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THREE MACWILLIAMS TYPE IDENTITIES AND QUANTUM ERROR-CORRECTING G-CODES

Chuangxun Cheng* and Xiaoguang Shang

Department of Mathematics, Nanjing University, Nanjing 210093, China

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Abstract. We prove three MacWilliams type identities for irreducible projective representations of finite groups. As an application, we deduce the MacWilliams identities of weight enumerators, double weight enumerators and complete weight enumerators for quantum error-correcting G-codes and obtain the Singleton bounds.

1. **Introduction.** In this paper, we prove three MacWilliams type identities for irreducible projective representations of finite groups via a detailed study of the matrix coefficients. Our motivation is from the MacWilliams identities for quantum error-correcting codes (abbreviated as QECCs) in various special settings (cf. [3, 4, 12, 15]). We give a uniform treatment for the earlier results in Section 3. The identities we prove have significance in two aspects. On one hand, they reveal relations of operators given by projective representations and are closely related to the Fourier analysis on finite groups. On the other hand, they have immediate applications in the study of QECCs and we deduce the MacWilliams identities of weight enumerators, double weight enumerators and complete weight enumerators for quantum error-correcting G-codes, where G is a certain finite group. Moreover we obtain the Singleton bounds for quantum error-correcting G-codes.

We fix some notation and explain our main results in the following. Let G be a finite group of order $\mathfrak{g}$. Let m and n be positive integers and let (ρ_i, V_i) $(1 \leq i \leq n)$ be m-dimensional unitary irreducible projective representations of G with multiplier α_i (cf. Definition 2.1). Assume that the subgroups $\mathrm{Ker}\rho_i$ have the same size s (cf. equation (3)). Let $P_1,\ P_2 \in \mathrm{End}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$. For each $1 \leq i \leq n$, fix $G_i \subseteq G$ a system of representatives of the quotient group $G/\mathrm{Ker}\rho_i$ such that the identity $1_G \in G$ is in G_i. Define $\mathcal{E}_n = \{\rho_1(g_1) \otimes \rho_2(g_2) \otimes \cdots \otimes \rho_n(g_n) \mid g_i \in G_i, 1 \leq i \leq n\}$. An element $E \in \mathcal{E}_n$ has *weight* t if $E = e_1 \otimes e_2 \otimes \cdots \otimes e_n$ and $|\{j : e_j \neq \mathrm{Id}\}| = t$. We denote the weight of E by $w(E)$.

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*Corresponding author: Chuangxun Cheng.

Definition 1.1. The *weight distributions* B_i and $B_i^{\perp}$ with respect to $\{(\rho_i, V_i)_{1\le i\le n}, P_1, P_2\}$ are defined by

$$B_i := \sum_{E\in\mathcal{E}_n, w(E)=i} \mathrm{Tr}(E^{-1}P_1)\mathrm{Tr}(EP_2),$$
$$B_i^{\perp} := \sum_{E\in\mathcal{E}_n, w(E)=i} \mathrm{Tr}(E^{-1}P_1EP_2),$$

and the *weight enumerators* with respect to $\{(\rho_i, V_i)_{1\le i\le n},\ P_1,\ P_2\}$ are defined by

$$f(x,y) := \sum_{i=0}^{n} B_i x^{n-i}y^i,$$
$$f^{\perp}(x,y) := \sum_{i=0}^{n} B_i^{\perp} x^{n-i}y^i.$$

We then have the following result.

Theorem 1.2. *Let $f,\ f^{\perp}$ be the weight enumerators with respect to $\{(\rho_i, V_i)_{1\le i\le n}, P_1, P_2\}$. Then*

$$f(x,y) = f^{\perp}(\frac{s^2m^2x + (\mathfrak{g}^2 - s^2m^2)y}{sm\mathfrak{g}}, \frac{sm(x-y))}{\mathfrak{g}}),$$

where $\mathfrak{g}$ is the order of G, m is the dimension of V_i and s is the size of $\mathrm{Ker}\rho_i$.

Assume now that all the (ρ_i, V_i) are the same, $P_1,\ P_2 \in \mathrm{End}(V_1^{\otimes n})$. Fix a bijection ι from G_1 to the set $\{1, 2, ..., \frac{\mathfrak{g}}{s}\} \subseteq \mathbb{Z}$ such that the identity 1_G corresponds to the integer 1. Let $\mathrm{IND}(n)$ be the set $\{J = (j_i) \in \mathbb{Z}_{\ge 0}^{\frac{\mathfrak{g}}{s}} \mid \sum_{i=1}^{\frac{\mathfrak{g}}{s}} j_i = n\}$. For $E = \rho_1(g_1)\otimes\rho_1(g_2)\otimes\cdots\otimes\rho_1(g_n) \in \mathcal{E}_n$ and $g \in G_1$, let $N_g(E)$ be the number $\sharp\{i \mid g_i = g,\ 1 \le i \le n\}$. We define an error set $E[J]$ associated to an index matrix $J = (j_{\iota(g)}) \in \mathrm{IND}(n)$ by

$$E[J] := \{E \in \mathcal{E}_n \mid N_g(E) = j_{\iota(g)}, \forall g \in G_1\}.$$

Definition 1.3. The *complete weight distributions* with respect to $\{(\rho_1, V_1),\ P_1,\ P_2\}$ are defined by

$$D_J := \sum_{E\in E[J]} \mathrm{Tr}(E^{-1}P_1)\mathrm{Tr}(EP_2),$$
$$D_J^{\perp} := \sum_{E\in E[J]} \mathrm{Tr}(E^{-1}P_1EP_2),$$

and the *complete weight enumerators* with respect to $\{(\rho_1, V_1),\ P_1,\ P_2\}$ are defined by

$$D(M) := \sum_{J=(j_{\iota(g)})\in\mathrm{IND}(n)} D_J M^J,$$
$$D^{\perp}(M) := \sum_{J=(j_{\iota(g)})\in\mathrm{IND}(n)} D_J^{\perp} M^J,$$

where $M = (M_g)_{g\in G_1}$ is a 1-by-$\frac{\mathfrak{g}}{s}$ matrix and $M^J = \prod_{g\in G_1} M_g^{j_{\iota(g)}}$.

We then have the following result.

Theorem 1.4. *Let G be an abelian group and let D, $D^{\perp}$ be the complete weight enumerators with respect to $\{(\rho_1, V_1),\ P_1,\ P_2\}$. Let $M = (M_g)_{g\in G_1}$ be a 1-by-$\frac{\mathfrak{g}}{s}$ matrix. Then*

$$D(M) = D^{\perp}(M^{\perp}), \tag{1}$$

where $M_g^{\perp} = \dfrac{sm}{\mathfrak{g}} \sum_{l\in G_1} \alpha_1(g^{-1}, l^{-1})\alpha_1^{-1}(l^{-1}, g^{-1})M_l$ *for all* $g \in G_1$ *and* $M^{\perp} = (M_g^{\perp})$.

