mathematics

IMPACT FACTOR 2.3

CITESCORE 4.0

Article

Prolongation Structure of a Development Equation and Its Darboux Transformation Solution

Lixiu Wang, Jihong Wang and Yangjie Jia

Special Issue
Symmetries of Integrable Systems, 2nd Edition

Edited by
Prof. Dr. Chuanzhong Li

MDPI

https://doi.org/10.3390/math13060921

mathematics

MDPI

Article

Prolongation Structure of a Development Equation and Its Darboux Transformation Solution

Lixiu Wang, Jihong Wang and Yangjie Jia *

School of Mathematics and Statistics, Qinghai Normal University, Xining 810008, China; wjh3207@126.com (J.W.)
* Correspondence: jiayangjie123@163.com

Abstract: This paper explores the third-order nonlinear coupled KdV equation utilizing prolongation structure theory and gauge transformation. By applying the prolongation structure method, we obtained an extended version of the equation. Starting from the Lax pairs of the equation, we successfully derived the corresponding Darboux transformation and Bäcklund transformation for this equation, which are fundamental to our solving process. Subsequently, we constructed and calculated the recursive operator for this equation, providing an effective approach to tackling complex problems within this domain. These results are crucial for advancing our understanding of the underlying principles of soliton theory and their implications on related natural phenomena. Our findings not only enrich the theoretical framework but also offer practical tools for further research in nonlinear wave dynamics.

Keywords: recursive operator; Bäcklund transformation; Darboux transformation; Lax pair; prolongation structure

MSC: 35Q51; 35Q53; 37K10; 20G43

1. Introduction

With the advancement of scientific and technological knowledge, the study of nonlinear partial differential equations has garnered considerable interest from the mathematical and physical sciences communities [1–3]. The phenomenon of isolated waves was first observed in 1834 by British scientist and marine engineer John Scott Russell in the Union Canal connecting Edinburgh and Glasgow, Scotland. Solitons, a broad class of solutions to nonlinear partial differential equations, display a wide array of distinctive characteristics and corresponding physical phenomena. Soliton theory finds extensive applications across various fields, including fluid mechanics, laser physics, classical field theory, biology, and nonlinear optics. Today, the theory of solitons and integrable systems stands as a pivotal area of research within nonlinear science, continuously evolving and expanding its scope. The exact solution of nonlinear partial differential equations is crucial for understanding a myriad of complex physical phenomena and addressing nonlinear engineering challenges. Recently, in the realm of continuous integrable systems, researchers have pinpointed three principal challenges [4–8]. The first involves the discrete lattice structures found in physical systems, ranging from simple to complex molecular or atomic arrangements. The second challenge highlights the necessity of discrete lattice structures for the general numerical computations of physical systems and related equations. Lastly, overcoming the limitations inherent in continuous integrable systems and solving integrability issues in practical applications poses the third major hurdle [9,10].

check for updates

Academic Editors: Luigi Rodino and Chuanzhong Li

Received: 8 December 2024
Revised: 16 January 2025
Accepted: 24 January 2025
Published: 11 March 2025

Citation: Wang, L.; Wang, J.; Jia, Y. Prolongation Structure of a Development Equation and Its Darboux Transformation Solution. *Mathematics* **2025**, *13*, 921. https://doi.org/10.3390/math13060921

In soliton theory, Sato theory plays an essential role in the study of the KP hierarchy and its associated hierarchies. The exploration of the BKP hierarchy, a sub-hierarchy of the KP hierarchy, has facilitated the gradual development and extensive application of exact solution methods for the KdV equation. Furthermore, numerous effective approaches have been developed to solve nonlinear evolution equations and to examine the physical properties of these solutions. These methods include the Hirota bilinear method, Darboux transformation, Bäcklund transformation, and KP reduction [11,12]. One of the focal points of this paper is the Bäcklund transformation, a method introduced by scholars to maintain consistency when studying surfaces with negative constant curvature in the context of the Sine–Gordon equation. This transformation enables the generation of new solutions to the equation based on an existing initial solution [13–15]. Additionally, Li Yishen developed Bäcklund transformations for nonlinear evolution equations through the application of gauge transformations. Moreover, in references [16–20], a close relationship between Bäcklund transformations and inverse scattering methods is highlighted, demonstrating their interconnected roles in solving and analyzing nonlinear evolution equations. Another key focus of this paper is the Darboux transformation, a potent method for solving integrable systems. It manifests in two types: differential type $T_d(q) = q\partial q^{-1}$ and integral type $T_i(r) = r^{-1}\partial^{-1} r$ [21–24]. In their research, He Jinsong and colleagues utilized the Darboux transformation operator generated by the adjoint wave functions of the constrained KP equations [25–28]. Among the myriad of soliton equations, the KdV equation was pioneering in introducing the property of recursive operators. With the progression of soliton theory in recent years, recursive operators have become fundamental to the concept of integrability. This evolution underscores their importance in both theoretical developments and practical applications within the field [29,30].

This paper utilizes the extension structure method to investigate the nonlinearly coupled KdV equation, providing a comprehensive insight into its structural characteristics. Through this approach, we have successfully derived Lax pairs, which are pivotal for understanding the intrinsic properties and dynamic behavior of solitons associated with these equations. Leveraging the derived Lax pairs, we proceed to formulate both the Darboux transformation and the Bäcklund transformation for the nonlinearly coupled KdV equation. These transformations not only facilitate the discovery of new solutions but also significantly enhance our analytical toolkit for exploring the equation's complexities. Furthermore, by constructing and analyzing the Lax operator, we identify the recursive operator pertinent to the equation. This recursive operator presents an effective mechanism for generating a plethora of new solutions, thereby deepening our comprehension of the nonlinearly coupled KdV equation's properties and broadening the scope for further explorations and applications.

2. The Extended Structure of Ohta-Hirota the Equation

This section primarily employs the theory of extension structures to solve a set of three coupled equations. Building upon Professor Jiayangjie′s previous research on Lax pairs, this study conducts an in-depth analysis and extends the work to further explore the methods of solving these equations and their applications.

