LECTURES IN THEORETICAL PHYSICS

Volume XII-B

HIGH ENERGY COLLISIONS OF ELEMENTARY PARTICLES

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PREFACE

Volume XII-B of the Lectures in Theoretical Physics contains the proceedings of the session on High Energy Collisions of Elementary Particles which was held simultaneously with the session on Mathematical Methods in Field Theory and Complex Analytic Varieties during the second part of the Twelfth Boulder Summer Institute for Theoretical Physics. It contains the text of all lectures and one seminar. The text of other seminars and discussions during the session have not been included.

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DAUGHTERS, CONSPIRACIES, TOLLER POLES: SOME PROBLEMS IN THE REGGEIZATION OF RELATIVISTIC PROCESSES†

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I. Introduction

Regge poles first made their appearance in physics within the context of potential scattering[1] in 1959. Soon afterwards a method was suggested for incorporating these ideas into relativistic scattering theory,[2] and this led to some remarkable experimental predictions.

In recent years, however, it has become clear that there are enormous difficulties involved in generalizing Regge poles from the realm of potential theory to that of the relativistic scattering of elementary particles. Only in the case of the elastic scattering of spinless particles is the generalization straightforward. In all other cases there arise subtle and intricate problems, the solution of which has involved many new concepts such as "daughter poles,"[3] "conspiracies,"[4] "Lorentz poles,"[5] etc. Perhaps the most fascinating and challenging of all these difficulties is the question of what happens at t = 0.

It will be the principal aim of these lectures to attempt to summarise, unify, and above all, simplify, the various attempts which have been made to deal with the problem of Regge behaviour at t = 0.

All these attempts fall basically into two classes, the analytic and the group theoretic, and it will become clear later that a full understanding of the relationship between these is not yet at hand, despite the vast effort that has gone into this problem during the past three or four years.

For this reason great emphasis will be placed upon a pedagogical approach to the problem. Whenever possible we shall try to look at the difficulties from several different angles, and we shall often

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†Presented at the INSTITUTE FOR THEORETICAL PHYSICS, University of Colorado, Summer 1969.

use rather heuristic methods to get a feeling for the essential aspects of the problem. The latter is absolutely necessary since the actual, realistic calculations are bogged down in a morass of technical notation. However it does mean that the reader who is interested in serious calculations in the field will have to refer to the original papers for the full details.

Since even our valiant attempt at a heuristic presentation is, when one looks at it, still somewhat bogged down in technicalities, we shall try to summarise here the overall picture as it now stands.

It is established beyond doubt that in order to ensure analyticity at $t = 0$ in inelastic reactions a given Regge pole must be accompanied by an infinite sequence of daughters. At $t = 0$ the daughter trajectories are separated from each other by one unit of angular momentum, i.e. $\alpha_n(0) = \alpha(0) - n$ for the nth daughter trajectory. The residue of each daughter is singular at $t = 0$, but the whole sequence of daughters plus parent produces a nonsingular function at $t = 0$. The coefficients of the most singular part of the daughter residues can be calculated explicitly. Also the slope of the daughter trajectories at $t = 0$ is given explicitly in terms of the slope of the parent.

Still within the framework of considerations of analyticity it is possible to characterize a Regge pole by a new "quantum" number M. Regge poles with $M = 0$ are the usual, old-fashioned type. Regge poles with $M \neq 0$ consist of a pair of poles, with opposite parity, with trajectories $\alpha_\pm(t)$ such that $\alpha_+(0) = \alpha_-(0)$. The pair of poles is said to conspire with each other. Each of the parents in the pair is accompanied by its own daughter sequence. Again explicit formulae for the singular part of the daughter residues are known. It can also be shown that the slopes and higher derivatives of the trajectories at $t = 0$ for the + and - families are equal, up to the $(M - 1)$th derivative. And the Mth derivatives, while not equal, are related to each other by an explicit formula. The daughter trajectories need not be parallel to each other or to the trajectory of their parent.

The quantum number M has a direct physical significance. If one considers the leading term (at high energies) of the s-channel helicity amplitude $f^{(s)}_{cd;ab}$ then for a Regge pole of type M, only the amplitudes with helicity flip $c - a = (d - b) = \pm M$ do not vanish as $t \to 0$. In old fashioned Regge Pole theory all amplitudes vanished at $t = 0$ except those with $c = a$ and $d = b$--a very restrictive situation.

The above mentioned formulae for residues and trajectories are all derived from a study of the analytic properties of the amplitudes for totally inelastic reactions of the type

$$A + B \to C + D$$

and pseudo-elastic reactions of the type

$$A + B \rightarrow A + D \ .$$

However the factorizability of Regge pole residues enables one to calculate the contribution of a Regge pole to an elastic reaction

$$A + B \rightarrow A + B$$

from a knowledge of its contribution to the reactions

$$A + B \rightarrow C + D$$

$$A + B \rightarrow A + D$$

$$A + B \rightarrow C + B$$

Thus one can calculate the slopes and residues for the Regge pole and its daughters for elastic reactions as well.

Now in elastic reactions there are no problems of analyticity at $t = 0$ and a single Regge pole gives an acceptable contribution. However there is an additional symmetry at $t = 0$ which is not satisfied by a single Regge pole. A totally different approach, based on group theoretical techniques, shows that this additional symmetry is satisfied by the contribution of one Toller pole, which is equivalent to an infinite sequence of Regge poles. It is a remarkable fact that the sequence of parent and daughter poles for elastic reactions, as deduced from inelastic reactions by factorisation, turns out to be precisely of the form of the infinite sequence which sums up to one Toller pole. A deep understanding of this extraordinary result is still lacking.

The exposition which follows leans heavily on several sources. The general introduction to the difficulties at $t = 0$, and to conspiracies follows the work of the author[4)] and some unpublished work of R. Omnes and the author.[6)] The method of obtaining a closed solution for the daughter residues is taken from the brilliant work of S. R. Cosslett,[7)] and from a more recent, and more general discussion of J. M. Wang and L. L. Wang.[8)] The introduction of the quantum number M is based on unpublished notes of the author, and is a generalization of the work of G. C. Fox, T. W. Rodgers and the author.[9)] The group theoretical development is based on the work of M. Toller,[5)] D. Z. Freedman and J. M. Wang,[10)] and R. Delbourgo, A. Salam and J. Strathdee.[11)]

In Sec. II we shall briefly review the canonical steps in the Reggeization of scattering amplitudes, and the origins of the difficulties which arise at $t = 0$.

Sec. III deals with inelastic reactions without spin and uses analytic methods to derive the properties of the daughters.

In Sec. IV spin is introduced, and the new phenomena associated with it are studied using analytic methods. Many properties of the daughters can be found by an extension of the methods used in Sec. III.

Sec. V is devoted to the group theoretical approach to elastic scattering at $t = 0$. Some mention is also made of attempts to generalize this approach to inelastic reactions and to $t \neq 0$.

The author is very much indebted to W. E. Brittin and K. T. Mahanthappa for their hospitality at the Boulder Summer Institute for Theoretical Physics.

II. The Origin of the Trouble at t = 0

Let us recapitulate very briefly the essential steps in the Reggeization of a relativistic process.

We are interested in the high energy behaviour of a process

$$A + B \to C + D \tag{II.1}$$

where the particles A, B, C, D have masses m_A, m_B ..., spins s_A, s_B... and four-momenta p_A, p_B.... The physical process (II.1) takes place in the s-channel, and is described by a helicity amplitude $f^{(s)}_{cd;ab}(s,t)$, where s and t are the Mandelstam variables

$$s = (p_A + p_B)^2 \quad ,$$

$$t = (p_A + p_C)^2 \quad . \tag{II.2}$$

Here s corresponds to the square of the center-of-mass-energy of process (II.1), and t to the square of the momentum transfer. High energies in (II.1) correspond to large values of s.

The main steps in the Reggeization of process (II.1) are the following:

a) Forget about process (II.1).

b) Instead study the crossed, t-channel process

$$\bar{D} + B \to C + \bar{A} \tag{II.3}$$

where e.g. $\bar{D}$ means the anti-particle of D. This process is described by the t-channel helicity amplitude $f^{(t)}_{\bar{c}a;\bar{d}b}(t,s)$ in which t now

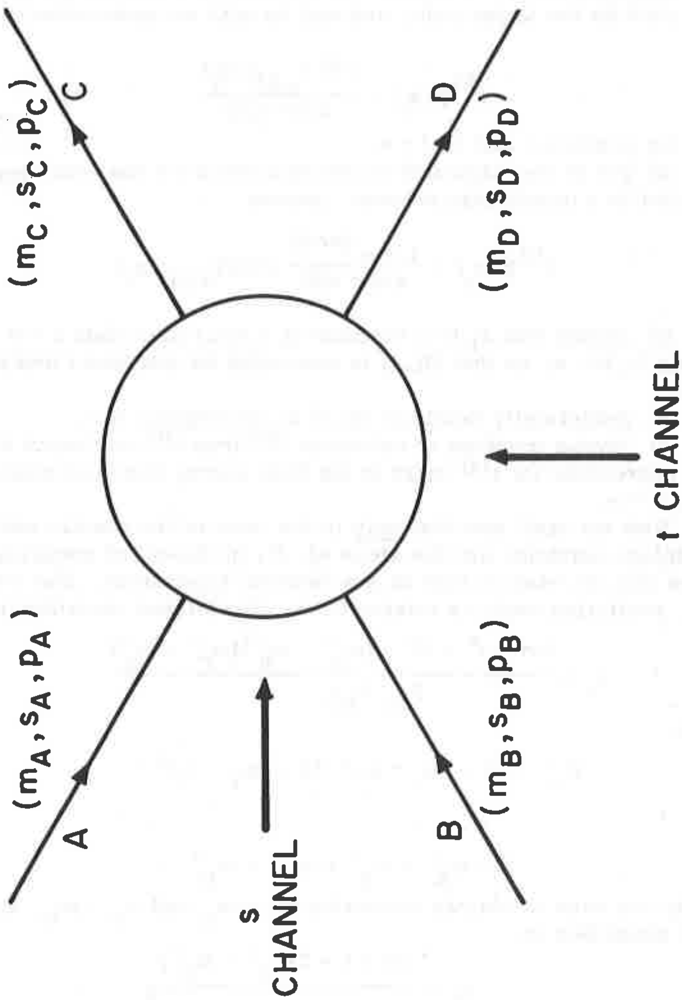

Fig. 1. Definition of the scattering channels.

plays the role of the square of the C.M. energy for (II.3), and s is the square of the momentum transfer.

c) Reggeize $f^{(t)}$ (considered as a function of t and z_t, the t-channel C.M. scattering angle) as though it were the scattering amplitude of a nonrelativistic potential problem, i.e., write a partial wave expansion for $f^{(t)}$, perform the Sommerfeld-Watson transformation, pick up the Regge pole, and end up with an expression of the form

$$f^{(t)}(t,z_t) = \frac{\beta(t)\, P_{\alpha(t)}(-z_t)}{\sin \pi\, \alpha(t)}$$

valid for positive t and $|z_t| \to \infty$.

d) Put in the signature factor to account for the exchange potential in a relativistic process, getting

$$f^{(t)}(t,z_t) = \frac{1 + e^{-i\pi\alpha(t)}}{\sin \pi\, \alpha(t)}\, \beta(t)\, P_{\alpha(t)}(-z_t) \quad . \qquad \text{(II.4)}$$

e) Notice that z_t is a function of s and t such that $s \to \infty$ implies $|z_t| \to \infty$, so that (II.4) is now valid for positive t and for $s \to \infty$.

f) Analytically continue (II.4) to the region $t \le 0$.

g) Invoke crossing to calculate $f^{(s)}$ from $f^{(t)}$ and hence arrive at an expression for $f^{(s)}$ valid in the high energy physical region $t \le 0$, $s \to \infty$.

Now we shall see that only in the case of the elastic scattering of spinless particles are the steps e), f), g) devoid of complications. To see this we have to look at the detailed kinematics. The t-channel C.M. scattering angle is related to the Mandelstam variables by

$$z_t = \frac{2st + t^2 - t\Sigma + (m_D^2 - m_B^2)(m_C^2 - m_A^2)}{\mathfrak{I}_{\bar{A}C}\, \mathfrak{I}_{B\bar{D}}} \qquad \text{(II.5)}$$

where

$$\mathfrak{I}_{ij}^2 = [t - (m_i + m_j)^2][t - (m_i - m_j)^2] \qquad \text{(II.6)}$$

and

$$\Sigma = m_A^2 + m_B^2 + m_C^2 + m_D^2 \quad .$$

Now in the case of elastic scattering $m_A = m_C$ and $m_B = m_D$, Eq. (II.5) simplifies to

$$z_t = \frac{2s + t - 2(m_A^2 + m_B^2)}{[(t - 4m_A^2)(t - 4m_B^2)]^{\frac{1}{2}}} \quad . \qquad \text{(II.7)}$$

Then we see that for elastic scattering $s \to \infty$ implies $|z_t| \to \infty$ for any fixed t. In particular, there is nothing special about the point t = 0, and step (e) is valid.

However in the general case given by Eq. (II.5), $s \to \infty$ implies $|z_t| \to \infty$ at fixed t, if and only if $t \neq 0$. At t = 0, on the contrary,

$$|z_t| = 1 \qquad \text{(II.8)}$$

independently of the value of s when $m_A \neq m_C$ and $m_B \neq m_D$.

If one pair of masses is equal, say $m_A = m_C$, then at t = 0

$$z_t = 0 \ . \qquad \text{(II.9)}$$

Thus for inelastic processes, step (e) breaks down at t = 0, and this discovery led to the question of whether Regge behaviour holds at t = 0 in inelastic processes. This is the first of the major problems in the Reggeization of relativistic processes, and it arises for inelastic processes even when the particles are spinless.

The second major problem arises for elastic processes when the particles have nonzero spin. Let us consider the consequences of step (g). If the particles have spin then $f^{(s)}$ is related to $f^{(t)}$ by the crossing matrix

$$f^{(s)}_{cd;ab} = M^{c'\bar{d}'\bar{a}'b'}_{cdab} \, f^{(t)}_{c'\bar{a}';\bar{d}'b'} \ . \qquad \text{(II.10)}$$

We shall see later that $f^{(s)}$ has to behave like

$$f^{(s)}_{cd;ab} \propto t^{\frac{1}{2}|(a-c)+(d-b)|} \quad \text{as } t \to 0 \qquad \text{(II.11)}$$

in order to conserve angular momentum for forward scattering. Then (II.11) and (II.10) imply that certain linear combinations of $f^{(t)}$ have to vanish at t = 0, i.e. the $f^{(t)}$ are correlated near t = 0. Since different Regge poles can contribute to the various $f^{(t)}$, this is tantamount to requiring correlations among sets of Regge poles. This is a most unexpected result, since one normally considers each Regge pole as an independent physical entity, and if these correlations exist they are of enormous physical significance.

From the above, we see that the standard method of Reggeization runs into serious difficulty at t = 0 for inelastic reactions and when the particles have nonzero spin. In the following we shall study in detail the attempts to overcome these difficulties.

III. Inelastic Reactions Without Spin

A. Heúristic Introduction

Since it is often convenient to discuss a reaction as viewed from the t-channel,it will be useful to introduce the following notation. If the physical reaction is elastic, i.e. $m_A = m_C$ and $m_B = m_D$, then in the t-channel the initial and final states comprise pairs of equal mass particles and the reaction will be said to be of the EE type. Similarly for pseudo-elastic reactions, $m_A = m_C$, $m_B \neq m_D$, we use the label UE and for totally inelastic reactions where $m_A \neq m_C$ and $m_B \neq m_D$ we use the label UU.

To see heuristically what is happening at t = 0 let us consider a spinless UU reaction.

We have seen from II.5 that at $t = 0$, $|z_t| = 1$ and hence $|z_t| \not\to \infty$ as $s \to \infty$. The question is: Does Regge behavior hold at $t = 0$? We shall show now that the result $|z_t| = 1$ at $t = 0$ is irrelevant to the above question.

There are two rather different methods of seeing this:

a) Since we are dealing with a spinless process,the scattering is described by one invariant amplitude A(s,t) which has Mandelstam analyticity. The helicity amplitudes in the s and t channels are then essentially identical to A, i.e.

$$f^{(s)}(s,z_s) \equiv f^{(t)}(t,z_t) \equiv A(s,t) \qquad \text{(III.1)}$$

(z_s is of course the cosine of the s-channel C.M. scattering angle).

Now $\mathscr{L}_{t\to 0} A(s,t)$ clearly exists and is some function of s, say

$$\mathscr{L}_{t\to 0} A(s,t) = L(s) \qquad \text{(III.2)}$$

From (III.1) then, also

$$\mathscr{L}_{t\to 0} f^{(t)}(t,z_t) = L(s) \qquad \text{(III.3)}$$

However, naively, using the fact that

$$\mathscr{L}_{t\to 0} z_t = 1$$

we get

$$\mathscr{L}_{t \to 0} f^{(t)}(t, z_t) = f^{(t)}(t = 0,\ z_t = 1)$$

$$= \text{constant} . \qquad \text{(III.4)}$$

contradicting (III.3).

The fallacy of course lies in assuming that if

$$f(x) \equiv g(\varphi(x))$$

then

$$\mathscr{L}_{x \to 0} f(x) = g\left(\mathscr{L}_{x \to 0} \varphi(x)\right)$$

a result which would only be true if the mapping $x \to \varphi(x)$ is nonsingular and well behaved at $x = 0$. This is not the case for the mapping

$$(s, t) \to (t, z_t)$$

which is certainly singular at $t = 0$ where it maps the whole s-plane into one point.

It should be noticed that the above does not depend on s being large, i.e. the trouble at $t = 0$ is here not essentially a problem of Reggeization. Any theory which insists on working with $f^{(t)}(t, z_t)$ will run into difficulty.

In summary, the function $f^{(t)}(t, z_t)$ evaluated at $t = 0$, $z_t = 1$ <u>has nothing to do with the physics at $t = 0$</u>. If we wish to use $f^{(t)}(t, z_t)$ to see what happens at $t = 0$ we must first undo the transformation $(s, t) \to (t, z_t)$ and write $f^{(t)}(t, z_t)$ in terms of nonsingular combinations of s and t <u>before</u> taking the limit $t \to 0$. The requirement that $f^{(t)}(t, z_t)$ be expressible in terms of analytic functions of s and t will constrain the possible functional dependence of $f^{(t)}$ on z_t, and these constraints will lead to the necessity of daughters when we Reggeize.

b) There is an alternate way to see that $f^{(t)}$ at $t = 0$, $z_t = 1$ has nothing to do with physics.

Consider the M-function

$$M(p_C, p_D; p_A, p_B) = \langle p_C p_D | S - 1 | p_A, p_B \rangle \qquad \text{(III.5)}$$

By the Lorentz invariance of S, for the spinless case, we have

$$M(\Lambda p_C, \Lambda p_D; \Lambda p_A, \Lambda p_B) = M(p_C, p_D; p_A, p_B) \qquad \text{(III.6)}$$

where Λ is the 4×4 matrix specifying any real Lorentz transformation. Assuming, as usual, that M is an analytic function of the components of the vectors which are its arguments, one generalizes (III.6) to hold also for complex matrices Λ which preserve the length of the 4-vectors p_i. In this way the analytic continuation which takes one from the t-channel C.M. to the s-channel C.M. can be effected by a complex Lorentz transformation.

Let us define

$$K = p_A - p_C = p_D - p_B$$

$$q = \tfrac{1}{2}(p_A + p_C)$$

$$q' = \tfrac{1}{2}(p_B + p_D) \qquad \text{(III.7)}$$

and note that

$$K^2 = t$$

$$q \cdot K = \tfrac{1}{2}(m_A^2 - m_C^2)$$

$$q' \cdot K = \tfrac{1}{2}(m_D^2 - m_B^2) \qquad \text{(III.8)}$$

We write M as a function of K, q, q′. Then by (III.6)

$$M(\Lambda K; \Lambda q, \Lambda q') = M(K; q, q') \qquad \text{(III.9)}$$

Now calculate K in the s-channel c.m. taking the Z axis along $\underline{K}$. One finds

$$K_{(s)} \equiv K\Big|_{\substack{\text{s channel}\\ \text{C.M.}}} = (\kappa, 0, 0, \sqrt{\kappa^2 - t}\,) \qquad \text{(III.10)}$$

where

$$\kappa = \frac{1}{2\sqrt{s}}\left(m_A^2 - m_C^2 + m_D^2 - m_B^2\right) \qquad \text{(III.11)}$$

Hence, at t = 0,

$$K_{(s)} = \kappa(1,0,0,1) \tag{III.12}$$

i.e. K is a light-like 4-vector at t = 0.

However, if one calculates K in the t-channel C.M., one finds

$$K_{(t)} \equiv K\Big|_{\substack{\text{t channel}\\ \text{C.M.}}} = (-\sqrt{t},0,0,0) \tag{III.13}$$

so that at t = 0

$$K_{(t)} = (0,0,0,0) \tag{III.14}$$

i.e. $K_{(t)}$ is a null 4-vector at t = 0.

Now

$$f^{(s)}(s,z_s) \equiv M(K_{(s)};q_{(s)},q'_{(s)}) \tag{III.15}$$

and

$$f^{(t)}(t,z_t) \equiv M(K_{(t)};\, q_{(t)},\, q'_{(t)})$$

Hence in order to have the crossing relation (which in the spinless case is trivial)

$$f^{(s)} = f^{(t)}$$

we require the existence of a Λ such that

$$K_{(s)} = \Lambda K_{(t)} \tag{III.16}$$

Clearly from (III.14) and (III.12) at t = 0, there is no Λ which can satisfy (III.16).

Thus there is no Lorentz transformation, real or complex, which relates the t-channel C.M. frame at t = 0 to the physical s-channel C.M. frame. So again we see that $f^{(t)}$ at t = 0, $z_t = 1$ is simply not related to the physics.

In summary, the fact that $|z_t| \not\to \infty$ as $s \to \infty$ at t = 0 has no bearing on whether or not Regge behaviour holds at t = 0.

Let us turn now to the question of how to calculate the scattering amplitude at t = 0, and of what constraints are forced onto the Regge poles by the demands of analyticity.

Suppose we have Regge behaviour for positive t, say $t > t_o$ where t_o is the threshold for physical t-channel reactions. Since A(s,t) is analytic and satisfies dispersion relations we can calculate A(s,t) for $t < t_o$ from our knowledge of A(s,t) for $t > t_o$ by using a fixed-s dispersion relation. For simplicity we take a very simple example

$$A(s,t) = \frac{1}{\pi}\int_{t_o}^{\infty} \frac{\operatorname{Im} A(s,t')}{t'-t}\, dt' \quad . \qquad \text{(III.17)}$$

We are interested in the behaviour as $s \to \infty$. So we shall feed into the integrand the high s behaviour of Im A(s,t). Notice that we only need this behaviour for $t \geq t_o$. Hence we take

$$\operatorname{Im} A(s,t) \underset{s\to\infty}{\approx} \operatorname{Im}\left\{\beta(t)\, P_{\alpha(t)}(-z_t)\right\} \quad . \qquad \text{(III.18)}$$

In (III.18) we have absorbed all inessential factors into $\beta(t)$.

Since for $t \neq 0$, $|z_t| \to \infty$ as $s \to \infty$ we can expand (III.18) in inverse powers of z_t:

$$\operatorname{Im} A(s,t) \underset{s\to\infty}{\approx} \operatorname{Im}\left\{\beta(t)\left[a_0 z_t^{\alpha} + a_2 z_t^{\alpha-2} + \ldots\right]\right\}$$

$$\underset{s\to\infty}{=} \operatorname{Im}\left\{\beta(t)\left[b_o (st)^{\alpha} + b_1 (st)^{\alpha-1} + b_2 (st)^{\alpha-2} + \ldots\right]\right\} \qquad \text{(III.19)}$$

where we have used the fact (see Eq. (II.5)) that $z_t \propto st$ for large s and $t \neq 0$. The coefficients a_i or b_i are irrelevant to the argument.

Now one can show from the definition of the Froissart-Gribov partial wave amplitude that for small momenta

$$\beta(t) \propto (p_{\bar{A}C}\, p_{\bar{D}B})^{\alpha(t)} \qquad \text{(III.20)}$$

where $p_{\bar{A}C}$, $p_{\bar{D}B}$ are the t-channel C.M. relative momenta. However,

$$p_{ij}^{\,2} = \frac{1}{4t}\, \mathfrak{T}_{ij}^{2} \qquad ij = \bar{A}C \text{ or } B\bar{D} \qquad \text{(III.21)}$$

so from (II.6), in the UU case,

$$p_{\bar{A}C} \propto \frac{1}{\sqrt{t}} \quad \text{etc.},$$

and hence

$$\beta(t) \propto t^{-\alpha} \tag{III.22}$$

Thus one puts

$$\beta(t) = t^{-\alpha}\,\bar{\beta}(t) \tag{III.23}$$

where $\bar{\beta}(t)$ is analytic at $t = 0$.†

Putting (III.23) into (III.19) gives the final form of the integrand

$$\operatorname{Im} A(s,t') \underset{s\to\infty}{\approx} \operatorname{Im}\left\{\bar{\beta}(t')\left[b_o\, s^{\alpha(t')} + \frac{b_1}{t'}\, s^{\alpha-1} + \frac{b_2}{t'^2}\, s^{\alpha-2} + \ldots\right]\right\} \tag{III.24}$$

Substituting into (III.17), opening up the contour, and using Cauchy's theorem to evaluate the integral, we pick up the residues of the poles at $t' = t$, <u>and</u> at $t' = 0$ for the non-leading terms. Hence we get

$$\begin{aligned} A(s,t) \approx\ & \bar{\beta}(t)\left[b_o(t)\, s^{\alpha(t)} + \frac{b_1(t)}{t}\, s^{\alpha(t)-1} + \frac{b_2(t)}{t^2}\, s^{\alpha(t)-2} + \ldots\right] \\ & - \frac{\bar{\beta}(0)b_1(0)}{t}\, s^{\alpha(0)-1} - \frac{\bar{\beta}(0)b_2(0)}{t^2}\, s^{\alpha(0)-2} \\ & - \frac{b_2(0)}{t^2}\left(\bar{\beta}'(0) + \alpha'(0)\bar{\beta}(0)\log s\right) s^{\alpha(0)-2} \ldots \end{aligned} \tag{III.25}$$

where $\bar{\beta}'$ and α' are derivatives of $\bar{\beta}$ and α with respect to t.

The result (III.25) is a remarkable one. First it is, of course, analytic at $t = 0$ despite the fact that individual terms blow up as $t \to 0$. This, of course, was guaranteed by the use of the dispersion relation (III.17), which is manifestly analytic at $t = 0$.

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†Actually it is incorrect to use (III.20) to deduce (III.22), since (III.20) is supposed to hold for small $p_{\bar{A}C}$ and $p_{\bar{D}B}$, whereas both momenta $\to \infty$ as $t \to 0$. In fact, in the present approach one need <u>not</u> even have $\bar{\beta}(t)$ analytic at $t = 0$ since no matter what happens to $\bar{\beta}(t)$ at $t = 0$ the dispersion relation produces an $A(s,t)$ which is guaranteed to be analytic at $t = 0$. However, for the purposes of explicit calculation (III.23) is most convenient.

Secondly, the leading term at $t = 0$ goes like $s^{\alpha(0)}$, which is the usual leading order Regge behaviour. Thus this method of reaching the point $t = 0$ successfully produces a uniform asymptotic behaviour as $t \to 0$.

However, one has in addition ended up with terms like $s^{\alpha(0)-n}$ which can be shown to correspond to the existence of fixed poles in the complex J plane at the infinite sequence of points $J = \alpha(0)-n$.† Thus we have what looks like a normal Regge pole plus a sequence of fixed poles. Note that this does not contradict our assumption (III.18) that Im A(s,t) is given by a single Regge pole, since all the fixed poles that have appeared are real, i.e. appear only in Re A(s,t). However it does show that it would have been inconsistent to assume that A(s,t) itself was given by just one Regge pole. In other words in the UU case a single Regge pole is not compatible with the dispersion relation (III.18).

If one is prepared to tolerate this infinite sequence of fixed poles then the amplitude given by (III.25) is quite acceptable. However fixed poles in the J-plane are generally considered taboo, since it can be shown that they contradict the partial wave unitarity condition,‡ provided that the unitarity condition can be analytically continued to the point in question. It is possible that the cut structure in the J-plane would prohibit this, thereby negating the argument against fixed poles, but this does not seem a very plausible assumption. Thus we must do something to get rid of the fixed poles. To remove them we invoke a sequence of Regge poles delicately chosen

†A Regge pole is, of course, a pole of the analytically continued partial wave amplitude at $J = \alpha(t)$ in the complex J-plane. It gives rise to a characteristic sequence $s^{\alpha(t)}, s^{\alpha(t)-1}, s^{\alpha(t)-2} \ldots$ as $s \to \infty$. Conversely one can show that each term of the form $s^{\alpha(0)-n}$ corresponds to a fixed, t-independent pole at $J = \alpha(0)-n$. In particular the sequence of terms $s^{\alpha(0)-1}, s^{\alpha(0)-2}, \ldots$ does not simply represent the power sequence which would correspond to one fixed pole at $J=\alpha(0)-1$.

‡The unitarity condition which holds initially for physical values of J, can be continued into the complex J-plane in the form of a discontinuity equation

$$f_J(t + i\varepsilon) - f_J(t - i\varepsilon) = 2i\rho\, f_J(t + i\varepsilon)\, f_J(t - i\varepsilon)$$

If $f_J(t)$ has fixed (t-independent) poles in the J-plane, say $f_J(t) \approx \frac{1}{J-\lambda}$ for $J \approx \lambda$ then the left and right hand sides of the discontinuity equation cannot balance as $J \to \lambda$. However, if the pole is at $J = \alpha(t)$ and if $\alpha(t + i\varepsilon) \neq \alpha(t - i\varepsilon)$ we have no contradiction since only one of the factors on the right hand side blows up as $J \to \alpha(t + i\varepsilon)$.

so as to cancel the fixed poles and at the same time to leave undisturbed the analyticity of $A(s,t)$ at $t = 0$.

Suppose therefore that there exists a second Regge pole (called the "first daughter") with trajectory function $\alpha_1(t)$ and residue $\beta_1(t)$. If we take

$$\beta_1(t) \propto t^{-\alpha_1} ,$$

its leading term will look like $\bar{\beta}_1(t)\, b_o(t)\, s^{\alpha_1(t)}$. However, to cancel the $1/t$ term in (III.25), the first daughter must have a more singular residue. Thus, if we put

$$\beta_1(t) = t^{-\alpha_1 - 1}\, \bar{\beta}_1(t) \tag{III.26}$$

then the contribution to $A(s,t)$ will look like†

$$\bar{\beta}_1(t) \frac{b_o(t)}{t} s^{\alpha_1(t)} + \frac{b_1(t)}{t^2} s^{\alpha_1(t)-1} + \ldots - \frac{\bar{\beta}_1(0)\, b_o(0)}{t} s^{\alpha_1(0)}$$
$$- \frac{\bar{\beta}_1(0)\, b_1(0)}{t^2} s^{\alpha_1(0)-1} - \frac{b_1(0)}{t}\left[\bar{\beta}_1{}'(0) + \alpha_1{}'(0)\, \bar{\beta}_1(0) \log s\right] s^{\alpha_1(0)-1} . \tag{III.27}$$

If we now take

$$\alpha_1(0) = \alpha(0) - 1 \tag{III.28}$$

and

$$\bar{\beta}(0)\, b_1(0) = -\bar{\beta}_1(0)\, b_o(0) \tag{III.29}$$

then adding (III.25) and (III.27) eliminates the fixed pole at $J = \alpha(0) - 1$. The leading terms left give

$$A(s,t) \approx \bar{\beta}(t)\, b_o(t)\, s^{\alpha(t)} + \frac{1}{t}\left\{\bar{\beta}(t)\, b_1(t)\, s^{\alpha(t)-1} + \bar{\beta}_1(t)\, b_o(t)\, s^{\alpha_1(t)}\right\} + \ldots \tag{III.30}$$

which, using (III.29), is analytic as $t \to 0$. Note that the cancellation is effected without any requirement on the slope $\alpha_1(t)$.

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†It is amusing to note that if we had given the parent a residue as singular as $\beta_1(t)$ then if $\alpha'(0) = 0$ we would still recover the standard Regge leading term at $t = 0$. Moreover we would end up not with a daughter sequence but simply with one fixed pole at $J = \alpha(0)$.

Now we can repeat the process, introducing a second daughter with $\alpha_2(t)$ and

$$\alpha_2(0) = \alpha(0) - 2$$

and a still more singular residue

$$\beta_2(t) = t^{-\alpha_2 - 2}\,\bar{\beta}_2(t) \quad .$$

Its leading terms will be

$$\frac{\bar{\beta}_2(t)\, b_o(t)}{t^2}\; s^{\alpha_2(t)} - \frac{\bar{\beta}_2(0)\, b_o(0)}{t^2}\; s^{\alpha_2(0)}$$

$$- \frac{b_o(0)}{t}\,[\bar{\beta}_2{}'(0) + \alpha_2{}'(0)\,\bar{\beta}_2(0)\log s]\; s^{\alpha_2(0)} - \ldots \qquad \text{(III.31)}$$

Putting together (III.31), (III.27), and (III.25), one sees that the cancellation requires

$$\bar{\beta}(0)\, b_2(0) + \bar{\beta}_1(0)\, b_1(0) + \bar{\beta}_2(0)\, b_2(0) = 0 \quad ,$$

$$\text{i.e.,}\; \bar{\beta}_2(0)\, b_o(0) = -\,\bar{\beta}(0)\left[b_2(0) - \frac{b_1^2(0)}{b_o(0)} \right] \quad , \qquad \text{(III.32)}$$

$$b_2(0)\,\bar{\beta}'(0) + b_1(0)\,\bar{\beta}_1{}'(0) + b_o(0)\,\bar{\beta}_2{}'(0) = 0 \quad , \qquad \text{(III.33)}$$

which yields $\bar{\beta}_2{}'(0)$ in terms of $\bar{\beta}'(0)$ and $\bar{\beta}_1{}'(0)$ and also

$$b_2(0)\,\bar{\beta}(0)\,\alpha'(0) + b_1(0)\,\alpha_1{}'(0)\,\bar{\beta}_1(0) + b_o(0)\,\alpha_2{}'(0)\,\bar{\beta}_2(0) = 0 \quad \text{(III.34)}$$

which gives $\alpha_2{}'(0)$ in terms of $\alpha'(0)$ and $\alpha_1{}'(0)$.

Clearly we can, in principle, continue this process. The slopes of the residues and trajectory of the first daughter are arbitrary, but thereafter the slopes for the second and higher daughters will be fixed. Indeed, the second derivatives for the third and higher daughters will be determined and so on. Note that in an expansion about $t = 0$, powers of log s will appear.

The above approach gives a very clear idea of the role of the daughters. However, it is not suitable for going much further. Thus we now turn to a closed method of handling the problem.

B. Formulation of the Daughter Prescription

Let $f(s,t)$ be analytic as $t \to 0$ and have an asymptotic expansion in $1/s$, or $1/s$ times powers of log s. We wish to represent $f(s,t)$ by functions of z_t in such a way that despite (II.8) or (II.9) the limit $t \to 0$ taken in the functions of z_t should correctly reproduce $f(s,t)$ as $t \to 0$.

The precise situation is sensitive to the masses. For the UU case let us fix $m_C > m_A$ and $m_D > m_B$. Then from (III.5),

$$\frac{z_t - 1}{2} = \mathscr{s}t + 0(t^2) \qquad \text{(III.35)}$$

where

$$\mathscr{s} = \frac{1}{(m_C^2 - m_A^2)(m_D^2 - m_B)}\left\{s + \frac{(m_A^2 - m_C^2 + m_D^2 - m_B^2)(m_A^2 m_D^2 - m_C^2 m_B^2)}{(m_C^2 - m_A^2)(m_D^2 - m_B^2)}\right\} \qquad \text{(III.36)}$$

and $\mathscr{s} \to \infty$ as $s \to \infty$.

For the UE case with $m_A = m_C$ and $m_D > m_B$,

$$z_t = \bar{\mathscr{s}}\sqrt{-t} + 0(t^{3/2}) \qquad \text{(III.37)}$$

where

$$\bar{\mathscr{s}} = \frac{2s - (2m_A^2 + m_B^2 + m_D^2)}{2m_A(m_B^2 - m_D^2)} \quad . \qquad \text{(III.38)}$$

Let us focus attention on the UU case. Suppose that

$$f(s,t) = \mathscr{s}^{\alpha(t)}\, t^N\, g(\mathscr{s},t) \qquad \text{(III.39)}$$

where

$$g(\mathscr{s},t) \approx \sum_{\mu,\nu,\sigma} g_{\mu\nu\sigma}\, \mathscr{s}^{-\mu}\, t^{\nu}\, (\log \mathscr{s})^{\sigma} \quad . \qquad \text{(III.40)}$$

Then if

$$f(s,t) = \hat{f}(t,x)$$

where $x = \frac{1}{2}(z_t - 1)$, we have

$$\hat{f}(t,x) \approx \left(\frac{x}{t}\right)^{\alpha} \sum_{\rho=N}^{\infty} t^{\rho} \varphi_{\rho}(x;\, x/t) \qquad \text{(III.41)}$$

where $\varphi_\rho(x;\, x/t)$ is a polynomial in $1/x$ of order ρ and a polynomial in $\log (x/t)$.

Conversely if

$$\hat{f}(t,x) = \left(\frac{x}{t}\right)^{\alpha} \Phi(t,x,x/t) \qquad \text{(III.42)}$$

and we want $\hat{f}(t,x) = f(s,t)$, we must ensure that: (i) $\Phi(t,u,v)$ has a Taylor series about $t = 0$ at fixed u and v, whose first term is t^N. (ii) The coefficient of t^ρ in the Taylor expansion of $\Phi(t,u,v)$ at fixed u, v, should at worst be a polynomial in $1/u$ of order ρ and a polynomial in log v.

These two points constitute a prescription for fixing the properties of the daughters so as to guarantee that the function $\hat{f}(t,x)$ is in fact an analytic function of t when considered as a function of s and t. However, it is unlikely that the above are necessary conditions. In what follows, $\Phi(t,x,x/t)$ will represent a sum over the parent and daughter amplitudes and the above conditions imposed on Φ will enforce certain relationships between the residues and trajectories of daughters and their parents.

In the UE case the result is modified as follows. If we put $z \equiv z_t$, and if

$$\hat{f}(t,z) = \left(\frac{z}{\sqrt{t}}\right)^{\alpha} \Phi(t,z,z/\sqrt{t}) \qquad \text{(III.43)}$$

then we must ensure that

$$\Phi(t,z,z/\sqrt{t}) = \Phi_1(t,z,z/\sqrt{t}) + \frac{\sqrt{t}}{z} \Phi_2(t,z,z/\sqrt{t}) \qquad \text{(III.44)}$$

where

$$\Phi_{1,2}(t,z,z/\sqrt{t}) = \sum_{\rho=N} t^{\rho} \varphi_{\rho_{1,2}}(z,z/\sqrt{t}) \qquad \text{(III.45)}$$

where $\varphi_{\rho_{1,2}}$ is a polynomial in $1/z^2$ of order ρ and a polynomial in $\log (z/\sqrt{t})$.

C. Solution of the Daughter Problem for the UU Case

It is preferable to use the Mandelstam form for the Regge pole contribution. This is obtained by replacing

$$\frac{P_\alpha(z)}{\sin \pi \alpha} \quad \text{by} \quad \frac{-\frac{1}{\pi}\, Q_{-\alpha-1}(z)}{\cos \pi \alpha}$$

in the usual formulae. Hence we have for the contribution of a single Regge pole

$$f^{(t)}(t, z_t) = \frac{2\alpha(t) + 1}{2} \, \frac{1 + \tau e^{-i\pi\alpha}}{\cos \pi \alpha} \, \beta(t) \, Q_{-\alpha-1}(-z_t) \qquad \text{(III.46)}$$

where $\tau = \pm 1$ is the signature.

The complete contribution of parent plus daughters will look like

$$f^{(t)}(t, z_t) = \sum_{n=0}^{\infty} \frac{2\alpha_n(t)+1}{2} \, \frac{1 + \tau_n e^{-i\pi\alpha_n}}{\cos \pi \alpha_n} \, \beta_n(t) \, Q_{-\alpha_n-1}(-z_t) \quad . \qquad \text{(III.47)}$$

We already know from the heuristic discussion in Sec. III.A that we must have

$$\alpha_n(0) = \alpha(0) - n \quad . \qquad \text{(III.48)}$$

We also saw that we should take

$$\beta_n(t) = t^{-\alpha_n(t)-n} \, \bar{\beta}_n(t) \qquad \text{(III.49)}$$

where $\bar{\beta}_n$ is analytic near $t = 0$.

We now see that because of the signature factor the cancellations will fail unless all the terms have the same phase as $t \to 0$. This requires choosing

$$\tau_n = (-1)^n \tau \quad , \qquad \text{(III.50)}$$

i.e., odd daughters have opposite signature to their parent.

We use

$$Q_{-\alpha-1}(-z) = \frac{1}{2} \frac{\Gamma^2(-\alpha)}{\Gamma(-2\alpha)} \left(\frac{1-z}{2}\right)^{\alpha} F(-\alpha, -\alpha; -2\alpha; \frac{2}{1-z}) \qquad \text{(III.51)}$$

and put

$$\frac{2\alpha_n(t)+1}{2\alpha(t)+1}\cdot\frac{\cos\pi\alpha}{\cos\pi\alpha_n}\cdot\frac{1+\tau_n e^{-i\pi\alpha_n}}{1+\tau e^{-i\pi\alpha}}\cdot\frac{\Gamma^2(-\alpha_n)}{\Gamma^2(-\alpha)}\cdot\frac{\Gamma(-2\alpha)}{\Gamma(-2\alpha_n)}\cdot\frac{\bar\beta_n(t)}{\bar\beta(t)}=b_n(t) \tag{III.52}$$

where clearly $b_o(t) \equiv 1$. The $b_n(t)$ thus relate the daughter's residue, etc., to the parent's.

Then (III.47) becomes

$$f^{(t)}(t,z_t)=\frac{2\alpha(t)+1}{2}\cdot\frac{1+\tau e^{-i\pi\alpha}}{\cos\pi\alpha}\cdot\frac{\Gamma^2(-\alpha)}{\Gamma(-2\alpha)}\cdot\bar\beta(t)\left(\frac{1-z_t}{2t}\right)^{\alpha}$$
$$\sum_{n=0}^{\infty} b_n(t)\left(\frac{1-z_t}{2}\right)^{-n}\left(\frac{2t}{1-z_t}\right)^{\alpha(t)-\alpha_n(t)-n} F(-\alpha_n,-\alpha_n;-2\alpha_n;\frac{2}{1-z_t}) . \tag{III.53}$$

We put

$$x=\tfrac{1}{2}(1-z_t) \quad ,$$
$$v=x/t \quad ,$$

and define

$$\hat f(t,x)=\left(\frac{x}{t}\right)^{\alpha}\sum_{n=0} b_n(t)\, x^{-n}\, v^{\alpha_n+n-\alpha}\, F(-\alpha_n,-\alpha_n;\,-2\alpha_n;\,\frac{1}{x}) \tag{III.54}$$

so that

$$f^{(t)}(t,z_t)=\frac{2\alpha(t)+1}{2}\cdot\frac{1+\tau e^{-i\pi\alpha}}{\cos\pi\alpha}\cdot\frac{\Gamma^2(-\alpha)}{\Gamma(-2\alpha)}\cdot\bar\beta(t)\,\hat f(t,x) \quad . \tag{III.55}$$

Now clearly (III.54) is in the form of (III.42), where we identify the sum in (III.54) as $\Phi(t,x,v)$. Thus the daughter properties will be determined by the requirement that

$$\frac{\partial^{\rho}}{\partial t^{\rho}}\sum_{n=0} b_n(t)\, v^{\alpha_n(t)+n-\alpha(t)}\, x^{-n} F(-\alpha_n(t),-\alpha_n(t);\,-2\alpha_n(t);\,\frac{1}{x})\Big|_{t=0}$$

$=$ polynomial in $\frac{1}{x}$ of order ρ, and polynomial in log v for

$\rho = 0, 1, 2, \ldots$. (III.56)

It will be assumed that the derivative can be carried through the summation sign. For $\rho = 0$ we get, using (III.48), the condition

$$\sum_{n=0} b_n(0)\, x^{-n}\, F\left(-\alpha(0)+n,\ -\alpha(0)+n;\ -2\alpha(0)+2n;\ \frac{1}{x}\right) = \text{constant}. \qquad \text{(III.57)}$$

A little thought shows that the only possible constant must be $b_o(0)$, which equals 1. Hence the $b_n(0)$ are determined by (III.57) with constant put equal to one on the right-hand side. The solution is†

$$b_n(0) = \frac{(-1)^n\, (-\alpha(0))_n\, (-\alpha(0))_n}{n!\, (-2\alpha(0) + n - 1)_n} \qquad \text{(III.58)}$$

where

$$(y)_n \equiv \frac{\Gamma(y+n)}{\Gamma(n)} \quad . \qquad \text{(III.59)}$$

Substituting $b_n(0)$ into (III.52) gives for the daughter residues in the UU case

$$\beta_n^{UU}(t) = t^{-\alpha_n(t)-n}\, \bar{\beta}_n^{UU}(t) \qquad \text{(III.60)}$$

with

$$\bar{\beta}_n^{UU}(0) = \frac{2n - 2\alpha(0) - 1}{n!} \cdot \frac{\Gamma(-2\alpha(0) + n - 1)}{\Gamma(-2\alpha)}\, \bar{\beta}^{UU}(0) \ . \qquad \text{(III.61)}$$

Thus the residue of the nth daughter at $t = 0$ is completely determined in terms of the parent's residue.

For $\rho = 1$ the differentiation with respect to t yields three kinds of terms, of which the most interesting comes from

$$\frac{\partial}{\partial t} v^{\alpha_n(t)+n-\alpha(t)} = [\alpha_n'(t) - \alpha'(t)] \log v \cdot v^{\alpha_n+n-\alpha} \quad .$$

---

†From Eq. (11), Chapter 4.3, of Reference 12, one can show that

$$\sum_{r=N}^{\infty} (-1)^r \frac{(a)_r\, (b)_r\, (c+r-1)_N (-r)_N}{r!\, (c+r-1)_r (a)_N (b)_N}\, z^r\, F(a+r,\ b+r;\ c+2r;\ z) = z^N$$

Equations (III.57) and (III.62) are special cases of this.

Since this is the only term involving log v we must have

$$\sum_{n=0} b_n(0)\,[\alpha_n'(0) - \alpha'(0)]\ \log v \cdot x^{-n}\, F(-\alpha_n, -\alpha_n; -2\alpha_n; \tfrac{1}{x})$$

$$= \text{polynomial of order one in } \frac{1}{x} \text{ and polynomial in } \log v. \tag{III.62}$$

Since the term on the left with n = 0 vanishes, it is clear that the right-hand side of (III.62) can only be

$$\frac{1}{x}\, b_1(0)\,[\alpha_1'(0) - \alpha'(0)]\ \log v \ .$$

From this it follows† that

$$\alpha_n'(0) - \alpha'(0) = \frac{(2\alpha(0) - n + 1)n}{2\alpha(0)}[\alpha_1'(0) - \alpha'(0)] \ . \tag{III.63}$$

Thus the slope of the trajectory of the nth daughter, for $n \geq 2$, is given completely in terms of the slope of the parent and first daughter trajectories.

It is interesting to note that for large enough n, $\alpha_n'(0) - \alpha'(0)$ is of opposite sign to $\alpha_1'(0) - \alpha'(0)$. This might have some bearing on large angle scattering. Also if the first daughter is parallel to its parent, then so are all the other daughters.

Clearly, by looking at the other terms with $\rho = 1$ we will be able to solve for $b_n'(0)$, i.e., for the slopes of the residue functions. Further, by looking at terms with $\rho > 1$ we will get information on higher derivatives of $\alpha_n(t)$ and $\bar{\beta}_n(t)$ at t = 0.