Finally we consider a particular case where all the ρ_i are the same and are given by the Weyl-Heisenberg representation. More precisely, let $(H, +, 0_H)$ be an abelian group with order m and $\widehat{H} = \mathrm{Hom}(H, \mathbb{C}^{\times})$ be its dual group with identity $1_{\widehat{H}}$. Fix a basis $\{x_h \mid h \in H\}$ of $\mathbb{C}^m$ indexed by elements of H. Let ρ be the Weyl-Heisenberg representation of $H \times \widehat{H}$ defined by

$$\begin{aligned} &\rho : H \times \widehat{H} \to \mathbf{U}(\mathbb{C}^m) \\ &\quad (a, \chi) \mapsto (x_h \mapsto \chi(h)x_{a+h},\ \forall\ h \in H). \end{aligned} \tag{2}$$

It is well-known that $(\rho, \mathbb{C}^m)$ is an irreducible faithful unitary projective representation of $H \times \widehat{H}$ (cf. [1, Exercise 4.1.8, Theorem 4.8.2]). For any $(a_1, \chi_1),\ (a_2, \chi_2) \in H \times \widehat{H}$, the multiplier is given by $\alpha((a_1, \chi_1),\ (a_2, \chi_2)) = \chi_1(a_2)$.

We consider the case that $G = H \times \widehat{H}$ and $(\rho_i, V_i) = (\rho, \mathbb{C}^m)$ for all $1 \le i \le n$. Let $P_1,\ P_2 \in \mathrm{End}((\mathbb{C}^m)^{\otimes n})$. For $E \in \mathcal{E}_n$, let

$$w_X(E) = \sum_{\substack{(a,\chi)\in G \\ a\neq 0_H}} N_{(a,\chi)}(E) \quad \text{and} \quad w_Z(E) = \sum_{\substack{(a,\chi)\in G \\ \chi\neq 1_{\widehat{H}}}} N_{(a,\chi)}(E),$$

and we call them the X*-weight* and the Z*-weight* of E respectively. Let $E[i, j] = \{E \in \mathcal{E}_n \mid w_X(E) = i, w_Z(E) = j\}$.

Definition 1.5. The *double weight distributions* with respect to $\{(\rho, \mathbb{C}^m),\ P_1,\ P_2\}$ are defined by

$$\begin{aligned} C_{i,j} &:= \sum_{E\in E[i,j]} \mathrm{Tr}(E^{-1}P_1)\mathrm{Tr}(EP_2), \\ C_{i,j}^{\perp} &:= \sum_{E\in E[i,j]} \mathrm{Tr}(E^{-1}P_1EP_2), \end{aligned}$$

and the *double weight enumerators* with respect to $\{(\rho, \mathbb{C}^m),\ P_1,\ P_2\}$ are defined by

$$\begin{aligned} C(X, Y, Z, W) &:= \sum_{i,j=0}^{n} C_{i,j}X^{n-i}Y^iZ^{n-j}W^j, \\ C^{\perp}(X, Y, Z, W) &:= \sum_{i,j=0}^{n} C_{i,j}^{\perp}X^{n-i}Y^iZ^{n-j}W^j. \end{aligned}$$

We then have the following result.

Theorem 1.6. *Let H be an abelian group and $G = H \times \widehat{H}$. Let C, $C^{\perp}$ be the double weight enumerators with respect to $\{(\rho, \mathbb{C}^m),\ P_1,\ P_2\}$. Then*

$$C(X, Y, Z, W) = C^{\perp}\left(\frac{Z + (m-1)W}{m}, \frac{Z - W}{m}, X + (m-1)Y, X - Y\right).$$

In Section 2.1, we review basic properties of projective representations and prove necessary identities of matrix coefficients. We then prove Theorems 1.2, 1.4 and 1.6 in Sections 2.2, 2.3 and 2.4 respectively.

In Section 3, we apply Theorems 1.2, 1.4 and 1.6 in the study of quantum error correcting G-codes. In particular, we deduce three versions of MacWilliams identities and obtain the Singleton bounds for quantum error correcting G-codes. As we shall see, the Singleton bounds depend only on the size of G. On the other hand, as explained in [9], it is meaningful to construct G-codes for various groups G.

Notation. Denote by $\mathbb{T}$ the set of complex numbers with modulus one. For a Hilbert space V, denote by $\mathbf{U}(V)$ the space of unitaty operators on V, $\mathbf{PU}(V)$ the quotient of $\mathbf{U}(V)$ by $\mathbb{T}$. Given $i \in \{1, 2, ..., m\}$ and $j \in \{1, 2, ..., m\}$, $E_{i,j} \in \mathbb{C}^{m\times m}$ is the matrix whose (i, j)-entry is 1 and other entries are zero.

2. The MacWilliams identities for projective representations.

In this section, we prove Theorems 1.2, 1.4 and 1.6. The main ingredients are certain identities of matrix coefficients of projective representations of finite groups, which we review/prove in Section 2.1 via Schur's lemma.

2.1. Projective representations of finite groups.

We recall the basic properties of projective representations of finite groups (cf. [2, 5]). Let G be a finite group with identity 1_G and let V be a finite dimensional $\mathbb{C}$-vector space. A *Schur multiplier* on G is a map $\alpha : G \times G \to \mathbb{T}$ such that $\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z)$ for all $x,\ y,\ z \in G$ and $\alpha(x, 1_G) = \alpha(1_G, x) = 1$ for all $x \in G$.

Definition 2.1. Let $\alpha \in Z^2(G, \mathbb{C}^\times)$ be a Schur multiplier. A map

$$\rho : G \to \mathrm{GL}(V)$$

is called a *projective representation of G with respect to α* (or an *α-representation*) if $\rho(x)\rho(y) = \alpha(x, y)\rho(xy)$ for all $x, y \in G$.

We denote this projective representation by (ρ, V, α) or (ρ, V). By the standard averaging argument, we may and do assume that ρ takes values in $\mathbf{U}(V)$ and call it a unitary projective representation.

Let $\pi : \mathbf{U}(V) \to \mathbf{PU}(V)$ be the natural homomorphism. If $\rho : G \to \mathbf{U}(V)$ is a projective representation of G, then $\pi \circ \rho$ is a homomorphism. We define the kernel of ρ by

$$\mathrm{Ker}\rho := \{g \in G \mid \rho(g) \in \mathbb{C}^\times \cdot 1_V\} = \mathrm{Ker}\pi \circ \rho. \tag{3}$$

If $\mathrm{Ker}\rho$ is $\{1_G\}$, then we call ρ a *faithful projective representation.*

Definition 2.2. A *sub projective representation* of a projective representation (ρ, V) is a vector subspace W of V which is stable under G, i.e $\rho(g)W \subset W$ for all $g \in G$. A projective representation is called *irreducible* if there is no proper nonzero sub projective representation of V.

As in linear representations case, we have Schur's lemma for projective representations (cf. [2, Lemma 2.1] and [14, Section 2.2]) .

Proposition 2.3 (Schur's lemma)**.** *Let $\rho^1 : G \to \mathbf{U}(V_1)$ and $\rho^2 : G \to \mathbf{U}(V_2)$ be two irreducible α-representations of G, and let h be a linear map from V_1 to V_2 such that $\rho^2(g) \circ h = h \circ \rho^1(g)$ for all $g \in G$. Then the following statements hold.*

1. *If ρ^1 and ρ^2 are not isomorphic, then $h = 0$.*
2. *If $V_1 = V_2$ and $\rho^1 = \rho^2$, then h is a homothety.*

Starting with Schur's lemma, one could deduce orthogonality relations of matrix coefficients of α-representations of finite groups. In particular, we have the following two results. Their proofs are similar as in linear representations case (cf. [14, Section 2.2]).

