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Abstract: A novel method for finding the eigenvalues of a Sturm–Liouville problem is developed. Following the minimalist approach, the problem is transformed to a single first-order differential equation with appropriate boundary conditions. Although the resulting equation is nonlinear, its form allows us to find the general solution by adding a second part to a particular solution. This splitting of the general solution into two parts involves the Schwarzian derivative: hence, the name of the approach. The eigenvalues that correspond to acceptable solutions can be found by requiring the second part to correct the asymptotically diverging behavior of the particular solution. The method can be applied to many different areas of physics, such as the Schrödinger equation in quantum mechanics and stability problems in fluid dynamics. Examples are presented.

Keywords: differential equations; Sturm–Liouville problem; analytical methods; mathematical physics; instabilities; fluid dynamics; hydrodynamics; magnetohydrodynamics

1. Introduction

Sturm–Liouville theory is of fundamental importance in many areas of physics and mathematics and has important applications in a variety of physical phenomena: quantum mechanics and stability problems in fluid mechanics, to name but two. Simple examples have become standard knowledge of a physicist, mathematician, or engineer and are taught at the undergraduate level, while more complicated cases are the subject of current research. Standard methods to find solutions have been developed, e.g., [1,2], but there is always room for improvement and better efficiency in finding the eigenvalues and the corresponding eigenfunctions.

A Sturm–Liouville problem corresponds to a second-order differential equation of the form

$$(pf')' + qf = 0,\tag{1}$$

where p and q are functions of the independent variable x (a prime denotes a derivative with respect to x), together with appropriate boundary conditions that the unknown function f must obey. These boundary conditions involve only the ratios f'/f at the extreme values of x enclosing the region of interest on the x axis. (Depending on the problem, this could be the whole axis $-\infty < x < \infty$, a semi-infinite interval $x_{\min} < x < \infty$ or $-\infty < x < x_{\max}$, or a finite portion $x_{\min} < x < x_{\max}$.) Note that the classical Sturm–Liouville problem is written as $(pf')' + qf = -\lambda wf$, where w is a function of x, and λ is the eigenvalue. However, we can include the right-hand side inside the last term of the left-hand side and write the equation as in (1). Our purpose anyway is to study the more general Sturm–Liouville problem in which the eigenvalue appears in a nonlinear way inside both q and p. This is the form arising from the linearization of equations describing stability problems in hydrodynamics and ideal magnetohydrodynamics; see, e.g., [3,4]. Motivated by the need to find an efficient way to solve these problems, trying to identify which factors determine their dispersion relation, and exploring ways to express the boundary conditions, we arrive at the novel approach presented in this paper, which applies to all kinds of Sturm–Liouville problems.

In Section 2, we extend the minimalist approach developed in Ref. [5] to any Sturm–Liouville problem. This provides a more economical way to solve the problem by transforming Equation (1) to a single first-order equation. In Section 3, we present a new method to split the solution into two parts and use them to conveniently express the boundary conditions corresponding to non-diverging solutions asymptotically. We dub the approach "Schwarzian" since the Schwarzian derivative and the fact that it remains unchanged under Möbius transformations are key ingredients of the developed method. In Section 4, we apply the method to examples from quantum mechanics. In Section 5, we discuss linear stability problems and present their Sturm–Liouville forms. An example for the stability of an astrophysical jet is presented in Section 6; we also use the new method developed in Ref. [5] for solving the complex equation resulting from the boundary conditions. Conclusions follow in Section 7.

2. The Minimalist Approach

Since Equation (1) is linear in f and the boundary conditions involve only the values of the ratios f'/f at the extreme values of x, we can reformulate the problem, following the minimalist approach of [5], using as unknown the ratio of f with its derivative or, equivalently, the function

$$F = \frac{pf'}{f}.$$ (2)

It is straightforward to show that Equation (1) is transformed to a first-order differential equation for F

$$F' = -\frac{F^2}{p} - q.$$ (3)

The formulation significantly simplifies the problem of finding the eigenvalues since it transforms the second-order original Sturm–Liouville equation to first-order. The new equation is non-linear, but this does not complicate things if one uses a shooting method to satisfy the boundary conditions; it only makes the procedure simpler by reducing the number of first-order differential equations by half (the original second-order equation can be thought as two first-order equations).

In some cases, especially when we integrate problems in real space, possible oscillations in f lead to F varying from infinity to infinity and back. One way to treat these cases and smoothly pass infinities at points where $f = 0$ is to substitute $F(x) = \cot \dfrac{\Phi(x)}{2}$. More generally, we can write

$$F(x) = F_1(x) + F_2(x) \cot \frac{\Phi(x)}{2}$$ (4)

with the $F_1(x)$ and $F_2(x)$ of our choice. In this way, the zeros of f correspond simply to Φ being an even multiple of π, and the solutions can be found without any numerical difficulty by integrating the differential equation

$$\Phi' = \frac{2F_1 F_2 + pF_2'}{pF_2} \sin \Phi - \frac{pq + pF_1' + F_1^2 - F_2^2}{pF_2} \cos \Phi + \frac{pq + pF_1' + F_1^2 + F_2^2}{pF_2}.$$ (5)

The boundary conditions for F are translated to boundary conditions for Φ, and they are satisfied for the eigenvalues of the problem. Having found an eigenvalue, one could return to the original equation or, simply, to Equation (2), which gives $\dfrac{f'}{f} = \dfrac{F}{p}$, and by integration find f. (Although F and Φ are uniquely defined for a particular eigenvalue, there is a free multiplication constant in the eigenfunction f, which is related to the normalization).

3. The Schwarzian Approach

3.1. The Schwarzian g Approach

We can once more reformulate the problem and write its general solution using the Schwarzian derivative. As will become clear later, this is particularly useful in cases where the problem extends to asymptotic regions, as the new approach is linked to conveniently satisfying the boundary conditions there by automatically choosing the non-diverging solution.

It is straightforward to show that Equation (1) can be transformed to a "variable frequency oscillator"

$$(f\sqrt{p})'' + \kappa^2(f\sqrt{p}) = 0 \tag{6}$$

with

$$\kappa^2 = \frac{q}{p} - \left(\frac{p'}{2p}\right)^2 - \left(\frac{p'}{2p}\right)'. \tag{7}$$

As explained in Appendix A, we can write its general solution as

$$f = \frac{C_1 g + C_2}{\sqrt{pg'}}, \tag{8}$$

where g is a particular solution of the Schwarz equation

$$\{g, x\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = 2\kappa^2, \tag{9}$$

involving the Schwarzian derivative $\{g, x\}$.

We can transform the latter to a system of first-order differential equations by writing the expression for $F = \dfrac{pf'}{f} = -\dfrac{pg''}{2g'} - \dfrac{p'}{2} + \dfrac{pg'}{g + C_2/C_1}$ (which is the general solution of Equation (3), since Equation (8) is the general solution of Equation (1)) as

$$F = F_p + \frac{e^{-2\Lambda}}{g + C_2/C_1}, \tag{10}$$

where $F_p = -\dfrac{pg''}{2g'} - \dfrac{p'}{2}$ and $e^{-2\Lambda} = pg'$. The last two equations together with Equation (9) give the system

$$F_p' = -\frac{F_p^2}{p} - q, \quad \Lambda' = \frac{F_p}{p}, \quad g' = \frac{e^{-2\Lambda}}{p}. \tag{11}$$

The solution of the Sturm–Liouville equation is

$$f = \frac{C_1 g + C_2}{e^{-\Lambda}}; \tag{12}$$

we remind the reader, though, that to find the eigenvalues, only F is needed.

Interestingly, the first from Equation (11) is the same as the original Equation (3), meaning that F_p is a particular solution of that equation and does not necessarily satisfy the boundary conditions. These conditions should be satisfied by the total F given by Equation (10), which consists of two parts. Its second part can be found using the second and third from Equation (11). We emphasize that only a particular solution is needed: the initial values of F_p, Λ, and g at some initial point of integration are completely free. For any choice of these conditions, we find F and apply the two boundary conditions at the ends of the region of interest. One of them specifies the free constant C_2/C_1 appearing in Equation (10), and the other gives the eigenvalues.

3.2. The Schwarzian Φ Approach

An alternative way to solve the problem—particularly useful if there are points where $f = 0$ (so F becomes infinity), but in general, with some potential connection to the phase of oscillations of the function f—is to replace $g + \dfrac{C_2}{C_1}$ with $\tan\dfrac{\Phi + C}{2}$ (and constant C).

Substituting g in the expressions above and using the chain rule (A7) with $\{g, \Phi\} = \dfrac{1}{2}$, we conclude that the general solution of the "variable frequency oscillator" is

$$f \propto \frac{1}{\sqrt{p\Phi'}} \sin \frac{\Phi + C}{2}, \tag{13}$$

and Φ is a particular solution of

$$\left\{ \tan \frac{\Phi + C}{2}, x \right\} \equiv \frac{\Phi'''}{\Phi'} - \frac{3\Phi''^2}{2\Phi'^2} + \frac{\Phi'^2}{2} = 2\kappa^2. \tag{14}$$

We can transform the latter to a system of first-order differential equations by writing the expression for $F = \dfrac{pf'}{f} = -\dfrac{p\Phi''}{2\Phi'} - \dfrac{p'}{2} + \dfrac{p\Phi'}{2} \cot \dfrac{\Phi + C}{2}$ as

$$F = F_1 + F_2 \cot \frac{\Phi + C}{2}, \tag{15}$$

where $F_1 = -\dfrac{p\Phi''}{2\Phi'} - \dfrac{p'}{2}$ and $F_2 = \dfrac{p\Phi'}{2}$. The last two equations together with Equation (14) give the system

$$F_1' = \frac{F_2^2}{p} - \frac{F_1^2}{p} - q, \quad F_2' = -\frac{2F_1 F_2}{p}, \quad \Phi' = \frac{2F_2}{p}. \tag{16}$$

The solution of the Sturm–Liouville equation is (with D a normalization constant)

$$f = \frac{D}{\sqrt{F_2}} \sin \frac{\Phi + C}{2}. \tag{17}$$

(We again remind the reader that to find the eigenvalues, only F is needed.) (Some caution is needed when one calculates the square root $\sqrt{F_2}$ in Equation (17). Since the constant D is arbitrary, we are free to chose, e.g., the principal square root. In cases, however, in which, as x varies, the $\text{Arg}[F_2]$ crosses the value π, we should change the branch to avoid a false discontinuity in $\Im[\sqrt{F_2}]$.)

The equations of the Schwarzian Φ approach are, of course, equivalent to the ones of the g approach; they correspond to the substitutions $g + \dfrac{C_2}{C_1} = \tan \dfrac{\Phi + C}{2}$, $e^{-2\Lambda} = \dfrac{F_2}{\cos^2 \dfrac{\Phi + C}{2}}$, and $F_p = F_1 - F_2 \tan \dfrac{\Phi + C}{2}$. Note also that $\tan \dfrac{\Phi + C}{2}$ can be seen as a Möbius transformation of $e^{-i\Phi}$, and the substitutions $g + \dfrac{C_2}{C_1} = e^{-i\Phi} - e^{iC}$, $e^{-2\Lambda} = -2iF_2 e^{-i\Phi}$, and $F_p = F_1 + iF_2$ lead also to the equations of the Schwarzian Φ approach.

