frame, in which case we denote $x^1 = \mathbf{x}$, $x^0 = t$ and $\Psi(x) = \psi(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}Et}$:

$$\left(i\gamma^1\partial_{\mathbf{x}} + \gamma^0 E - m\mathbb{1}\right)\psi(\mathbf{x}) = \gamma^0 V(\mathbf{x})\,\psi(\mathbf{x}) = \mathcal{I}(\mathbf{x})\,\psi(\mathbf{x}), \tag{3}$$

with (2) now depending only on the spatial coordinate $\mathbf{x}$ in the laboratory frame.

Assuming that the *regular* vector field $A(x)$ satisfies the $(1+1)$-dimensional Maxwell's equations, the component $A_1(x)$ in (2) can always be eliminated in a given Lorentz frame by making a suitable gauge transformation through a smooth function $\alpha(x)$ (i.e. $\psi(x) \to \mathrm{e}^{\mathrm{i}\alpha(x)}\psi(x)$ and $A_\mu \to A_\mu + \partial_\mu\alpha(x)$) [39, 60]. With this, the most general *regular* relativistic interaction in $(1+1)$-dimensions is given, in the laboratory reference frame, by

$$\left(i\gamma^1\partial_{\mathbf{x}} + \gamma^0 E - m\mathbb{1}\right)\psi(\mathbf{x}) = \left[B(\mathbf{x})\,\mathbb{1} + A_0(\mathbf{x})\,\gamma^0 + i\,W(\mathbf{x})\,\gamma^5\right]\psi(\mathbf{x}), \tag{4}$$

and it is completely determined by specifying the *three* real *regular functions* $B(\mathbf{x}), A_0(\mathbf{x})$ and $W(\mathbf{x})$. However, as we will see in section 3, a point 'magnetic' potential has non-trivial effects, and cannot be eliminated by a unitary transformation.

2.1. Interactions with a point singularity

Let us now consider the case in which the time-independent interaction is singular, i.e. it has at least one non-integrable point singularity in the laboratory frame. In this case, $\mathcal{I}(\mathbf{x})$ is not regular and has meaning only as a *singular distribution*; hence, for consistency, the Dirac spinor $\psi(\mathbf{x})$ must also be treated as a distribution-valued spinor. However, it is well-known that the product of two arbitrary distributions cannot be defined in general and, therefore, the interaction term in the right hand side of (3) is not well defined when the potential is a singular distribution [59]. In this work, we consider the space of distributions of slow growth, $\mathcal{S}'(\mathbb{R})$, also known as *tempered distributions*. These are continuous linear functionals defined on the Schwartz's space of test functions $\mathcal{S}(\mathbb{R})$, formed by complex functions of one real variable $\phi(x)$ that are infinitely smooth and such that they and all their derivatives decrease faster than any power of $\frac{1}{|x|}$ when $|x| \to \infty$. For the basic concepts of Schwartz's distribution theory needed in this work, we refer the reader to the appendix 1 in [57], and for more details see [59].

A key concept in the distributional approach followed here is that of the *order* of a distribution. It is a well-known result in the Schwartz's theory of distributions that, on any closed finite interval $K \subset \mathbb{R}$, any distribution $d(x)$ is the $(r+2)$th distributional derivative ($r \in \mathbb{Z}$) of

a continuous function whose first derivative (in the ordinary sense) is itself not continuous on K.[3] In this statement we are allowing distributional derivatives of negative order, which are interpreted as indefinite integrals in the distributional sense. This result can be extended to allow $r = -\infty$, in which case the distribution $d(x)$ corresponds to an infinitely smooth function on K. The integer (or $-\infty$) number r is called the *order of the distribution* $d(x)$, and we denote it by r_d.[4] For example, on any closed finite interval K around the origin the Heaviside theta distribution $\theta(x)$ is the first (distributional) derivative of a function that is continuous, but whose ordinary derivative is discontinuous; thus, $r_\theta = -1$ on K. Also, the Dirac delta distribution $\delta(x)$ is the distributional derivative of the θ distribution, and thus $r_\delta = 0$ on K. We denote by $d^{(n)}$ the nth order distributional derivative of a distribution d, with $n \in \mathbb{Z}$. As a general result, on a closed finite interval K, the order of $d^{(n)}$ is $r_d + n$. For instance, $r_{\delta^{(n)}} = n$ on K, if K contains the origin. It follows that when the order of a distribution is $\leqslant -2$, the distribution is *regular* on K, and when the order is $\geqslant 0$ it is *singular*. Except for some pathological examples, when the order of a distribution is -1 the distribution is regular[5].

By following [49], with a singular $\mathcal{I}(\mathbf{x})$ we must replace the ill-defined product of distributions on the right hand side of (3) by a well-defined *interaction distribution* $D[\psi](\mathbf{x})$ (a 2×1 matrix), such that the stationary Dirac equation becomes the distributional equation

$$\left(i\gamma^1 \partial_{\mathbf{x}} + \gamma^0 E - m\mathbb{1}\right) \psi(\mathbf{x}) = D[\psi](\mathbf{x}). \tag{5}$$

For a potential $V(\mathbf{x}) = \gamma^0 \mathcal{I}(\mathbf{x})$ that has singularities in a *finite* set of points σ, the interaction distribution $D[\psi](\mathbf{x})$ is required to satisfy the following conditions [49, 57]:

- **(R1)** The distribution $D[\psi](\mathbf{x})$, defined on the entire real line, must coincide with $\mathcal{I}(\mathbf{x})\psi(\mathbf{x})$ on $\mathbb{R}\backslash\sigma$.
- **(R2)** The components of the Dirac spinor $\psi(\mathbf{x})$ must correspond to regular distributions in the entire real line (i.e. they must be slow growth functions and locally integrable in the Lebesgue sense). Thus, the *order* r_ψ of the spinor components is bounded from above by $r_\psi \leqslant -1$.
- **(R3)** The distribution $D[\psi](\mathbf{x})$ must be such that *the current density is conserved* across the singular points σ.

The above requirements can be justified as follows. Condition **R1** guarantees that the problem is well-posed and corresponds to the original system in the complement of the set σ of singular points, by imposing that $D[\psi](\mathbf{x})$ is always a well-defined *distribution* giving the interaction between the Dirac's particle and the external singular potential [59, 63]. Since the solutions we seek for the stationary Dirac equation (5) will be the spatial components of the generalized (in the sense of distributions) energy eigenstates, requirement **R2** is a necessary (albeit not sufficient) condition to later obtain square integrable solutions of the time-dependent Dirac equation by superposing these generalized eigenstates. Requirement **R3** is generally associated with the physical interpretation of the current density as a probability flux in the domain

[3] The equality between a distribution and a continuous function must be understood as an equality *almost everywhere*.

[4] Unfortunately, the definition of the order (also referred to as 'singular order' by some authors) is not consensual in the literature. Here we follow the definition of the classic monograph by Zemanian [59].

[5] An example of a singular distribution having order -1 is the distributional derivative of the Cantor function, since this distributional derivative is not identically zero and is concentrated on a set of Lebesgue measure zero (the Cantor set); therefore, it cannot be represented by a locally summable ordinary function on any closed set containing the Cantor set.

of validity of relativistic quantum mechanics [64]. From the three requirements above we will find below that the generalized energy eigenstates must satisfy b.c. at the singular points that are the same ones that characterize the states in the domain of a self-adjoint Dirac Hamiltonian perturbed by PIs supported on σ.[6]

Now we restrict ourselves to the case of interest in this work, that is, a single singularity at the origin. Thus, the interaction in which we are interested vanishes everywhere except at $\sigma = \{0\}$, and **R1** implies that the support of $D[\psi](\mathbf{x})$ is concentrated at the origin, i.e. $\mathrm{supp}\, D[\psi](\mathbf{x}) = \{0\}$. From the fact that **R2** requires $r_\psi \leqslant -1$ and that taking the derivative increases the order of the resulting distribution by one [57, 59], it follows immediately that the order of the distribution $D[\psi](\mathbf{x})$ is $r_D \leqslant 0$, since balancing the order of the distributions in both sides of the equation (5) implies that r_D must be equal to the order of the derivative of the wave function (spinor). On the other hand, given that $D[\psi](\mathbf{x})$ is singular, we must have $r_D \geqslant 0$ and, therefore, the order of the interaction distribution is $r_D = 0$. Let us now use these requirements to find the most general distribution $D[\psi](\mathbf{x})$ corresponding to an RPI.

From the Schwartz's theory of distributions, it follows that any distribution concentrated at a single point must be a linear combination of the delta distribution $\delta(\mathbf{x})$ and its derivatives, with the order of the highest delta derivative determined by the order of the distribution (see theorem A.1 in [57]). This result, together with the requirements **R1** and **R2**, implies

$$D\left[\psi\right](\mathbf{x}) = \Omega\left[\psi\right]\delta\left(\mathbf{x}\right), \tag{6}$$

where $\Omega[\psi]$ is a 2×1 complex matrix, independent of the coordinate $\mathbf{x}$, which remains to be determined and whose components are functionals of ψ. It follows immediately that the two complex components of $\Omega[\psi]$ determine completely the interaction distribution $D[\psi](\mathbf{x})$.