Corollary 2.4. *Let h be a linear map from V_1 to V_2, and define*

$$h^0 = \frac{1}{\mathfrak{g}} \sum_{g \in G} (\rho^2(g))^{-1} h \rho^1(g).$$

Then the following two statements hold.

1. *If ρ^1 and ρ^2 are not isomorphic, then $h^0 = 0$.*
2. *If $V_1 = V_2$ and $\rho^1 = \rho^2$, let G_1 be a system of representatives of the quotient group $G/\mathrm{Ker}\rho^1$. Then $h^0 = \frac{s}{\mathfrak{g}} \sum_{g \in G_1} (\rho^1(g))^{-1} h \rho^1(g)$ and h^0 is a homothety of ratio $\frac{1}{m}\mathrm{Tr}(h)$, where $m = \dim V_1$ and $s = \sharp\mathrm{Ker}\rho^1$.*

Assume that ρ^1, ρ^2 and h are given in matrix form

$$\rho^1(g) = (r_{i_1 j_1}(g)), \rho^2(g) = (r_{i_2 j_2}(g)), h = (x_{j_2 j_1}).$$

Let $h^0 = \frac{1}{\mathfrak{g}} \sum_{g \in G} (\rho^2(g))^{-1} h \rho^1(g)$. If we write $h^0 = (x^0_{i_2 i_1})$, then

$$x^0_{i_2 i_1} = \frac{1}{\mathfrak{g}} \sum_{g \in G} \sum_{j_2, j_1} r^*_{i_2 j_2}(g) x_{j_2 j_1} r_{j_1 i_1}(g),$$

where $(\rho^2(g))^{-1} = (r^*_{i_2 j_2}(g))$. Therefore we have the following result.

Corollary 2.5. *With the notation as above, the following statements hold.*

1. *If ρ^1 and ρ^2 are not isomorphic, then*
$$\frac{1}{\mathfrak{g}} \sum_{g \in G} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = 0$$
for arbitrary i_1, i_2, j_1, j_2.
2. *If $V_1 = V_2$ and $\rho^1 = \rho^2$, then*

$$\frac{1}{\mathfrak{g}} \sum_{g \in G} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = \frac{s}{\mathfrak{g}} \sum_{g \in G_1} r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) = \begin{cases} \frac{1}{m} & \text{if } j_1 = j_2 \text{ and } i_1 = i_2, \\ 0 & \text{otherwise,} \end{cases}$$

where m is the dimension of V_1, $s = \sharp\mathrm{Ker}\rho^1$ and $\mathfrak{g}$ is the order of G.

The following result is a twisted version of Corollary 2.5, which is trivial in linear representations case. It is the key ingredient in the proof of Theorem 1.4.

Corollary 2.6. *With the notation as above, let G be an abelian group and fix an $l \in G$. In the case of* Corollary 2.5(2)*, we have*

$$\begin{aligned} &\frac{1}{\mathfrak{g}} \sum_{g \in G} \alpha(g^{-1}, l^{-1}) \alpha^{-1}(l^{-1}, g^{-1}) r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) \\ =&\frac{s}{\mathfrak{g}} \sum_{g \in G_1} \alpha(g^{-1}, l^{-1}) \alpha^{-1}(l^{-1}, g^{-1}) r^*_{i_2 j_2}(g) r_{j_1 i_1}(g) \\ =&\frac{1}{m} r^*_{j_1 j_2}(l) r_{i_2 i_1}(l) \end{aligned}$$

for arbitrary i_1, i_2, j_1, j_2.

Proof. Replacing h with $h' = \rho^1(l^{-1})h$ in Corollary 2.4, we have

$$\frac{1}{\mathfrak{g}}\sum_{g\in G}(\rho^1(g))^{-1}\rho^1(l^{-1})h\rho^1(g) = \frac{s}{\mathfrak{g}}\sum_{g\in G_1}(\rho^1(g))^{-1}\rho^1(l^{-1})h\rho^1(g)$$
$$= (\frac{1}{m})\mathrm{Tr}(\rho^1(l^{-1})h)\cdot \mathrm{Id}.$$

Note that $\rho^1(g)^{-1}\rho^1(l^{-1}) = \alpha(g^{-1}, l^{-1})\alpha^{-1}(l^{-1}, g^{-1})\rho^1(l^{-1})\rho^1(g)^{-1}$ for any $g \in G$ since G is abelian, we have

$$\begin{aligned}&\frac{1}{\mathfrak{g}}\sum_{g\in G}\alpha(g^{-1}, l^{-1})\alpha^{-1}(l^{-1}, g^{-1})(\rho^1(g))^{-1}h\rho^1(g)\\ =&(\frac{1}{m})(\rho^1(l^{-1}))^{-1}\mathrm{Tr}(\rho^1(l^{-1})h)\\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h)),\end{aligned}$$

and

$$\begin{aligned}&\frac{s}{\mathfrak{g}}\sum_{g\in G_1}\alpha(g^{-1}, l^{-1})\alpha^{-1}(l^{-1}, g^{-1})(\rho^1(g))^{-1}h\rho^1(g)\\ =&(\frac{1}{m})(\rho^1(l^{-1}))^{-1}\mathrm{Tr}(\rho^1(l^{-1})h)\\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h)).\end{aligned}$$

So

$$\begin{aligned}&\frac{1}{\mathfrak{g}}\sum_{g\in G}\alpha(g^{-1}, l^{-1})\alpha^{-1}(l^{-1}, g^{-1})(\rho^1(g))^{-1}h\rho^1(g)\\ =&\frac{s}{\mathfrak{g}}\sum_{g\in G_1}\alpha(g^{-1}, l^{-1})\alpha^{-1}(l^{-1}, g^{-1})(\rho^1(g))^{-1}h\rho^1(g)\\ =&(\frac{1}{m})(\rho^1(l)\mathrm{Tr}(\rho^1(l)^{-1}h)).\end{aligned}$$

Let h go through the matrices $E_{j_1j_2}$, we obtain the expected identities by comparing the entries of the matrices on both sides of the above equation. □

2.2. **Proof of Theorem 1.2.** For the convenience of the readers, we restate Theorem 1.2 as follows.