D. Solution of the Daughter Problem for the UE Case

From Eq. (III.37) we see that z_t itself, rather than $z_t - 1$, is in this case the most suitable variable to deal with. Hence we use

$$Q_{-\alpha-1}(-z) = \frac{(-2)^{\alpha}\sqrt{\pi}\,\Gamma(-\alpha)}{\Gamma(-\alpha+\frac{1}{2})}\ z^{\alpha}\, F\left(-\frac{\alpha}{2}+\frac{1}{2}, -\frac{\alpha}{2}; -\alpha+\frac{1}{2}; \frac{1}{z^2}\right) \ . \tag{III.64}$$

Since F depends only on z^2, we will require only even daughters. We begin with the sequence (III.47), summed over n even, and put

$$\beta_n(t) = t^{-\frac{1}{2}(\alpha_n(t)+n)}\,\bar{\beta}_n(t) \tag{III.65}$$

†See footnote on previous page.

and

$$\frac{e^{i\pi\alpha_n}}{e^{i\pi\alpha}} \cdot \frac{\Gamma(-\alpha_n)}{\Gamma(-\alpha)} \cdot \frac{\Gamma(-\alpha+\frac{1}{2})}{\Gamma(-\alpha_n+\frac{1}{2})} \cdot \frac{1+\tau e^{-i\pi\alpha_n}}{1+\tau e^{-i\pi\alpha}} \cdot \frac{\cos\pi\alpha}{\cos\pi\alpha_n} \cdot \frac{2^{\alpha_n}}{2^{\alpha}} \cdot \frac{\bar{\beta}_n(t)}{\bar{\beta}(t)} = b_n(t) \tag{III.66}$$

with

$$b_o(t) \equiv 1 .$$

The daughter sequence is now

$$f^{(t)}(t,z_t) = \frac{2\alpha(t)+1}{2} \cdot \frac{1+\tau e^{-i\pi\alpha}}{\cos\pi\alpha} \cdot \frac{\sqrt{\pi}\,(-2)^{\alpha}\,\Gamma(-\alpha)}{\Gamma(-\alpha+\frac{1}{2})}\,\bar{\beta}(t)\left(\frac{z_t}{\sqrt{t}}\right)^{\alpha}$$
$$\sum_{n=0,2,\ldots} b_n(t)\left(\frac{z_t}{\sqrt{t}}\right)^{\alpha_n+n-\alpha} z_t^{-n}\, F\left(-\frac{\alpha_n}{2}+\tfrac{1}{2},\ -\frac{\alpha_n}{2};\ -\alpha_n+\tfrac{1}{2};\ \frac{1}{z_t^2}\right) . \tag{III.67}$$

We put

$$x = z_t ,$$
$$v = x/\sqrt{t} ,$$

and define

$$\hat{f}(t,x) = \left(\frac{x}{\sqrt{t}}\right)^{\alpha} \sum_{n=0,2,\ldots} b_n(t)x^{-n}\, v^{\alpha_n+n-\alpha}\, F\left(-\frac{\alpha_n}{2}+\tfrac{1}{2},-\frac{\alpha_n}{2};\ -\alpha_n+\tfrac{1}{2};\frac{1}{x^2}\right) . \tag{III.68}$$

Analogously to (III.56) and the arguments that follow it, we require, for $\rho = 0$,

$$\sum_{n=0,2,\ldots} b_n(0)x^{-n}\, F\left(-\tfrac{1}{2}(\alpha(0)-n-1),\ -\tfrac{1}{2}(\alpha(0)-n);\ -\alpha(0)+n+\tfrac{1}{2};\ \frac{1}{x^2}\right) = 1 . \tag{III.69}$$

By putting $n = 2m$, this reduces to the same problem as (III.57) and the solution is

$$b_{2m}(0) = \frac{(-1)^m \left(\frac{1-\alpha}{2}\right)_m \left(-\frac{\alpha}{2}\right)_m}{m!\,(-\frac{1}{2} - \alpha + m)_m} \,.$$

Substituting into (III.66), we have for the daughter residues in the UE case

$$\beta_n^{UE}(t) = t^{\frac{\alpha_n}{2} - \frac{n}{2}} \bar{\beta}_n^{UE}(t) \tag{III.70}$$

with

$$\bar{\beta}_{2m}^{UE}(0) = \frac{(-1)^m}{m!} \cdot \frac{(2m - \alpha(0) - \frac{1}{2})\,\Gamma(m - \alpha - \frac{1}{2})}{\Gamma(\frac{1}{2} - \alpha)} \cdot \bar{\beta}^{UE}(0) \quad . \tag{III.71}$$

Now in Sec. III.C we derived a formula for the slope of the daughter trajectories. Since the trajectory is a property of the Regge pole itself, it must turn out that if we calculate the trajectory slopes in the UE case we should find the same result. If not, then the whole scheme is inconsistent. On the other hand, the universal property of a Regge pole comes from the unitarity condition, which links different processes together, and we have nowhere made use of this condition. Hence it is by no means obvious that the slopes calculated in the UE case will be compatible with (III.63).

Let us therefore take $\rho = 1$ and look at the equation governing the slopes. One has

$$\sum_{n=0,2,\ldots} b_n(0)[\alpha_n{}'(0) - \alpha'(0)] \log v \quad x^{-n}$$

$$F\left(-\tfrac{1}{2}(\alpha-n-1),\ -\tfrac{1}{2}(\alpha-n);\ -\alpha+n+\tfrac{1}{2};\ \frac{1}{x^2}\right)$$

$$= \frac{b_2(0)}{x^2}[\alpha_2{}'(0) - \alpha'(0)] \log v$$

which yields

$$\alpha_{2m}{}'(0) - \alpha'(0) = \frac{(2\alpha-2m+1)\,m}{2\alpha - 1}[\alpha_2{}'(0) - \alpha'(0)] \,. \tag{III.72}$$

Iterating (III.63) once, and putting $n = 2m$ gives exact agreement with (III.72).

Hence, for some non-obvious reasons, the UU and UE daughter sequences are consistent.

E. Daughters in the EE Case

Since in this case the mapping

$$(s,t) \to (t, z_t)$$

is nonsingular at $t = 0$, there is no need at all for daughters. One single Regge pole gives a contribution to $f^{(t)}(t, z_t)$ which is analytic at $t = 0$. However, if the daughters exist, then we must examine their role in the EE case.

By the factorization theorem we can calculate the EE residues from the known UU and UE ones. We have for the 2m-th daughter

$$\beta^{EE}_{2m}(t)\, \beta^{UU}_{2m}(t) = [\beta^{UE}_{2m}(t)]^2 \qquad \text{(III.73)}$$

Using (III.60) and (III.70) gives

$$\beta^{EE}_{2m}(t)\, t^{-\alpha_{2m} - 2m}\, \bar{\beta}^{UU}_{2m}(t) = t^{-\alpha_{2m} - 2m}\, [\bar{\beta}^{UE}_{2m}(t)]$$

showing that $\beta^{EE}(t)$ is regular at $t = 0$ and given by

$$\beta^{EE}_{2m}(0) = \frac{[\bar{\beta}^{UE}_{2m}(0)]^2}{\bar{\beta}^{UU}_{2m}(0)} \quad . \qquad \text{(III.74)}$$

Substituting (III.61) and (III.71), we find eventually

$$\beta^{EE}_{2m}(0) = \frac{(2m - \alpha - \frac{1}{2})\, \Gamma(m + \frac{1}{2})\, \Gamma(m - \alpha - \frac{1}{2})}{m!\, \Gamma(m - \alpha)} \, \frac{\Gamma(-\alpha)}{4\sqrt{\pi}\, \Gamma(\frac{1}{2} - \alpha)} \, \beta^{EE}(0) \ . \qquad \text{(III.75)}$$

Now we shall see later on, in the group theoretical approach to EE scattering, that a single Toller pole (which Toller insists on calling a Lorentz pole) gives rise to an infinite sequence of Regge poles at $t = 0$, spaced by integers, as in the above daughter sequences. It is a remarkable fact that the formula (III.75) for the residue of the 2m-th daughter agrees with the corresponding formula (see (V.56) and Ref. 8) derived from the Toller pole. Thus the requirement of analyticity in UU and UE reactions, plus factorization, leads to an EE sequence of daughters which just sums to one Toller pole. A full understanding of this phenomenon is not yet at hand.

There are several points in the above which require further clarification.

We have discussed only even daughters, i.e., trajectories at $\alpha(0) - 2m$ for the UE and EE cases, whereas the UU case had daughters at $\alpha(0) - n$. The reason is that odd and even daughters cannot both couple to an equal mass vertex, assuming that at the E vertex we have either $\bar{A} = C$ or $A = C$.

For example, if $A = C$ we have a coupling to a particle-antiparticle pair. Suppose they are bosons. Then if the Regge pole has isospin T and signature $\tau = (-1)^J$, we must have $\tau(-1)^T = +1$ to satisfy the Pauli principle. Now the whole family has the same T value, but the daughters have alternating values of τ. Hence there are two possibilities:

(i) If the parent couples to the E vertex, then its odd daughters will completely decouple at the E vertex. This is the situation which corresponds to the above analysis.

(ii) If the parent, and hence the even daughters, cannot couple to the E vertex, then the sequence in the UE case will be

$$\sum_{n=1,3,5,\ldots} \beta_n^{UE}(t)\,\frac{1+\tau_1 e^{-i\pi\alpha_n(t)}}{\cos\pi\alpha_n(t)}\,Q_{-\alpha_n(t)-1}(-z_t) \tag{III.76}$$

and we must take

$$\beta_1^{UE}(t) \propto t^{-\frac{1}{2}\alpha_1} \tag{III.77}$$

to get a finite result at $t = 0$ (cf. (III.70)). Then by factorization we will have

$$\beta_1^{EE}(t) = \frac{[\beta_1^{UE}(t)]^2}{\beta_1^{UU}(t)} \propto \frac{t^{-\alpha_1}}{t^{-\alpha_1 - 1}} \propto t\ . \tag{III.78}$$

Similarly,

$$\beta_{2n+1}^{EE}(t) \propto t\ .$$

Thus, in this case we have only odd daughters coupled to the EE reaction and they all decouple as $t \to 0$. Hence the whole sequence decouples at $t = 0$.

IV. The Effect of Spin

When the external particles have nonzero spin the situation becomes very much more interesting. As mentioned in the Introduction, even in the EE case, where there are no singularities at $t = 0$, the effect of spin, working via the conservation of angular momentum in the s-channel, forces certain linear combinations of t-channel helicity amplitudes to vanish as $t \to 0$ at certain prescribed rates. This, in turn, requires the existence of correlated Regge poles and results in a rather unexpected spin dependence of the scattering amplitudes as $t \to 0$.

Consider forward scattering

$$A + B \to C + D$$

in the C.M. of the s-channel, and let the incoming particles have helicity a, b, and the outgoing ones c, d. Since all particles are moving along in one direction, say the z-axis, the orbital angular momentum of each of them, being perpendicular to their direction of motion, has no component along OZ. Hence if we consider the conservation of J_z, the z-component of the total angular momentum, we have:†

$$\text{Initially:} \quad J_z = a - b \ ,$$

$$\text{Finally:} \quad J_z = c - d \ .$$

So conservation of J_z at $\theta_s = 0$ requires

$$a - b = c - d \ .$$

In other words, the s-channel helicity amplitude for an arbitrary transition $f^{(s)}_{cd;ab}(s, \theta_s)$ must satisfy

$$f^{(s)}_{cd;ab}(s, \theta_s = 0) = 0 \text{ unless } a - c = b - d \ . \qquad \text{(IV.1)}$$

Now one can refine the above argument to show that the larger we make $|(a - b) - (c - d)|$ the faster $f^{(s)}_{cd;ab} \to 0$ as $\theta_s \to 0$.

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†Remember that according to the Jacob-Wick convention, B's helicity is its spin projection along its direction of motion, i.e., in the minus z direction if A is moving in the plus z direction.

The result is

$$f^{(s)}_{cd;ab}(s,\theta_s) \propto (\sin\theta_s/2)^{|(a-c)-(b-d)|} \text{ as } \theta_s \to 0. \qquad (IV.2)$$

One (non-rigorous) way to see this is to note that in the Jacob-Wick partial wave expansion of $f^{(s)}$:

$$f^{(s)}_{cd;ab}(s,\theta_s) = \sum_J (2J+1)\, f^J_{cd;ab}(s)\, d^J_{\tilde\lambda\tilde\mu}(\theta_s) \qquad (IV.3)$$

where $\tilde\lambda = a-b$; $\tilde\mu = c-d$; each $d^J_{\tilde\lambda\tilde\mu}$ has in it a factor

$$\left(\frac{1-z_s}{2}\right)^{\frac{1}{2}|\lambda-\mu|} = (\sin\theta_s/2)^{|\lambda-\mu|} \qquad (IV.4)$$

where $\lambda = a-c$, $\mu = b-d$ which immediately gives (IV.2).†

Now θ_s is given by

$$\sin\theta_s = \frac{2[s\,\varphi(s,t)]^{\frac{1}{2}}}{g_{AB}\, g_{CD}}, \quad 0 \le \theta_s \le \pi \qquad (IV.5)$$

where $\varphi(s,t)$ is the usual function specifying the boundaries of the physical regions:

$$\varphi(s,t) = st[\Sigma-s-t] - s(m_B^2-m_D^2)(m_A^2-m_C^2) - t(m_A^2-m_B^2)(m_C^2-m_D^2)$$

$$- (m_A^2 m_D^2 - m_B^2 m_C^2)(m_A^2 - m_C^2 + m_D^2 - m_B^2) \qquad (IV.6)$$

and

$$g_{ij}^2 = [s-(m_i-m_j)^2][s-(m_i+m_j)^2]. \qquad (IV.7)$$

Notice that in an EE reaction, and only in this case, i.e., if $m_A = m_C$ and $m_B = m_D$, we have

$$\sin\theta_s \propto t^{\frac{1}{2}}. \qquad (IV.8)$$

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†One can derive (IV.2) directly from the covariance conditions discussed in Sec. V.

Hence, from (IV.2), for EE reactions,

$$f^{EE(s)}_{cd;ab}(s,\theta_s) \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda-\mu|} \quad . \tag{IV.9}$$

This implies, via crossing, that

$$\sum M^{c'\bar{d}'\bar{a}'b'}_{cdab} \; f^{(t)}_{c'\bar{a}';\bar{d}'b'}(t,z_t) \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda-\mu|} \; . \tag{IV.10}$$

These are referred to as equations of constraint.

It would appear from the above that the condition (IV.2) is only relevant to the behavior at t = 0 in the EE case, in the form of (IV.9) or (IV.10). However, we are studying asymptotic behavior, and it is legitimate to ask about the behavior of the leading term in s as s → ∞.

Provided we keep only the leading term in s at fixed nonzero t, we have from (IV.4) that

$$\sin\theta_s \approx 2\sqrt{-t/s} \tag{IV.11}$$

independently of the masses, i.e., to leading order

$$\sin\theta_s \underset{t\to 0}{\propto} t^{\frac{1}{2}} \; .$$

Thus in leading order in s, there is no distinction between the various mass situations and in all cases

$$f^{(s)}_{cd;ab}(s,\theta_s) \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda-\mu|} \quad \text{(leading order in s)}. \tag{IV.12}$$

A word of caution is needed in connection with (IV.4) and (IV.12). The rate of vanishing as t → 0 as given represents a minimum rate. Amplitudes can vanish faster, and indeed do so in various models. However they may not vanish less rapidly without violating analyticity. The behavior (IV.4) and (IV.12) has been called the "kinematically normal behavior" or k.n.b.

The problems we had earlier in the UU and UE spinless cases, concerning the analyticity of the nonleading terms in the asymptotic expansion, will again appear when spin is present; and we shall deal with this in Sec. IV.C. However there are entirely new features which emerge when spin is present and it is fortunate that they can be understood in some detail simply by looking at the leading term as $s \to \infty$. Thus we shall begin by considering only the leading term in the contribution of a Regge pole to $f^{(s)}$.

A. Spin: Treatment to Leading Order in s

When spin is present, the contribution of a single bosonic (or even Fermion number) Regge pole to the t-channel helicity amplitudes is given by[13]

$$f^{(t)}_{c\bar{a};\bar{d}b}(t,z_t) = \frac{2\alpha+1}{2} \cdot \frac{1+\tau e^{i\pi\alpha}}{\cos\pi\alpha} \cdot (-1)^{\lambda'} \beta_{c\bar{a};\bar{d}b}\, e^{-\alpha-1}_{-\lambda',\mu'}(-z_t) \tag{IV.13}$$

where

$$\lambda' = \bar{d} - b \quad ,$$

$$\mu' = c - \bar{a} \quad ,$$

and the functions $e^{-\alpha-1}_{-\lambda',\mu'}$ are analogous to the $Q_{-\alpha-1}$. To specify them, define

$$m' = \max\{|\lambda'|;|\mu'|\} ,$$

$$n' = \min\{|\lambda'|;|\mu'|\} ,$$

$$\mathscr{A} = \text{sign}(\lambda'\mu') \tag{IV.14}$$

Then

$$(-1)^{\lambda'} e^{-\alpha-1}_{-\lambda',\mu'}(-z) = \frac{\zeta(\lambda',\mu')}{2\Gamma(-2\alpha)}\{\Gamma(m'-\alpha)\Gamma(-m'-\alpha)\Gamma(n'-\alpha)\Gamma(-n'-\alpha)\}^{\frac{1}{2}}$$
$$\times\left(\frac{1-z}{2}\right)^{\frac{1}{2}|\lambda'-\mu'|}\left(\frac{1+z}{2}\right)^{\frac{1}{2}|\lambda'+\mu'|}$$
$$\times\left(\frac{1-z}{2}\right)^{\alpha-m'} F\left(-\alpha+m',-\alpha+\mathscr{A}n';-2\alpha;\frac{2}{1-z}\right) \tag{IV.15}$$

where

$$\zeta(\lambda',\mu') = (-1)^{\lambda'} \quad \text{if } \lambda'<\mu' ,$$
$$= -(-1)^{\mu'} \quad \text{if } \lambda'>\mu' . \tag{IV.16}$$

It is worth noting the trivial but useful formulae

$$|\lambda' + \mu'| = m' + \not{n}' \quad ,$$

$$|\lambda' - \mu'| = m' - \not{n}' \quad . \tag{IV.17}$$

Let us redefine the residue so as to absorb several factors. We put

$$\beta = t^{-\alpha}\,\bar{\beta} \qquad \text{in the UU case,}$$

$$\beta = t^{-\alpha/2}\,\bar{\beta} \qquad \text{in the UE case,}$$

$$\beta = \bar{\beta} \qquad \text{in the EE case,} \tag{IV.18}$$

and define

$$\gamma_{c\bar{a};\bar{d}b}(t) = \frac{2\alpha+1}{2}\,\frac{1+\tau e^{i\pi\alpha}}{\cos\pi\alpha}\left\{\Gamma(m'-\alpha)\Gamma(-m'-\alpha)\Gamma(n'-\alpha)\Gamma(-n'-\alpha)\right\}^{\frac{1}{2}}$$

$$\times\frac{1}{2\Gamma(-2\alpha)}\,\bar{\beta}_{c\bar{a};\bar{d}b} \quad . \tag{IV.19}$$

Note that γ is still factorizable if $\bar{\beta}$ is. Then

$$f^{(t)}_{c\bar{a};\bar{d}b}(t,z_t) = \zeta(\lambda',\mu')\,L(t)\,\gamma_{c\bar{a};\bar{d}b}(t)\left(\frac{1+z_t}{1-z_t}\right)^{\frac{1}{2}(m'+\not{n}')}$$

$$\left(\frac{1-z_t}{2}\right)^{\alpha} F\left(-\alpha+m',\,-\alpha+\not{n}';\,-2\alpha;\,\frac{2}{1-z_t}\right) \tag{IV.20}$$

where

$$L(t) = t^{-\alpha},\ t^{-\alpha/2},\ 1 \ \text{for UU, UE, EE respectively.} \tag{IV.21}$$

Now the above refers to the contribution of one single Regge pole with definite signature τ and parity P. So strictly speaking, γ should be labelled $\gamma_{c\bar{a};\bar{d}b}(\tau,P)$. We shall need the following very important symmetry property of γ:†

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†This symmetry which was derived in Ref. 4 can be seen as follows. Define helicity states which have a definite parity

$$|J;\bar{d}b;\sigma\rangle = \tfrac{1}{2}\left\{|J;\bar{d}b\rangle + \sigma\,\zeta_D\,\zeta_B(-1)^{S_D+S_B}\,|J;-\bar{d}-b\rangle\right\}$$

(continued on following page)

$$\gamma_{c\bar{a};-\bar{d}-b}(\tau,P) = \tau\, P\, \zeta_{\bar{D}}\zeta_B(-1)^{S_D+S_B}\,\gamma_{c\bar{a};\bar{d}b}(\tau,P) \qquad \text{(IV.22)}$$

where $\zeta_{\bar{D}}$, ζ_B are the intrinsic parities of $\bar{D}$ and B and S_D, S_B their spins.

Let us put

$$\xi_{\bar{D}B} = \zeta_{\bar{D}}\,\zeta_B(-1)^{S_{\bar{D}}+S_B} \quad . \qquad \text{(IV.23)}$$

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with $\sigma = \pm 1$. Then under the operation of parity $\hat{\mathcal{P}}$:

$$\hat{\mathcal{P}}|J;\bar{d}b;\sigma\rangle = \sigma(-1)^J\,|J;\bar{d}b;\sigma\rangle$$

So these states have a definite parity $(-1)^J\sigma$.

Then the partial wave amplitude

$$T^J_{c\bar{a};\bar{d}b} = \langle J;\, c\bar{a}|T|J;\,\bar{d}b\rangle$$

can be written in terms of partial wave amplitudes

$$T^{J,\sigma}_{c\bar{a};\bar{d}b} = \langle J;\, c\bar{a};\,\sigma|T|J;\,\bar{d}b;\,\sigma\rangle$$

corresponding to transitions of definite J and P, as follows:

$$T^J_{c\bar{a};\bar{d}b} = \tfrac{1}{2}\left(T^{J,\sigma}_{c\bar{a};\bar{d}b} + T^{J,-\sigma}_{c\bar{a};\bar{d}b}\right)$$

Under Reggeization a Regge pole of parity P and signature τ will appear only as a pole in the amplitude with $\sigma = \tau P$.

Now since from its definition

$$|J;\, -\bar{d}-b;\,\sigma\rangle = \sigma\zeta_D\zeta_B(-1)^{S_D+S_B}\,|J;\,\bar{d}b;\,\sigma\rangle$$

we have

$$T^{J,\sigma}_{c\bar{a};-\bar{d}-b} = \sigma\zeta_D\zeta_B(-1)^{S_D+S_B}\,T^{J,\sigma}_{c\bar{a};\bar{d}b}$$

and since $\beta_{c\bar{a};\bar{d}b}$ is the residue of T, we will have

$$\beta^{\tau,P}_{c\bar{a};-\bar{d}-b} = \tau\, P\, \zeta_D\zeta_B(-1)^{S_D+S_B}\,\beta^{\tau,P}_{c\bar{a};\bar{d}b}$$

and γ then has the same property.

It should be stressed that in any parity conserving theory we would have a relation of the type

$$\gamma_{-c-\bar{a};-\bar{d}-b} = \text{phase} \times \gamma_{c\bar{a};\bar{d}b} \ .$$

However, (IV.22) is much stronger, and is a direct consequence of the exchange of one P value in the t-channel.

Note also that if we utilize factorization to write

$$\gamma_{c\bar{a};\bar{d}b} = \gamma_{c\bar{a}}\,\gamma_{\bar{d}b} \tag{IV.24}$$

then also

$$\gamma_{-\bar{d}-b} = \tau\, P\, \xi_{\bar{D}B}\, \gamma_{\bar{d}b} \ . \tag{IV.25}$$

Let us now calculate the leading order term as $s \to \infty$. We do this at fixed, nonzero t, so that $s \to \infty$ corresponds to $z_t \to \infty$. Firstly, from (IV.20) and (IV.24), to leading order

$$f^{(t)}_{ca;db} \underset{s\to\infty}{\sim} \pm \gamma_{ca}\,\gamma_{db}\, e^{\frac{i\pi}{2}\lambda'}\, e^{-\frac{i\pi}{2}\mu'}\, s^{\alpha(t)} \ . \tag{IV.26}$$

Next, to calculate the leading term in $f^{(s)}$ we must use the crossing matrices keeping only their leading term. This is obtained by putting $s = \infty$ in the expression for the crossing angles and the resulting expressions are then independent of the external masses. Using (IV.26) and (IV.22), one can then show that to leading order in s

$$f^{(s)}_{c-\bar{d};\bar{a}-b} = \tau\, P(-1)^{\bar{d}-b}\, \xi_{\bar{D}B}\, f^{(s)}_{c\bar{d};\bar{a}b} \ . \tag{IV.27}$$

This is a fundamental result, for we shall see that (IV.27) is incompatible with (IV.12).

Suppose at first that we are exchanging one Regge pole; so that the phase factor in (IV.27) is fixed. Then from (IV.12) and (IV.4),

$$f^{(s)}_{c-\bar{d};\bar{a}-b} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda+\mu|} \tag{IV.28}$$

and therefore, using (IV.27) also,

$$f^{(s)}_{cd;ab} = \tau\, P\, \xi_{DB} (-1)^{\bar{d}-b}\, f^{(s)}_{c-\bar{d};\bar{a}-b} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda+\mu|} . \qquad (IV.29)$$

But by (IV.12) directly

$$f^{(s)}_{c\bar{d};\bar{a}b} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda-\mu|} \qquad (IV.30)$$

which contradicts (IV.29) unless μ or λ or both happen to be zero.

The only way to make (IV.28) and (IV.12) consistent with each other is to make both $f^{(s)}_{c\bar{d};\bar{a}b}$ and $f^{(s)}_{c-\bar{d};\bar{a}-b}$ vanish at the faster of the two rates, i.e., we must take

$$f^{(s)}_{cd;ab} \underset{t\to 0}{\propto} t^{\frac{1}{2}(|\lambda| + |\mu|)} \qquad (IV.31)$$

Hence the spin dependence of $f^{(s)}$, in a one pole model, is much more restricted than the most generally allowed type given in (IV.12).

A classic example of this phenomenon occurs in nucleon-nucleon scattering. There, the amplitude

$$\varphi_2 = f^{(s)}_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}}$$

has (a-c) - (b-d) = 0, so that φ_2 can go to a constant as $t \to 0$. However, in a one pole model, by (IV.31), we would get

$$\varphi_2 \propto t \quad .$$

The highly restricted spin behavior (IV.31) seems rather unrealistic, and certainly at present day energies does not correspond to experiment. Thus we must try to get a less restrictive behavior by taking a model in which two Regge poles are exchanged. Let us call the poles (1) and (2). Then, to leading order in s, we have

$$f^{(s)}_{cd;ab} = f^{(s)(1)}_{cd;ab} + f^{(s)(2)}_{cd;ab} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda-\mu|} \qquad (IV.32)$$

and, using (IV.27),

$$f^{(s)}_{c-d;a-b} = (-1)^{d-b}\,\xi_{\bar{D}B}\left\{(\tau P)_1\, f^{(s)^{(1)}}_{cd;ab} + (\tau P)_2 f^{(s)^{(2)}}_{cd;ab}\right\} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda+\mu|} . \tag{IV.33}$$

Clearly if $(\tau P)_1 = (\tau P)_2$ we are back where we started and (IV.33) will contradict (IV.32). But if $(\tau P)_1 = -(\tau P)_2$ then (IV.33) is equivalent to

$$f^{(s)^{(1)}}_{cd;ab} - f^{(s)^{(2)}}_{cd;ab} \underset{t\to 0}{\propto} t^{\frac{1}{2}|\lambda+\mu|} \tag{IV.34}$$

which does not contradict (IV.32).

Without loss of generality, let us take $(\tau P)_1 = +1$, $(\tau P)_2 = -1$ and relabel the poles + and −. As $s \to \infty$, we have

$$f^{(s)^{+}} \sim s^{\alpha_+(t)} ,$$

$$f^{(s)^{-}} \sim s^{\alpha_-(t)} , \tag{IV.35}$$

where $\alpha_\pm(t)$ are the trajectories of the ± type poles. Since (IV.32) and (IV.34) have to remain compatible as s varies, we clearly need

$$\alpha_+(0) = \alpha_-(0) . \tag{IV.36}$$

Thus in order to break away from the restrictive behavior (IV.31), we require the existence of a second Regge pole, a conspirator, with opposite τP, and whose trajectory satisfies (IV.36). This is often called a "parity doublet" conspiracy.

It might be hoped that now that we have introduced a conspirator, $f^{(s)}$ can have the most general allowed behavior as $t \to 0$, as given in (IV.12). We shall see that on the contrary the behavior is still highly restricted, though very different from (IV.31).

Let us first note that the crossing matrix M completely factorizes:

$$M^{c'\bar{d}'\bar{a}'b'}_{cdab}(s=\infty,t) = M^{c'\bar{a}'}_{ca}(\infty,t)\; M^{\bar{d}'b'}_{db}(\infty,t) .$$

Then if we define a kind of s-channel residue by

$$\gamma^{(s)}_{ca}(t) = M^{c'\bar{a}'}_{ca}(\infty,t)\; e^{\frac{i\pi}{2}(c'-\bar{a}')}\,\gamma_{c'\bar{a}'}(t) \tag{IV.37}$$

and similarly $\gamma_{db}^{(s)}$, then in leading order, from (IV.26),

$$f_{cd;ab}^{(s)} \approx (-1)^{d-b}\, \gamma_{ca}^{(s)}(t)\, \gamma_{db}^{(s)}(t)\, s^{\alpha(t)} \quad . \qquad \text{(IV.38)}$$

Thus the helicity dependence of $f^{(s)}$ factorizes to leading order in s. We shall now see that Eqs. (IV.32) and (IV.34) are incompatible with this factorizability.

Adding and subtracting (IV.32) and (IV.34) yields

$$f_{cd;ab}^{(s)+} \propto t^{\frac{1}{2}|\lambda-\mu|} + t^{\frac{1}{2}|\lambda+\mu|} \quad ,$$

$$f_{cd;ab}^{(s)-} \propto t^{\frac{1}{2}|\lambda-\mu|} - t^{\frac{1}{2}|\lambda+\mu|} \quad . \qquad \text{(IV.39)}$$

Thus both f^{+} and f^{-} will have the same dominant behavior as $t \to 0$ and this will be given by that term in (IV.39) which has the smaller exponent. We can summarize the situation as follows: Define

$$m = \max\left\{|\lambda|;\, |\mu|\right\} \quad ,$$

$$n = \min\left\{|\lambda|;\, |\mu|\right\} \quad ,$$

$$\sigma = \text{sign}\,(\lambda\,\mu) \quad . \qquad \text{(IV.40)}$$

Then

$$f_{cd;ab}^{(s)+} = \sigma f_{cd;ab}^{(s)-} \underset{t \to 0}{\propto} t^{\frac{1}{2}(m-n)} \quad . \qquad \text{(IV.41)}$$

Clearly this behavior is not factorizable. So $f^{(s)+}$ and $f^{(s)-}$ cannot have the general behavior given by (IV.41), which would in turn have given $f^{(s)}$ in (IV.32) the most generally allowed behavior.

We now wish to find the most general behavior for $f^{(s)\pm}$ which will be compatible with (IV.32) and (IV.34). We put for each of the poles

$$f_{cd;ab}^{(s)} \underset{t \to 0}{\propto} t^{\frac{1}{2}\{|\lambda| - g(\lambda) + |\mu| - g(\mu)\}} \quad . \qquad \text{(IV.42)}$$

This is manifestly factorizable, and since g is unspecified, perfectly general. If $g(\lambda) = 0$ we have the one-pole behavior (IV.31). So we wish to try to choose $g(\lambda)$ as large and positive as possible, thereby getting away as far as possible from the restrictive one-pole behavior.

From (IV.41) and (IV.42), $g(\lambda)$ must be such that

$$|\lambda| + |\mu| - g(\lambda) - g(\mu) \geq \mathfrak{m} - \mathfrak{n} \qquad \text{(IV.43)}$$

or

$$g(\lambda) + g(\mu) \leq 2\mathfrak{n} \quad . \qquad \text{(IV.44)}$$

We now show that this equation plus the principle of making $g(\lambda)$ as large and positive as possible allows us to specify $g(\lambda)$ uniquely. We shall construct $g(\lambda)$ step by step and the procedure is illustrated in Fig. 2.

(i) Put $\lambda = \mu$. Then (IV.44) implies

$$g(\lambda) \leq |\lambda| \quad \text{for all } \lambda \ . \qquad \text{(IV.45)}$$

Try to take $g(\lambda) = |\lambda|$ for some value of λ, say $\lambda = M$, in order to make g large; i.e.,

$$g(M) = |M| \quad . \qquad \text{(IV.46)}$$

(ii) Put $\lambda = M$, $\mu = -M$. Then by (IV.44) and (IV.46),

$$g(-M) \leq |M| \quad .$$

Thus we can take also $g(-M) = |M|$.

(iii) Put $\lambda = M$, $\mu = |M| + n$ $(n > 0)$. Then (IV.44) gives

$$g(M) + g(|M| + n) \leq 2\,|M|$$

or, using (IV.46),

$$g(|M| + n) \leq |M| \quad . \qquad \text{(IV.47)}$$

Similarly, putting $\lambda = M$, $\mu = -|M| - n$ $(n > 0)$ gives

$$g(-|M| - n) \leq |M| \quad . \qquad \text{(IV.48)}$$

(iv) Put $\lambda = M$, $\mu = |M| - n \geq 0$ $(n > 0)$. Then from (IV.44),

$$g(M) + g(|M| - n) \leq 2(|M| - n)$$

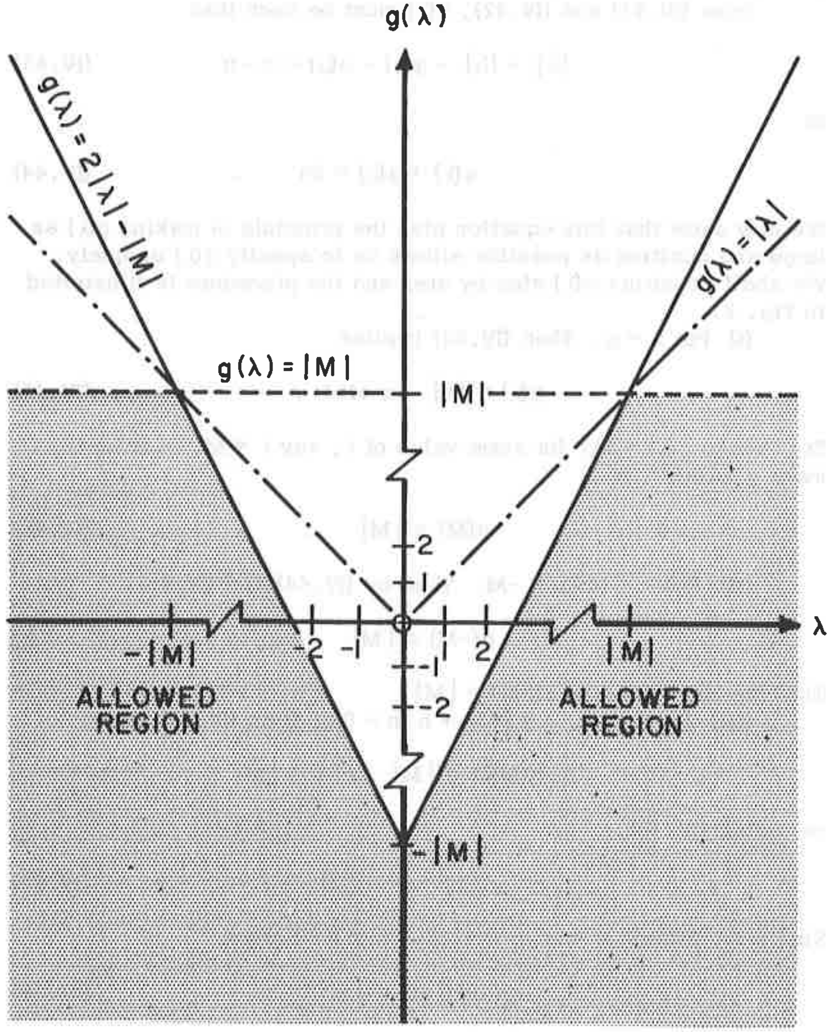

Fig. 2. Optimal solution for g(λ).

or

$$g(|M| - n) \leq |M| - 2n$$

which we can rewrite as

$$g(\mu) \leq 2\mu - |M| \quad \text{for } 0 \leq \mu \leq |M| . \tag{IV.49}$$

Similarly,

$$g(\mu) \leq -2\mu - |M| \quad \text{for } -|M| \leq \mu \leq 0 . \tag{IV.50}$$

Hence we see that:

(a) There exist at most two values of λ, $\lambda = \pm M$ at which $g(\lambda) = |\lambda|$.

(b) Once the value M is chosen, and this is arbitrary, then $g(\lambda)$ has to lie in the shaded region of Fig. 2 for all λ.

Clearly, the optimal choice we can make is to take $g(\lambda)$ as given by the boundary curve of the shaded region in Fig. 2. This corresponds to choosing equality in Eqs. (IV.47-50).

Hence the optimal choice is

$$g(\lambda) = g(|\lambda|) \tag{IV.51}$$

with

$$g(\lambda) = M \qquad \text{for } |\lambda| > M,$$

$$g(M) = M \qquad ,$$

$$g(\lambda) = 2|\lambda| - M \qquad \text{for } |\lambda| < M, \tag{IV.52}$$

where M is now a positive integer. This for any M is optimal in the sense that $g(\lambda)$ is as large as possible for all λ.

Thus the optimal behavior for $f^{(s)}$ is characterized by an integer M. This M is a kind of quantum number attached to the pair of conspiring Regge poles. It will turn out in the group theoretical analysis of EE reactions at $t = 0$, that M can be identified with one of the labels of a Toller pole in the O(3,1) expansion of the scattering amplitude. However, as introduced above, M appears to have a more general significance, and plays a role even in the UU and UE cases where the O(3,1) analysis is inapplicable.

Substituting (IV.52) into (IV.42) gives

$$f^{(s)^{\pm}}_{cd;ab} \underset{t\to 0}{\propto} t^{\frac{1}{2}|M-|\lambda||} \cdot t^{\frac{1}{2}|M-|\mu||} . \qquad \text{(IV.53)}$$

This gives some physical insight into the meaning of M. Only those amplitudes with $\lambda = \pm M$ and $\mu = \pm M$ survive as $t \to 0$. In other words, when a Regge pole of type M couples to a vertex which has a spin flip λ, as viewed from the s-channel C.M., then as $t \to 0$ only those vertices with $\lambda = \pm M$ can go to a nonzero constant.

In the above, we constructed $g(\lambda)$ to ensure that (IV.41) is not violated. However, this is not yet sufficient to satisfy both (IV.39) and (IV.40). For we must also have

$$f^{(s)^{+}}_{cd;ab} - \epsilon f^{(s)^{-}}_{cd;ab} \underset{t\to 0}{\propto} t^{\frac{1}{2}(\mathfrak{m}+\mathfrak{n})} . \qquad \text{(IV.54)}$$

Let us put, for the leading term in s,

$$f^{(s)^{\pm}}_{cd;ab} = t^{\frac{1}{2}\{|M-|\lambda||+|M-|\mu||\}} \hat{\beta}^{\pm}_{cd;ab}(t)\, s^{\alpha_{\pm}(t)} \qquad \text{(IV.55)}$$

where $\hat{\beta}(t)$ is analytic, nonzero at $t = 0$. Then (IV.54) implies

$$\hat{\beta}^{+}_{cd;ab}(t)\, s^{\alpha_{+}(t)} - \epsilon \hat{\beta}^{-}_{cd;ab}(t)\, s^{\alpha_{-}(t)} \propto t^{\nu_M} \qquad \text{(IV.56)}$$

where

$$\nu_M = \tfrac{1}{2}\left\{\mathfrak{m} + \mathfrak{n} - |M - |\lambda|| - |M - |\mu||\right\} . \qquad \text{(IV.57)}$$

Now in certain situations $\nu_M \le 0$. In these cases (IV.56) puts no restrictions on the $\hat{\beta}^{\pm}$. But if $|\lambda| + |\mu| > M$ then one can show that $\nu_M > 0$ and in these cases $\hat{\beta}^{+}$ is related to $\hat{\beta}^{-}$. The maximum value of ν_M is M and this is attained whenever both $|\lambda| \ge M$ and $|\mu| \ge M$. Choosing helicities in this range, (IV.56) implies that

$$\frac{\partial^m}{\partial t^m}\left(\hat{\beta}^{+}_{cd;ab}(t)\, s^{\alpha_{+}(t)}\right)\Bigg|_{t=0} = \epsilon \frac{\partial^m}{\partial t^m}\left(\hat{\beta}^{-}_{cd;ab}(t)\, s^{\alpha_{-}(t)}\right)\Bigg|_{t=0} ,$$

$$m = 0, 1, 2, \ldots M-1 \qquad \text{(IV.58)}$$

Since these have to hold for arbitrary large s, it follows, on differentiating the product, that also

$$\frac{d^m}{dt^m}\,\alpha_+(t)\bigg|_{t=0} = \frac{d^m}{dt^m}\,\alpha_-(t)\bigg|_{t=0}, \quad m = 0, 1, 2, \ldots M-1. \qquad (IV.59)$$

So for a conspiracy of the optimal type characterized by M, the first M-1 derivatives of the trajectories $\alpha_+(t)$, $\alpha_-(t)$ must be equal at $t = 0$. Similar results hold for those $\hat{\beta}^{\pm}_{cd;ab}$ whose helicity labels satisfy $|\lambda| \geq M$, $|\mu| \geq M$.

This completes the specification of the relationship between the conspiring poles. They are now guaranteed to give amplitudes $f^{(s)\pm}$ that satisfy both (IV.39) and (IV.40).

Let us now see how the complete amplitude $f^{(s)} = f^{(s)+} + f^{(s)-}$ behaves as $t \to 0$. Using (IV.53) and (IV.56), we get finally

$$f^{(s)}_{cd;ab} \underset{t \to 0}{\propto} t^{\frac{1}{2}X} \,.\, t^{\frac{1}{2}(\mathfrak{m} - \sigma \mathfrak{n})} \qquad (IV.60)$$

where

$$\begin{aligned} X &= (1 + \sigma)(\mathfrak{n} - M) && \text{for } \mathfrak{n} \geq M \\ &= 0 && \text{for } \mathfrak{n} \leq M \leq \mathfrak{m} \\ &= (1 + \sigma)(M - \mathfrak{m}) && \text{for } \mathfrak{m} \leq M \leq \mathfrak{m} + \mathfrak{n} \\ &= 2(M - \mathfrak{m}) + (\sigma - 1)\mathfrak{n} && \text{for } \mathfrak{m} + \mathfrak{n} \leq M \,. \qquad (IV.70) \end{aligned}$$

Note that the most generally allowed behavior is just (see (IV.12))

$$t^{\frac{1}{2}(\mathfrak{m} - \sigma \mathfrak{n})}$$

so X measures the deviations from this. Note that in general $X \neq 0$, so we have failed to produce the most general behavior. The reason for this failure can be traced to the property of factorizability.

Now in the above, we were working to leading order in s. However, in the Regge pole model, the residue function <u>multiplies</u> the function of s (or z_t) which has the asymptotic expansion in powers of 1/s and therefore the coefficient of the leading term in s is the true residue function (aside from trivial factors). Thus we can use the leading order treatment to determine the behavior of the residue function.

B. Behavior of the Residue Functions as $t \to 0$

We have above the behavior as $t \to 0$ of the leading term for large s of $f^{(s)}$. We can now invert the crossing matrix and calculate the behavior as $t \to 0$ of the leading term for large s of $f^{(t)}$ considered as a function of s and t. However, the crossing matrices are sensitive to the external masses, so the calculation has to be done separately for U and E vertices.

Consider first the case $m_C \neq m_A$. Noting that the crossing angles χ behave as follows:

$$\begin{array}{l} \chi_A(\infty,t) \to 0 \\ \qquad\qquad\qquad \text{as } t \to 0 \\ \chi_C(\infty,t) \to 0 \end{array} \tag{IV.71}$$

we get that

$$M_{ca}^{c'\bar{a}'}(\infty,t) \underset{t\to 0}{\propto} t^{\frac{1}{2}\{|c'-c| + |\bar{a}'-a|\}} \quad . \tag{IV.72}$$

Then, inverting (IV.37) and using (IV.53), we get

$$\overset{U}{\gamma}{}^{\pm}_{c'\bar{a}'}(t) \underset{t\to 0}{\propto} t^{\frac{1}{2}|M - |c'-\bar{a}'||} \tag{IV.73}$$

for the factorized piece of the t-channel residue for a U vertex (see (IV.19) and (IV.24)).

A similar result holds for $\gamma^{\pm}_{\bar{d}b}$ if $m_D \neq m_B$.

Also from (IV.54) it follows that for a UU reaction

$$\gamma^{+}_{c'\bar{a}';\bar{d}'b'} + \not{\sigma}'\gamma^{-}_{c'\bar{a}';\bar{d}'b'} \propto t^{\frac{1}{2}(|\lambda'| + |\mu'|)}$$

$$\text{for } |\lambda'| + |\mu'| \geq M \quad . \tag{IV.74}$$

Otherwise, if the helicities are outside this range, there is no special relationship between the ± residues. (The plus sign in (IV.74) comes about because $\chi_B(\infty,t)$ and $\chi_D(\infty,t) \to \pi$ as $t \to 0$.)

Consider now the case $m_A = m_C$. Then χ_A and $\chi_C \to \pi/2$ as $t \to 0$. Inverting (IV.37), one can then pick out the dominant term in the sum for $\overset{E}{\gamma}{}^{\pm}_{c'\bar{a}'}$, which is the term with $|c-a| = M$. This would then give $\gamma^{\pm}_{c'\bar{a}'} \to$ constant as $t \to 0$. However there is a complication owing to the fact that the sum runs over both $\gamma^{(s)}_{ca}$ and $\gamma^{(s)}_{-c-a}$ and the symmetry of $\gamma^{(s)}$ under this reversal of helicities (cf., (IV.27) and (IV.38)) then leads to certain cancellations in the sum.

The overall result is

$$\gamma^{E(\tau P)}_{c'\bar{a}'}(t) \xrightarrow[t\to 0]{} \text{constant} \qquad \text{if} \quad \tau P(-1)^{M+c'-\bar{a}'} = 1$$

$$\propto \sqrt{t} \qquad \text{if} \quad \tau P(-1)^{M+c'-\bar{a}'} = -1$$

or, more neatly,

$$\gamma^{E(\tau P)}_{c'\bar{a}'}(t) \underset{t\to 0}{\propto} t^{\frac{1}{4}\{1-\tau P(-1)^{M+c'-\bar{a}'}\}} \quad . \tag{IV.75}$$

Notice that this implies that only one of the members of the conspiring pair can couple at t = 0 to an E-type vertex.

Unfortunately there are two complications which modify the range of validity of (IV.75):

(i) Let us call the common spin at the E vertex $s_E = s_A = s_C$. Then the maximum value that $|c-a|$ can ever achieve is $2s_E$. Hence if $M > 2s_E$, then our argument above fails and we cannot pick up a term with $|c-a| = M$ in the sum. The best we can do is to pick up the term with $|c-a| = 2s_E$, which then give an additional factor $t^{\frac{1}{2}|M-2s_E|}$. Thus

$$\gamma^{E(\tau P)}_{c'\bar{a}'} \underset{t\to 0}{\propto} t^{\frac{1}{2}|M-2s_E|} \cdot t^{\frac{1}{4}\{1-\tau P(-1)^{2s_E+c'-\bar{a}'}\}} \text{ if } M > 2s_E \; . \tag{IV.76}$$

(ii) When M = 0, (IV.75) holds only for Regge poles with $PG(-1)^T = +1$ or $PC = +1$, whichever is applicable in the given reaction. For poles with $PG(-1)^T = -1$ or $PC = -1$ and $M = 0$, one has instead of (IV.75)

$$\gamma^{E(\tau P)}_{c'\bar{a}'}(t) \propto \sqrt{t} \quad \text{if} \quad \tau P(-1)^{c'-\bar{a}'} = -1$$

$$\propto t \quad \text{if} \quad \tau P(-1)^{c'-a'} = +1 \; . \tag{IV.77}$$

The readon for the latter behavior is that there is an additional symmetry at an E vertex

$$\gamma^{(s)}_{ca}(t) = \begin{pmatrix} PG(-1)^T \\ \text{or} \\ PC \end{pmatrix} (-1)^{a-c}\, \gamma^{(s)}_{ac}(t) \qquad \text{(IV.78)}$$

which makes the dominant terms in the sum, which for $M = 0$ are the terms $\gamma^{(s)}_{aa}(t)$, identically zero when $PG(-1)^T$ or PC equal -1. The above gives a complete specification of the behavior of all residue functions as $t \to 0$. To get this, we used an optimal behavior for $g(\lambda)$. It is amusing to note that if we had chosen a less than optimal solution for $g(\lambda)$, but one which nevertheless retained the property that there exists a value $\lambda = M$ at which $g(M) = |M|$, then we would have got the same behavior as above for the t-channel residues. This gives an even more important role to M than before. Once there exists a number M such that the contribution of the Regge pole to $f^{(s)}$ does not vanish as $t \to 0$ when $|\lambda| = |\mu| = M$, then the entire behavior of the residues as $t \to 0$ is determined.

Working to leading order in s, we have succeeded in obtaining the behavior of the residues for small t. We can now go on to study the question of the analyticity of the non-leading terms and to see how the daughter properties must be modified due to the presence of spin.

C. Spin: Treatment to All Orders in s

1. The UU Case.

From (IV.20) we see that every term in the Regge expansion of $f^{(t)}$ will have in it a factor

$$\left(\frac{1 + z_t}{1 - z_t}\right)^{\frac{1}{2}(m' + \not{s}' n')} .$$

Thus we prefer to work with the amplitude

$$\hat{f}_{c\bar{a};\bar{d}b}(t, z_t) \equiv \left(\frac{1 + z_t}{1 - z_t}\right)^{-\frac{1}{2}(m' + \not{s}' n')} f^{(t)}_{c\bar{a};\bar{d}b}(t, z_t) . \qquad \text{(IV.79)}$$

The contribution of a single Regge pole to $\hat{f}$ is then

$$\hat{f}_{\mu',\lambda'}(t,z_t) = \zeta(\lambda',\mu')\,\gamma_{\mu';\lambda'}(t)\,L(t)\left(\frac{1-z_t}{2}\right)^{\alpha}$$

$$\times\, F\left(-\alpha+\mathfrak{m}',\ -\alpha+\sigma'\mathfrak{n}';\ -2\alpha;\ \frac{2}{1-z_t}\right) \qquad \text{(IV.80)}$$

where we have written

$$\gamma_{c\bar{a};\bar{d}b} \equiv \gamma_{\mu'\lambda'} \quad .$$

The procedure for fixing the properties of the daughters will be as follows. First we consider $\hat{f}$ as a function of the variables (s,t) and establish its behavior near t = 0. Then we write a daughter sequence for $\hat{f}$ and arrange the daughter residues, etc., so as to ensure that the Regge model for $\hat{f}$ does not violate the specified behavior near t = 0.

From (III.35), (IV.79), and the fact that the UU crossing matrix is neither singular nor vanishing as $t \to 0$ at fixed s, one can establish that the most general allowed behavior of $\hat{f}$ near t = 0 is

$$\hat{f}_{\mu';\lambda'} \underset{t\to 0}{\propto} t^{\frac{1}{2}(\mathfrak{m}'+\sigma'\mathfrak{n}')} \quad . \qquad \text{(IV.81)}$$

It can, of course, vanish faster as $t \to 0$ in practice, but it may not vanish more slowly.

