

Degenerate wave-like solutions to the Dirac equation for massive particles

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received 15 December 2022; accepted in final form 11 May 2023

published online 30 May 2023

Abstract – In this work we provide a novel class of degenerate solutions to the Dirac equation for massive particles, where the rotation of the spin of the particles is synchronized with the rotation of the magnetic field of the wave-like electromagnetic fields corresponding to these solutions. We show that the state of the particles does not depend on the intensity of the electromagnetic fields, but only on their frequency, which is proportional to the mass of the particles and lies in the region of Gamma/X-rays for typical elementary charged particles, such as electrons and protons. We have also calculated the electric current density corresponding to the electromagnetic 4-potentials connected to the degenerate solutions and found that it has the same spatial and temporal dependence on the electromagnetic fields, rotating at an exceptionally high frequency. This result indicates that the degenerate states may occur at locations where matter collapses, *e.g.*, in the central region of a black hole. Finally, we have calculated the spin of the particles described by degenerate spinors and found that it rotates in synchronization with the magnetic field and the current density.



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Introduction. – In a recent article [1] we have shown that all solutions to the Dirac equation,

$$i\gamma^\mu \partial_\mu \Psi + a_\mu \gamma^\mu \Psi - m\Psi = 0, \quad (1)$$

satisfying the conditions $\Psi^\dagger \gamma^0 \Psi = 0$ and $\Psi^T \gamma^2 \Psi \neq 0$, where γ^μ are the standard Dirac matrices and $\gamma = \gamma^0 + i\gamma^1 \gamma^2 \gamma^3$ are degenerate, corresponding to an infinite number of electromagnetic 4-potentials A_μ , explicitly calculated in theorem 5.4 in [1]. In the Dirac equation, m and q are the mass and the electric charge of the particle, respectively, and $a_\mu = qA_\mu$. It should also be noted that eq. (1) is written in natural units, where both the speed of light in vacuum c and the reduced Planck constant \hbar are equal to one. Furthermore, in [1] we have shown that all solutions to the Weyl equation are degenerate, corresponding to an infinite number of electromagnetic 4-potentials, explicitly calculated in theorem 3.1 in [1]. Some very interesting

properties of Weyl particles, mainly regarding their control and localization, are discussed in [2,3].

As was also shown in [1], the degenerate solutions in the case of free Dirac particles correspond to massless particles, except for particle-antiparticle pairs. However, the net charge of the particle-antiparticle pair is zero, and consequently the degeneracy is not particularly meaningful from a practical point of view. Additionally, in a recent work [4] we have shown that degenerate solutions for massive particles can exist in potential barriers. However, these solutions involve real exponential terms and, consequently, they cannot describe the state of particles in free space. Finally, it should be mentioned that in [5] we provide a general method for obtaining degenerate solutions to the Dirac and Weyl equations, both for massive and massless particles.

Degenerate wave-like solutions to the Dirac equation and the corresponding 4-potentials. –

In this work we investigate the existence of degenerate

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$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\frac{\tan(\alpha + \beta) \left(2 \sin(\alpha - \beta) \left(\frac{\partial h}{\partial z} - g \right) + (\sin(2\alpha) - \sin(2\beta)) \frac{\partial h}{\partial t} + m(\sin(2\alpha) + \sin(2\beta)) \right)}{\cos(2\alpha) - \cos(2\beta)} \\ -\sec(\alpha + \beta) \cos d \left(\sin(\alpha + \beta) \left(\frac{\partial h}{\partial z} - g \right) + 2m \cos \alpha \cos \beta \csc(\alpha - \beta) \right) + \frac{\partial h}{\partial x} \\ \frac{1}{2} \csc(\alpha - \beta) \sec(\alpha + \beta) \sin d \left((\cos(2\alpha) - \cos(2\beta)) \left(\frac{\partial h}{\partial z} - g \right) - 4m \cos \alpha \cos \beta \right) + \frac{\partial h}{\partial y} \\ g \end{pmatrix}. \quad (4)$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -\frac{\tan(\alpha + \beta) \left((\sin(2\alpha) - \sin(2\beta)) \frac{\partial h}{\partial t} + m(\sin(2\alpha) + \sin(2\beta)) \right)}{\cos(2\alpha) - \cos(2\beta)} \\ -2m \cos \alpha \cos \beta \csc(\alpha - \beta) \sec(\alpha + \beta) \cos d + \frac{\partial h}{\partial x} \\ -2m \cos \alpha \cos \beta \csc(\alpha - \beta) \sec(\alpha + \beta) \sin d + \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial z} \end{pmatrix}, \quad (7)$$