Theorem 2.7 (Theorem 1.2)**.** *Let G be a finite group of order $\mathfrak{g}$. For each $1 \le i \le n$, let (ρ_i, V_i) be an irreducible m-dimensional projective representation of G. Assume that* $\mathrm{Ker}\rho_i$ *have the same size, say s. Let $P_1,\ P_2 \in \mathrm{End}(V_1 \otimes V_2 \otimes \cdots \otimes V_n)$. Let $f,\ f^{\perp}$ be the weight enumerators with respect to $\{(\rho_i, V_i)_{1\le i\le n},\ P_1,\ P_2\}$. Then*

$$f(x, y) = f^{\perp}(\frac{s^2m^2x + (\mathfrak{g}^2 - s^2m^2)y}{sm\mathfrak{g}}, \frac{sm(x - y)}{\mathfrak{g}}). \tag{4}$$

Proof. For $E \in \mathcal{E}_n$, say $E = \rho_1(g_1) \otimes ... \otimes \rho_n(g_n),\ g_t \in G_t\ (1 \le t \le n)$, if we fix a basis of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, then we have the matrix form of E and $P_1,\ P_2$:

$$E = (e_{ij}),\ E^{-1} = (e^*_{ij}),\ P_1 = (p^1_{ij}),\ P_2 = (p^2_{ij})\ (1 \le i, j \le m^n).$$

Via the Kronecker product of matrices, we may write

$$e_{ij} = (\rho_1(g_1))_{i_1j_1} \otimes ... \otimes (\rho_n(g_n))_{i_nj_n} = \prod_{t=1}^{n} (\rho_t(g_t))_{i_tj_t}.$$

In this way, $f(x,y)$ can be written as

$$\begin{aligned} f(x,y) &= \sum_{t=0}^{n} x^{n-t}y^t \sum_{\substack{E\in\mathcal{E}_n \\ w(E)=t}} \sum_{i,j,k,l} e^*_{ij}p^1_{ji}e_{kl}p^2_{lk} \\ &= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in\mathcal{E}_n} e^*_{ij}e_{kl}x^{n-w(E)}y^{w(E)} \\ &= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in\mathcal{E}_n} (\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t}x^{n-w(E)}y^{w(E)}). \end{aligned}$$

For $1 \le t \le n$, let

$$b_{i_tj_tk_tl_t}(x,y) = x(I)_{i_tj_t}(I)_{k_tl_t} + y(\sum_{g\in G_t\setminus\{1_G\}} (\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}).$$

It is easy to verify that

$$\begin{aligned} &\sum_{E\in\mathcal{E}_n} (\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t}x^{n-w(E)}y^{w(E)}) \\ =&\sum_{E\in\mathcal{E}_n} (\prod_{\substack{t=1 \\ g_t\neq 1_G}}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t}y^{w(E)} \cdot \prod_{\substack{t=1 \\ g_t=1_G}}^{n} (\rho_t(g_t)^{-1})_{i_tj_t}(\rho_t(g_t))_{k_tl_t}x^{n-w(E)}) \\ =&\prod_{t=1}^{n} b_{i_tj_tk_tl_t}(x,y). \end{aligned}$$

Similarly,

$$\begin{aligned} f^{\perp}(x,y) &= \sum_{t=0}^{n} x^{n-t}y^t \sum_{\substack{E\in\mathcal{E}_n \\ w(E)=t}} \sum_{i,j,k,l} e^*_{kj}p^1_{ji}e_{il}p^2_{lk} \\ &= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in\mathcal{E}_n} e^*_{kj}e_{il}x^{n-w(E)}y^{w(E)} \\ &= \sum_{i,j,k,l} p^1_{ji}p^2_{lk} \sum_{E\in\mathcal{E}_n} (\prod_{t=1}^{n} (\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t}x^{n-w(E)}y^{w(E)}). \end{aligned}$$

For $1 \le t \le n$, let

$$b^{\perp}_{i_tj_tk_tl_t}(x,y) = x(I)_{k_tj_t}(I)_{i_tl_t} + y(\sum_{g\in G_t\setminus\{1_G\}} (\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}).$$

Then

$$\sum_{E\in\mathcal{E}_n}(\prod_{t=1}^{n}(\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t}x^{n-w(E)}y^{w(E)})$$
$$=\sum_{E\in\mathcal{E}_n}(\prod_{\substack{t=1\\g_t\neq 1_G}}^{n}(\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t}y^{w(E)}\cdot\prod_{\substack{t=1\\g_t=1_G}}^{n}(\rho_t(g_t)^{-1})_{k_tj_t}(\rho_t(g_t))_{i_tl_t}x^{n-w(E)})$$
$$=\prod_{t=1}^{n}b^{\perp}_{i_tj_tk_tl_t}(x,y).$$

Therefore to prove the theorem, it suffices to show that

$$b_{i_tj_tk_tl_t}(x,y)=b^{\perp}_{i_tj_tk_tl_t}(\frac{s^2m^2x+(\mathfrak{g}^2-s^2m^2)y)}{sm\mathfrak{g}},\frac{sm(x-y)}{\mathfrak{g}}),\ \text{for all } 1\le t\le n.$$

Considering the projective representations (ρ_t,V_t), we obtain the following identities from Corollary 2.5:

$$\frac{s}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}=\frac{1}{m}(I)_{k_tj_t}(I)_{i_tl_t},$$
$$\frac{s}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}=\frac{1}{m}(I)_{i_tj_t}(I)_{k_tl_t}.$$

Thus

$$\frac{sx}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}=\frac{x}{m}(I)_{k_tj_t}(I)_{i_tl_t},$$
$$\frac{sy}{\mathfrak{g}}\sum_{g\in G_t}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}=\frac{y}{m}(I)_{i_tj_t}(I)_{k_tl_t}.$$

We then obtain that

$$(\frac{sx}{\mathfrak{g}}-\frac{y}{m})(I)_{i_tj_t}(I)_{k_tl_t}+\frac{sx}{\mathfrak{g}}\sum_{\substack{g\in G_t\\g\neq 1_G}}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}$$
$$=(\frac{x}{m}-\frac{sy}{\mathfrak{g}})(I)_{k_tj_t}(I)_{i_tl_t}+\frac{-sy}{\mathfrak{g}}\sum_{\substack{g\in G_t\\g\neq 1_G}}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}.$$

Let $\frac{sx}{\mathfrak{g}}-\frac{y}{m}=X, \frac{sx}{\mathfrak{g}}=Y$, we have

$$X(I)_{i_tj_t}(I)_{k_tl_t}+Y\sum_{\substack{g\in G_t\\g\neq 1_G}}(\rho_t(g)^{-1})_{i_tj_t}(\rho_t(g))_{k_tl_t}$$
$$=\frac{s^2m^2X+(\mathfrak{g}^2-s^2m^2)Y}{sm\mathfrak{g}}(I)_{k_tj_t}(I)_{i_tl_t}+\frac{sm(X-Y)}{\mathfrak{g}}\sum_{\substack{g\in G_t\\g\neq 1_G}}(\rho_t(g)^{-1})_{k_tj_t}(\rho_t(g))_{i_tl_t}.$$

Therefore

$$b_{i_tj_tk_tl_t}(X,Y)=b^{\perp}_{i_tj_tk_tl_t}(\frac{s^2m^2X+(\mathfrak{g}^2-s^2m^2)Y)}{sm\mathfrak{g}},\frac{sm(X-Y)}{\mathfrak{g}}),\quad 1\le t\le n.$$

This completes the proof. □

2.3. **Proof of Theorem 1.4.** We prove Theorem 1.4 in this section.