To proceed in the determination of $D[\psi](\mathbf{x})$, we now consider the distributional Dirac equation (5), with $D[\psi](\mathbf{x})$ given by (6), and take an indefinite distributional integral on both sides of the equation, obtaining

$$i\gamma^1\psi\left(\mathbf{x}\right) + \left(\gamma^0 E - m\mathbb{1}\right)\psi^{(-1)}\left(\mathbf{x}\right) = \Omega\left[\psi\right]\theta\left(\mathbf{x}\right) + c, \tag{7}$$

where $\psi^{(-1)}$ is a primitive of ψ, $\theta(\mathbf{x})$ is the Heaviside step distribution (a primitive of the delta distribution), and c is an arbitrary constant 2×1 matrix. Since the order of the distribution ψ is $r_\psi = -1$, its primitive $\psi^{(-1)}$ has order -2 and, thus, its components correspond to functions that are continuous but not ordinarily differentiable at the origin [57]. From the above result, it follows that the components of the column matrix in the rhs of (7) are ordinary functions that have finite lateral limits when $\mathbf{x}\to 0^{\pm}$. As a consequence of this fact, the lhs of (7) must have finite lateral limits at the origin too, and we conclude that the term $i\gamma^1\psi(\mathbf{x})$ must have finite lateral limits around the origin, given by (note that lateral limits of distributions can be defined even at the singular points, if they exist. See [65–67])

$$i\gamma^1\left[\psi\left(0^+\right) - \psi\left(0^-\right)\right] = \Omega\left[\psi\right]. \tag{8}$$

Equation (8) shows that the interaction completely determines the jump of the Dirac spinor at the singular point, and vice-versa. This is, as expected, in agreement with the method of SAE. An important advantage of the present distributional approach is that it establishes a

[6] When the energy eigenstates are not square-integrable, as it is the case of scattering eigenstates, we may take square integrable linear superpositions of them; these superpositions will satisfy exactly the same b.c.'s, since, as we will see, the parameters that characterize them are independent of the energy.

unique relationship between the *explicit* distribution $D[\psi](\mathbf{x})$, defined on the entire real axis, and the permitted b.c. Additionally, as we will see below, the distributional approach allows us to identify which physical singular fields (point potentials) will produce a given admissible b.c. around that point, a desirable feature for a physical model. Furthermore, the explicit form of the interaction distribution $D[\psi](\mathbf{x})$ will facilitate the investigation of the interaction's properties, such as its symmetries and Lorentz covariance.

In the stationary case, the requirement of current conservation (**R3**) across the singular point implies that the spatial component of the current $j^1(\mathbf{x})$ [with $j^\mu(\mathbf{x}) = \psi^\dagger(\mathbf{x})\gamma^0\gamma^\mu\psi(\mathbf{x})$] must be continuous at $\mathbf{x} = 0$, that is[7],

$$\psi^\dagger\left(0^+\right)\sigma_1\psi\left(0^+\right) = \psi^\dagger\left(0^-\right)\sigma_1\psi\left(0^-\right). \tag{9}$$

That (9) can be satisfied is guaranteed by the fact that $\psi(\mathbf{x})$ has finite limits when $\mathbf{x} \to 0^\pm$, as demonstrated after equation (7). In order to impose (9), it is convenient to introduce the following column vectors:

$$V_1[\psi] \equiv M_1\,\psi\left(0^+\right) + M_2\,\psi\left(0^-\right), \tag{10}$$

$$V_2[\psi] \equiv M_2\,\psi\left(0^+\right) + M_1\,\psi\left(0^-\right), \tag{11}$$

where we introduced the matrices

$$M_1 \equiv \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad M_2 \equiv \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \tag{12}$$

which satisfy

$$M_1\,\sigma_1\,M_1^{\mathrm{T}} = 2\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{13}$$

$$M_2\,\sigma_1\,M_2^{\mathrm{T}} = 2\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \tag{14}$$

$$M_1\,\sigma_1\,M_2^{\mathrm{T}} = 0_{2\times 2}, \tag{15}$$

where 'T' denotes matrix transposition. The relations (10) and (11) can be inverted to write the lateral limits of ψ as (in what follows, to simplify the notation, we will sometimes omit the functional dependence of $V_1[\psi]$ and $V_2[\psi]$ on ψ)

$$\psi\left(0^+\right) = \frac{1}{2}\left[M_1^{\mathrm{T}}\,V_1 + M_2^{\mathrm{T}}\,V_2\right] \tag{16}$$

$$\psi\left(0^-\right) = \frac{1}{2}\left[M_2^{\mathrm{T}}\,V_1 + M_1^{\mathrm{T}}\,V_2\right]. \tag{17}$$

Now, by using (16) and (17) and the properties (13)–(15), the requirement of conservation of current across the origin, equation (9), can be written as

$$V_1^\dagger\,V_1 = V_2^\dagger\,V_2. \tag{18}$$

[7] In the context of the Schwartz's theory of distributions, a distribution is said to be continuous in some open interval (or in the entire real axis) if it coincides with a regular distribution defined by an ordinary continuous function in the same interval [57].

This condition implies that there exists a unitary matrix [49, 68]

$$U = \mathrm{e}^{\mathrm{i}\theta} \begin{pmatrix} z & w \\ -w^* & z^* \end{pmatrix}, \quad |z|^2 + |w|^2 = 1, \quad \theta \in [0, \pi) \tag{19}$$

such that

$$V_1 = UV_2. \tag{20}$$

The unitary matrix U, which has only *four independent real parameters*, completely characterizes the most general one-dimensional RPI satisfying the requirements **R1**–**R3**.

Substituting (16) and (17) into (8), and using (20), we can write the coefficient $\Omega[\psi]$, which determines the interaction distribution, as

$$\Omega[\psi] = \frac{i}{2} \left(\mathbb{1} + \gamma^1\right) (U - \mathbb{1}) V_2[\psi] = \frac{i}{2} \left(\mathbb{1} + \gamma^1\right) \left(\mathbb{1} - U^\dagger\right) V_1[\psi]. \tag{21}$$

Thus, from the distributional approach it follows that the most general RPI constitute *a four-parameter family of PIs*, whose *interaction distribution* (6) can be written as

$$D[\psi](x) = \frac{i}{2} \left(\mathbb{1} + \gamma^1\right) (U - \mathbb{1}) V_2[\psi] \, \delta(x). \tag{22}$$

Observe that, from the definitions (10) and (11), $V_1[\psi]$ and $V_2[\psi]$ are linear functionals of ψ; thus it immediately follows that the distribution $D[\psi](x)$ also depends linearly on the spinor ψ, as required by the superposition principle in quantum mechanics. This eliminates the need for the explicit requirement of linearity used in [49, 57], reducing the conditions necessary for obtaining a well-defined distributional theory to the optimal set **R1**–**R3**.

Now we turn back to equation (20) and substitute V_1 and V_2 in terms of the boundary values $\psi(0^\pm)$, according to (10) and (11). After rearranging the terms, we obtain

$$R^+ \psi\left(0^+\right) = R^- \psi\left(0^-\right), \tag{23}$$

where we have defined

$$R^- \equiv M_2 - UM_1 = \begin{pmatrix} -z\mathrm{e}^{\mathrm{i}\theta} & -z\mathrm{e}^{\mathrm{i}\theta} \\ w^*\mathrm{e}^{\mathrm{i}\theta}+1 & w^*\mathrm{e}^{\mathrm{i}\theta}-1 \end{pmatrix}, \tag{24}$$

$$R^+ \equiv UM_2 - M_1 = \begin{pmatrix} w\mathrm{e}^{\mathrm{i}\theta}-1 & -w\mathrm{e}^{\mathrm{i}\theta}-1 \\ z^*\mathrm{e}^{\mathrm{i}\theta} & -z^*\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}. \tag{25}$$

It follows immediately that $\det R^- = 2z\mathrm{e}^{\mathrm{i}\theta}$ and $\det R^+ = 2z^*\mathrm{e}^{\mathrm{i}\theta}$. Therefore, both matrices $R^\pm$ are invertible if, and only if, $z \neq 0$. Below we will consider the two possible cases separately.

2.1.1. Impermeable PIs. Let us first consider the case $z = 0$. Under this condition the number of independent real parameters characterizing the interaction is reduced to only two, and from (23)–(25) we obtain

$$R^{+}\psi\left(0^{+}\right) = 0 = R^{-}\psi\left(0^{-}\right). \tag{26}$$

Therefore, in this case, the wave function to the right of the origin (the singular point) is completely independent of the wave function to the left of it. The singular point is said to be an *impermeable barrier*.