The Ohta–Hirota coupling KdV equation is given by:

$$\begin{cases} v_t + \left(v_{xx} + 3v^2 + 3vu^2 + 3uu_{xx}\right)_x = 0, \\ u_t + \left(u_{xx} + 3vu + u^3\right)_x = 0, \end{cases} \tag{1}$$

First, introduce new variables such that $v_x = w$, $v_{xx} = w_x = z$, $u_x = p$, and $u_{xx} = p_x = q$. Consequently, the aforementioned system of partial differential equations can be reformulated in the following equivalent form:

$$\begin{cases} v_x = w, \\ u_x = p, \\ v_{xx} = w_x = z, \\ u_{xx} = p_x = q, \\ u_t + q_x + 3uw + 3vp + 3u^2 p = 0, \\ v_t + z_x + 6vw + 3u^2 w + 6uvp + 3pq + 3uq_x = 0. \end{cases} \tag{2}$$

To proceed, it is necessary to define a set of exterior differential 2-forms on the manifold. $M = \{x, t, u, v, w, z, p, q\}$.

$$\begin{cases} \alpha_1 = dt \wedge du + dx \wedge dt\ p, \\ \alpha_2 = dt \wedge dp + dx \wedge dt\ q, \\ \alpha_3 = dt \wedge dv + dx \wedge dt\ w, \\ \alpha_4 = dt \wedge dw + dx \wedge dt\ z, \\ \alpha_6 = dx \wedge du - dt \wedge dq + dx \wedge dt(3uw + 3vp + 3u^2 p), \\ \alpha_5 = dx \wedge dv - dt \wedge dz - dt \wedge dq(3u) + dx \wedge dt(6vw + 3u^2 w + 6uvp + 3pq). \end{cases} \tag{3}$$

Here, d denotes the exterior derivative and $\wedge$ denotes the exterior product. Upon applying the exterior differentiation to Equation (3), the result is:

$$\begin{cases} d\alpha_1 = dx \wedge \alpha_2, \\ d\alpha_2 = -dx \wedge \alpha_6, \\ d\alpha_3 = dx \wedge \alpha_4, \\ d\alpha_4 = dx \wedge \alpha_5 + 3u\alpha_6, \\ d\alpha_5 = dx \wedge [\alpha_4(6v + 3u^2) + \alpha_2 6uv + \alpha_3 \alpha_2 \alpha_6], \\ d\alpha_6 = dx \wedge [\alpha_4 3u + \alpha_2(3v + 3u^2)]. \end{cases} \tag{4}$$

Therefore, the set $M = \{\alpha_1, \alpha_2 \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ forms a closed ideal on the manifold. When each 2-form α_i is restricted to the solution manifold $S = \{x, t, u(x,t), v(x,t), w(x,t), z(x,t), p(x,t), q(x,t)\}$, it vanishes. Consequently, the equations of system (1) can be derived.

In this section, we introduce the n-1 forms ω^k:

$$w^k = dy^k + F^k(x, t, u, v, w, z, p, q, y^i)dx + G^k(x, t, u, v, w, z, p, q, y^i)dt, \quad (k = 1, \cdots, n) \tag{5}$$

The variable y^i is a continuation variable that needs to form a new closed ideal with α_i and requires ω^k to satisfy the condition.

$$d\omega^k = \sum_{j=1}^{4} f_j^i \alpha^j + \sum^{n} \eta_j^i \wedge \omega^j, \tag{6}$$

Among them, f_j^i is a 0-form and η_j^i is a 1-form. From this, we can derive the system of partial differential equations (PDEs) satisfied by $F^k(u, v, w, z, p, q, y^i)$ and $G^k(u, v, w, z, p, q, y^i)$.

$$F_p = F_q = F_w = F_z = 0, \tag{7}$$

$$G_q = -F_u - 3uF_v, \quad G_z = -F_v, \tag{8}$$

$$pG_u + wG_v + qG_p + zG_w + (3uw + 3vp + 3u^2p)F_u + (6vw + 3u^2w + 6uvp + 3pq)F_v + [G, F] = 0. \tag{9}$$

Upon integrating Equation (8) with respect to q and φ, we obtain:

$$G = -qF_u - 3uqF_v - zF_v + G_1(u, v, w, p, y^i) \tag{10}$$

With respect to substituting (10) into (9), the following can be derived.

$$G_1 = \frac{1}{2}p^2F_{uu}\frac{3}{2}p^2uF_{vu} + pwF_{uv} + p[F_u, F] + 3up[F_v, F] + \frac{1}{2}w^2F_{vv} + w[F_v, F] + G_2(u, v, y^i) \tag{11}$$

In relation to substituting (11) into (10), the following equations can be derived.

$$\begin{cases} [F_{vu}, F] = 0, \quad [G_2, F] = 0, \\ 2[F_v, F] + 2u[F_u, F_v] = 0, \\ F_{uuu} = F_{uv} = F_{uuv} = F_{vvv} = F_{uvv} = F_{vv} = 0, \\ G_{2v} = -3uF_u - 6vF_v - 3u^2F_v, \\ G_{2u} = 3vF_u - 3u^2F_u - 6uvF_v - [[F_u, F], F] - 3u[[F_v, F], F]. \end{cases} \tag{12}$$

By (9)–(12),

$$F = X_1 + uX_2 + vX_3 + u^2X_4, \tag{13}$$

Here, each $X_i (i = 1, 2, 3, 4)$ depends solely on the continuation variable y^j. Further, let $[X_2, X_1] = X_5$, $[X_3, X_1] = X_6$, $[X_3, X_2] = X_7$. Upon substituting F into Equation (12) and applying the Jacobi identity, the following relationship can be derived:

$$\begin{cases} [X_4, X_1] = [X_3, X_1] = 0, \quad [X_4, X_3] = 0, \\ [X_5, X_4] - [X_5, X_3] = 0, \quad [X_6, X_3] = -[X_6, X_4], \\ [X_4, X_2] + 2[X_3, X_2] = 0, \quad [X_5, X_3] = [X_6, X_2] + [X_7, X_1], \\ G_2 = -(3uv + u^3)X_2 - 3(u^2v + v^2)X_3 - (u^4 + 4u^2v)X_4 - uv[X_5, X_3] - u[X_5, X_1] \\ \qquad -\frac{1}{2}u^2[X_5, X_2] - \frac{1}{2}u^2[X_6, X_1] - \frac{1}{2}v^2[X_6, X_3] - v[X_6, X_1] + X_0. \end{cases} \tag{14}$$