We shall treat the case of one parent (M = 0) separately from the case where there are two conspiring parents (M ≥ 1).

(a) One Parent: M = 0. The parent pole gives a contribution to $\hat{f}$ which has an overall t-dependent factor which behaves like (see (IV.73))

$$t^{\frac{1}{2}(\mathfrak{m}'+\mathfrak{n}')} \text{ as } t \to 0 .$$

If we choose λ', μ' so that $\sigma' = +1$, then this behavior is as singular as is allowed, and the daughter sequence must sum to a function which is at worst constant as $t \to 0$.

Let us introduce daughters with residues

$$\beta^{(n)}_{\mu';\lambda'}(t) = t^{-\alpha_n - n}\,\bar{\beta}^{(n)}_{\mu',\lambda'}(t) \qquad \text{(IV.82)}$$

in analogy with (III.60) for the spinless case.

If we define $\gamma^{(n)}_{\mu',\lambda'}(t)$ in terms of $\bar{\beta}^{(n)}_{\mu',\lambda'}(t)$ by a formula analogous to (IV.19) in which α is replaced by α_n, then the contribution of the nth daughter to $\hat{f}$ is

$$\hat{f}_{\mu';\lambda'}(t,z_t) = \zeta(\lambda',\mu')\,\gamma^{(n)}_{\mu',\lambda'}(t)\,L_n(t)\left(\frac{1-z_t}{2}\right)^{\alpha_n}$$

$$\times\, F\left(-\alpha_n + \mathcal{M}',\ -\alpha_n + \epsilon'\mathcal{N}';\ -2\alpha_n;\ \frac{2}{1-z_t}\right) \qquad \text{(IV.83)}$$

where in the UU case

$$L_n(t) = t^{-\alpha_n - n}\,. \qquad \text{(IV.84)}$$

It is clear that the daughter residues will have to have the same overall helicity dependent t-factor as the parent. Thus we put

$$\gamma^{(n)}_{\mu';\lambda'}(t) = t^{\frac{1}{2}(|\lambda'| + |\mu'|)}\,\bar{\gamma}^{(n)}_{\mu',\lambda'}(t), \quad n = 0, 1, 2, \ldots \qquad \text{(IV.85)}$$

where $\bar{\gamma}^{(n)}$ is analytic at $t = 0$.

Lastly, defining

$$\bar{\gamma}^{(n)}_{\mu',\lambda'}(t) = b^{(n)}_{\mu',\lambda'}(t)\,\bar{\gamma}_{\mu',\lambda'}(t) \qquad \text{(IV.86)}$$

the full daughter sequence will be

$$\hat{f}_{\mu';\lambda'}(t,z_t) = \zeta(\lambda',\mu')\,t^{\frac{1}{2}(|\lambda'|+|\mu'|)}\,\bar{\gamma}_{\mu',\lambda'}(t)\left(\frac{x}{t}\right)^{\alpha}$$

$$\times \sum_{n=0}^{\infty} b^{(n)}_{\mu',\lambda'}(t)\,x^{-n}\,v^{\alpha_n+n-\alpha}\,F\left(-\alpha_n+\mathcal{M}', -\alpha_n+\epsilon'\mathcal{N}'; -2\alpha_n; \frac{1}{x}\right) \qquad \text{(IV.87)}$$

where as earlier

$$x = \tfrac{1}{2}(1 - z_t)\,,$$

$$v = x/t\,. \qquad \text{(IV.88)}$$

Taking then λ', μ' such that $\epsilon' = +1$, the series in (IV.87) will have to satisfy the same conditions as the series in (III.56); i.e., its ρ-th partial derivative with respect to t must yield at worst a polynomial of order ρ in 1/x and a polynomial in log v.

For ρ = 0, we now find

$$b^{(n)}_{\mu',\lambda'}(0) = \frac{(-1)^n (-\alpha(0)+m')_n (-\alpha(0)+n')_n}{n!\,(-2\alpha(0)+n-1)_n} \qquad \text{(IV.89)}$$

from which the $\beta^{(n)}_{\mu';\lambda'}$ can be calculated via (IV.86), (IV.82), and (III.60).

Now the above was derived for the situation $\mathscr{I}' = +1$. In the case $\mathscr{I}' = -1$, the t-dependence of the Regge term vanishes more quickly than generally required, i.e., as

$$t^{\frac{1}{2}(m'+n')} \quad \text{compared with} \quad t^{\frac{1}{2}(m'-n')} \quad .$$

Thus the daughter sequence can sum to a function which diverges at most like $t^{-n'}$ as $t \to 0$. In other words, its ρ-th derivative can sum to a polynomial in 1/x of order ρ + n′.

Now in order that a cancellation which is effective for certain μ′, λ′ remain effective when say λ′ → −λ′ it is necessary that the parent and all daughters have the same symmetry under λ′ → −λ′. By (IV.86) this will be achieved if $b^{(n)}_{\mu',\lambda'} = b^{(n)}_{\mu',-\lambda'}$. Thus our result cannot depend on the sign of $\mathscr{I}'$ and (IV.89) must hold also for $\mathscr{I}' = -1$. To check this, let us choose $\mathscr{I}' = -1$, take ρ = 0, substitute (IV.89) into our sequence and see what emerges. One has then a series

$$\sum_{n=0}^{\infty} b^{(n)}_{\mu',\lambda'}(0)\, x^{-n}\, F(-\alpha + n + m', -\alpha + n - n'; -2\alpha + 2n; \frac{1}{x})$$

which can be shown to sum to†

†From Eq. (11), Chapter 4.3 of Ref. 12, and using Eq. (3) of Chapter 4.4, one can show that

$$F(A, b; A+c-a; z) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{(b)_r (a-A)_r (c-a)_r}{(c+r-1)_r (c-a+A)_r} z^r F(a+r, b+r; c+2r; z)$$

from which the result needed follows.

$$F(-n', -\alpha + m'; -\alpha; \frac{1}{x})$$

which is a polynomial of order n' in $1/x$. So everything is consistent between the case $\mathscr{s}' = \pm 1$ for $\rho = 0$. We have not checked this consistency for $\rho > 1$, but there is little doubt that it will hold.

(b) Conspiring Parents: $M \geq 1$. Now $\hat{f}$ is given, to start with, by the sum of the contributions of the two parents. Moreover, to leading order in s, we will have

$$\hat{f}_{\mu';\lambda'} \propto \gamma^{+}_{\mu';\lambda'} s^{\alpha_+} + \gamma^{-}_{\mu';\lambda'} s^{\alpha_-}$$

and the relationship (IV.74) and (IV.59) between the ± poles will guarantee that $\hat{f}_{\mu';\lambda'}$ does not have a more singular behavior than (IV.80).

When the daughters are introduced, each daughter sequence will be designed to cancel unwanted singularities in the non-leading terms coming from its own parent. However, the two sequences will have to cooperate term by term to ensure that $\hat{f}$ is not too singular. To ensure this, we take analogously to (IV.74) and (IV.59),

$$\gamma^{(n)+}_{\mu';\lambda'}(t)\, s^{\alpha_n^{+}(t)} + \mathscr{s}' \gamma^{(n)-}_{\mu';\lambda'}(t)\, s^{\alpha_n^{-}(t)} \underset{t \to 0}{\propto} t^{\frac{1}{2}(m' + n')} \qquad \text{(IV.90)}$$

where $\gamma^{(n)\pm}$ are defined in terms of $\bar{\beta}^{(n)\pm}$ analogously to (IV.19) and where we have taken

$$\beta^{(n)\pm}_{\mu';\lambda'}(t) = t^{-\alpha_n^{\pm} - n}\, \bar{\beta}^{(n)\pm}_{\mu';\lambda'}(t) \qquad \text{(IV.91)}$$

in analogy with (III.60). The analogue of (IV.85) in this case, using (IV.73), is to put

$$\gamma^{(n)\pm}_{\mu';\lambda'}(t) = t^{\frac{1}{2}\{|M-m'| + |M-n'|\}}\, \bar{\gamma}^{(n)\pm}_{\mu'\lambda'}(t) \qquad \text{(IV.92)}$$

where $\bar{\gamma}^{(n)\pm}$ is analytic at $t = 0$.

Then (IV.74) implies that

$$\frac{d^m}{dt^m}\left(\bar{\gamma}^{(n)+}_{\mu'\lambda'} + \mathscr{s}'\, \bar{\gamma}^{(n)-}_{\mu'\lambda'}\right) = 0 \quad \text{for } m = 0, 1, 2, \ldots (\nu_M' - 1) \qquad \text{(IV.93)}$$

where

$$\nu_M' = \tfrac{1}{2}\{m' + n' - |M - m'| - |M - n'|\} . \quad \text{(IV.94)}$$

Now it follows from (IV.74) and (IV.59) that if $\varphi^{(n)}_{\mu'\lambda'}(\alpha(t); x)$ is <u>any</u> function analytic as a function of α as $t \to 0$, then

$$\left\{\gamma^{(n)^+}_{\mu'\lambda'}(t)\, \varphi^{(n)}_{\mu'\lambda'}(\alpha^+(t); x) + \sigma' \gamma^{(n)^-}_{\mu'\lambda'}(t)\, \varphi^{(n)}_{\mu'\lambda'}(\alpha^-(t); x)\right\}$$

$$= t^{\frac{1}{2}(m'+n')}\, C^{(n)}_{\mu'\lambda'}(t;x) \quad \text{(IV.95)}$$

where $C^{(n)}_{\mu'\lambda'}(t;x)$ is analytic and nonzero at $t = 0$ and is given at $t = 0$ by

$$C^{(n)}_{\mu'\lambda'}(0;x) = \frac{1}{[\frac{1}{2}(m'+n')]!} \left.\frac{\partial^{\frac{1}{2}(m'+n')}}{\partial t^{\frac{1}{2}(m'+n')}} \{ \quad \}\right|_{t=0} . \quad \text{(IV.96)}$$

Carrying out the differentiation and using (IV.74) and (IV.59), one gets

$$C^{(n)}_{\mu'\lambda'}(0;x) = \frac{1}{\nu_M'!} \left\{ \varphi^{(n)}_{\mu'\lambda'}(\alpha(0);x) \frac{\partial^{\nu_M'}}{\partial t^{\nu_M'}} \left[\bar{\gamma}^{(n)^+}_{\mu'\lambda'}(t) + \sigma' \bar{\gamma}^{(n)^-}_{\mu'\lambda'}(t)\right]\right|_{t=0}$$

$$\left. + \bar{\gamma}^{(n)^+}_{\mu'\lambda'}(0) \frac{\partial^{\nu_M'}}{\partial t^{\nu_M'}} \left[\varphi^{(n)}_{\mu'\lambda'}(\alpha^+(t),x) - \varphi^{(n)}_{\mu'\lambda'}(\alpha^-(t),x)\right]\right|_{t=0} \Bigg\} . \quad \text{(IV.97)}$$

Let us now construct the daughter sequences. We have, ignoring the irrelevant factor $\zeta(\lambda',\mu')$,

$$\hat{f}_{\mu';\lambda'}(t,z_t) = \left(\frac{x}{t}\right)^{\alpha^+} \sum_{n=0} \gamma^{(n)^+}_{\mu'\lambda'}(t)\, x^{-n}\, v^{\alpha_n^+ + n - \alpha^+}$$

$$F(-\alpha_n^+ + m', -\alpha_n^+ + \sigma' n'; -2\alpha_n^+; \tfrac{1}{x}) + \left(\frac{x}{t}\right)^{\alpha^-} \sum_{n=0} ((+) \to (-)) . \quad \text{(IV.98)}$$

For each n value we can identify the factors multiplying $\gamma^{(n)\pm}$ as functions $\varphi^{(n)}(\alpha\pm)$ of the type discussed above. Hence (IV.95) will hold for each pair of terms in the series. Thus (IV.98) becomes

$$\hat{f}_{\mu';\lambda'}(t,z_t) = (1-\mathscr{A}')\left(\frac{x}{t}\right)^{\alpha^+} t^{\frac{1}{2}\{|M-m'|+|M-n'|\}}\, \bar{\gamma}^{+}_{\mu'\lambda'}(t)$$
$$\times \sum_{n=0} b^{(n)}_{\mu'\lambda'}(t)\, x^{-n}\, v^{\alpha_n^+ + n - \alpha^+}\, F(-\alpha_n^+ + m', -\alpha_n^+ + \mathscr{A}' n'; -2\alpha_n^+; \frac{1}{x})$$
$$+ t^{\frac{1}{2}(m'+n')} \sum_{n=0} C^{(n)}_{\mu'\lambda'}(t,x) \qquad \text{(IV.100)}$$

where we have

$$b^{(n)}_{\mu'\lambda'}(t) = \bar{\gamma}^{(n)+}_{\mu'\lambda'}(t) / \bar{\gamma}^{+}_{\mu'\lambda'}(t) \qquad \text{(IV.101)}$$

and $C^{(n)}_{\mu'\lambda'}(0;x)$ is given by (IV.97) with

$$\varphi^{(n)}_{\mu'\lambda'}(\alpha;x,v) = x^{-n}\, v^{\alpha_n + n}\, F(-\alpha_n + m',\ -\alpha_n + \mathscr{A}' n';\ -2\alpha_n;\ \frac{1}{x}) \ . \qquad \text{(IV.102)}$$

We now choose μ', λ' so that $\mathscr{A}' = -1$. We then further restrict μ', λ' to the region $m' \geq M \geq n'$ so as to make the t factor in (IV.100) as singular as possible. In this region, the factor looks like

$$t^{\frac{1}{2}(m'-n')}$$

which from (IV.81) is already as singular as $\hat{f}$ can be. Thus by similar arguments to those used previously, we choose the $b^{(n)}(0)$ so that

$$\sum_{n=0} b^{(n)}_{\mu'\lambda'}(0)\, x^{-n}\, F(-\alpha+n+m',\ -\alpha+n-n';\ -2\alpha+2n;\ \frac{1}{x}) = 1$$

yielding

$$b^{(n)}_{\mu'\lambda'}(0) = \frac{(-1)^n(-\alpha(0)+m')_n\ (-\alpha(0)-n')_n}{n!\ (-2\alpha(0)+n-1)_n}$$
$$\text{for } m' \geq M \geq n' \ . \qquad \text{(IV.103)}$$

Also we shall have to require that

$$\sum_{n=0} C^{(n)}_{\mu'\lambda'}(0;x) = \text{polynomial in } 1/x \text{ of order } \mathfrak{n}' . \qquad (IV.104)$$

The latter requirement clearly does not interfere with our determination of the $b^{(n)}$, since by (IV.97) it involves derivatives of the $b^{(n)}$. We shall not analyze (IV.104) further at this stage.

Once we have the $b^{(n)}_{\mu'\lambda'}(0)$ for $\mathscr{s}' = -1$ and the range of μ', λ' as indicated in (IV.103), we can invoke factorization to find all the other $b^{(n)}_{\mu'\lambda'}$. Firstly, as discussed earlier, $b^{(n)}_{\mu'\lambda'}$ cannot depend on $\mathscr{s}'$. Thus (IV.103) holds for both $\mathscr{s}' = \pm 1$.

Secondly, to find $b^{(n)}_{\mu'\lambda'}$, when $\mathfrak{m}', \mathfrak{n}' > M$, we can use factorization in the form

$$\gamma_{\lambda'\mu'}\, \gamma_{MM} = \gamma_{\lambda' M}\, \gamma_{M\mu'} \qquad (IV.105)$$

since MM, λ'M, and Mμ' all fall in the range where (IV.103) holds. Hence one finds

$$b^{(n)}_{\mu'\lambda'}(0) = \frac{(-1)^n\, (-\alpha(0) + \mathfrak{m}')_n\, (-\alpha(0) - M)_n\, (-\alpha(0) + \mathfrak{n}')_n}{n!\, (-2\alpha(0)+n-1)_n\, (-\alpha(0)+M)_n}$$

$$\text{for } \mathfrak{m}', \mathfrak{n}' \geq M . \qquad (IV.106)$$

Finally, in an analogous fashion, if $\mathfrak{m}', \mathfrak{n}' < M$, we get

$$b^{(n)}_{\mu'\lambda'}(0) = \frac{(-1)^n\, (-\alpha(0) - \mathfrak{m}')_n\, (-\alpha(0) + M)_n\, (-\alpha(0) - \mathfrak{n}')_n}{n!\, (-2\alpha(0) + n - 1)_n\, (-\alpha(0) - M)_n}$$

$$\mathfrak{m}', \mathfrak{n}' \leq M . \qquad (IV.107)$$

This completes the specification of the $b^{(n)}_{\mu'\lambda'}(0)$ for all λ', μ'. However, we found the $b^{(n)}$ by using (IV.100) in a special region of helicities and using factorization. So we have to go back to (IV.100), insert our values of $b^{(n)}$ as given by (IV.103), (IV.106), and (IV.107) and check that in all cases we do not violate (IV.81).

For the case $\mathscr{s}' = -1$, one finds

$$\sum_{n=0} b^{(n)}_{\mu'\lambda'}(0)\, x^{-n}\, F(-\alpha + n + \mathfrak{m}', -\alpha + n - \mathfrak{n}'; -2\alpha + 2n; \tfrac{1}{x})$$

$$= \begin{cases} F(M - \mathfrak{n}', -\alpha + \mathfrak{m}'; -\alpha + M; \frac{1}{x}) & \text{for } \mathfrak{n}' \geq M \\ F(\mathfrak{m}' - M, -\alpha - \mathfrak{n}'; -\alpha - M; \frac{1}{x}) & \text{for } \mathfrak{m}' \leq M \end{cases} \qquad \text{(IV.108)}$$

which are polynomials in $1/x$ of order $\mathfrak{n}' - M$ and $M - \mathfrak{m}'$ respectively. Then we shall have from (IV.100) and (IV.108),

$$\hat{f}_{\mu'\lambda'} \propto \begin{cases} t^{\frac{1}{2}(\mathfrak{m}' + \mathfrak{n}' - 2M)} \cdot t^{M - \mathfrak{n}'} & \text{for } \mathfrak{n}' \geq M \\ t^{\frac{1}{2}(2M - \mathfrak{m}' - \mathfrak{n}')} \cdot t^{\mathfrak{m}' - M} & \text{for } \mathfrak{m}' \leq M \end{cases}$$

In both cases,

$$\hat{f}_{\mu'\lambda'} \propto t^{\frac{1}{2}(\mathfrak{m}' - \mathfrak{n})}$$

which does not violate (IV.81). Of course, one still has to adjust the $C^{(n)}_{\mu'\lambda'}$, but this will not interfere with the determination of the $b^{(n)}_{\mu'\lambda'}(0)$.

For the case $\mathscr{o}' = +1$, the series involving $b^{(n)}_{\mu'\lambda'}(t)$ does not contribute to $\hat{f}_{\mu'\lambda'}$ at all, and the non-violation of (IV.81) can be ensured by suitably adjusting the $C^{(n)}_{\mu'\lambda'}$. For example, when $\mathfrak{m}' \geq M \geq \mathfrak{n}'$, $\hat{f}_{\mu'\lambda'}$ must be no more singular than

$$t^{\frac{1}{2}(\mathfrak{m}' + \mathfrak{n}')}$$

implying that

$$\sum_{n=0} C^{(n)}_{\mu'\lambda'}(0;x) = \text{constant in } x \text{ and polynomial in } \log v,$$

$$(\mathscr{o}' = +1;\ \mathfrak{m}' \geq M \geq \mathfrak{n}') \,. \qquad \text{(IV.109)}$$

It turns out that this equation contains an interesting new piece of information on the slopes of the trajectories, as we shall see later.

(c) The Slopes of the Daughter Trajectories. Taking $\rho = 1$, i.e., considering the first partial derivative of (IV.100) with respect to t, and isolating the terms proportional to log v (as was done in (III.62)), we find for the (+) family:

$$\alpha_n{}'^{+}(0) - \alpha'^{+}(0) = \frac{n(2\alpha(0) - n + 1)}{2\alpha(0)}[\alpha_1{}'^{+}(0) - \alpha'^{+}(0)] \tag{IV.110}$$

giving the slope at $t = 0$ of all the + daughters. This formula is identical with the spinless result (III.63).

To find the slopes for the (-) family, remember that we have from (IV.90)

$$\left.\frac{d^m}{dt^m}\alpha_n^{+}(t)\right|_{t=0} = \left.\frac{d^m}{dt^m}\alpha_n^{-}(t)\right|_{t=0}, \quad m = 0, 1, 2, \ldots (M-1).$$

Thus provided $M \geq 2$, (IV.110) will hold also for the (-) family and $\alpha_n{}'^{+}(0) = \alpha_n{}'^{-}(0)$. To find the slopes of the (-) family when $M = 1$, let us return to Eq. (IV.109). In detail, it reads

$$\sum_{n=0} x^{-n} F(-\alpha + n + \mathfrak{m}', -\alpha + n + \mathfrak{n}'; -2\alpha + 2n; \tfrac{1}{x}) \, x$$

$$\times \frac{d^{\mathfrak{n}'}}{dt^{\mathfrak{n}'}}\left[b^{(n)+}_{\mu'\lambda'}(t) + b^{(n)-}_{\mu'\lambda'}(t)\right]\Big|_{t=0} + \sum_{n=0} b^{(n)+}_{\mu'\lambda'}(0)\, x^{-n}$$

$$\times \frac{d^{\mathfrak{n}'}}{dt^{\mathfrak{n}'}}\left\{v^{\alpha_n^{+}+n} F(-\alpha_n^{+}+\mathfrak{m}', -\alpha_n^{+}+\mathfrak{n}'; -2\alpha_n^{+}; \tfrac{1}{x})\right\}\Big|_{t=0}$$

$$= \text{constant in } x \text{ and polynomial in } \log v. \tag{IV.111}$$

In carrying out the differentiation there is only one term proportional to $\log v$. Isolating this term and choosing $\mathfrak{n}' = \mathfrak{m}' = M$, one has to have

$$\sum_{n=0} b^{(n)}_{MM}(0) \frac{d^M}{dt^M}\left[\alpha_n^{+}(t) - \alpha_n^{-}(t)\right]\Big|_{t=0} x^{-n}$$

$$F(-\alpha + n + M, -\alpha + n + M; -2\alpha + n; \tfrac{1}{x}) = \frac{d^M}{dt^M}\left[\alpha^{+}(t) - \alpha^{-}(t)\right]\Big|_{t=0} \tag{IV.112}$$

which yields

$$\frac{d^M}{dt^M}\left[\alpha_n^{\;+}(t) - \alpha_n^{\;-}(t)\right]\Bigg|_{t=0} = \frac{(-\alpha(0)+M)_n}{(-\alpha(0)-M)_n}\frac{d^M}{dt^M}\left[\alpha^+(t)-\alpha^-(t)\right]\Bigg|_{t=0} \quad . \tag{IV.113}$$

Thus the difference between the M-th derivatives of the ± daughters is determined by the difference between the M-th derivatives of the ± parents, at $t = 0$. This result holds for all M.

However, we can also use it when $M = 1$ in conjunction with (IV.110) to determine the slope of the (-) type daughters. Clearly if we had done everything in terms of the (-) rather than (+) family we would have found that (IV.10) holds also for the (-) family.

Thus for $M \geq 2$ all slopes are determined in terms of $\alpha(0)$ and $\alpha'(0)$, whereas for $M = 1$ the slopes depend on $\alpha(0)$, $\alpha'^{+}(0)$, and $\alpha'^{-}(0)$.

2. The UE Case

The situation in this case is very much more complicated, and it turns out to be more convenient to work with $f^{(t)}$ rather than $\hat{f}$. We put $m_B = m_D = m$; $s_B = s_D = s_E$ and ask how singular $f^{(t)}$ can be as $t \to 0$. The situation here is very different from the previous cases since in this case the crossing angles χ_B, χ_D blow up as $t \to 0$. So if the s-channel amplitudes are assumed analytic at $t = 0$, by crossing the $f^{(t)}$ will be quite singular. We have

$$\cos\chi_B = \frac{1}{\mathcal{S}_{AB}\,\mathcal{T}_{B\bar{D}}}\left\{t(s + m^2 - m_A^{\;2}) - 2m^2(m_C^{\;2} - m_A^{\;2})\right\} \quad ,$$

$$\cos\chi_D = \frac{1}{\mathcal{S}_{CD}\,\mathcal{T}_{B\bar{D}}}\left\{-t(s + m^2 - m_C^{\;2}) - 2m^2(m_C^{\;2} - m_A^{\;2})\right\} \quad , \tag{IV.114}$$

where, in this case,

$$\mathcal{T}_{B\bar{D}} = \left\{t(t - 4m^2)\right\}^{\frac{1}{2}} \tag{IV.115}$$

so that

$$\cos\chi_B \approx \cos\chi_D \underset{t\to 0}{\propto} t^{-\frac{1}{2}} \quad . \tag{IV.116}$$

Since the crossing matrices involve rotation functions like $d^{s_E}_{b'b}(\chi_B)$ we shall have, using

$$d^{s_E}_{b'b}(\chi_B) \propto (\cos\chi_B)^{s_E} \quad \text{for} \quad \cos\chi_B \to \infty$$

that

$$f^{(t)}_{\mu'\lambda'} \underset{t\to 0}{\propto} t^{-s_E} \tag{IV.117}$$

independently of the helicities λ', μ'. This is then the most singular behavior allowed for $f^{(t)}$.

Let us now look at the Regge model for $f^{(t)}$. We shall only treat the case $M > 0$, since $M = 0$ follows trivially from it.

Let us write the contribution of each Regge pole to $f^{(t)}$ as

$$f^{(t)}_{ca;db}(t,z_t) = (-1)^{\lambda'}\, t^{-\frac{\alpha}{2}}\, \mathfrak{B}_{ca;db}(t)\, e^{-\alpha-1}_{-\lambda',\mu'}(-z_t) \tag{IV.118}$$

where

$$\mathfrak{B}_{ca;db} = \frac{2\alpha+1}{2} \cdot \frac{1+\tau e^{i\pi\alpha}}{\cos\pi\alpha} \cdot t^{\frac{\alpha}{2}}\, \beta_{ca;db} \;.$$

Using the factorizability of β, and remembering that particles A and C are at the U vertex, B and D at the E vertex, we write

$$\mathfrak{B}^{UE}_{ca;db}(t) = \mathfrak{B}^{U}_{ca}(t)\; \mathfrak{B}^{E}_{db}(t) \;. \tag{IV.119}$$

Now from the leading order treatment, (IV.73), we know how γ^U behaves as $t \to 0$, from which it follows that

$$\mathfrak{B}^{U}_{\mu'}(t) \underset{t\to 0}{\propto} t^{\frac{1}{2}|M-|\mu'||} \tag{IV.120}$$

and that for $m_C > m_A$,

$$\mathfrak{B}^{U+}_{\mu'}(t) - \text{sign}(\mu')\, \mathfrak{B}^{U-}_{\mu'}(t) \propto t^{\frac{1}{2}|M+|\mu'||} \tag{IV.121}$$

which can be derived from (IV.56).

Also from (IV.75) we know the behavior of γ^E from which follows

$$\mathfrak{B}^{E\pm}_{\lambda'}(t) \propto t^{\frac{1}{4}\left\{1 \mp (-1)^{M+\lambda'}\right\}} \quad \text{for} \quad M < 2s_E \;. \tag{IV.122}$$

Combining (IV.122) and (IV.120), we get

$$\beta^{UE\pm}_{\mu'\lambda'}(t) \propto t^{\frac{1}{2}|M-|\mu'||} \cdot t^{\frac{1}{4}\left\{1 \mp (-1)^{M+\lambda'}\right\}} \qquad \text{(IV.123)}$$

and (IV.121) does not give any relation between β^+ and β^-.

The daughter sequences are set up as usual. We take

$$\beta^{(n)\pm}_{\mu';\lambda'}(t) = \frac{2\alpha_n^{\pm}+1}{2} \cdot \frac{1+\tau_n^{\pm} e^{i\pi\alpha_n^{\pm}}}{\cos \pi \alpha_n^{\pm}} \cdot t^{\frac{\alpha_n^{\pm}}{2}+\frac{n}{2}}\, \beta^{(n)\pm}_{\mu';\lambda'}(t) \qquad \text{(IV.124)}$$

and define

$$\beta^{(n)(\tau P)_n}_{\mu';\lambda'}(t) = t^{\frac{1}{2}|M-|\mu'||} \cdot t^{\frac{1}{4}\left\{1-(\tau P)_n (-1)^{M+\lambda'+n}\right\}}\, B^{(n)(\tau P)_n}_{\mu'\lambda'}(t) \qquad \text{(IV.125)}$$

where $B^{(n)}$ is analytic and nonzero at $t = 0$. The justification for (IV.125) is as follows. Firstly we know that $\tau_n = (-1)^n\tau$. Secondly, from a study of the UU case one can show that one must have $P_n = (-1)^n P$ which then makes $(\tau P)_n = \tau P$. That the latter was necessary was seen from the fact that $(\tau P)_n$ controls the effect on the daughter residues of $\lambda \to -\lambda$. Thus if a daughter sequence is effecting a certain cancellation of the singular terms of its parent for some value of λ, then the sequence must transform in the same way as the parent under $\lambda \to -\lambda$ in order to continue to effect the cancellation.

Once we have $(\tau P)_n = \tau P$ it is then necessary to have the factor $(-1)^n$ in (IV.125) but the reason is subtle and will only emerge after Eq. (IV.147). In the meantime, let us take (IV.125) as correct and study its consequences.

We see that odd daughters of a given parent will differ from it by a factor $\sqrt{t}$, but will have the same factor as the other parent and its even daughters. Thus the cancellation of the singular nonleading terms of a given parent is achieved by a collaboration between its own even daughters and the odd daughters of its conspirator parent. The sequence will then look like

$$f^{(t)}_{\mu';\lambda'} = t^{\frac{1}{2}|M-|\mu'||}\left\{t^{\frac{1}{4}(1-(-1)^{M+\lambda'})}\left[\sum_{n=0,2,4,\ldots} B^{(n)+}_{\mu'\lambda',\ldots} + \sum_{n=1,3,5,\ldots} B^{(n)-}_{\mu'\lambda',\ldots}\right]\right.$$

$$\left. + t^{\frac{1}{4}(1+(-1)^{M+\lambda'})}\left[\sum_{n=0,2,4,\ldots} B^{(n)-}_{\mu'\lambda',\ldots} + \sum_{n=1,3,5,\ldots} B^{(n)+}_{\mu'\lambda',\ldots}\right]\right\} \qquad \text{(IV.126)}$$

Let us write (IV.126) in the form

$$f^{(t)}_{\mu';\lambda'} = t^{\frac{1}{2}|M-|\mu'||}\left\{ t^{\frac{1}{4}(1-(-1)^{M+\lambda'})}\ \Sigma_{(1)}(t;z,v) + t^{\frac{1}{4}(1+(-1)^{M+\lambda'})}\ \Sigma_{(2)}(t;z,v)\right\} \qquad \text{(IV.127)}$$

where $z \equiv z_t$, $v = z/\sqrt{t}$.

Since for any λ' one of the t factors multiplying $\Sigma_{(1)(2)}$ will be a constant, the overall t factor in (IV.127) is just $t^{\frac{1}{2}|M-|\mu'||}$ which is always less singular than the allowed behavior (IV.117). Take as an example that λ' is such that

$$(-1)^{M+\lambda'} = -1 \quad .$$

Then we require

$$\frac{\partial^\rho}{\partial\sqrt{t}^\rho}\ \Sigma_{(1)}(t;z,v)\Big|_{t=0} = \text{polynomial in } 1/z \text{ of order } 2s_E - \tfrac{1}{2}|M-|\mu'||+\rho \qquad \text{(IV.128)}$$

and

$$\frac{\partial^\rho}{\partial\sqrt{t}^\rho}\ \Sigma_{(2)}(t;z,v)\Big|_{t=0} = \text{polynomial in } 1/z \text{ of order } 2s_E - \tfrac{1}{2}|M-|\mu'||+\rho+1. \qquad \text{(IV.129)}$$

Thus even when $\rho = 0$ the sums are allowed to be polynomials, whereas in all previous cases when we looked at $\rho = 0$ the sums had to yield a constant. Hence in this case we cannot find the $B^{(n)\pm}_{\mu\lambda}(0)$ uniquely from (IV.128), (IV.129).

In order to actually find the $B^{(n)\pm}(0)$, we have to get much more detailed information on the structure of $f^{(t)}(t,z)$, which in turn depends on the detailed structure of $f^{(s)}$.

Now by arguments similar to those used in obtaining (IV.2), one can write

$$f^{(s)}_{cd;ab}(s,z_s) = (1-z_s)^{\frac{1}{2}|\lambda-\mu|}\ (1+z_s)^{\frac{1}{2}|\lambda+\mu|}\ \bar{f}^{(s)}_{cd;ab}(s,t) \quad (IV.130)$$

in which $\bar{f}^{(s)}$ is analytic at $t = 0$. We shall write $f^{(s)}(s,z_s)$ as a function of (t,z_t) utilizing the additional information about the structure of $f^{(s)}$ as given by (IV.130). We take the expression (II.5) for z_t and solve for s getting, when $m_B = m_D = m$,

$$s = \frac{m(m_A^2 - m_C^2)}{\sqrt{-t}}\ z_t + \frac{\sqrt{-t}}{2}\Sigma + 0(t^{3/2}) \quad . \quad (IV.131)$$

Then substitute for s in the formula for z_s:

$$z_s = \frac{1}{\mathcal{S}_{AB}\,\mathcal{S}_{CD}}[s^2 - s\Sigma + 2st] \quad (IV.132)$$

where $\mathcal{S}_{ij}$ is given by (IV.7). One has

$$\mathcal{S}_{AB} = \frac{m|m_A^2 - m_C^2|}{\sqrt{-t}}\ z_t\left\{1 + 0(t^{\frac{1}{2}})\right\} \quad (IV.133)$$

and similarly for $\mathcal{S}_{CD}$. Thus

$$z_s = 1 + 0(t^{\frac{1}{2}}) \quad . \quad (IV.134)$$

Putting this into (IV.130) we thus see that working at fixed z_t and expanding in powers of t yields a behavior for $f^{(s)}$ as $t \to 0$ which is identical with the behavior deduced in the leading order in s treatment (cf. (IV.12)).

Next we take the expressions for the crossing angles (IV.114) and substitute for s using (IV.131). We get

$$\cos\chi_B = \frac{\text{sign}\,(m_C - m_A)}{z_t} + 0(t^{\frac{1}{2}}) \quad (IV.135)$$

and a similar expression for $\cos \chi_D$.

For the other crossing angles we get

$$\cos \chi_A = \text{sign}\,(m_C - m_A) + 0(t^{\frac{1}{2}}) \tag{IV.136}$$

and similarly for $\cos \chi_C$. Using (IV.135), (IV.136), (IV.55) and the inverse crossing relations, we get for $f^{(t)}$:

$$f^{(t)}_{\mu';\lambda'}(t,z_t) = s^{\alpha(0)} \sum_{\substack{a,b\\c,d}} \delta_{a\bar{a}'}\, \delta_{cc'}\, d^{s_E}_{b'b}(1/z_t)\, d^{s_E}_{\bar{d}'d}(1/z_t) \times$$

$$\times \left\{ t^{\frac{1}{2}(|M-|a-c||+|M-|b-d||)} \left[\hat{\beta}^{+}_{cd;ab}(0) + \hat{\beta}^{-}_{cd;ab}(0)\right] + t^{\frac{1}{2}|(a-c)-(b-d)|}\, 0(1/s) \right\} + \text{higher order terms in } t \tag{IV.137}$$

where we have used our knowledge (cf. (IV.55)) of the specific form of the leading term of $f^{(s)}$. Thus

$$f^{(t)}_{\mu';\lambda'}(t,z_t) = s^{\alpha(0)} \sum_{b,d} d^{s_E}_{b'b}(z_t^{-1})\, d^{s_E}_{\bar{d}'d}(z_t^{-1}) \left\{ t^{\frac{1}{2}(|M-|\mu||+|M-|\mu'||)} \times \right.$$

$$\left. \times \left[\hat{\beta}^{+}_{c'd;\bar{a}'b} + \hat{\beta}^{-}_{c'd;\bar{a}'b}\right] + t^{\frac{1}{2}|\mu'+\mu|}\, 0(\sqrt{t}/z) \right\} + \ldots \tag{IV.138}$$

The leading terms in the sums, as $t \to 0$, will have behavior

$$t^{\frac{1}{2}|M-|\mu'||} + 0(\sqrt{t}/z) + \ldots$$

Since $f^{(t)}$ has a factor $t^{\frac{1}{2}|M-|\mu'||}$ in it and since we do not know the structure of the other terms on the right-hand side of (IV.138), we cannot determine the polynomials to which $\Sigma_{(1),(2)}$ must sum, if $|\mu'| \neq M$. However, if $\mu' = \pm M$ then the only term on the right-hand side of (IV.138) which goes as a constant is the first one, whose structure we know. Thus for say $\mu' = M$ we must require

$$\Sigma_{(1)} \text{ or } \Sigma_{(2)} \propto \sum_{b-d=\pm M} d^{s_E}_{b'b}(z^{-1})\, d^{s_E}_{\bar{d}'d}(z^{-1}) \left[\hat{\beta}^{+}_{c'd;\bar{a}'b} + \hat{\beta}^{-}_{c'd;\bar{a}'b}\right]$$

$$\text{for } (-1)^{\lambda'+M} = 1 \text{ or } -1 \quad . \tag{IV.139}$$

But again the right-hand side of (IV.139) is an arbitrary polynomial, since we do not know how $\hat{\beta}_{c'd;\bar{a}'b}$ depends on b and d. So even in the case $\mu' = \pm M$ the $B^{(n)\pm}$ are undetermined.

We shall see that the best we can do is to determine certain linear combinations of the $B^{(n)}$. Firstly we decompose the product of d functions into irreducible components. We take

$$d^{s_E}_{b'b}(z^{-1})\, d^{s_E}_{\bar{d}'d}(z^{-1}) = (-1)^{b'-b}\, d^{s_E}_{-b',-b}(z^{-1})\, d^{s_E}_{\bar{d}'d}(z^{-1})$$

$$= (-1)^{b'-b} \sum_{s} C(s_E, s_E, s;\, \bar{d}', -b')\, C(s_E, s_E, s'd, -b)\, d^{s}_{\lambda',\mu}(z^{-1}) \,. \tag{IV.140}$$

Now define for $s' \geq \lambda'$,

$$\tilde{f}^{(t)}_{\mu';s'\lambda'} = \sum_{\substack{\bar{d}',b' \\ (\bar{d}'-b'=\lambda')}} (-1)^{b'-s_E}\, C(s_E, s_E, s';\bar{d}', -b')\, f^{(t)}_{\mu';\bar{d}'b'} \,. \tag{IV.141}$$

Substituting (IV.140) into (IV.138) and then computing $\tilde{f}^{(t)}$, we get

$$\tilde{f}^{(t)}_{\mu';s',\lambda'} = \sum_{b,d} d^{s'}_{\lambda'\mu}(z^{-1})\, (-1)^{b-s_E}\, C(s_E, s_E, s';\, d, -b) \times$$

$$\times \left\{ t^{\frac{1}{2}(|M-|\mu||+|M-|\mu'||)} \left[\hat{\beta}^{+}_{c'd;a'b} + \hat{\beta}^{-}_{c'd;a'b}\right] + t^{\frac{1}{2}|\mu'+\mu|}\, 0(\sqrt{t}/z) + \ldots \right\} \tag{IV.142}$$

Writing $\sum_{b,d} = \sum_{\mu} \sum_{d-b=\mu}$ and defining

$$\tilde{\hat{\beta}}_{\mu'\mu} = \sum_{d-b=\mu} (-1)^{b-s_E} C(s_E, s_E, s'; d, -b)\, \hat{\beta}_{c'd;\bar{a}'b} \tag{IV.143}$$

we get finally

$$\tilde{f}^{(t)}_{\mu';s'\lambda'} = s^{\alpha(0)} \sum_{\mu} d^{s'}_{\lambda'\mu}(z^{-1}) \Big\{ t^{\frac{1}{2}(|M-|\mu||+|M-|\mu'||)} \times$$

$$\times \left[\tilde{\hat{\beta}}^{+}_{\mu'\mu} + \tilde{\hat{\beta}}^{-}_{\mu'\mu}\right] + t^{\frac{1}{2}|\mu'+\mu|}\, 0(\sqrt{t}/z) + \ldots \Big\} \tag{IV.144}$$

The Regge pole expression for $\tilde{f}^{(t)}$ will be obtained by replacing $B^{(n)\pm}_{\mu'\lambda'}$ by

$$\tilde{B}^{(n)\pm}_{\mu';s'\lambda'} = \sum_{\substack{\bar{d}',b' \\ (\bar{d}'-b'=\lambda')}} (-1)^{b'-s_E} C(s_E, s_E, s'; \bar{d}', -b')\, B^{(n)\pm}_{\mu';\bar{d}'b'} \tag{IV.145}$$

inside $\Sigma_{(1)}$ and $\Sigma_{(2)}$. We label these new sequences $\tilde{\Sigma}_{(1)}$ and $\tilde{\Sigma}_{(2)}$.

We now choose $\mu' = M$ and repeating the argument which led to (IV.139), we require

$$\tilde{\Sigma}_{(1)} \text{ or } \tilde{\Sigma}_{(2)} = \sum_{\mu=\pm M} d^{s'}_{\lambda'\mu}(z^{-1}) \left[\tilde{\hat{\beta}}^{+}_{M\mu} + \tilde{\hat{\beta}}^{-}_{M\mu}\right] \text{ for } (-1)^{M+\lambda'} = 1 \text{ or } -1 . \tag{IV.146}$$

Using (IV.56) and (IV.143) we have that

$$\tilde{\hat{\beta}}^{+}_{M\pm M} = \pm\tilde{\hat{\beta}}^{-}_{M\pm M}$$

so (IV.146) becomes finally

$$\tilde{\Sigma}_{(1)} \text{ or } \tilde{\Sigma}_{(2)} \propto d^{s'}_{\lambda' M}(z^{-1}) \text{ for } \mu' = M \text{ as } (-1)^{M+\lambda'} = 1 \text{ or } -1 . \tag{IV.147}$$

More explicitly, e.g., if $(-1)^{\lambda'+M} = +1$, then (IV.147) reads

$$\left(\sum_{n=0,2,4,\ldots} \tilde{B}^{(n)+}_{M;s'\lambda'}(0) + \sum_{n=1,3,5,\ldots} \tilde{B}^{(n)-}_{M;s'\lambda'}(0)\right)\left(\frac{1-z}{2z}\right)^{\alpha}\left(\frac{2}{1-z}\right)^{n} \times$$

$$\times F(-\alpha+n+m', -\alpha+n+n'; -2\alpha+2n; \frac{2}{1-z}) \propto d^{s'}_{\lambda',M}(z^{-1}) \quad \text{(IV.148)}$$

which determines the $\tilde{B}^{(n)+}$ for n even and the $\tilde{B}^{(n)-}$ for n odd. A similar expression holds for $(-1)^{\lambda'+M} = -1$.

The explicit solution of (IV.148) is much more difficult than previous cases and has been given in Ref. 8.

Let us now see why the factor $(-1)^n$ is necessary in (IV.125). Consider, e.g., the case $(-1)^{\lambda'+M} = +1$. If we now look at the case $\mu' = -M$ then the right-hand side of (IV.147) will involve a sequence in $1/z$ which can be obtained from the case $\mu' = M$ by changing the sign of the odd powers of $1/z$ in $d^{s'}_{\lambda' M}(z^{-1})$. On the other hand, changing μ' from M to $-M$ on the left-hand side of (IV.147) causes the odd powers of $1/z$ in the even family members, but the even powers of $1/z$ in the odd daughters, to change sign. Thus to attain an overall change of sign of <u>all</u> odd powers of $1/z$ on the left-hand side the odd daughters must change, in addition, by an overall minus sign relative to the + parent and its even daughters. Hence the odd daughters involved in the cancellation must have opposite τP to the parent whose singularities they are cancelling. The factor $(-1)^n$ precisely guarantees this.

Notice that in contrast to all previous cases, we will not be able to use factorization to find the UE residues for $\mu' \neq \pm M$ from those with $\mu' = \pm M$. For we would need to use, e.g.,

$$\tilde{B}_{\mu';s'\lambda'}\,\tilde{B}_{M;s''M} = \tilde{B}_{\mu';s''M}\,\tilde{B}_{M;s'\lambda'}$$

which would be of no help.

However, factorization does give information about EE processes.

3. <u>The EE Case</u>

To fit in with the above notation, we write the contribution of each Regge pole to $f^{(t)}$ as

$$f^{(t)}_{c\bar{a};\bar{d}b}(t,z_t) = (-1)^{\lambda'}\,\mathcal{B}_{c\bar{a};\bar{d}b}(t)\; e^{-\alpha-1}_{-\lambda',\mu'}(-z_t) \qquad \text{(IV.149)}$$

where

$$\mathcal{B}_{c\bar{a};\bar{d}b} = \frac{2\alpha+1}{2} \, \frac{1+\tau e^{i\pi\alpha}}{\cos\pi\alpha} \, \beta_{c\bar{a};\bar{d}b}$$

and then put

$$\mathcal{B}^{(\tau P)}_{\mu';\lambda'} = t^{\frac{1}{4}\left\{1+\tau P(-1)^{M+\lambda'}\right\}} \cdot t^{\frac{1}{4}\left\{1+\tau P(-1)^{M+\mu'}\right\}} \, B^{(\tau P)}_{\mu',\lambda'} \, . \qquad \text{(IV.150)}$$

Finally we define

$$\tilde{f}^{(t)}_{s''\mu';s'\lambda'} = \sum_{\substack{\bar{d}',b' \\ (\bar{d}'-b'=\lambda')}} \; \sum_{\substack{\bar{a}',c' \\ (c'-\bar{a}'=\mu')}} (-1)^{b'-s_B} (-1)^{\bar{a}'-s_A}$$

$$C(s_B,s_B,s';\bar{d}',-b')\, C(s_A,s_A,s'';c',-\bar{a}')\, f^{(t)}_{c'\bar{a}';\bar{d}'b'} \qquad \text{(IV.151)}$$

where we have taken $s_A = s_C$ and $s_B = s_D$.

The Reggeized form for $\tilde{f}^{(t)}$ is given by replacing $B_{\mu'\lambda'}$ by

$$\tilde{B}_{s''\mu';s'\lambda'} = \sum_{\substack{\bar{d}',b' \\ (\bar{d}'-b'=\lambda')}} \; \sum_{\substack{\bar{a}',c' \\ (c'-\bar{a}'=\mu')}} (-1)^{b'-s_B} (-1)^{\bar{a}'-s_A}$$

$$C(s_B,s_B,s';\bar{d}',-b')\, C(s_A,s_A,s'';c',-\bar{a}')\, B_{c'\bar{a}';\bar{d}'b'} \qquad \text{(IV.152)}$$

in (IV.150) and (IV.149).

If we now define a modified UU residue $B^{UU}_{\mu'\lambda'}$ by

$$\frac{2\alpha+1}{2} \, \frac{1+\tau e^{i\pi\alpha}}{\cos\pi\alpha} \, t^{\alpha} \beta^{UU}_{\mu'\lambda'} = t^{\frac{1}{2}|M-|\mu'||} \cdot t^{\frac{1}{2}|M-|\lambda'||} \, B^{UU}_{\mu'\lambda'} \, , \qquad \text{(IV.153)}$$

then the factorization theorem gives

$$\tilde{B}^{EE}_{s''\mu';s'\lambda'} \, B^{UU}_{\nu\rho} = \tilde{B}^{UE}_{\nu;s'\lambda'} \, \tilde{B}^{EU}_{s''\mu';\rho} \, . \qquad \text{(IV.154)}$$

Hence, from our knowledge of B^{UU} and $\tilde{B}^{EU}$, we can calculate $\tilde{B}^{EE}$. The actual calculation is very complicated and can be found in Ref. 8. It is also shown there that the EE residues thus evaluated are consistent with the EE residues of the Regge poles contained in the expansion of a Toller pole (cf. Sec. V.).

V. The Group Theoretical Approach

A. Introduction

Consider first the spinless case. Let us examine the role of group theory in giving us the usual partial wave expansion which is the basis of the standard Reggeization procedure.