It is interesting to note the connection of the Schwarzian Φ approach with the one described at the end of Section 2. If in that approach we substitute $\Phi \to \Phi + C$ and require the Φ' as given by Equation (5) to be independent of Φ (i.e., choose F_1 and F_2 such that the coefficients of $\sin\Phi$ and $\cos\Phi$ are zero), we arrive at the Schwarzian Φ approach. Similarly, the substitution $F = F_p + \dfrac{e^{-2\Lambda}}{g}$ in Equation (3) gives $g' =$

$\dfrac{e^{-2\Lambda}}{p} + \left(-2\Lambda' + \dfrac{2F_p}{p}\right)g + \left(q + F_p' + \dfrac{F_p^2}{p}\right)\dfrac{g^2}{e^{-2\Lambda}}$. The replacement $g \to g + C_2/C_1$, and most importantly, the requirement that g' be independent of g, leads to the Schwarzian g approach and Equation (11).

3.3. Non-Diverging Asymptotic Solutions

Besides offering a way to express the general solution of a Sturm–Liouville problem, the Schwarzian approach can be used to choose the non-diverging asymptotically solution of a physical problem. In the g approach, this is achieved through the minimalist approach (solve the solution for F) and the splitting of F into two parts, as shown in Equation (10).

Suppose that we examine the behavior of the solution near $x = +\infty$. The two independent solutions of Equation (1), or equivalently, of Equation (6), behave as $\dfrac{1}{\sqrt{p_\infty}} e^{\pm i\kappa_\infty x} = \dfrac{1}{\sqrt{p_\infty}} e^{\pm i\Re\kappa_\infty x} e^{\mp \Im\kappa_\infty x}$. One of them is diverging and dominates any linear combination in the general solution. This is the case for the part F_p of the solution that by itself satisfies the original Equation (3); see the first from Equation (11). The diverging solution corresponds to a positive real part of F_p/p, and thus, $e^{-2\Lambda}$ decreases exponentially according to the middle from Equation (11). Although $e^{-2\Lambda}$ vanishes asymptotically, the freedom to choose the additive constant C_2/C_1 in Equation (10)—essentially, the fact that the Schwarzian derivative remains the same under Möbius transformations—allows us to make the second part of the solution important. By choosing $g + C_2/C_1 \to 0$, this second part $\dfrac{e^{-2\Lambda}}{g + C_2/C_1}$ becomes $\dfrac{0}{0}$, and by applying L'Hôpital's rule, we find that it equals $\dfrac{-2\Lambda' e^{-2\Lambda}}{g'} = -2F_p$.

The resulting sum follows exactly the non-diverging branch since $F_p + \dfrac{e^{-2\Lambda}}{g + C_2/C_1} \to -F_p$.

It is instructive to exactly solve the case where $p = 1$ and q is a constant, corresponding to an oscillator with complex constant "frequency" $\kappa = \sqrt{q}$. Without loss of generality, we can choose the root with a positive $\Im\kappa$. A particular solution of Equation (11) is $F_p = -\kappa \tan(\kappa x)$, $e^{-2\Lambda} = \dfrac{1}{\cos^2(\kappa x)}$, $g = \dfrac{\tan(\kappa x)}{\kappa}$. Asymptotically, $\lim\limits_{x\to+\infty} g = \dfrac{i}{\kappa}$. Choosing the additive constant such that $\dfrac{i}{\kappa} + \dfrac{C_2}{C_1} = 0$, we find $F = F_p + \dfrac{e^{-2\Lambda}}{g - i/\kappa} = i\kappa$. We see that although the random combination of the independent solutions corresponds to F_p and the diverging solution $F_p = \dfrac{pf'}{f} \Rightarrow f = \exp\left(\int F_p dx\right) \propto \cos(\kappa x)$, the solution corresponding to $F = i\kappa$ is $F = \dfrac{pf'}{f} \Rightarrow f = \exp\left(\int F dx\right) \propto e^{i\kappa x}$ and has the physically acceptable behavior $\lim\limits_{x\to+\infty} f = 0$. (The general solution of Equation (11) is $F_p = -\kappa \tan(\kappa x + D_1)$, $e^{-2\Lambda} = \dfrac{D_2}{\cos^2(\kappa x + D_1)}$, $g = D_2 \dfrac{\tan(\kappa x + D_1)}{\kappa} + D_3$. Repeating the process, we again find $F = i\kappa$, confirming that only a particular solution is needed, not the general solution.)

The above can be applied to other asymptotic regimes, e.g., to $x \to -\infty$. They can also be applied in the asymptotic regimes $\varpi \to \infty$ and $\varpi \to 0$ in cylindrical coordinates.

Similarly to the Schwarzian g approach in the Φ approach, Equation (15) can be written

$$F = (F_1 + iF_2) + \dfrac{2iF_2}{e^{i(\Phi+C)} - 1}, \tag{18}$$

and this can be seen as a way to split the solution into two parts. The first part, $F_1 + iF_2$, satisfies by itself the original Equation (3) if Equations (16) hold and corresponds to the diverging solution asymptotically. The second part, $\dfrac{2iF_2}{e^{i(\Phi+C)} - 1}$, can be used to find the non-diverging solution by choosing the constant C such that $e^{i(\Phi+C)} = 1$ asymptotically.

Had we chosen the Schwarzian Φ approach to solve the oscillator problem (with constant $p = 1$, $q = \kappa^2$, and $\Im\kappa > 0$), starting the integration from some point with arbitrary conditions, the solution would behave asymptotically as $\Phi \approx De^{2i\kappa x} + D_0$, $F_1 \approx -i\kappa$, $F_2 \approx i\kappa De^{2i\kappa x}$. (The general solution is $\cot\dfrac{\Phi + D_3}{2} = D_2 \cot(\kappa x + D_1)$.) The resulting $F = -i\kappa + i\kappa De^{2i\kappa x} \cot\dfrac{De^{2i\kappa x} + D_0 + C}{2}$, for $D_0 + C = 0$, would have given the physically acceptable solution $F = i\kappa$.

3.4. The Quantization Condition

Suppose we follow the Schwarzian Φ approach (the steps are trivially similar if we follow the Schwarzian g approach). Solving Equation (14) with arbitrary initial conditions, we find a sort of background for the solution for F. (We remind the reader that to find a particular solution is enough: there is no need to find the general solution.) This is because the expression for F given by Equation (15) depends on the free constant C, so by appropriately choosing this constant, we can satisfy one of the boundary conditions of the problem.

There is always a second boundary condition to satisfy, and this can be done with the shooting method, i.e., change the eigenvalue until the second condition is also satisfied (for each eigenvalue, the constant C is found from the first boundary condition).

It is of particular importance to discuss the case where one (or both) boundary is at infinity. As already discussed, the second-order differential Equation (1) has two asymptotic solutions: one diverging and unphysical and another finite and physically acceptable. Their superposition is diverging, and for this reason, it is nontrivial to find the acceptable solution. The Schwarzian approach solves the problem by splitting the solution F into two parts; see Equation (15). The part F_1 is dominated by the asymptotically diverging behavior. The inclusion of the second part of the solution $F_2 \cot\dfrac{\Phi + C}{2}$ is the way to correct things and have the sum follow the physically acceptable, finite solution. Although F_2 vanishes asymptotically (if $F_1 = pf_1'/f_1$, we get $F_2 \propto 1/f_1^2$ from the middle from Equation (16), and thus, if f_1 diverges, F_2 vanishes), this is achieved by requiring infinite $\cot\dfrac{\Phi + C}{2}$, i.e., have a "quantization" condition of $\Phi + C$ being equal to an even multiple of π asymptotically.

From a different point of view related to causality, if we want the two boundary conditions to "communicate", the term in Equation (15) that includes C, which contains the information of the first boundary condition, should remain present at the location of the second boundary condition. If Φ' vanishes, this is possible only if $\cot\dfrac{\Phi + C}{2}$ diverges.

Another way to reach the same result is by observing Equations (13) or (17). If Φ' vanishes asymptotically, the only way to have finite f is by requiring $\sin\dfrac{\Phi + C}{2}$ to be zero as well.

Lastly, looking Equation (6), the second part remains important and corrects the diverging behavior of the first if, asymptotically, $e^{i(\Phi + C)} = 1$.

For all these reasons, the related boundary condition is:

$$\text{if } \Phi'_{\text{BC}} = 0 \text{ then } \Phi_{\text{BC}} + C = 2n\pi, n \in \mathbb{Z}. \tag{19}$$

If the two boundaries are at $\pm\infty$, then the quantization condition is that $\Phi + C$ is equal to an even multiple of π at both ends, so the difference is also an even multiple of π. Instead of both conditions, the eigenvalues can be found using the following single quantization condition:

$$\text{if } \Phi'|_{1,2} = 0 \text{ then } \Phi|_2 - \Phi|_1 = 2n\pi, n \in \mathbb{Z}. \tag{20}$$

The corresponding quantization conditions in the Schwarzian g approach are:

$$\text{if } g'_{\text{BC}} = 0 \text{ then } g_{\text{BC}} + C_2/C_1 = 0, \tag{21}$$

$$\text{if } g'|_{1,2} = 0 \text{ then } g|_2 - g|_1 = 0. \tag{22}$$

The quantization conditions, and the Schwarzian approaches in general, offer a convenient way to solve the problem exactly in cases where the regime of interest extends to asymptotic regions where all functions approach constant values. This applies to $x = \pm\infty$ in Cartesian geometry, but it also applies to $\varpi \to 0, \infty$ in cylindrical geometry and to $r \to 0, \infty$ in spherical geometry (with ϖ and r the cylindrical and spherical radius, respectively). When the regime of interest does not include asymptotic regions, it is sufficient to follow the minimalist approach of Section 2. Of course Schwarzian approaches still can be used, but the boundary conditions are $F|_{1,2} = F_{\text{BC}}|_{1,2}$ at the two extreme values of the independent variable (x or ϖ or r) and not the quantization conditions.

4. Examples in Quantum Mechanics

4.1. Quantum Morse Potential

The Schrödinger equation $-\dfrac{\hbar^2}{2m}\Psi'' + (V - E)\Psi = 0$ is a Sturm–Liouville problem, and with proper normalization, the Morse potential corresponds to the choice $p = 1$, $q = \varepsilon - \lambda^2(1 - e^{-x})^2$ (the origin has been transferred to the minimum of the potential). The eigenvalues can be easily found following the Scharzian Φ approach, i.e., integrating Equation (16) starting from $x = 0$, moving toward positive and negative x, and requiring the difference $\dfrac{\Phi|_{+\infty} - \Phi|_{-\infty}}{2\pi}$ to be an integer.