Concerning the substitution of G_2 into the equation $[G_2, F] = 0$, we then obtain the following relationship:

$$\begin{cases} [X_0, X_1] = 0, \\ [[X_5, X_1], X_1] + [X_0, X_2] = 0, \\ [[X_6, X_1], X_1] - [X_0, X_3] = 0, \\ 2[[X_5, X_3], X_3] + [[X_6, X_3], X_2] = 0. \\ 3X_6 + \frac{1}{2}[[X_6, X_3], X_1] + [[X_6, X_1], X_3] = 0, \\ X_6 - \frac{1}{2}[[X_5, X_2], X_4] - \frac{1}{2}[[X_6, X_1], X_4] = 0, \\ [[X_6, X_3], X_3] = [[X_6, X_3], X_4] = [[X_5, X_3], X_4] = 0, \\ 3X_5 + [[X_5, X_3], X_1] + [[X_6, X_1], X_2] + [[X_5, X_1], X_3] = 0, \\ X_5 + \frac{1}{2}[[X_5, X_2], X_2] + \frac{1}{2}[[X_6, X_1], X_2] + [[X_5, X_1], X_4] = 0, \\ \frac{1}{2}[[X_5, X_2], X_1] + \frac{1}{2}[[X_6, X_1], X_1] + [[X_5, X_1], X_2] - [X_0, X_4] = 0, \\ X_6 - \frac{1}{2}[[X_5, X_2], X_3] - \frac{1}{2}[[X_6, X_1], X_3] - [[X_5, X_3], X_2] - [[X_6, X_1], X_4] = 0. \end{cases} \tag{15}$$

Let $[X_5, X_1] = X_6$, $[X_5, X_2] = X_9$, $[X_5, X_3] = X_{10}$, $[X_6, X_1] = X_{11}$, $[X_6, X_3] = X_{12}$. Then, from Equation (15), we obtain the following relations:

$$\begin{cases} [X_8, X_2] = [X_9, X_1] = 0, \\ [X_9, X_3] = [X_7, X_5] + [X_{10}, X_2], \\ [X_9, X_4] = 2[X_5, X_7] - [X_{10}, X_2], \\ [X_{11}, X_2] = [X_{10}, X_1] + [X_5, X_6], \\ [X_8, X_3] = -[X_8, X_4] = [X_6, X_5] + [X_{10}, X_1], \\ [X_{11}, X_3] = -[X_{11}, X_4] + [X_{12}, X_1], [X_{10}, X_3] = -[X_{10}, X_4]. \end{cases} \tag{16}$$

From Equations (15) and (16), we derive the following relationship:

$$\begin{cases} [X_{10}, X_1] = -X_9, \quad [X_{12}, X_1] = -2X_6, \\ 3[X_8, X_3] = [X_9, X_2], \\ [X_9, X_3] = [X_9, X_4] = 0, \\ [X_{10}, X_2] = [X_{10}, X_3] = [X_{10}, X_4] = 0, \\ [X_{12}, X_2] = [X_{12}, X_3] = [X_{12}, X_4] = 0. \end{cases} \tag{17}$$

Given the generator set from Equation (17) as $\{X_i = 1, 2, 3, \cdots, 12\}$, let us consider $X_7 = 0$, Consequently, we find that $X_{12} = 2X_4$, $X_4 = -X_3$,and the commutation relations $[X_6, X_3] = -2X_3$ and $[X_3, X_1] = X_6$. Here, X_3 is identified as a nilpotent element, while X_6 serves as a central element.

Concerning Equation (1), it has been determined that the following scaling symmetry applies, where λ denotes the spectral parameter:

$$x \to \lambda^{-1}x, \quad t \to \lambda^{-3}t, \quad u \to \lambda^2 u, \quad v \to \lambda^2 v, \tag{18}$$

If ω^i needs to remain invariant under the aforementioned scaling transformation, then we have:

$$F \to \lambda F, \quad G \to \lambda^3 G. \tag{19}$$

And X_i satisfies

$$\begin{cases} X_0 \to \lambda^3 X_0, \quad X_1 \to \lambda X_1, \quad X_2 \to X_2, \quad X_3 \to \lambda^{-1} X_3, \\ X_4 \to \lambda^{-1} X_4, \quad X_5 \to \lambda X_5, \quad X_6 \to X_6, \quad X_7 \to \lambda^{-1} X_7. \end{cases} \tag{20}$$

Using the basis of generators $X_i, (i = 1, 2, 3, 4)$, which are defined by the commutation relations $[X_6, X_3] = -2X_3$ and $[X_3, X_1] = X_6$, we embed the extended structure y into the *Lie* algebra $sl(n+1, C)$. When $n = 1$, it is evident that $sl(2, C)$ is not a suitable choice. Subsequent calculations indicate that $sl(3, C)$ is the optimal selection.

Assume the following relations :$X_{10} = -X_2, X_{12} = -2X_3, X_3 = e_-, X_6 = h_1$.

Then, we have $X_0 = \lambda^3(h_1 + 2h_2)$, $X_1 = e_- + (h_1 + 2h_2)$, $X_2 = X_5 = X_8 = X_9 = X_{10} = 0$

The matrix representations of the nilpotent element X_3 and the central element X_6 are presented below:

$$e_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

Substitute the aforementioned generated elements into Equations (10), (11), (13)–(15). Consequently, the concrete expressions for F and G have been derived.

$$F = \begin{pmatrix} \lambda & 0 & u^2 - v \\ 0 & -2\lambda & 0 \\ 1 & 0 & \lambda \end{pmatrix}.$$
$$G = \begin{pmatrix} -up - r + \lambda^3 & 0 & uq + s + p^2 - u^4 - u^2v + 2v^2 \\ 0 & -2\lambda^3 & 0 \\ -u^2 - 2v & 0 & up + r + \lambda^3 \end{pmatrix}. \tag{22}$$

Here, $p = \phi_x$, $q = p_x = \phi_{xx}$. If we require that $\omega^k|_U = 0$, we can obtain the *Lax* pair representation of the Ohta-Hirota Equation (1):

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_x = -F \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}, \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}_t = -G \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix}. \tag{23}$$

The original Equation (1) can be readily derived from the compatibility relation $y_{xt} = y_{tx}$. The Lax pair of equations was successfully identified through a drag structure, representing a crucial step in solving the nonlinear coupled KdV equation and laying a foundation for subsequent analysis. Additionally, this methodology offers the potential for gaining a comprehensive understanding of the underlying equation structure.