Let $M(p_C, p_{\bar{A}}; p_{\bar{D}}, p_B)$ be the scattering amplitude for the t-channel process

$$\bar{D} + B \rightarrow C + \bar{A} \quad .$$

The invariance of the scattering operator under Lorentz transformations tells us that if the scattering is viewed in a Lorentz transformed frame where the momenta have the values

$$p_i' = \Lambda\, p_i$$

then, for the spinless case, we have the covariance condition

$$M(\Lambda p_C, \Lambda p_{\bar{A}}; \Lambda p_{\bar{D}}, \Lambda p_B) = M(p_C, p_{\bar{A}}; p_{\bar{D}}, p_B) \quad . \tag{V.1}$$

Hence we can evaluate M in any frame we choose, provided it can be reached by a Lorentz transformation from the frame in which we wish to know the scattering amplitude. Because of energy-momentum conservation, we can take M to depend on the three vectors:

$$\begin{aligned} P &= p_{\bar{D}} = p_B = p_C + p_{\bar{A}} \qquad ; \quad P^2 = t \geq 0 \;, \\ \wp &= \tfrac{1}{2}(p_{\bar{D}} - p_B) \;, \\ \wp' &= \tfrac{1}{2}(p_C - p_{\bar{A}}) \;. \end{aligned} \tag{V.2}$$

By (V.1) we have

$$M(\Lambda P; \Lambda\wp', \Lambda\wp) = M(P; \wp', \wp) \quad . \tag{V.3}$$

Now P is time-like. Hence we can always find a Λ such that

$$\Lambda P = P^{(t)} = (\sqrt{t}, 0, 0, 0) \quad . \tag{V.4}$$

In this reference frame, which is, of course, the t-channel C.M. frame, we have

$$p_{\bar{D}} = (E_{\bar{D}}, \underline{p}) \qquad p_B = (E_B, -\underline{p})$$
$$p_C = (E_C, \underline{p}') \qquad p_{\bar{A}} = (E_{\bar{A}}, -\underline{p}')$$

where the E_i are known functions of t and the particle masses only. We have now

$$\wp^{(t)} = \left(\frac{E_{\bar{D}} - E_B}{2}, \underline{p} \right) \quad ,$$
$$\wp'^{(t)} = \left(\frac{E_C - E_{\bar{A}}}{2}, \underline{p}' \right) \quad , \tag{V.4a}$$

and we can write

$$M(P^{(t)}, \wp^{(t)}, \wp'^{(t)}) = f^{(t)}(t; \underline{p}, \underline{p}') \quad . \tag{V.5}$$

Now note that for any rotation R,

$$R\, P^{(t)} = P^{(t)} \quad . \tag{V.6}$$

So the covariance condition (V.3) gives

$$M(P^{(t)}; R\wp^{(t)}, R\wp'^{(t)}) = M(P^{(t)}; \wp^{(t)}, \wp'^{(t)})$$

or by (V.5),

$$f^{(t)}(t; R\underline{p}, R\underline{p}') = f^{(t)}(t; \underline{p}, \underline{p}') \tag{V.7}$$

since

$$R\wp^{(t)} = \left(\frac{E_B - E_{\bar{D}}}{2}, R\,\underline{p} \right) \qquad \text{etc.}$$

An immediate conclusion is that at fixed t, $f^{(t)}$ is a function only of the scalar product $\underline{p} \cdot \underline{p}'$. Since $|\underline{p}|$ and $|\underline{p}'|$ are calculable in terms of t, we can say that $f^{(t)}$ is a function only of t and the angle between $\underline{p}$ and $\underline{p}'$.

An alternative way of looking at this is to put

$$\underline{p} = R_{\underline{p}} \hat{\underline{p}}_z \quad ,$$
$$\underline{p}' = R_{\underline{p}'} \hat{\underline{p}}'_z \quad , \qquad (V.8)$$

where, e.g.,

$$\hat{\underline{p}}_z = (0, 0, |\underline{p}|)$$

is along the z-axis, and to use (V.7) to write

$$f^{(t)}(t; \underline{p}, \underline{p}') = f^{(t)}(t; \hat{\underline{p}}_z, R_{\underline{p}'\underline{p}} \hat{\underline{p}}'_z) \qquad (V.9)$$

where

$$R_{\underline{p}'\underline{p}} = R_{\underline{p}}^{-1} R_{\underline{p}'}$$

is the rotation which takes $\underline{p}$ into the direction of $\underline{p}'$. Since $\hat{\underline{p}}_z$ and $\hat{\underline{p}}'_z$ are functions of t, we have that

$$f^{(t)}(t; \underline{p}, \underline{p}') = f^{(t)}(t; R_{\underline{p}'\underline{p}}) \quad , \qquad (V.10)$$

i.e., $f^{(t)}$ is defined at fixed t as a function on the rotation group. It may therefore be expanded in terms of the representations functions of the rotation group and this leads in the usual way to the Jacob-Wick partial wave expansion.

All the above is very well known. We have repeated it just in order to emphasize: (a) the group theoretical aspects of the steps involved, and (b) that a functional approach is possible without need of talking about intermediate states which are eigenstates of J (this will be a great help later on).

Let us notice that in fact $f^{(t)}$ is also a function of the masses; in the sense that all momenta satisfy $p_i^2 = m_i^2$. From (V.2) it follows that

$$\not{p} \cdot P = \tfrac{1}{2}(m_D^2 - m_B^2) \quad ,$$

and

$$\not{p}' \cdot P = \tfrac{1}{2}(m_C^2 - m_A^2) \quad . \qquad (V.11)$$

Thus at fixed P the range of variation of $\not{p}$ and $\not{p}'$ is restricted by (V.11). In particular, since

$$R\not{p}^{(t)} \cdot P^{(t)} = \not{p}^{(t)} \cdot R^{-1} P^{(t)} = \not{p}^{(t)} \cdot P^{(t)} \tag{V.12}$$

it is permissible and consistent to rotate $\not{p}^{(t)}$ at fixed $P^{(t)}$, as is needed in (V.7), while always satisfying (V.11).

Now notice that at t = 0,

$$P^{(t)} \; (t = 0) = (0, \; 0, \; 0, \; 0)$$

is a null vector. In this case, for any $\Lambda \in O(3,1)$ we will have

$$\Lambda \; P^{(t)} \; (t = 0) = P^{(t)} \; (t = 0)$$

and therefore by (V.3),

$$M(P^{(t)}; \Lambda\not{p}^{(t)}, \Lambda\not{p}'^{(t)})_{t=0} = M(P^{(t)}; \not{p}^{(t)}, \not{p}'^{(t)})_{t=0} \; . \tag{V.13}$$

By the same arguments as above, we can now conclude that at t = 0, M is a function only of the Lorentz scalar product $\not{p} \cdot \not{p}'$.

However, from (V.11), if $m_D \neq m_B$ or $m_C \neq m_A$, we see that the components of $\not{p}$ and $\not{p}'$ are infinite, which indicates that something peculiar is happening, as was discussed in Sec. III.A(b). Since for the UU and UE cases at t = 0, P is actually light-like and not null, we concluded that covariance does not permit us to find M by evaluating it in the t-channel C.M. frame. Thus the additional symmetry at t = 0 is only relevant for EE processes.

In this case one can restate (V.13) in a form analogous to (V.10), i.e.,

$$f^{(t)}(t = 0; \not{p}^{(t)}, \not{p}'^{(t)}) = f^{(t)}(t = 0; \Lambda_{\not{p}'\not{p}}) \tag{V.14}$$

where $\Lambda_{\not{p}'\not{p}}$ is the Lorentz transformation that takes $\not{p}^{(t)}$ into the direction of $\not{p}'^{(t)}$. Thus $f^{(t)}$ at t = 0 is defined as a function on the homogeneous Lorentz group and can be expanded in terms of representation functions of O(3,1), giving rise to the Toller expansion.

Note that since

$$\not{p} \cdot \not{p}' \equiv \underline{p} \cdot \underline{p}'$$

we do not expect any serious difference between using the Toller expansion or the usual partial wave expansion for the spinless case.

B. The EE Case at t = 0 with Spin

Let $J^{\mu\nu}$ be the usual generators of the homogeneous Lorentz group O(3,1), and define

$$J_i = \tfrac{1}{2}\,\epsilon_{0ijk}\,J^{jk} \quad ,$$
$$K_i = J_{0i} \quad . \tag{V.15}$$

The states of a particle of momentum $\underline{p}$, spin s, and helicity λ are defined by

$$|p,s,\lambda\rangle = e^{-i\varphi J_3}\, e^{-i\theta J_2}\, e^{i\varphi J_3}\, e^{-i\alpha K_3}\,|\hat{p},s,\lambda\rangle = U(L_p)|\hat{p},s,\lambda\rangle \tag{V.16}$$

where

$$\hat{p} = (m,\ 0,\ 0,\ 0) \quad ,$$

$$p = m(\cosh\alpha,\ \sinh\alpha\,\sin\theta\,\cos\varphi,\ \sinh\alpha\,\sin\theta\,\sin\varphi,\ \sinh\alpha\,\cos\theta),$$

and

$$p = L_p\,\hat{p} \quad .$$

The state $|\hat{p},s,\lambda\rangle$ represents a particle at rest with spin projection λ along some fixed z-axis.

Under a Lorentz transformation, these states have the complicated transformation law:

$$U(\Lambda)|p,s,\lambda\rangle = \sum_{\lambda'} D^{s}_{\lambda'\lambda}\,(L^{-1}_{\Lambda p}\,\Lambda\,L_p)\,|\Lambda p,s,\lambda'\rangle \tag{V.17}$$

where $L^{-1}_{\Lambda p}\,\Lambda\,L_p$ is the Wigner rotation, and where $L_{\Lambda p}\,\hat{p} = \Lambda p$.

The M function is defined as the matrix elements of the S operator in a basis designed to obviate the complicated transformation law (V.17). We thus define "states"

$$|p,s,\mu\} = \sum_{\lambda} D^{os}_{\lambda\mu}\,(L_p^{-1})\,|p,s,\lambda\rangle \tag{V.18}$$

where D^{os} is a finite-dimensional representation function of O(3,1). These states have a simple transformation law:

$$U(\Lambda)\,|p,s,\mu\} = \sum_{\mu'} D^{os}_{\mu'\mu}(\Lambda)\,|\Lambda p,s,\mu'\} \quad , \qquad (V.19)$$

i.e., they transform according to the (o,s) representation of O(3,1). Note, however, that the states $|p,s,\mu\}$ are still eigenstates of the momentum operators $\hat{P}_\nu$, and hence they cannot at the same time be eigenstates of the Casimir operators of O(3,1).

We similarly define[14)]

$$[p,s,\mu| = \sum_\lambda D^{os}_{\mu\lambda}(L_p)\,\langle p,s,\lambda| \quad . \qquad (V.20)$$

Then we define an M function by

$$M_{ca;db}(p_C,p_A;\,p_D,p_B) \equiv [p_C,s_f,c;\,p_A,s_f,a|T|p_D,s_i,d;\,p_B,s_i,b\}$$

$$= \sum_{\substack{c'a'\\d'b'}} D^{os_f}_{cc'}(L_{p_C})\,D^{os_f}_{aa'}(L_{p_A})\,D^{os_i}_{d'd}(L^{-1}_{p_D})\,D^{os_i}_{b'b}(L^{-1}_{p_B})$$

$$\langle p_C,s_f,c';p_A,s_f,a'|T|p_D,s_i,d';p_B,s_i,b'\rangle \qquad (V.21)$$

where we have put $s_A = s_C = s_f$, $s_B = s_D = s_i$ and where the T-matrix element on the right is in the helicity representation, but does not have the additional phase used by Jacob and Wick. Then M satisfies the covariance condition

$$M_{c\bar{a};\bar{d}b}(\Lambda p_C,\,\Lambda p_{\bar{A}};\,\Lambda p_{\bar{D}},\,\Lambda p_B)$$

$$= \sum_{\substack{c',\bar{a}'\\ \bar{d}',b'}} D^{os_f}_{cc'}(\Lambda)\,D^{os_f}_{\bar{a}\bar{a}'}(\Lambda)\,D^{os_i}_{\bar{d}'\bar{d}}(\Lambda^{-1})\,D^{os_i}_{b'b}(\Lambda^{-1})$$

$$M_{c'\bar{a}';\bar{d}'b}(p_C,\,p_{\bar{A}};\,p_{\bar{D}},\,p_B) \quad . \qquad (V.22)$$

We can now couple the spins of C, $\bar{A}$ and $\bar{D}$, B. We also change to the variables $P, \wp, \wp'$ (see Eq. (V.2)). Thus we define

$$\mathfrak{m}_{J'm';Jm}(P;\wp',\wp) = \sum_{c,\bar{a},\bar{d},b} C(s_f, s_f, J'; c, \bar{a}, m')$$

$$C(s_i, s_i, J; \bar{d}, b, m)\, M_{c\bar{a};\bar{d}b}(P;\wp',\wp) \quad . \qquad (V.23)$$

The covariance condition (V.22) now reads

$$\mathfrak{m}_{J'm';Jm}(\Lambda P; \Lambda\wp', \Lambda\wp) = \sum_{n,n'} D^{OJ'}_{m',n'}(\Lambda) D^{OJ}_{nm}(\Lambda^{-1}) \mathfrak{m}_{J'n';Jn}(P;\wp',\wp) \ . \qquad (V.24)$$

Note that $\mathfrak{m}$ can be thought of formally as the matrix element of a P-dependent T operator:

$$\mathfrak{m} = [\wp', J', m' \,|\, T(P) \,|\, \wp, J, m\} \qquad (V.25)$$

with the requirement, to satisfy (V.24), that

$$U(\Lambda)\ T(P)\ U^{-1}(\Lambda) = T(\Lambda P) \ . \qquad (V.26)$$

Now we saw earlier that at t = 0, in the t-channel C.M.,

$$P \equiv P^{(t)} = (0, 0, 0, 0) \quad .$$

Hence, in <u>any</u> Lorentz frame, we will have

$$P = (0, 0, 0, 0) \quad .$$

Thus

$$\mathfrak{m}_{J'm';Jm}(P;\wp',\wp)_{t=0} = \mathfrak{m}_{J'm';Jm}(0;\wp',\wp) \quad . \qquad (V.27)$$

By going to the t-channel C.M. we see that at t = 0 ,

$$\wp^{(t)} = (0, 0, 0, i\, m_i) \quad ,$$

$$\wp'^{(t)} = (0, \underline{p}') \quad , \qquad (V.28)$$

where $\underline{p}'^2 = -m_f^2$ and where we have put $m_A = m_C = m_f$, $m_B = m_D = m_i$.

Thus in any frame, at t = 0,

$$\mathcal{p}^2 = m_i^2 \quad ,$$

$$\mathcal{p}'^2 = m_f^2 \quad . \qquad \text{(V.29)}$$

We now wish to write $\mathcal{p}, \mathcal{p}'$ in terms of some standard vectors, thereby defining $\mathfrak{m}$ as a function of the transformation which takes the standard vectors into $\mathcal{p}, \mathcal{p}'$. The choice of standard vectors seems to be fairly arbitrary. A simple choice is

$$\hat{\mathcal{p}} = (m_i, 0, 0, 0) \quad ,$$

$$\hat{\mathcal{p}}' = (m_f, 0, 0, 0) \quad , \qquad \text{(V.30)}$$

in which case we will have

$$\mathcal{p} = \mathcal{L}_{\mathcal{p}} \hat{\mathcal{p}} \quad ,$$

$$\mathcal{p}' = \mathcal{L}_{\mathcal{p}'} \hat{\mathcal{p}}' \quad , \qquad \text{(V.31)}$$

where $\mathcal{L}_{\mathcal{p}}, \mathcal{L}_{\mathcal{p}'}$ are clearly not real Lorentz transformations. For example, in the t-channel C.M. frame, where $\mathcal{p}, \mathcal{p}'$ are given by (V.28), we will have

$$\mathcal{L}_{\mathcal{p}^{(t)}} = L_z(-i\pi/2) \quad , \qquad \text{(V.32)}$$

i.e., a boost in the z-direction through an imaginary angle $\alpha = -i\pi/2$, i.e., a real rotation in the zt plane.

The covariance condition (V.24) holds also for complex Lorentz transformations, so we have

$$\mathfrak{m}_{J'm';Jm}(0; \mathcal{p}', \mathcal{p}) = \sum_{n,n'} D^{OJ'}_{m'n'}(\mathcal{L}_{\mathcal{p}'}) \, D^{OJ}_{n,m}(\mathcal{L}_{\mathcal{p}}^{-1})$$

$$\mathfrak{m}_{J'n';Jn}(0; \hat{\mathcal{p}}', \mathcal{L}_{\mathcal{p}'\mathcal{p}} \hat{\mathcal{p}}) \qquad \text{(V.33)}$$

where

$$\mathcal{L}_{\mathcal{p}'\mathcal{p}} \equiv \mathcal{L}^{-1}_{\mathcal{p}'} \mathcal{L}_{\mathcal{p}} \quad .$$

Finally, we define the function

$$\mathfrak{F}_{J'm';Jm}(\Lambda) = \sum_n D^{OJ}_{nm}(\Lambda) \, \mathfrak{m}_{J'm';Jn}(0; \mathcal{p}', \Lambda \mathcal{p}) \quad . \qquad \text{(V.34)}$$

This function has simple properties and can be interpreted formally as

$$\mathfrak{F}_{J'm';Jm}(\Lambda) = [\hat{\not p}', P=0, J', m' | T\, U(\Lambda) | \hat{\not p}, P=0, J, m\} . \quad (V.35)$$

Note that $\mathfrak{F}(\Lambda)$ is defined for all Λ in the complex Lorentz group. Note also that the interpretation (V.35) is formal in the sense that the states involved have to be considered as analytic continuations of the actual physical states. To do this rigorously is difficult and it is therefore more convenient to consider $\mathfrak{F}$ simply as a function of its arguments, and not to emphasize its interpretations in terms of matrix elements of operators.

From (V.33) and (V.34) we have the right and left covariance properties of $\mathfrak{F}$:

$$\mathfrak{F}_{J'm';Jm}(\Lambda R) = \sum_n D^J_{nm}(R)\, \mathfrak{F}_{J'm';Jn}(\Lambda) \quad ,$$

$$\mathfrak{F}_{J'm';Jm}(R\Lambda) = \sum_{n'} D^{J'}_{m'n'}(R)\, \mathfrak{F}_{J'n';Jm}(\Lambda) \quad , \quad (V.36)$$

where R is any rotation. The transformations which constitute the right and left covariance groups are, respectively, the intersection of the groups of transformations which leave both P and $\hat{\not p}$ and both P and $\hat{\not p}'$ invariant.

It is the satisfaction of these covariance conditions which distinguishes the Toller expansion from the Regge expansion. It can be shown that these covariance conditions ensure that the constraint equations (IV.10) are satisfied at $t = 0$, and we shall see later that each term in the Toller expansion separately satisfies them. Thus a model or an approximation involving one or a few terms in the Toller expansion will preserve the vital property (V.36).

The choice of expansion for $\mathfrak{F}(\Lambda)$ is dictated by the desire that each term in its expansion should possess the essential symmetry properties of $\mathfrak{F}$. Consider, for example, what happens for $t > 0$.

We could define the analogue of $\mathfrak{F}(\Lambda)$. It would be, in the t-channel C.M.,

$$\mathfrak{F}_{J'm';Jm}(t;R) = [\not p'^{(t)}, P^{(t)}, J', m' | T\, U(R) | \not p^{(t)}, P^{(t)}, J, m\}$$

where $P^{(t)}$, etc., are given by (V.4) and (V.4a). The covariance groups of $\mathfrak{F}(t;R)$ are now limited to rotations about the z axis, and one has, e.g.,

$$\mathfrak{F}_{J'm';Jm}(t;\, R\, R_z(\varphi)) = e^{i\varphi m}\, \mathfrak{F}_{J'm';Jm}(t;R) \quad .$$

The standard partial wave expansion for $\mathfrak{F}(t;R)$ would be

$$\mathfrak{F}_{J'm';Jm}(t;R) = \sum_j \mathfrak{F}^j_{J',J}(t)\, d^j_{m'm}(R)$$

and since

$$d^j_{m'm}(R\, R_z(\varphi)) = e^{i\varphi m}\, d^j_{m'm}(R)$$

we see that each term in the expansion satisfies the covariance property of $\mathfrak{F}(t;R)$.

If we now go to $t = 0$ in this expansion, it is impossible to satisfy the covariance conditions of $\mathfrak{F}(\Lambda)$ since $d^j_{m'm}$ is not even defined for the transformations of the covariance group, which at $t = 0$ becomes $O(2,1)$.

We thus need an expansion based on a group which is large enough to accommodate the full symmetry properties of $\mathfrak{F}(\Lambda)$. The precise choice of group depends upon the choice of standard vector used in defining $\mathfrak{F}(\Lambda)$. With our choice and working in the t-channel C.M., the covariance groups are the rotations $O(3)$, and $\mathfrak{F}$ is a function of

$$\mathcal{L}_{p'^{(t)}\, p^{(t)}} = \mathcal{L}_{p'}(t)\mathcal{L}_{p}(t)$$

where

$$\mathcal{L}_{p}(t) = L_z(-i\pi/2) \quad ,$$

$$\mathcal{L}_{p'}(t) = R_{p'(t) \leftarrow p(t)}\; L_z(-i\pi/2) \quad , \qquad \text{(V.37)}$$

where R is the rotation from the direction of $p^{(t)}$ to that of $p'^{(t)}$.

Thus

$$\mathcal{L}_{p'^{(t)}\, p^{(t)}} = L_z(i\pi/2)\, R^{-1}_{p'^{(t)} \leftarrow p^{(t)}}\; L_z(-i\pi/2) \quad . \qquad \text{(V.38)}$$

Hence, the simplest choice† is the group O(4) which is isomorphic to the group composed of the elements of O(3) together with the boosts corresponding to imaginary velocities.

Now, in general, if one has a group G with elements {g} and representation functions $\mathcal{D}^C_{mm'}(g)$, then if certain conditions are satisfied, one can expand functions f(g) by

$$f(g) = \int \sum_{m,m'} f^C_{m,m'} \mathcal{D}^C_{m,m'}(g)\, dC \qquad (V.39)$$

and the inversion is given by

$$f^C_{m,m'} = \int \mathcal{D}^{C*}_{m,m'}(g)\, f(g)\, dg \quad . \qquad (V.40)$$

Note that the possibility of inversion, i.e., of finding the expansion coefficients $f^C_{m,m'}$, depends upon f(g) being defined on the whole group G. Thus we cannot use too large a group for the expansion.

In our case the group is O(4), and we therefore expand $\mathcal{F}(\Lambda)$ in terms of the representation functions of O(4).

The unitary, irreducible representations are written $\mathcal{D}^{j_o\sigma}_{jm;j'm'}(\Lambda)$ and have the following main properties: If $\Lambda = R$, an ordinary rotation,

$$\mathcal{D}^{j_o\sigma}_{jm;j'm'}(R) = \delta_{jj'}\, D^j_{mm'}(R) \qquad (V.41)$$

where D^j is the usual representation function of O(3). If $\Lambda = L_z(i\alpha) = R_{tz}(\alpha)$,

$$\mathcal{D}^{j_o\sigma}_{jm;j'm'}(L_z(i\alpha)) = \delta_{mm'}\, d^{j_o\sigma}_{jmj'}(i\alpha) \qquad (V.42)$$

where the $d^{j_o\sigma}$ are known functions.

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† In Toller's work (Ref. 5) the group O(3,1) is used. This is a more natural group to use than O(4) but it has the complication of being noncompact, and this enormously increases the mathematical difficulties involved. The use of O(4) corresponds to the treatment of Freedman and Wang (Ref. 10).

The parameters j_o, σ run over the range:†

$$j_o = 0, \pm 1, \pm 2, \ldots \quad ,$$
$$\sigma = 1, 2, 3, \ldots . \quad ,$$
$$j, j' = |j_o|, |j_o| + 1, \ldots (\sigma - 1) \quad . \tag{V.43}$$

To include parity, one must extend the representations. The parity operator $\hat{P}$ has the following commutation relations with the group generators:

$$\hat{P} J_{rs} \hat{P}^{-1} = J_{rs} \quad ,$$
$$\hat{P} J_{or} \hat{P}^{-1} = -J_{or} \quad . \tag{V.44}$$

As a result, the parity operator changes the sign of j_o when operating on the basis states $|j_o\ \sigma; j\ m\}$ which furnish the representations $\mathcal{D}^{j_o \sigma}$. There will then be two situations:

If $j_o = 0$, we can take

$$\hat{P}\,|j_o{=}0, \sigma; j\ m\} = \mathcal{P}(-1)^j\,|j_o{=}0, \sigma; j\ m\} \text{ with } \mathcal{P} = \pm 1$$

and hence we can represent the space inversion element $\Lambda_{\mathcal{J}}$ for $j_o = 0$ by‡

$$\mathcal{D}^{\mathcal{P};\sigma}_{jm;j'm'}(\Lambda_{\mathcal{J}}) = \mathcal{P}(-1)^j\,\delta_{jj'}\,\delta_{mm'} \quad . \tag{V.45}$$

If $j_o \neq 0$, we can take

$$\hat{P}\,|j_o\ \sigma; j\ m\} = \mathcal{P}(-1)^{j+M}\,|-j_o\ \sigma; j\ m\}$$

where $M = |j_o|$.

If we put $j_o = qM$, $q = \pm 1$, and label the states $|M, \sigma, q; j\ m\}$, then $\hat{P}$ has the effect of changing $q \to -q$. Hence we will have representation functions for $j_o \neq 0$,

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†The ranges given correspond to the group O(4). To get half-integer angular momenta, one must consider the group SU(2) ⊗ SU(2).

‡The phases used here and in (V.46) are chosen to agree with Toller's work (Ref. 5).

$$\mathcal{D}^{M\sigma}_{jmq;j'm'q'}(\Lambda_{\mathscr{P}}) = (-1)^{j+M}\,\delta_{q,-q'}\,\delta_{jj'}\,\delta_{mm'}\,q^{2M}. \qquad (V.46)$$

For group elements Λ not involving any space inversion, we have

$$\mathcal{D}^{\mathscr{P};\sigma}_{jm;j'm'}(\Lambda) = \mathcal{D}^{0\sigma}_{jm;j'm'}(\Lambda) \qquad ,$$

$$\mathcal{D}^{M\sigma}_{jmq;\,j'm'q'}(\Lambda) = \delta_{qq'}\,\mathcal{D}^{qM,\sigma}_{jm;j'm'}(\Lambda) \quad . \qquad (V.47)$$

We can now apply the general expansion formula (V.39) and (V.40) to our case. Initially, we have

$$\mathfrak{F}_{J'm';Jm}(\Lambda) = \sum_{\sigma}\sum_{\substack{j'n'\\jn}}\sum_{\mathscr{P}} \mathfrak{F}^{\mathscr{P};\sigma}_{j'n';jn}(J',m';J,m)\,\mathcal{D}^{\mathscr{P};\sigma}_{j'n';jn}(\Lambda)$$

$$+\sum_{M>0}\sum_{\sigma}\sum_{\substack{j'n'q'\\jnq}} \mathfrak{F}^{M,\sigma}_{j'n'q',jnq}(J'm';Jm)\,\mathcal{D}^{M,\sigma}_{j'n'q';\,jnq}(\Lambda) \qquad (V.48)$$

and the inversion formula, e.g.,

$$\mathfrak{F}^{M,\sigma}_{j'n'q';jnq}(J'm';\,Jm) = \int_{O(4)} \mathfrak{F}_{J'm';Jm}(\Lambda)\,\mathcal{D}^{M\sigma *}_{j'n'q';\,jnq}(\Lambda)\,d\Lambda \qquad (V.49)$$

where the symbol $\int d\Lambda$ means integration over the group using an invariant measure.

Fortunately, many of the plethora of labels in (V.48) and (V.49) are redundant. One can show that the integrals (V.49) vanish unless $j' = J'$, $m' = n'$, $J = n$, and $m = n$. Moreover, one can show that $\mathfrak{F}^{M,\sigma}_{j'n'q';jnq}$ is independent of n and n′. All these results follow from the use of the covariance conditions (V.36) which $\mathfrak{F}(\Lambda)$ satisfies.[†]

[†]For example, every Λ in O(4) can be written as $\Lambda = R_1\,\Lambda_{zt}\,R_2$ where R_1 and R_2 are ordinary rotations and Λ_{zt} is a rotation in the zt plane. We can then use (V.36) to write

$$\mathfrak{F}_{J'm';Jm}(\Lambda) = \mathfrak{F}_{J'm';Jm}(R_1\,\Lambda_{zt}\,R_2)$$

$$= \sum_{n,n'} D^{J'}_{m'n'}(R_1)\,D^{J}_{nm}(R_2)\,\mathfrak{F}_{J'n';Jn}(\Lambda_{zt}) \quad .$$

Substituting this into (V.49) and performing the integrations over the ordinary rotations then yields the results quoted.

Thus the final expansion is much simpler:

$$\mathfrak{F}_{J'm';Jm}(\Lambda) = \sum_{\sigma,\mathcal{P}} \mathfrak{F}^{\mathcal{P};\sigma}_{J';J} \, \mathfrak{D}^{\mathcal{P};\sigma}_{J'm';Jm}(\Lambda)$$

$$+ \sum_{M>0} \sum_{\sigma,q,q'} \mathfrak{F}^{M,\sigma}_{J'q';Jq} \, \mathfrak{D}^{M,\sigma}_{J'm'q';Jmq}(\Lambda) . \qquad (V.50)$$

Note, as mentioned earlier, that <u>each term</u> in the expansion satisfies the covariance conditions of $\mathfrak{F}(\Lambda)$. For example,

$$\mathfrak{D}^{\mathcal{P};\sigma}_{J'm';Jm}(\Lambda R) = \sum_{J'',M''} \mathfrak{D}^{\mathcal{P};\sigma}_{J'm';J''m''}(\Lambda) \, \mathfrak{D}^{\mathcal{P};\sigma}_{J''m'';Jm}(R)$$

$$= \sum_{n} \mathfrak{D}^{\mathcal{P};\sigma}_{J'm';Jn}(\Lambda) \, D^{J}_{nm}(R)$$

which is in accord with (V.36).

Up to now we have not considered the consequences of the invariance of the S matrix under space inversion. This leads to further covariance properties for $\mathfrak{F}(\Lambda)$ under the transformation $\Lambda \to \Lambda_{\mathscr{I}} \Lambda$ or $\Lambda \to \Lambda \Lambda_{\mathscr{I}}$ where, as earlier, $\Lambda_{\mathscr{I}}$ is the group element corresponding to a space inversion.

For example, one has

$$\mathfrak{F}_{J'm';Jm}(\Lambda \Lambda_{\mathscr{I}}) = \zeta_{\bar{D}} \zeta_B \, \mathfrak{F}_{J'm';Jm}(\Lambda) \qquad (V.51)$$

where the ζ's are the intrinsic parities of $\bar{D}$ and B.

If we consider a term in the expansion (V.50) with $j_0 = 0$, then since by (V.45),

$$\mathfrak{D}^{\mathcal{P};\sigma}_{J'm';Jm}(\Lambda \Lambda_{\mathscr{I}}) = \mathcal{P}(-1)^J \, \mathfrak{D}^{\mathcal{P};\sigma}_{J'm';Jm}(\Lambda)$$

we see that we must have

$$\mathfrak{F}^{\mathcal{P};\sigma}_{J';J} = 0 \quad \text{if} \quad \mathcal{P}(-1)^J \neq \zeta_{\bar{D}} \zeta_B ,$$

i.e.,

$$\mathfrak{F}^{\mathcal{P};\sigma}_{J';J} \neq 0 \ \text{ only if } \ \mathcal{P} = (-1)^J \zeta_{\bar{D}} \zeta_B . \qquad (V.52)$$

Similarly, one requires $\mathcal{P} = (-1)^J \zeta_{\bar{A}} \zeta_C$.

For the terms in (V.50) with $j_o \neq 0$, we have from (V.46),

$$\mathfrak{s}^{M,\sigma}_{J'm'q';Jmq}(\Lambda\,\Lambda_{\mathcal{P}}) = (-1)^{J+M}\, \mathfrak{s}^{M,\sigma}_{J'm'q';Jm-q}(\Lambda)\cdot q^{2M}$$

and therefore we must have

$$\mathfrak{F}^{M,\sigma}_{J'q';J-q} = \zeta_{\bar{D}}\,\zeta_B\,(-1)^{J+M}\, q^{2M}\, \mathfrak{F}^{M,\sigma}_{J'q';Jq} \tag{V.53}$$

and similarly,

$$\mathfrak{F}^{M,\sigma}_{J'-q';Jq} = \zeta_{\bar{A}}\,\zeta_C\,(-1)^{J'+M}\, q'^{2M}\, \mathfrak{F}^{M,\sigma}_{J'q';Jq}\,.$$

C. Toller Poles

The expansion (V.50) is Tollerized in a manner analogous to the usual Reggeization procedure. The sum over σ is written as a contour integral, the contour is opened á la Sommerfeld-Watson, and the poles of $\mathfrak{F}^{\mathcal{P};\sigma}$ and $\mathfrak{F}^{M,\sigma}$ in the complex σ plane are picked up.

The representation functions appearing in (V.50) are of the form

$$\mathfrak{s}^{M,\sigma}_{J'm';Jm}(\mathcal{L}_{\mathcal{P}'(t)\,\mathcal{P}(t)}) = \sum_{J''=M}^{\sigma-1} d^{M\sigma}_{J'm'J''}(i\pi/2)\, D^{J''}_{m'm}(R^{-1})\, d^{M\sigma}_{J''mJ}(-i\pi/2) \tag{V.54}$$

where we have used (V.38) and (V.43) and have written R for $R_{\mathcal{P}'(t)\leftarrow\mathcal{P}(t)}$.

When σ becomes complex, the sum in (V.54) is meaningless. Hence we first write (V.54) as

$$\mathfrak{s}^{M,\sigma}_{J'm';Jm}(\mathcal{L}_{\mathcal{P}'(t)\,\mathcal{P}(t)}) = \sum_{N=0}^{\infty} d^{M\sigma}_{J',m',\sigma-1-N}(i\pi/2)\, D^{\sigma-1-N}_{m'm}(R^{-1})\; d^{M\sigma}_{\sigma-1-N,m,J}(-i\pi/2) \tag{V.55}$$

and this expression can be analytically continued in σ. The use of (V.55) amounts to a choice of a particular continuation.

We now assume that $\mathfrak{F}^{\mathcal{P};\sigma}_{J';J}$ and $\mathfrak{F}^{M,\sigma}_{J'q';Jq}$ have poles in the complex σ plane whose positions are independent of J, J', q, q'. This assumption ensures that the contribution of each pole individually satisfies the covariance requirements of $\mathfrak{F}(\Lambda)$. Note in particular that the covariance under space inversion, (V.53), forces the same pole into both $\mathfrak{F}^{M,\sigma}_{J'q';Jq}$ and $\mathfrak{F}^{M,\sigma}_{J'q';J-q}$. Thus for $M \neq 0$ there are effectively two coincident poles playing a role together. This is the analogue of the parity-doubling which was found necessary for Regge poles characterized by $M \neq 0$ (cf. Sec. IV.A, in particular Eq. (IV.35), (IV.36) and discussion thereafter).

If we consider a pole at $\sigma = \alpha + 1$, say in $\mathfrak{F}^{\mathcal{P};\sigma}_{J',J}$, with residue $\beta_{J';J}$, then we will have

$$\mathfrak{F}_{J'm';Jm}(\mathcal{L}_{p'(t)\,p(t)}) \propto \beta^{\mathcal{P}}_{J';J} \sum_n d^{0,\alpha+1}_{J',m',\alpha_n}(i\pi/2)\, d^{\alpha_n}_{mm'}(z_t)\, d^{0,\alpha+1}_{\alpha_n,m,J}(-i\pi/2) \tag{V.56}$$

where $\alpha_n = \alpha - n$. If the pole is in $\mathfrak{F}^{M,\sigma}_{J'q';Jq}$ we would have

$$\mathfrak{F}_{J'm';Jm}(\mathcal{L}_{p'(t)\,p(t)}) \propto \sum_q \beta^{M}_{J'q;Jq} \sum_n d^{qM,\alpha+1}_{J',m',\alpha_n}(i\pi/2)\, d^{\alpha_n}_{mm'}(z_t)\, d^{qM,\alpha+1}_{\alpha_n,m,J}(-i\pi/2) \quad . \tag{V.57}$$

Now $\mathfrak{F}_{J'm';Jm}(\mathcal{L}_{p'(t)\,p(t)})$ is essentially the irreducible t-channel helicity amplitude $\tilde{f}^{(t)}_{J'm';Jm}$ (defined in (IV.151)) evaluated at $t = 0$.[†] Thus (V.56), (V.57) are precisely in the form of a sequence of Regge poles, corresponding to the parent and daughter sequence in the Regge expansion of $\tilde{f}^{(t)}$. The Regge residues can be identified as the coefficients of $d^{\alpha_n}_{mm'}(z_t)$ in the sequence. Note that the requirement that the Regge residues factorize forces the Toller residues to be factorizable.

[†] One has

$$\mathfrak{F}_{J'm';Jm}(\mathcal{L}_{p'(t)\,p(t)}) = (-1)^{s_i+s_f}\, \tilde{f}^{(t)}_{J'm';Jm}$$

where $\tilde{f}^{(t)}$ is defined in (IV.151).

The contributions of a single Toller pole, as given in (V.56) and (V.57), appear as an infinite sum over Regge pole contributions. This, however, is not the form in which Toller presents his results. We thus recast (V.56), (V.57) using an identity proved by Bitar and Tindle,[15] namely:

$$\sum_n d^{M,\alpha+1}_{J'm'\alpha_n}(\delta')\, d^{\alpha_n}_{m'm}(\theta)\, d^{M,\alpha+1}_{\alpha_n mJ}(\delta) = \sum_\mu d^J_{m\mu}(\psi)\, d^{J'}_{\mu m'}(\psi')\, d^{M,\alpha+1}_{J\mu J'}(\gamma) \qquad (V.60)$$

where

$$\sin\psi = \frac{\sinh\delta' \sin\theta}{\sinh\gamma} ,$$

$$\sin\psi' = \frac{\sinh\delta \sin\theta}{\sinh\gamma} ,$$

$$\cos\psi = \frac{\cosh\delta \sinh\delta' \cos\theta + \sinh\delta \cosh\delta'}{\sinh\gamma} ,$$

$$\cos\psi' = \frac{\cosh\delta' \sinh\delta \cos\theta + \sinh\delta' \cosh\delta}{\sinh\gamma} ,$$

and

$$\cosh\gamma = \cosh\delta' \cosh\delta + \sinh\delta' \sinh\delta \cos\theta .$$

In our case,

$$\theta = \theta_t , \qquad \delta = -i\pi/2 , \qquad \delta' = i\pi/2$$

and hence,

$$\cosh\gamma = z_t ,$$

$$\psi = \pi/2 ,$$

$$\psi' = -\pi/2 .$$

Hence, for example, for a Toller pole with $M \neq 0$ we get

$$\mathcal{F}_{J'm';Jm}(\mathcal{M}_{\mathcal{P}}(t)\,\mathcal{P}(t)) \propto \sum_\mu d^J_{m\mu}(\pi/2)\, d^{J'}_{\mu m'}(-\pi/2) \left[d^{M,\alpha+1}_{J\mu J'}(z_t) + \zeta\, d^{-M,\alpha+1}_{J\mu J'}(z_t) \right] \qquad (V.61)$$

where $\zeta = \zeta_{\bar{D}} \zeta_B \zeta_C \zeta_{\bar{A}} (-1)^{J+J'}$.

The asymptotic behavior of the $d^{M,\alpha+1}$ functions is

$$d^{M,\alpha+1}_{J\mu J'}(z_t) \underset{z_t \to \infty}{\approx} |z_t|^{\alpha-|M-\mu|} \quad . \tag{V.62}$$

Hence, independently of J, J', m, m', the t-channel helicity amplitude at t = 0 has leading behavior $|z_t|^\alpha$.

Let us finally calculate the s-channel C.M. helicity amplitude. It is simplest to calculate $\tilde{f}^{(s)}$, the s-channel analogue of $\tilde{f}^{(t)}$, defined by

$$\tilde{f}^{(s)}_{J'n';Jn} = \sum_{\substack{a,b\\c,d}} (-1)^{b-s_B} (-1)^{a-s_A} C(s_B\, s_B\, J';\, d,\, -b,\, n')$$

$$C(s_A\, s_A\, s;\, c,\, -a,\, n)\, f^{(s)}_{cd;ab} \quad . \tag{V.63}$$

The crossing relations then give

$$\tilde{f}^{(s)}_{J'n';Jn} = \sum_{m,m'} d^{J'}_{m'n'}(\pi/2)\, d^{J}_{mn}(\pi/2)\, \tilde{f}^{(t)}_{J'm';Jm} \quad . \tag{V.64}$$

The constraint condition at t = 0 (cf. (IV.9)) requires that

$$\tilde{f}^{(s)}_{J'n';Jn} \propto \delta_{n'n} \quad .$$

Substituting (V.61) into (V.64), we get

$$\tilde{f}^{(s)}_{J'n';Jn} \propto \delta_{n'n} \left\{ d^{M,\alpha+1}_{JnJ'}(z_t) + \zeta\, d^{-M,\alpha+1}_{JnJ'}(z_t) \right\} \quad .$$

Hence the constraint at t = 0 is automatically satisfied by the Toller pole expression.

Moreover, from (V.62), we see that for a Toller pole characterized by M,

$$f^{(s)}_{J'n';Jn} \approx \delta_{n'n} |z_t|^{\alpha-|M-|n||} \tag{V.65}$$

and only the amplitudes with n = n' = ±M have the asymptotic behavior $|z_t|^\alpha$. This is in agreement with the results obtained to leading order in Sec. IV.A.

Thus the group theoretic treatment, using O(4), which is only valid at $t = 0$, in EE cases, gives results which are consistent with those obtained from a study of the UU and UE cases. This is a remarkable result and it would be very interesting to have a deeper understanding of this fact.

D. Group Theoretic Methods in the UU and UE Cases

Several attempts have been made to utilize expansions based on O(3,1) or O(4) for UU and UE cases where there is no genuine additional symmetry at $t = 0$. We shall give a brief resumé of these ideas and to be specific we shall restrict ourselves to UU reactions.

One can work with a function analogous to $\mathfrak{F}_{J'm';Jm}(\Lambda)$ which will have two additional labels:

$$\Delta = m_D^2 - m_B^2 \quad ,$$

$$\Delta' = m_C^2 - m_A^2 \quad ,$$

to remind us that the masses are not equal. Let us simply write this function as $\mathfrak{F}(\Lambda; \Delta, \Delta')$.

Now $\mathfrak{F}(\Lambda; \Delta, \Delta')$ is defined only for those transformations which leave P invariant. In this case, at $t = 0$, P is no longer a null vector, as it was the EE case, but is now a light-like vector. Thus we can only give a meaning to $\mathfrak{F}$ when Λ is an element of E_2, the group which leaves the vector (1, 0, 0, 1) invariant. Thus if we proceed to expand $\mathfrak{F}$ in terms of O(3,1) or O(4) representation functions

$$\mathfrak{F}(\Lambda; \Delta, \Delta') = \sum \mathfrak{F}^{M\sigma}(\Delta, \Delta')\, \mathfrak{D}^{M\sigma}(\Lambda)$$

it will not be possible to determine the coefficients $\mathfrak{F}^{M\sigma}(\Delta, \Delta')$, since the inversion formula, (V.40), involves integrating Λ over the whole group O(3,1) or O(4).

However, when $\Delta = \Delta' = 0$, the coefficients are determinable and one can make models in which the Δ, Δ' dependence is either ignored or put in explicitly, but arbitrarily.

Alternatively, one can hope that given the function $\mathfrak{F}(\Lambda; \Delta, \Delta')$ defined for $\Lambda \in E_2$ there exist well-behaved analytic continuations of $\mathfrak{F}$ in the variable Λ onto the whole complex Lorentz group.

In the same spirit one can consider $t \neq 0$ in the EE, UE, or UU cases, where again there is no O(3,1) symmetry, and nevertheless try to make an expansion based on the O(3,1) representation functions. One has now a function $\mathfrak{F}(t; \Lambda; \Delta, \Delta')$ and expansion coefficients

$\mathfrak{F}^{M\sigma}(t; \Delta, \Delta')$ determinable only at $t = \Delta = \Delta' = 0$. Again, the behavior away from this point can be put in in some model dependent fashion. It should be stressed that the use of this type of expansion involves very strong assumptions, far beyond those needed in the EE case at $t = 0$.

In any event, ignoring the dubious nature of the assumptions involved, one will have an expansion of the schematic form:

$$\mathfrak{F}_{J'm';Jm}(t; \Lambda; \Delta, \Delta') \approx \sum_{\sigma, M} \sum_{j, j'} \mathfrak{F}^{M,\sigma}_{jj'}(t; \Delta, \Delta'; J'm', Jm) \, \mathfrak{D}^{M,\sigma}_{j'm';jm}(\Lambda) \, . \tag{V.66}$$

Notice that here we have a sum over j, j'. This is because the covariance group is not large enough to force $j = J$, $j' = J'$, as it did in the EE case at $t = 0$.

One can now proceed to Tollerize or Reggeize (V.66). One assumes that the position of the poles in $\mathfrak{F}^{M,\sigma}_{jj'}(t; \Delta, \Delta'; J'm', Jm)$, considered as a function in the complex σ plane, depends only on M and t. The contribution of one of these generalized Toller poles then looks very much like (V.56) or (V.57) for arbitrary t, where now the residues are functions of t. It can be seen that this formulation leads to daughters which are separated from each other by integer values for <u>all t</u>. For example, if the parent trajectory is considered to be linear, then all daughter trajectories are parallel to it. Since this is a much more restricted sequence of daughters than required in the analytic solution of the daughter problem, we have to conclude that the assumptions used in this approach are much too strong.

Nevertheless, the above approach is useful in that it provides a daughter sequence with good analytic properties and in that it sheds some light on the structure of the singular daughter residues. To see this, note that the physical scattering is given when $\Lambda = \Lambda_{p'p}$. It is easy to show that independently of the choice of standard vectors $\hat{p}, \hat{p}'$, if $\Lambda_{p'p}$ is decomposed into rotations and a real boost along the z axis, then the boost angle is always given by

$$\cosh \alpha = \frac{p \cdot p'}{|p||p'|}$$

Thus

$$\cosh \alpha = \frac{u - s}{\left\{\left[2m_D^2 + 2m_B^2 - t\right]\left[2m_C^2 + 2m_A^2 - t\right]\right\}^{\frac{1}{2}}} \tag{V.67}$$

and this is well behaved in the region $t \to 0$, so there should be no problems of analyticity for a generalized Toller pole contribution.

As an example, consider a UU reaction with t small and positive. We work in the t-channel C.M. and define the standard vectors

$$\hat{\not p} = (|\not p|, 0, 0, 0) \quad ,$$
$$\hat{\not p}' = (|\not p'|, 0, 0, 0) \quad , \qquad (V.68)$$

where $|\not p|$ and $|\not p'|$ are finite as $t \to 0$.

We now have $\Lambda = \mathcal{L}_{\not p'(t)\not p(t)} = \mathcal{L}^{-1}_{\not p'(t)} \mathcal{L}_{\not p(t)}$ where

$$\mathcal{L}_{\not p(t)} = L_{\not p(t)} \quad ,$$
$$\mathcal{L}_{\not p'(t)} = R_{\not p'(t) \leftarrow \not p(t)} \cdot L_{\not p'(t)} \quad , \qquad (V.69)$$

where the L^s are boosts in the zt plane such that

$$L_{\not p(t)} (|\not p|, 0, 0, 0) = \left(\frac{E_D - E_B}{2}, 0, 0, |\underline{p}|\right) \quad ,$$

and

$$L_{\not p'(t)} (|\not p'|, 0, 0, 0) = \left(\frac{E_C - E_A}{2}, 0, 0, |\underline{p}'|\right) \quad , \qquad (V.70)$$

where $|\underline{p}|$, $|\underline{p}'|$ are given by (III.21) and (II.6).

Hence we can again decompose

$$\mathcal{D}^{M,\sigma}_{j'm';jm}(L^{-1}_{\not p'(t)} R_{\not p'(t) \leftarrow \not p(t)} L_{\not p(t)})$$
$$= \sum_{j''} d^{M,\sigma}_{j'm'j''} (L^{-1}_{\not p'(t)})\, D^{j''}_{m'm} (R)\, d^{M,\sigma}_{j''mj}(L_{\not p(t)}) \qquad (V.71)$$

and defining $f^{j''}_{J'm';Jm}(t)$ as the coefficient of $D^{j''}(R)$ in (V.66), i.e.,

$$f^{j''}_{J'm';Jm}(t) = \sum_{\substack{M,\sigma \\ j,j'}} \mathfrak{F}^{M,\sigma}_{j'j}(t;\, J'm';Jm)\, d^{M,\sigma}_{j'm'j''} (L^{-1}_{\not p'(t)})\, d^{M,\sigma}_{j''mj} (L_{\not p(t)}) \qquad (V.72)$$

we end up with (V.66) in the form

$$\mathfrak{F}_{J'm';Jm}(t; \mathcal{L}_{\rho'}(t)\,_{\rho}(t)) = \sum_j f^j_{J'm';Jm}(t)\, D^j_{m'm'}(z_t) \quad (V.73)$$

which is the form of the usual partial wave expansion in the t-channel C.M.

We can now see clearly that <u>each</u> partial wave amplitude f^j t) is singular as $t \to 0$ in order to ensure that the sum (V.73) is analytic as $t \to 0$. The precise form of the singularity is shown in (V.72), since $L_{\rho}(t)$ and $L_{\rho'}(t)$ become singular as $t \to 0$ in order to satisfy (V.70). (Remember that $|\rho|$ is finite, but $|\underline{p}| \to \infty$ when $t \to 0$.)

Lastly let us just mention a less ambitious approach to $t \neq 0$ in which one <u>first</u> expands $\mathfrak{F}(t; \Lambda)$ for small t and then applies the group theoretical analysis to each coefficient of t in the expansion.[11] This seems to be a more realistic approach than the generalized expansions used above, but unfortunately there seem to be ambiguities in its formulation which may, in principle, be unavoidable.

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RECENT DEVELOPMENTS IN HIGH ENERGY PHYSICS†‡

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"All changes trying, he will take the form of ev'ry reptile
on the earth, will seem a river now, and now devouring
fire but hold him ye, and grasp him still the more."
-- Homer, The Odyssey

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Introduction

The interest of a large part of the community of elementary particle physicists has been focused in recent times on the Veneziano model. The reason for such widespread interest is that this model, although admittedly still unsatisfactory from many points of view, already contains many of the properties that one believes to be true in general and is at the same time sufficiently simple to allow explicit calculations.

Although it is certainly premature to say whether or not a new chapter of physics has begun, in the hope that this may be so, it is probably useful to present a panorama as complete as possible of the main developments that have lately occurred in the field of high energy strong interaction dynamics. Hopefully, the present lectures should provide a sufficient basis for an outsider to be able to follow further developments in the field.

In the first part of these lectures, we shall review some of the most significant steps that have taken place starting from the work on finite energy sum rules. In the second part, we will mostly be interested in discussing the various physical and mathematical properties of the Veneziano model and only very briefly comment on its many applications and various generalizations.