Figure 1 shows the numerical results for $\lambda = 5$. Integration in the region $-7 < x < 15$ is sufficient to give the eigenvalues (we need to resolve the region in which F_2 remains practically nonzero and not the whole x axis). The results agree with the known expression for the eigenvalues $\varepsilon = \lambda^2 - (\lambda - n - 1/2)^2$ with $n = 0, 1, 2, \dots$ and positive $\lambda - n - 1/2$ (corresponding to bound states with energy less than V_∞); see, e.g., [6]. The quantum number n corresponds to the integer $\dfrac{\Phi|_{+\infty} - \Phi|_{-\infty}}{2\pi} - 1$.
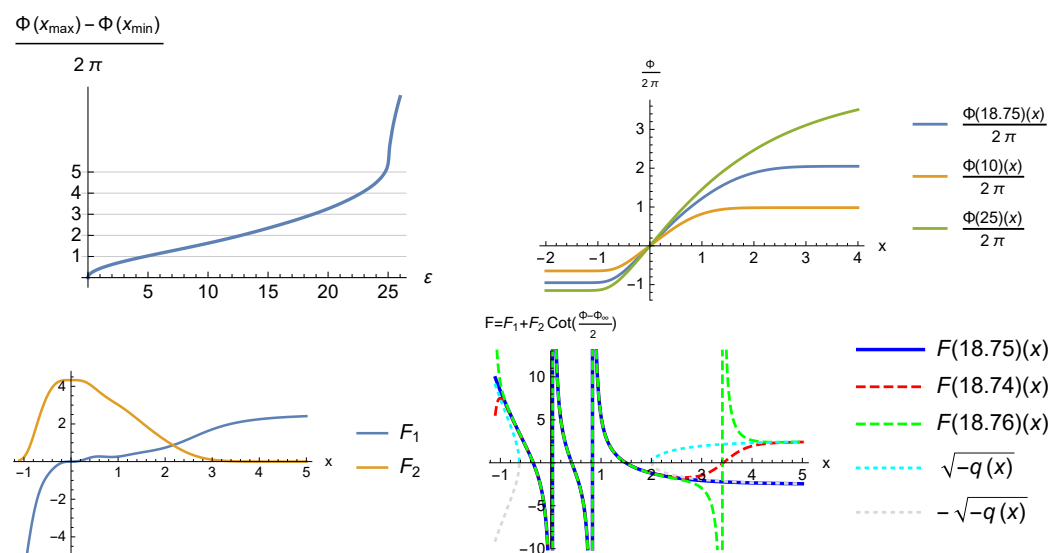

Figure 1. Solution for the Morse potential with $\lambda = 5$ for initial conditions $\Phi(0) = 0$, $F_1(0) = 0$, $F_2(0) = \sqrt{q(0)}$. Top left panel: the value of $\dfrac{\Phi|_{+\infty} - \Phi|_{-\infty}}{2\pi}$ as a function of ε (integer values correspond to the accepted eigenvalues). Top right panel: typical behavior of $\Phi(x)$ (shown for various values of the eigenvalue ε). Bottom left panel: the functions F_1 and F_2 for the third eigenvalue $\varepsilon = 18.75$ (corresponding to $n = 2$). Bottom right panel: the function F for the eigenvalue $\varepsilon = 18.75$ and two neighboring values of ε. The function F approaches asymptotically one of the $\pm\sqrt{-q}$, which are also shown. At large x, the blue curve follows the acceptable branch with $f'/f \approx -\sqrt{-q(x)}$, while the red and green curves follow the unphysical branch with $f'/f \approx \sqrt{-q(x)}$.

The bottom right panel of Figure 1 shows the function F. Asymptotically, according to Equation (3), we expect the function F to approach either $+\sqrt{-pq}$ or $-\sqrt{-pq}$. As $x \to \infty$, these correspond to positive/negative f'/f, i.e., an exponentially increasing/decreasing f, respectively. Any superposition of the two solutions of the second-order differential Equation (1) is dominated by the exponentially increasing part and is unacceptable as a physical solution of the problem. The acceptable solution corresponds to cases where the exponentially increasing part is absent, and this is what happens for the eigenvalues. Indeed, as shown in the figure, the blue curve, which corresponds to the eigenvalue $n = 2$, follows the acceptable branch $F \to -\sqrt{-q}$, while the red and green curves, which correspond to slightly smaller and larger values of ε, respectively, follow the unphysical branch $F \to +\sqrt{-q}$. Similarly, we understand the behavior of F in the asymptotic regime $x \to -\infty$.

The bottom left panel of Figure 1 shows the functions F_1 and F_2. The former corresponds to a superposition that includes the unphysical branch and behaves asymptotically as $\approx +\dfrac{x}{|x|}\sqrt{-pq}$, while the latter approaches zero at $x \to \pm\infty$. Nevertheless, its contribution is crucial and results in $F = F_1 + F_2 \cot \dfrac{\Phi + C}{2}$ following the physically acceptable branch if ε is an eigenvalue.

The eigenvalues do not depend on the initial conditions at $x = 0$. Our choice in Figure 1 was $F_1(0) = 0$, $F_2(0) = \sqrt{q(0)}$ such as $F_1'(0) = 0$, and thus, the value of Φ' remains as close to a constant as possible ,since $\Phi''(0) = 0$ and $\Phi'''(0) = 0$. In the general case, the choice $\Phi'' = 0$, $\Phi''' = 0$ at the initial point is possible and corresponds to initial values $F_1 = -\dfrac{p'}{2}$, $F_2 = p\kappa$. This is not necessary, but it is a convenient choice as it avoids as much as possible oscillations in the functions $F_{1,2}$. In the shown case, the oscillations in F are solely due to the tan function. On the contrary, Figure 2 shows the numerical results for different initial conditions, and we see that $F_{1,2}$ oscillate. The end result in the function F and the eigenvalues are the same, as expected, and are independent on the initial conditions of the integration. Since F is the same as in Figure 1, we show in the bottom right panel of Figure 2 the eigenfunction f (with arbitrary normalization). The blue curve corresponds to the eigenvalue, while the red and green curves correspond to two neighboring values of ε (clearly, f diverges at $x \to \pm\infty$ if ε is not an eigenvalue).
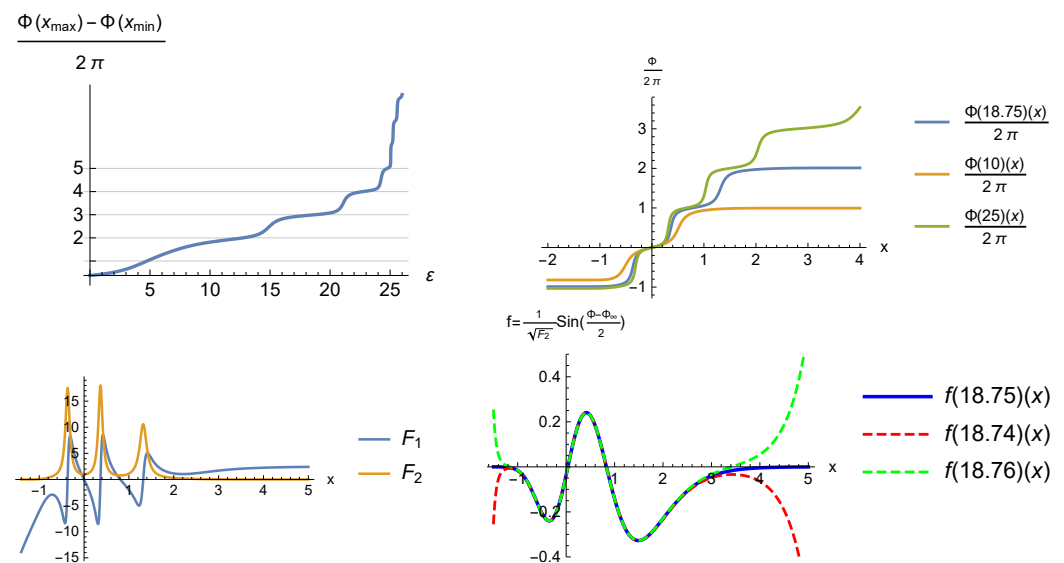

Figure 2. Same as Figure 1 but for a boundary condition $F_2(0) = 1$. The functions $\Phi(x)$, $F_1(x)$, $F_2(x)$ are affected by the choices of the conditions at $x = 0$, but the difference $\Phi|_{+\infty} - \Phi|_{-\infty}$ and the function $F(x)$ are the same for the same eigenvalue ε. The eigenfunction $f(x)$ (with arbitrary normalization) for the eigenvalue $\varepsilon = 18.75$ and two neighboring values of ε is shown in the bottom right panel.

The same results can be found using the Schwarzian g approach. The problem of the oscillations that make F infinity at points where $f = 0$ can be circumvented by working in the complex domain. Figure 3 shows the results of the integration of Equation (11) in the interval $-7 < x < 15$ with initial conditions $g(0) = 0$, $\Lambda(0) = 0$, and $F_p(0) = i\sqrt{q(0)}$. The resulting F, given by Equation (10) with $C_2/C_1 = -g|_{-\infty}$, is the same as the one found using the Φ approach. The eigenfunction f (given by Equation (12)) is also the same, apart from a complex multiplication constant.
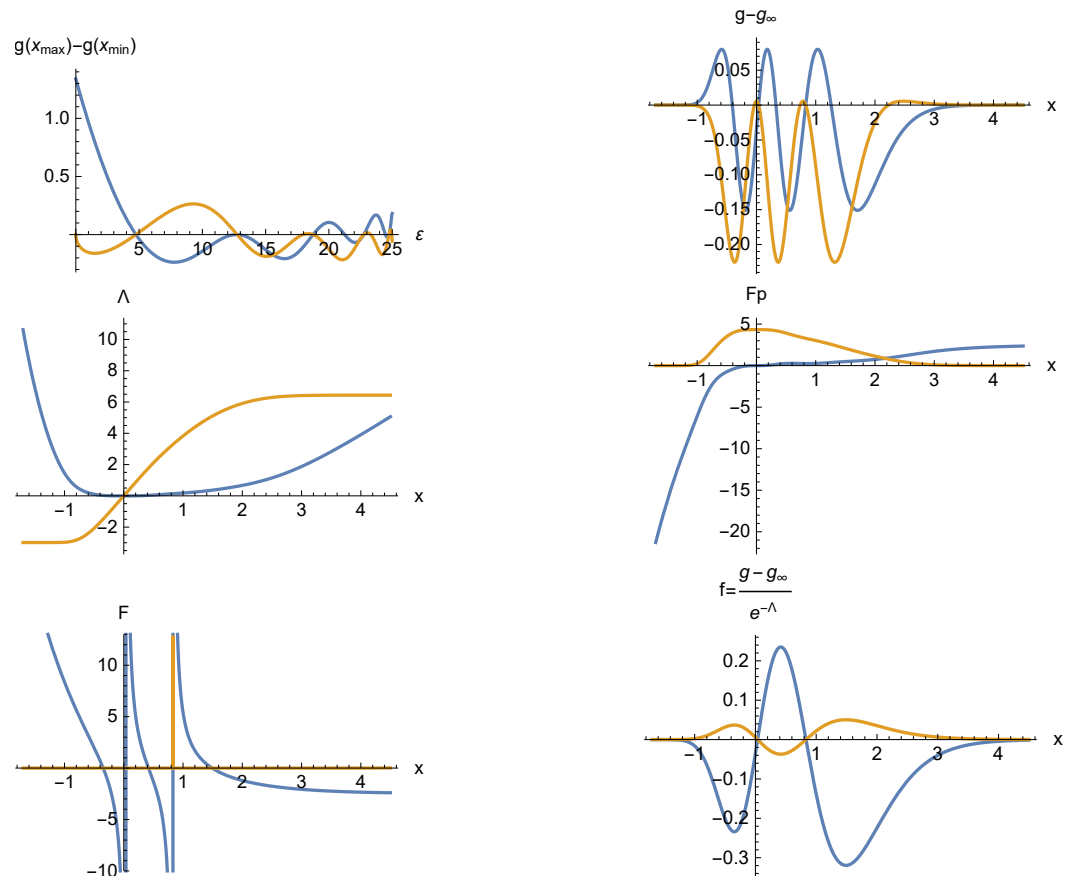