3. Bäcklund Transformation and Darboux Transformation of Ohta–Hirota Equation

The Lax pair for the Ohta–Hirota equation has been obtained. To ascertain the integrability of the Ohta–Hirota equation, we will now seek the Bäcklund transformation and Darboux transformation for this equation. The spatial part of the spectral problem given in Formula (1) will continue to be studied, starting from the obtained Lax pair:

$$\phi_x = M\phi, \quad \phi = (\phi_1, \phi_2)^T, \tag{24}$$

In the following equations, λ represents a constant spectral parameter, and the matrix T is the canonical transformation matrix, which performs the canonical transformation on ϕ

$$\tilde{\phi} = T\phi, \quad T = T(u, v, \tilde{u}, \tilde{v}, \lambda) \tag{25}$$

The spectral problem satisfies the same form as that of Equation (1), namely:

$$\tilde{\phi_x} = \tilde{M}\tilde{\phi}, \qquad \widetilde{M} = \begin{pmatrix} \lambda & 0 & \widetilde{u}^2 - \widetilde{v} \\ 0 & -2\lambda & 0 \\ 1 & 0 & 1 \end{pmatrix} \tag{26}$$

where $\tilde{u}$ and $\tilde{v}$ represent another set of solutions. The compatibility condition for the linear system is given by:

$$T_x + TM - \widetilde{M}T = 0 \tag{27}$$

The spatial part of the Bäcklund transformation is provided, and it is determined that the canonical transformation with respect to λ is linear. Consequently, the corresponding Bäcklund transformation can be derived.

First, let us write out M and M̃ separately:

$$M = \lambda F + G, \quad M = \begin{pmatrix} \lambda & 0 & u^2 - v \\ 0 & -2\lambda & 0 \\ 1 & 0 & \lambda \end{pmatrix}. \tag{28}$$

$$\tilde{M} = \lambda \tilde{F} + \tilde{G}, \quad \tilde{M} = \begin{pmatrix} \lambda & 0 & \tilde{u^2} - \tilde{v} \\ 0 & -2\lambda & 0 \\ 1 & 0 & \lambda \end{pmatrix}. \tag{29}$$

Assuming that T is of the form $T = \lambda H + I$, we substitute this expression into the compatibility condition:

$$T_x + TM - \widetilde{M}T = 0 \tag{30}$$

By performing the calculation and comparing the coefficients of like terms in λ, the results are as follows:

$$O(\lambda^2) : [H, F] = 0 \tag{31}$$

$$O(\lambda^1) : H_x + HG + IF - \tilde{F}I - \tilde{G}H = 0 \tag{32}$$

$$O(\lambda^0) : I_x + IG - \tilde{G}I = 0 \tag{33}$$

These equations correspond to the following conditions on the elements of

$$h_{12} = h_{21} = h_{23} = h_{32} = 0 \tag{34}$$

$$\begin{cases} I_{12} = I_{21} = I_{23} = I_{32} = 0, \quad h_{22,x} = 0, \\ h_{11,x} + h_{13} - h_{31}(\widetilde{u^2} - \widetilde{v}) = 0, \\ h_{13,x} + h_{11}(u^2 - v) - h_{33}(\widetilde{u^2} - \widetilde{v}) = 0, \\ h_{31,x} + h_{33} - h_{11} = 0, \\ h_{33,x} + h_{31}(u^2 - v) - h_{13} = 0, \end{cases} \tag{35}$$

$$\begin{cases} I_{11,x} + I_{13} - I_{31}(\widetilde{u^2} - \widetilde{v}) = 0, \\ I_{13,x} + I_{11}(u^2 - v) - I_{33}(\widetilde{u^2} - \widetilde{v}) = 0, \\ I_{31,x} + I_{33} - I_{11} = 0, \\ I_{33,x} + I_{31}(u^2 - v) - I_{13} = 0, \end{cases} \tag{36}$$

Let $q = u^2 - v$. Rearranging and transforming Equation (35) yields:

$$(h_{11}h_{33} - h_{13}h_{31})_x = 0 \tag{37}$$

From this, it can be inferred that $T = \lambda H + I$ is a constant matrix, and from this, it can be inferred that h_{31} is a constant, and knowing that $h_{11} = h_{33}$, along with $q = \widetilde{q}$ and $F = \widetilde{F}$, $G = \widetilde{G}$, we obtain:

$$M = \widetilde{M} \tag{38}$$

Moreover, since the I matrix is not a constant matrix, the relationships can be derived by transforming Equation (36):

$$(I_{11}+I_{33})_x=0,\quad (I_{11}-I_{33})(1+q)=(I_{31}-I_{13})_x \tag{39}$$

Regarding the I matrix, the values of I_{11} and I_{33} merit separate consideration. The H matrix is subdivided into eight categories to examine its constituent elements. Once the H matrix has been determined, further analysis is conducted on the I matrix, which is divided into six categories for discussion.

It can be observed that the transformation matrix T presents a variety of cases. Initially, let $q=u^2-v$, and then proceed with a categorized discussion. When $h_{22}=1$ and $I_{22}=1$, the subsequent discussion focuses on the specific values taken by $h_{11}=h_{33}$ and h_{31}, respectively. (1.1) When $h_{31}=0,\ h_{13}=0,\ h_{11}=h_{33}=0,\ I_{11}=1,\ I_{33}=0,$ $\tilde{\phi}=T\phi$

$$T=\begin{pmatrix} 1 & 0 & -\frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ -\frac{2q}{q_x} & 0 & 1 \end{pmatrix}$$

(1.2) When $h_{31}=0,\ h_{13}=0,\ h_{11}=h_{33}=0,\ I_{11}=0,\ I_{33}=1,\tilde{\phi}=T\phi$

$$T=\begin{pmatrix} 1 & 0 & \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ \frac{2q}{q_x} & 0 & 1 \end{pmatrix}$$

(1.3) When $h_{31}=0,\ h_{13}=0,\ h_{11}=h_{33}=0,\ I_{11}=1,\ I_{33}=1,\tilde{\phi}=T\phi$

$$T=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(2.1) When $h_{31}=0,h_{13}=0,h_{11}=h_{33}=1,I_{11}=1,I_{33}=0,\tilde{\phi}=T\phi$

$$T=\begin{pmatrix} \lambda+1 & 0 & -\frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ -\frac{2q}{q_x} & 0 & \lambda+1 \end{pmatrix}$$

(2.2) When $h_{31}=0,\ h_{13}=0,\ h_{11}=h_{33}=1,\ I_{11}=0,\ I_{33}=1,\tilde{\phi}=T\phi$

$$T=\begin{pmatrix} \lambda+1 & 0 & \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ \frac{2q}{q_x} & 0 & \lambda+1 \end{pmatrix}$$