I. Finite Energy Sum Rules and Further Developments

A. Finite energy sum rules

The finite energy sum rules (FESR) of Dolen, Horn and Schmid[1] are an almost immediate generalization of the superconvergence sum rules of de Alfaro et al[2] and are, therefore, a consequence of analyticity. Their advantage is that they allow one to study nonsuperconvergent amplitudes much in the same way as one would study superconvergent amplitudes since they put on equal footing all Regge poles irrespective of whether their intercept is $\gtrless -1$ (which is the critical value for writing a superconvergence sum rule). One of the major developments of FESR is the idea of duality (which was proposed in slightly different form and with different motivations by other authors[3]).

An indirect consequence of FESR can also be considered the Veneziano model[4] and a new form of the bootstrap idea (see Sec. I.B).

The basic assumptions in writing a FESR are that the amplitude a) satisfies a dispersion relation and b) can be expanded as a sum of Regge poles at high energies and as a sum of resonances at low energies. We parametrize an individual Regge pole as

$$R_{\pm} = \beta(t) \frac{\pm 1 - e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)\, \Gamma(1 + \alpha(t))} \nu^{\alpha(t)} \qquad \text{(I.1)}$$

where ± characterizes the signature and, because of its crossing symmetry, the variable

$$\nu = \frac{s - u}{2M} \qquad \text{(I.2)}$$

is used. For the sake of completeness, we briefly review next the usual derivation of a FESR. This will allow us to discuss the basic assumptions that one uses as a starting point for further developments.

The assertion is now that if an amplitude $F(s,t)$ is well represented by Regge poles for a certain $\nu \geq N$ then within the same approximation we have the FESR

$$S_n(N) = \int_0^N \nu^n \frac{\operatorname{Im} F(s,t)}{N^{n+1}} d\nu = \sum \frac{\beta N^{\alpha}}{(\alpha + n + 1)\, \Gamma(\alpha + 1)} \qquad \text{(I.3)}$$

where $\alpha \equiv \alpha(t)$ and the sum is over Regge poles. The integration is over the right hand cut in s and includes the Born term.

We can begin by noticing in the above formula one ambiguity which will play a major role in future developments, namely, where can we safely cut off the integral in (I.3)? In other words, what criterion (if any) can be used to determine the value of N for which the above approximation holds?

To show how (I.3) comes about, let us start from an amplitude $F(\nu)$ which is antisymmetric and satisfies an unsubtracted dispersion relation. By using crossing symmetry we can then write

$$F(\nu) = \frac{2\nu}{\pi} \int_0^{\infty} \frac{\operatorname{Im} F(\nu')}{\nu'^2 - \nu^2} d\nu' \, . \qquad \text{(I.4)}$$

If the leading Regge pole in the asymptotic expansion of $F(\nu)$ has $\operatorname{Re} \alpha < -1$, then we have a superconvergence relation

$$\int_0^{\infty} \mathrm{Im}\, F(\nu)\, d\nu = 0\ . \tag{I.5}$$

However, if the leading Regge pole is above -1 (but below 1), we can write (I.1) as

$$R_+(\nu) = \frac{2\nu}{\pi} \int_0^{\infty} \frac{\beta}{\Gamma(\alpha + 1)} \frac{\nu'^{\alpha}}{\nu'^2 - \nu^2}\, d\nu' \qquad (\alpha < 1) \tag{I.6}$$

and subtract it off the complete amplitude. This process can be repeated as many times as needed to arrive at a superconvergent amplitude after sufficiently many Regge poles have been subtracted out so that, without any loss of generality, we can write

$$\int_0^{\infty} d\nu\, \mathrm{Im}[F - R] = 0 \tag{I.7}$$

Let us now label with i all Regge poles such that $\alpha_i(0) > -1$, with j those such that $\alpha_j(0) < -1$ and, finally, with k those corresponding to $\alpha_k(0) = -1$. Therefore, (I.7) can be written explicitly as

$$\int_0^{\infty} \left[\mathrm{Im}\, F - \sum_i \frac{\beta_i}{\Gamma(\alpha_i + 1)} \nu^{\alpha_i}\right] d\nu = \beta_k\ . \tag{I.8}$$

Notice that each integral in the l.h.s. diverges if taken separately. We are now going to assume that we can cut off the integral at some suitable value $\nu_{max} = N$ and attribute the (vanishing) high energy tail of the integrand to the Regge poles with $\alpha_j < -1$. (This essentially amounts to assuming that, for a sufficiently large ν, the amplitude can be approximated by a sum of pure Regge poles without background.) Thus

$$\int_0^{N} \left[\mathrm{Im}\, F - \sum_i \frac{\beta_i \nu^{\alpha_i}}{\Gamma(\alpha_i + 1)}\right] d\nu + \int_N^{\infty} \sum_j \frac{\beta_j}{\Gamma(\alpha_j + 1)} \nu^{\alpha_j}\, d\nu = \beta_k \tag{I.9}$$

All integrals are now separately convergent and we get the FESR

$$S(N) \equiv \int_0^{N} \frac{\mathrm{Im}\, F}{N}\, d\nu = \sum_i \frac{\beta_i N^{\alpha_i}}{\Gamma(\alpha_i + 2)} + \sum_j \frac{\beta_j N^{\alpha_j}}{\Gamma(\alpha_j + 2)} + \frac{\beta_k}{N} =$$

$$= \sum_{\text{all } \alpha} \frac{\beta N^{\alpha}}{\Gamma(\alpha + 2)} \tag{I.10}$$

The general FESR of an arbitrary moment n can be established in a completely analogous way. Notice that in (I.3) or (I.10) there is no further reference to any special role played by the value $\alpha_k = -1$ which appears critically in the derivation of superconvergence sum rules. Also, the latter are obtained from (I.10) by letting $N \to \infty$ if all $\alpha < -1$. However, whereas for $N \to \infty$ ($\alpha < -1$) we recover again the exact expression (I.5), Eq. (I.10) or (I.3) are not exact in that we have already supposed that N is so large that for $\nu > N$ we can approximate F with a (finite) sum of Regge poles.

One can similarly derive (formally) FESR for negative n to get

$$\int_0^N \frac{\operatorname{Im} F(\nu)}{\nu^{m+1}}\, d\nu - \sum \frac{\beta N^{\alpha-m}}{\Gamma(\alpha+1)(\alpha-m)} = \frac{F^{(m)}(0)}{m!} \quad (m \geq 0)$$
$$= 0 \qquad m < 0 \qquad \text{(I.11)}$$

The above formula makes sense so long as $\left.\frac{\operatorname{Im} F}{\nu^m}\right|_{\nu=0}$ is zero. For m = 0, in particular, if $\alpha \geq 0$, we used a subtraction constant (scattering length). This is, essentially, the argument used by Igi[5] to establish the existence of the P′ trajectory.

The literature concerning FESR in their various aspects has boomed tremendously in the last years and it is practically impossible to give a complete list of references. We can distinguish not less than five major developments that have occurred as more or less direct consequences of FESR:

i) Use of low energy data on πN and KN to study P, P′, ρ, N_α and A_2 contributions together with an analysis of NN data to determine intercepts of P′, ω, ρ and A_2 trajectories together with the study of the relative importance of the Pomeranchukon and other trajectories in Compton scattering.[6]

ii) Use of photoproduction data to study π and A_2 contributions.[7]

iii) Derivation of continuous moment sum rules.[8] These are obtained considering dispersion integrals for either $(\nu^2 - \mu^2)^\gamma e^{-i\pi\gamma} F(\nu)$ or $\nu^\gamma e^{-\frac{i\pi}{2}\gamma} F(\nu)$ in which γ is considered as a continuous parameter. The use of these new sum rules allows one to get a continuous curve S_γ instead of the discrete points S_n. Furthermore, one can now introduce both the real and imaginary part of the amplitude into the game.

iv) Veneziano model.[4] This is not, strictly speaking, a direct consequence of FESR but it is hard to see how this model could have been devised without all the background represented by the results on FESR.

v) FESR allow a revival of bootstrap ideas. This possibility was already mentioned in Ref. 1 and gave rise to many different applications.[3],[9)-16] The application[11],[12] of these new bootstrap techniques which we will briefly consider here does not make use of the full content of FESR, but rather of that particular aspect of it which is presently called "duality" and which we shall discuss at length later on.

The original bootstrap scheme suggested in Ref. 1 was to use FESR to bootstrap trajectories in the crossed channel and to calculate resonance widths by saturating FESR with resonant states. Mandelstam[9)] was, however, the first to suggest the viability of the narrow resonance approximation (NRA) bootstrap procedure by showing how the ρ can bootstrap itself in a frame in which FESR are used with a finite number of Regge poles. Freund[10)] showed that one can bootstrap ρ and P' from the πN spectrum while this cannot be done for the Pomeranchuk. A correlation of this phenomenon with other effects was later noticed by Harari[17)] and will be discussed later on (Sec. I.G.).

Of a somewhat different nature is the bootstrap mechanism proposed by Chew and Pignotti.[15)] These authors, in fact, argue that since there exists a connection between a peripheral (crossed channel) and a resonance (direct channel) effect, the explanation of the A_1 as a peripheral reaction (Deck effect) or as a true resonance, would amount to the same, and one should not count both these effects as independent ones.

Another interesting line of attack to the bootstrap problem has been proposed by Chu et al[16)] in which some of the previous simplifying assumptions (zero width, linear trajectories) are relaxed. The numerical results are perhaps not very conclusive.

In the present lectures we will only discuss briefly the bootstrap of Ref. 11 and 12, both because this seems to be the most natural development of FESR and also because it leads in a very straightforward way to the Veneziano model. We will, however, not discuss the preliminary point of why Regge behavior and crossing symmetry require indefinitely rising trajectories, also because it is still controversial how these trajectories should be asymptotically rising. Claims have been made both in favor of a linear[18)] and of a square root[19)] behavior (to within logarithmic factors in both cases).

We will now very briefly discuss[20)] why a NRA cannot, strictly speaking, be consistent with a FESR. First, notice that a NRA requires that we can write

$$\text{Im}\, F(s,t) = \sum_{\ell} (2\ell + 1)\, \beta(m_\ell^2)\, P_\ell\left(1 + \frac{2t}{m_\ell^2 - \Sigma}\right) \delta(\ell - \alpha(s)) \qquad (I.12)$$

so that Eq. (I.3) (with one Regge pole) becomes

$$\sum_{\ell=0}^{\alpha(N)} (2\ell + 1)\beta(m_\ell^2)\, m_\ell^{2n}\, P_\ell\left(1 + \frac{2t}{m_\ell^2 - \Sigma}\right) \sim \beta(t)\frac{N^{\alpha(t)+n+1}}{(\alpha(t)+n+1)\Gamma(\alpha(t)+1)} \tag{I.13}$$

Taken literally, the above equality is impossible since the r.h.s. is a smooth function of N whereas the l.h.s. increases discontinuously with N. We may, however, assume that Eq. (I.13) must be valid only at those N values for which $\alpha(N)$ is integer with a smooth interpolating function in between. Then, we want to solve for $\beta(t)$ once $\beta(m_\ell^2)$ are given for discrete values of ℓ as $\ell \to \infty$. Taking the difference in (I.13)

$$(2\ell + 3)\, \beta(m_{\ell+1}^2)\, P_{\ell+1}\left(1 + \frac{2t}{m_\ell^2 - \Sigma}\right) m_{\ell+1}^{2n} \underset{\ell\to\infty}{\sim} \beta(t)\frac{(m_{\ell+1}^2)^{\alpha(t)+n+1} - (m_\ell^2)^{\alpha(t)+n+1}}{[\alpha(t) + n + 1]\Gamma(1 + \alpha(t))} \quad . \tag{I.14}$$

Using the asymptotic form $(t \neq 0)$

$$P_{\ell+1}\left(1 + \frac{2t}{m_\ell^2 - \Sigma}\right) \underset{\ell\to\infty}{\sim} \left(\frac{m_\ell}{4\pi\ell\,|t|^{\frac{1}{2}}}\right)^{\frac{1}{2}} \exp\left[2\ell\sqrt{\frac{|t|}{m_\ell^2}}\right]$$

we find

$$\frac{\beta(m_{\ell+1}^2)}{\beta(t)} \underset{\ell\to\infty}{\sim} \frac{|t|^{\frac{1}{4}}\left[(m_{\ell+1}^2)^{\alpha(t)+n+1} - (m_\ell^2)^{\alpha(t)+n+1}\right]}{(\pi\ell m_\ell)^{\frac{1}{2}}\,(m_{\ell+1}^2)^n\,(\alpha(t) + n + 1)\,\Gamma(\alpha(t)+1)\exp\left[2\ell\sqrt{\frac{|t|}{m_\ell^2}}\right]} \tag{I.15}$$

which shows that the l.h.s. is in the form of a product of a function of ℓ times a function of t whereas the r.h.s. is not.

Finally, the validity of a NRA can also be questioned on an empirical ground since most of the baryon resonances do not have a negligibly narrow width. What is more important, however, is that the widths of baryon resonances on given Regge families seem to grow as $\sqrt{s}$ (see Fig. 1).

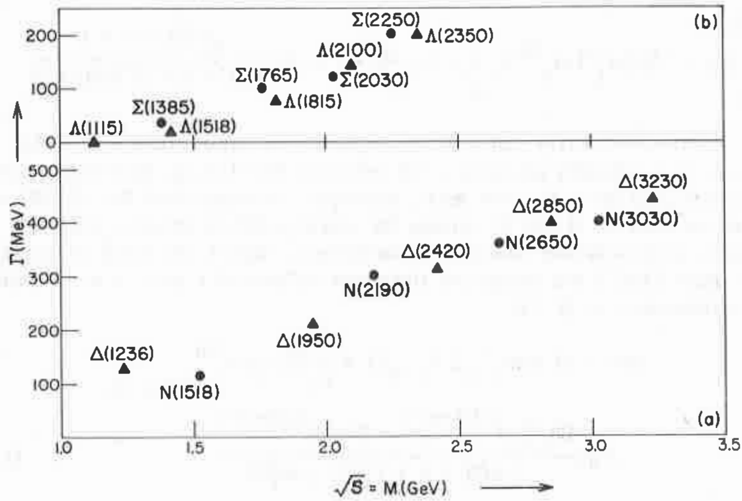

Fig. 1. Plot of the widths of the N_γ and Δ_δ resonances (1.a) and of the Y_0^* and Y_1^* resonances lying on the exchange degenerate $\Lambda(1115)$ and $\Sigma^*(1385)$ Regge trajectories. A $\sqrt{s}$ growth is well exhibited.

B. The New Bootstrap

The novelty in the bootstrap approach represented by FESR is that low energy effects can be used to predict Regge parameters.

The question now becomes very drastically dependent on what value one should use for the cut off N. The point is that we want to saturate the low energy integral in (I.3) by a small number of resonances in order to introduce a number of parameters not so large as to make the result doubtful. So N cannot be too large, typically 1 or 2 GeV. This, however, means that we are going to use the Regge approximation in a region which nobody dared before to consider accessible to a Regge pole analysis[21)] since there still are many resonances. However, if we forget the approximations needed in deriving (I.3) and take the latter literally, then we would conclude that the Regge pole fit extrapolated to intermediate energies should reproduce the amplitude integrated from 0 to N (this is what is usually called the "averaged" amplitude). The only blemish in this argument is that this will be true so long as the approximation used in deriving (I.3) is good and this explicitly assumed that we already were in the

Regge regime. Conversely, if the approximation is good we are already in the Regge domain and the above conclusions must hold. A clear cut example[1),23)] in which the conditions required above seem to be met is the difference of total cross sections $[\sigma_t(\pi^+p) - \sigma_t(\pi^-p)]$ (see Fig. 2). The charge exchange reaction $K^-p \to \bar{K}_0 n$ has been analyzed in the same spirit by T. Lasinski. The preliminary conclusion is that the fit is qualitatively good (although statistically rather poor). We shall return to this point in Sec. I.G. (see also Ref. 56).

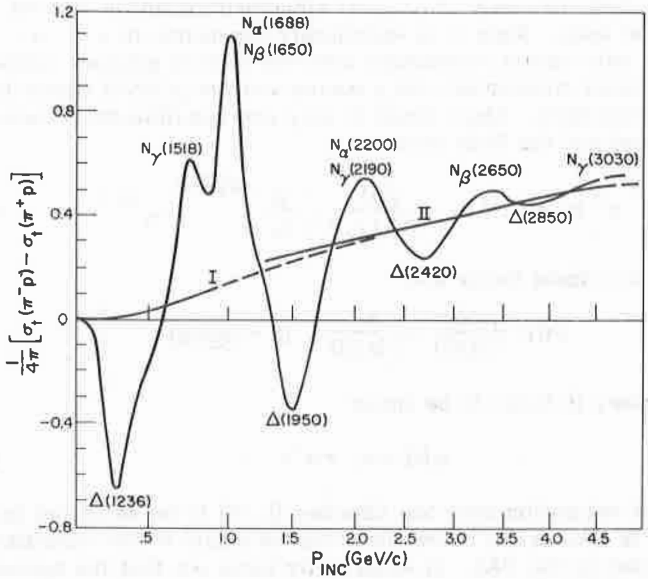

Fig. 2. Plot of the difference of π^-p and π^+p total cross sections. Curve I represents the low energy nonresonant amplitude as obtained from phase shifts and curve II is the extrapolation of the ρ Regge trajectory contribution (from Ref. 23).

The above startling hypothesis that the Regge pole fit extrapolated from the high energy down to the intermediate energy region equals some local average of the scattering amplitude is nowadays referred to as the "Dolen-Horn-Schmid duality." We shall return to it in Sec. I.F.

Let us now look in detail at the analysis of $\pi\pi \to \pi\omega$ which was made by Ademollo et al.[11),12)] The choice is due to the fact that one single invariant amplitude T crossing symmetric in s, u, t describes the process in which only I = 1, G = $\pm$, odd J normal parity resonances are allowed. We define the invariant amplitude A(s,t,u) in terms of the T-matrix as

$$T = \epsilon_{\mu\nu\rho\sigma}\, e_{\mu}\, P_{1\nu}\, P_{2\rho}\, P_{3\sigma}\, A(s,t,u) \tag{I.16}$$

where P_i are the pion four momenta, e_μ is the ω polarization vector.

A(s,t,u) has only dynamical singularities and is free of kinematical ones. Also, it is completely symmetric in s, t, u.

The only known resonances with the correct quantum numbers for the present problem are the ρ meson and the g(1650) meson lying on the ρ trajectory. Also, there is only one possible Regge pole (the ρ trajectory) and the FESR becomes

$$\int_0^N \nu^n \, \mathrm{Im}\, A(\nu,t)\, d\nu = \frac{\bar{\beta}(t)}{\alpha(t)+n}\left(\frac{N}{\nu_0}\right)^{\alpha(t)+n} \nu_0^{\,n+1} \tag{I.17}$$

where ν_0 is a scale factor and

$$\bar{\beta}(t) = \frac{\beta(t)}{\Gamma(\alpha(t))} \simeq \frac{\beta}{\Gamma(\alpha(t))}\,; \quad (\beta = \text{const}).$$

The trajectory is taken to be linear

$$\alpha(t) = \alpha_0 + \alpha' t \quad . \tag{I.18}$$

In the first saturation step one chooses (I.17) to be saturated by the ρ only (N is thus taken below the g-meson mass) whose contribution is calculated in the NRA. It empirically turns out that the optimum choice for N is half-way between the ρ and g mass. Evaluating Eq. (I.17) one gets (with n = 0)

$$2m_\rho^2 - 3m_\pi^2 - m_\omega^2 + t = \frac{\alpha(t)}{\alpha'}\, \Phi_1(\alpha)\left[\frac{1}{2\nu_0\alpha'}\right]^{\alpha-1} \tag{I.19}$$

where

$$\Phi_1(\alpha) = \left[\frac{\alpha+2}{2}\right]^{\alpha+1}\left[\Gamma(\alpha+2)\right]^{-1} \tag{I.20}$$

The l.h.s. of (I.19) vanishes at $t = m_\omega^2 - 2m_\rho^2 + 3m_\pi^2$ and so α(t) must have a zero at $t \simeq -.53\ (\mathrm{GeV}/c)^2$ which is just what one finds in πN charge exchange. Imposing the above condition in the linear approximation for α(t), one finds the consistency equation

$$\Phi_1(\alpha) = (2\nu_0\alpha')^{\alpha-1} \tag{I.21}$$

which for

$$2\nu_0\alpha' \simeq 1$$

is very well satisfied for $-1.5 \le t \le .5$ $(\mathrm{GeV}/c)^2$.

One can then go one step further and saturate (I.17) with both ρ and g mesons. It is found that the position of the dip does not change very much (it is now at $t \simeq -.58$ $(\mathrm{GeV}/c)^2$) and that the equation is satisfied (at least approximately) for a larger interval of negative t.

In general, if r resonances lying on a Regge trajectory are used, the self consistency condition becomes

$$\Phi_r(\alpha) = 1 \tag{I.22}$$

with

$$\Phi_r(\alpha) = \frac{\Gamma(2r-1)}{\Gamma(2r+\alpha)}\left[\frac{\alpha+4r-2}{2}\right]^{\alpha+1} \tag{I.23}$$

The case r = 3 is given in Fig. 3 which shows how well (I.22) is satisfied for a rather large t interval. Furthermore, from (I.23) it is manifest that for any fixed α

$$\lim_{r\to\infty} \Phi_r(\alpha) = 1 \quad .$$

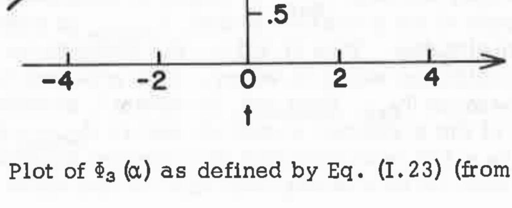

Fig. 3. Plot of $\Phi_3(\alpha)$ as defined by Eq. (I.23) (from Ref. 11).

The very successful picture described above is, however, partly spoiled by the conclusions reached in Ref. 11 when trying to saturate the bootstrap equations with more and more resonances. It is found, in fact, that with increasing $|t|$ one cannot saturate the resonance side to the Regge term since the latter becomes increasingly large compared to the former. In other words, the bootstrap cannot selfsustain with one single trajectory.

A possible way out was proposed[12)] after the observation of Schmid[13)] that the partial wave projection of a Regge pole gives rise to loops that look very much similar to those obtained in the phase shift analysis of resonances. We shall return to this point later on (Sec. I.E.) but, following this observation, it was suggested[12)] that the bootstrap program previously outlined could be accomplished by taking into account parallel daughter trajectories.

Although the results are not as conclusive as one would like and many points still need clarification, the above example is certainly a very successful example of the new bootstrap ideas previously discussed.

C. FESR vs. the Interference Model

We can now discuss the advantages of the FESR as compared to the so-called interference model[24)] which we shall refer to as the RIM (for Regge interference model) but not to confuse it with the DIM (diffraction interference model) of Ref. 22.

The general argument brought against the RIM[1),23),25)] is that if we use a formula of the form

$$F = F_{Regge} + F_{Res.} \tag{I.24}$$

the tail of $F_{Res.}$ superimposes to the Regge term (which is already supposed to give the asymptotic form) to yield double counting. Conversely, the Regge term continued to low energies gives again a contribution superimposed to the resonance term and thus double counting.

Occasionally, the above criticism is rephrased trying to give it a more stringent meaning on a theoretical basis but, in our opinion, it really only confuses the issues. In one way of saying it, one would argue that whereas F_{Res} is suited to describe s-channel amplitudes (because of its s-channel poles), F_{Regge} is suited to describe t-channel amplitudes. Thus (I.24) is like summing up s-channel and t-channel amplitudes which is wrong. This argument is, however, fallacious because $F_{Res.}$ does not, in general, provide a complete description of the s-channel amplitude and so F_{Regge} does not, in general, give a full representation of t-channel amplitudes. In either case there ought to be a background term for the above argument to hold through and in fact it was just the presence of such a background term that motivated the authors of Ref. 25 to use parametrizations of the form (I.24).

In another way of arguing, one would start from the observation that the partial wave expansion on the one hand and the Regge expansion on the other hand are two complete descriptions of the same amplitude and conclude that one should not use (I.24). Again the same kind of fallacy as before is met here since one should first prove that there is no background in either one of the two representations since neither a Regge expansion nor a resonance expansion is, in general, complete from a mathematical point of view.

The point, however, remains that the first objection we mentioned, about double counting, is certainly valid when using (I.24) (unless further specification is given concerning the behavior of each term). This can be given a better qualitative, if not quantitative meaning, if we retain the basic assumption already made in Sec. I.B. that we can neglect background contribution in the Regge pole fit already in the region of 1, 2 GeV. If this assumption is made, then FESR tell us that the sum of Regge terms alone gives a fit to the smoothed out experimental curve. Under these conditions, Eq. (I.24) would count essentially twice the contribution of a resonance, once in the explicit term F_{Res}, and another time in F_{Regge} which "knows" already of the averaged value of the resonance (or at least of part of it). Notice, however, how the argument in its prediction of double counting depends on having completely neglected any background (this was assumed to derive Eq. (I.3)). If, however, this background is not completely negligible (or, rather, if the Regge and the resonant background do not cancel exactly), then we can say, at best, that there is a "larger than one" counting in writing down Eq. (I.24) but also a "less than double" counting.

To avoid the above double counting problem, the authors of Ref. 1 suggest that instead of (I.24) one should write

$$F = F_{Regge} + F_{Res} - \langle F_{Res} \rangle \tag{I.25}$$

where the last term is supposed to remove the discrepancy that arises from adding the asymptotic tail of the resonances together with the Regge term. At the same time this term serves the purpose of removing the contribution of the Regge pole fit extrapolated at low and intermediate energies where, according to FESR this term already represented some sort of "averaged amplitude" (see Fig. 1).

It is now clear that if all resonances enter with the same sign, then the inclusion of the term $\langle F_{Res} \rangle$ is rather important whereas if they alternate in sign and have comparable strength, then $\langle F_{Res} \rangle \simeq 0$. Finally, if it were true that the resonant background was totally negligible in writing a partial wave expansion, then we would have

$$F_{Regge} \simeq \langle F_{Res} \rangle \qquad (I.26)$$

and conversely if the Regge background was totally negligible, then

$$F_{Res} \simeq \langle F_{Res} \rangle \ . \qquad (I.27)$$

The theoretical implications of (I.25) (which constitutes what is called Dolen-Horn-Schmid duality) and of (I.26), (I.27) will be discussed in Sec. I.F.

We want now to examine briefly the evidence in favor of (I.25) as compared to (I.24). Crucial tests to check whether (I.25) is a good substitute for (I.24) are cases in which resonances occur with the same sign; we next discuss a few of the examples given in Ref. 1.

i) $\underline{\text{Im}\, A'^{(+)}(k,0)/k}$. This is the average of $\pi^{\pm}p$ total cross sections. Extrapolating the Regge fits down to $k \sim 1$ GeV/c one gets somewhere in between 35 to 40 mb whereas the experimental average is (37 ± 7) mb in which the error gives the size of the resonance enhancements over the background. Thus the extrapolated Regge fit already saturates the averaged amplitude and there is no room left for the resonances to contribute whereas around this energy value there are at least four resonances amounting to over 25 mb.

ii) $\underline{\text{Im}\, \nu\, B^{(-)}(k,0)}$. In Fig. 4 the amplitude is given as reconstructed from phase shifts data and the Regge pole fit is also shown. It appears that the reconstructed amplitude is smaller than the one obtained from resonances only and thus the Regge contribution cannot represent the background term since they would be of opposite sign.

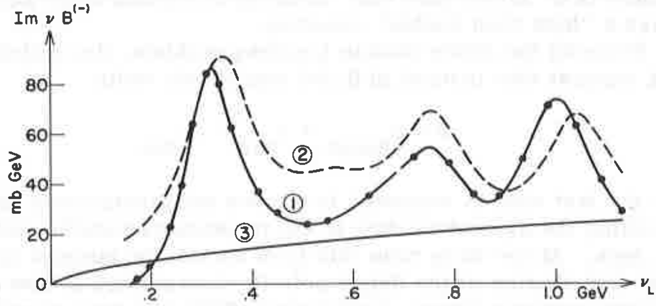

Fig. 4. Value of ν Im $B^{(-)}(\nu,0)$ as reconstructed from phase shift (curve I) and as calculated from a resonance model (curve II) and from Regge poles (curve III) (from Ref. 1).

iii) $\pi^+ p$ backward differential cross section. According to the well known argument of Barger and Cline,[24] backward $\pi^+ p$ scattering is largely saturated by the direct channel resonance below 4 GeV/c. On the other hand it has also been shown[26] that the Regge fit alone extrapolated down to the same energy interval accounts for most of the data. In this case, therefore, not only is (I.24) ruled out because it would lead to a very severe double counting, but (I.26) and (I.27) seem to hold. Due to the absence of diffraction (which should be negligibly small in the backward direction) this example is also particularly crucial to check whether resonances alone can describe entirely the angular distribution in the backward scattering region. Preliminary results[27] seem to provide a positive answer to such a question provided resonances on several Δ trajectories are taken into account.

The previous are examples in which the predictions of the FESR quite sharply contrast those of the RIM. Other less unambiguous tests have been suggested[1] where the inherent ambiguity stems from the fact that not all resonances contribute with the same sign and large cancellations occur. Such is the case of $\pi^- p$ backward scattering which was fitted both with the RIM,[24] with Regge poles only[26] and with a pure resonances model.[28]

In conclusion, we can say that the inadequacy of the conventional RIM model seems fairly well established. Its possible modifications to avoid double counting will be discussed in Sec. I.J.

In Sec. I.G. we will also shortly review the way in which the DIM[22] would differ from the RIM with respect to the previous problem of double counting.

D. Schmid Loops

Recently, Schmid[13] showed that the partial wave projection (in the direct channel) of a Regge amplitude of the form

$$A = \beta \left(\frac{\nu}{\nu_0}\right)^{\alpha(t)-1} \frac{1 - e^{-i\pi\alpha(t)}}{\sin \pi\alpha(t)} \tag{I.28}$$

gives rise to loops in the Argand diagram (Fig. 5) of the familiar structure that one sees when analyzing resonances. He therefore offered the interpretation that these loops be associated with direct channel resonances.

If one defines a resonance by the requirement that:

i) A resonance leads to an energy variation of the resonating phase shift which describes a circle in the Argand diagram of the real vs. the imaginary part of the corresponding partial wave (the radius of the

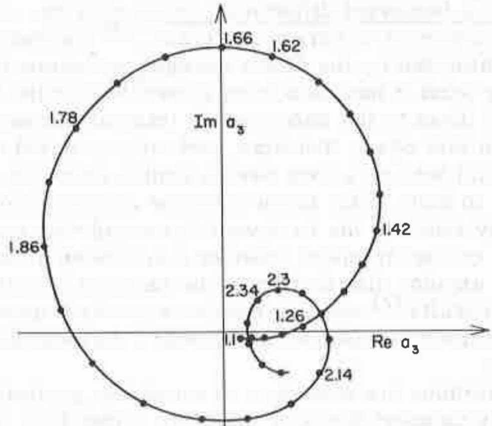

Fig. 5. Argand diagram for $\ell = 3$ for the reaction $\pi\pi \to \pi\omega$ (from Ref. 12); dots along the curve denote masses in GeV.

circle being the elasticity of the resonance) and if one further assumes that the converse is true, i.e.
ii) Every loop in the Argand diagram is a resonance; then Schmid's conclusion that the loops of Ref. 13 are associated with resonances is an inescapable consequence. This would be so in spite of the fact that the (traditionally) more familiar property of a resonance, i.e. a pole in the second energy sheet, is completely absent from (I.28). The explanation for the absence of such a pole-aspect would be that Eq. (I.28) is already an asymptotic expansion which does not have entire recollection of all the properties of a resonance but only of some. From this point of view, it may be interesting to recall that it has been shown recently[29)] that the slope of the small angle angular distributions is a rather sensitive indicator of resonances. Here also, the pole aspect is totally absent.

It may be, however, that conditions i), ii) are not really enough to guarantee that a resonance is being seen. For one thing, for example, Schmid loops do not give rise to any even minimal bump in either angular distribution or cross section (since the various partial waves compensate each other). Also, by unitarity true resonances must occur in all processes with the same direct (s) channel quantum numbers whereas Schmid loops are due to t-channel exchanges.

The question, therefore, arises of how can two different Regge poles give rise to the same set of resonances (example, $\pi\pi \to \pi\pi$ and $\pi\pi \to N\bar{N}$ have the same s-channel but different t-channel quantum numbers). The above objections[30] are of a different relevance and the first is, really, the very crucial obstacle to believing that these loops are really resonances. The second objection could be met by actually assuming that this situation is an ideal key to the new bootstrap. Furthermore, a specific example[12] has shown that the partial wave analysis of $\pi\pi \to \pi\omega$, $\pi\pi \to \pi H_{\lambda=0}$ and $\pi\pi \to \pi H_{\lambda=1}$ (λ being the helicity of the H(990) meson) shows the same loop structure in all three channels in spite of the difference in their spin structure. However, the same trajectory contributes here so that the argument is not conclusive. Another objection[31] is that in partial wave projecting (I.28) one not only finds the loop for $\ell_0 \simeq s_0$ discussed in Ref. 13 but one finds (infinitely) many more for $\ell > s_0$. This means that even if we identify a given loop at a given energy with an experimentally observed resonance occurring at a given angular momentum, we would in addition have infinitely many other resonances at higher value of angular momentum (ancestors). Whereas the appearance of these ancestors does not, strictly speaking, conflict with any theoretical principle, their existence would, certainly, lead to a drastic modification of what one intuitively believes to be a resonance. It is, however, to be noticed that the phenomenon of resonances occurring at the same energy but different angular momenta is not ruled out on experimental basis.

It has been suggested[32] that ancestors should be included in the error bars since one may argue that their effect becomes negligibly small and that they appear as the effect of the (small) violation of unitarity occurring in a Regge pole treatment. However, although it is true that each single ancestor gives a small contribution, this may not be the case[33] when one has infinitely many ancestors.

Other objections[34] to the interpretation of Schmid's loops as resonances have been raised by various authors. For instance, not only should there be resonances of low mass and very high spin, but also it is easy to obtain loops that rotate clockwise by combining two or more Regge poles. Also, Kreps and Logan[34] have analyzed $\pi^- p \to \pi^0 n$ and concluded that there is a lack of correspondence between Schmid's loops and experimental resonances contrary to Lipshutz's claim[35] that most resonances and loops can be identified.

Probably the best way to exhibit the ambiguity associated with Schmid's interpretation of his analysis is seen if one replaces the energy dependent formula (I.28) with the purely t-dependent factor

$$A = e^{i\pi\alpha(t)}; \quad \alpha(t) = a + bt.$$

In this case (see Fig. 6) we still have the same kind of loops as before.[36] We must therefore conclude that the entire "loop" structure associated with a Regge pole can be traced back to, essentially, the signature factor. Since the latter was originally introduced because of crossing symmetry arguments, it is hard to see its possible connection with resonances. On the other hand, as will be discussed below (Sec. I.I.) it is just the fact that these loops are

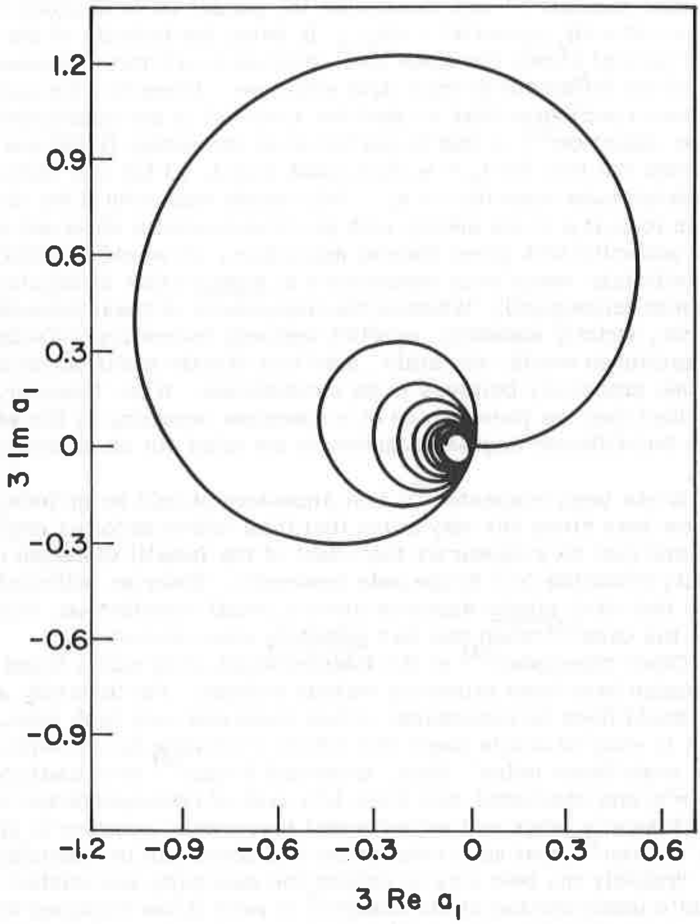

Fig. 6. Argand diagram for $\ell = 1$ from the amplitude $A = e^{i\pi\alpha(t)}$ (from Ref. 36).

connected to the signature factor that allows one[13] to give an indirect support to the idea of duality through the mechanism of exchange degeneracy.

A final word of caution seems, finally, necessary in taking as absolute truth the result of a phase shift analysis since it is well known that, for instance, several resonances could be very close together without one being able to resolve them through a partial wave decomposition. Evidence for a resonance coming from partial wave analysis should, in general, be taken with some care unless independent supporting evidence can be obtained.

In conclusion, as we have seen, almost every argument discussed before can be seen in a light either supporting or casting doubts on the validity of Schmid's conjecture. The least one can say is that the subject is, certainly, very controversial, and although Schmid's conjecture opens an entirely new field, it must be realized that the acceptance of it is close to being an act of faith. Possibly, one could rephrase St. Augustine by saying that,"He who believes will understand."

The acceptance of conditions i), ii) as defining a resonance (which is now dissociated from bump effects[37]) has very interesting consequences which will be discussed in the next sections and which give rise to the new dynamical mechanism of duality.

E. Classical Interpretation of Regge Poles

In order to appreciate better what is so new about the Schmid conjecture, we recall here the usual interpretation that was given of a Regge pole. Because of the nonrelativistic origin of a Regge pole, and of its subsequent interpretation in relativistic terms, traditionally, a Regge pole which determines the leading asymptotic behavior in the direct channel, has been associated to the exchanges that occur in the crossed channel. This interpretation has been considerably strengthened by the work of Fubini and coworkers[38] on the so called multiperipheral model where it was shown that the series of ladder diagrams in the crossed t-channel has the typical $s^{\alpha(t)}$ behavior as $s \to \infty$.

More recently, it has been shown by Van Hove[39] that a tower of single particles exchanges in the crossed channel can, again, give an $s^{\alpha(t)}$ behavior at large energies, reinforcing the conclusions drawn previously. This is, also, the motivation for an interference model.

It should be stressed that in the "old fashioned" interpretation of Regge poles discussed above, the conclusion is that a Regge pole is a manifestation of the crossed channel exchanges, which is, therefore, free of direct channel resonances but not of direct channel branch points. This point is important to make because soon we will particularly be interested in cases in which the only singularities are poles. To see the relevance of the above point we first give a

Definition: We shall say that an amplitude F(s,t) is Regge behaved if its asymptotic behavior as $|s| \to \infty$ is $\beta(t)\, s^{\alpha(t)}$ uniformly[40] in the entire complex s plane except, possibly, for an (arbitrary) direction

$$F(s,t) \underset{|s| \to \infty}{\to} \beta(t)\, s^{\alpha(t)}$$

$$|\arg(se^{i\theta})| \geq \epsilon > 0.$$

$$0 \leq \theta \leq 2\pi; \quad \theta \text{ fixed.} \qquad \text{(I.29)}$$

We can then state the following:[41]

Theorem I: No entire function of finite order can have an asymptotic power-like behavior along every direction except possibly one. If this one direction is not excluded then, as an immediate consequence of Picard's theorem on essential singularities, we have:

Theorem II: The only entire function that has a uniform power-like asymptotic behavior is a polynomial.

We do not know of any comparable theorem for functions of infinite order.[42]

Using Theorem I we can say that given an amplitude A(s,t) which is purely meromorphic as a function of both $\alpha(s)$ and $\alpha(t)$[43] and is Regge behaved in both s and t, the part which contains only crossed crossed (t) channel poles (and is thus entire in $\alpha(s)$) cannot be Regge behaved in s except, possibly, if it is infinite order.[44),45] This, however, does not apply when there are cuts. Therefore, the above argument simply means that, according to the traditional interpretation of Regge poles we would not have said anything like "the crossed channel poles build up the direct channel asymptotic Regge behavior" but rather that the Regge pole term corresponds to crossed channel exchanges and knows nothing about direct channel exchanges (but must certainly have direct channel cuts).

The revolutionary idea contained in Schmid's conjecture (Sec. I.D.) can now be seen to be a modification of the above picture in which one would say that a Regge pole knows not only about the crossed channel, but also about the direct channel exchanges.

F. Duality

In the very recent times an extraordinarily large literature has grown centered about the new concept of "duality" in spite of the fact that this concept seems very difficult to define unambiguously. As a consequence, the range of definitions of duality is extremely broad going from the requirement of cyclic symmetry between the external identical particles (which guarantees crossing symmetry but it is not demanded by it) to the requirement that the sum of resonances in the direct channel is the one that gives rise to the Regge poles. A whole spectrum of definitions exist in between but the most commonly used definition seems to be[46)] that the direct channel resonances "build up" to the asymptotic Regge behavior. This is a local generalization of the original "averaged" statement contained in Ref. 1.

We first attempt a classification of the definitions that seem more commonly used.

A) The amplitude i) obeys crossing symmetry requirements, ii) results from the sum of infinitely many poles in each channel, iii) is Regge behaved in each channel. This can be rephrased by saying that we can indifferently sum the spectrum of resonances in either the direct or the crossed channel. Each expansion is complete and Regge behaved.

B) At intermediate energies the sum of direct channel resonances smoothed out coincides with the extrapolated Regge behavior. Alternatively, the latter gives in a sense a semilocal average of the resonance peaks.

C) The sum of direct channel resonances asymptotically builds up (at least in part) to the asymptotic Regge behavior.

It is clear that there is a considerable overlapping between the above definitions. We will now briefly discuss them to point out some of their inherent ambiguities. Also, for reasons to be discussed below, we will assume that the Pomeranchuk term is excluded from our present considerations.

First, we notice that definition A) is, probably, the broadest among the various definitions given above. Not only does it leave room for some generalized interference model[47),48)] but it may be compatible with Van Hove's and Durand's model[39)] and it may ultimately reduce to the usual mechanism by which Regge poles were discovered in potential scattering. In the second formulation of A) it is understood that the sum of resonances in different channels may converge in different domains so that such formulation can be given a well defined meaning only if we can sum the series of resonances in closed form and analytically continue it.

We would like to emphasize that in definition A) the essentially new ingredient as compared to the pre-duality models is the requirement that there are infinitely many poles and that Regge behavior (as defined by (I.29)) holds with respect to each variable. It is the latter condition that makes the difference with a phenomenological interference model but, for any practical purpose, a "duality model" can then be disguised as a "generalized interference model"[47),48)] even in the case of meromorphic amplitudes as the specific example of the Veneziano model will show.[44)] Notice that it is condition (ii) in the definition A) that provides the new dynamical assumption (see, however, Ref. 22) because no set of poles forms, mathematically speaking, a complete basis. This dynamical assumption becomes then, most naturally, the foundation of the new bootstrap outlined in Sec. I.B.

Definition B) which we shall refer to as "weak duality"[44)] is the same definition which arises from the context of FESR as mentioned already in Sec. I.C. It is an inescapable consequence of the same hypotheses that led us to (I.3) (that is, neglect of the high energy background contribution to the asymptotic Regge expansion) plus the further assumption that the low energy contribution (the l.h.s. in (I.3)) can be well approximated by pure resonant effects. The latter is still another assumption since mathematically speaking, no set of poles represents a complete basis as already mentioned. Therefore, weak (or Dolen-Horn-Schmid) duality can be mathematically stated by Eq. (I.25). The only visible trouble in the above definition lies in its ambiguity concerning the "semilocal average" represented by $\langle F_{Res} \rangle$. It is also worth noticing explicitly that nothing in either definition B) or Eq. (I.25) distinguishes the situation in which the asymptotic behavior is originated by crossed channel poles from the one in which neither cross nor direct channel poles are responsible for the high energy behavior). As in the case of definition A), therefore, weak duality could still be reconciled with the familiar mechanism of potential scattering. Furthermore, its distinction from A) is that it does not explicitly require an infinity of poles nor does it enforce crossing symmetry. The latter is a consequence of the way FESR were derived starting from a one-dimensional dispersion relation.

Definition C) which we shall refer to as strong duality,[44)] rules out the old fashioned potential scattering mechanism and requires that the sum of direct channel resonances builds up locally to the crossed t-channel Regge pole. Suggestions that this could be a possible mechanism are found in Refs. 3, 4, 13, and 46.

It is quite clear that strong duality contains weak duality as the particular case in which Eqs. (I.26), (I.27) simultaneously hold and this explains the adopted terminology.

In Refs. 49 and 50 collective sets of references are given in which definitions of weak and strong duality, respectively, are either given or implied (these are certainly not complete references).

It is easy to convince oneself that definition C) is a most deceptive one and that if one takes it at face value it essentially cannot be given a definite and unambiguous meaning.[44] This is so unless it is interpreted in the sense of def. A) (in which case, however, the statement that direct s-channel poles "build up" to the s-Regge behavior cannot be distinguished from the equally meaningless statement that crossed t-channel poles build up to the s-Regge behavior).

The first problem opened by def. C) is that mathematically speaking the concept that the poles of a function should determine its asymptotic behavior does not hold (for instance, for a purely meromorphic function the role of its zeros is equally important). If we want, however, to insist on def. C) as a new dynamical request and we want it not to coincide with A), the only way is to assume that the s-Regge behavior comes uniquely from that part of the amplitude that contains only s-channel poles. This implies that one can write

$$F(s,t) = F^{(s)}(s,t) + F^{(t)}(s,t) \tag{I.30}$$

where $F^{(s)}$ contains only s-channel poles and is Regge behaved according to our definition (I.29). For purely meromorphic amplitudes, this model would require $F^{(s)}(s,t)$ to be entire in t. Also $F^{(t)}(s,t)$ would have to be entire in s and bounded by Regge behavior as $|s| \to \infty$. Theorem I of Sec. I.E. above, however, guarantees that the latter requirement is excluded for functions of finite order and type. Unless the latter case holds we can conclude that def. C) makes no sense in mathematical terms for amplitudes meromorphic in $\alpha(s)$ and $\alpha(t)$. If Regge cuts are allowed, theorem I does not hold any more but in this case there is no way to give a meaning to def. C).

It should be noticed that the difficulty associated with def. C) goes essentially back to Mittag-Leffler theorem[51] on functions with infinite isolated singularities. This theorem states that such functions are determined by their poles and residues only to within an entire function. In physical terms, this problem reflects itself in the ambiguities inherent in the use of the word "resonance." The only information (coupling constant or residue) that we have about a resonance is in the neighborhood of the pole which is, in fact, in a region inaccessible to experiment. The latter point adds still more ambiguity (to the determination of the width of the resonance) but, even if we could measure the residue just at the pole, this would essentially give no information on the structure of the resonance away from it. In other words, we could modify the form of a resonance in an arbitrary way provided we would not alter the residue and the position of the pole. This fact is, actually, more or less consciously used by every phenomenologist when he adjusts the energy dependence or the centrifugal barrier or the tail of the resonance to get a better fit. However, what Mittag-Leffler theorem means is that this freedom in changing the form of a resonance away from the pole is a mathematical and not a physical ambiguity. The only principle that could help in reducing the above ambiguity would be inelastic unitarity which is, unfortunately, intractable.

In the following, when using the word duality, unless otherwise stated, we will always refer to def. A).

It should, finally, be emphasized that if this property of duality holds, the ultimate consequence of it would be that the forces (i.e. the crossed channel exchanges) would be determined once the direct channel singularities are given;[52] this is, of course, the content of bootstrap.

G. The Pomeranchukon Diffraction Interference Model

It has been pointed out by Freund[10] and Harari[17] that the Pomeranchukon seems to play a rather special role and should in fact be absent from all considerations made above. The reason for this is that whereas "ordinary" Regge trajectories can be bootstrapped using the resonance approximation to FESR, this seems not to be the case for the Pomeranchukon so that it has been suggested that it should be built from the nonresonating backgroup.

Among the arguments brought against considering the Pomeranchukon as an ordinary Regge trajectory are:

a) there is no conclusive evidence of particles lying on such trajectory;

b) the slope would be essentially different from that of all Regge trajectories (i.e. much flatter and not inconsistent with zero);

c) the only "simple" dynamical origin that one can conceive for the Pomeranchuk is in terms of diffraction.

Other strange properties of the world of high energy physics which are not, probably, distinct from the above uncertainty on P are:

A) K^+p, K^+n, pp and pn total cross sections are essentially constant from 2 to 20 GeV/c contrary to what happens for K^-p, K^-n, $\bar{p}p$, $\bar{p}n$, $\pi^\pm p$,[53]

B) K^+p and pp angular distributions are essentially structureless and do not show secondary maxima;

C) the above channels are, exactly, those for which well established resonances do not exist in the s-channel at low energies whereas all other channels (K^-p, $p\bar{p}$, $\pi^\pm p$) appear filled with "low" energy s-channel resonances.

The suggestion that the Pomeranchukon is made of the nonresonating background (contrary to the program outlined in the previous sections for all other Regge poles), is meant to be the way out to both sets of phenomena a) ÷ c) and A) ÷ C).