Figure 3. Solution for the Morse potential using the Schwarzian g approach. In all panels, the real part of the functions are shown with blue lines, and the imaginary parts are shown with orange lines. The eigenvalues for which $g|_{+\infty} - g|_{-\infty} = 0$ (practically, $g(x_{\max}) - g(x_{\min}) = 0$ with $x_{\min} = -7$, $x_{\max} = 15$) can be seen in the top left panel. The other panels correspond to the solution for the third eigenvalue $\varepsilon = 18.75$.

4.2. Quantum Harmonic Oscillator

The Schrödinger equation for a harmonic potential corresponds to the choice $p = 1$, $q = 2\varepsilon - x^2$ (with proper normalization). Following the Schwarzian Φ approach, integrating Equation (16) starting from $x = 0$ and requiring the difference $\dfrac{\Phi|_{+\infty} - \Phi|_{-\infty}}{2\pi}$ to be an integer, we find the expected eigenvalues $\varepsilon = n + 1/2$ as shown in Figure 4.
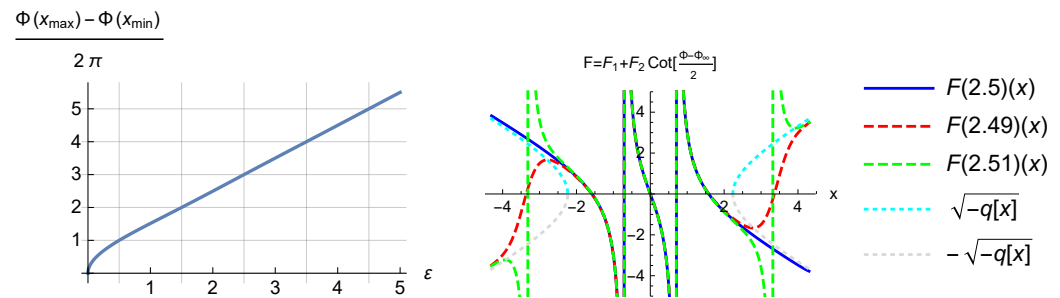
Figure 4. Solution for the quantum harmonic oscillator. Left panel: the value of $\dfrac{\Phi|_{+\infty} - \Phi|_{-\infty}}{2\pi}$ as a function of the eigenvalue. Right panel: the function F for the eigenvalue $\varepsilon = 2.5$ (in blue) and two neighboring values (in red and green). The function F approaches asymptotically one of the $\pm\sqrt{-q}$, which are also shown. The blue curve follows the acceptable branch with $f'/f \approx -x$ at large $|x|$, while the red and green curves follow the unphysical branch with $f'/f \approx x$.

4.3. Paine Problem

This is a test spectral problem corresponding to $p = 1$, $q = \lambda - \dfrac{1}{(x + 0.1)^2}$ with boundary conditions $f(0) = f(\pi) = 0$; see Ref. [2]. Since the region of interest does not contain asymptotic regimes, it is sufficient to use the minimalist approach (and not the Schwarzian). To avoid infinities at possible points where $f = 0$, especially since we work in real space, it is better to use the method described at the end of Section 2. The boundary conditions are $F(0) = F(\pi) = \infty$. Adopting the simplest possible choice $F_1(x) = 0$, $F_2(x) = 1$, the boundary conditions are $\cot\dfrac{\Phi(0)}{2} = \cot\dfrac{\Phi(\pi)}{2} = \infty$. Equation (5) can be integrated in the interval $x \in [0, \pi]$ starting with $\Phi(0) = 0$. The second condition corresponds to $\Phi(\pi) = 2n\pi$ with integer n: a condition that gives the eigenvalues. The numerical results are shown in Figure 5. The eigenvalues are $\lambda \approx 1.51987, 4.94331, 10.2847,$ $17.5599, 26.7828, 37.9643, 51.1131, 66.2361, 83.3385, 102.424, 123.497, 146.558, 171.611,$ $198.657, \ldots$
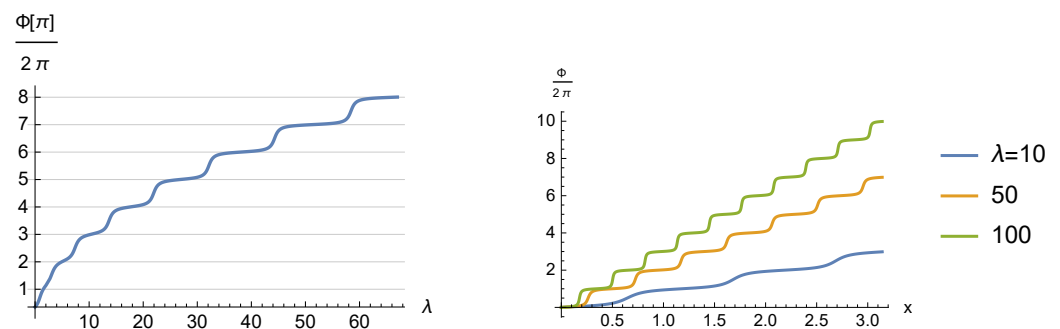

Figure 5. Left panel: the value of $\dfrac{\Phi(\pi)}{2\pi}$ as a function of the eigenvalue. Right panel: typical behavior of $\Phi(x)$ (shown for various values of the eigenvalue λ).

5. Stability Problems

The stability problems in fluid or plasma dynamics can be formulated using two functions: y_1, related to the Lagrangian displacement of fluid elements, and y_2, related to the perturbation of the total pressure. The corresponding system of differential equations can be found, e.g., in Equation (27) of Ref. [4]

$$\frac{d}{d\varpi}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\mathcal{D}}\begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} \\ \mathcal{F}_{21} & \mathcal{F}_{22} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0. \tag{23}$$

This applies to non-relativistic and relativistic ideal magnetohydrodynamic cases in cylindrical geometry in which the unperturbed state depends only on the cylindrical radius

ϖ; a similar system of equations exists in other geometries as well, e.g., in Cartesian geometry with the unperturbed state depending on x (actually, the equations of [4] can be directly simplified to that case if we ignore terms that depend on the curvature of the coordinate system).

Instead of system (23) of the two first-order differential equations for y_1 and y_2, it is equivalent to work with one second-order differential equation for either y_1 or y_2, i.e., use either Equations (32) or (33) of [4]:

$$y_1'' + \left[\frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{\mathcal{D}} + \frac{\mathcal{F}_{12}}{\mathcal{D}} \left(\frac{\mathcal{D}}{\mathcal{F}_{12}} \right)' \right] y_1' + \left[\frac{\mathcal{F}_{11}\mathcal{F}_{22} - \mathcal{F}_{12}\mathcal{F}_{21}}{\mathcal{D}^2} + \frac{\mathcal{F}_{12}}{\mathcal{D}} \left(\frac{\mathcal{F}_{11}}{\mathcal{F}_{12}} \right)' \right] y_1 = 0, \quad y_2 = -\frac{\mathcal{D}y_1' + \mathcal{F}_{11}y_1}{\mathcal{F}_{12}}, \tag{24}$$

$$y_2'' + \left[\frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{\mathcal{D}} + \frac{\mathcal{F}_{21}}{\mathcal{D}} \left(\frac{\mathcal{D}}{\mathcal{F}_{21}} \right)' \right] y_2' + \left[\frac{\mathcal{F}_{11}\mathcal{F}_{22} - \mathcal{F}_{12}\mathcal{F}_{21}}{\mathcal{D}^2} + \frac{\mathcal{F}_{21}}{\mathcal{D}} \left(\frac{\mathcal{F}_{22}}{\mathcal{F}_{21}} \right)' \right] y_2 = 0, \quad y_1 = -\frac{\mathcal{D}y_2' + \mathcal{F}_{22}y_2}{\mathcal{F}_{21}}. \tag{25}$$

As shown in Appendix B, these can be written as a Sturm–Liouville problem and solved using a Schwarzian approach.

There are two essential differences compared to the classical Sturm–Liouville problems, of e.g., quantum mechanics, for the examples examined so far. The first is that the eigenvalue enters nonlinearly in the problem and in both functions p and q. The second is that we work in the complex domain, i.e., all known and unknown functions are complex. This is unavoidable if we are interested in finding unstable modes, in which case the eigenvalue is a complex number. (One can follow either a temporal or a spatial approach, which correspond to a real wavevector and a complex frequency or a real frequency and a complex wavevector, respectively). The methods remain the same though.

There are many variants of the procedure: working with y_1 and the Schwarzian g approach, with y_1 and the Schwarzian Φ approach, with y_2 and the Schwarzian g approach, or with y_2 and the Schwarzian Φ approach. All are equivalent to each other. One could think that the Schwarzian Φ approaches describe more naturally oscillations of the eigenfunctions, but this is not the case. In the complex domain, the oscillations do not lead to zeros of the unknown complex function, since this requires its real and imaginary parts to vanish simultaneously. On the contrary, g approaches are slightly preferable because the corresponding differential equations are simpler and, more importantly, the quantization condition does not involve a trigonometric function (as a result, it does not contain an arbitrary integer). In real space, as in most examples presented so far, the g approach is problematic because the function F becomes infinite at some points. In the complex domain though, this happens only at poles, which are automatically circumvented during the numerical integration. (When we integrate a differential equation in the complex domain, we need infinite accuracy to hit a pole: something impossible numerically. For this reason, the integration passes without problem close to poles. See a related discussion in Appendix C of [5].)