(2.3) When $h_{31} = 0,\ h_{13} = 0,\ h_{11} = h_{33} = 1,\ I_{11} = 1,\ I_{33} = 1, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+1 & 0 \\ 0 & 0 & \lambda+1 \end{pmatrix}$$

(3.1) When $h_{31} = 1, h_{13} = q,\ h_{11} = h_{33} = 0,\ I_{11} = 1,\ I_{33} = 0, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} 1 & 0 & q - \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ q - \frac{2q}{q_x} & 0 & 1 \end{pmatrix}$$

(3.2) When $h_{31} = 1,\ h_{13} = q,\ h_{11} = h_{33} = 0, I_{11} = 0, I_{33} = 1, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} 1 & 0 & q + \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ q + \frac{2q}{q_x} & 0 & 1 \end{pmatrix}$$

(3.3) When $h_{31} = 1,\ h_{13} = q,\ h_{11} = h_{33} = 0,\ I_{11} = 1,\ I_{33} = 1, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} 1 & 0 & q \\ 0 & \lambda+1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(4.1) When $h_{31} = 1,\ h_{13} = q,\ h_{11} = h_{33} = 1,\ I_{11} = 1,\ I_{33} = 0, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} \lambda+1 & 0 & q - \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ q - \frac{2q^2}{q_x} & 0 & \lambda+1 \end{pmatrix}$$

(4.2) When $h_{31} = 1,\ h_{13} = q,\ h_{11} = h_{33} = 1,\ I_{11} = 0,\ I_{33} = 1, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} \lambda+1 & 0 & q + \frac{2q^2}{q_x} \\ 0 & \lambda+1 & 0 \\ q + \frac{2q^2}{q_x} & 0 & \lambda+1 \end{pmatrix}$$

(4.3) When $h_{31} = 1,\ h_{13} = q,\ h_{11} = h_{33} = 1,\ I_{11} = 1, I_{33} = 1, \tilde{\phi} = T\phi$

$$T = \begin{pmatrix} \lambda+1 & 0 & q \\ 0 & \lambda+1 & 0 \\ 1 & 0 & \lambda+1 \end{pmatrix}$$

At this point, Equation (27) $T_x + TM - \widetilde{M}T = 0$ is equivalent to the Bäcklund transformation. However, it should be noted that there are other cases to consider, such as $h_{22} = 0,\ I_{22} = 1; h_{22} = 1,\ I_{22} = 0$. These different cases will result in distinct gauge transformation matrices, thereby yielding multiple solutions to the spectral problem.

Furthermore, we examine the Darboux transformation of the Ohta–Hirota equation, with a specific focus on the scenario where $q = \widetilde{q}$ in the subsequent Darboux transformation.

$$\begin{cases} u^2 - v = \widetilde{u^2} - \widetilde{v}, \\ \widetilde{\psi} = T\psi \end{cases} \tag{40}$$

In this context, the matrix T represents the transformation derived from the Bäcklund transformation, obtained through the application of a canonical transformation. It can be demonstrated that $\widetilde{\psi}$ satisfies the following equation: $\widetilde{\psi_x} = \widetilde{M}\widetilde{\psi}$. Thus far, we have completed the Bäcklund transformation for the Ohta–Hirota equation and conducted an analysis of the Darboux transformation. Based on the identified Lax pairs, we performed systematic computations of both the Darboux and the Bäcklund transformation. Through these transformations, we have successfully constructed a series of new solutions, providing us with a diverse range of solution structures. These solutions not only verify the theoretical correctness of the underlying theory but also facilitate further analysis of the properties of integrable systems. Additionally, they offer substantial support for practical applications by enabling deeper insights into the behavior of these systems.

By solving nonlinear integrable equations, we obtained Lax pairs. On this basis, we further constructed Bäcklund and Darboux transformations. As an essential tool for studying nonlinear partial differential equations, Lax pairs provide a profound understanding of the intrinsic structure of systems. Using these transformations, not only can new solutions be generated, but they also reveal the underlying symmetries and conservation laws of the original equations. This process not only lays a solid foundation for the discussion of recursive operators in Chapter 4 but also offers us a unique perspective to delve into the issues of locality and non-locality of transformations and operators. Specifically, we will analyze in detail how these characteristics influence the behavior of the operators used and their significance in different application contexts. For instance, local properties are typically associated with differential operators, reflecting the behavior of the system near a point; non-local properties, on the other hand, may involve integral operators or other global effects that capture interactions over a broader range within the system. Furthermore, we will explore how these concepts help us to more comprehensively understand and articulate the theoretical framework related to recursive operators. In this context, recursive operators play a crucial role, as they not only generate an infinite number of conservation laws but also reveal the deeper structural features of the system. Through a detailed examination of locality and non-locality, we can better comprehend the potential applications of recursive operators in various physical models and how they facilitate the development of analytical solutions for nonlinear systems. In conclusion, our study aims to provide a more robust theoretical foundation for the research on recursive operators by systematically integrating Lax pairs, Bäcklund transformations, and Darboux transformations, thereby advancing the field.

4. Recursive Operator of Ohta–Hirota Equation

Significant progress has been made in the construction and calculation of recursive operators. The introduction of the recursive operator facilitates a more efficient solution for the Hamiltonian structure of the nonlinear coupled KdV equation. Based on the previously obtained matrix form, we provide the recursion relation between u_{tm} and u_{tm+1} for the coupled KdV equation under the n-reduction condition. Specifically, our goal is to find an operator R_n such that the equation $u_{tm+1} = R_n u_{tm}$ holds.

Calculate $L = \partial^2 + u\partial^{-1}v\partial$, thereinto $u_{tm} = (L^m)_{\geq 1}(u), v_{tm} = -\partial^{-1}(L^m)^*_{\geq 1}$, $u_{tm+1} = (L^{m+1})_{\geq 1}(u)$, Let $(L^m)_{<1} = a_1 + a_2\partial^{-1} + \cdots$ the following calculation $(L^{m+1})_{\geq 1}(u)$.

$$\begin{aligned}(L^{m+1})_{\geq 1}(u) &= (L \cdot L^m)_{\geq 1}(u) = L_{\geq 1} \cdot L^m_{\geq 1}(u) + (L_{\geq 1} \cdot (L^m_{\geq 1})_{\geq 1}(u) + (L_{<1} \cdot (L^m)_{\geq 1})_{\geq 1}(u) \\ &= \partial^2 u_{tm} + \partial^2(a_1 + a_2\partial^{-1} + ...)(u) + (L_{<1} \cdot (L^m)_{\geq 1})_{\geq 1}(u)\end{aligned} \tag{41}$$

According to the above formulas, through analysis, we first calculate the coefficient of a_1 by taking ∂^0 at both ends of L_{tm}.