First, one notices that P dominates at large s and small t irrespective of whether there are resonances or not so that it is hard to correlate P to the presence of s-channel resonances. Assuming that it is related to nonresonating (diffractive) effects, this implies that if we start from a FESR (I.3)

$$\int_0^N \nu^n \, \mathrm{Im}\, F \, d\nu = \sum \beta_i(t) \frac{N^{\alpha_i+n+1}}{\alpha_i + n + 1}$$

and we split F into a "resonant" and a "background" part, we can write

$$\int_0^N \nu^n \, \mathrm{Im}\, F_{bg} \, d\nu = \beta_P(t) \frac{N^{\alpha_P(t)+n+1}}{\alpha_P(t) + n + 1} \qquad (I.31)$$

$$\int_0^N \nu^n \, \mathrm{Im}\, F_{res} \, d\nu = \sum_{i \neq P} \beta_i(t) \frac{N^{\alpha_i(t)+n+1}}{\alpha_i(t) + n + 1} \qquad (I.32)$$

When $\mathrm{Im}\, F_{res} \simeq 0$ for $-\infty < \nu < \infty$, we are led, for sufficiently large N (say $\gtrsim$ 2 GeV), to

$$\sum_{i \neq P} \beta_i(t) \frac{N^{\alpha_i(t)+n+1}}{\alpha_i(t) + n + 1} \simeq 0 \qquad (I.33)$$

Furthermore, if we choose a reaction such that the t-channel quantum numbers prevent P from contributing, then we expect the amplitude F to be real.

Under these conditions, and assuming the validity of Schmid conjecture, we would have the following picture:

i) all total cross sections of reactions for which no important resonances are known should be essentially constant; this accounts for point A) above. Conversely, total cross sections for processes with many resonances should still decrease with energy eventually to reach the Pomeranchuk limit.

ii) In view of the absence of I = 2 resonances, $\sigma_{tot}(\pi^+\pi^+)$ should be constant and if the $\pi\pi$ amplitude is parametrized with P, P', ρ, this leads to

$$\alpha_\rho = \alpha_{P'} \qquad (I.34)$$

$$(\gamma_{\rho^o\pi^+\pi^-})^2 = (\gamma_{P'\pi^+\pi^-})^2 . \tag{I.35}$$

Whereas (I.34) is well satisfied, there seem to be no data inconsistent with (I.35). Similar analysis for πK, KK and KN gives

$$\alpha_\rho = \alpha_{A_2}$$

$$\alpha_\omega = \alpha_{P'}$$

$$\alpha_{K*} = \alpha_{K**} \tag{I.36}$$

(with corresponding relations between residues).

iii) All high energy KN, NN reactions in which P cannot be exchanged must have a real amplitude. This is, of course, a prediction very hard to test and agrees with the absorptive model prediction (although it is less restrictive).

iv) If one parametrizes pp and K^+p in terms of Reggė poles

$$F_{pp} = T_P + T_{P'} - T_\rho - T_\omega + T_{A_2}$$

$$F_{K^+p} = T_P + T_{P'} + T_\rho - T_\omega - T_{A_2} \tag{I.37}$$

and makes use of the exchange degeneracy (I.34) and (I.36), one finds that, aside from T_P, the amplitude for $pp \to pp$ and $K^+p \to K^+p$ using the parametrization (I.1) is proportional to

$$\frac{\alpha_\rho}{\sin \pi\alpha_\rho}$$

and therefore there is no zero at $\alpha_\rho = 0$. This in turn means that F_{pp} and F_{K^+p} angular distributions have no dips and are essentially structureless.[54] On the other hand, for K^-p, for instance, we have

$$F_{K^-p} = T_P + T_{P'} + T_\rho + T_\omega + T_{A_2} . \tag{I.38}$$

Aside from T_P, the amplitude is therefore proportional to

$$\alpha_\rho \frac{e^{-i\pi\alpha_\rho}}{\sin \pi\alpha_\rho}$$

whose imaginary part vanishes at $\alpha_\rho = 0$ where there is therefore a dip (similar conclusions hold for $\pi^{\pm}p$).

This accounts for point B) above and C) is taken care of by construction.

The picture proposed above, while explaining A), B), C), is consistent with the interpretation of the Pomeranchukon as a diffractive effect and models to construct it along this line have been proposed.[55)]

If the above discussion is correct, we should be able to use a modified interference model[17)] in which P is added to resonance contributions without incurring any double counting. It is clear that this modified interference model is nothing more than the diffractive interference model[22)] (DIM) which should, accordingly, be free of double counting troubles. Whether or not this is actually so, clearly depends on the specific parametrization chosen but it should be stressed that the problem of double counting is certainly absent if one uses the procedure of Ref. 22 of fitting the data by determining the resonance parameters through the fit itself rather than taking them from the tables. It is comforting that these two sets of values for the resonance parameters are very close.[22)] The model has been successfully applied to reproduce K^-p elastic data in the intermediate energy region (see Fig. 7). Recently it has been used to give a fairly successful fit[56)] to $K^-p \rightarrow K^-p$ and $K^-p \rightarrow \bar{K}_0 n$ angular distributions together with $K^-p \rightarrow K^-p$ polarization data from 1 GeV/c to about 3 GeV/c for the K^- lab. momentum. Furthermore, the model has been used to fit πp data.[57)] The model has also been tested in forward πN[58)] finding good agreement with the data. This result has, however, been questioned by Dance and Shaw[59)] who found that, in the same case considered in Ref. 58, the DIM fails badly and the data can reasonably well be reproduced by a simple isobar model (although the discrepancy increases with increasing energy). Much to the same conclusion come the authors of Ref. 60 who conclude that at much higher energies than those considered in Ref. 58, the DIM fails unless a large number of (as yet undiscovered) resonances is found.

The seemingly paradoxical conclusion that one can draw from the above discussion is that the result depends largely on the authors. This is perhaps not so surprising if we keep in mind the discussion of Sec. I.F. on the ambiguities inherently associated with the concept of resonance. For instance, the discrepancy between the results of Ref. 58 and 59 is, essentially, due to the differences in the parametrization of the resonances.

The conjecture that P is solely due to nonresonating background has been analyzed by Rosner[61)] who has shown that the system that arises is inconsistent. He showed that if one assumes σ_t to be flat

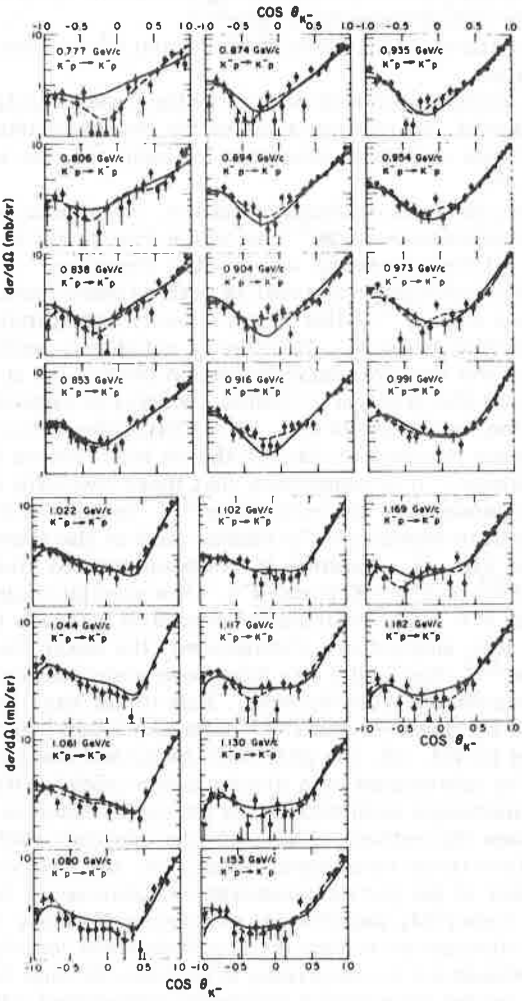

Fig. 7. $K^-p \to K^-p$ differential cross section from .7 to 1.2 GeV/c. The broken line curves are fits to the data using a Legendre expansion up to sixth order and the continuous line is the result of the fit with the diffractive interference model (from Ref. 22).

in all meson-meson, meson-baryon and baryon-baryon channels that are thought to lack resonances (i.e. channels outside 1 and 8 for MM, 1, 8 and 10 for MB and all BB) and one makes, besides factorizability, the additional assumptions that channels with direct channel resonances have a monotomically decreasing cross section, and f′ and φ decouple from S = 0 particles, then as a consequence, for instance, $\sigma_t(\bar{\Sigma}^+ p)$ is energy dependent whereas $\sigma_t(\bar{\Omega}^+ p)$ is flat. The conclusion reached in Ref. 61 is that there must exist enhancements in channels with unusual quantum numbers (exotic resonances). These exotic resonances should manifest themselves in $B\bar{B}$ systems.

The fact that not even $p\bar{p}$ does show any prominent resonance structure may, however, mean that these exotic resonances may be very hard to discover. Furthermore, it has been pointed out by Pinsky[62] that another way out of the difficulty mentioned before is to assume that there exist Regge cuts together with Regge poles. The larger number of parameters thus introduced does, essentially, leave freedom enough to solve the problem without the need for exotic resonances.

H. Graphical Duality

It has been recently suggested[63] that one can use a graphical version of the quark model[64] to give a visualization of duality. Duality here is taken according to definition A) of Sec. I.F. Since there is no way of putting this graphical form into an analytic structure, there is, however, no way of checking that Regge asymptotic behavior holds.

We assume that all incoming and outgoing particles as well as the poles in all channels are not exotic so that they can be represented by a three-quark or a quark-antiquark system. We will say, following Harari's definition[63] that duality appears if the process is given in terms of "legal diagrams." The rules for drawing a legal diagram are, in turn, extremely simple:

i) There are three types of quarks p, n, λ that do not change identity; every external baryon is made with three quark lines running in the same direction and every meson is made with two quark lines running in opposite directions;

ii) in any baryonic channel we can cut the diagram into two by cutting only three quark lines (and not, $4q + \bar{q}$ etc.); in any mesonic channel we can cut the diagram in two by intersecting only two lines.

In Fig. 8 the first three examples represent "legal diagrams" whereas the fourth is "illegal" since the $B\bar{B}$ channel requires intersecting four lines.

The physical assumptions to give meaning to those diagrams are that all baryons lie in the 1, 8, 10 SU(3) multiplets and all mesons

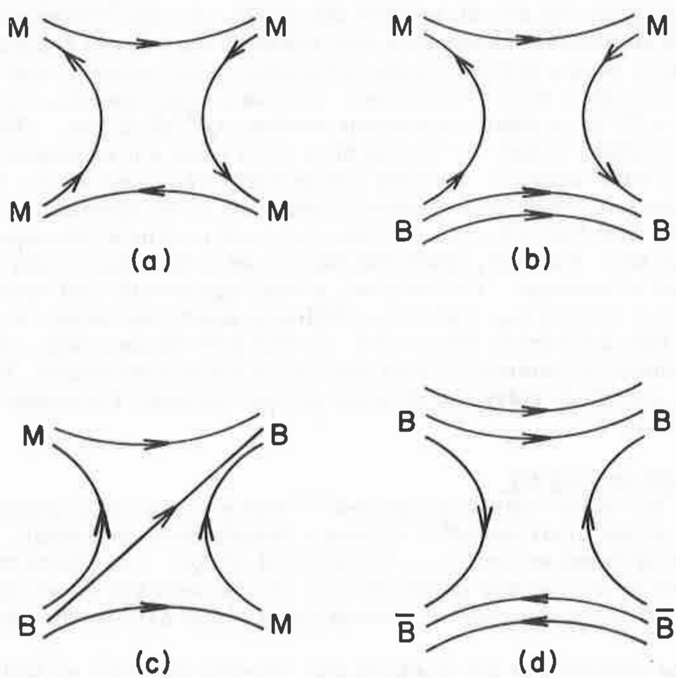

Fig. 8. "Legal" quark diagrams for (a) meson-meson scattering, (b,c) meson-baryon scattering; (d) is an "illegal" diagram for baryon-antibaryon scattering.

in the 1 and 8 multiplets. Furthermore it is assumed again that, aside possibly for the Pomeranchukon contribution, the scattering amplitude is the sum of single-particles states.

Duality is supposed to manifest itself if one can assume that one can describe the entire scattering as sum of either one-particles states in the direct channel or the crossed channel (in agreement with definition A) of Sec. I.F.).

One immediate consequence of the above discussion is found in the confirmation of the prediction of exotic resonances in the $B\bar{B}$ system. As seen in fact from Fig. 8d, in at least one of the s and t channels we must cut $2\bar{q} + 2q$ lines contrary to the previous rules.

Assuming now that the complete duality program can be accomplished, one would approximate the imaginary part of the amplitude by resonances (the real part may receive contribution from far away resonances and is scarcely affected by the nearby resonances because it vanishes just at resonance). Therefore, if a process does not exhibit direct channel resonances, its amplitude (aside from the Pomeranchukon) will be real. The real part will in turn vanish only if both the s and u channels lack resonances.

For detailed predictions following from this graphical method, we refer to the original papers.[63]

I. Experimental Support for the Principle of Duality

The most interesting theoretical consequence of duality is, perhaps, the possibility of a completely self-consistent bootstrap calculation in which the knowledge of either the direct or the crossed channel poles provides all the needed information. Due to the large arbitrariness in the parametrization of a resonance, this program needs a large number of confirmations before it can be taken as a practical dynamical scheme. Therefore, a less ambitious approach is probably needed to find some support for the idea of duality. Recalling the previous developments, we see that the necessity of excluding the Pomeranchuk from the duality game was a most important (and still rather mysterious) step. This has led also very naturally to the prediction of exotic resonances whose discovery would be a strong (although indirect) support for the idea of duality. Unfortunately, as the example of $p\bar{p} \rightarrow p\bar{p}$ teaches us, it may be very difficult (if at all possible) to reach this sort of confirmation.

However, the very general exchange degeneracy[65] previously noted (I.34), (I.36) may also be taken to within the limits of its experimental validity, as an indirect support for duality. The argument is, essentially, the same already given by Schmid[13] that if odd signature meson trajectories (ω, Φ, ρ etc.) and even signature trajectories (A_2, P, P$'$etc.) are exchange degenerate, then the signature factor in K^+p is real (and so is in pp) whereas is complex in K^-p (and $\bar{p}p$). According to the discussion of Sec. I.D., therefore, the former channels cannot give rise to any loop in the Argand diagram while the latter can. The Regge trajectories that are known to be exchange degenerate are Y^*, Λ, and the meson trajectories (see Fig. 9). N_α and N_γ trajectories are only partially degenerate.[32),66] These trajectories will have, in general, different residue functions but their slopes are essentially the same. Thus, there is a large economy of parameters which would be surprising if considered as a mere accident whereas it can simply be a reflection of the validity of the bootstrap program.[67] As pointed out before, in fact, it provides an

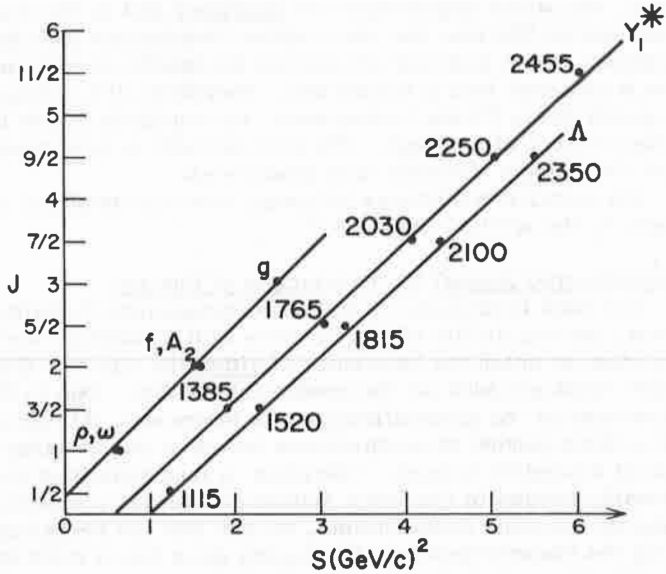

Fig. 9. Plot of the exchange degenerate meson (ρ, ω, P′, A_2) and baryon (Λ, Y_1^*) trajectories.

explanation for the absence of structure in both total cross section[13] and angular distributions for pp and K^+p and in $pp \to \pi^+ d$.[66] At the same time it also gives an indication that the I = 2, ππ phase shifts are negative.[68] The implication of exchange degeneracy on the hadron spectrum has also been discussed.[69]

Direct, although not conclusive, confirmations of the duality idea can, on the other hand, be considered the fits with resonances only to a) $\sigma_{tot}(\pi^- p) - \sigma_{tot}(\pi^+ p)$ (see Fig. 1 and Ref. 1, 23); b) $K^- p \to \bar{K}_0 n$ (Ref. 24); c) backward $\pi^+ p \to \pi^+ p$ (Ref. 25, 26); d) backward angular distribution $\pi^+ p \to \pi^+ p$ (Ref. 27); e) backward $\pi^- p \to \pi^- p$ (Ref. 28). Also, in the same category, we have the fits with the DIM (Ref. 22, 56, 57, 58). As mentioned, these fits are not free of ambiguities[59),60)] essentially due to the large number of parameters and freedom in parametrization of resonances.

More confirmations either direct or indirect are needed (along the above lines and also in the frame of multiperipheral Regge models) before definite conclusions on the validity of the principle of duality can be drawn.

J. Generalized Interference Model (GIM)

Alessandrini, Amati, and Squires[47]) have argued that the duality program as outlined in Sec. I.F. (according to definition C) and the interpretation of Schmid's loops as resonances (Sec. I.D.) run against a few conceptual difficulties in that if the Regge amplitude indeed results from a superposition of direct channel resonances $\sum_i g_i P_{\ell_i}(z_s)(s - s_i)^{-1}$ that gives

$$A_t^R = \sum_j \beta_j(t)\, s^{\alpha_j(t)} \frac{e^{-i\pi\alpha_j} \pm 1}{\Gamma(\alpha_j(t)) \sin \pi\alpha_j(t)} \tag{I.39}$$

then:

a) The partial wave amplitudes arising from the r.h.s. of (I.39) do not have the poles in the second sheet that one would associate with a genuine resonance.

b) The amplitude A_t^R represented by the r.h.s. of (I.39) does not have the experimentally observed peaks.

c) The partial wave projection of the r.h.s. of (I.39) leads to ancestors in positions unlikely to correspond to resonances.

These objections have been considered in Sec. I.D. when discussing Schmid's loops. The point here is that in Ref. 47 it is suggested that Schmid's loops have essentially nothing to do with resonances and their occasional coincidence with the actual position of resonances is a dynamical coincidence that can be attributed to the universality of all Regge trajectories (except the Pomeranchukon), and which can be traced back to the fact that all Regge trajectories have the same slope and that it is the signature factor which is the one responsible for the appearance of Schmid loops. As a consequence, they suggest adding the direct channel poles to A_t^R and writing

$$A = A_{\text{Regge}} + \sum_i \frac{g_i P_{\ell_i}(z_s)}{s - s_i} \; . \tag{I.40}$$

In order to avoid double counting, it is assumed that the parameters in (I.40) should not be taken from the tables but used as free parameters to determine the resonance parameters. Thus, since A_{Regge} has partial waves which are rapidly varying functions of energy, if it happens that a Schmid loop is in phase with a resonance, then the parameter g_i will not be simply related to the elasticity of the resonance and may even be complex or negative. By construction, there would therefore be no double counting.

The bootstrap program that was one of the main features of the duality program would still be possible. In fact, the last term on the r.h.s. of (I.40) can be continued a la Van Hove[39] to the physical region of the t-channel writing

$$A_s^R \simeq \sum_j \beta_j(s)\, t^{\alpha_j(s)} \frac{e^{-i\pi\alpha_j(s)} \pm 1}{\Gamma(\alpha_j(s)) \sin \pi\alpha_j(s)} \quad . \tag{I.41}$$

Alternatively one could write

$$A \simeq \sum_i \frac{g_i P_{\ell_i}(z_s)}{s - s_i} + \sum_j \frac{g_j P_{\ell_j}(z_t)}{t - t_j} + \sum_k \frac{g_k P_{\ell_k}(z_u)}{u - u_k} \tag{I.42}$$

where the g's are sufficiently well behaved so that
i) the sum over Legendre polynomials of physical argument ($|z| \leq 1$) converges in such a way that a limited number of resonances is needed;
ii) the sum over Legendre polynomials outside the physical z domain can be summed with a Sommerfeld-Watson transform to give a Regge pole.

In this case a finite number or even one Regge trajectory could reproduce itself whereas if a Regge pole in the s-channel must be generated by t and/or u Regge poles, this can necessarily happen only if there are infinitely many poles (so that the corresponding series can diverge). It should also be noticed that, as proved in Ref. 70 (see also Sec. I.A. and Ref. 20), for a sum of narrow resonances to reproduce an $s^{\alpha(t)}$ behavior, an infinite number of Regge trajectories is needed. Even when the narrow resonance approximation is removed, it appears very difficult to saturate a Regge pole behavior with bona fide resonances.[71]

The model discussed in this section is, really, only a simple minded model for a background. The major shortcoming of the model is that this background appears as a fairly rapidly varying function and this implies, as already discussed, that the parameters do not directly reflect the residues and positions of the poles which come out only after the dynamical analysis outlined before has been performed.

Several examples of GIM have been explicitly constructed.[72]

It has been recently shown by Jengo[48] that an amplitude can, under very general assumptions, be decomposed in the form of a GIM in which double counting is avoided by making sure that the direct channel resonances do not contribute to the high energy part of an

FESR in the sense that they are contained in a term which is strictly superconvergent in the sum rules of any moment. Furthermore, as will be discussed later (part II), it has been shown[44] that even the Veneziano model (i.e. a purely meromorphic amplitude exhibiting duality according to def. A of Sec. I.F.) can be cast into the form of a GIM. Thus, it is not unreasonable to conjecture that GIM and dual models are actually the same thing. In fact, Def. A of Sec. I.F. is in no way contradictory to the definition of the GIM or to Eq. (I.42). The only difference would be that in a dual model one would not explicitly decompose the amplitude as one would do in a GIM. Should this conjecture turn out to be generally true, one would conclude that a lot of very heated controversial statements between partisans of dual and GIM models have been rather fruitless.

In partial support of our conjecture, we notice that the GIM has the same kind of ambiguities encountered when discussing the duality program. These ambiguities are most evidently displayed by (I.42) and are, once again, implicit in Mittag-Leffler's theorem.[51] They can, as we already discussed, be summarized by saying that one could add or subtract entire functions to the pole terms of Eq. (I.42). A different way of stating this ambiguity is found in the result of Atkinson et al[73] that if a saturation of superconvergence is given with a tower of infinitely many resonances, other infinite saturations of this superconvergence problem are also possible.

Finally, it must be mentioned that the form of GIM derived by Okubo et al[48] has been obtained using dispersion relations as a starting point (just like FESR) and does in fact display certain duality properties. This makes the parallelism between dual models and GIM even more stringent.

II. The Veneziano Model and Its Properties

A. Preliminaries to the Veneziano Model

Recently, an extraordinary interest has arisen in connection with with the Veneziano model.[4] This interest is due to many combined factors. First of all the Veneziano model (V.M. hereafter) displays in a beautifully simple fashion most properties that one would like to attribute to an amplitude according to the discussion of part I. Second, the model is at the same time sufficiently simple for practical computational purposes and for a complete investigation of its mathematical properties but is already sufficiently complicated to provide examples of the ambiguities we have discussed before and to clarify many physical aspects of the program outlined in part I. Third, the model can be generalized in many respects. Last, and more important, the model seems to have the unprecedented virtue that it also works.

By this we do not mean to say that there are no problems (just the contrary), but that the general picture that emerges from it agrees substantially with our present experimental knowledge. However, not only is there some confusion about what its merits are (on a purely theoretical ground) but there are also quite a few points on which one would like to improve it. For instance, its most noticeable and seemingly incurable defect is its intrinsic violation of unitarity; also, its analytic properties are not exactly what one would like them to be. In spite (and partly because) of the above points, it is easy to predict that the V.M. will be the natural arena for theoretical physics in the near future although it is at present impossible to foresee whether it will represent a fundamental first step toward a new chapter in the understanding of strong interaction physics.

In the following (Secs. II.B. to II.K.) we will discuss the V.M. in its various aspects both from a physical and from a mathematical point of view trying to point out both its positive as well as its unsatisfactory properties. Sec. II.L. will be devoted to a very brief qualitative discussion of the successes met in applying the V.M. together with a few words of caution against excessive optimism in the interpretation of these successes. In Sec. II.M. we will then list and briefly discuss the many generalizations that have been proposed in the literature.

It will appear that there is a large disproportion between the time devoted to the discussion of the properties of the V.M. on the one hand and of its applications and generalizations on the other hand. The point, however, is that the subjects covered in both Secs. II.L. and II.M. would in themselves warrant a new entire chapter and this would make the present notes excessively lengthy. Furthermore the arguments of Secs . II.L. and II.M. are at the same time the most controversial and the ones in which things are moving particularly fast, so that any conclusion drawn now may be subjected to a drastic revision very soon. The properties of the V.M., on the contrary, seem by now sufficiently well established (although, admittedly, not yet in every respect) so that the disproportion mentioned above is somewhat justified.

B. Derivation of the V.M.

The original derivation of the V.M.[4] was actually a brilliant extrapolation from the work of Ademollo et al,[11] devised to give a crossing symmetric content to the bootstrap model discussed in Sec. I.B. We remember that it was suggested that a good parametrization for $A(s,t,u)$ for $\pi\pi \to \pi\omega$ is

$$A(s,t,u) \underset{s\to\infty}{\simeq} \frac{\bar{\beta}}{\pi}\Gamma(1-\alpha(t))(-\alpha(s))^{\alpha(t)-1} + (s \leftrightarrow u) \qquad \text{(II.1)}$$

with $\bar{\beta}$ = const. Veneziano suggested that (II.1) should be replaced by

$$A(s,t,u) \simeq \frac{\bar{\beta}}{\pi}\left[\frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} + \begin{array}{l}\text{symmetric}\\ \text{permutations}\end{array}\right] \qquad \text{(II.2)}$$

which reduces to (II.1) as $s \to \infty$ at fixed t (provided $\alpha(s) \underset{s\to\infty}{\to} \infty$) and treats in a completely crossing symmetric way the s and t channels. In the following we shall use the notation

$$V(s,t) \equiv \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} . \qquad \text{(II.3)}$$

Since the Veneziano amplitude is a beta function

$$V(s,t) = B(1-\alpha(s),\ 1-\alpha(t)) \qquad \text{(II.4)}$$

from the well known properties of beta functions, we see that (II.3) exhibits an infinite set of simple poles in both s and t channels at every value for which

$$\alpha(s) = n \ \text{ or } \ \alpha(t) = m \quad (\underset{m}{n} = 1, 2, \ldots) \ . \qquad \text{(II.5)}$$

Double poles, however, never appear since if both conditions (II.5) are satisfied, then

$$\alpha(s) + \alpha(t) = n + m$$

and the gamma function at the denominator gives a zero.

Before discussing the various properties of (II.3), we want to exhibit a "derivation" of it. Actually, a more appropriate wording would be that we want to give some plausibility arguments to show how (II.3) can be introduced.

Suppose we want to write an amplitude which possesses an infinite number of poles in the s-channel in an integral form. One possible way[74] is to write

$$A(s,t) = \int_0^1 dv\, v^{-\alpha(s)}\, f(v,s,t) \qquad \text{(II.6)}$$

where f(v,s,t) is regular at v = 0. Eq. (II.6) is well defined for any Re $\alpha(s) < 1$. When $0 < \text{Re}\,\alpha(s) < 1$, we can perform an integration by part and get

$$A(s,t) = \frac{v^{1-\alpha(s)}\, f(v,s,t)}{1-\alpha(s)}\Bigg|_{v=0}^{v=1} - \frac{1}{1-\alpha(s)}\int_0^1 v^{1-\alpha(s)} f'_v(v,s,t)\,dv$$

$$= \frac{f(1,s,t)}{1-\alpha(s)} - \frac{1}{1-\alpha(s)}\int_0^1 v^{1-\alpha(s)} f'_v(v;s,t)\,dv\ . \qquad \text{(II.7)}$$

Now the integral at the r.h.s. of (II.7) is defined for Re $\alpha(s) < 2$ and we have explicitly exhibited the singularity that prevented us from using (II.6) beyond the point $\alpha(s) = 1$, namely, a (simple) pole at that point. Therefore, provided f(1,s,t) be regular at $\alpha(s) = 1$, we can use (II.7). Clearly we can push the domain of validity of (II.7) arbitrarily to the right in the complex $\alpha(s)$ plane provided that f(v,s,t) is differentiable an arbitrary number of times and its derivatives are regular at v = 0 and v = 1.

If we now want to crossing symmetrize between the t and the s channel, one possible way to do it (but certainly not the only one) is to write

$$f(v,s,t) = (1-v)^{-\alpha(t)}\, g(v;\, s,t) \qquad \text{(II.8)}$$

where, by requiring

$$g(v;\, s,t) = g(1-v;\, t,s) \qquad \text{(II.9)}$$

we make A(s,t) = A(t,s) . Eq. (II.6) becomes then

$$A(s,t) = \int_0^1 dv\, v^{-\alpha(s)}(1-v)^{-\alpha(t)} g(v;\, s,t)\,. \qquad \text{(II.10)}$$

We can then repeat the above argument to show that A(s,t) as defined in (II.10) has an infinity of (crossing symmetric) simple poles in the s and t channels at the points (II.5) provided that g(v; s,t) satisfies (II.9) and is regular (together with all its derivatives) at v = 0 and v = 1.

In particular, if we choose

$$g(v;\, s,t) \equiv 1 \qquad \text{(II.11)}$$

we get, as a special case, exactly the Veneziano amplitude since (II.10) appears now to be the integral form of the Euler B-function

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 dv\, v^{x-1}(1-v)^{y-1} \quad . \qquad \mathrm{Re}(x,y) > 0 \tag{II.12}$$

Another "derivation" of (II.3) has, again, been given by Veneziano[75] by using the Khuri expansion.

It should also be mentioned that Schmid[32] has shown that formulating superconvergence at infinitely many discrete t-values one can construct an amplitude which coincides with (II.3). By construction, however, this solution depends crucially on the various assumptions that are made so that, although very elaborate and ingenious, we do not think that this method sheds very much light on the question of how fundamental or unique Eq. (II.3) may be. This conclusion is reinforced by the analysis of West[76] who has considered the same problem discussed by Schmid[32] under somewhat different assumptions.

Clearly, the example represented by (II.3) can be adapted to describe essentially any invariant amplitude in which the quantum numbers of the channels are given. This can be seen by suitably modifying (II.3) as follows

$$V^p_{nm}(s,t) = \frac{\Gamma(n-\alpha(s))\,\Gamma(m-\alpha(t))}{\Gamma(p-\alpha(s)-\alpha(t))} \tag{II.13}$$

where, for what was said previously (to avoid the appearance of double poles and of ancestors) one must assume

$$\max(m,n) \le p \le m+n \quad (m,n \ge 0). \tag{II.14}$$

In Eq. (II.13), m, n and p are integer or half-integer positive numbers (according to whether the corresponding Regge trajectory will be a meson or a fermion trajectory). The particular example of Eq. (II.2) provides a crossing symmetric amplitude, but one can similarly construct amplitudes that satisfy general crossing symmetry requirements. This has been done for a number of physical processes.[77]

Crossing is actually one of the most appealing features of (II.3) since this is the first time that an explicit, very simple, completely crossing symmetric amplitude has been written down without having to crossing symmetrize a posteriori. Because of this explicit crossing symmetry we can concentrate on the properties of (II.2) in one given channel and they will be valid in every channel.

C. Asymptotic Behavior

If we assume that

$$|\alpha(s)| \underset{|s|\to\infty}{\to} \infty \qquad \text{(II.15)}$$

then, at fixed t, and for any s such that

$$|\arg \alpha(s)| \geq \varepsilon \qquad \varepsilon > 0 \qquad \text{(II.16)}$$

we have, using Stirling formula

$$V(s,t) \underset{|s|\to\infty}{\to} \Gamma(1-\alpha(t))[-\alpha(s)]^{\alpha(t)-1} . \qquad \text{(II.17)}$$

We therefore come to the conclusion that the V.M. is asymptotically Regge behaved (according to the definition of Sec. I.E.) provided only that the Regge trajectory $\alpha(s)$ is, asymptotically, linear in s

$$\alpha(s) \underset{|s|\to\infty}{\to} 0(s) . \qquad \text{(II.18)}$$

Due to the combined properties discussed so far: i) crossing symmetry, ii) poles in all channels, iii) Regge behavior, we immediately conclude that Veneziano's amplitude has duality according to definition A) of Sec. I.F. We shall discuss later whether or not duality according to definitions B) and C) is also a property of (II.2).

If one next considers what happens when both $\alpha(s)$ and $\alpha(t)$ tend to infinity (asymptotic behavior $|s|\to\infty$ at fixed angle), assuming (II.18) to hold and applying Stirling formula again (with the same limitation (II.16)) one finds

$$\lim_{\substack{s\to+\infty \\ t\to-\infty}} V(s,t) = 0(e^{-s\ \text{const}}) . \qquad \text{(II.19)}$$

The above behavior (which is a strict consequence of the linear growth of the Regge trajectory) is a somewhat unsatisfactory prediction of the model. This is not so much so because of the fixed angle bound of Cerulus and Martin[78)] predicting that an amplitude cannot decrease faster than an exponential in $\sqrt{s}$[79)] (up to logarithmic factors) since this bound is derived under the assumption of analyticity conditions that are not satisfied by the V.M. Rather, a more compelling reason is that Eq. (II.19) shows a too fast rate of decrease than allowed by experiments. The plot of α(s) given in Fig. 10 (see Ref. 80) is obtained by fitting p-p angular distribution at high energy with a Regge form. A deviation from a straight line is definitely evident for $s < -6 (\mathrm{GeV/c})^2$. This is, on the other hand, simply the effect of Orear's fit[81] at large angles.

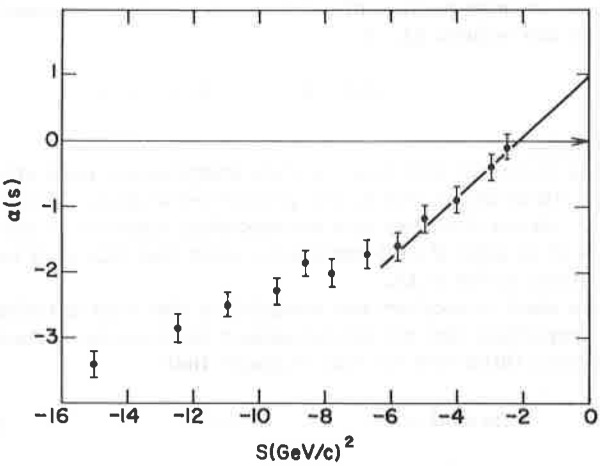

Fig. 10. Plot of α(s) in the negative s region is derived from high energy p-p elastic data (from Ref. 80).

Whereas one could argue that the evidence mentioned above comes from elastic scattering data and the V.M. cannot contribute to the Pomeranchukon (see Sec. I.G.) so that a direct comparison is not fair, the point is that the diffractive contribution is expected to be negligible in the large angle region. Furthermore, it appears that the large angle behavior of the V.M. would be the correct one should the "effective" Regge trajectory α(s) deviate asymptotically from a straight line toward a $\sqrt{s}$ behavior as suggested by many authors.[19]

D. Analyticity

Here and in the following the word analyticity will never be used in the conventional sense in which it has often been used for scattering amplitudes in physics, i.e. referring to the property of satisfying a dispersion relation or a Mandelstam representation. The amplitude (II.2) cannot, strictly speaking, satisfy any dispersion relation or Mandelstam representation because it is a purely meromorphic function of α(s), α(t) and α(u); furthermore, it is badly behaved in the unphysical domain of the s - u - t Mandelstam plane for $s \to +\infty$ and $t/s \to$ positive constant. However, the fact that an amplitude does not satisfy Mandelstam representation does not mean that it cannot have good analytic properties.

With the above clarification, the V.M. has a priori rather nice analytic properties since the following structure holds:

i) we have an infinite number of resonances in every channel corresponding to the values (II.5)

$$\alpha(s) = n \qquad n = 1,2,3,\ldots \qquad \text{(II.5)}$$

provided $\alpha(s) \underset{s\to\infty}{\to} \infty$,

ii) (II.3) may have, in principle, a very complicated cuts structure provided this structure is due to the properties of $\alpha(s)$. This is quite clear since a gamma function is a meromorphic function of its argument and therefore it is only if $\alpha(s)$ possesses cuts that this will be true for the amplitude in the V.M.

We now want to explore the possibility that $\alpha(s)$ satisfies the analyticity properties that we would expect on physical grounds. If Σ is the physical threshold we will suppose that

$$\mathrm{Im}\,\alpha(s) = 0 \qquad s \le \Sigma \qquad \text{(II.20a)}$$

$$\mathrm{Im}\,\alpha(s) > 0 \qquad s > \Sigma \qquad \text{(II.20b)}$$

$$\mathrm{Re}\,\alpha(s) \underset{s\to\pm\infty}{\to} \pm\infty\,, \qquad \alpha(0) \le 1 \qquad \text{(II.20c)}$$

Condition (II.20b) is the requirement that the total width of a resonance be positive. Condition (II.20c) guarantees that infinitely many "true" resonances are found (whenever Re $\alpha(s)$ crosses a positive integer) and that in the negative s region Re $\alpha(s)$ does not oscillate to infinity. In fact we can as well suppose that Re $\alpha(s)$ has no zeros on the negative s axis. Under these conditions $\alpha(s)$ is proportional to a Herglotz function and can therefore be written as

$$\alpha(s) = R(s)\,H(s) \qquad \text{(II.21)}$$

where R(s) is a polynomial and H is a Herglotz function

$$H(s) = H(0) + s\left\{A + \int_{\Sigma}^{\infty} \frac{\mathrm{Im}\,\alpha(s')}{s'(s'-s)}ds'\right\} \qquad \text{(II.22)}$$

with H(0) and A real (A ≥ 0). Therefore, if one takes the simplest choice

$$R(s) = \text{const} \tag{II.23}$$

one would be led to the conclusion that, unless A = 0

$$\lim_{|s|\to\infty} \alpha(s) = As \tag{II.24}$$

in agreement with that required by (II.20) which was needed to obtain Regge behavior. Notice, however, that the above argument is only a plausibility argument in favor of an asymptotically linear behavior since nothing prevents A from being zero. In this case, whereas $\alpha(s)$ would still be linear around s = 0, it would increase less than linearly as $|s| \to \infty$, but it can be shown[82)] that the deviation from linearity is very small.

E. Unitarity and the Structure of Resonances

Unitarity is the most troublesome aspect of the V.M. to the extent that in order to avoid its violation either one has to give up the analytic properties that one would expect an amplitude to display, or else one must allow ancestors to appear.

To see how this comes about, we have to give a closer look to the resonance structure and to the ensuing analytic properties of the V.M.

Let us consider the resonance at $\alpha(s)=n+1$. Remembering that at z = -n one has

$$\Gamma(z) \underset{z\to -n}{\simeq} \frac{(-1)^n}{n!} \frac{1}{z+n} \tag{II.25}$$

the residue of V(s,t) at $\alpha(s) = (n+1)$ is

$$R_n(t) = \frac{\Gamma(1-\alpha(t))}{\Gamma(1-n-\alpha(t))} \frac{(-1)^n}{n} . \tag{II.26}$$

Using

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n \frac{\Gamma(n+1-z)}{\Gamma(1-z)} \tag{II.27}$$

one finally finds

$$R_n(t) = \frac{\Gamma(n+\alpha(t))}{\Gamma(\alpha(t))} \frac{1}{n!} . \tag{II.28}$$

Notice that if we interpret $\alpha(s)$ as the Regge trajectory, we would like to require that the residue at the resonance $\alpha(s) = n + 1$ be a polynomial in t at most of the same order. A look at (II.28) shows that $R_n(t)$ is indeed a polynomial in t of order n if and only if $\alpha(t)$ is strictly linear. Whereas this is compatible with (II.18) which was needed to get Regge asymptotics, it is incompatible with the analyticity requirement discussed in Sec. II.E. since no strictly linear amplitude can satisfy a dispersion relation of the form (II.22). One could get away by giving up the dispersion integral for the Regge trajectory and therefore assume that the latter is indeed strictly linear

$$\alpha(s) = \alpha + \alpha' s \equiv a + bs. \tag{II.29}$$

Under these circumstances, however, we cannot but violate unitarity a priori. Unitarity, in fact, requires the residue $R_n(t)$ to be a real polynomial in t. This is possible only if both α and α' are both real. But, if this is so, the condition

$$\alpha(s) = n \tag{II.30}$$

means that the whole spectrum of the V.M. does not consist of bona fide resonances but is made of poles on the real s axis. As discussed in part I, this situation is referred to as "narrow resonance approximation"[9] (NRA). From the point of view of unitarity, the trouble is that this corresponds to having a situation in which the imaginary part of the resonance, i.e. the total width of the resonance is zero, whereas the residue at the pole, i.e. the partial width is nonzero. This is, clearly, a violation of unitarity and in fact, for what was said before, this violation inherently occurs in every NRA.

Essentially we can summarize things as follows:

a) we can let $\alpha(s)$ obey (II.22) so as not to spoil analyticity; this turns out not to be a straightforward point at all[83] and we shall discuss how one can do this in the following. In this case, $\alpha(s)$ cannot be purely linear and as a consequence, ancestors must appear.

b) we can assume $\alpha(s)$ not to satisfy (II.22) and to be linear with complex coefficients. In this case, the total width is nonzero but the partial width is complex (ghosts) and unitarity is violated.

c) we can assume $\alpha(s)$ to be strictly linear with real coefficients (in this case if the residues of the Legendre polynomials are positive, there are no ghosts). The total width of the resonance is, however, zero (narrow resonance approximation), and unitarity is, again, violated.

In either case b) or c), both unitarity and analyticity are lost (the latter in the sense that there are no cuts and the amplitude is, strictly speaking, a purely meromorphic function of s, t, u).

Normally, in discussing or applying the V.M., possibility c) above is the one considered and we now want to discuss it a bit more in detail. In this case, not only the V.M. is, as mentioned above, a meromorphic function but there also follows a fairly unpleasant consequence concerning the region of validity in which the Regge asymptotic behavior takes place. To apply Stirling formula, in fact, limitation (II.16) must be imposed. However, if $\alpha(s)$ is real, (II.16) implies that the Regge asymptotic behavior of V.M. holds along every direction in the complex s plane except on the real axis (and on an arbitrarily small cone centered around it). We are thus in the situation in which, given an information on the real s axis (where physics occurs) we extrapolate from it in order to obtain a model in which this behavior holds uniformly, just to find ourselves in the condition that the only domain where this model does not reproduce the wanted behavior, is the one which we used to proceed to our extrapolation. It is clear, however, that the above difficulty does not represent, in practice, a great obstacle to a determined physicist.

In principle, the above objections can be circumvented by saying that in the NRA the imaginary part of the amplitude is an infinite sum of Dirac delta functions and that the cut along the real axis has been replaced by an infinity of poles. Due to the structure of the model, this prescription is, clearly, a modification of the model itself which to some extent spoils the crossing properties of the original amplitude in that one has to select a channel. If, for instance, we take s to be positive, then the prescription would require that

$$\text{Im } V_s(s,t) = \pi \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha(t))}{\Gamma(\alpha(t))\; n!}\, \delta(n+1-\alpha(s))$$

$$\text{Re } V_s(s,t) = \sum_{n=0} \frac{\Gamma(n+\alpha(t))}{\Gamma(\alpha(t))\; n!}\; \frac{P}{n+1-\alpha(s)} . \qquad \text{(II.31)}$$

The modified amplitude $V_s(s,t)$ defined in (II.31) would now, formally, satisfy a dispersion relation.

From the point of view taken above, therefore, one would say that the exclusion of the real axis from the asymptotic behavior corresponds to the recollection of a cut in the amplitude. The point, however, remains that the above procedure does indeed give rise to a somewhat different model than represented by (II.2) which is purely meromorphic function of $\alpha(s)$.

Yet another aspect of the violation of unitarity in the V.M. will be discussed in Sec. II.J. in connection with the appearance of fixed poles singularities.

Before ending this section, we want to comment on the general implications of the inherent violation of unitarity in the V.M. It has been argued[67] that after so many essentially fruitless attempts of explicitly enforcing unitarity (in some approximate way) in the field of high energy physics, it may be reasonable to start from a model in which unitarity is violated a priori but crossing is preserved. This may well be so and this attitude has led one to consider the Veneziano amplitude as some sort of Born approximation which should then be unitarized.[84],[85] The solution of this problem would essentially amount to carrying on the "nonlinear" part of the program outlined in Ref. 75. The fact that unitarity is, intuitively, so very important in the high energy domain, on the other hand, may cast doubts on the final possibilities of success of such a program. It should also be noticed that the present situation is in agreement with the general frustrating fact that in practice it seems impossible to satisfy at the same time unitarity and crossing. By this we do not mean that this is actually impossible since it has been proved[86] that, at least under certain assumptions, unitarity and crossing are simultaneously compatible. What we mean is that so far no example of a model has been produced in which it is manifestly obvious that unitarity and crossing can both be satisfied at the same time. The reason for this is that it is hard[87],[89] to guarantee crossing symmetry when the only explicit formalism in which one can check that the unitarity limit is not violated is the partial wave expansion. Thus, although it is a conceivable program (the one of unitarizing the V.M. in its partial wave expansion), the risk is that in this way crossing symmetry is lost; furthermore, there is no guarantee that the asymptotic Regge behavior will be preserved.[89]

F. Daughters. Decoupling of the Odd Daughters.

As noticed in Sec. II.E., the residue $R_n(t)$ as given by (II.28), together with (II.29), is a polynomial in t of order n which is just what we would expect from the identification of $\alpha(s)$ as the Regge trajectory. The fact that all powers of t from n to zero are present means that beside the leading Regge trajectory (parent) there are (parallel) daughters spaced of one unit of angular momentum. The usual analyticity arguments or O(4) symmetry arguments would in the present case require the daughters to be spaced of two units of angular momentum. It should be remarked that there is in principle nothing against these odd trajectories. In fact, according to the general decomposition of $\pi\pi \to \pi\omega$, the only condition that must be satisfied is that the invariant amplitude A(s,t,u) be crossing symmetric which is the case for (II.2). More explicitly, one notices that if $\alpha(s) = n + 1$, the two resonant terms in (II.2) [that is V(s,t) and V(s,u)] have the overall residue

$$R_n(t) + R_n(u) = \frac{1}{n!}\left[\frac{\Gamma(n+\alpha(t))}{\Gamma(\alpha(t))} + \frac{\Gamma(n+\alpha(u))}{\Gamma(\alpha(u))}\right] . \qquad \text{(II.32)}$$

Eq. (II.39) is a polynomial of order n in both t and u which, if we write t and u in terms of cos θ_s (θ_s scattering angle in the channel in which s is the energy variable) is a polynomial in $\cos^2 \theta_s$ of maximum order $\left[\frac{n}{2}\right]$ ($\left[\frac{n}{2}\right]$ denotes the largest entire number contained in n/2). This is, of course, just what we expect for a crossing symmetric amplitude and there is a priori nothing wrong in the presence of odd daughters.

It has been, however, shown by Veneziano[4] how a simple device can altogether get rid of these odd trajectories. Suppose again $\alpha(s)$ linear and let us consider the first "unwanted" pole at even integer α's, i.e. $\alpha(s) = 2$. According to (II.32) the residue in this case is

$$R_2 = \alpha(t) + \alpha(u) \qquad (\text{for } \alpha(s) = 2) \qquad \text{(II.33)}$$

and the pole is absent if we demand that this residue is zero. This can be satisfied by imposing that

$$\alpha(s) + \alpha(t) + \alpha(s) = 2. \qquad \text{(II.34)}$$

The very remarkable thing happens that condition (II.34) not only takes care of eliminating the pole at $\alpha(s) = 2$ but removes also <u>all</u> subsequent even resonances $\alpha(s) = 2n$ and thus completely decouples the odd daughter trajectories. An interesting consequence of (II.34) in the specific case of $\pi\pi \to \pi\omega$ is that it leads to the prediction that

$$\alpha[-2m_\rho^2 + m_\omega^2 + 3m_\pi^2] = \alpha(-.53(\text{GeV}/c)^2) = 0 . \qquad \text{(II.35)}$$

This condition was already derived[11] in Sec. I.B. as a consequence of FESR and already commented upon.

Applying the same technique and the same constraint (II.34) to the case of $\pi\eta \to \pi\rho$ (in which the only trajectory in either s or u is the A_2 and the only in the t channel is the ρ) we get

$$\alpha_{A_2}(s) + \alpha_{A_2}(u) + \alpha_\rho(t) = 2 \qquad \text{(II.36)}$$

which demands

$$\alpha_{\rho}' = \alpha_{A_2}' . \qquad \text{(II.37)}$$

Furthermore using $m_{\rho}^2 \simeq .6(\mathrm{GeV}/c)^2$ in (II.36) one gets

$$\alpha_{A_2}(0) = 1 - \frac{1}{2}\,\frac{3m_{\rho}^2 - m_{\omega}^2 - m_{\pi}^2 + m_{\eta}^2}{3m_{\rho}^2 - m_{\omega}^2 - 3m_{\pi}^2} \simeq .36 \qquad \text{(II.38)}$$

which predicts

$$m_{A_2} \simeq 1350 \text{ MeV} . \qquad \text{(II.39)}$$

Similar applications of constraints of the form (II.34) have been considered by other authors.[90]

An important byproduct of constraints of the form (II.34) arises now if we consider the entire amplitude in the V.M. According to (II.2) this reads

$$A(s,t,u) = V(s,t) + V(u,t) + V(s,u). \qquad \text{(II.40)}$$

Let us consider the limits $s \to \infty$ at fixed t. In this case, since

$$s + t + u = \Sigma$$

it is seen that $u \to -\infty$ and therefore if V(s,t) is Regge behaved (with the restriction (II.16)), so is V(u,t). The question remains, however, of what is the behavior of V(s,u). This depends crucially on the relative growth of $\alpha(u)$ and $\alpha(s)$. However, if (II.34) holds, we can rewrite

$$V(s,u) = \frac{\sin \pi\alpha(t)}{\sin \pi\alpha(u)}\, V(s,t) \qquad \text{(II.41)}$$

so that V(s,u) is, in this case, Regge behaved to the same extent of V(s,t).[91]

It is also easily checked that for the Regge behavior of the entire amplitude (II.40) not to be spoiled, it is actually sufficient to replace (II.34) with the less restrictive condition

$$\alpha(s) + \alpha(t) + \alpha(u) = c \geq 2 \qquad \text{(II.42)}$$

and, strictly speaking, it is enough that

$$\lim_{\substack{|s|\to\infty \\ |\arg s|>\epsilon \\ t \text{ fixed}}} [\alpha(s) + \alpha(u)] = \text{const} \ . \tag{II.43}$$

The above argument implies that a constraint on the trajectories of the form (II.34), (II.42) can be necessary to give a well defined meaning to the asymptotic behavior of the entire amplitude.