As explained in [5], only the ratio $Y = \dfrac{y_1}{y_2}$ is involved in the boundary conditions that determine the dispersion relation of a stability problem, not the functions y_1, y_2 separately. The function Y is always continuous, even at interfaces (discontinuities) of the unperturbed state. For this reason, if we choose the Sturm–Liouville for y_1, it is advantageous to replace y_1'/y_1 with the continuous function Y using the second from Equation (24). Similarly, if we choose the Sturm–Liouville for y_2, it is advantageous to replace y_2'/y_2 with the continuous function Y using the second from Equation (25). Essentially, we follow the minimalist approach of Ref. [5] and work with Y alone by integrating the equation

$$\frac{dY}{d\varpi} = \frac{\mathcal{F}_{21}}{\mathcal{D}} Y^2 + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}} Y - \frac{\mathcal{F}_{12}}{\mathcal{D}}. \tag{26}$$

The Schwarzian approach splits the Y function in two parts and helps with expressing the asymptotic boundary conditions in a convenient way.

Details and all the needed equations for all variants can be found in Appendix B. The asymptotic behavior near the axis is discussed in Appendix C. According to this, the formulation with y_2 should be avoided in the case $m = 0$ when the quantization condition is applied on the axis. All other variants are equivalently applicable.

Here, we summarize the procedure for two variants in order to use them in an example case.

5.1. Formulation with y_1 and the Schwarzian Φ Approach

In this case, the ratio Y and the eigenfunctions are given by

$$\frac{1}{Y} = \mathcal{Y}_4 - \mathcal{Y}_3 \cot \frac{\Phi_1 + C}{2}, \quad y_1 \propto \frac{1}{\sqrt{\mathcal{Y}_3 \, e^{\int \frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{\mathcal{D}} d\varpi}}} \sin \frac{\Phi_1 + C}{2}, \quad y_2 = \frac{1}{Y} y_1, \tag{27}$$

where Φ_1, $\mathcal{Y}_3$, and $\mathcal{Y}_4$ are a particular solution (with arbitrary initial conditions) of the system of Equations

$$\mathcal{Y}_4' = -\frac{\mathcal{F}_{21}}{\mathcal{D}} - \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}} \mathcal{Y}_4 + \left(\mathcal{Y}_4^2 - \mathcal{Y}_3^2\right)\frac{\mathcal{F}_{12}}{\mathcal{D}}, \quad \mathcal{Y}_3' = 2\mathcal{Y}_3 \mathcal{Y}_4 \frac{\mathcal{F}_{12}}{\mathcal{D}} + \frac{\mathcal{F}_{11} - \mathcal{F}_{22}}{\mathcal{D}} \mathcal{Y}_3, \quad \Phi_1' = 2\mathcal{Y}_3 \frac{\mathcal{F}_{12}}{\mathcal{D}}. \tag{28}$$

The application to each problem with specific boundary conditions is obvious:

- Suppose we have a jet in $0 \leq \varpi \leq \varpi_j$ with known Y_{BC} at ϖ_j^+. We integrate Equation (28) starting from ϖ_j with arbitrary conditions for Φ_1, $\mathcal{Y}_3$, $\mathcal{Y}_4$, and the free additive constant C chosen as $Y|_{\varpi_j} = Y_{BC}$ is satisfied, with $Y|_{\varpi_j}$ given by the first from Equation (27). The eigenvalues are found from the quantization condition on the axis $\sin \frac{\Phi_{1axis} + C}{2} = 0$ (the condition $\mathcal{Y}_3 \to 0$ on the axis will be automatically satisfied by the solution of the differential equations). Note that in case of internal discontinuities of the unperturbed state inside the jet, we keep integrating the equations passing each of them while keeping Φ_1, $\mathcal{Y}_3$, $\mathcal{Y}_4$ continuous.
- Suppose we want to find numerically the solution for $0 \leq \varpi \leq \infty$. We integrate Equation (28) starting from an arbitrary point with arbitrary conditions for Φ_1, $\mathcal{Y}_3$, $\mathcal{Y}_4$ toward both directions, i.e., toward $\varpi = 0$ and toward $\varpi = \infty$. The quantization condition $\sin \frac{\Phi_{1\infty} - \Phi_{1axis}}{2} = 0$ gives the eigenvalues. If we additionally need to find the eigenfunctions, we specify the constant C from either $\sin \frac{\Phi_{1\infty} + C}{2} = 0$ or $\sin \frac{\Phi_{1axis} + C}{2} = 0$.

5.2. Formulation with y_1 and the Schwarzian g Approach

In this case, the ratio Y and the eigenfunctions are given by

$$\frac{1}{Y} = \mathcal{Y}_4 - \frac{e^{-2\mathcal{Y}_3}}{g_1 + C_2/C_1}, \quad y_1 \propto \frac{g_1 + C_2/C_1}{e^{-\mathcal{Y}_3 + \int \frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{2\mathcal{D}} d\varpi}}, \quad y_2 = \frac{1}{Y} y_1, \tag{29}$$

where $\mathcal{Y}_4$, $\mathcal{Y}_3$, and g_1 are a particular solution (with arbitrary initial conditions) of the Equations

$$\mathcal{Y}_4' = -\frac{\mathcal{F}_{21}}{\mathcal{D}} - \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}} \mathcal{Y}_4 + \frac{\mathcal{F}_{12}}{\mathcal{D}} \mathcal{Y}_4^2, \quad \mathcal{Y}_3' = -\mathcal{Y}_4 \frac{\mathcal{F}_{12}}{\mathcal{D}} + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{2\mathcal{D}}, \quad g_1' = \frac{\mathcal{F}_{12}}{\mathcal{D}} e^{-2\mathcal{Y}_3}. \tag{30}$$

The application to each problem with specific boundary conditions can be done following the same steps as in the Φ approach, with the boundary conditions on the axis

and at infinity replaced by $g_{1axis} + C_2/C_1 = 0$, and $g_{1\infty} + C_2/C_1 = 0$, respectively. (If both apply, the eigenvalues can be found by the single condition $g_{1\infty} - g_{1axis} = 0$.)

6. An Example for the Stability of Astrophysical Jets

In this section, we apply the Schwarzian approach to the stability of a particular non-relativistic jet model considered by Cohn in Ref. [7]. The reason behind this choice is mostly connected to the fact that there are analytical solutions for some choices of the parameters, and we can use them to test the results of the new method.

In the unperturbed state, we assume that there is a uniform cylindrical jet of radius ϖ_j consisting of a hydrodynamic fluid with constant polytropic index $\Gamma = 5/3$, density ρ_j, sound velocity c_{sj}, and pressure $P_j = \dfrac{\rho_j c_{sj}^2}{\Gamma}$ moving with bulk velocity $V_j \hat{z} = M c_{sj} \hat{z}$ ($\hat{z}$ is the symmetry axis, and we use cylindrical coordinates ϖ, ϕ, z). The environment of the jet is assumed to be a cold, static, ideal, magnetized plasma with constant density $\rho_e = \eta \rho_j$ and an azimuthal magnetic field $\boldsymbol{B} = \dfrac{I}{\varpi} \hat{\phi}$ with constant I. This field corresponds to a surface current at the interface $\varpi = \varpi_j$ whose value is connected to the internal pressure through the pressure balance $P_j = \dfrac{I^2}{2\varpi_j^2}$.

We use units for which lengths are measured in ϖ_j, wavelengths in $1/\varpi_j$, velocities in c_{sj}, frequencies and growth rates in c_{sj}/ϖ_j, and densities in ρ_j, and a factor $\sqrt{4\pi}$ is absorbed in the magnetic field (Lorentz–Heaviside units). The expressions of the various $\mathcal{F}_{ij}/\mathcal{D}$ needed for this non-relativistic ideal magnetohydrodynamic stability problem can be found in Section 5.6 of Ref. [4] and are summarized in Appendix D.

The dimensionless parameters that fully determine the unperturbed state are the Mach number M in the jet (velocity of the jet in units of the sound velocity in the jet) and the density ratio η (density of the environment over the density of the jet). We assume in the following $M = 1$ and $\eta = 0.01$.

Suppose we follow the temporal approach and need to find the eigenvalues and eigenfunctions of unstable modes ($\propto e^{i(m\phi + kz - \omega t)}$ with $\Im \omega > 0$) corresponding to wavelength $k = \pi$, $m = 0$. Applying the Schwarzian g approach, we start the integration of Equation (30) from the jet surface toward smaller and larger ϖ. The eigenvalues ω are the roots of the quantization condition $g_{1\infty} - g_{1axis} = 0$. Following the method presented in Section 3 of Ref. [5], the roots of this complex function can be found through the isocontours of $\Psi = \text{Arg}[g_{1\infty} - g_{1axis}]$ in the ω plane: the so-called Spectral Web. This term was introduced by [3] for a similar map showing "solution paths" and "conjugate paths", which correspond to particular isocontours of Ψ, and was generalized by [5] to the map of Ψ in which the roots can be distinguished from poles. The roots are seen as positive line charges in this map, i.e., points of discontinuities around which the Ψ increases as we move counterclockwise. Poles of the complex function are also seen in the map as negative line charges, and they can be distinguished from the roots because around poles, the Ψ decreases as we move counterclockwise.

The left upper panel of Figure 6 shows the resulting Spectral Web. Evidently, there is one root $\omega \approx 3.08 + i1.97$. (Note that any choice of initial conditions at the starting point of the integration leads to the same result for the roots, i.e., the eigenvalues, although the Spectral Web and the positions of poles—if they exist—in general differ).