Theorem 2.8 (Theorem 1.4)**.** *Let G be an abelian group with order $\mathfrak{g}$, and let (ρ_1, V_1) be an irreducible m-dimensional projective representation of G with $\sharp\mathrm{Ker}\rho_1 = s$. Let $P_1,\ P_2 \in \mathrm{End}(V_1^{\otimes n})$, and let $D,\ D^\perp$ be the complete weight enumerators with respect to $\{(\rho_1, V_1),\ P_1,\ P_2\}$. Let $M = (M_g)_{g\in G_1}$ be a 1-by-$\frac{\mathfrak{g}}{s}$ matrix. Then we have*

$$D(M) = D^\perp(M^\perp), \tag{5}$$

where $M_g^\perp = \dfrac{sm}{\mathfrak{g}} \sum_{l\in G_1} \alpha_1(g^{-1}, l^{-1})\alpha_1^{-1}(l^{-1}, g^{-1})M_l$ for all $g \in G_1$ and $M^\perp = (M_g^\perp)$.

Proof. We use the notation in Section 2.2. Direct computation shows that

$$\begin{aligned}
D(M) &= \sum_{J=(j_{\iota(g)})\in \mathrm{IND}(n)} D_J M^J \\
&= \sum_{J=(j_{\iota(g)})\in \mathrm{IND}(n)} \prod_{g\in G_1} M_g^{j_{\iota(g)}} \sum_{E\in E[J]} \sum_{i,j,k,l} e_{ij}^* p_{ji}^1 e_{kl} p_{lk}^2 \\
&= \sum_{i,j,k,l} p_{ji}^1 p_{lk}^2 \sum_{E\in\mathcal{E}_n} e_{ij}^* e_{kl} \prod_{g\in G_1} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p_{ji}^1 p_{lk}^2 \sum_{E\in\mathcal{E}_n} \prod_{t=1}^{n} (\rho_1(g_t)^{-1})_{i_t j_t} (\rho_1(g_t))_{k_t l_t} \prod_{g\in G_1} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p_{ji}^1 p_{lk}^2 \sum_{E\in\mathcal{E}_n} \prod_{g\in G_1} \prod_{\substack{t=1\\ g_t=g}}^{n} (\rho_1(g_t)^{-1})_{i_t j_t} (\rho_1(g_t))_{k_t l_t} M_g^{N_g(E)} \\
&= \sum_{i,j,k,l} p_{ji}^1 p_{lk}^2 \prod_{t=1}^{n} d_t(M),
\end{aligned}$$

where

$$d_t(M) = \sum_{g\in G_1} (\rho_1(g)^{-1})_{i_t j_t} (\rho_1(g))_{k_t l_t} M_g.$$

Similarly we have

$$D^\perp(M^\perp) = \sum_{i,j,k,l} p_{ji}^1 p_{lk}^2 \prod_{t=1}^{n} d_t^\perp(M^\perp),$$

where

$$d_t^\perp(M^\perp) = \sum_{g\in G_1} (\rho_1(g)^{-1})_{k_t j_t} (\rho_1(g))_{i_t l_t} M_g^\perp.$$

Let $M_g^\perp = \dfrac{sm}{\mathfrak{g}} \sum_{l\in G_1} \alpha_1(g^{-1}, l^{-1})\alpha_1^{-1}(l^{-1}, g^{-1})M_l$ for all $g \in G_1$. By Corollary 2.6, we have $d_t^\perp(M^\perp) = d_t(M)$ for every t, the theorem then follows. □

We remark that Corollary 2.6 does not hold for non-abelian groups. This is the reason that for the complete weight enumerators we only consider abelian groups.

2.4. **Proof of Theorem 1.6.** In this section we prove Theorem 1.6 by adapting the strategy from the proof of [4, Theorem 4]. We explain the relation between Theorem 1.4 and Theorem 1.6, and then deduce the expected identity easily. For convenience, we change the notation a little bit. Let $\mathrm{IND}(n)$ be $\{J = (j_{\lambda,\mu}) \in \mathbb{Z}_{\geq 0}^{m\times m} \mid \sum_{\lambda,\mu=1}^{m} j_{\lambda,\mu} = n\}$ and fix two bijections from H, $\widehat{H}$ to the set $\{1, 2, ..., m\} \subseteq \mathbb{Z}$, such that the group units correspond to integer 1. They induce a bijection ι from G to $\{(\lambda, \mu) \mid \lambda, \mu \in \{1, 2, ..., m\}\}$. Then variables of the complete weight enumerators can be written as an m-by-m matrix $M = (M_{\lambda,\mu})$.

Lemma 2.9. *Let H be an abelian group and $G = H\times\widehat{H}$. Let (G, ρ) be the projective representation defined by equation (2) and $P_1,\ P_2 \in \mathrm{End}((\mathbb{C}^m)^{\otimes n})$. Let $f,\ f^\perp$ be the weight enumerators, $C,\ C^\perp$ be the double weight enumerators, $D,\ D^\perp$ be the complete weight enumerators with respect to $\{(\rho, \mathbb{C}^m),\ P_1,\ P_2\}$. Then the following identities hold:*

$$\begin{aligned} f(X, Y) &= D(\Phi(X, Y)),\\ f^\perp(X, Y) &= D^\perp(\Phi(X, Y)),\\ C(X, Y, Z, W) &= D(\Psi(X, Y, Z, W)),\\ C^\perp(X, Y, Z, W) &= D^\perp(\Psi(X, Y, Z, W)), \end{aligned} \tag{6}$$

where $\Phi(X, Y)$ is the matrix

$$\begin{pmatrix} X & Y & Y & \cdots & Y\\ Y & Y & Y & \cdots & Y\\ Y & Y & Y & \cdots & Y\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ Y & Y & Y & \cdots & Y \end{pmatrix}$$

and $\Psi(X, Y, Z, W)$ is the matrix

$$\begin{pmatrix} XZ & XW & XW & \cdots & XW\\ YZ & YW & YW & \cdots & YW\\ YZ & YW & YW & \cdots & YW\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ YZ & YW & YW & \cdots & YW \end{pmatrix}.$$

Proof. For $J = (j_{\iota(a,\chi)}) \in \mathrm{IND}(n)$, let $|J_1| := \sum_{\substack{(a,\chi)\in G\\ a\neq 0_H}} j_{\iota(a,\chi)}$, $|J_2| := \sum_{\substack{(a,\chi)\in G\\ \chi\neq 1_{\widehat{H}}}} j_{\iota(a,\chi)}$, $|J_0| := \sum_{\substack{(a,\chi)\in G\\ (a,\chi)\neq (0_H, 1_{\widehat{H}})}} j_{\iota(a,\chi)}$. From the definition of the X-weight and the Z-weight, for $0 \leq i, j \leq n$, we have

$$\begin{aligned} B_l &= \sum_{\substack{J\in \mathrm{IND}(n)\\ |J_0|=l}} D_J,\\ B_l^\perp &= \sum_{\substack{J\in \mathrm{IND}(n)\\ |J_0|=l}} D_J^\perp,\\ C_{i,j} &= \sum_{\substack{J\in \mathrm{IND}(n)\\ |J_1|=i, |J_2|=j}} D_J, \end{aligned}$$

$$C_{i,j}^{\perp} = \sum_{\substack{J \in \mathrm{IND}(n) \\ |J_1|=i, |J_2|=j}} D_J^{\perp}.$$