As it will turn out, the validity of constraints between the Regge trajectories in the various channels will be crucial in the problem of the compatibility of the V.M. in which the analyticity is restored by letting $\alpha(s)$ satisfy a dispersion relation (see Sec. II.H.).

It should also be mentioned that other ways have been suggested to decouple the odd trajectories by either (i) adding nonleading terms[92] (satellites) or (ii) modifying the model itself in a more drastic way.[93),94),95),96] These modified forms could provide alternative possibilities to the V.M. when constraints of the form (II.34) lead to predictions that do not agree with experiment (such is the case for $\pi\eta' \to \pi\rho$, or $\pi\pi \to \pi H$; also, the corresponding condition[93] for $\pi\pi \to \pi\pi$ is only satisfied within 10% if one uses for the ρ trajectory the same parameters derived from the case of $\pi\pi \to \pi\omega$). Furthermore the hope is, of course, that drastic modifications of the V.M. can improve the theoretical situation also from other points of view.

G. Positivity Condition

For an elastic scattering, unitarity requires the residues at the resonances to be positive. Such a positivity condition although less restrictive than the unitarity requirement which, in addition, demands positive total widths etc., should nevertheless be satisfied in a physically satisfactory model. That this is not a trivial condition has been shown by Oehme in trying to saturate superconvergence relations.[97]

The analysis of the positivity condition in the V.M. has been carried out by several authors and here we will only state the results. First, a numerical analysis[98] has shown that for the leading term of the V.M. in $\pi\pi$ scattering

$$V_{\pi\pi} = \frac{\Gamma(1 - \alpha(s))\,\Gamma(1 - \alpha(t))}{\Gamma(1 - \alpha(s) - \alpha(t))} \tag{II.44}$$

there are no ghost states among all daughters up to the 50th recurrence (for linear trajectories whose intercept is $\alpha(0) \geq \frac{1}{2}$).

Other authors[99] have analyzed the positivity condition within the general context of the properties of the V.M. in the complex angular momentum with results similar to those of Ref. 99. Oehme has shown[100] that, with the exception of the lowest term (II.44) individual higher Veneziano terms of the general form (II.14) have infinitely many negative residues. Finally, the general result has been proved[101] for the leading term (II.44) that all residues are asymptotically positive provided only that

$$\alpha(0) > \tfrac{1}{3} \quad . \tag{II.45}$$

This means that, at most, (II.44) will have a finite number of ghosts.

H. The V.M. and Complex Regge Trajectories

The problem of whether or not one can modify the structure of the V.M. to have at the same time analyticity and nonautomatic violation of unitarity has been considered by a few authors.[83),95),102] In particular in this section we want to discuss what would happen if we retained the V.M. and allowed $\alpha(s)$ to become complex and satisfy a dispersion relation (II.22). This has been the attitude taken by Roskies.[83] (See also Ref. 103.) Because of what was said in Sec. II.E. it is quite obvious that even if we succeed in removing the violation of analyticity, and having finite total widths, there still would be the problem that ancestors should appear. One may, however, argue that this is preferable to the appearance of resonances having zero total widths but finite partial widths.

The trouble with letting $\alpha(s)$ satisfy a dispersion integral is that, clearly, one cannot have a relation of the form (II.34) any more. As a consequence, it may happen that the third term $V(s,u)$ in (II.40) becomes unbounded as $s \to \infty$ at fixed t. In fact, it turns out that $\mathrm{Im}\,\alpha(s)$ is very strongly constrained in order to avoid that $V(s,u)$ will diverge. The following theorem holds:[83]

Theorem. Suppose

(a)
$$\alpha(s) = a + bs + \frac{s}{\pi}\int_{\Sigma}^{\infty} \frac{\mathrm{Im}\,\alpha(s')}{s'(s'-s)}\, ds' \tag{II.46}$$

(b)
$$\mathrm{Im}\,\alpha(s) \underset{s\to+\infty}{\to} +\infty \quad . \tag{II.47}$$

(c) For some μ, $0 < \mu < 1$

$$\frac{\mathrm{Im}\,\alpha(s)}{s^{1-\mu}} \to 0 \quad \text{as } s \to \infty \; ; \tag{II.48}$$

(d)
$$I(s) \equiv \frac{\mathrm{Im}\,\alpha(s)}{s^{1-\mu}} \tag{II.49}$$

satisfies a smoothness condition of the form

$$|I(s_1) - I(s_2)| \le C|s_1 - s_2|^{\gamma} \tag{II.50}$$

for some C, $\gamma > 0$ when $|s_1 - s_2| \le 1$.

If we now denote

$$g(s) = \alpha(s) + \alpha(u) \ , \tag{II.51}$$

then the following is true

(A) $$|\mathrm{Re}\ g(s)| < C^1 s^{1-\mu}\ \ell n\ s \quad \text{as } s \to \infty \tag{II.52}$$

(B) There exists a $k > 0$ and a sequence s_n for which

$$-\frac{\mathrm{Re}\ g(s_n)}{\mathrm{Im}\ \alpha(s_n)} \ge k \ . \tag{II.53}$$

We refer for the rather lengthy proof of this theorem to the original paper[83] and simply notice that the consequence of it is that one finds that the V.M. is still Regge behaved only if

$$\lim_{s\to\infty} \frac{\mathrm{Im}\ \alpha(s)}{s^{1-\mu}} = +\infty \quad \text{for all positive } \mu \tag{II.54}$$

but

$$\int^{\infty} \frac{\mathrm{Im}\ \alpha(s)}{s^2}\, ds < \infty \ . \tag{II.55}$$

Under the above conditions, the poles move away from the first sheet and the amplitude is still Regge behaved in every direction of the s-plane outside the cone (II.16).

An example of $\alpha(s)$ for which the conditions discussed before hold is

$$\mathrm{Im}\ \alpha(s) \underset{s\to\infty}{\to} \frac{s}{(\ell n\ s)^{\nu}} \quad \nu > 1 \ . \tag{II.56}$$

Inserting (II.56) into (II.46) one gets

$$\alpha(s) \underset{s\to\infty}{\simeq} bs + \frac{s}{\pi}\ \frac{1}{\nu - 1}\ (\ell n\ s)^{1-\nu} + \ldots \tag{II.57}$$

and the amplitude behaves as

$$A(s,t,u) \simeq [\alpha(s)]^{\alpha(t)} \simeq (bs)^{\alpha(t)} \left[1 + \frac{\alpha(t)}{\pi b(\nu - 1)} (\ell n\, s)^{1-\nu} + \ldots \right] \tag{II.58}$$

i.e. consists of powers of s with logarithmic corrections. As shown long ago by Freund and Oehme[104] the term with logarithms arises from cuts in the complex angular momentum whereas the first term is due to Regge poles. Thus, when $\mathrm{Im}\,\alpha \neq 0$, besides Regge poles at

$$\ell = \alpha(s) - n \qquad n = 0, 1, 2, \ldots$$

the model has cuts which end at each of the Regge poles.

As already noticed, we also have ancestors because now the residue $R_n(t)$ (II.28) is not a polynomial of t any more. It is argued in Ref. 83 that although these ancestors lie on the Regge trajectories $\ell = \alpha(s) + n$, they are not really Regge poles in the usual sense since they will not contribute to the leading behavior.

More specifically, the argument goes that at a given energy s_n not all the infinite partial waves that resonate do couple strongly. In the expansion of the residue $R_n(t)$ in Legendre polynomials

$$R_n(t) = \Sigma\, C_{\ell n}\, P_\ell(z) \tag{II.59}$$

the coefficients decrease very rapidly (with an exponential law) with increasing ℓ at fixed n. Furthermore, one has that $C_{\ell n}$ is maximum for $\ell \propto n^{\frac{1}{2}}$. This observation could also, incidentally, reconcile the present trend of linear growth of Regge trajectories with the previously noted deflection to a square root asymptotic behavior in the sense that the latter would be exhibited by some sort of "effective" Regge trajectory.

If we retain the usual definition of the width of the resonances in terms of the trajectories on which they lie

$$\Gamma(s) = \frac{\mathrm{Im}\,\alpha(s)}{\sqrt{s}\,\frac{d\,\mathrm{Re}\,\alpha}{ds}} \tag{II.60}$$

we see that in the present case this implies

$$\Gamma \simeq \frac{s^{\frac{1}{2}}}{(\ell n\, s)^{\nu}} \,. \tag{II.61}$$

It is amusing to note that the behavior (II.61) is extremely close to the empirical one suggested by the plot of Fig. 1.

Other examples have been suggested[95),102)] in which the same qualitative situation as in (II.61) arises in the sense that the "width" of the resonances, far from being zero, increases rapidly with energy. This may actually represent an indication that as the energy increases, the effect of a resonance becomes less and less pronounced so that, for every practical purpose, their effect washes out to some smooth background.

I. Duality and the Interference Model in the V.M.

As previously noted, the V.M. does certainly possess duality if def. A of Sec. I.F. is used. This is, however, not so surprising since, in fact, this definition has been given with an eye on the V.M. As remarked before, the key dynamical ingredient in this definition of duality is the assumption that infinitely many resonances appear and act as if they could represent a complete set in some way. Without this, duality would be indistinguishable from crossing symmetry and Regge behavior. With the above qualification, one should perhaps be explicitly cautioned that def. A of Sec. I.F. of duality may reveal still too limited when trying to construct an amplitude that also satisfies unitarity and normal analyticity requirements. In these conditions, probably, a somewhat less restrictive definition of duality may be needed. However, special care in devising such definition of duality was given to avoid any specific commitment as to whether the asymptotic behavior should arise from poles only. One may therefore hope that such a definition could still be effective also for amplitudes which are not purely meromorphic.

We now want to briefly discuss whether the stronger definition C) of duality can be applied namely, whether it is true or not in the V.M. that it is the direct channel poles that build up to the asymptotic Regge behavior. That this sentence cannot be given any well defined meaning was the content of Ref. 44 and was already discussed in Sec. I.F. Summarizing very briefly, the argument was that so long as direct and crossed channel poles coexist together one has equal rights to attribute the asymptotic behavior to either set or to a combination of them or, more probably, to the effect of very many properties of the function under consideration (poles, zeros, etc.). The stronger statement was, however, proved in Ref. 44 (see also Ref. 45) that one could not give a meaningful content to def. C of Sec. I.E. unless one could prove the existence of an entire function (of s) of infinite order which was also Regge behaved (according to the definition of Sec. I.E.) as $|s| \to \infty$. Even if such a function would exist, however, def. C would still be ambiguous.

In Ref. 44 the above results are proved by showing that the V.M. can generally be cast in the form of a generalized interference model.[47),48)] To show this, consider for instance

$$V(s,t) = \frac{\Gamma(1-\alpha(s))\,\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))} \tag{II.62}$$

(the same considerations apply to V(t,u) and V(u,s)) and let us use the integral representation

$$V(s,t) = \int_0^1 dx\, x^{-\alpha(s)}\,(1-x)^{-\alpha(t)}\,, \tag{II.63}$$

valid for $\mathrm{Re}[\alpha(s),\,\alpha(t)] < 1$. As noted previously, the s-channel poles (direct channel poles) are associated with the lower limit of integration whereas the t-channel poles (crossed channel poles) come from the upper limit of integration. Therefore, if we split

$$V(s,t) = V_s(s,t) + V_t(s,t) \tag{II.64}$$

with

$$V_s(s,t) = \int_0^a dx\, x^{-\alpha(s)}(1-x)^{-\alpha(s)} \equiv$$

$$\equiv \frac{a^{1-\alpha(s)}}{1-\alpha(s)}\; {}_2F_1(\alpha(t),\,1-\alpha(s);\,2-\alpha(s);\,a) \tag{II.65}$$

and

$$V_t(s,t) = \int_0^{1-a} dx\, x^{-\alpha(t)}(1-x)^{-\alpha(s)} \equiv$$

$$\equiv \frac{(1-a)^{1-\alpha(t)}}{1-\alpha(t)}\; {}_2F_1(\alpha(s),\,1-\alpha(t);\,2-\alpha(t);\,1-a) \tag{II.66}$$

(${}_2F_1$ being the usual Gauss hypergeometric series), we see that $V_s(s,t)$ contains only direct-channel poles and $V_t(s,t)$ contains only crossed -channel poles. The parameter a is completely at our disposal provided we do not choose it real and negative or real positive ≥ 1 where the hypergeometric functions in (II.65), (II.66) have cuts. If we can choose a such that for $|\alpha(s)| \to \infty$, $\mathrm{Re}\,\alpha(s) > 0$, $V_t(s,t)$ gives the entire asymptotic behavior (II.17) whereas $V_s(s,t)$ goes there exponentially to zero, the decomposition (II.64) makes the V.M. indistinguishable from a generalized interference model.

Recalling that[105)]

$$\lim_{\beta\to\infty} F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-\beta z)^{-\alpha}\left[1+0\left(\frac{1}{|\beta z|}\right)\right] + \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{\beta z}(\beta z)^{\alpha-\gamma}\left[1+0\left(\frac{1}{|\beta z|}\right)\right] \quad \text{(II.67)}$$

we see that provided

$$\mathrm{Re}\,[(1-a)\,\alpha(s)] < 0 \quad , \quad \text{(II.68)}$$

the asymptotic behavior of (II.66) is exactly given by (II.17). In the same case (and with the further limitation given by (II.16)) one can also check that $V_s(s,t)$ goes to zero.[106)] Notice that if $\Psi \equiv \arg(1-a)$, condition (II.68) ensures that $V_t(s,t)$ gives the asymptotic behavior in the open half plane

$$\frac{\pi}{2} - \Psi < \arg\alpha(s) < \frac{3\pi}{2} - \Psi \quad \text{(II.69)}$$

In particular, if Ψ is chosen close to π, we find that (II.64) provides a decomposition of the V.M. in the form of an interference model in the open half plane $\mathrm{Re}\,\alpha(s) > 0$ with the wedge $|\mathrm{Re}\,\alpha(s)| < \epsilon$ (ϵ arbitrarily small) excluded.

This result clarifies the close connection between models showing duality (according to def. A of Sec. I.F.) and generalized interference models and, combined with the findings of Jengo[48)] and of Hsu, Mohapatra and Okubo,[48)] gives also a very strong support to the conjecture of Sec. I.J. of an equivalence of the form "duality" $\Longrightarrow$ generalized interference models." It is not, however, obvious that the arrow in the above equivalence statement can be reverted.

It should explicitly be noticed also that Coulter, Ma and Shaw[48)] use the V.M. as a guide to suggest an interference model which is slightly different from the one discussed here. Taking the limit $|s|\to\infty$ at fixed t, they essentially replace the term V(s,t) (containing s-channel resonances) with resonant terms but retain the asymptotic contribution coming from V(u,t) since this is essentially real (as $s \to +\infty$) and cannot give rise to any loops in the Argand diagram. This procedure not only very heavily relies on the validity of Schmid's conjecture (Sec. I.E.) but appears also rather arbitrary. The application of the above prescription to the fit of $\pi^- p \to \pi^- p$ backward scattering seems, on the other hand, rather encouraging.

We finally want to see whether the V.M. can be said to satisfy duality at least according to def. B of Sec. I.F., i.e. in the sense of satisfying FESR. Clearly, strictly speaking this is not possible if we consider the amplitude of the V.M. as a meromorphic function. Veneziano,[4)] however, has shown that (II.2) can be made to satisfy superconvergency relations if interpreted in the sense of a NRA according to the discussion at the end of Sec. II.E. (see (II.13)). To see this the following steps are needed: first one takes a smoothed Regge form for Im A(s,t)

$$\text{Im}\,A_{\text{Regge}}(s,t) \underset{s\to\infty}{\sim} \beta\, \frac{\Gamma(\alpha(s)+\alpha(t)-1)}{\Gamma(\alpha(t))\,\Gamma(\alpha(s))} \tag{II.70}$$

and one verifies that

$$\text{Im}\,A_{\text{Regge}} \simeq \frac{\beta}{\Gamma(\alpha(t))}\left[\alpha(s)+\frac{\alpha(t)-2}{2}\right]^{\alpha(t)-1} \tag{II.71}$$

coincides with (II.70) up to the second leading term.

The first moment sum rule reads ($\nu = \frac{s-u}{4}$)

$$\int_0^N d\nu\, \nu\, \text{Im}\,A(\nu,t) = \frac{\beta}{\Gamma(\alpha(t))}\, N^a\, \frac{(2\alpha' N)^{\alpha(t)-1}}{\alpha(t)+1}\ . \tag{II.72}$$

The last step needed is to assume that (according to (II.31)), for for s positive V(s,t) + V(u,s) give

$$\text{Im}\,A(\nu,t) = -\beta \sum_{n=1}^{\infty} (-1)^n\, \frac{\Gamma(1-\alpha(t))\,\delta(s-sn)}{\alpha'\Gamma(n)\,\Gamma(2-n-\alpha(t))} + (t \leftrightarrow u)\ . \tag{II.73}$$

Setting N midway between the nth and the (n + 1)-th resonance we now get

$$\sum_{n=0}^{m} \frac{(\alpha+4n)\,\Gamma(\alpha+2n)}{\Gamma(\alpha+1)\,\Gamma(2n+1)} = \frac{\Gamma(\alpha+2+2m)}{\Gamma(2+\alpha)\,\Gamma(2m+1)}\,\Phi_{m+1}(\alpha) \tag{II.74}$$

where Φ_m is the same function already encountered in Sec. I.B.

$$\Phi_{m+1} = \frac{\Gamma(2m+1)}{\Gamma(\alpha+2+2m)}\left(\frac{\alpha+2+4m}{2}\right)^{\alpha+1}. \tag{II.75}$$

As in Sec. I.B. the consistency condition requires

$$\Phi_{m+1}(\alpha) = 1 \tag{II.76}$$

which is rather well satisfied for $\alpha(t) < 2m$ and becomes a strict equality as $m \to \infty$ at fixed $\alpha(t)$. Also, if $\alpha(t) < -1$, from (II.74) we obtain the usual superconvergence sum rule.

We therefore conclude that, to the extent to which the modifications of the V.M. leading to (II.70), (II.71) and to (II.73) are accepted, the V.M. satisfies FESR and therefore also duality according to def. B of Sec. I.F. is in a way a property of the model. Remembering, however, the remarks made in Sec. II.E. on the NRA interpretation of the V.M. we see also that def. B of duality (Sec. I.F.) is literally not applicable to the V.M. Notice also that in the previous argument the term V(u,t) has been completely ignored; furthermore, the disagreement between (II.71) and (II.70) rapidly increases as we move away from the asymptotic region.

K. Angular Momentum Properties of the V.M.

The properties of the V.M. in the complex angular momentum plane and in the Lorentz plane have been studied by a number of authors.[99),107),108),109),110),111),112),113)]

We shall in the following consider the specific example of linear trajectories (II.29) and concentrate on the case of $\pi\pi$ elastic scattering for which the building block is given in Eq. (II.44). The results can be stated as follows:

i) the partial waves have, for physical ℓ, the correct threshold behavior;

ii) the positivity condition (see Sec. II.G) holds if $\alpha(0) \geq \frac{1}{2}$;

iii) there are in the complex ℓ plane, infinitely many Regge poles with parallel trajectories spaced by one unit of angular momentum;

iv) there also are fixed poles[108),109)] at the negative integers (nonsense wrong signature integers, according to a somewhat accepted terminology);

v) partial waves do exhibit some sort of duality[110)] in the sense that the contribution from the Regge amplitude in the crossed channel is roughly the same as that of the direct channel pole in calculating low energy quantities;

vi) in the Lorentz plane there also is an infinite sequence of Lorentz poles and, again, fixed poles at negative integers.

We now turn to the analysis of points iii) and iv) above for which we use[107)] the expansion

$$V(s,t) = -\frac{1}{\Gamma(\alpha(s))}\sum_{n=0}^{\infty}\frac{\Gamma(n+1+\alpha(s))}{\Gamma(n+1)}\,\frac{1}{n+1-\alpha(t)} \qquad \text{(II.77)}$$

which, together with

$$\alpha(t) = \alpha + \alpha' t \qquad (\alpha' > 0)$$

$$s = 4(k^2 + \mu^2)$$

$$t = -2k^2(1 - z)$$

$$z_n(s) = 1 + \frac{n + 1 - \alpha}{2k^2\alpha'} \tag{II.78}$$

can be rewritten as

$$V(s,t) = \frac{1}{\Gamma(\alpha(s))} \frac{1}{2\alpha' k^2} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \alpha(s))}{\Gamma(n + 1)} \frac{1}{z - z_n(s)} . \tag{II.79}$$

The partial wave projection of (II.79) is now

$$f_\ell(s) = \frac{1}{2\alpha' k^2 \Gamma(\alpha(s))} \sum_{n=0}^{\infty} \frac{\Gamma(n + 1 + \alpha(s))}{\Gamma(n + 1)} Q_\ell(z_n(s)) \tag{II.80}$$

which is the Froissart-Gribov expression for the continued partial wave. Therefore, (II.80), wherever it converges, provides a unique continuation to $f_\ell(s)$. To find the region of convergence of (II.80), we set

$$f_\ell(s) = \sum_{n=0}^{\infty} f_{\ell n}(s) \tag{II.81}$$

and note that for $n \to \infty$

$$f_{\ell n}(s) \sim \frac{\beta_0(s)}{(n + 1)^{\ell + 1 - \alpha(s)}} + \frac{\beta_1(s)}{(n + 1)^{\ell + 2 - \alpha(s)}} + \dots \tag{II.82}$$

where the first coefficients are given by

$$\beta_0(s) = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\ell + 1)}{\Gamma(\ell + 3/2)} (\alpha' k^2)^\ell \frac{1}{\Gamma(\alpha(s))}$$

$$\beta_1(s) = \beta_0(s)\left[\frac{\alpha(s)(\alpha(s) - 1)}{2} - (\ell + 1)(2\alpha' k^2 - \alpha)\right] \tag{II.83}$$

Eq.(II.80) shows that $f_\ell(s)$ satisfies the usual threshold conditions wherever it converges and (II.83) shows that $f_\ell(s)$ is a holomorphic function of ℓ for

$$\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha(s) \tag{II.84}$$

and provided

$$\ell \neq -m \qquad (m = 1, 2, \ldots) \tag{II.85}$$

The continuation below (II.84) can be performed by writing

$$f_\ell(s) = g_{\ell N}(s) + \sum_{n=0}^{\infty} \left[f_{\ell n} - \sum_{m=0}^{N-1} \frac{\beta_m(s)}{(n+1)^{\ell + m + 1 - \alpha(s)}} \right] \tag{II.86}$$

with

$$g_{\ell N}(s) = \sum_{m=0}^{N-1} \beta_m(s) \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\ell + m + 1 - \alpha(s)}} =$$

$$= \sum_{m=0}^{N-1} \beta_m(s)\, \zeta(\ell + m + 1 - \alpha(s)) \; . \tag{II.87}$$

In (II.87), $\zeta(x)$ is the Riemann zeta function.[114] The convergence of the series at the r.h.s. of (II.86) is now for

$$\mathrm{Re}\,\ell > \mathrm{Re}\,\alpha(s) - N \tag{II.88}$$

so that $f_\ell(s)$ is now the sum of a regular function (the series) plus a finite sum of zeta-functions. Due to the analytic properties of the zeta function whose only singularity is a first order pole with unit residue at unit argument, we conclude that the only singularities of $g_{\ell N}(s)$ are poles at

$$\ell = \alpha(s) - m \qquad (m = 0, 1, 2, \ldots) \tag{II.89}$$

with residue $\beta_m(s)$. These residues have been computed[107] and their expression is rather involved analytically

$$\beta_m(s) = \tfrac{1}{2} \cos \pi\, \alpha(s)\, e^{i\pi\alpha(s)} \sum_{p=0}^{m} \frac{a_p(s)}{p!}\, 2^{p - \alpha(s) - \frac{1}{2}} \; .$$

$$\cdot\, \Gamma(p - \alpha(s) - \tfrac{1}{2})\, C_{m-p}^{p - \alpha(s) - \frac{1}{2}}(0) \tag{II.90}$$

where

$$a_p(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \alpha(s)(2\alpha' k^2)^{\alpha(s)} \frac{d^p}{dy^p} \cdot$$

$$\cdot \left[e^{z_0(s)y} \left(\frac{1 - e^{y/2\alpha' k^2}}{y/2\alpha' k^2} \right)^{-\alpha(s) - 1} \right]_{y=0} .$$

$z_0(s)$ was defined in (II.78) and $C_k^{\lambda}(0)$ are Gegenbauer polynomials of zero argument.

In Ref. 109 the problem of fixed point singularities is specifically considered and it is shown that there are infinitely many fixed poles at negative integers arising solely from the V(s,u) term in the complete Veneziano amplitude. This is at first sight very surprising since these nonsense wrong signature poles are usually related to the presence of the third double spectral function[115)] ($\rho(s,u)$ in the specific case we are discussing) which is by construction absent from the Veneziano model. One therefore must conclude that the third spectral function is not a necessary but only a sufficient condition for fixed poles to appear.

In Ref. 109 it is also shown that it is possible to modify the V.M. in such a way as to kill <u>all</u> the fixed point singularities by adding nonleading satellite terms to (II.44) only at the price of removing completely V(s,u).

As it is known, fixed poles conflict with unitarity so that their presence implies a further violation of unitarity besides the ones discussed in Sec. II.E.

As a final comment, we notice that an analysis of the same kind as the one briefly discussed before leads to very similar conclusions in analyzing the V.M. in its Lorentz plane properties[111),112)] and we refer to the original papers for a complete discussion of this problem.

K. Miscellaneous Properties

In the previous sections we have discussed most of the better understood properties of the V.M. Many are still, however, left out and we shall just mention a few here without entering too much into details.

a) Exchange degeneracy. It is rather easy to convince oneself that the condition that certain channels are free of resonances demands an exchange degeneracy in the V.M. (the argument is, essentially, the same already given in Part I).

First notice that, for linear trajectories, the slopes must be the same or else there would be physical regions in the fixed angle limit where the Veneziano amplitude

$$V^p_{nm}(\alpha_i(s), \alpha_j(t)) = \frac{\Gamma(n - \alpha_i(s))\,\Gamma(m - \alpha_j(t))}{\Gamma(p - \alpha_i(s) - \alpha_j(t))} \qquad (II.92)$$

would tend to infinity. The intercepts, however, may be different.

Next, if we consider the I = 2 s-channel $\pi\pi \to \pi\pi$ amplitude

$$\begin{aligned} A^2(s,t,u) = {} & V^1_{11}(\alpha_\rho(s), \alpha_\rho(u)) - V^1_{11}(\alpha_\rho(s) - \alpha_f(u)) \\ & - V^1_{11}(\alpha_\rho(s), \alpha_f(t)) + V^1_{11}(\alpha_\rho(s), \alpha_\rho(t)) - \\ & - V^1_{11}(\alpha_\rho(u), \alpha_f(t)) - V^1_{11}(\alpha_\rho(t), \alpha_f(u)) \end{aligned} \qquad (II.93)$$

it is clear that we must have

$$\alpha_\rho(s) = \alpha(s) \qquad (II.94)$$

in order to guarantee that there are no s-channel poles.

Similar considerations can be made for the other cases where no direct-channel resonances are known. It has, however, been pointed out[116] that such a strict exchange degeneracy would, for instance, imply the impossibility of $\pi\Sigma$ scattering. Furthermore exchange degeneracy seems so far well obeyed by mesons but not so much by baryons so that one may be forced to introduce satellites in the V.M.

b) Parity doubling. Similar conclusions as previously obtained on the need to introduce satellites are reached when one tries to avoid the unpleasant feature of parity doubling in the case of reaction processes involving two or more spin particles.[117]

c) Factorization. This is one of the properties of the V.M. that has not been thoroughly investigated until recently. The most comprehensive analysis has been given by Fubini and Veneziano[85] (see also Ref. 118, 119) who considered the structure of the residue at each pole by decomposing it into the minimum number of linearly independent terms which would, separately, be in a factorized form. This decomposition is shown to be independent of the number of initial and final lines (see Sec. II.M. for the discussion of the generalization of the V.M. to the many point functions). The number of linearly independent terms denote the degeneracy of the state and, essentially, "count" the number of states one has to deal with. If $s_n = \alpha + \beta n$ is the position of the pole, the number of terms increases like $\exp[a\sqrt{n}]$.

This explosive proliferation of states is attributed to the many body nature of the problem. A further complication of the problem is represented by the appearance of ghost states connected with the difficulty discussed in Sec. II.G.

The factorization procedure discussed in Ref. 85 appears as a necessary preliminary step to the unitarization program attempted in Ref. 84.

In a somewhat less sophisticated approach to the problem, Freund[118] has shown that higher and higher order terms in the V.M. are needed to ensure factorization of the parent Regge trajectory and of the first few daughters.

d) Uniqueness. From the discussion of the previous sections it is quite obvious that there can be no answer to the question of to what extent the V.M. can provide a unique parametrization to a scattering amplitude unless one makes very definite assumptions on what properties one wants to attribute to such an amplitude. For instance, one may want to allow a superposition of infinitely many terms of a Veneziano type but in such a way that there are no ghosts, no parity doubling, no fixed poles and factorization is obeyed together with crossing and Regge behavior. No such program, to the best of our knowledge, has been shown to be feasible although many authors have variously commented on the uniqueness of the Veneziano representation.[32),119),120),121)] A fortiori, the conclusion[122)] that the V.M. must be considered more fundamental than a model should be taken with some reservation.

L. Remarks on the Applications of the V.M.

It seems essentially impossible to report in any simple and coherent form on all the applications that have been given of the V.M. These applications[77),89),90),110),116),122),123)] mainly deal with i) predictions of coupling constants and comparison with experiment, ii) low energy effects and connections with chiral symmetry, iii) analysis of scattering problems and high energy predictions.

The general panorama that emerges from the analysis of the various applications of the V.M. is that an overwhelming majority of results seem to lend support to the validity of the simple V.M. as a lowest order approximation to nature in describing both low as well as high energy effects (the former better than the latter). Words of caution are, however, not absent[110),116)] and these appear especially relevant[110)] in relation to the exciting possibility of connections between the V.M. and chiral symmetry.[122)] Furthermore, as mentioned above, the largest number of successes obtain in the low energy domain. Whereas this is not so surprising since violations of unitarity are expected to play a major role especially at high energies, it also opens the question of which among the aspects of the the V.M. is likely to be mostly responsible for the agreement that one obtains with experiment. It would, clearly, be desirable that the "dynamical" aspects of the V.M. should be the ones that are at work and by dynamical we mean the duality aspect (according to def. A of Sec. I.F.). Some doubts on this possibility are, however, cast by the results of Ref. 124 where it is shown that all the predictions obtained by Lovelace[122)] from the V.M. can essentially be reproduced with a simple isobaric model which is crossing symmetric and, to some extent, remodeled on the low energy expression that one derives from the V.M.

We conclude this section with an explicit example[125)] which is very much instructive on how one can be deceived when drawing general conclusions from a specific model.

It has been recently shown by Martin[126)] starting from a dispersion relation approach, using some unitarity (positivity of the spectral function) and applying crossing symmetry in a very smart way, that the following inequalities hold for the s-wave of $\pi_0\pi_0 \to \pi_0\pi_0$

$$f_0(3.205) > f_0(.2134) > f_0(2.9863) \qquad (II.95)$$

(the values in parenthesis are squared c.m. energies in units of $m_\pi^2 = 1$). Furthermore, Martin's procedure also shows that the last inequality in (II.95) is a very tight one.

We can now ask ourselves what result we would get using the V.M. since everything is now fixed.

If A^0 and A^2 are the I = 0, 2 isospin amplitudes in the s-channel, one has

$$A_{\pi_0\pi_0}(s,t,u) = \tfrac{2}{3}A^2 + \tfrac{1}{3}A^0 \propto$$

$$\propto -V^1_{11}(\alpha_\rho(t), \alpha_\rho(u)) - \tfrac{1}{2}[V^1_{11}(\alpha_\rho(s), \alpha_\rho(t)) + V^1_{11}(\alpha_\rho(s), \alpha_\rho(u))] \qquad (II.96)$$

Taking linear Regge trajectories and performing an expansion of (II.96) up to terms of the order $(s/m_\rho^2)^2$ and partial wave projecting the $\ell = 0$ contribution in the s-channel, one obtains

$$f_0(s) = c_1 + c_2(5s^2 - 16s) \qquad (II.97)$$

where c_1 and c_2 are given in terms of $\alpha_\rho(0)$ and $\alpha'_\rho(0)$ ($c_2 > 0$). From (II.97) we find

$$f_0(s_1) - f_0(s_2) = 5c_2(-s_1 + s_2)(-(s_1 + s_2) + 3.2) \qquad (II.98)$$

Using $s_1 = .2134$ and $s_2 = 2.9863$ we see that the last factor at the r.h.s. of (II.98) reduces to .0003 which means that the rightmost inequality derived by Martin (II.95) is actually satisfied with the equality sign. This agreement between the prediction of the V.M. and the results derived by Martin in a completely different context is certainly striking and one may therefore wonder whether this agreement is not the result of something more basic than the specific model.

To show that this is so, let us write the most general crossing symmetric amplitude that one can write in the second order of (s/m_ρ^2). This is seen to be

$$A(s,t,u) = a + b(s^2 + t^2 + u^2) + c(st + su + ut) =$$

$$= a + (2b - c)[s^2 + t^2 + st - 4(s + t)] \qquad (II.99)$$

where the second line follows from $s + t + u = 4$. The $\ell = 0$ partial wave projection of (II.99) gives

$$f_0(s) = a - \frac{8}{3}(2b - c) + \frac{1}{6}(2b - c)(5s^2 - 16s) \qquad (II.100)$$

The above formula shows that the striking agreement of (II.98) with (II.95) holds for every model in which $2b - c > 0$ independent of any specific detailed dynamical property of the model such as poles, Regge behavior and so on and is a mere consequence of the assumed crossing symmetry.

M. Generalizations of the V.M.

Very many generalizations of the V.M. have been proposed and we can distinguish several different kinds of generalizations.

Aside from the modifications which are in the form of a generalized interference model[70),72)] and from the work of Khuri,[3)] the simplest kind of generalizations of the V.M. have usually been motivated by the desire to improve the V.M. in some of its aspects.

The work of Roskies[83)] has been largely discussed already as an attempt (see also Ref. 103) to incorporate analyticity and unitarity in the V.M. without altering its structure.

In Ref. 92 (see also Ref. 127), it is shown that one can use a superposition of the form

$$A(s,t) = \sum_r a_r V_{rr}^r (s,t) \qquad (II.101)$$

(where $V^p_{nm}(s,t)$ is as defined in (II.13)) in order to achieve decoupling of daughters without having to use constraint conditions of the form (II.34). The coefficients a_r in (II.101) are chosen as to eliminate alternate trajectories and not to spoil the asymptotic Regge behavior. This leads to the closed form

$$A(s,t) = B(-\alpha(s), -\alpha(t))\; {}_3F_2[-\alpha(s), -\alpha(t), -\delta; -\tfrac{1}{2}(\alpha(s)+\alpha(t)), -\tfrac{1}{2}(\alpha(s)+\alpha(t) - 1); \tfrac{1}{4}] \qquad \text{(II.102)}$$

where B is the Euler beta function and ${}_3F_2$ is the generalized hypergeometric series. The parameter δ is given by

$$\delta = \tfrac{1}{2}(4\alpha'\mu^2 + 3\alpha + 1)$$

(where $\alpha(s) = \alpha + \alpha' s$).

A somewhat formally analogous attitude is taken in Ref. 128 in order to modify the large angle behavior of the V.M. which, as discussed in Sec. II.C, does not agree with experimental findings. The form used in Ref. 128 is, however, very much different, in practice, from the one of Ref. 92.

In order to remove the poles of the V.M. from the first to the second sheet, Martin[129)] suggested treating (II.2), (II.3) as a distribution (see (II.31)). Accordingly, he wrote

$$\bar{V}(\alpha(s), \alpha(t)) = \int_{x_m}^{1} \varphi(x)\, V(x\alpha(s), x\alpha(t))\, dx \qquad \text{(II.103)}$$

where $\varphi(x)$ is an arbitrary meromorphic function of x, positive in the interval $x_m \div 1$ which vanishes at both ends of integration. In this way, the positivity is retained (if present in the original Veneziano amplitude) and the poles move to the second sheet. Regge behavior is, however, lost. The form (II.103) represents thus an alternative to Roskies' proposal (Sec. II.H) in which the appearance of ancestors is traded for the loss of Regge behavior. A somewhat similar attitude is taken by Huang.[130)]

In the context of a more radical modification, it has been shown[95)] that a (crossing symmetrized) sum of terms proportional to

$$A(s,t) \propto e^{i\frac{\pi}{2}\alpha(t)}\; {}_2F_1\left(-\frac{\alpha(t)}{2}, a - bs; 1 - \frac{\alpha(t)}{2}; \frac{s}{\Sigma}\right) \qquad \text{(II.104)}$$

has i) the correct analytic properties (cuts); ii) the poles at integer values of α spaced of two units (no need for killing of unwanted daughters); iii) positive residues at resonance provided only the parameters a, b are positive; iv) Regge-like asymptotic behavior. Also, $\alpha(t)$ can be assumed: v) to satisfy a dispersion relation such that vi) the total width of the resonance is positive and vii) $\alpha(t)$ has an asymptotic square root type behavior[19] so that the large angle behavior is of the correct form. However, (II.104) has also cuts in the angular momentum plane and its low energy behavior is very hard to reconcile with Adler's self consistency condition contrary to what happens for the V.M.[122]

Another modification, in the form of an infinite product, leading to nonlinear trajectories with Regge behavior and poles with polynomial residues has been given by Coon.[131] Also a more ambitious program with the Mandelstam representation as a goal has been suggested.[102]

A triple product of gamma functions representation has been suggested by Virasoro[93] (see also Ref. 94). For the case of $\pi\pi \to \pi\omega$, this reads

$$A(s,t,u) = \beta \frac{\Gamma(\frac{1-\alpha(s)}{2})\Gamma(\frac{1-\alpha(t)}{2})\Gamma(\frac{1-\alpha(u)}{2})}{\Gamma(\frac{2-\alpha(s)-\alpha(t)}{2})\Gamma(\frac{2-\alpha(t)-\alpha(u)}{2})\Gamma(\frac{2-\alpha(u)-\alpha(s)}{2})} \tag{II.105}$$

In this representation the decoupling of odd daughters is automatic, complete s,t,u crossing symmetry is displayed by one single term, no fixed poles seem to appear, and furthermore, Eq. (II.105) reduces to the Veneziano amplitude (II.2), (II.3) if the supplementary condition (II.34) is imposed. Comparatively little attention has been devoted to the Virasoro amplitude (see, however, Ref. 132) and, probably more work is needed because of its very appealing features.

It should, however, be noticed that the explicit asymptotic behavior of (II.105) has not been investigated in detail in the case of nonlinear trajectories and does not seem easy to reconcile with Regge behavior. Also, the form (II.105) cannot be written in the form of an infinite product (contrary to what happens in the case of the V.M.). Finally, one can notice that if $\alpha(s)$ and/or $\alpha(t)$ are positive odd integers, (II.105) has a simple pole but no pole is present if all $\alpha(s)$, $\alpha(t)$ and $\alpha(u)$ are positive integers. This is, again, in contrast to what happens in the V.M. However, if (II.105) is multiplied by $\Gamma(\frac{3}{2}-\alpha(s)/2-\alpha(t)/2-\alpha(u)/2)$ all the previous troubles disappear and furthermore the original formula is essentially unmodified if the trajectory is linear. A new set of poles is, however, introduced.

Mandelstam,[133] finally, has given a generalization which embodies both the Veneziano and the Virasoro forms as special cases. This is in the form of a double integral

$$A(s,t,u) = \int dx\, dy \left\{\frac{1-x}{y(2-x-y)}\right\}^{\nu_1} \left\{\frac{1-y}{x(2-x-y)}\right\}^{\nu_2} \cdot \left\{\frac{x+y-1}{xy}\right\}^{\nu_3} x^{-2-\alpha(s)}\, y^{-2-\alpha(t)} (2-x-y)^{-2-\alpha(u)} \tag{II.106}$$

where the range of integration is

$$x < 1,\ y < 1,\ x + y > 1 \tag{II.107}$$

and ν_1, ν_2, ν_3 are parameters that in limiting cases reproduce either (II.3) or (II.105).

In a totally different philosophy, Bardakci and Ruegg[134] on the one hand and Virasoro[135] on the other hand have generalized the V.M. to the five point function by making use of an extension of the integral representation (II.12) for the beta function. The result is

$$V_5(s,t,u) = \int_0^1 \int_0^1 \frac{du_i\, du_j}{1-u_i u_j} u_1^{-1-\alpha_{12}}\, u_2^{-1-\alpha_{23}}\, u_3^{-1-\alpha_{34}}\, u_4^{-1-\alpha_{45}}\, u_5^{-1-\alpha_{51}} \tag{II.108}$$

where $\alpha_{12} = \alpha + \alpha' s_{12}$ etc. (the indices labeling the corresponding particles) and indices i, j denote any two nonsuccessive integers (counting 6 and 1 as equivalent). The variables u_i satisfy the constraints

$$u_i = 1 - u_{i-1}\, u_{i+1} \qquad i = 1, \ldots 5$$
$$u_6 = u_1 \tag{II.109}$$

It can be checked that only three of the five equations (II.109) are linearly independent so that there are only two effective integration variables in (II.108). The latter can then be rewritten as

$$V_5(s,t,u) = \int_0^1 \int_0^1 du_1\, du_4\, u_1^{-1-\alpha_{12}}\, u_4^{-1-\alpha_{45}} \left(\frac{1-u_1}{1-u_1 u_4}\right)^{-1-\alpha_{23}} \left(\frac{1-u_4}{1-u_1 u_4}\right)^{-1-\alpha_{34}} (1-u_1 u_4)^{-2-\alpha_{15}} \tag{II.110}$$

This formula has simple poles in all channels and reggeizes both in the single and double Regge limits in all channels. The application of this model to determine coupling constants has proved rather successful.[136] The results of Ref. 134, 135 have been further extended to the general case of N point functions by various authors. authors.[137] Much work seems, however, to be still needed on this subject both in a detailed analysis of the asymptotic behavior and to extend these results to the case of particles with spin. The only partial result so far reported in the latter direction refers to the case of $\pi\pi \to \pi S$ where S has arbitrary spin and parity.[138] We should also mention that a generalization of the N point function to incorporate isospin has also been discussed.[139]

A very ambitious program has been started by Kikkawa, Sakita and Virasoro[84),140] in which the V.M. is used as an input to be treated as a Born term. A technique based on an extensive use of dual diagrams (both in the sense of projective geometry as well as in the sense discussed in I.F) is introduced in which the contribution of intermediate states is taken by means of Feynman-like diagrams.

The final aim in this approach is to obtain some sort of unitarity corrections; it is proved, in fact, that the sum of these diagrams gives rise to a trajectory which is no longer linear. The difficulties associated with this program are, however formidable especially when nonplanar diagrams (generating Regge cuts) are taken into account.

It is especially in the context of this attempt to unitarize the Veneziano amplitude that the work of Fubini and Veneziano[85] on the factorization properties seems very relevant. Only future investigation will, however, tell what will be the probabilities of success in the development of such a program.

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REMARKS ON DIFFRACTION SCATTERING†‡

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I. Introduction

There exist at present, and have for several years, two entirely different phenomenological descriptions of high-energy elastic scattering (i.e., of diffraction scattering; and indeed these independent phenomenologies apply to diffraction dissociation processes as well). One of these is the Regge pole model; the other may conveniently be called the "classical" model, and it really includes a class of models which have in common a picture in which diffraction scattering is simply the collision of two fuzzy round balls of a given finite radius.

Both models satisfy the requirements that the total cross section becomes constant at high energies, and in addition the forward elastic amplitudes, in both cases, asymptotically become pure imaginary. Experimentally, of course, constant total cross sections seem clearly to be required, and the real parts of scattering amplitudes are certainly much smaller than the imaginary parts, for high energies, and may well disappear altogether in the truly asymptotic region. Hence both of these facets of the two models are consistent with present data.

The fundamental difference between the two models lies in the "shrinkage" of the diffraction peak: In the Regge case, the high-energy elastic scattering amplitude T(s,t) has the form[1]

$$T(s,t) \Rightarrow \frac{\beta(t)}{\sin \pi\,\alpha(t)} \left(\frac{s}{s_o}\right)^{\alpha_P(t)} \left(\frac{1+e^{-i\pi\,\alpha_P(t)}}{2}\right)$$

and hence the elastic differential cross section approaches

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$$\frac{d\sigma}{dt} \xrightarrow[s\to\infty]{} \frac{1}{16\pi} \left| f(t)\, e^{(\alpha_P(t)-1)\log s/s_o} \right|^2 \qquad (I.1)$$

where

$$f(t) = \frac{\beta(t)}{\sin \pi\, \alpha(t)} \cdot \frac{1 + e^{-i\pi\, \alpha_P(t)}}{2\, s_o} \quad .$$

$\alpha_P(t)$ is called the Pomeranchuk trajectory (we assume $\alpha_P(0) = 1$); s is, as usual, the total center-of-mass energy squared and t is the invariant momentum transfer. f(t) is some function of t and s_o is some (arbitrary) constant.

For small t, $\alpha_P(t) - 1 \approx t\,\alpha_P{}'(0)$, so that near forward directions

$$\frac{d\sigma}{dt} \approx |f(0)|^2 \; e^{2\alpha_P{}'(0)\, t \log s/s_o} \quad . \qquad (I.2)$$

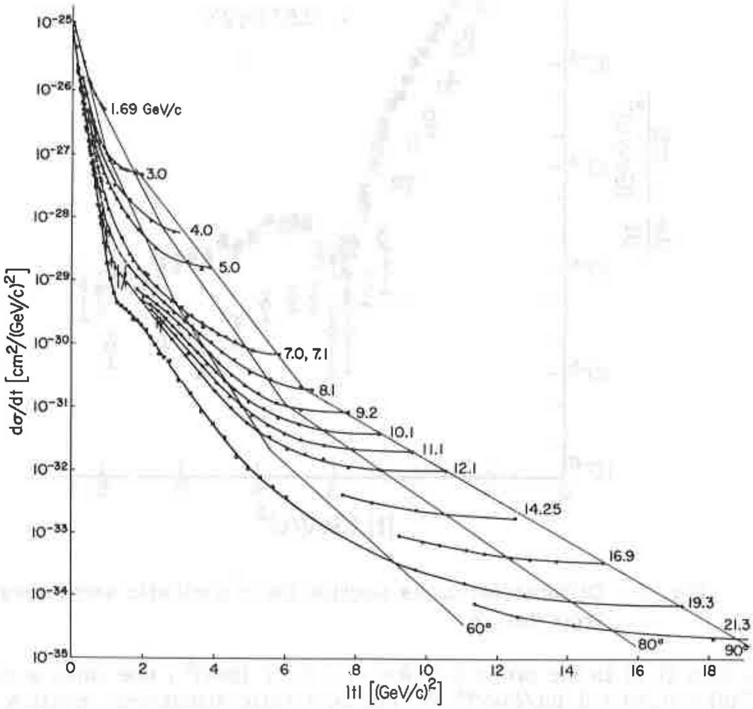
The cross section thus has an exponential forward peak in momentum transfer, but the peak shrinks logarithmically with increasing energy.

The "classical" model, on the other hand, has simply the form $T(s,t) \to -is\, f(t)$ and hence

$$\frac{d\sigma}{dt} \xrightarrow[s\to\infty]{} \frac{1}{16\pi} |f(t)|^2 \quad . \qquad (I.3)$$

At present, both of these models are phenomenological, in that neither can be derived from any "fundamental" theory. One does not know whether conventional field theory, or "S-matrix theory," or the bootstrap theory, or anything else, leads to either (I.1), or to (I.3), or to something else.

Experiment also does not, as yet, clearly distinguish between the two models.[2] It seems that π^+p and π^-p elastic scattering develop a saturated non-shrinking forward peak at high energies, and that the size in momentum transfer of this saturated peak grows as the energy increases. The experimental differential cross section for π^-p is shown in Figure 2. In the case of pp scattering, for energies between 10 and 30 BeV (s = 20 to 60 BeV^2) the same situation seems to prevail, as is shown in Figure 1. At higher energies, however, from 30 up to 70 BeV (s = 60 to 140 BeV^2) the forward peak in pp elastic scattering seems to shrink again, and in fact if one fits the cross section with



Proton-proton elastic scattering cross sections $d\sigma/dt$ as functions of $|t|$. The curves joining the experimental points are hand-drawn to guide the eye. The loci of cross sections for fixed c.m.s. scattering angle are indicated for 60°, 80° and 90°.