On the right side of the origin, by choosing $w = \mathrm{e}^{-\mathrm{i}\varphi}$ (since now $|w| = 1$), the condition (26) can be written as

$$\left(\; -\mathrm{e}^{\mathrm{i}(\theta-\varphi)}+1 \, , \quad \mathrm{e}^{\mathrm{i}(\theta-\varphi)}+1 \; \right)\psi\left(0^{+}\right) = 0. \tag{27}$$

Then, at this side of the origin, the interaction is completely characterized by just *one independent parameter* $(\theta - \varphi)$. By using a convenient reparameterization, the above condition can be rewritten as:

$$Q^{+}\psi\left(0^{+}\right) = 0, \tag{28}$$

where the row matrix Q^{+} is defined by

$$Q^{+} \equiv \left(1, \, ih^{+}\right), \qquad h^{+} \in \mathbb{R} \cup \{\infty\}, \tag{29}$$

with $h^{+} \equiv \cot(\frac{\varphi-\theta}{2})$.

Similarly, on the left side of the origin, we can write (with the reparameterization $h^{-} \equiv \tan(\frac{\varphi+\theta}{2})$)

$$Q^{-}\psi\left(0^{-}\right) = 0, \tag{30}$$

with

$$Q^{-} \equiv \left(1, \, ih^{-}\right), \qquad h^{-} \in \mathbb{R} \cup \{\infty\}. \tag{31}$$

In (29) and (30) the infinite value was included in the range of the parameters $h^{\pm}$ to allow the situation in which the lower component vanishes, but the upper component is finite (and arbitrary).

Equations (28) and (30) can be cast in the form

$$R_0^{+}\psi\left(0^{+}\right) + R_0^{-}\psi\left(0^{-}\right) = 0, \tag{32}$$

$$R_0^{+} = \begin{pmatrix} 1 & ih^{+} \\ 0 & 0 \end{pmatrix}, \qquad R_0^{-} = \begin{pmatrix} 0 & 0 \\ 1 & ih^{-} \end{pmatrix}.$$

Thus, for impermeable PIs, the distribution $D[\psi](x)$ (given by (6) and (8)) can be characterized by only *two independent parameters*, $h^{\pm}$, which set the constraints satisfied by the wave function at each side of the origin via (28)–(30). This subfamily of PIs does not contain the null interaction (free particle), since for any choice of the parameters $h^{\pm}$ the wave function at the right of the origin can always be specified independently of the wave function at the left. As we will see below, the limit of no interaction is contained in the case $z \neq 0$. It should be noted that in the non-relativistic case impermeable interactions have been investigated in detail in [69] (see also [29]).

2.1.2. Permeable interactions. Let us now consider the situation $z \neq 0$, in which case both matrices $R^{\pm}$ in (24) and (25) are invertible. By multiplying (23) from the left by $(R^{+})^{-1}$ we can rewrite the b.c. as the following linear relationship between $\psi(0^{+})$ and $\psi(0^{-})$:

$$\psi\left(0^{+}\right) = \Lambda\,\psi\left(0^{-}\right), \tag{33}$$

where $\Lambda \equiv (R^{+})^{-1} R^{-}$ is given by

$$\Lambda = \mathrm{e}^{\mathrm{i}\arg z}\begin{pmatrix} \frac{\cos\theta+\Re(w)}{|z|} & i\,\frac{\sin\theta-\Im(w)}{|z|} \\ i\,\frac{\sin\theta+\Im(w)}{|z|} & \frac{\cos\theta-\Re(w)}{|z|} \end{pmatrix}, \tag{34}$$

with $\det\Lambda = \mathrm{e}^{2\mathrm{i}\arg z}$. In the expression above, $\Im(\cdot)$ and $\Re(\cdot)$ indicate the imaginary and real parts of a complex number, respectively. The matrix Λ has four independent real parameters (a direct consequence of the fact that there were only four real independent parameters in the unitary matrix U); thus, it can be reparameterized as follows

$$\Lambda = \mathrm{e}^{\mathrm{i}\varphi}\begin{pmatrix} a & ib \\ -ic & d \end{pmatrix}, \quad ad - bc = 1, \quad \varphi \in [0,\pi), \tag{35}$$

where a, b, c, d are real parameters (the restriction in the range of φ to $[0,\pi)$ is to ensure uniqueness).

The results (32) and (33) for the b.c., obtained by the distributional approach, for both permeable and impermeable interactions, coincide with those obtained from the SAE approach, in which all the SAEs of the free Dirac Hamiltonian defined on $\mathbb{R}\backslash\{0\}$ are characterized by the those b.c. the Dirac spinor must satisfy at the origin, as we already anticipated. However, the distributional approach provides additional information about the interaction, namely, *it determines explicitly the interaction distribution concentrated at the origin that results in those b.c.*

3. The physical parameters and their relationships to the Λ-matrix parameters

In this section we will write the interaction distribution $D[\psi](\mathbf{x}) = \Omega[\psi]\delta(\mathbf{x})$ in a form that is more suitable to investigate the Lorentz covariance and other symmetries.

First we observe that it is convenient to decompose $\psi(0^{\pm})$ in the equations (32) and (33) in terms of the *difference* and the *sum* of the boundary values $\psi(0^{\pm})$, which form two linearly independent combinations of $\psi(0^{\pm})$:

$$\psi\left(0^{\pm}\right) = \frac{1}{2}\left[\psi\left(0^{+}\right) + \psi\left(0^{-}\right)\right] \pm \frac{1}{2}\left[\psi\left(0^{+}\right) - \psi\left(0^{-}\right)\right]. \tag{36}$$

Let us first consider impermeable interactions. By substituting equation (36) into (32) and rearranging the terms, we obtain:

$$\left(R_0^{+} - R_0^{-}\right)\left[\psi\left(0^{+}\right) - \psi\left(0^{-}\right)\right] = -\left(R_0^{+} + R_0^{-}\right)\left[\psi\left(0^{+}\right) + \psi\left(0^{-}\right)\right],$$

which can be rewritten as

$$G\,i\gamma^{1}\left[\psi\left(0^{+}\right) - \psi\left(0^{-}\right)\right] = i\left(R_0^{+} + R_0^{-}\right)\left[\psi\left(0^{+}\right) + \psi\left(0^{-}\right)\right], \tag{37}$$

with

$$G = \begin{pmatrix} -ih_+ & 1 \\ ih_- & -1 \end{pmatrix}. \tag{38}$$

Now, if

$$h_- - h_+ \neq 0, \tag{39}$$

then G is invertible, and we can multiply (37) from the left by G^{-1}. Therefore, from (8), we obtain

$$\begin{aligned} \Omega[\psi] &= i\gamma^1 \left[\psi\left(0^+\right) - \psi\left(0^-\right)\right] \\ &= \frac{2}{h_- - h_+} \begin{pmatrix} 2 & i(h_- + h_+) \\ i(h_- + h_+) & -2h_- h_+ \end{pmatrix} \frac{\psi(0^+) + \psi(0^-)}{2} \\ &= \left(B\mathbb{1} + A_0\gamma^0 + A_1\gamma^1 + iW\gamma^5\right) \frac{\psi(0^+) + \psi(0^-)}{2}, \end{aligned} \tag{40}$$

where we make the reparameterization

$$\begin{aligned} B &= \frac{2(1 - h_- h_+)}{h_- - h_+} \\ A_0 &= \frac{2(1 + h_- h_+)}{h_- - h_+} \\ A_1 &= 0 \\ W &= \frac{2(h_- + h_+)}{h_- - h_+}. \end{aligned} \tag{41}$$

This set of equations implies that

$$A_0^2 - B^2 - W^2 + 4 = 0. \tag{42}$$

The trivial parameter $A_1 = 0$ was introduced for later convenience, since we will interpret these real parameters in terms of strengh of singular point potentials. From this new set of parameters, the impermeable barrier is characterized by two independent parameters and a choice of sign in the last equation. We observe that, if condition (39) is not satisfied, some (or all) these new parameters are infinite, but they may still characterize an impermeable barrier if we allow the parameters to approach infinite in a way that condition (42) is still satisfied.