Hence the identities hold and the lemma follows. □

Theorem 2.10 (Theorem 1.6)**.** *In the setting of Lemma 2.9, we have*

$$C(X,Y,Z,W) = C^{\perp}\left(\frac{Z+(m-1)W}{m}, \frac{Z-W}{m}, X+(m-1)Y, X-Y\right).$$

Proof. Let $\Psi^{\perp}(X,Y,Z,W) = (M_{(a,\chi)}^{\perp}(X,Y,Z,W))$ be the matrix associated with $\Psi(X,Y,Z,W)$. The entries $M_{(a,\chi)}^{\perp}(X,Y,Z,W)$ can be calculated from equation (5):

$$\begin{aligned} M_{(a,\chi)}^{\perp}(X,Y,Z,W) =& \frac{1}{m}(XZ + \sum_{\substack{\chi_0 \in \widehat{H} \\ \chi_0 \neq 1_{\widehat{H}}}} \chi_0^{-1}(a)XW + \sum_{\substack{b \in H \\ b \neq 0_H}} \chi(b)YZ \\ &+ \sum_{\substack{b \neq 0_H, \chi_0 \neq 1_{\widehat{H}} \\ (b,\chi_0) \in G}} \chi(b)\chi_0^{-1}(a)YW). \end{aligned} \tag{7}$$

It follows from the properties of characters that

$$\begin{aligned} M_{(1,1)}^{\perp} &= \frac{1}{m}(X+(m-1)Y)(Z+(m-1)W), \\ M_{(1,\chi)}^{\perp} &= \frac{1}{m}(X-Y)(Z+(m-1)W),\ \chi \neq 1_{\widehat{H}}, \\ M_{(a,1)}^{\perp} &= \frac{1}{m}(X+(m-1)Y)(Z-W),\ a \neq 0_H, \\ M_{(a,\chi)}^{\perp} &= \frac{1}{m}(X-Y)(Z-W),\ a \neq 0_H,\ \chi \neq 1_{\widehat{H}}. \end{aligned} \tag{8}$$

From equations (6), (7) and (8) we see that

$$\begin{aligned} C(X,Y,Z,W) &= D(\Psi(X,Y,Z,W)) \\ &= D^{\perp}(\Psi^{\perp}(X,Y,Z,W)) \\ &= C^{\perp}\left(\frac{Z+(m-1)W}{m}, \frac{Z-W}{m}, X+(m-1)Y, X-Y\right). \end{aligned}$$

This completes the proof. □

3. The MacWilliams identities for quantum error-correcting G-codes.

In the classical coding theory, the MacWilliams identities establish a relation between the weight enumerators/ the complete weight enumerators of a code C and its dual code $C^{\perp}$. For quantum error-correcting codes (abbreviated as QECCs), an analogous concept of weight enumerators was introduced in 1997 by Shor and Laflamme [15] for binary quantum error-correcting codes and the MacWilliams identity was also deduced. Rains [12] later generalized this result to general QECCs. Hu-Yang-Yau [3, 4] introduced the notions of double weight enumerator and the complete weight enumerator and then generalized the MacWilliams identities for complete weight enumerators from classical coding theory to the quantum case, both for binary and non-binary codes. Using the generalized MacWilliams identities, they derived new Singleton-type and Hamming-type bounds for asymmetric quantum codes, which are refined versions of the classical results. We refer to [3, 4] for more information on the development of this topic. Note that the codes studied in [3, 4] are defined over finite fields.

In the following we prove three versions of quantum MacWilliams identities for general QECCs, not necessarily defined over finite fields, and explain that these are special cases of Theorems 1.2, 1.4 and 1.6. We review the necessary notions of quantum codes and refer to [7, 8, 10, 11] for more information.

In this section, $|a\rangle$, $|b\rangle$ denote complex vectors, and $\langle a|$, $\langle b|$ denote their conjugate transposes. For an operator E, $E|a\rangle$ denotes the operation of E on $|a\rangle$, E^* denotes the complex conjugation of E. Denote by $\langle a|b\rangle$ the usual inner product of $|a\rangle$ and $|b\rangle$ in complex vector spaces. The symbol $\delta_{i,j}$ denotes the Kronecker delta.

Fix positive integers m, n, let $\mathcal{H} = \mathbb{C}^m$ and a *quantum error-correcting code* of length n is a subspace $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$. Let $\mathcal{E}_n$ be a set of unitary linear operators on $\mathcal{H}^{\otimes n}$. We say that $\mathcal{Q}$ can detect an error $E \in \mathcal{E}_n$ if for any $c_1, c_2 \in \mathcal{Q}$, the condition $\langle c_1|E|c_2\rangle = \lambda_E \langle c_1|c_2\rangle$ holds for some $\lambda_E \in \mathbb{C}$ depending only on E. We say that a QECC $\mathcal{Q}$ is $\mathcal{E}_n$*-correcting* if for an orthogonal basis $\{|a_i\rangle\}_i$ of $\mathcal{Q}$ and every $A,\ B \in \mathcal{E}_n$, we have $\langle a_i|A^*B|a_j\rangle = \lambda_{A,B}\delta_{i,j}$ for some $\lambda_{A,B} \in \mathbb{C}$ depending on A and B. Let $P_{\mathcal{Q}} \in \mathrm{End}(\mathcal{H}^{\otimes n})$ be the orthogonal projection operator of $\mathcal{H}^{\otimes n}$ onto $\mathcal{Q}$, then the condition can be restated in the form

$$P_{\mathcal{Q}} A^* B P_{\mathcal{Q}} = \alpha_{ij} P_{\mathcal{Q}},$$

for some Hermitian matrix $\alpha = (\alpha_{ij})$.

Let $\mathcal{E}$ be a set of unitary operators on $\mathcal{H}$ and $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. We say that $E \in \mathcal{E}_n$ has *weight* t if $E = A_1 \otimes A_2 \otimes ... \otimes A_n$ where $A_j \in \mathcal{E}, 1 \le j \le n$, and $|\{j : A_j \notin \mathbb{C}^\times \cdot \mathrm{Id}\}| = t$. Denote by $w(E)$ the weight of E.

Let $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$ be a QECC with dimension K. The *minimal distance* of the QECC $\mathcal{Q}$ with error set $\mathcal{E}$ is the maximal integer $d_{\mathcal{E}}$ such that $\mathcal{Q}$ can detect all errors in $\mathcal{E}_n$ of weight less than $d_{\mathcal{E}}$. Then we say that $\mathcal{Q}$ has parameters $((n, K, d_{\mathcal{E}}))_m$.

Let $\mathcal{Q} \subset \mathcal{H}^{\otimes n}$ be a QECC, we say that $\mathcal{Q}$ can correct all errors in $\mathcal{E}_n$ of weight $\le l$ $(0 \le l \le n)$ if $\mathcal{Q}$ is $\mathcal{E}_n^l$-correcting, where $\mathcal{E}_n^l = \{E \in \mathcal{E}_n, w(E) \le l\}$.

Remark 3.1. When $m = q$ for some q that is a power of a prime p, we say that $\mathcal{Q}$ is defined over the finite field $\mathbb{F}_q$. In this case, one could construct meaningful QECCs in various ways (cf. [6]).