solutions for massive Dirac particles involving only complex exponential terms, which are well defined throughout space and time. For this purpose, we have used the following general form of degenerate spinors:

$$\Psi = \begin{pmatrix} \cos \alpha \exp(ih_1) \\ \sin \alpha \exp(ih_2) \\ \cos \beta \exp(ih_1) \\ \sin \beta \exp(ih_2) \end{pmatrix}, \quad (2)$$

where h_1, h_2 are arbitrary real functions of the spatial coordinates and time and α, β are real constants. Using the above ansatz and requiring Ψ to be solution to the Dirac equation for real 4-potentials, it is found that all spinors of the form

$$\Psi = c_1 \exp(ih) \begin{pmatrix} \cos \alpha \\ \sin \alpha \exp(id) \\ \cos \beta \\ \sin \beta \exp(id) \end{pmatrix} \quad (3)$$

are degenerate, satisfying the Dirac equation for the following 4-potentials:

see eq. (4) above

Here, c_1 is an arbitrary complex constant, h, g are arbitrary real functions of the spatial coordinates and time and

$$d = \frac{4m[t - z \cos(\alpha + \beta)]}{\cos(2\alpha) - \cos(2\beta)}. \quad (5)$$

In the above expressions we have also supposed that $\cos(2\alpha) - \cos(2\beta) \neq 0$, $\cos(\alpha + \beta) \neq 0$, $\sin(\alpha - \beta) \neq 0$,

implying that

$$\alpha \pm \beta \neq n\pi \quad \text{and} \quad \alpha + \beta \neq n\pi + \pi/2, n \in \mathbb{Z}. \quad (6)$$

An important characteristic of the degenerate spinors given by eq. (3) is that they can describe particles of any mass, including massless particles. In addition, the aforementioned solutions correspond to particles in non-localized states, which can exist throughout space and time without any restriction. In contrast, the degenerate solutions in [4] describe particles existing only in classically forbidden regions.

Another interesting remark is that the 4-potentials given by eq. (4) can be substantially simplified setting $g = (\partial h / \partial z)$. In this case they take the form

see eq. (7) above

which is more convenient for studying the physical properties of the degenerate solutions and the corresponding electromagnetic fields. Specifically, it can be easily verified that the 4-potentials given by eq. (7) become zero assuming that

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial y} = \frac{\partial h}{\partial z} = 0, \quad (8)$$

$$\alpha = n\pi + \frac{\pi}{2} \quad \text{or} \quad \beta = n\pi + \frac{\pi}{2}, n \in \mathbb{Z} \quad (9)$$

and

$$\frac{\partial h}{\partial t} = -m \frac{\sin(2\alpha) + \sin(2\beta)}{\sin(2\alpha) - \sin(2\beta)}. \quad (10)$$

Here, it should be mentioned that the conditions described by eq. (6) ensure that $\sin(2\alpha) - \sin(2\beta) \neq 0$. Thus,

if eqs. (8)–(10) are valid, the degenerate spinors given by eq. (3) are solutions to the Dirac equation corresponding to zero electromagnetic 4-potential and field. As an example, the following spinors,

$$\begin{aligned}\Psi_0 &= c_1 \exp(imt) \begin{pmatrix} 0 \\ \exp(id) \\ \cos \beta \\ \sin \beta \exp(id) \end{pmatrix}, \\ \Psi'_0 &= c_1 \exp(-imt) \begin{pmatrix} \cos \alpha \\ \sin \alpha \exp(id) \\ 0 \\ \exp(id) \end{pmatrix},\end{aligned}\quad (11)$$

describe particles that could exist in a region of space free of electromagnetic fields.

The electromagnetic fields corresponding to the degenerate solutions and some important remarks.

– The electromagnetic fields corresponding to the 4-potentials given by eq. (7) are given by the following formulae, in Gaussian units [6,7]:

$$\begin{aligned}\mathbf{E} &= -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \\ &= \frac{4m^2}{q} \cos \alpha \cos \beta \csc^2(\alpha - \beta) \csc(\alpha + \beta) \\ &\quad \times \sec(\alpha + \beta) (-\sin d \mathbf{i} + \cos d \mathbf{j}),\end{aligned}\quad (12)$$

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} = -\frac{4m^2}{q} \cos \alpha \cos \beta \csc^2(\alpha - \beta) \\ &\quad \times \csc(\alpha + \beta) (\cos d \mathbf{i} + \sin d \mathbf{j}),\end{aligned}\quad (13)$$

where $\varphi = a_0/q$ is the electric potential and $\mathbf{A} = -(1/q)(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k})$ is the magnetic vector potential. It should also be reminded that the above equations are expressed in the natural system of units, where $\hbar = c = 1$.