Given $L = \partial^2 + u\partial^{-1}v\partial$, the coefficient on the left side is $(u\partial^{-1}v\partial)_{tm} = \partial_{tm}(u\partial^{-1}v\partial)$.

On the right-hand side, we have:

$$\begin{aligned}&= -[a_1 + a_2\partial^{-1} + ..., \partial^2 + u\partial^{-1}v\partial] = -[a_1, \partial^2] = -a_1\partial^2 + \partial^2 a_1 \\ &= -a_1\partial^2 + (a_1)_{xx} + 2(a_1)_x\partial + a_1\partial^2 \\ &= (a_1)_{xx} + 2(a_1)_x\partial.\end{aligned} \tag{42}$$

Therefore, the coefficient of ∂^0 generated on the right side is $(a_1)_{xx}$. By equating the coefficients on both sides, we obtain $\partial_{tm}(u\partial^{-1}v\partial) = (a_1)_{xx}$, from which we can deduce:

$$\begin{aligned}a_1 &= \int\int \partial_{tm}(u\partial^{-1}v\partial)dx_1dx_2 \\ &= \int\int u_{tm}\partial^{-1}v\partial dx_1dx_2 + \int\int u\partial^{-1}v_{tm}\partial dx_1dx_2 \\ &= \partial^{-2}u_{tm}\partial^{-1}v\partial + \partial^{-2}u\partial^{-1}v_{tm}\partial\end{aligned} \tag{43}$$

Also, because $\partial^{-1}v = v\partial^{-1} - v_x\partial^{-2} + v_{xx}\partial^{-3} + \cdots$, there are $\partial^{-1}v_{tm} = v_m\partial^{-1} - v_{mx}\partial^{-2} + v_{mxx}\partial^{-3} + \cdots$. Therefore, $\partial_{tm}(u\partial^{-1}v\partial) = u_{tm}(v\partial^{-1} - v_x\partial^{-2} + v_{xx}\partial^{-3} + ...)\partial + u(v_m\partial^{-1} - v_{mx}\partial^{-2} + v_{mxx}\partial^{-3} + ...)\partial$. That is, the coefficient of ∂^0 on the left side is $u_{tm}v + uv_{tm}$.

In summary:

$$\begin{aligned}(L^{m+1})_{\geq 1}(u) &= \partial^{-2}u_{tm}v + \partial^{-2}uv_{tm}\partial u \\ &= u_x\partial^{-2}u_{tm}v + u_x\partial^{-2}uv_{tm}\end{aligned} \tag{44}$$

On account of:

$$\begin{aligned}(L_{<1} \cdot (L^m)_{\geq 1})_{\geq 1} &= (L_{<1} \cdot (L^m)_{\geq 1})_{\geq 0} - (L_{<1} \cdot (L^m)_{\geq 1})_{[0]} \\ &= (u\partial^{-1}v\partial B_m)_{\geq 0} = u\partial^{-1}v\partial B_m - (u\partial^{-1}v\partial B_m)_{<0} \\ &= u\partial^{-1}v\partial B_m - u\partial^{-1}(\partial B)^*_m(v) \\ &= u\partial^{-1}v\partial B_m + u\partial^{-1}B^*_m\partial(v)\end{aligned} \tag{45}$$

So, we can calculate:

$$\begin{aligned}(L_{<1}\cdot(L^m)_{\geq 1})_{\geq 1}(u) &= (u\partial^{-1}v\partial B_m + u\partial^{-1}B_m^*\partial(v))(u)\\ &= (u\partial^{-1}v\partial(L^m)_{\geq 1} - u\partial^{-1}(L^m)_{\geq 1}^*\partial(v))(u)\\ &= u\partial^{-1}v\partial u_{tm} + uv_{tm}(u)\\ &= u\partial^{-1}v\partial u_{tm} + u^2v_{tm}\end{aligned} \tag{46}$$

All in all, it can be calculated:

$$\begin{aligned}u_{tm+1} = (L^{m+1})_{\geq 1}(u) &= \partial^2 u_{tm} + \partial^2(a_1 + a_2\partial^{-1} + \dots)(u) + (L_{<1}\cdot(L^m)_{\geq 1})_{\geq 1}(u)\\ &= (\partial^2 + u_x\partial^{-2}v + u\partial^{-1}v\partial, u_x\partial^{-2}u + u^2)\end{aligned} \tag{47}$$

$$\begin{pmatrix} u_{tm} \\ v_{tm} \end{pmatrix} \triangleq \begin{pmatrix} R_{11} & R_{12} \end{pmatrix}\begin{pmatrix} u_{tm} \\ v_{tm} \end{pmatrix} \tag{48}$$

We can use the same method to find R_{21}, R_{22}, and the calculation can be obtained: $v_{tm+1} = -\partial^{-1}(L^{m+1})_{\geq 1}^*\partial(v)$

$$(L^{m+1})_{\geq 1} = (L^m\cdot L)_{\geq 1} = (L^m_{\geq 1}\cdot L_{\geq 1})_{\geq 1} + (L^m_{\geq 1}\cdot L_{<1})_{\geq 1} + (L^m_{<1}\cdot L_{\geq 1})_{\geq 1} + (L^m_{<1}\cdot L_{<1})_{\geq 1} \tag{49}$$

The first item is awarded: $(L^m_{\geq 1}\cdot L_{\geq 1})_{\geq 1} = L^m_{\geq 1}\cdot L_{\geq 1} = B_m\cdot\partial^2$

On account of $(L^m_{\geq 1}\cdot L_{\geq 1})^*_{\geq 1} = -\partial^2\cdot B_m$,

$$\begin{aligned}v_{tm+1} &= -\partial^{-1}(L^{m+1})^*_{\geq 1}\partial(v)\\ &= -\partial^{-1}(-\partial^2 B_m^*)\partial(v) = \partial B_m^*\partial(v)\\ &= \partial B_m^*(\partial v) = v_{tm}\end{aligned} \tag{50}$$