We now consider a permeable barrier. Substituting (36) into (33), and after some elementary matrix manipulations, we arrive at

$$Ki\gamma^1 \left[\psi\left(0^+\right) - \psi\left(0^-\right)\right] = 2(\Lambda - \mathbb{1}) \frac{\psi(0^+) + \psi(0^-)}{2}, \tag{43}$$

with

$$K = i(\mathbb{1} + \Lambda)\gamma^1. \tag{44}$$

The matrix K is invertible if

$$2\cos\varphi + a + d \neq 0 \tag{45}$$

and, in this case, we can multiply (43) from the left by K^{-1}. After using (8), we obtain:

$$\begin{aligned} D[\psi](\mathbf{x}) &= \Omega[\psi]\,\delta(\mathbf{x}) \\ &= 2i\gamma^1(\Lambda+\mathbb{1})^{-1}(\Lambda-\mathbb{1})\,\delta(\mathbf{x})\,\frac{\psi(0^+)+\psi(0^-)}{2} \qquad (46) \\ &\equiv \left(B\mathbb{1}+A_\mu\gamma^\mu+iW\gamma^5\right)\delta(\mathbf{x})\,\frac{\psi(0^+)+\psi(0^-)}{2}, \qquad (47) \end{aligned}$$

where we made the following identifications:

$$B = \frac{2(c+b)}{2\cos\varphi+d+a}, \tag{48}$$

$$A_0 = \frac{2(c-b)}{2\cos\varphi+d+a}, \tag{49}$$

$$A_1 = \frac{-4\sin\varphi}{2\cos\varphi+d+a}, \tag{50}$$

$$W = \frac{2(d-a)}{2\cos\varphi+d+a}. \tag{51}$$

In the above equations, the real quantities B, A_μ and W are completely determined by the set of four Λ-matrix parameters, and they correspond to the strengths of the scalar, vector, and a pseudoscalar point potential, respectively, as we will see in the next section.

Let us now investigate under which conditions relations (48)–(51) are invertible, so that we can establish a unique correspondence among the parameters B, A_0, A_1, W, and the Λ-matrix parameters. To this end, let us define

$$\begin{aligned} F &\equiv B\mathbb{1}+A_0\gamma^0+A_1\gamma^1+iW\gamma^5 \\ &= \begin{pmatrix} B+A_0 & A_1+iW \\ -A_1+iW & B-A_0 \end{pmatrix}, \end{aligned} \tag{52}$$

which, from (46) and (47), satisfies

$$\Lambda\left(2\mathbb{1}-i\gamma^1F\right) = 2\mathbb{1}+i\gamma^1F. \tag{53}$$

This equation can be solved for the matrix Λ if, and only if,

$$\det\left(2\mathbb{1}-i\gamma^1F\right) = 4-B^2-W^2+A_0^2-A_1^2+4iA_1 \neq 0, \tag{54}$$

which is equivalent to the condition

$$A_1 \neq 0 \quad \text{or} \quad A_0^2-B^2-W^2+4 \neq 0. \tag{55}$$

These two conditions are nothing but the *permeability conditions*, since if both were simultaneously violated the point barrier would be impermeable, as immediately seen from (42) and

the condition $A_1 = 0$ in (41). We observe, for instance, that $A_1 \neq 0$ is a *sufficient condition* for permeability. Assuming that condition (55) holds, we find

$$\Lambda = \left(2\mathbb{1} + i\gamma^1 F\right)\left(2\mathbb{1} - i\gamma^1 F\right)^{-1}, \tag{56}$$

which, after substituting (52), yields

$$\Lambda = \frac{e^{i\arg\left(B^2+W^2-4-A_0^2+A_1^2+4iA_1\right)}}{\sqrt{\left(B^2+W^2-4-A_0^2+A_1^2\right)^2+16A_1^2}} \begin{pmatrix} A_0^2-A_1^2-B^2-(W-2)^2 & 4i\,(A_0-B) \\ 4i\,(A_0+B) & A_0^2-A_1^2-B^2-(W+2)^2 \end{pmatrix}. \tag{57}$$

Since in (35) we defined the parameter φ in the interval $[0,\pi)$, we can now identify the parameters φ, a, b, c, d of the matrix Λ in terms of the parameters B, W, A_0, and A_1 as

$$\varphi = \tan^{-1}\left(\frac{4A_1}{B^2+W^2-4-A_0^2+A_1^2}\right) \qquad \varphi \in [0,\pi), \tag{58}$$

$$a = \pm \frac{A_0^2-A_1^2-B^2-(W-2)^2}{\sqrt{\left(B^2+W^2-4-A_0^2+A_1^2\right)^2+16A_1^2}}, \tag{59}$$

$$b = \pm \frac{4\,(A_0-B)}{\sqrt{\left(B^2+W^2-4-A_0^2+A_1^2\right)^2+16A_1^2}}, \tag{60}$$

$$c = \pm \frac{-4\,(A_0+B)}{\sqrt{\left(B^2+W^2-4-A_0^2+A_1^2\right)^2+16A_1^2}}, \tag{61}$$

$$d = \pm \frac{A_0^2-A_1^2-B^2-(W+2)^2}{\sqrt{\left(B^2+W^2-4-A_0^2+A_1^2\right)^2+16A_1^2}}, \tag{62}$$

where in the above expressions the plus (minus) sign must be taken if $A_1 > 0$ ($A_1 < 0$); if $A_1 = 0$ we must take the same sign of $B^2 + W^2 - 4 - A_0^2$. It is easy to verify that the condition $ad - bc = 1$ is automatically satisfied by (59)–(62). In conclusion, under the permeability conditions (55), the set of parameters B, W, A_0, and A_1 determines, in a unique way, the set of Λ-matrix parameters a, b, c, d, and φ, together with the condition $ad - bc = 1$. Conversely, under (45), the set of Λ-matrix parameters a, b, c, d, and φ, with $ad - bc = 1$, also determines univocally the parameters B, W, A_0, and A_1. Similarly to what happens in the impermeable case, we note that if condition (45) is not satisfied, some (or all) the new parameters will be infinite. However, they still may represent a permeable barrier if the permeability condition (55) still holds, under an appropriate limiting process.

There have been claims in the literature that the phase φ and, consequently, also the magnetic potential parameter A_1 (given (58) and, in the non-relativistic case, [29]), are *trivial* in the stationary case [30]. Although, for the Schrödinger theory Golovaty [25] has recently shown, using a norm-convergent regularization, an example to the contrary—including the fact that the magnetic potential can affect all parameters in the b.c. Λ-matrix. Our results for the relativistic case (58)–(62), clarify the effect of the magnetic potential parameter by establishing, in an explicit and general way, how A_1 affects *all* the b.c. parameters of the Λ-matrix. In addition, and most importantly, (58) and (55) firmly establish the *non-triviality* of A_1, after all, it follows

immediately from (55) that it is sufficient to have $A_1 \neq 0$ for the interaction to be permeable. The non triviality of the singular magnetic point potential in (47) is reflected in the fact that, in contrast with a regular one dimensional magnetic potential, it *cannot* be 'gauged away' by a unitary transformation of the spinors.

3.1. Lorentz covariant form of Dirac equation with a general PI

The time independent Dirac equation (5), with the most general interaction distribution given by (47), is given in the laboratory frame, defined as the frame in which the singular point is at rest. In this reference frame, the two-velocity of the singular point is given by $u \equiv u^\mu = (1,0)$ and its space location can be covariantly specified by noticing that the expression $\epsilon_{\mu\nu}\, u^\mu x^\nu = 0$, where $\epsilon_{\mu\nu}$ is the Levi–Civita anti-symmetric tensor, reduces to $x^1 \equiv \mathbf{x} = 0$. Thus, the Dirac delta appearing in (47) can be written covariantly as $\delta\left(x^1\right) = \delta\left(\epsilon_{\mu\nu} u^\mu x^\nu\right)$. Then, multiplying equation (5) by $\mathrm{e}^{-\mathrm{i}Ex^0}$ and recalling that the time dependent Dirac spinor for a fixed energy E is given, in the laboratory frame, by $\Psi(x) \equiv \Psi\left(x^0, x^1\right) = \psi\left(x^1\right) \mathrm{e}^{-\mathrm{i}Ex^0}$, we obtain

$$\begin{aligned}\left(i\gamma^\mu \partial_\mu - m\mathbb{1}\right)\Psi(x) &\equiv D_{\mathrm{c}}[\Psi](x) \\ &= \left(B\mathbb{1} + A_0\gamma^0 + A_1\gamma^1 + iW\gamma^5\right)\delta\left(\epsilon_{\mu\nu}u^\nu x^\nu\right)\frac{\Psi^+ + \Psi^-}{2},\end{aligned} \tag{63}$$

where $D_{\mathrm{c}}[\Psi](x)$ stands for the covariant interaction distribution, and we have introduced the notation

$$\Psi^\pm \equiv \Psi(x)\big|_{\epsilon_{\mu\nu}\, u^\mu x^\nu = 0^\pm}, \tag{64}$$

with the expressions at the rhs of the above equations interpreted as lateral limits with respect to the world line of the singularity. Therefore, equation (63) can be written in any Lorentz frame as

$$\left(i\gamma^\mu \partial_\mu - m\mathbb{1}\right)\Psi(x) = D_{\mathrm{c}}[\Psi](x) = \mathcal{I}_{\mathrm{c}}(x)\,\frac{\Psi^+ + \Psi^-}{2}, \tag{65}$$

where the manifestly covariant singular distribution $\mathcal{I}_{\mathrm{c}}(x)$ is given by

$$\mathcal{I}_{\mathrm{c}}(x) = \left(B\mathbb{1} + A_\mu\gamma^\mu + iW\gamma^5\right)\delta\left(\epsilon_{\mu\nu}u^\mu x^\nu\right). \tag{66}$$

The Lorentz covariance of equation (65), with $\mathcal{I}_{\mathrm{c}}(x)$ given by (66), requires that the real parameters B, W, A_0 and A_1 transform respectively as a scalar, a pseudoscalar, and the two components of a Lorentz two-vector. Thus, for a distribution $\mathcal{I}_{\mathrm{c}}(x) = \gamma^0 V_{\mathrm{c}}(x)$ described by a *singular point potential* $V_{\mathrm{c}}(x)$ concentrated on the word line of the singularity, the parameters B, W, A_0 and A_1 are just the *strengths of the physical singular point potentials*.