The left middle panel of Figure 6 shows the Spectral Web resulting from the boundary condition $Y|_{\varpi = \varpi_i^+} - Y|_{\varpi = \varpi_i^-} = 0$, which is simply the continuity of Y at the jet surface, using the analytical expressions that exist for this particular model. (The analytical solution for the jet interior $\varpi < \varpi_j$ corresponding to the finite eigenfunction on the axis is $Y = -\dfrac{\lambda \varpi}{\rho_j \omega_0^2} \dfrac{J_1(\lambda \varpi)}{J_0(\lambda \varpi)}$, where $\lambda = \sqrt{\omega_0^2/c_{sa}^2 - k^2}$, and $\omega_0 = \omega - kV_j$. For the environment $\varpi > \varpi_j$, the solution that represents an outgoing wave with a decreasing amplitude has

$$Y = \frac{\omega^4}{I^2}\left[-\frac{z^2 U}{(4-z)U + 2z\,dU/dz} - 1\right]^{-1},\ \text{where } z = \frac{-i\omega}{|I|/\sqrt{\rho_e}}\omega^2 \text{ and } U(a,b,z) \text{ is Tricomi's}$$

(confluent hypergeometric) function satisfying the Kummer equation $z\dfrac{d^2 U}{dz^2} + (b-z)\dfrac{dU}{dz} - aU = 0$ with $a = \dfrac{3}{2} + i\dfrac{k^2|I|/\sqrt{\rho_e}}{4\omega}$, and $b = 3$. In addition to Ref. [7], details for the interior and exterior solutions, respectively, can be found in Sections 5.1 and 5.4.2 of [4] (adapting these results to the non-relativistic present case).) The resulting eigenvalue is identical to the one found using the Schwarzian approach.
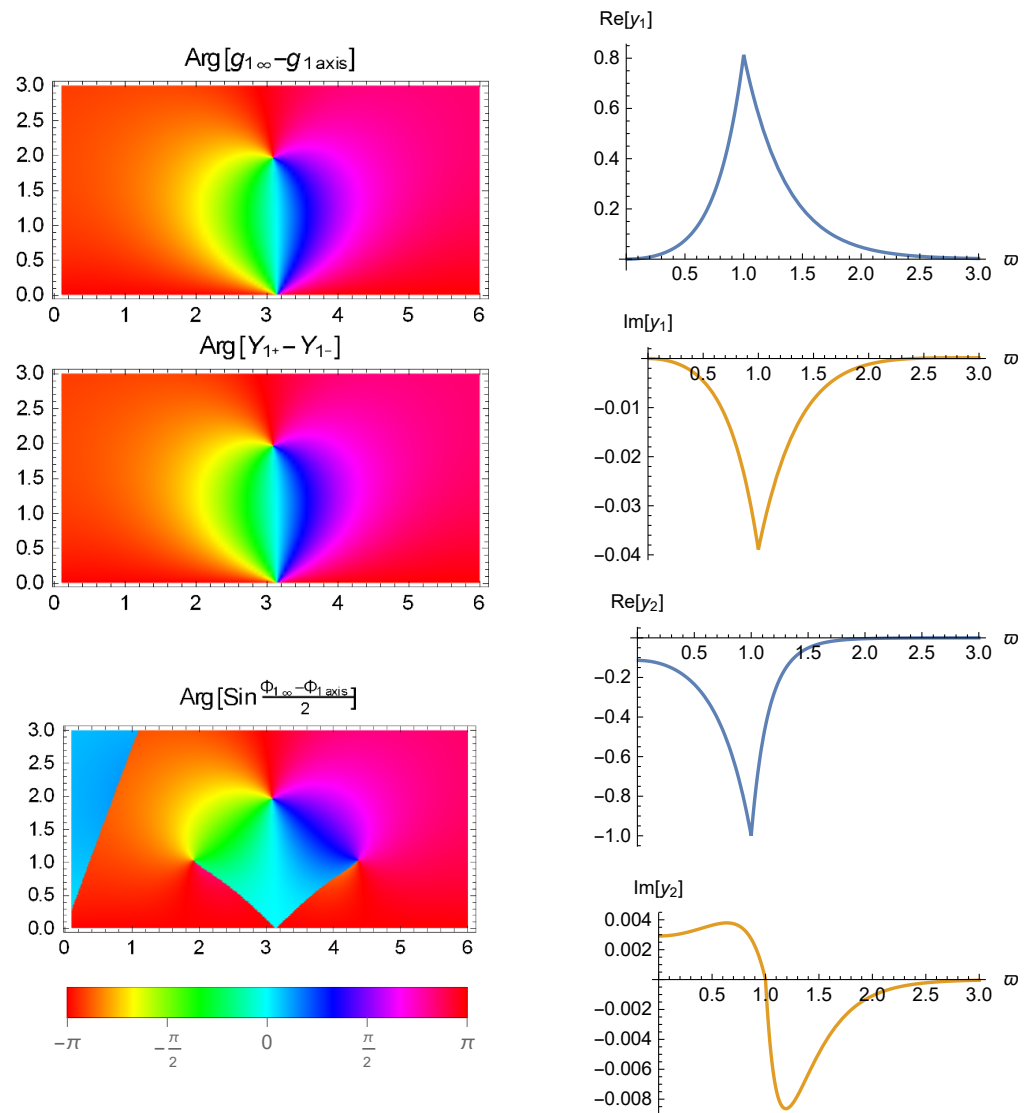

Figure 6. Left column, upper panel: the Spectral Web for the Schwarzian g approach starting the integration from the jet surface with initial conditions $g|_{\varpi=1} = 0$, $\mathcal{Y}_3|_{\varpi=1} = 0$, $\mathcal{Y}_4|_{\varpi=1} = 0$. Left column, middle panel: the Spectral Web using the analytical expressions of Y inside and outside the jet. Left column, lower panel: the Spectral Web for the Schwarzian Φ approach starting the integration from the jet surface with initial conditions $\Phi_1|_{\varpi=1} = 0$, $\mathcal{Y}_3|_{\varpi=1} = 1$, $\mathcal{Y}_4|_{\varpi=1} = 0$. Right column: the eigenfunctions for the eigenvalue $\omega \approx 3.08 + i1.97$ with arbitrary normalization (we can freely multiply both $y_1 = \Re y_1 + i\Im y_1$ and $y_2 = \Re y_2 + i\Im y_2$ with the same arbitrary complex constant—their ratio Y is uniquely defined).

The eigenfunction can be directly found from Equation (29) by knowing the eigenvalue and choosing $C_2/C_1 = -g_{1axis}$. The result is shown in the right column of Figure 6.

Note that in both asymptotic regimes, near the axis and at large distances, the function e^{-y_3} vanishes and the eigenfunctions approach constant values. Thus, it is sufficient to perform the integration in the region where these functions vary significantly. For the case shown in Figure 6, it was done in the interval $0.01 < \varpi < 10$.

Applying also the Schwarzian Φ approach, the eigenvalues ω are the roots of the quantization condition $\sin \dfrac{\Phi_{1\infty} - \Phi_{1axis}}{2} = 0$. The bottom left panel of Figure 6 shows the corresponding Spectral Web. Apart from the root at $\omega \approx 3.08 + i1.97$, in this case, there are two poles and some curves of discontinuities that correspond to crossing of poles during the integration. The existence and the positions of the poles and the curves of discontinuity depend on the initial conditions, but the root is always the same.

The eigenfunction Y corresponding to a particular eigenvalue is the same no matter which approach we use to find it. The eigenfunction y_1 is also the same apart from a multiplication complex constant, which is free due to the linearity of the problem. The same multiplication constant appears in y_2: in agreement with the ratio $Y = y_1/y_2$ being uniquely determined.

Repeating the process for other values of k, we find the dispersion relation shown in the left panel of Figure 7.
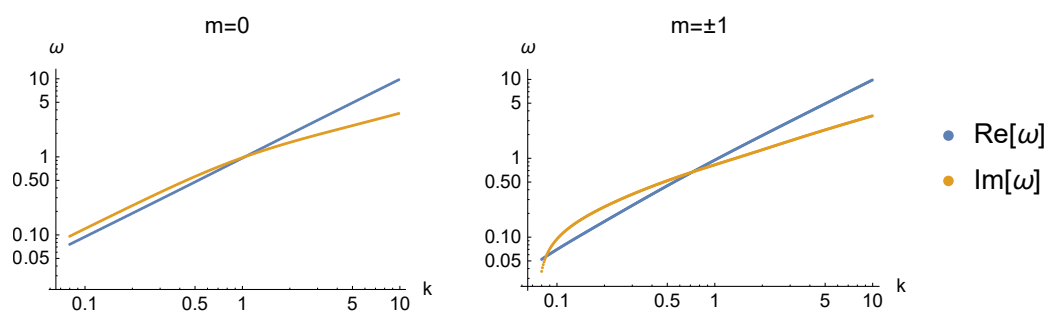

Figure 7. The dispersion relation for the model with $M \equiv \dfrac{V_j}{c_{sj}} = 1$, $\eta \equiv \dfrac{\rho_e}{\rho_j} = 0.01$, and $m = 0$ (**left**) and $m = \pm 1$ (**right**).

Using the Schwarzian approach, we can extend the results in the parametric space to values for which no analytical solutions exist. It is beyond the scope of this paper to carry out this interesting task and analyze the physics of the solutions; we only show the dispersion relation for the $m = \pm 1$ non-axisymmetric modes in the right panel of Figure 7 (the result is independent of the sign of m in this particular case).

7. Conclusions

In this work, we develop a novel way to solve Sturm–Liouville problems by following the minimalist/Schwarzian approach, which is particularly useful when the boundaries are at asymptotic distances. The minimalist approach is a first step toward economy and efficiency since it reduces the differential equation to first-order. The second step is the Schwarzian approach. Although the resulting differential equation is nonlinear, the properties of the Schwarzian derivatives allow us to split its general solution into two parts. The first part is any particular solution, and the form of the second part can be used to explore under which conditions the sum is non-diverging asymptotically.

Essentially, the Schwarzian approach finds the non-diverging solution from any particular solution and, simultaneously, the eigenvalues for which this is possible. It can be seen as an alternative to other methods to choose the non-diverging solution of a second-order linear differential equation, e.g., the termination of series found using the Frobenius method in quantum mechanics, which, however, can be applied to simple cases only. The new approach can be applied to any case and to stability problems as well, in which the eigenvalue enters nonlinearly in both functions p and q. The application of the method requires numerical integration of ordinary differential equations; in cases where the boundaries are

at asymptotic distances, it is sufficient to continue the integration as long as the resulting unknown functions vary significantly.

Two Schwarzian approaches are presented, the g and Φ, that correspond to different ways to split the general solution into two parts. Although they are mathematically equivalent, they have some practical differences. The g approach is simpler, but it cannot trivially handle the infinities in f'/f when the function f is real and oscillates, as in the examples presented in relation to the Schrödinger equation in quantum mechanics. Nevertheless, as we saw in the example for the quantum Morse potential, it can still be used by working in the complex domain, even though the eigenvalues are real. When f is a complex function, as in stability problems, there is no problem with infinities, since zero for a denominator (pole of f'/f) requires fine-tuning to make both parts, real and imaginary, vanish simultaneously.

The Φ approach can be seen as a transformation of the g approach, but it successfully handles the mentioned infinities in f'/f when the function f is real and oscillates. One could think that the function Φ is connected with the phase of the oscillations. This is true in some cases, but not always. The form of the corresponding eigenfunction $\propto \dfrac{1}{\sqrt{F_2}}\sin\dfrac{\Phi+C}{2}$ represents the superposition of two oppositely moving waves, but in general, the amplitude is highly variable, so the result is not a standing wave in general. It is an open question if other Schwarzian approaches, corresponding to different transformations and different ways of splitting the general solution into two, can be more closely related to the oscillations of the eigenfunctions, especially in stability problems. Another interesting question is related to the freedom to choose the initial condition that determines the particular solution. Is there an optimal choice?

Besides questions related to the deeper understanding of the symmetries behind the quantization condition and the somewhat unexpected existence of the method itself, its practical use is obvious. It helps with solving difficult problems with relative ease without needing to work the asymptotic behaviors, e.g., near the symmetry axis and near infinity in cylindrical jet problems, and without the need to integrate the equations at very small and very large distances.

Appendix A. Properties of the Schwarzian Derivative

The Schwarzian derivative of a function $g(x)$—in general, complex and with a complex argument—is defined as

$$\{g, x\} \equiv \frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2, \tag{A1}$$

where prime denotes the derivative with respect to x.