An interesting remark is that the electromagnetic fields given by eqs. (12), (13) resemble a circularly polarized plane wave propagating along the $+z$ -direction with Poynting vector

$$\begin{aligned}\mathbf{S} &= \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} = \\ &= \frac{4m^4}{\pi q^2} \cos^2 \alpha \cos^2 \beta \csc^4(\alpha - \beta) \csc^2(\alpha + \beta) \sec(\alpha + \beta) \mathbf{k}.\end{aligned}\quad (14)$$

In addition, according to theorem 5.4 in [1], the spinors in eq. (3) will also be solutions to the Dirac equation for an

infinite number of 4-potentials, given by the formula

$$b_\mu = a_\mu + s \kappa_\mu, \quad (15)$$

where

$$\begin{aligned}(\kappa_0, \kappa_1, \kappa_2, \kappa_3) &= \\ &\left(1, -\frac{\Psi^T \gamma^0 \gamma^1 \gamma^2 \Psi}{\Psi^T \gamma^2 \Psi}, -\frac{\Psi^T \gamma^0 \Psi}{\Psi^T \gamma^2 \Psi}, \frac{\Psi^T \gamma^0 \gamma^2 \gamma^3 \Psi}{\Psi^T \gamma^2 \Psi} \right) \\ &= (1, -\sin(\alpha + \beta) \cos d, -\sin(\alpha + \beta) \sin d, \\ &\quad -\cos(\alpha + \beta))\end{aligned}\quad (16)$$

and s is an arbitrary real function of the spatial coordinates and time.

The electromagnetic fields corresponding to the 4-potentials $b_\mu - a_\mu = \kappa_\mu s$ are the following:

$$\begin{aligned}\mathbf{E}_s &= -\left(2ms_q \csc(\alpha - \beta) \sin d + \sin(\alpha + \beta) \right. \\ &\quad \times \cos d \frac{\partial s_q}{\partial t} + \frac{\partial s_q}{\partial x} \Big) \mathbf{i} + \left(2ms_q \csc(\alpha - \beta) \cos d \right. \\ &\quad \left. - \sin(\alpha + \beta) \sin d \frac{\partial s_q}{\partial t} - \frac{\partial s_q}{\partial y} \right) \mathbf{j} \\ &\quad - \left(\cos(\alpha + \beta) \frac{\partial s_q}{\partial t} + \frac{\partial s_q}{\partial z} \right) \mathbf{k},\end{aligned}\quad (17)$$

$$\begin{aligned}\mathbf{B}_s &= \left(-\sin(\alpha + \beta) \sin d \frac{\partial s_q}{\partial z} + \cos(\alpha + \beta) \right. \\ &\quad \times \left(-2ms_q \csc(\alpha - \beta) \cos d + \frac{\partial s_q}{\partial y} \right) \Big) \mathbf{i} \\ &\quad + \left(\sin(\alpha + \beta) \cos d \frac{\partial s_q}{\partial z} - \cos(\alpha + \beta) \right. \\ &\quad \times \left(2ms_q \csc(\alpha - \beta) \sin d + \frac{\partial s_q}{\partial x} \right) \Big) \mathbf{j} \\ &\quad + \sin(\alpha + \beta) \left(-\cos d \frac{\partial s_q}{\partial y} + \sin d \frac{\partial s_q}{\partial x} \right) \mathbf{k}.\end{aligned}\quad (18)$$

In the above formulae $s_q = s/q$. Consequently, particles described by the degenerate spinors given by eq. (3) have the remarkable property to exist in the same quantum state in the wide variety of electromagnetic fields described by eqs. (12), (13) and (17), (18). In addition, it should also be noted that the factor d , given by eq. (5), is also involved in the electromagnetic fields \mathbf{E}_s , \mathbf{B}_s and, consequently, they are expected to have similar properties with the fields described by eqs. (12), (13).