We should emphasize that the relationships (48)–(51) and (58)–(62) between the physical parameters (i.e. strengths of the Lorentz singular potentials) and the Λ-matrix parameters were obtained in the laboratory frame (where the singular point is at rest at the origin). Those relationships can be generalized to an arbitrary Lorentz frame by a transformation (boost) from the laboratory frame (unprimed) to a reference frame (primed) moving with speed v to the *left* of the laboratory, as given by [64]

$$x' = \begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = Lx = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix}\begin{pmatrix} x^0 \\ x^1 \end{pmatrix}, \qquad \tanh\omega = -v. \tag{67}$$

$$\psi'(x') = S_L \psi(x), \qquad S_L = \mathbb{1}\cosh\frac{\omega}{2} + \gamma^5 \sinh\frac{\omega}{2}. \tag{68}$$

Under this transformation, the singular point moves to the *right* with velocity v. Now the b.c., in any Lorentz frame, can be obtained from the knowledge of the Λ-matrix, which gives the b.c. in the laboratory frame, as

$$S_L \Psi\left(x^0, 0^+\right) = \left(S_L \Lambda S_L^{-1}\right) S_L \Psi\left(x^0, 0^-\right), \tag{69}$$

$$\Psi'\left(x'^0, v x'^0 + 0^+\right) = \Lambda_L \Psi'\left(x'^0, v x'^0 + 0^-\right), \tag{70}$$

$$\Psi'^{+} = \Lambda_L \Psi'^{-}, \tag{71}$$

where $\Psi'^{\pm}$ were introduced in (64) and we defined Λ_L, giving the b.c. around the world line of the singular point in an arbitrary Lorentz frame, as

$$\Lambda_L \equiv S_L \Lambda S_L^{-1}, \tag{72}$$

which, as expected, depends on the velocity of the singular point. Thus, (72) and (57) allow us to express the relationships between the physical parameters and the Λ-matrix parameters in any Lorentz frame, if these relationships are known in the laboratory frame. In this work, for simplicity, we will always express these parameters in the laboratory frame.

Finally, the condition of permeability in terms of the physical strengths can now be written in covariant form by multiplying (53) at the left by S_L and at the right by S_L^{-1}, which does not change the determinant (54), and results in the covariantly permeability condition :

$$\epsilon_{\mu\nu} u^\mu A^\nu \neq 0 \qquad \text{or} \qquad 4 - B^2 - W^2 + A_\mu A^\mu \neq 0. \tag{73}$$

3.2. *The non-relativistic limit*

Let us now consider the non-relativistic limit of our results in order to discuss, in terms of the physical parameters, several particular non-RPIs that have been studied in the literature in terms of the Λ-matrix parameters.

To obtain the non-relativistic limit of the Dirac equation with PIs, we once again consider the time independent equation (47), which is given in the laboratory frame. In terms of the components u and v of the time independent spinor $\psi(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}))^{\mathrm{T}}$, that equation is written as

$$\left(i\gamma^1 \frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} + \gamma^0 E - m\right) \begin{pmatrix} u(\mathbf{x}) \\ v(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} B + A_0 & iW + A_1 \\ iW - A_1 & B - A_0 \end{pmatrix} \delta(\mathbf{x}) \begin{pmatrix} \frac{u^+ + u^-}{2} \\ \frac{v^+ + v^-}{2} \end{pmatrix}. \tag{74}$$

where we used the notation $u^\pm \equiv u(0^\pm)$ and $v^\pm \equiv v(0^\pm)$. The above matrix equation corresponds to the following two equations for the components $u(x)$ and $v(x)$:

$$(E - m)u + iv' = \left[(B + A_0)\left(\frac{u^+ + u^-}{2}\right) + (iW + A_1)\left(\frac{v^+ + v^-}{2}\right)\right]\delta(\mathbf{x}), \tag{75}$$

$$-(E + m)v - iu' = \left[(iW - A_1)\left(\frac{u^+ + u^-}{2}\right) + (B - A_0)\left(\frac{v^+ + v^-}{2}\right)\right]\delta(\mathbf{x}). \tag{76}$$

In the non-relativistic limit

$$E = m + \epsilon, \qquad E + m \cong 2m, \tag{77}$$

where ϵ is the non-relativistic energy. In this limit, $u(\mathbf{x})$ and $v(\mathbf{x})$ are the large and the small components of the Dirac spinor, respectively[8]. Outside the origin, the spinor ψ satisfies the free equation, and the small component $v(\mathbf{x})$ is essentially the derivative of the large component, as it is seen from equation (76) for $\mathbf{x} \neq 0$. Thus, in the non-relativistic limit we have

$$v^{\pm} = -\frac{i}{2m} u'^{\pm}, \quad \text{where} \quad u'^{\pm} \equiv u'\left(0^{\pm}\right). \tag{78}$$

By isolating $v(\mathbf{x})$ in (76), substituting it into (75), taking the non-relativistic limit (77), and using (78), we obtain a Schrödinger equation for the large component $u(\mathbf{x})$:

$$\begin{aligned} -\frac{1}{2m} u'' - \epsilon u = & \left[-(B + A_0) \left(\frac{u^+ + u^-}{2} \right) - \frac{(W - iA_1)}{2m} \left(\frac{u'^+ + u'^-}{2} \right) \right] \delta(\mathbf{x}) \\ & + \left[\left(\frac{W + iA_1}{2m} \right) \left(\frac{u^+ + u^-}{2} \right) - \frac{(B - A_0)}{4m^2} \left(\frac{u'^+ + u'^-}{2} \right) \right] \delta'(\mathbf{x}), \end{aligned} \tag{79}$$

where $\delta'(\mathbf{x})$ is the distributional derivative of the Dirac delta $\delta(\mathbf{x})$. We can express the b.c. that the non-relativistic wave function $u(\mathbf{x})$ and its derivative $u'(\mathbf{x})$ must satisfy around the origin by simply writing (33) in terms of the spinor components $u^{\pm}$ and $v^{\pm}$ and by using the non-relativistic expression (78), obtaining

$$\begin{pmatrix} u^+ \\ u'^+ \end{pmatrix} = \Lambda_n \begin{pmatrix} u^- \\ u'^- \end{pmatrix}, \quad \text{where} \quad \Lambda_n \equiv \mathrm{e}^{\mathrm{i}\varphi_n} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \mathrm{e}^{\mathrm{i}\varphi} \begin{pmatrix} a & \frac{b}{2m} \\ 2mc & d \end{pmatrix}, \tag{80}$$

where the subscript 'n' indicates the non-relativistic parameters. From the above, we see that there is a one-to-one relationship between the non-relativistic and the relativistic parameters: $\varphi_n = \varphi$, $a_n = a$, $d_n = d$, $b_n = \frac{b}{2m}$, and $c_n = 2mc$ [36, 49]. The above non-relativistic b.c. can be promptly obtained in terms of the strengths of the singular potentials, by simply expressing the (relativistic) parameters φ, a, b, c, d in (80) in terms of the strengths (58)–(62).

Let us now consider some particular cases of one parameter non-RPIs and express them in terms of the (relativistic) physical strengths, by using (80) and (48)–(51).