An important property of this derivative is that if G_1 and G_2 are two linearly independent solutions of

$$G'' + \kappa^2 G = 0 \tag{A2}$$

with κ a function of x, then their quotient $g = \dfrac{G_1}{G_2}$ satisfies the Schwarz equation

$$\{g, x\} = 2\kappa^2. \tag{A3}$$

Conversely, if g is a solution of $\{g, x\} = 2\kappa^2$, then $G_1 \propto \dfrac{1}{\sqrt{(1/g)'}}$ and $G_2 \propto \dfrac{1}{\sqrt{g'}}$ are two independent solutions of $G'' + \kappa^2 G = 0$. Thus, the general solution of Equation (A2) is

$$G = \frac{C_1 g + C_2}{\sqrt{g'}}. \tag{A4}$$

The proof can be done as in Ref. [8]. Direct substitution shows that $G_{1,2}$ are solutions of Equation (A2). Also, by substitution, we find that their Wronskian $G_1 G_2' - G_2 G_1'$ is constant, so they are indeed linearly independent.

In fact, every second-order linear differential equation can be written in the form of Equation (1) and can be transformed to a "variable frequency oscillator" in the form of Equation (6), and its general solution can be written through a particular solution of the equation involving the Schwarzian, as in Equation (8).

Some other useful properties follow (a, b, c, d are complex constants):

$$\left\{ \frac{ax + b}{cx + d}, x \right\} = 0 ; \tag{A5}$$

$$\left\{ \frac{ag + b}{cg + d}, x \right\} = \{g, x\} \tag{A6}$$

(a Möbius transformation leaves the Schwarzian unchanged);

$$\{g, x\} = \{g, \Phi\} \left(\frac{d\Phi}{dx} \right)^2 + \{\Phi, x\} \tag{A7}$$

(can be thought as a chain rule for $\{g \circ \Phi, x\}$);

$$\{g, x\} = \left\{ g, \frac{ax + b}{cx + d} \right\} \frac{(ad - bc)^2}{(cx + d)^4} \tag{A8}$$

(the chain rule for $\Phi = \dfrac{ax + b}{cx + d}$).

Appendix B. Sturm–Liouville Formulation for Stability Problems

In the next two subsections of this Appendix, we analyze the Sturm–Liouville formulation for the functions y_1 and y_2 separately. These are thought of as functions of the cylindrical radius ϖ, and a prime denotes the derivative with respect to that variable.

Appendix B.1. Formulation with y_1

Equation (24) is equivalent to a Sturm–Liouville problem for y_1:

$$y_1'' + 2\gamma_1 y_1' + b_1 y_1 = 0 \Leftrightarrow \left(e^{\int 2\gamma_1 d\varpi} y_1' \right)' + e^{\int 2\gamma_1 d\varpi} b_1 y_1 = 0 , \tag{A9}$$

$$\gamma_1 = \frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{2\mathcal{D}} + \frac{\mathcal{F}_{12}}{2\mathcal{D}} \left(\frac{\mathcal{D}}{\mathcal{F}_{12}} \right)', \quad b_1 = \frac{\mathcal{F}_{11} \mathcal{F}_{22} - \mathcal{F}_{12} \mathcal{F}_{21}}{\mathcal{D}^2} + \frac{\mathcal{F}_{12}}{\mathcal{D}} \left(\frac{\mathcal{F}_{11}}{\mathcal{F}_{12}} \right)', \tag{A10}$$

and the expression for y_2 (as function of y_1) is

$$y_2 = -\frac{\mathcal{D}}{\mathcal{F}_{12}} y_1' - \frac{\mathcal{F}_{11}}{\mathcal{F}_{12}} y_1 . \tag{A11}$$

The corresponding "variable frequency oscillator" is

$$\left(e^{\int \gamma_1 d\varpi} y_1 \right)'' + \kappa_1^2 \left(e^{\int \gamma_1 d\varpi} y_1 \right) = 0 , \quad \kappa_1^2 = b_1 - \gamma_1^2 - \gamma_1' . \tag{A12}$$

Its solution in the Schwarzian g approach is

$$y_1 = \frac{C_1 g_1 + C_2}{\sqrt{g_1' \dfrac{\mathcal{D}}{\mathcal{F}_{12}} e^{\int \frac{\mathcal{F}_{11}+\mathcal{F}_{22}}{\mathcal{D}} d\varpi}}}, \quad \{g_1, \varpi\} = \frac{g_1'''}{g_1'} - \frac{3}{2}\left(\frac{g_1''}{g_1'}\right)^2 = 2\kappa_1^2. \tag{A13}$$

Writing $\dfrac{1}{Y} = \dfrac{y_2}{y_1} = -\dfrac{\mathcal{D}}{\mathcal{F}_{12}}\dfrac{y_1'}{y_1} - \dfrac{\mathcal{F}_{11}}{\mathcal{F}_{12}}$ as

$$\frac{1}{Y} = \mathcal{Y}_4 - \frac{e^{-2\mathcal{Y}_3}}{g_1 + C_2/C_1}, \tag{A14}$$

we end up with the following system:

$$\mathcal{Y}_4' = -\frac{\mathcal{F}_{21}}{\mathcal{D}} - \frac{\mathcal{F}_{22}-\mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}_4 + \frac{\mathcal{F}_{12}}{\mathcal{D}}\mathcal{Y}_4^2, \quad \mathcal{Y}_3' = -\mathcal{Y}_4\frac{\mathcal{F}_{12}}{\mathcal{D}} + \frac{\mathcal{F}_{22}-\mathcal{F}_{11}}{2\mathcal{D}}, \quad g_1' = \frac{\mathcal{F}_{12}}{\mathcal{D}}e^{-2\mathcal{Y}_3}. \tag{A15}$$

Similarly, in the Schwarzian Φ approach,

$$y_1 \propto \frac{1}{\sqrt{\Phi_1' \dfrac{\mathcal{D}}{\mathcal{F}_{12}} e^{\int \frac{\mathcal{F}_{11}+\mathcal{F}_{22}}{\mathcal{D}} d\varpi}}} \sin\frac{\Phi_1 + C}{2}, \quad \frac{\Phi_1'''}{\Phi_1'} - \frac{3\Phi_1''^2}{2\Phi_1'^2} + \frac{\Phi_1'^2}{2} = 2\kappa_1^2, \tag{A16}$$

and by writing $\dfrac{1}{Y} = \dfrac{y_2}{y_1} = -\dfrac{\mathcal{D}}{\mathcal{F}_{12}}\dfrac{y_1'}{y_1} - \dfrac{\mathcal{F}_{11}}{\mathcal{F}_{12}}$ as

$$\frac{1}{Y} = \mathcal{Y}_4 - \mathcal{Y}_3 \cot\frac{\Phi_1 + C}{2}, \tag{A17}$$

we end up with the following system:

$$\mathcal{Y}_4' = -\frac{\mathcal{F}_{21}}{\mathcal{D}} - \frac{\mathcal{F}_{22}-\mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}_4 + \left(\mathcal{Y}_4^2 - \mathcal{Y}_3^2\right)\frac{\mathcal{F}_{12}}{\mathcal{D}}, \quad \mathcal{Y}_3' = 2\mathcal{Y}_3\mathcal{Y}_4\frac{\mathcal{F}_{12}}{\mathcal{D}} + \frac{\mathcal{F}_{11}-\mathcal{F}_{22}}{\mathcal{D}}\mathcal{Y}_3, \quad \Phi_1' = 2\mathcal{Y}_3\frac{\mathcal{F}_{12}}{\mathcal{D}}. \tag{A18}$$

Appendix B.2. Formulation with y_2

Equation (25) is equivalent to a Sturm–Liouville problem for y_2:

$$y_2'' + 2\gamma_2 y_2' + b_2 y_2 = 0 \Leftrightarrow \left(e^{\int 2\gamma_2 d\varpi}y_2'\right)' + e^{\int 2\gamma_2 d\varpi}b_2 y_2 = 0, \tag{A19}$$

$$\gamma_2 = \frac{\mathcal{F}_{11}+\mathcal{F}_{22}}{2\mathcal{D}} + \frac{\mathcal{F}_{21}}{2\mathcal{D}}\left(\frac{\mathcal{D}}{\mathcal{F}_{21}}\right)', \quad b_2 = \frac{\mathcal{F}_{11}\mathcal{F}_{22}-\mathcal{F}_{12}\mathcal{F}_{21}}{\mathcal{D}^2} + \frac{\mathcal{F}_{21}}{\mathcal{D}}\left(\frac{\mathcal{F}_{22}}{\mathcal{F}_{21}}\right)', \tag{A20}$$

and the expression for y_1 (as function of y_2) is

$$y_1 = -\frac{\mathcal{D}}{\mathcal{F}_{21}}y_2' - \frac{\mathcal{F}_{22}}{\mathcal{F}_{21}}y_2. \tag{A21}$$

The corresponding "variable frequency oscillator" is

$$\left(e^{\int \gamma_2 d\varpi}y_2\right)'' + \kappa_2^2\left(e^{\int \gamma_2 d\varpi}y_2\right) = 0, \quad \kappa_2^2 = b_2 - \gamma_2^2 - \gamma_2'. \tag{A22}$$

Its solution in the Schwarzian g approach is

$$y_2 = \frac{C_1 g_2 + C_2}{\sqrt{g_2' \dfrac{\mathcal{D}}{\mathcal{F}_{21}} e^{\int \frac{\mathcal{F}_{11}+\mathcal{F}_{22}}{\mathcal{D}} d\varpi}}}, \quad \{g_2, \varpi\} = \frac{g_2'''}{g_2'} - \frac{3}{2}\left(\frac{g_2''}{g_2'}\right)^2 = 2\kappa_2^2. \tag{A23}$$

Writing $Y = \dfrac{y_1}{y_2} = -\dfrac{\mathcal{D}}{\mathcal{F}_{21}} \dfrac{y_2'}{y_2} - \dfrac{\mathcal{F}_{22}}{\mathcal{F}_{21}}$ as

$$Y = \mathcal{Y}_2 - \frac{e^{-2\mathcal{Y}_1}}{g_2 + C_2/C_1}, \tag{A24}$$

we end up with the following system:

$$\mathcal{Y}_2' = \frac{\mathcal{F}_{21}}{\mathcal{D}}\mathcal{Y}_2^2 + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}_2 - \frac{\mathcal{F}_{12}}{\mathcal{D}}, \quad \mathcal{Y}_1' = -\mathcal{Y}_2\frac{\mathcal{F}_{21}}{\mathcal{D}} - \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{2\mathcal{D}}, \quad g_2' = \frac{\mathcal{F}_{21}}{\mathcal{D}}e^{-2\mathcal{Y}_1}. \tag{A25}$$