As an example, we suppose that the arbitrary function s is constant. Then, the electromagnetic fields corresponding to the 4-potentials b_μ take the following form:

$$\begin{aligned}\mathbf{E}_{t,w} &= \frac{2m}{q} \csc(\alpha - \beta) (2m \cos \alpha \cos \beta \csc(\alpha - \beta) \\ &\quad \times \csc(\alpha + \beta) \sec(\alpha + \beta) + s) (-\sin d \mathbf{i} + \cos d \mathbf{j}),\end{aligned}\quad (19)$$

$$\mathbf{B}_{t,w} = -\frac{2m}{q} \csc(\alpha - \beta) (2m \cos \alpha \cos \beta \csc^2(\alpha - \beta) \times \csc(\alpha + \beta) + s \cos(\alpha + \beta)) (\cos d\mathbf{i} + \sin d\mathbf{j}), \quad (20)$$

having the same spatial and temporal dependence on the electromagnetic fields given by eqs. (12), (13). This practically means that the state of the particles does not depend on the magnitude of the fields, but only on their spatial and temporal dependence, given by the function

$$d = \frac{4m[t - z \cos(\alpha + \beta)]}{\cos(2\alpha) - \cos(2\beta)} = \omega_d t - k_d z, \quad (21)$$

where

$$\omega_d = \frac{4m}{\cos(2\alpha) - \cos(2\beta)} \quad (22)$$

and

$$k_d = \frac{4m \cos(\alpha + \beta)}{\cos(2\alpha) - \cos(2\beta)} \quad (23)$$

are constants related to the angular frequency and the wave number, respectively. It should also be noted that the phase velocity,

$$v_{ph} = \frac{\omega_d}{k_d} = \sec(\alpha + \beta), \quad (24)$$

is higher than the speed of light ($c = 1$ in natural units). However, this does not violate the special theory of relativity since a sinusoidal wave with a unique frequency does not transmit any information. It is reminded that the phase velocity of an electromagnetic wave propagating through a medium can exceed the speed of light in vacuum, as it happens in most glasses at X-ray frequencies [8] and in unmagnetized plasmas [9].

Another interesting remark is that the frequency of these wave-like fields depends on the mass of the particles. In more detail, in SI units, the factor $4m$ in eq. (22) becomes $4mc^2/\hbar$ and consequently the frequency of the oscillation is given by the formula

$$f_d(\text{SI}) = \frac{\omega_d(\text{SI})}{2\pi} = \frac{4mc^2}{h} \frac{1}{\cos(2\alpha) - \cos(2\beta)}. \quad (25)$$

For example, in the case of electrons ($m_e = 9.109 \times 10^{-31} \text{ kg}$) the frequency of the oscillation becomes

$$f_d(\text{SI}) = \frac{4.95 \times 10^{20}}{\cos(2\alpha) - \cos(2\beta)} \text{ Hz}, \quad (26)$$

corresponding to photons with energy higher than 2.05 MeV, in the region of Gamma/X-rays. Furthermore, in the case of heavier particles, *e.g.*, protons ($m_p = 1.673 \times 10^{-27} \text{ kg}$), the oscillation frequency takes much higher values, above $9.09 \times 10^{23} \text{ Hz}$, corresponding to photons with extremely high energy, higher than 3.75 GeV.

Here, it should be noted that the electric charge ρ and current \mathbf{J} densities corresponding to the electromagnetic

4-potentials and fields described in this section, can be calculated through the inhomogeneous Maxwell's equations [6,7]:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (27)$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{J}, \quad (28)$$

which can also be written in terms of the electromagnetic potentials, as

$$\nabla^2 \varphi + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -4\pi\rho, \quad (29)$$

$$\left(\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right) - \nabla \left(\nabla \cdot \mathbf{A} + \frac{\partial \varphi}{\partial t} \right) = -4\pi\mathbf{J}. \quad (30)$$

Consequently, in principle, any differentiable real functions of the spatial coordinates and time can be used to describe the electric and magnetic potential in a region of space where the electric charge and current densities are given by the above formulae. In practice, the required charge and current densities can be created using appropriate distributions of positive and negative electric charges moving at suitable velocities. For example, in an electric conductor, the net electric charge density is zero since the negative charge of the electrons is compensated by the positive charge of the ions of the lattice. However, the current density can be set to any desired form using the appropriate voltage.