(i) **The *delta* interaction.** This interaction is characterized by the following choices for the Λ_n-matrix parameters: $\varphi_n = b_n = 0, a_n = d_n = 1$ and $c_n \neq 0$ [27, 32, 71]. From (80) and (48)-(51) this interaction is given by the following physical potential strengths:

$$W = A_1 = 0 \quad \text{and} \quad B = A_0 \neq 0. \tag{81}$$

Thus, this interaction is obtained by taking an equal mixing of the electric and the scalar point potentials. The corresponding Schrödinger equation (79) is

$$-\frac{1}{2m} u'' - \epsilon u = -(B + A_0)\, \overline{u(0)}\, \delta(\mathbf{x}), \quad \text{with} \quad B = A_0, \tag{82}$$

[8] The non-relativistic limit of the Dirac equation in (1+1) dimensions is just a Schrödinger equation for one of the two components of the Dirac spinor, since there is no spin in only one space dimension [70].

where we introduced the notation $\overline{u(0)} = \frac{u(0^+)+u(0^-)}{2}$. The non-relativistic matrix Λ_n can also be written in terms of the physical strengths by using (57) and (80):

$$\Lambda_n = \begin{pmatrix} 1 & 0 \\ 2m(B+A_0) & 1 \end{pmatrix} \quad \text{with} \quad B = A_0. \tag{83}$$

(ii) **The *delta prime* interaction** (also referred to as ***non-local δ′* interaction**). This interaction is defined by the following Λ_n-parameters: $\varphi_n = c_n = 0$, $a_n = d_n = 1$ and b_n arbitrary [32, 71–73]. From (80) and (48)–(51) it corresponds to the following physical strengths:

$$W = A_1 = 0 \quad \text{and} \quad B = -A_0 \neq 0, \tag{84}$$

and are thus obtained from an inverted mixing of the electrostatic and scalar point potentials. The corresponding Schrödinger equation is

$$-\frac{1}{2m}u'' - \epsilon u = -\frac{(B-A_0)}{4m^2}\overline{u'(0)}\,\delta'(\mathbf{x}), \quad \text{with} \quad B = -A_0, \tag{85}$$

where $\overline{u'(0)} = \frac{u'(0^+)+u'(0^-)}{2}$, and the Λ_n matrix, in terms of the physical strengths, is

$$\Lambda_n = \begin{pmatrix} 1 & \frac{B-A_0}{2m} \\ 0 & 1 \end{pmatrix} \quad \text{with} \quad B = -A_0. \tag{86}$$

Note that our identification of the pure delta and the delta prime as, respectively, an equal and an inverted mixing of the scalar and electrostatic point potential was obtained previously by the authors of [18], which used a limiting (regularization) procedure starting from a regular scalar plus a vector (electrostatic) potential. However, our results for the point potentials differ from theirs in the more general case in which $B \neq \pm A_0$.

(iii) **The *local delta prime* interaction**. This interaction corresponds to the following Λ parameters: $d_n = \frac{1}{a_n}, \varphi_n = b_n = c_n = 0$ [71, 73][9]. The physical strengths of the potentials producing this interaction are

$$B = A_0 = A_1 = 0, \quad W \neq 0. \tag{87}$$

Thus, the local delta prime is produced by a pure pseudoscalar point potential. The corresponding Schrödinger equation is

$$-\frac{1}{2m}u'' - \epsilon u = \frac{W}{2m}\left[-\overline{u'(0)}\,\delta(\mathbf{x}) + \overline{u(0)}\,\delta'(\mathbf{x})\right], \tag{88}$$

and the Λ_n matrix is

$$\Lambda_n = \begin{pmatrix} \frac{2-W}{2+W} & 0 \\ 0 & \frac{2+W}{2-W} \end{pmatrix} \quad \text{with} \quad W \neq \pm 2. \tag{89}$$

[9] In [73] this interaction is denoted by the symbol $\delta^{(1)}$.

If $W=\pm 2$ the point potential would be an impermeable barrier (see equation (55)). It is worth to note that, *if* $u(\mathbf{x})$ and $u'(\mathbf{x})$ were *continuous* at the origin, then the distributional product $\frac{W}{2m}u(\mathbf{x})\delta'(\mathbf{x})$ would be well defined and equal to $\frac{W}{2m}\left[-u'(0)\,\delta(\mathbf{x})+u(0)\,\delta'(\mathbf{x})\right]$. This is *not* the case here, since both u and u' are discontinuous at $x=0$, due to the presence of both the δ and δ' terms in the rhs of (88). However, one may consider the rhs of (88) as a 'natural' way to interpret the ill-defined product $u(\mathbf{x})\delta'(\mathbf{x})$ and, in fact, in [29] the author obtained this expression as a result of a general *prescription* to build a non standard theory of distributions with discontinuous test functions, which loses several desirable properties of the Schwartz distributions [58]. In the present work, the rhs of (88) emerges as *a direct consequence* of the approach we are adopting, which is based on the *standard* Schwartz's theory of distributions, and that has no ill-defined products in the intermediary steps; therefore, we *do not* use ad hoc prescriptions to cure ill-defined products of distributions in any step of our calculations.

(iv) **The '*singular gauge field*' interaction.** This interaction is characterized by the following Λ_n parameters [73]: $[0,\pi)\ni\varphi\neq 0$, $a=d=\pm 1$, $b=c=0$, and corresponds to the following physical strengths:

$$B=A_0=W=0,\quad A_1\neq 0. \tag{90}$$

Thus, this PI is produced by a pure 'magnetic' point potential. The associated Λ_n matrix, in terms of physical strengths, is

$$\Lambda_n=\mathrm{e}^{\mathrm{i}\arg\left(A_1^2-4+4iA_1\right)}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}. \tag{91}$$

The examples above are just the most commonly used non-RPI. But *any* combination of the four physical Lorentz point potentials will give a well-defined RPI which, in turn, has a well-defined non-relativistic limit given by the rhs of (79)—and the corresponding Λ_n matrix can always be written in terms of the physical strengths by using (57) and (80). Finally, it should be noted that the relativistic interactions whose non-relativistic limit resuts in the δ and δ' have been investigated, *in terms of b.c. parameters*, in [38].

4. Symmetries of PIs under $\mathcal{P},\mathcal{T}$ and $\mathcal{C}$

In this section we will study the behavior of the distribution $D_{\mathrm{c}}[\Psi](x)$ in the covariant Dirac equation (65) under transformations of space reflections ($\mathcal{P}$), time reversal ($\mathcal{T}$) and charge conjugation ($\mathcal{C}$), and investigate how the requirement of *invariance* of $D_{\mathrm{c}}[\Psi](x)$ under these transformations restricts the possibilities for the physical strengths (and, accordingly, for the Λ-matrix parameters).

Given a general symmetry transformation $\mathcal{U}$ (where $\mathcal{U}$ stands for $\mathcal{P},\mathcal{T}$ or $\mathcal{C}$), the *invariance* of $D_{\mathrm{c}}[\Psi](x)$ under $\mathcal{U}$ requires the invariance of the physical parameters that define the interaction. Under the transformation $\mathcal{U}$, the coordinates, spinor and the covariant interaction distribution change, respectively, as

$$\begin{aligned}&\mathcal{U}:x\mapsto x';\\ &\mathcal{U}:\Psi(x)\mapsto\Psi^{U}(x');\\ &\mathcal{U}:D_{\mathrm{c}}[\Psi](x)\mapsto D_{\mathrm{c}}^{U}\left[\Psi^{U}\right](x').\end{aligned} \tag{92}$$

Then, requiring the *invariance* of the covariant Dirac equation (65) under $\mathcal{U}$ is equivalent to require

$$D_{\mathrm{c}}^{U}\left[\Psi^{U}\right](x') = D_{\mathrm{c}}\left[\Psi^{U}\right](x'), \tag{93}$$

i.e. the transformed spinor must satisfy the same equation, *with the same values of the physical strengths*.

The *invariance* property under the improper transformation $\mathcal{U}$ generally holds in a given (arbitrary) Lorentz frame, and a boost to another Lorentz frame may break this invariance. Here, for simplicity, we will restrict the symmetry analysis to the laboratory frame.

4.1. Space reflection

To obtain the operator performing a space reflection on the distributional spinor $\Psi(x)$ one can proceed in much the same way as for the usual Dirac equation, obtaining the condition $P\gamma^{\mu}P^{-1} = g^{\mu\mu}\gamma^{\mu}$ (no summation—P indicates the matrix implementing the space reflection transformation, itself indicated by $\mathcal{P}$). It follows that, up to an irrelevant phase factor, the matrix P is [39]

$$P = \gamma^{0}. \tag{94}$$

Then, denoting the transformed coordinates by a prime and the transformed spinor and interaction distribution by a superscript P, we have

$$\mathcal{P}:\ \left(x^{0},x^{1}\right) \mapsto \left(x'^{0},x'^{1}\right) = \left(x^{0},-x^{1}\right), \tag{95}$$

$$\mathcal{P}:\ \Psi(x) \mapsto \Psi^{\mathrm{P}}(x') = \gamma^{0}\Psi(x) = \gamma^{0}\Psi\left(x'^{0},-x'^{1}\right), \tag{96}$$

$$\mathcal{P}:\ D_{\mathrm{c}}[\Psi](x) \mapsto D_{\mathrm{c}}^{\mathrm{P}}\left[\Psi^{\mathrm{P}}\right](x') = \gamma^{0}D_{\mathrm{c}}[\Psi]\left(x'^{0},-x'^{1}\right), \tag{97}$$

which, together with (65) end (66), imply

$$\mathcal{P}:\ \mathcal{I}_{\mathrm{c}}(x) \mapsto \mathcal{I}_{\mathrm{c}}^{\mathrm{P}}(x') = \gamma^{0}\mathcal{I}_{\mathrm{c}}\left(x'^{0},-x'^{1}\right)\gamma^{0}. \tag{98}$$