On account of

$$\begin{aligned}(L^m_{\geq 1}\cdot L_{<1})_{\geq 1} &= (B_m u\partial^{-1}v\partial)_{\geq 1} = B_m u\partial^{-1}v\partial - (B_m u\partial^{-1}v\partial)_{<0}\\ &= (B_m u\partial^{-1}(\partial v - v_x))_{<0} = (B_m uv - B_m u\partial^{-1}v_x)_{<0} = -(B_m u\partial^{-1}v_x)_{<0}\\ &= -B_m(u)\partial^{-1}v_x = -u_{tm}\partial^{-1}v_x\end{aligned} \tag{51}$$

as a result:

$$\begin{aligned}&-\partial^{-1}(B_m u\partial^{-1}v\partial + u_{tm}\partial^{-1}v_x\partial(v))\\ &= -\partial^{-1}(\partial v\partial^{-1}u(B_m)^* - v_x\partial^{-1}u_{tm})\partial(v)\\ &= -\partial^{-1}\partial v\partial^{-1}u(B_m)^*)\partial(v) + \partial^{-1}v_x\partial^{-1}u_{tm}\partial(v)\\ &= v\partial^{-1}u\partial v_{tm} + \partial^{-1}v_x\partial^{-1}u_{tm}v_x\end{aligned} \tag{52}$$

The third part of the equation, first of all to $L^m_{<1} = a_0 + a_1\partial^1 + a_2\partial^2 + \cdots$ available:

$$(L^m_{<1}\cdot L_{<1})_{\geq 1} = ((a_0 + a_1\partial^{-1} + a_2\partial^{-2} + \dots)\partial^2)_{\geq 1} = a_0\partial^2 \tag{53}$$

For the following calculation: a_0 mean $L_{tm} = [(L^m)_{\geq 1}, L] = -[(L^m)_{<1}, L]$, take the coefficient of ∂^0 at both ends.

$$L_{tm} = (u\partial^{-1}v\partial)_{tm} = u_{tm}\partial^{-1}v\partial + u\partial^{-1}v_{tm}\partial \tag{54}$$

$$\begin{aligned}-[(L^m)_{<1}, L] &= -[a_0 + a_1\partial^{-1} + a_2\partial^{-2} + \dots, \partial^2 + u\partial^{-1}v\partial] = -[a_0, \partial^2]\\ &= -a_0\partial^2 + \partial^2 a_0 + 2(a_0)_x\partial + (a_0)_{xx}\end{aligned} \tag{55}$$

The coefficient of ∂^0 on the left is $u_{tm}v + uv_{tm}$. The coefficient of ∂^0 on the right is:

$$(a_0)_{xx} = u_{tm}v + uv_{tm} \tag{56}$$

It is possible to accrue points:

$$a_0 = \partial^{-2}u_{tm}v + \partial^{-2}uv_{tm} \tag{57}$$

There is:

$$\begin{aligned}
&-\partial^{-1}(a_0\partial)^*\partial(v) \\
&= (\partial^{-2}u_{tm}v + \partial^{-2}uv_{tm})\partial(v) \\
&= \partial^{-2}u_{tm}v\partial(v) + \partial^{-2}uv_{tm}\partial(v) \\
&= \partial^{-2}u_{tm}vv_x + \partial^{-2}uv_{tm}v_x
\end{aligned} \tag{58}$$

In conclusion, the recursive operator of the Ohta–Hirota equation can be written as follows:

$$\begin{pmatrix} u_{tm} \\ v_{tm} \end{pmatrix} \triangleq \begin{pmatrix} \partial^2 + u_x\partial^{-2}v + u\partial^{-1}v\partial & u_x\partial^{-2}u + u^2 \\ \partial^{-1}v_x\partial^{-1}v_x + \partial^{-2}vv_x & v\partial^{-1}u\partial + \partial^{-2}uv_x \end{pmatrix} \begin{pmatrix} u_{tm} \\ v_{tm} \end{pmatrix} \tag{59}$$

In this section, we constructed the Lax operator and performed the necessary calculations to obtain the recursive operator. Subsequent analysis led to deriving the recursive operator for the Ohta–Hirota equation. Significant progress has been made in both the construction and computation of recursive operators. By constructing these recursive operators, we can efficiently generate higher-order solutions and further analyze the complex behavior of integrable systems. This section not only verifies the effectiveness of recursive operators but also introduces new tools and methods for studying integrable systems, thereby greatly advancing research in this field.

5. Summary and Conclusions

The research outlined in this paper is segmented into four principal sections.

Part 1: This section offers an extensive review of the evolution of the solitary wave equation, culminating in the exploration of a third-order block-coupled partial differential equation. Utilizing foundational principles from extension structures theory, our primary aim is to delve into the integrability properties of the third-order coupled KdV equation.

Part 2: Transitioning into the second part, we leverage pertinent theories from prolongation structure theory alongside the specified Ohta–Hirota equation to derive the corresponding Lax pair. This step is pivotal for subsequent analyses.

Part 3: With the derived Lax pair as our foundation, we apply a canonical transformation to the spectral problem at hand. By rigorously employing compatibility conditions associated with this transformation, we meticulously examine both the Bäcklund and Darboux transformations. This comprehensive approach ensures our findings are characterized by exceptional precision and depth.

Part 4: The final segment focuses on exploring the intrinsic characteristics of the Ohta–Hirota equation, alongside a detailed analysis of its recurrence operator. Through the application of the Lax operator obtained from the Lax equation, we conduct an exhaustive investigation to uncover the nuances of the recurrence operator specific to the Ohta–Hirota equation.

This paper primarily examines prolongation structures, Bäcklund transformations, Darboux transformations, and recursive operators as they relate to the Ohta–Hirota equation. However, several areas warrant further research and discussion: Firstly, for integrable equations, it is crucial to devise a straightforward yet effective method to achieve representation via prolongation structures. Developing such an approach will enhance our understanding and facilitate broader applications. Secondly, the relationship between matrix linear spectral problems and prolongation structure theory requires deeper exploration. Investigating this connection could yield significant insights into both theoretical frameworks and their practical implications. Lastly, the Hamiltonian structure of integrable equations should be studied in greater depth through the lens of prolongation structure theory. A thorough analysis in this area could provide valuable perspectives on the integrability and structural properties of these equations.

Author Contributions: Investigation, J.W.; Writing—original draft, L.W.; Writing—review & editing, Y.J. All authors have read and agreed to the published version of the manuscript.

Funding: This work was supported by the partially supported by National Natural Science Foundation of China (Grant No.12261072).