Definition 3.2. Let G be a finite group and $\rho : G \to \mathbf{U}(\mathcal{H})$ be an irreducible faithful α-representation. Let $\mathcal{E} = \{\rho(g) \mid g \in G\}$ and $\mathcal{E}_n = \mathcal{E}^{\otimes n}$. A QECC $\mathcal{Q} \subset \mathcal{H}^n$ with error set $\mathcal{E}$ is called a *quantum error-correcting G-code*.

In the following, we deduce the MacWilliams identities for quantum error-correcting G-codes.

3.1. The MacWilliams identity for weight enumerators.

Definition 3.3. Let $(\rho, \mathcal{H})$ be an irreducible faithful unitary projective representation of a finite group G and let $\mathcal{E} = \{\rho(g) \mid g \in G\}$. Let $\mathcal{E}_n = \mathcal{E}^{\otimes n} = \{\rho(g_1) \otimes \rho(g_2) \otimes \cdots \otimes \rho(g_n) \mid g_t \in G,\ 1 \le t \le n\}$. Let $\mathcal{Q}$ be a quantum error-correcting G-code of dimension K, and let

$$B_i = \frac{1}{K^2} \sum_{E \in \mathcal{E}_n, w(E) = i} \mathrm{Tr}(E^{-1} P_{\mathcal{Q}}) \mathrm{Tr}(E P_{\mathcal{Q}}),$$

$$B_i^{\perp} = \frac{1}{K} \sum_{E \in \mathcal{E}_n, w(E) = i} \mathrm{Tr}(E^{-1} P_{\mathcal{Q}} E P_{\mathcal{Q}}),$$

$$D_J = \frac{1}{K^2} \sum_{E \in E[J]} \mathrm{Tr}(E^{-1} P_\mathcal{Q}) \mathrm{Tr}(E P_\mathcal{Q}),$$

$$D_J^\perp = \frac{1}{K} \sum_{E \in E[J]} \mathrm{Tr}(E^{-1} P_\mathcal{Q} E P_\mathcal{Q}).$$

The *weight enumerators* of $\mathcal{Q}$ are defined by

$$f_\mathcal{Q}(x, y) = \sum_{i=0}^{n} B_i x^{n-i} y^i,$$

$$f_\mathcal{Q}^\perp(x, y) = \sum_{i=0}^{n} B_i^\perp x^{n-i} y^i,$$

and the *complete weight enumerators* of $\mathcal{Q}$ are defined by

$$D_\mathcal{Q}(M) = \sum_{J \in \mathrm{IND}(n)} D_J M^J,$$

$$D_\mathcal{Q}^\perp(M) = \sum_{J \in \mathrm{IND}(n)} D_J^\perp M^J,$$

where $E[J]$, $\mathrm{IND}(n)$ are defined in the paragraph before Definition 1.3.

Theorem 3.4 (Weight enumerators)**.** *With the notation as above, we have*

$$f_\mathcal{Q}(x, y) = \frac{1}{K} f_\mathcal{Q}^\perp\left(\frac{m^2 x + (\mathfrak{g}^2 - m^2) y}{m\mathfrak{g}}, \frac{m(x - y)}{\mathfrak{g}}\right),$$

where m is the dimension of $\mathcal{H}$, $\mathfrak{g}$ is the order of G.

Proof. This is a special case of Theorem 1.2 where $(\rho_i, V_i) = (\rho, \mathcal{H})$ and $P_1 = P_2 = P_\mathcal{Q}$. □

Remark 3.5. The set $\mathcal{E} = \{\rho(g) \mid g \in G\}$ forms a *nice error bases* (cf. [7]) if and only if $\mathfrak{g}$ is a square and ρ is a unitary irreducible faithful projective representation of G with degree $\mathfrak{g}^{1/2}$. In this case the direct sum of $\mathfrak{g}^{1/2}$ copies of $\rho : G \to \mathbf{U}(\mathcal{H})$ is the regular α-representation, i.e. the α-representation of G on the vector space $\mathbb{C}[G]$ with $r(g)h = \alpha(g, h)gh$. Hence $\mathrm{Tr}(\rho(g)) = 0$ if $g \in G$ is not the identity element and $\{\rho(g) \mid g \in G\}$ forms an orthogonal basis of $\mathrm{End}(\mathcal{H})$. Then Theorem 3.4 is a special case of [12, Theorem 19]. We remark that Theorem 3.4 focuses on operators from projective representations without other conditions on the number of operators and [12, Theorem 19] focuses on operators that form a basis of $\mathrm{End}(\mathcal{H})$ without other conditions on their relations. The proof of Theorem 3.4 and the proof of [12, Theorem 19] are different.

By the same argument as in [15], we have the following result.

Theorem 3.6. *For a quantum error-correcting G-code $\mathcal{Q}$ and the corresponding real numbers B_i, $B_i^\perp$ $(0 \leq i \leq n)$, we have*

1. *$B_0 = B_0^\perp = 1$, $B_i^\perp \geq B_i \geq 0$ $(0 \leq i \leq n)$.*
2. *If there exists $t \leq n - 1$, such that $B_i^\perp = B_i$ $(0 \leq i \leq t)$, and $B_{t+1}^\perp > B_{t+1}$, then the minimal distance d is $t + 1$.*

For quantum error-correcting G-codes, the MacWilliams identities give necessary conditions for their existence. Moreover they also give the bound of minimum distance. The binary version of the quantum Singleton bound was first proved by Knill and Laflamme in [10], and later generalized by Rains using the quantum MacWilliams identities in [13, Theorem 2]. By Theorem 3.4 and the same argument of [6, Corollary 28], we obtain the Singleton bound for quantum G-codes.

Theorem 3.7 (Quantum Singleton Bound)**.** *If $\mathcal{Q}$ is a quantum error-correcting G-code with parameters $((n, K, d))_m$, then*

$$K \leq (\frac{\mathfrak{g}}{m})^{n-2d+2}.$$

Proof. We sketch the proof following the strategy in [6]. Let n and l be positive integers. Recall that the Krawtchouk polynomials $P_t(x)$ for $n,\ l$ are defined by the formula

$$P_t(x) := \sum_{j=0}^{t} (-1)^j (l^2-1)^{t-j} \binom{x}{j}\binom{n-x}{t-j},$$

where $0 \leq t \leq n$. Let $f_{\mathcal{Q}}(x,y) = \sum_{i=0}^{n} B_i x^{n-i} y^i$ and $f_{\mathcal{Q}}^{\perp}(x,y) = \sum_{i=0}^{n} B_i^{\perp} x^{n-i} y^i$ be the weight enumerators of $\mathcal{Q}$ with parameters $((n, K, d))_m$. By the MacWilliams identity in Theorem 3.4, we have

$$\begin{aligned}\sum_{i=0}^{n} B_t x^{n-t} y^t =& \frac{1}{K}\sum_{i=0}^{n} B_i^{\perp} (\frac{m^2 x + (\mathfrak{g}^2 - m^2) y}{m\mathfrak{g}})^{n-i} (\frac{m(x-y)}{\mathfrak{g}})^i \\ =& \frac{m^n}{K\mathfrak{g}^n}\sum_{i=0}^{n} B_i^{\perp} \sum_{\lambda,\mu\geq 0} \binom{n-i}{\lambda}\binom{i}{\mu} (x)^{n-i-\lambda} (((\frac{\mathfrak{g}}{m})^2 - 1) y)^{\lambda} (x)^{i-\mu} (-y)^{\mu}.\end{aligned}$$