Similarly, in the Schwarzian Φ approach

$$y_2 \propto \frac{1}{\sqrt{\Phi_2'\dfrac{\mathcal{D}}{\mathcal{F}_{21}}e^{\int \frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{\mathcal{D}}d\varpi}}} \sin\frac{\Phi_2 + C}{2}, \quad \frac{\Phi_2'''}{\Phi_2'} - \frac{3\Phi_2''^2}{2\Phi_2'^2} + \frac{\Phi_2'^2}{2} = 2\kappa_2^2, \tag{A26}$$

and by writing $Y = \dfrac{y_1}{y_2} = -\dfrac{\mathcal{D}}{\mathcal{F}_{21}} \dfrac{y_2'}{y_2} - \dfrac{\mathcal{F}_{22}}{\mathcal{F}_{21}}$ as

$$Y = \mathcal{Y}_2 - \mathcal{Y}_1 \cot\frac{\Phi_2 + C}{2}, \tag{A27}$$

we end up with the following system:

$$\mathcal{Y}_2' = \left(\mathcal{Y}_2^2 - \mathcal{Y}_1^2\right)\frac{\mathcal{F}_{21}}{\mathcal{D}} + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}_2 - \frac{\mathcal{F}_{12}}{\mathcal{D}}, \quad \mathcal{Y}_1' = 2\mathcal{Y}_1\mathcal{Y}_2\frac{\mathcal{F}_{21}}{\mathcal{D}} + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}_1, \quad \Phi_2' = 2\mathcal{Y}_1\frac{\mathcal{F}_{21}}{\mathcal{D}}. \tag{A28}$$

The formulations with y_1 and y_2 are, of course, equivalent. Actually, by replacing $\mathcal{Y}_1 = -\mathcal{Y}\sin\dfrac{\Phi_1 - \Phi_2}{2}$, $\mathcal{Y}_2 = \mathcal{Y}\cos\dfrac{\Phi_1 - \Phi_2}{2}$ in Equation (A27), we find Equation (A17), with $\mathcal{Y}_3 = \dfrac{1}{\mathcal{Y}}\sin\dfrac{\Phi_1 - \Phi_2}{2}$, $\mathcal{Y}_4 = \dfrac{1}{\mathcal{Y}}\cos\dfrac{\Phi_1 - \Phi_2}{2}$ (this corresponds to a Möbius transformation that leaves the Schwarzian unaffected).

So both formulations are equivalent with the following symmetrical expressions:

$$Y = \mathcal{Y}\frac{\sin\dfrac{\Phi_1 + C}{2}}{\sin\dfrac{\Phi_2 + C}{2}}, \quad y_2 = \frac{D\sin\dfrac{\Phi_2 + C}{2}}{\sqrt{e^{\int \frac{\mathcal{F}_{11} + \mathcal{F}_{22}}{\mathcal{D}}d\varpi}\mathcal{Y}\sin\dfrac{\Phi_1 - \Phi_2}{2}}}, \quad y_1 = Yy_2, \tag{A29}$$

$$\Phi_1' = 2\frac{\mathcal{F}_{12}}{\mathcal{D}}\frac{1}{\mathcal{Y}}\sin\frac{\Phi_1 - \Phi_2}{2}, \quad \Phi_2' = -2\frac{\mathcal{F}_{21}}{\mathcal{D}}\mathcal{Y}\sin\frac{\Phi_1 - \Phi_2}{2}, \tag{A30}$$

$$\mathcal{Y}' = \left(\frac{\mathcal{F}_{21}}{\mathcal{D}}\mathcal{Y}^2 - \frac{\mathcal{F}_{12}}{\mathcal{D}}\right)\cos\frac{\Phi_1 - \Phi_2}{2} + \frac{\mathcal{F}_{22} - \mathcal{F}_{11}}{\mathcal{D}}\mathcal{Y}. \tag{A31}$$

Asymptotically, $\sin\dfrac{\Phi_1 - \Phi_2}{2} \to 0$, and the quantization condition is $\sin\dfrac{\Phi_1 + C}{2} \to 0$ (the seemingly equivalent $\sin\dfrac{\Phi_2 + C}{2} \to 0$ is not accurate in cases with $m = 0$ when the quantization condition is applied on the axis, as discussed in Appendix C).

Appendix C. The Stability Problem near the Symmetry Axis $\varpi = 0$

- For $m \neq 0$ near the axis, all limits $d_{ij} = \lim\limits_{\varpi \to 0} \dfrac{\varpi \mathcal{F}_{ij}}{\mathcal{D}}$ are constants (given in Appendix B of [4]), and the relations $d_{22} = -d_{11}$, $d_{11}^2 + d_{12}d_{21} = m^2$ hold.

Using these, we can find approximate expressions for the functions given by Equation (A10) $\varpi\gamma_1 \approx \frac{1}{2}$, $\varpi^2 b_1 \approx -m^2$, $\varpi^2\kappa_1^2 \approx \frac{1}{4} - m^2$. The solutions of Equations (A13) and (A16) behave as $g_1' \propto \varpi^{2|m|-1}$, $\Phi_1' \propto \varpi^{2|m|-1}$: approaching zero as $\varpi \to 0$. The approximate solution of Equation (A18) is $\Phi_1 \approx \Phi_{1axis} + C_0\varpi^{2|m|}$, $\mathcal{Y}_3 \approx \frac{C_0|m|}{d_{12}}\varpi^{2|m|}$, $\mathcal{Y}_4 \approx \frac{|m| - d_{11}}{d_{12}}$, and thus, $\frac{1}{Y} \approx -\frac{C_0|m|}{2d_{12}}\varpi^{2|m|} \cot \frac{\Phi_{1axis} + C + C_0\varpi^{2|m|}}{2} + \frac{|m| - d_{11}}{d_{12}}$. For all values of $\Phi_{1axis} + C$ that do not satisfy the quantization condition, the result is the unphysical branch $\frac{1}{Y} \approx \frac{|m| - d_{11}}{d_{12}}$. For $\Phi_{1axis} + C = 2n\pi$ though, the resulting $\frac{1}{Y} \approx -\frac{|m| + d_{11}}{d_{12}}$ is the acceptable solution (see Equation (5) in [5]).

Similarly, we can find approximate expressions for the functions given by Equation (A20): $\varpi\gamma_2 \approx \frac{1}{2}$, $\varpi^2 b_2 \approx -m^2$, $\varpi^2\kappa_2^2 \approx \frac{1}{4} - m^2$. The solutions of Equations (A23) and (A26) behave as $g_2' \propto \varpi^{2|m|-1}$, $\Phi_2' \propto \varpi^{2|m|-1}$: approaching zero as $\varpi \to 0$.

- For $m = 0$, near the axis, the constant limits are $b_{11} = \lim_{\varpi \to 0} \frac{\mathcal{F}_{11}}{\varpi\mathcal{D}}$, $b_{12} = \lim_{\varpi \to 0} \frac{\mathcal{F}_{12}}{\varpi\mathcal{D}}$, $b_{21} = \lim_{\varpi \to 0} \frac{\varpi\mathcal{F}_{21}}{\mathcal{D}}$, $b_{22} = \lim_{\varpi \to 0} \frac{\mathcal{F}_{22}}{\varpi\mathcal{D}}$ (see Appendix B of [4]).

  Using these, we can find approximate expressions for the functions given by Equation (A10) $\varpi\gamma_1 \approx -\frac{1}{2}$, $b_1 \approx -b_{12}b_{21}$, $\varpi^2\kappa_1^2 \approx -\frac{3}{4}$, and for the solutions of Equations (A13) and (A16): $g_1' \propto \varpi$, $\Phi_1' \propto \varpi$ (approaching zero as $\varpi \to 0$). The approximate solution of Equation (A18) is $\Phi_1 \approx \Phi_{1axis} + C_0\varpi^2$, $\mathcal{Y}_3 \approx \frac{C_0}{b_{12}}$, $\mathcal{Y}_4 \approx -b_{21}\ln\varpi$, and for $\Phi_{1axis} + C_0 = 2n\pi$, the resulting $\frac{1}{Y} \approx -\frac{2}{b_{12}\varpi^2}$ is the acceptable solution (see Equation (6) in [5]).

  Similarly, we can find approximate expressions for the functions given by Equation (A20): $\varpi\gamma_2 \approx \frac{1}{2}$, $b_2 \approx -b_{12}b_{21} + 2b_{22}$, $\varpi^2\kappa_2^2 \approx \frac{1}{4}$. However, in this case, the solution of Equation (A23) $g_2' \propto \varpi^{-1}$ does not approach zero as $\varpi \to 0$. So this method should be avoided when using the quantization condition on the axis.

Appendix D. The Stability Problem for Non-relativistic Cylindrical Jets

If the unperturbed sate has density $\rho_0(\varpi)$, pressure $P_0(\varpi)$, bulk velocity $V_0(\varpi)\hat{z}$, and magnetic field $\boldsymbol{B}_0 = B_{0z}(\varpi)\hat{z} + B_{0\phi}(\varpi)\hat{\phi}$ satisfying the equilibrium condition

$$\frac{dP_0}{d\varpi} + \frac{d}{d\varpi}\left(\frac{B_{0z}^2}{2}\right) + \frac{1}{\varpi^2}\frac{d}{d\varpi}\left(\frac{\varpi^2 B_{0\phi}^2}{2}\right) = 0, \tag{A32}$$

the linearization of the ideal magnetohydrodynamic equations lead to system (23) with

$$-\varpi\frac{\mathcal{F}_{11}}{\mathcal{D}} = \frac{B_{0\phi}^2\tilde{\kappa}^2 + 2B_{0\phi}k(B_{0\phi}k - B_{0z}m/\varpi)}{\rho_0\omega_{co}^2 - (\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)^2}, \quad \frac{\mathcal{F}_{22}}{\mathcal{D}} = -\frac{\mathcal{F}_{11}}{\mathcal{D}}, \tag{A33}$$

$$\varpi\frac{\mathcal{F}_{12}}{\mathcal{D}} = \frac{\tilde{\kappa}^2\varpi^2}{\rho_0\omega_{co}^2 - (\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)^2}, \quad -\varpi\frac{\mathcal{F}_{21}}{\mathcal{D}} = \rho_0\omega_{co}^2 - (\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)^2 + \frac{B_{0\phi}^2}{\varpi^2}\frac{B_{0\phi}^2\tilde{\kappa}^2 - 4B_{0z}k(\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)}{\rho_0\omega_{co}^2 - (\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)^2}, \tag{A34}$$

$$\tilde{\kappa}^2 = \frac{\rho_0\omega_{co}^4}{(\rho_0 c_s^2 + B_0^2)\omega_{co}^2 - c_s^2(\boldsymbol{k}_{co}\cdot\boldsymbol{B}_0)^2} - k_{co}^2, \quad c_s = \sqrt{\frac{\Gamma P_0}{\rho_0}}, \quad \boldsymbol{k}_{co} = k\hat{z} + \frac{m}{\varpi}\hat{\phi}, \quad \omega_{co} = \omega - kV_0. \tag{A35}$$

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