Data Availability Statement: Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflicts of Interest: The authors have no conflicts to disclose.

References

1. Rusell, J. Peakon and solitary wave solutions for the modified Fornberg-Whitham equation using simplest equation method. *Int. J. Math. Comput. Sci.* **2019**, *14*, 635–645.
2. Li, C. N= 2 supersymmetric extension on multi-component D type Drinfeld-Sokolov hierarchy. *Phys. Lett. B* **2024**, *855*, 138771. [CrossRef]
3. Yang, G.Y.; Liu, Q.P. A Darboux transformation for the coupled Kadomtsev-Petviashvili. *Chin. Phys. Lett.* **2008**, *25*, 1.
4. Li, C.; Mironov, A.; Orlov, A.Y. Hopf link invariants and integrable hierarchies. *Phys. Lett. B* **2025**, *860*, 139170. [CrossRef]
5. Ablowitz, M.J. *Solitons, Nonlinear Evolution Equations and Inverse Scattering*; Cambridge University Press: Cambridge, UK, 1992; pp. 70–152
6. Hirota, R. *The Direct Method in Soliton Theory*; Cambridge University Press: Cambridge, UK, 2004; pp. 1–58
7. Matveev, V. B.; Salle, M.A. *Darboux Transformation and Solitons*; Springer: Berlin/Heidelberg, Germany, 1991; 120p.
8. Rao, J.G.; Cheng, Y.; He, J.S. Rational and semi-rational solution of the nonlocal Davey-Stewardton. *Stud. Appl. Math.* **2017**, *139*, 568–598. [CrossRef]
9. Estabrook, F.B.; Wahlquist, H.D. Prolongation structures of nonlinear evolution equations. *J. Math. Phys.* **1975**, *16*, 1–7. [CrossRef]
10. Estabrook, F.B.; Wahlquist, H.D. Prolongation structures of nonlinear evolution equations II. *J. Math. Phys.* **1976**, *17*, 1293–1297. [CrossRef]
11. Zhao, W.Z.; Bai, Y.Q.; Wu, K. Generalized inhomogeneous Heisenberg ferromagnet model and generalized nonlinear Schrodinger equation. *Phys. Lett.* **2006**, *3*, 52–64. [CrossRef]
12. Chandan, K.D.; Chowdhury, A.R. On the prolongation structure and integrability of HNLS equation. *Chaos Solitons Fractals* **2001**, *12*, 2081–2086.
13. Ito, M. Symmetries and conservation laws of a coupled nonlinear wave equation. *Phys. Lett. A* **1982**, *91*, 335–343. [CrossRef]
14. Bogolyubov, N.N.; Prikarpatskii, A.K. Complete integrability of nonlinear Ito and Benney-Kaup systems: A gradient algorithm and the Lax representation. *Teor. Mat. Fiz.* **1986**, *67*, 410–425. [CrossRef]
15. Alfinito, E.; Grassi, V.; Leo, R.A. Equations of the reaction-diffusion type with a loop algebra structure. *Inverse Probl.* **1997**, *14*, 1387–1393. [CrossRef]

16. Gilson, C.R.; Nimmo, J.J.C.; Ohta, Y. Quasideterminant solutions of a non-Abelian Hirota—CMiwa equation. *J. Phys. A* **2007**, *76*, 24–35.
17. Wang, D.S.; Liu, J.; Zhang, Z.F. Integrability and equivalence relationships of six integrable coupled Korteweg—Cde Vries equations. *Math. Methods Appl. Sci.* **2016**, *39*, 3516–3530. [CrossRef]
18. Wang, D.S.; Zhang, Z.F. On the integrability of the generalized Fisher-type nonlinear diffusion equations. *J. Phys. A Math. Theor.* **2009**, *42*, 035209. [CrossRef]
19. Geng, X.G.; Wu, Y.T. From the special 2 + 1 Toda lattice to the Kadomtsev-Petviashvili equation. *J. Math. Phys.* **1997**, *38*, 3069–3077. [CrossRef]
20. Ma, W.X. Integrable couplings and matrix loop algebras. In Proceedings of the AIP Conference Proceedings, Tampa, FL, USA, 9–13 March 2013; Volume 1562.
21. Ma, W.X. Type $(-\lambda, -\lambda^*)$ reduced nonlocal integrable mKdV equations and their soliton solutions. *Appl. Math. Lett* **2022**, *131*, 108074. [CrossRef]
22. Ma, W.X. Nonlocal integrable mKdV equations by two nonlocal reductions and their soliton solutions. *J. Geom. Phys*. **2022**, *177*, 104522. [CrossRef]
23. Lou, S.Y. Ren-integrable and ren-symmetric integrable systems, communications in theoretical physics. *Commun. Theor. Phys*. **2024**, *76*, 035006. [CrossRef]
24. Hu, X.B.; Li, Y. Nonlinear superposition formulae of the Ito equation and a model equation for shallow water waves. *J. Phys. Math. Gen*. **1991**, *24*, 1979–1986. [CrossRef]
25. Liu, Q.P. Hamiltonian structures for Ito's equation. *Phys. Lett*. **2000**, *277*, 31–34. [CrossRef]
26. Hirota, R.; Hu, X.B.; Tang, X.Y. A vector potential KdV equation and vector Ito equation; soliton solutions, bilinear Backlund transformations and Lax pairs. *J. Math. Anal. Appl*. **2003**, *288*, 326–348. [CrossRef]
27. Lou, S.Y. Extensions of dark KdV equations: Nonhomogeneous classifications, bosonizations of fermionic systems and supersymmetric dark systems. *Phys. D Nonlinear Phenom.* **2024**, *464*, 134199. [CrossRef]
28. Arguelles, C.R.; Becerra-Vergara, E.A.; Rueda, J.A.; Ruffini, R. Fermionic dark matter: Physics, astrophysics and cosmology Universe. *Universe* **2023**, *9*, 197. [CrossRef]
29. Wang, D.; Koussour, M.; Malik, A.; Myrzakulov, N.; Mustafa, G. Observational constraints on a logarithmic scalar field dark energy model and black hole mass evolution in the universe. *Eur. Phys. J.* **2023**, *83*, 670. [CrossRef]
30. Flambaum, V.V.; Liu, X.; Samsonov, I.; Wu, L.; Zhu, B. Probing supersymmetry breaking scale with atomic clocks. *Nucl. Phys.* **2023**, *993*, 116260. [CrossRef]