Comparing the coefficients, we have

$$\begin{aligned}B_t =& \frac{m^n}{K\mathfrak{g}^n}\sum_{i=0}^{n} B_i^{\perp} \sum_{\substack{\lambda,\mu\geq 0 \\ \lambda+\mu=t}} \binom{n-i}{\lambda}\binom{i}{\mu} ((\frac{\mathfrak{g}}{m})^2 - 1)^{\lambda} (-1)^{\mu} \\ =& \frac{m^n}{K\mathfrak{g}^n}\sum_{i=0}^{n} B_i^{\perp} P_t(i),\end{aligned}$$

where $P_t(x)$ are the Krawtchouk polynomials for $n,\ \frac{g}{m}$. Similarly, we have

$$B_t^{\perp} = \frac{m^n K}{\mathfrak{g}^n}\sum_{i=0}^{n} P_t(i) B_i.$$

The rest of the proof is exactly the same as the proof of [6, Corollary 28] and the theorem follows. ☐

3.2. **The MacWilliams identity for complete weight enumerators.** In [3, 4], Hu-Yang-Yau proved the MacWilliams identity for double weight enumerators and the complete weight enumerators for binary and non-binary quantum codes. One could generalize them easily to G-codes and the results in [3, 4] correspond to $G = \mathbb{F}_2 \times \mathbb{F}_2$ and $G = \mathbb{F}_q \times \mathbb{F}_q$ respectively in our setting.

Theorem 3.8 (Complete enumerators)**.** *With the notation as in* Definition 3.3*, let G be an abelian group with order $\mathfrak{g}$. For a quantum error-correcting G-code $\mathcal{Q}$ with parameters $((n, K, d))_m$ we have*

$$D_{\mathcal{Q}}(M) = \frac{1}{K} D_{\mathcal{Q}}^{\perp}(M^{\perp}),$$

where $M_g^{\perp} = \frac{m}{\mathfrak{g}} \sum_{l\in G} \alpha(g^{-1}, l^{-1}) \alpha^{-1}(l^{-1}, g^{-1}) M_l$, for all $g \in G$ and $M^{\perp} = (M_g^{\perp})$.

Proof. This is a special case of Theorem 1.4 where $(\rho_i, V_i) = (\rho, \mathcal{H})$ and $P_1 = P_2 = P_{\mathcal{Q}}$. ∎

3.3. The MacWilliams identity for double weight enumerators.

Definition 3.9. Let H be an abelian group and $(\rho, \mathcal{H})$ be the Weyl-Heisenberg representation of $G = H \times \widehat{H}$ as in equation (2). Let $\mathcal{Q}$ be a quantum error-correcting G-code with parameters $((n, K, d_{\mathcal{E}}))_m$, and let

$$C_{i,j} := \frac{1}{K^2} \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}})\mathrm{Tr}(EP_{\mathcal{Q}}),$$
$$C^{\perp}_{i,j} := \frac{1}{K} \sum_{E \in E[i,j]} \mathrm{Tr}(E^{-1}P_{\mathcal{Q}}EP_{\mathcal{Q}}).$$

The *double weight enumerators* of $\mathcal{Q}$ are defined by

$$C(X,Y,Z,W) := \sum_{i,j=0}^{n} C_{i,j} X^{n-i}Y^iZ^{n-j}W^j,$$
$$C^{\perp}(X,Y,Z,W) := \sum_{i,j=0}^{n} C^{\perp}_{i,j} X^{n-i}Y^iZ^{n-j}W^j.$$

The following result follows from Theorem 1.6.

Theorem 3.10 (Double weight enumerators)**.** *The double weight enumerators of $\mathcal{Q}$ satisfy the following identity*

$$C(X,Y,Z,W) = \frac{1}{K} C^{\perp}(\frac{Z+(m-1)W}{m}, \frac{Z-W}{m}, X+(m-1)Y, X-Y).$$

As in [4], we introduce the asymmetric quantum G-codes.

Definition 3.11. With the notation in Section 1. Let H be an abelian group and $(\rho, \mathcal{H})$ be the Weyl-Heisenberg representation of $G = H \times \widehat{H}$ as in equation (2). Given a quantum error-correcting G-code $\mathcal{Q}$, let d_X and d_Z be the maximum integers such that each error $E \in E[i,j]$ with $i < d_X, j < d_Z$ is detected by $\mathcal{Q}$, then we call $\mathcal{Q}$ an *asymmetric quantum G-code* with parameters $((n, K, d_Z/d_X))_m$.

The following theorem can be deduced in the same way as in [4, Theorems 1, 2, 6]. Once we have the identity in Theorem 3.10, the argument does not involve the group structure of G and is exactly the same as the argument in [4], we omit the details.

Theorem 3.12. *Let $\mathcal{Q}$ be an asymmetric quantum G-code with double weight distributions $C_{i,j}$, $C^{\perp}_{i,j}$ and parameters $((n, K, d_Z/d_X))_m$, then*

1. *$C^{\perp}_{i,j} \geq C_{i,j} \geq 0$ for $0 \leq i,j \leq n$, and $C_{0,0} = C^{\perp}_{0,0} = 1$.*
2. *If t_X, t_Z are the two largest integers such that $C_{i,j} = C^{\perp}_{i,j}$ for $i < t_X$ and $j < t_Z$, then $d_X = t_X$ and $d_Z = t_Z$.*
3. *(Singleton Bound) $K \leq m^{n+2-d_X-d_Z}$.*
4. *(Hamming Bound) $K \leq m^{n(1-H(\frac{\delta_X}{2})-H(\frac{\delta_Z}{2})+o(1))}$. Here $o(1)$ tends to zero as n goes to infinity, $\delta_X = \frac{d_X}{n}$ and $\delta_Z = \frac{d_Z}{n}$ satisfy $0 \leq \delta_X \leq \frac{1}{5}$ and $0 \leq \delta_Z \leq \frac{1}{5}$ respectively, $H(x)$ is a function defined by*

$$H(x) = x\log_m(m-1) - x\log_m x - (1-x)\log_m(1-x),\ 0 \leq x \leq 1.$$

Remark 3.13. As mentioned in Remark 3.5, if the set $\mathcal{E} = \{\rho(g), g \in G\}$ forms a *nice error bases* (cf. [7]), then ρ is a unitary irreducible faithful projective representation of G of degree $\mathfrak{g}^{1/2}$. In this case Theorem 3.7 has a simple form. In [9], Knill discussed the construction of quantum codes based on nice error bases and obtained certain equivalent characterizations for nice error bases. There are examples where the nice error bases occur with non-abelian group G (cf. the list in [7]). The above results show that the Singleton bound and the Hamming bound of quantum G-codes depend only on the size of G. This is closely related to the question motivated by [9, Theorem 3.4].

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