From the above transformation rules, and using (64), it follows that

$$\mathcal{P}:\ \Psi^{\pm} \mapsto \Psi^{\mathrm{P}\mp} = \gamma^{0}\Psi^{\pm}, \tag{99}$$

$$\mathcal{P}:\ \mathcal{I}_{\mathrm{c}}(x) \mapsto \mathcal{I}_{\mathrm{c}}^{\mathrm{P}}(x') = \left[B\mathbb{1} + A_{0}\gamma^{0} - A_{1}\gamma^{1} - iW\gamma^{5}\right]\delta\left(\epsilon_{\mu\nu}u'^{\mu}x'^{\nu}\right), \tag{100}$$

where we have used $\epsilon_{\mu\nu}u'^{\mu}x'^{\nu} = -\epsilon_{\mu\nu}u^{\mu}x^{\nu}$ and the fact that the delta is an even distribution. Therefore, the interaction distribution $D_{\mathrm{c}}[\Psi](x)$ transforms as

$$\mathcal{P}:\ D_{\mathrm{c}}[\Psi](x) \mapsto D_{\mathrm{c}}^{\mathrm{P}}\left[\Psi^{\mathrm{P}}\right](x') = \left[B^{\mathrm{P}}\mathbb{1} + A_{0}^{\mathrm{P}}\gamma^{0} + A_{1}^{\mathrm{P}}\gamma^{1} + iW^{\mathrm{P}}\gamma^{5}\right]\delta\left(\epsilon_{\mu\nu}u'^{\mu}x'^{\nu}\right)\frac{\Psi^{\mathrm{P}+} + \Psi^{\mathrm{P}-}}{2}, \tag{101}$$

and the transformed physical strengths are

$$B^{\mathrm{P}} = B, \qquad A_{0}^{\mathrm{P}} = A_{0}, \qquad A_{1}^{\mathrm{P}} = -A_{1} \quad \text{and} \quad W^{\mathrm{P}} = -W. \tag{102}$$

We will consider two cases in what regards space reflection.

(i) ***Even interactions.*** From (102), it follows that a PI will be *invariant* under a parity transformation if

$$A_1 = W = 0. \tag{103}$$

The interaction is thus specified by just two physical parameters, namely, the strengths B and A_0 of a scalar and of an electrostatic point potential, respectively. In this case, Ψ and Ψ^{P} are solutions of the same distributional Dirac equation, with an interaction term $D_{\mathrm{c}}[\Psi](x)$ fully specified by

$$D_{\mathrm{c}}^{\mathrm{even}}(x) = \left(B\mathbb{1} + A_0\gamma^0\right)\delta\left(\epsilon_{\mu\nu}u^\mu x^\nu\right)\frac{\Psi^+ + \Psi^-}{2}. \tag{104}$$

Assuming that condition (103) is satisfied in the laboratory frame, and using (58)–(62), we obtain the corresponding constraints on the parameters of the matrix Λ as

$$\varphi = 0 \qquad \text{and} \qquad a = d, \tag{105}$$

which, due to the condition $ad - bc = 1$, results in two independent parameters [8, 49]. The Λ matrix (57), in terms of the physical strengths, is

$$\Lambda^{\mathrm{even}} = \frac{1}{-A_0^2 + B^2 - 4}\begin{pmatrix} A_0^2 - B^2 - 4 & 4i\,(A_0 - B) \\ 4i\,(A_0 + B) & A_0^2 - B^2 - 4 \end{pmatrix}. \tag{106}$$

The corresponding non-relativistic matrix $\Lambda_n^{\mathrm{even}}$ has essentially the same form, and can be obtained straightforwardly from (80). This case includes both the non-relativistic 'delta' and the 'non-local delta prime' interactions, given in (83) and (86), as special cases (respectively the equal and inverted mixes of A_0 and B). Notice that, despite its name, the non-local delta prime interaction is in fact an *even* interaction.

(ii) ***Odd interactions.*** Similarly to the regular case, an interaction is said to be *odd* under parity if the distribution $D_{\mathrm{c}}[\Psi](x)$ *is not* invariant under parity, but instead it satisfies

$$\mathcal{P} : D_{\mathrm{c}}[\Psi](x) \mapsto D_{\mathrm{c}}^{\mathrm{P}}\left[\Psi^{\mathrm{P}}\right](x') = -D_{\mathrm{c}}\left[\Psi^{\mathrm{P}}\right](x'), \tag{107}$$

or, equivalently,

$$\mathcal{P} : \mathcal{I}_{\mathrm{c}}(x) \mapsto \mathcal{I}_{\mathrm{c}}^{\mathrm{P}}(x') = -\mathcal{I}_{\mathrm{c}}(x'). \tag{108}$$

From (102), this condition will be satisfied if, and only if,

$$B = A_0 = 0. \tag{109}$$

Thus, we conclude that such interactions are characterized by only two Lorentz point potentials, namely, the strengths of the pseudo-scalar point potential W and that of the 'magnetic' point potential A^1—that is, the most general odd point potential is such that

$$D_{\mathrm{c}}^{\mathrm{odd}}(x) = \left(A_1\gamma^1 + iW\gamma^5\right)\delta\left(\epsilon_{\mu\nu}u^\mu x^\nu\right). \tag{110}$$

Again, assuming that condition (109) holds in the laboratory frame, from (58)–(62) it implies the following restrictions on the Λ parameters

$$b = c = 0, \qquad ad = 1, \tag{111}$$

and one may choose the two independent Λ-matrix parameters to be, for instance, φ and d (with $d \neq 0$ and $d \neq \infty$ for permeable interactions or, equivalently, $A_1 \neq 0$ or $W \neq \pm 2$)[10]. From (57), we obtain the Λ matrix for odd interactions in terms of the physical parameters as

$$\Lambda^{\text{odd}} = \frac{\mathrm{e}^{\mathrm{i}\arg\left(-4+W^2+A_1^2+4\mathrm{i}A_1\right)}}{\sqrt{\left(-4+W^2+A_1^2\right)^2+16A_1^2}} \begin{pmatrix} -A_1^2-(W-2)^2 & 0 \\ 0 & -A_1^2-(W+2)^2 \end{pmatrix}. \tag{112}$$

The non-relativistic limit of the subfamily of odd interactions above include, as special cases, both the 'local delta prime' ($A_1 = 0, W \neq 0$) and the 'singular gauge field' ($W = 0, A_1 \neq 0$) interactions, given by (89) and (91), respectively. The fact that the local delta prime, associated to the 'pure pseudoscalar' relativistic point potential, is odd under parity reinforces the suggestion in section 3.2.iii that this interaction has several natural properties that one would expect from an interaction associated to the derivative of a Dirac delta distribution—although, it should be stressed, it is not equivalent to a derivative of the delta distribution, since, as we have seen above, there are several other non equivalent possibilities to construct odd PIs by combining pseudoscalar and magnetic point potentials.

The conditions for even and odd interactions derived above can be restated in terms of the properties of the matrix Λ (and the corresponding b.c.) as

$$\textbf{Even interactions}: \qquad \gamma^0\Lambda\gamma^0 = \Lambda^{-1} \implies \psi'\left(0^+\right) = \Lambda\,\psi'\left(0^-\right), \tag{113}$$

$$\textbf{Odd interactions}: \qquad \gamma^0\Lambda\gamma^0 = \Lambda \implies \psi'\left(0^+\right) = \Lambda^{-1}\psi'\left(0^-\right). \tag{114}$$

Finally we should notice that the existence of odd (relativistic and non-relativistic) PIs demonstrated here contradicts previous statements in the literature, including by some of us (see, e.g. [49])[11]. It should be noted that space inversion in relativistic case was also considered in [38], where the *invariance* of the whole Hamiltonian was investigated and, consequently, only even interactions were obtained.

4.2. Time reversal

The time reversal transformation is *defined* by [38, 64]

$$\mathcal{T}: \left(x'^0, x'^1\right) \mapsto \left(-x^0, x^1\right), \tag{115}$$

$$\mathcal{T}: \Psi(x) \mapsto \Psi^{\mathrm{T}}(x') = T\Psi^*(x), \tag{116}$$

where T is the matrix operator defining the time reversal transformation, and the same superscript denotes the time-reversed quantities; '$*$' stands for complex conjugation. The Dirac equation will be covariant under $\mathcal{T}$ if

$$T\gamma^0T^{-1} = \gamma^0 \quad \text{and} \quad T\gamma^1T^{-1} = -\gamma^1,$$

[10] By requiring invariance of the Schrödinger operator under Weyl scale transformations, the authors of [73] obtained the same restriction on the corresponding non-relativistic Λ_n parameters.