The above equations can be used to calculate the electric charge and current densities corresponding to the wave-like electromagnetic fields given by eqs. (12), (13), obtaining that

$$\begin{aligned} \rho &= 0, \\ \mathbf{J} &= -\frac{m^3}{\pi q} \cos \alpha \cos \beta \csc^3(\alpha - \beta) \\ &\quad \times \sec(\alpha + \beta) (\cos d\mathbf{i} + \sin d\mathbf{j}). \end{aligned} \quad (31)$$

From the above equation it becomes clear that the current density has the same spatial and temporal dependence with the electromagnetic field, described by the parameter d , given by eq. (21). This practically means that the vector of the current density rotates at an exceptionally high frequency, given by eqs. (25), (26). Consequently, the degenerate spinors presented in this article are expected to describe particles in regions of space where electric charges rotate at particularly high speeds. As can be easily deduced through the conservation law of angular momentum, this is expected to occur in regions of space where matter collapses, *e.g.*, in the central region of a black hole. Furthermore, the fact that the quantum state of particles described by degenerate spinors is extremely robust under a wide variety of electromagnetic perturbations may imply that particles tend to maintain their quantum state as they travel across the cosmos through a black hole. Obviously, no one knows what happens at the center of a black hole,

so the above argument is valid only in the case that the particles are not destroyed, but re-emerge in another region of spacetime, as suggested by some theories [10]. Furthermore, to study in more detail, the behavior of Dirac particles in the region of a black hole, one should consider the Dirac equation in curved spacetimes [11–14]. This is quite an interesting topic and will be addressed in future works.

Additionally, it should be noted that, setting $\cos(2\alpha) - \cos(2\beta) = 2/n$, $n \in \mathbb{Z}$, in eq. (25), the energy of the photons becomes exactly equal to the one required for the production of n particle-antiparticle pairs. Consequently, the degenerate spinors given by eq. (3) could also be related to the Schwinger effect [15–21].

Finally, it is particularly interesting to study the spin of the particles described by the degenerate spinors given by eq. (3). Specifically, the expected values of the projections of the spin of the particles along the x -, y -, and z -axes are given by the following formulae [22,23]:

$$S_x = \frac{i}{2} \Psi^\dagger \gamma^2 \gamma^3 \Psi = \frac{|c_1|^2}{2} (\sin(2\alpha) + \sin(2\beta)) \cos d, \quad (32)$$

$$S_y = \frac{i}{2} \Psi^\dagger \gamma^3 \gamma^1 \Psi = \frac{|c_1|^2}{2} (\sin(2\alpha) + \sin(2\beta)) \sin d, \quad (33)$$

$$S_z = \frac{i}{2} \Psi^\dagger \gamma^1 \gamma^2 \Psi = \frac{|c_1|^2}{2} (\cos(2\alpha) + \cos(2\beta)). \quad (34)$$

From the above expressions it becomes clear that the expected value of the projection of the spin of the particles on the x - y plane rotates in synchronization with the magnetic field of the wave-like electromagnetic fields given by eqs. (12), (13) and (19), (20). Therefore, the synchronization between the spin of the particles and the wave-like electromagnetic fields can be regarded as a key feature of the degenerate solutions presented in this article. Furthermore, under the conditions described by eqs. (8)–(10), *e.g.*, in the case of the degenerate spinors given by eq. (11), the rotation of the spin of the particles occurs even in the absence of an electromagnetic field. Thus, it can be considered that the electromagnetic fields should be synchronized to the rotation of the spin of the particles and not the opposite.

Conclusions. – In conclusion, we have provided a novel class of degenerate solutions to the Dirac equation for massive particles, where the key feature is the synchronization between the rotation of the spin of the particles and the magnetic field of the wave-like electromagnetic fields corresponding to these solutions. We have shown that the frequency of these wave-like electromagnetic fields depends on the mass of the particles and lies in the region of Gamma/X-rays for typical subatomic particles, such as electrons, protons, etc. Another interesting characteristic of these fields is that their phase velocity is higher than the speed of light in vacuum, which does not violate the special

theory of relativity, since a sinusoidal wave with a single frequency does not transmit any information. We have also calculated the electric current density corresponding to the electromagnetic 4-potentials and fields associated with the degenerate solutions and found that it rotates at an exceptionally high frequency, in synchronization with the magnetic field. This result indicates that this situation may occur in regions of space where matter collapses, *e.g.*, in the central region of a black hole. Finally, we have shown that the spin of the particles described by degenerate spinors rotates in synchronization with the magnetic field and the electric current density.

Data availability statement: The data that support the findings of this study are available upon reasonable request from the authors.

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