[11] In [49] the parity symmetry was investigated in terms of the Λ-matrix parameters. The definition of even interactions given there is equivalent to (113), but for odd interactions the definition given in [49] needs to be reformulated: equation (114) is the correct way to characterize odd point interactions in terms of the Λ-parameters.

which imply that $T=\gamma^0$, apart from an irrelevant phase. Since under $\mathcal{T}$ we have $\epsilon_{\mu\nu}u^\mu x^\nu = \epsilon_{\mu\nu}u'^\mu x'^\nu$ and $\Psi^{T\pm}=\gamma^0\Psi^{*\pm}$, it follows that

$$\mathcal{T}: D_{\rm c}\left[\Psi\right](x)\mapsto D_{\rm c}^{\rm T}\left[\Psi^{\rm T}\right](x')=\left[B^{\rm T}\mathbb{1}+A_0^{\rm T}\gamma^0+A_1^{\rm T}\gamma^1+iW^{\rm T}\gamma^5\right]\delta\left(\epsilon_{\mu\nu}u'^\mu x'^\nu\right)\frac{\Psi^{T+}+\Psi^{T-}}{2}, \tag{117}$$

where the transformed physical strengths are

$$B^{\rm T}=B,\qquad A_0^{\rm T}=A_0,\qquad A_1^{\rm T}=-A_1\quad\text{and}\quad W^{\rm T}=W. \tag{118}$$

Therefore, the interaction is *invariant* under time reversal if, and only if,

$$A_1=0, \tag{119}$$

And the subfamily of all time reversal invariant PIs is given, in a Lorentz frame in which (119) holds, by

$$D_{\rm c}^{\rm tr}(x)=\left(B\mathbb{1}+A_0\gamma^0+iW\gamma^5\right)\delta\left(\epsilon_{\mu\nu}u^\mu x^\nu\right)\frac{\Psi^{+}+\Psi^{-}}{2}. \tag{120}$$

Assuming that the interaction is invariant by time reversal in the laboratory frame, from (58)–(62) this implies that the only constraint on the Λ-matrix parameters is

$$\varphi=0. \tag{121}$$

In terms of the physical strengths, we have

$$\Lambda^{\rm tr}=\frac{1}{B^2+W^2-4-A_0^2}\begin{pmatrix}A_0^2-B^2-(W-2)^2 & 4i\,(A_0-B)\\ 4i\,(A_0+B) & A_0^2-B^2-(W+2)^2\end{pmatrix}, \tag{122}$$

which agrees with the result of [73] when expressed in terms of the Λ_n-matrix parameters in the non-relativistic limit.

4.3. Charge conjugation

Charge conjugation in the Dirac equation is an internal symmetry which transforms electrons in positrons, and *vice-versa*. It is defined as [64]

$$\mathcal{C}:\Psi(x)\mapsto\Psi^{\rm C}(x)=C\gamma^0\Psi^*(x), \tag{123}$$

where C is a matrix and the superscript C indicates charge conjugated quantities. Then, covariance of the Dirac equation under charge conjugation implies

$$C\gamma^0=-\gamma^0C\quad\text{and}\quad C\gamma^1=\gamma^1C,$$

which results in $C=\gamma^5$. Accordingly, the physical potential strengths must transform as[12]

$$B^{\rm C}=B,\quad A_0^{\rm C}=-A_0,\quad A_1^{\rm C}=-A_1\quad\text{and}\quad W^{\rm C}=-W. \tag{124}$$

[12] In this work we absorb all the coupling constants (charges) into the definitions of the external potential strengths. Therefore, to each spinor Ψ describing an *electron* in the external point potential $\gamma^0\mathcal{I}_{\rm c}=\gamma^0\left(B\mathbb{1}+A_\mu\gamma^\mu+iW\gamma^5\right)\delta\left(\epsilon_{\mu\nu}u^\mu x^\nu\right)$ there corresponds a spinor $\Psi^{\rm C}$ describing a *positron* in the *same* point potential, and this spinor is identical to the spinor describing an *electron* in the *transformed* potential (124).

Invariance of the PI under charge conjugation implies that

$$A_0 = A_1 = W = 0, \qquad B \neq 0, \tag{125}$$

and the potential must be a pure scalar point potential, given by

$$D_{\mathrm{c}}^{\mathrm{cc}}[\Psi](x) = B\,\delta\left(\epsilon_{\mu\nu}u^{\mu}x^{\nu}\right)\frac{\Psi^{+}+\Psi^{-}}{2}. \tag{126}$$

Assuming this invariance holds in the laboratory frame, we have

$$\varphi = 0, \quad a = d, \quad b = c, \quad a^2 - b^2 = 1,$$

$$\Lambda^{\mathrm{cc}} = \frac{1}{B^2-4}\begin{pmatrix} -B^2-4 & -4iB \\ 4iB & -B^2-4 \end{pmatrix}.$$

Our results for interactions that are invariant under $\mathcal{P}$ and $\mathcal{T}$ transformations coincide, in the non-relativistic limit, with the results of [73], in which the authors investigated how the invariance of Schrödinger operators under $\mathcal{P}, \mathcal{T}$ restricts the general PIs by using a non-standard distributional approach based on discontinuous test functions [29]. It should be stressed that the distributional approach adopted here is, instead, based on the standard Schwartz's theory of distributions, and it allows a clear identification of the physical meaning of the parameters of the RPIs as the strengths of four Lorentz point potentials.

5. Conclusions

In this work we have used the approach, based on Schwartz theory of distributions, proposed in [49] to present a fully relativistic study of PIs in one dimensional quantum mechanics. Consequently, we revisited the set of conditions necessary to obtain a well-defined distributional theory, and reduced it to the more economical set of requirements **R1–R3**. We have shown that this approach leads to the same four-parameter family of interactions characterizing all the SAEs of the free Dirac Hamiltonian. We considered permeable and impermeable PIs and, after writing the Dirac equation in a manifestly covariant form, we identified—under the condition that (45) and (39) hold—a one-to-one relationship between the usual set of the abstract parameters giving the b.c. that the spinor must satisfy at the origin and the set of four physical parameters giving the strengths of the four Lorentz point potentials characterizing the interaction, namely a scalar, the two components of a vector, and a pseudoscalar potential. These relationships allow us to assign physical meaning to the b.c. and, thus, may provide additional information to help decide which SAEs (b.c.) should be used in particular physical problems, since it is well-known that different extensions may lead to different physical properties, such as different spectra (e.g. [57, 74–77] and references therein).

By considering the non-relativistic limit we have shown how some widely used non-RPIs can be interpreted in terms of the four physical point potentials. In particular, we identified the specific combinations of the physical parameters that generate the delta interaction (an equal mixing of an electric and a scalar point potential), the so-called delta prime, in its 'local' (a pure pseudoscalar point potential) and 'non-local' (an inverted mixing of an electric and a scalar point potential) versions, and the 'pure gauge' interaction (a pure 'magnetic' point potential).

The investigation of the symmetries of PIs is greatly facilitated in the distributional approach, since the interaction distribution is given explicitly in terms of the Lorentz point potentials and the lateral limits of the Dirac spinor around the singular point. Thus, we studied

how the most general PI behaves under the transformations of space reflections, time reversal and charge conjugation, and we have shown how the requirements of symmetry under these transformations restrict the parameters of the interaction. Thus, by specifying the symmetry requirements we were able to particularize the subfamilies of RPI satisfying such requirements, and to obtain their physical interpretation in terms of the four Lorentz point potentials.

A significant result obtained was the characterization of odd PIs (see equation (114)), which were demonstrated to form a two-parameter subfamily of PIs, characterized by the pseudoscalar (W) and the 'magnetic' (A_1) potential strengths. In addition, we have identified the 'local delta prime', characterized by a single pseudoscalar point potential, as the 'natural' *extension* of the derivative of a delta potential in the non-relativistic limit, due to the formal similarity between the rhs of equation (88) and the product of the derivative of the delta distribution with a smooth function.

Finally, our results help to clarify the physical relevance of the phase φ, which is one of the Λ-matrix parameters (see equation (35)) and has been the focus of some scrutiny in the literature of non-RPI. For instance, in [78] it is claimed that the phase parameter is not redundant in non-stationary problems, whereas in the stationary case it corresponds to a gauge transformation, and thus has no physical consequences (see also [29]). Similarly, the authors of [30, 79] argue that only three of the four Λ_n parameters are physically relevant, excluded the phase, since the latter does not affect probabilities and expectation values of observables (at least for the one-body problem [31]). Our results, however, show that φ is *not* physically irrelevant, since (58) implies that $\varphi = 0$ if, and only if, $A_1 = 0$. But $A_1 \neq 0$ (hence $\varphi \neq 0$) is a *sufficient* condition for the permeability of the PI at the origin due to the condition (55). Note that the *singular* interaction strength A_1, in contradistinction to a regular one-dimensional field $A_1(x)$, cannot be eliminated by a gauge transformation (even allowing for a discontinuous local phase function). This, of course, raises the question of the physical interpretation of this singular magnetic potential in Dirac's theory, which we leave for a future work.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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