Fig. 1. Differential cross section for pp elastic scattering as a function of t for various values of p_{lab}. From J. V. Allaby et al., Physics Letters <u>28B</u>, 67 (1968).

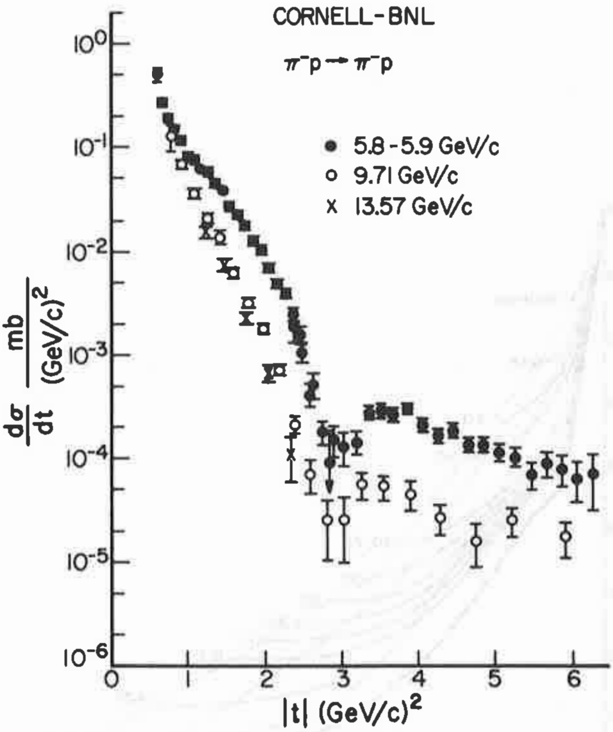


Fig. 2. Differential cross section for $\pi^- p$ elastic scattering. From Ref. 2.

the form (I.2) in the range $0.008 < -t < 0.12$ $(BeV)^2$, one finds a slope $\alpha_P'(0) = 0.40 \pm 0.09/(BeV)^2$.[9] For $\bar{p}p$ elastic scattering, again a non-shrinking peak appears, at energies $s = 16$ and 32 $(BeV)^2$. (In fact, here the peak may even anti-shrink a bit.) The cross section is shown in Fig. 3. Elastic K^+p and K^-p scattering has also been measured at energies up to $s = 27$ $(BeV)^2$; the cross section for K^-p is shown in Fig. 4. Here again, the forward peak anti-shrinks somewhat.

The experimental evidence, therefore, on the face of it would seem to favor the form (I.3). However, the data is still all at finite energy, and it may still be possible to fit it with the Regge model, if

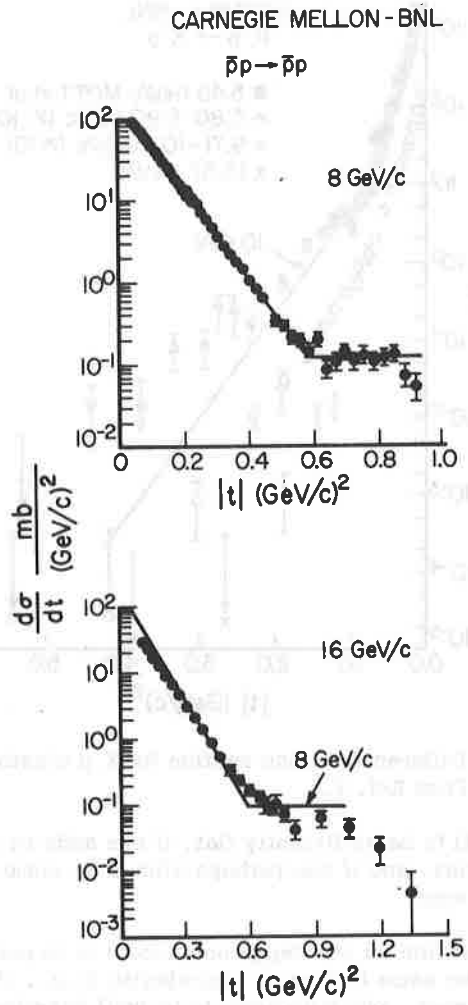


Fig. 3. Differential cross section for $\bar{p}p$ elastic scattering. From Ref. 2.

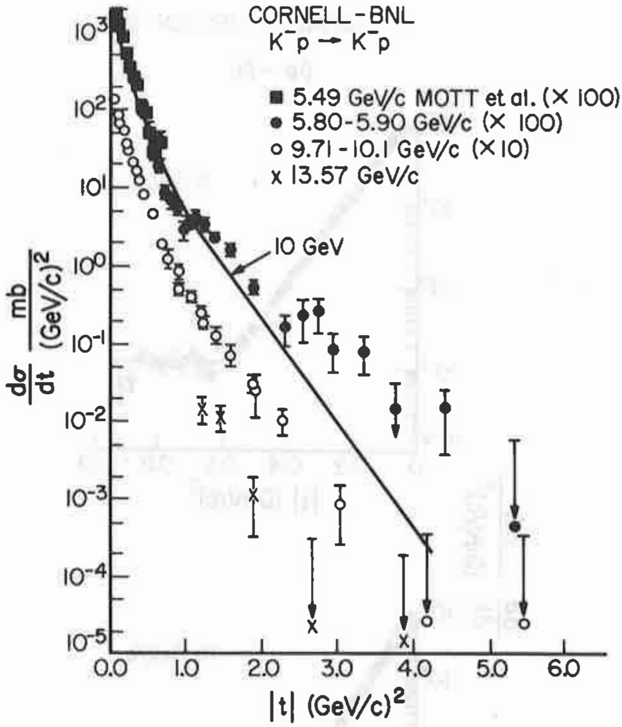


Fig. 4. Differential cross section for K^-p elastic scattering. From Ref. 2.

one allows $\alpha_P(t)$ to be sufficiently flat, if one adds in some other Regge trajectories, and if one perhaps allows for some other terms such as Regge cuts.

The motivation for the Regge pole model is to put diffraction scattering on the same footing as non-elastic (e.g., charge exchange) two body processes, which do seem to be well described by forms like (1) (but, of course, with different trajectories $\alpha(t)$).

The "rules" of Regge theory, briefly, are as follows.[1)]

Each Regge trajectory is associated with a given channel (i.e., a given set of quantum numbers such as isospin, strangeness, etc.), and in addition each trajectory is associated with a given signature (even or odd). In any two body process in which some channel can be exchanged, each trajectory associated with that channel contributes, at high energy, a term

$$\frac{\beta(t)}{\sin \pi \alpha(t)} \quad (s/s_o)^{\alpha(t)} \left(\frac{1 \pm e^{-i\pi \alpha(t)}}{2} \right) \qquad (I.4)$$

to the amplitude. (Here the ± is chosen according to whether the trajectory carries even or odd signature.)

Furthermore, at any positive value t at which an even (odd) signature trajectory $\alpha(t)$ takes on an even (odd) integer value (half-integer for fermion channels), then there is a particle in the associated channel of that integer spin and mass $\sqrt{t}$; i.e., $\alpha(M^2) = J$. These particles are said to lie on the trajectory $\alpha(t)$.

Now, to incorporate diffraction scattering into the general Regge picture, one invents a trajectory $\alpha_P(t)$, called the Pomeranchuk trajectory, of even signature and associated with the vacuum channel (i.e., the channel with no quantum numbers). It is this channel which is exchanged in diffraction, or diffraction dissociation, processes. The choice $\alpha_P(0) = 1$ is usually (though not always) made, because, using (I.4), its contribution to the elastic scattering amplitude at t = 0 and large s is then

$$T(s,t) = - i\, \beta(0)/2\ (s/s_o) \quad ; \qquad (I.5)$$

that is, pure imaginary and proportional to s. This insures that, by the optical theorem, $\sigma_T(s) \rightarrow \beta(0)/2s_o$ = const. at large s. Even signature is required to avoid the existence of a massless spin one particle.

The existence of the trajectory $\alpha_P(t)$ then leads directly to (1).

We may next ask what particles lie on $\alpha_P(t)$. These will be particles with no quantum numbers and spins of 0^+, 2^+, 4^+,... etc. The 0^+ particle would occur at $t < 0$, i.e., would have imaginary mass, since $\alpha(0) = 1$; hence, it is presumably absent. Candidates for the 2^+ particles might be $f_o(1260)$, or $f_o'(1515)$.

Other Regge trajectories seem experimentally to be very closely straight lines with slopes near one per $(BeV)^2$.[3)] If $\alpha_P(t)$ is also nearly straight, its slope would be ~ 0.63 if f_o were on it, or ~ 0.43 if f_o' were on it.

Now if $\alpha_P(t)$ really has a slope of this order, then the form (I.1) cannot by itself fit the high-energy elastic scattering data. It is believed, however, that there is at least one other trajectory, called the P', in the vacuum channel, and at the energies available it will also contribute to the elastic amplitude. In addition, other trajectories, such as ρ, can have effects too. The form (1), then, while valid at every large s, is too simple at presently available energies, and should be replaced by

$$T(s,t) = \frac{\beta_P(t)}{\sin\pi\,\alpha_P(t)}\,(s/s_o)^{\alpha_P(t)}\,\frac{1+e^{-i\pi\,\alpha_P(t)}}{2}$$

$$+\ (\text{same with } P \to P')$$

$$+\ (\text{same with } P \to \rho,\ + \text{opposite signature}).$$

With this amplitude, and appropriate choices of the functions $\beta(t)$, it is possible to fit all elastic scattering data.[4] However, in order to fit the lack of observed shrinkage, a small slope for P is required, in the vicinity of $\alpha_P'(0) \approx 0.3$ or less.[4] This corresponds more closely to the situation where f_o' lies on P than where f_o lies on P. Presumably, then, f_o lies on P′, and if this is the case, and if $\alpha_P'(0) \sim \frac{1}{2}$, the slope of the P′ is $0.95/(\text{BeV})^2$ which is close to the slopes of other Regge trajectories. There is a minor difficulty with this assignment, however. At $t = 0$, the P must be a pure SU(3) singlet, since it is associated with the vacuum channel. The f_o', however, seems to be a mixture of singlet and octet with a considerable amount of octet in it. This requires the mixing angle to vary considerably between $t = 0$ and $t = (1515\ \text{BeV})^2$, which is certainly not impossible, but not wholly pleasant either.

One then ends up with a situation in which all Regge poles have similar slopes ($\sim 1/(\text{BeV})^2$) except P, which has a much smaller slope ($\lesssim 0.3/(\text{BeV})^2$; perhaps $\sim 0.1/(\text{BeV})^2$).

To summarize, it is possible to fit the elastic cross sections with the Regge pole model, at the price (which violates the original purpose of the Regge model) of making the P rather peculiar--namely a lot flatter--than all other known trajectories.

Finally, it is also important to keep in mind the fact that all these fits are quite fuzzy. There are so many parameters available that none of them are very precisely fixed by the data, and by the same token, it is unlikely that enough data will ever exist to eliminate the Regge model with 100% certainty. This is reflected in the fact that a large range of slopes for the Pomeranchuk trajectory have been used in different fits to the same data, varying all the way from 0 to $0.7/(\text{BeV})^2$.

The motivation for the classical model is entirely the opposite from that of the Regge model. Whereas one may well accept the Regge description of non-diffractive processes, one abandons the Pomeranchuk trajectory and ascribes elastic scattering, and diffraction dissociation reactions, to an entirely different mechanism.

Let us outline the arguments used to derive the form (3); as will be seen, these are very much based on a simple physical picture of elementary particles, and have as a result a considerable intuitive appeal.[5)]

We write, at large s,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s^2} \; |T(s,t)|^2 \quad . \tag{I.6}$$

The amplitude T(s,t) has a partial wave expansion

$$T(s,t) = \sum_{\ell} (2\ell+1) \, P_{\ell}(\cos\theta) \, T(s,\ell) \tag{I.7}$$

where, for large s,

$$T(s,\ell) = -16\pi \frac{\eta_{\ell}(s)\, e^{2i\delta_{\ell}(s)} - 1}{2i} \quad . \tag{I.8}$$

It is convenient to rewrite Eq. (I.7) in the impact parameter form.[6)]

$$T(s,t) = \frac{s}{2} \int_0^{\infty} b\,db \, J_0(b\sqrt{-t}) \, T(s,\ell) \tag{I.9}$$

where $(2\ell+1) = \sqrt{s}\, b$.

Now what happens physically at large s? Intuitively, we might expect the phase shift δ to vanish, and the absorption η to depend only on b, the impact parameter, and not explicitly on the energy. Thus we have

$$T(s,\ell) = -16\pi \frac{\eta(b) - 1}{2i} \tag{I.10}$$

and hence

$$T(s,t) = 4\pi i s \int_0^{\infty} b\,db \, J_0(b\sqrt{-t}) \, (\eta(b) - 1)$$

$$\equiv i\, s\, f(t) \tag{I.11}$$

where

$$f(t) = 4\pi \int_0^{\infty} b\,db \, J_0(b\sqrt{-t}) \, (1 - \eta(b)) \quad . \tag{I.12}$$

Hence we have, at large s, the form

$$\frac{d\sigma}{dt} \to \frac{1}{16\pi} \,|f(t)|^2 \quad . \qquad (I.13)$$

This form really depends only on the belief that η is a function just of b. Physically, this follows if one thinks of the scattering as if it were from a particle made up of material which absorbs a constant amount per unit volume, and has some well-defined geometrical shape which is independent of the energy of the scattering particle.

A further hint that (I.11) is the proper asymptotic form is provided by studies of the high-energy limit of quantum electrodynamics (quantum electrodynamics is, after all, the source of all of our beliefs about everything), where it is claimed[7] that -is f(t) is indeed found to hold. (However, something must alter this form in quantum electrodynamics if t is positive, as is explained in detail in the following section of these lectures.)

Now, the question arises, how different are the Regge and "classical" models really? Are they compatible with each other?

Superficially, the answer is certainly yes. The Pomeranchuk Regge pole term in the scattering amplitude looks like

$$T(s,t) = \frac{\beta_P(t)}{\sin \pi \alpha_P(t)} \; (s/s_o)^{\alpha_P(t)} \left(\frac{1+e^{-i\pi \alpha_P(t)}}{2} \right) , \qquad (I.14)$$

and this is supposed to dominate the amplitude at large s. Now if $\alpha_P(t) \to 1$, for all t, we have

$$T(s,t) \to -\, i \,\beta_P(t)/2 \;(s/s_o) \qquad (I.15)$$

so that if we identify

$$f(t) = \beta_P(t)/2s_o \qquad (I.16)$$

we have precisely the "classical" form. Thus it seems that the classical model is simply a special case of the Regge theory, one in which the Pomeranchuk trajectory is precisely flat. This would also mean that no particles lie on $\alpha_P(t)$, so that the embarrassment of trying to decide whether f_o, or f_o', or whatever, is on the Pomeranchuk trajectory disappears.

This would appear to be a happy resolution of the conflict between the two models, but unfortunately it's too easy. It turns out, as we'll see in the following section, that the "classical" result $T(s,t) \to -is\, f(t)$ is very hard to reconcile with the general principles of field theory (or S-matrix theory if you prefer), so that a flat Pomeranchon is not allowed.

One is thus faced with a conflict: The "classical" model fits the data better, and is intuitively very appealing, but (almost) contradicts general principle, while the Regge model satisfies all general principles, but isn't anywhere near as good a phenomenological description of the experimental situation, unless rather peculiar behavior is assigned to the Pomeranchon.

In conclusion, and for the sake of completeness, we should at least mention the existence of a third class of models, which lies somewhere between the two we have been discussing so far. These are the so-called hybrid models. On the simplest level, they consist merely of saying that the scattering amplitude is the sum of the diffraction scattering of the "classical" model plus some (non-Pomeranchuk) Regge poles, such as the P′. Thus

$$T(s,t) \to -is\, f(t) + \frac{\beta_{P'}(t)}{\sin \pi \alpha_{P'}(t)} \left(\frac{1 + e^{-i\pi \alpha_{P'}(t)}}{2} \right) \left(\frac{s}{s_0} \right)^{\alpha_{P'}(t)}$$
$$+ \text{(possibly other Regge poles)} \quad . \qquad \text{(I.17)}$$

Clearly, such models still abandon the truly high-energy behavior to the simple form (I.11). The additional Regge terms are only present as finite energy corrections.

On a more sophisticated level, the hybrid models attempt to unitarize the expression (I.17). In effect, this amounts to allowing repeated Regge "exchanges" in the scattering amplitude. As is well known, this generates Regge cuts, so one now has an amplitude with $-is\, f(t)$, with assorted Regge poles, and with Regge cuts as well. Evidently, one now has a considerable amount of freedom in fitting data, and so it is perhaps not surprising that the hybrid models agree reasonably well with quite a broad range of data. (For further details, and in particular for a list of references, see Ref. 3.) Nevertheless, the <u>asymptotic</u> amplitude is still just given by (I.11). If one attempts to replace $-is\, f(t)$ by a Pomeranchon with a large ($\sim 1/(\text{BeV})^2$) slope, the arithmetic of the hybrid models may still be carried through, and one now has a normal Regge model but with Regge cuts from repeated Regge pole exchange included too. Now, however, the agreement with experiment is much less impressive -- in other words, the addition of Regge cuts to the "normal" Regge model is not enough to save the experimental situation.

To conclude this introductory section, we shall henceforth accept the experimental preference for something like the simple $-is\, f(t)$ form as convincing, and therefore we shall next concern ourselves with its theoretical implications.

II. Compatibility With Unitarity and Analyticity

As we have seen, in many ways, both experimental and theoretical, an appealing model is that an elastic amplitude approaches

$$T(s,t) \rightarrow -is\, f(t) \qquad (II.1)$$

at large s and fixed (negative) t. Now on rather general grounds we expect $T(s,t)$ to have very restrictive analyticity properties, which suggest that the form (II.1) should apply outside the physical ($t < 0$) region for the scattering as well, and in fact should continue to hold for positive t. If (II.1) holds beyond the first threshold in the t-channel (presumably at $t = 4m_\pi^2$), then t-channel unitarity is violated, as was first noticed by Gribov.[1]

To understand Gribov's argument, and to incorporate what we believe to be the analyticity properties of $T(s,t)$, let us write the amplitude at large s in the form

$$T(s,t) = T^+(x_t,t) + T^+(-x_t,t) \qquad (II.2)$$

where $x_t = 1 + s/2q_t^2$, and where

$$T^+(x_t,t) = a + b\, x_t + \frac{x_t^2}{\pi} \int_{x_o(t)}^{\infty} \frac{A(z,t)}{z^2(z - x_t)}\, dz\,. \qquad (II.3)$$

Here, a and b are arbitrary functions of t. We choose to use x_t and t as variables instead of s and t; we believe that $T(s,t)$ is, at large s, symmetric in the interchange of s and u and hence even in x_t (recall $s = -2q_t^2(1 - x_t)$, $u = -2q_t^2(1 + x_t)$); we make the assumption that $T(s,t)$ is an analytic function of s (or x_t) for fixed t, with the usual cuts, and we subtract the dispersion relation (II.3) twice to allow for asymptotic behavior in s like that given by (II.1).

Now let us suppose $A(z,t) \rightarrow z\, F(t)$ as $z \rightarrow \infty$. Then we have, as $x_t \rightarrow \infty$,

$$T^+(x_t,t) \to a + b\,x_t - \frac{x_t\,F(t)}{\pi}\;\log\left(\frac{x_o(t) - x_t}{x_o(t)}\right) \tag{II.4}$$

and hence, as $s \to \infty$,

$$T(s,t) \to \frac{x_t\,F(t)}{\pi}\;\log\,(-\,x_t/x_o) + \frac{x_t\,F(t)}{\pi}\;\log\,(x_t/x_o)$$

$$= i\,x_t\,F(t) = i\,s/2q_t^2\;F(t) = -\,is\,f(t) \quad . \tag{II.5}$$

Thus the required asymptotic form is obtained, and obtained, we note, for all t.

In particular, it applies for $4m_\pi^2 < t <$ first inelastic threshold. In this region, unitarity requires

$$\mathrm{Im}\;T(s,t) = \rho(t)\int_{-1}^{1} dx_1 \int_{-1}^{1} dx_2\;\frac{\theta(1 - x^2 - x_1^2 - x_2^2 - 2x\,x_1\,x_2)}{\sqrt{1 - x^2 - x_1^2 - x_2^2 - 2x\,x_1\,x_2}}$$

$$T^*(s_1\,,\,t)\;T(s_2\,,\,t) \quad , \tag{II.6}$$

where $\rho(t) = = 1/32\pi^2\sqrt{t - 4m_\pi^2/t}$, and where $x_{1,2} = 1 + s_{1,2}/2q_t^2$. This relation is valid in the physical region for the t-channel; that is, for $-1 < x,\ x_1,\ x_2 < 1$. It may, however, be continued out of this region in to the region where $x > 1$, and thus into the region where s is large. To carry out the continuation, note the identity

$$\frac{\theta(1 - x^2 - x_1^2 - x_2^2 + 2x\,x_1\,x_2)}{\sqrt{1 - x^2 - x_1^2 - x_2^2 + 2x\,x_1\,x_2}} = \frac{\pi}{2}\sum_\ell\;(2\ell+1)\;P_\ell(x)\;P_\ell(x_1)\;P_\ell(x_2). \tag{II.7}$$

With this relation, with Eqs. (II.2) and (II.3), and with the well-known fact that

$$\tfrac{1}{2}\int_{-1}^{1}\frac{dx}{z - x}\;P_\ell(x) = Q_\ell(z) \quad , \tag{II.8}$$

it is easy to show that the unitarity relation (II.6) may be rewritten

$$\mathrm{Im}\;T(s,t) = \bar{a}(t) + \frac{4}{\pi}\,\rho(t)\int_1^\infty dz_1 \int_1^\infty dz_2\;A(z_1\;t)\;A(z_2\;t)$$

$$\sum_\ell\;(2\ell+1)(P_\ell(x) + P_\ell(-x))\;Q_\ell(z_1)\;Q_\ell(z_2) \quad . \tag{II.9}$$

The function $\bar{a}(t)$ appears due to the fact that the dispersion relation for $T(s,t)$ was twice subtracted, and can easily be expressed in terms of $a(t)$. Since $a(t)$ was arbitrary, so is $\bar{a}(t)$.

Finally, we note another identity, namely:

$$\sum_{\ell}(2\ell+1)\, P_\ell(x)\, Q_\ell(z_1)\, Q_\ell(z_2) = \frac{1}{\pi}\int_1^\infty \frac{dz}{z-x} \frac{\theta(z-z_>)}{\sqrt{z^2+z_1{}^2+z_2{}^2-1-2z_1\, z_2\, z}}\ . \tag{II.10}$$

Here $z_>$ is the largest root of the denominator:

$$z_> = z_1\, z_2 + \sqrt{z_1{}^2-1}\,\sqrt{z_2{}^2-1}\ .$$

Making use of this, we have

$$\operatorname{Im} T(s,t) = \frac{1}{\pi}\int_1^\infty \frac{\rho(z,t)}{z-x}\, dz + \frac{1}{\pi}\int_1^\infty \frac{\rho(z,t)}{z+x}\, dz \tag{II.11}$$

where

$$\rho(z,t) = \frac{4}{\pi}\,\rho(t)\iint dz_1\, dz_2\, \frac{\theta(z-z_>)}{z^2+z_1{}^2+z_2{}^2-1-2z\, z_1\, z_2} A(z_1\, t)\, A(z_2\, t)\ . \tag{II.12}$$

Now, if the asymptotic form (II.1) is to hold, we should expect

$$\rho(z,t) \to z \operatorname{Im} F(t) \tag{II.13}$$

as $z \to \infty$. On the other hand, we know

$$A(z,t) \to z\, F(t) \tag{II.14}$$

so we can also calculate how $\rho(z,t)$ in fact behaves from (II.12). At large z, the dominant contribution to the integral in (II.12) comes from large z_1 and z_2, so the asymptotic form (II.14) can be used. We then have

$$\rho(z,t) \xrightarrow[z\to\infty]{} \frac{4}{\pi}\,\rho(t)(F(t))^2 \int_1^z dz_2 \int_1^{z/2z_2} dz_1 \frac{z_1\, z_2}{\sqrt{2z\, z_2\left(\frac{z}{2z_2}-z_1\right)}}$$

$$\to \frac{4}{\pi}\,\rho(t)\,(F(t))^2\,\frac{z}{3}\int_1^z \frac{dz_2}{z_2}$$

$$\to \frac{4}{3\pi}\,\rho(t)\,(F(t))^2\, z\log z \tag{II.15}$$

We are thus led to a contradiction; hence the assumed asymptotic behavior is inconsistent with t-channel unitarity.

The same argument may be rephrased, in what is perhaps a more transparent way, in terms of t-channel partial waves. From (II.3) we find that the partial wave amplitude is

$$T^+(t,\ell) = \tfrac{1}{2}\int_{-1}^{1} dx_t\, T^+(x_t,t)\, P_\ell(x_t)$$

$$= \left(a(t) - \frac{1}{\pi}\int_{x_o(t)}^{\infty} \frac{A(z,t)}{z}\, dz\right)\delta_{\ell o}$$

$$+ \tfrac{1}{3}\left(b(t) - \frac{1}{\pi}\int_{x_o(t)}^{\infty} \frac{A(z,t)}{z^2}\, dz\right)\delta_{\ell 1} + \frac{1}{\pi}\int_{x_o(t)}^{\infty} A(z,t) Q_\ell(z)\, dz\ . \qquad \text{(II.16)}$$

(Note that if $A(z,t) \to z\,F(t)$, the separate terms on the right-hand side are singular at $\ell = 0, 1$, but the entire combination is nevertheless finite.)

The last term on the right-hand side of (II.16) can be analytically continued into the complex ℓ plane. The T^+ partial wave amplitude is thus a smooth piece, plus Kronecker deltas $\delta_{\ell o}$ and $\delta_{\ell 1}$. The coefficients of the Kronecker delta depend on the subtraction constants a and b; the purely Regge theory corresponds to the case where a and b are such as to cancel the integrals in the coefficients of the Kronecker deltas, leaving a purely smooth function of ℓ: In pure Reggeism,

$$T^+(t,\ell) = \frac{1}{\pi}\int_{x_o(t)}^{\infty} A(z,t)\, Q_\ell(z)\, dz\ . \qquad \text{(II.17)}$$

Now, for our case, we want to assume $A(z,t) \to z\,F(t)$ as $z \to \infty$. Then near $\ell = 1$, we find

$$T^+(t,\ell) = \frac{F(t)}{3\pi}\ \frac{1}{\ell - 1}\ . \qquad \text{(II.18)}$$

Exactly at $\ell = 1$, on the other hand, T is finite:

$$T^+(t,1) = \tfrac{1}{3}\, b(t)\ . \qquad \text{(II.19)}$$

Now let's impose t-channel elastic unitarity. This says that for $4m_\pi^2 < t <$ inelastic threshold,

$$T^+(t,\ell) - T^+(t,\ell^*)^* = 2i\,\rho(t)\,T^+(t,\ell)\,T^+(t,\ell^*)^* \qquad \text{(II.20)}$$

for any complex ℓ. Near $\ell = 1$, the left-hand side is $\sim 1/\ell-1$, but the right-hand side is $\sim (1/\ell-1)^2$. Even though this form for T is not valid exactly at $\ell = 1$, it is valid arbitrarily close to it, so there is a contradiction.

The physics here is exactly the same as that in Gribov's argument, and in words, is expressed by saying that fixed poles in ℓ (viz., $1/\ell-1$) are incompatible with unitarity.[2] The fact that the pole is not strictly present (because of (II.19)) is not really relevant to the argument.

In any event, we are faced with a paradox. The form (II.1) is incompatible with general principles -- if it holds not only for $t < 0$ but also for $t > 4m_\pi^2$. How can one escape the difficulty? One way is to say that somehow the form (II.1) fails at $t = 4m_\pi^2$. For example, a Regge cut could pass through 1 at $t = 4m_\pi^2$ and take over the asymptotic behavior for this and larger values of t.[2] The difficulty with this is that any sign of this cut is not present experimentally; furthermore, such cuts would have to conveniently appear at each t-channel threshold.

Another possibility is that in the limit $s \to \infty$ the analyticity properties of T(s,t) fail, so that, for example, f(t) is replaced by $f(t)\,\theta(4m_\pi^2 - t)$. Here the difficulty is that no model with this feature has been constructed, and it may in fact be impossible to do so.

An altogether more pleasant way out of Gribov's difficulty is the following.[3]

It is well known that a Regge trajectory $\alpha(t)$ has branch points in t at physical thresholds of all channels to which the trajectory couples. It may be, however, that every trajectory has a branch point at $t = 0$ as well, and that each trajectory is real only between $t = 0$ and the lowest available threshold, for the following reason.

In potential theory, when two Regge trajectories $\alpha(t)$ and $\alpha_1(t)$ collide (that is, when there is a value t_1 such that $\alpha(t_1) = \alpha_1(t_1)$), then both trajectories develop a square root branch point at $t = t_1$.[4] Let us suppose this is also true in the relativistic theory. If a trajectory $\alpha(t)$ collides with a set of trajectories $\alpha_i(t)$ at values $t = t_i$ (that is, if $\alpha(t_i) = \alpha_i(t_i)$), then $\alpha(t)$ has square root branch point at each t_i, so that $\alpha(t)$ has a set of singularities like $\sum_i C_i\sqrt{t - t_i}$.

A Regge cut is just a continuous superposition of Regge poles; hence if a trajectory $\alpha(t)$ collides with a Regge cut we may plausibly anticipate that $\alpha(t)$ has a continuous set of singularities of the form

$$\int_{-\infty}^{t_o} C(t')\sqrt{t-t'}\ dt' \quad ,$$

where t_o is the value of t at which $\alpha(t)$ collides with the leading edge of the cut.

Now any Regge trajectory, when coupled with the Pomeranchon $\alpha_P(t)$, generates a Regge cut, and if $\alpha_P(0) = 1$ the trajectory crosses the leading edge of this cut at $t = 0$.[5] Therefore, we expect any Regge trajectory to develop a singularity at t = 0 of the form

$$\int_{-\infty}^{t_o} C(t')\sqrt{t-t'}\ dt'$$

If $C(t')$ behaves like any integer power $(t')^n$ near $t' = 0$, then the singularity in $\alpha(t)$ is like $\sqrt{t}\, t^{n+1}$, i.e., a square root branch point.

It is interesting to note that attempts to calculate a Regge trajectory dynamically, using methods which are sufficiently sophisticated to generate Regge cuts, also seem to yield the result that the trajectory is complex for $t < 0$.[6]

For the understanding of elastic scattering the left-hand cut in a trajectory can have great significance. For suppose the Pomeranchuk trajectory $\alpha_P(t)$ has a square root branch point at $t = 0$, so that

$$\alpha_P(t) = 1 + \sqrt{t}\, g(t) \quad ,$$

and suppose $g(t)$ is analytic with only the usual right-hand cut in t. Then $g(t)$ is real for $t < 4m_\pi^2$, and hence Re $\alpha_P(t) = 1$ precisely for $t < 0$; i.e.,

$$\alpha_P(t) = 1 + i\ \mathrm{Im}\ \alpha_P(t)$$

for $t < 0$. Then the usual Regge pole theory tells us that, at large s and negative t, the contribution of the Pomeranchon itself approaches

$$T(s,t) \to \frac{i\,\beta_P(t)}{\sinh \pi\ \mathrm{Im}\ \alpha_P(t)}\ (s/s_o)\ e^{i\ \mathrm{Im}\ \alpha_P(t) \log s/s_o} \left(\frac{1-e^{\pi\ \mathrm{Im}\ \alpha_P(t)}}{2}\right)$$

$$= -is\ f(t)\, e^{i\ \mathrm{Im}\ \alpha_P(t) \log s/s_o} \qquad \text{(II.21)}$$

where

$$f(t) = \frac{\beta_P(t)}{s_o} \frac{e^{\pi \operatorname{Im} \alpha_P(t)}}{e^{\pi \operatorname{Im} \alpha_P(t)} - e^{-\pi \operatorname{Im} \alpha_P(t)}} \quad . \tag{II.22}$$

To this, of course, must be added the contribution of the cut with which the pole collided. This is not precisely of the form (II.1); however, the resulting differential cross section is still given by

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} |f(t)|^2$$

so that no shrinkage of diffraction peaks exists. On the other hand, since $\operatorname{Im} \alpha_P(t) \neq 0$ for $t > 4m_\pi^2$, Gribov's paradox is resolved.

Incidentally, if $\operatorname{Im} \alpha_P(t) \sim Ct$ $(C > 0)$, then from (II.22) we have

$$f(t) \rightarrow -\frac{\beta_P(t)}{s_o} e^{2\pi Ct}$$

for large negative t. Thus we find that an exponential behavior for f(t) is consistent with only a slow variation of $\beta_P(t)$.

One further constraint follows from t-channel unitarity, and that is the following. The partial wave unitarity relation, for complex ℓ, and in the t-channel elastic region, appears in Eq. (II.20). This unitarity equation has a pole on both sides, at $\ell = \alpha_P(t)$. If we equate the residues of this pole, we obtain

$$1 = 2i\,\rho(t)\, T^*(t, \alpha_P^*(t)) \quad . \tag{II.23}$$

Now one part of T(s,t) is given by (II.21), and this contributes to $T(t,\ell)$ a term

$$T(t,\ell) = \frac{\bar{\beta}_P(t)}{\ell - \alpha_P(t)} \tag{II.24}$$

where

$$\bar{\beta}_P(t) = -\frac{1}{2\sqrt{\pi}} \frac{\Gamma(\alpha_P(t)+1)}{\Gamma(\alpha_P(t)+\frac{3}{2})} \left(-\frac{q_t^2}{s_o}\right)^{\alpha_P(t)} \left(\frac{\beta_P(t)}{2\alpha_P(t)+1}\right). \tag{II.25}$$

For small $\operatorname{Im} \alpha_P(t)$, this term will dominate $T^*(t, \alpha_P^*(t))$. Thus we obtain the result that

$$1 = 2i\,\rho(t)\,\frac{\bar{\beta}_P^{\,*}(t)}{\alpha_P(t) - \alpha_P^{\,*}(t)} \quad ,$$

or

$$\mathrm{Im}\,\alpha_P(t) = \rho(t)\,\bar{\beta}_P(t) \quad , \tag{II.26}$$

and this is valid if Im $\alpha_P(t)$ is small, in particular near threshold.[7] Unfortunately, we learn nothing from t-channel unitarity about Im $\alpha_P(t)$, or $f(t)$, for $t < 0$, which is the region of primary interest for us.

This pretty much exhausts what we can get from t-channel unitarity, and we must turn to s-channel unitarity for further information.

The first question which arises is whether the form (II.21) is compatible with s-channel unitarity. It is not easy to answer this question clearly, since at large s, s-channel unitarity is highly inelastic and hence is quite complicated. It is necessary to assume a model for high energy n-particle production amplitudes in order to test (II.21). The most natural such model to take is the multi-Regge pole model,[8] in which the amplitude for n particle production is written

$$T_{2\to n} = \beta_1(t_1)\,R(t_1)\left(\frac{s_{12}}{s_o}\right)^{\alpha_1(t_1)}\beta_2(t_1,t_2)\,R(t_2)\left(\frac{s_{23}}{s_o}\right)^{\alpha_2(t_2)}\cdots$$

$$\cdots R(t_{n-1})\left(\frac{s_{n-1,n}}{s_o}\right)^{\alpha_{n-1}(t_{n-1})}\beta_{n-1}(t_{n-1}) \tag{II.27}$$

where

$$R(t_i) = \frac{1 \pm e^{-i\pi\,\alpha(t_i)}}{2\,\sin\pi\,\alpha(t_i)} \quad ,$$

where

$$t_i = (q_i - p)^2 \quad ,$$

where

$$s_{i,i+1} = (q_i + q_{i+1})^2 \quad ,$$

and where the momenta for the process $2 \to n$ are labelled $p + p' \to q_1 + q_2 + \ldots + q_n$, as shown in Figure 5. The form (II.27) is,

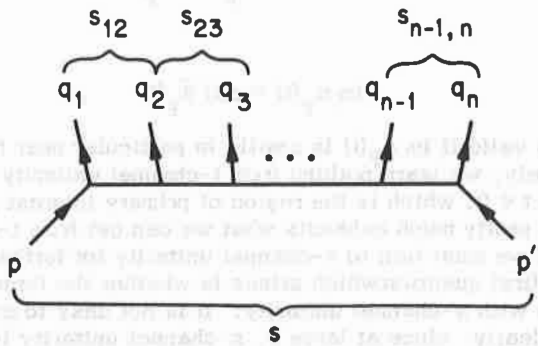


Fig. 5. Multi-Regge graph describing the process $p + p' \to q_1 + \ldots + q_n$.

according to the multi-Regge model, valid when all t_i are small and all pair energies $s_{i,i+1}$ are large.

The multi-Regge form, assuming that it is valid over all of phase space, and with certain assumptions about the behavior of the $\beta_i(t_{i-1}, t_i)$, has been used to test s-channel unitarity; and within these assumptions the form given in (II.21) is inconsistent when all $\alpha_i(t_i)$ are chosen to be $\alpha_P(t_i)$ with $\alpha_P(0) = 1$.[9)]

However, the authors of Ref. 9 point out that several things might invalidate their result. Among these is the possibility that the Pomeranchuk trajectory cannot be dealt with by itself, without also including the effects of its associated ℓ-plane cut. Our Pomeranchuk trajectory has a non-zero imaginary part for negative t only because of the existence of its associated cut. Without the cut, $\alpha_P(t) = 1$ for all t, which violates t-channel unitarity as well as s-channel unitarity. Thus in our case the ℓ-plane cut is inseparable from the trajectory itself, and we may hope that everything will in fact turn out to be consistent with s-channel unitarity, when the cut is included too.

An additional possibility is that for some as yet entirely un-understood reason, there is a rule that the Pomeranchuk trajectory cannot be exchanged more than once in the multi-Regge model. If this were the case, the calculation in Ref. 9 is clearly irrelevant. It is interesting that the existing experimental evidence seems to support such a rule.[10]

We therefore conclude that a Pomeranchuk Regge trajectory, whose real part is identically one for $t < 0$ (but whose imaginary part varies) can be consistent with all known requirements.

The next question is, how do we calculate the functions $\mathrm{Im}\, \alpha_P(t)$ and $f(t)$ appearing in Eq. (II.21)? At present, there is no clean way to do this, but certain approximate methods do exist which at least begin to answer part of this question, and we shall turn to these next.

III. Calculations of f(t)

We learned in the last section that the asymptotic form $T \rightarrow -is\, f(t)$ was not allowed, but that the modified form given in Eq. (II.21) was. Evidently, if $\mathrm{Im}\, \alpha$ is small, the first form will be a good approximation in the $t < 0$ region.

If we assume, then, this asymptotic behavior, can we calculate $f(t)$ and/or $\mathrm{Im}\, \alpha(t)$ from anything? The first source of information on these functions is s-channel unitarity, and we shall begin with a discussion of this.

Let us recapitulate our choice of normalization. The scattering amplitude $T(s,t)$ is chosen so that, at large s,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi} \left| \frac{T(s,t)}{s} \right|^2 \tag{III.1}$$

and s-channel unitarity then reads, at large s and negative t,

$$\mathrm{Im}\, T(s,t) = (\mathrm{Im}\, T(s,t))_{\text{inelastic}}$$

$$- \frac{1}{16\pi^3 s} \iint \frac{\theta(2t\, t_1 + 2t\, t_2 + 2t_1 t_2 - t^2 - t_1^2 - t_2^2)}{\sqrt{2t\, t_1 + 2t\, t_2 + 2t_1 t_2 - t^2 - t_1^2 - t_2^2}}\, dt_1\, dt_2$$

$$T(s,t_1)\, T^*(s,t_2) \quad . \tag{III.2}$$

Here, $(\mathrm{Im}\, T)_{\text{inelastic}}$ refers to the contribution to Im T of all inelastic intermediate states in the unitarity sum. The optical theorem relates $\mathrm{Im}\, T(s,o)$ to the total cross section: Again for large s,

$$\mathrm{Im}\, T(s,o) = -s\, \sigma_T(s) \quad . \tag{III.3}$$

Finally let us write the partial wave expansion

$$T(s,t) = \sum_{\ell} (2\ell+1) P_{\ell}(\cos\theta)\, T(s,\ell) \qquad \text{(III.4)}$$

where the partial wave amplitude is

$$T(s,\ell) = -16\pi \frac{E}{p} (\eta_{\ell}(s)\, e^{2i\delta_{\ell}(s)} - 1)/2i \quad . \qquad \text{(III.5)}$$

So much for normalizations. Now let us assume that

$$T(s,t) \to -is\, f(t)\, e^{i \,\mathrm{Im}\, \alpha(t) \log s/s_o} + \text{cut contribution.} \qquad \text{(III.6)}$$

as $s \to \infty$, t fixed and negative. Thus

$$\frac{d\sigma}{dt} \to \frac{1}{16\pi} |f(t)|^2 \qquad \text{(III.7)}$$

and, since

$$\mathrm{Im}\, T(s,t) = -s\, f(t) \cos(\mathrm{Im}\,\alpha(t) \log s/s_o) \qquad \text{(III.8)}$$

we have

$$\sigma_T(s) \to f(0) \quad . \qquad \text{(III.9)}$$

Now let us insert our ansatz into s-channel unitarity, Eq. (III.2). We have, ignoring for the moment the cut contribution,

$$s\, f(t) \cos(\mathrm{Im}\,\alpha(t) \log s/s_o) = -(\mathrm{Im}\, T(s,t))_{\text{inelastic}}$$

$$+ \frac{s}{16\pi^2} \iint \frac{dt_1\, dt_2}{\sqrt{\;\;''\;\;}} f(t_1) f(t_2)\, e^{i(\mathrm{Im}\,\alpha(t_1) - \mathrm{Im}\,\alpha(t_2)) \log s/s_o} \quad . \qquad \text{(III.10)}$$

If we furthermore assume that Im $\alpha(t)$ is small for $t < 0$, and that

$$(\mathrm{Im}\, T(s,t))_{\text{inelastic}} \to -s\, f_o(t) + 0(\mathrm{Im}\,\alpha(t)) \qquad \text{(III.11)}$$

then we get, approximately,

$$f(t) = f_o(t) + \frac{1}{16\pi^2} \iint \frac{dt_1\, dt_2}{\sqrt{\;\;''\;\;}} f(t_1) f(t_2) + 0(\mathrm{Im}\,\alpha(t)) \quad . \qquad \text{(III.12)}$$

This equation has, of course, been written down many times before[1)] (apart from the +0(Im α(t))). But while it looks attractive, it obviously does not solve anything but merely moves the difficulties into $f_o(t)$, or, rather, into $(\mathrm{Im}\ T)_{inelastic}$. Some theoretical models for f_o will be discussed later, but before doing that, it may be of interest to use Eq. (III.12) and experimental knowledge of f(t) to find what experiment says $f_o(t)$ looks like.

The best available high energy elastic data is for pp scattering; the experimental results are shown in Fig. 1.[2] In principle, one can use Eq. (III.7) and this data to extract f(t); however, this is possible only if the experimental data are truly from the asymptotic region. From the figure, it is clear that at larger values of t, $d\sigma/dt$ is still s-dependent, so if our assumptions are at all valid, at these larger t's one is not yet asymptotic. For small t, on the other hand, no s-dependence is evident, so here one may hope Eq. (III.7) applies. Out to $t \sim -2\ (\mathrm{BeV})^2$, then, we can (hopefully) get f(t) from the data, and hence $f_o(t)$ from Eq. (III.12). These two functions are displayed in Fig. 6. The crucial thing to notice is that $f_o(t)$ changes sign near $t \approx -0.7\ (\mathrm{BeV})^2$; this feature is a consequence of the rapid falloff of f(t), and results from the fact that the integral in Eq. (III.12) cannot fall as fast as f itself, so that it eventually overtakes f, thereby making f_o negative. The behavior of f_o for larger (negative) t is, as mentioned earlier, unreliable because the data are not yet asymptotic. (A method of extracting the truly asymptotic $d\sigma/dt$ from measured $d\sigma/dt$ for large but not asymptotic s would obviously be of great value.)

An additional way of comparing Eq. (III.12) with experiment is worth mentioning at this point. We had found $\sigma_T = f(0)$. Equation (III.12) allows us to break up σ_T into the elastic and inelastic total cross sections, by writing

$$\sigma_T = \sigma_T(\text{elastic}) + \sigma_T(\text{inelastic}) \qquad \text{(III.13)}$$

and we have

$$\sigma_T(\text{inelastic}) = f_o(0) \qquad \text{(III.14)}$$

while

$$\sigma_T(\text{elastic}) = \left(\frac{1}{16\pi^2} \iint \frac{dt_1\ dt_2\ f(t_1)\ f(t_2)}{\sqrt{\qquad ''\qquad}} \right) \Bigg|_{t=0} \qquad \text{(III.15)}$$

Any theoretical model for f_o, together with a solution of (III.12), then permits the prediction not only of $d\sigma/dt$ but also of $\sigma_T(\text{elastic})/\sigma_T$. This ratio is well measured in both pp and πp scattering; the results are 0.24 ± 0.01 for pp and 0.16 ± 0.01 for πp.

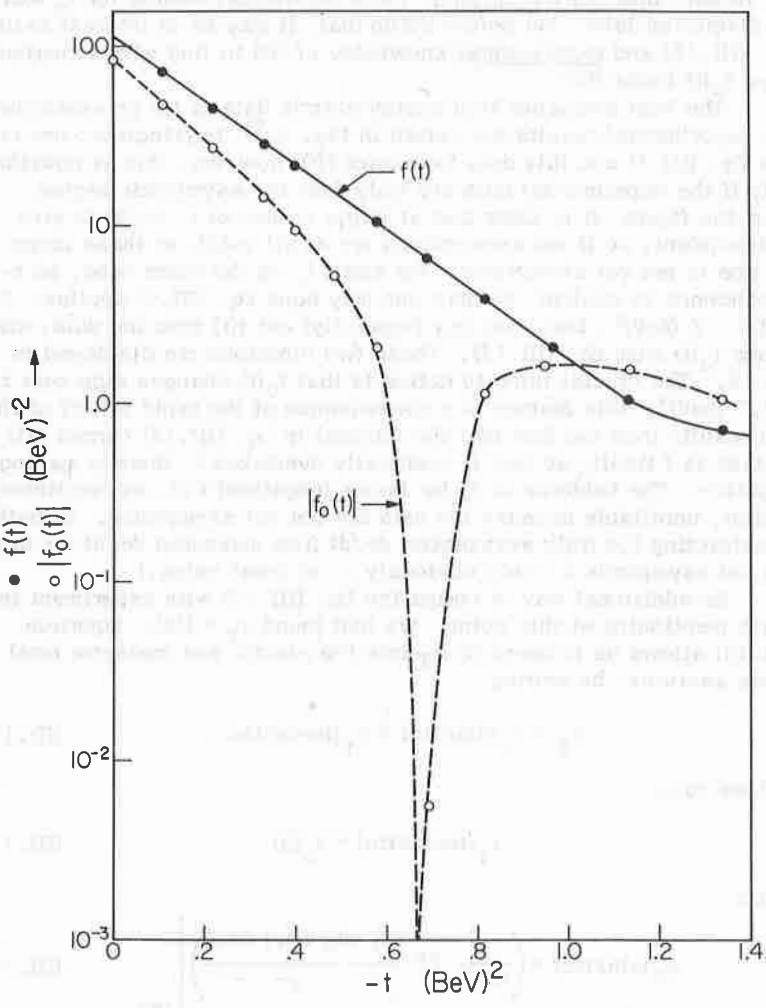


Fig. 6. The functions f(t) and $f_0(t)$, calculated from the pp elastic data of J. V. Allaby et al, Physics Letters 28B, 67 (1968), at $p_{lab} = 19.3$ $(BeV)^2$.

There is a model due to Van Hove based on Eq. (III.12), which it may be of interest to describe.[3] This model assumes, somewhat arbitrarily, that f_o is simply an exponential:

$$f_o(t) = B_1 e^{A_1 t} \quad . \tag{III.16}$$

Since this f_o has no sign change, we know already that it cannot fit the experimental data; nevertheless, it is amusing to carry out the solution of Eq. (III.12) for this case.

Let's try a solution for f(t) of the form

$$f(t) = \sum_{n=1}^{\infty} B_n e^{A_n t} \quad . \tag{III.17}$$

We substitute this in Eq. (III.12), and note that

$$\iint \frac{dt_1\, dt_2}{\sqrt{\quad''\quad}} e^{A_n t_1 + A_m t_2} = \frac{\pi}{A_n + A_m} e^{(A_n A_m / A_n + A_m)t} \quad . \tag{III.18}$$

Thus we have

$$\sum_{n=1}^{\infty} B_n e^{A_n t} = B_1 e^{A_1 t} + \frac{1}{16\pi} \sum_{n,m=1}^{\infty} \frac{B_n B_m}{A_n + A_m} e^{(A_n A_m / A_n + A_m)t} \quad . \tag{III.19}$$

Evidently this is solved by

$$A_n = \frac{A_m A_{n-m}}{A_m + A_{n-m}} \quad , \quad \text{or} \quad A_n = A_1 / n \tag{III.20}$$

and

$$B_n = \frac{1}{16\pi} \sum_{m=1}^{n-1} \frac{B_m B_{n-m}}{A_n + A_m} = \frac{1}{16\pi A_1} \sum_{m=1}^{n-1} \frac{m(n-m)}{n} B_m B_{n-m} \quad . \tag{III.21}$$

Let us now define

$$B_n = (16\pi A_1 / n) C_n \quad . \tag{III.22}$$

Then

$$C_n = \sum_{m=1}^{n-1} C_m C_{n-m} \tag{III.23}$$

and

$$f(t) = 16\pi A_1 \sum_{n=1}^{\infty} \frac{C_n}{n} e^{A_1 t/n} \quad . \tag{III.24}$$

Define

$$G(x) = \sum_{n=1}^{\infty} \frac{C_n}{n} e^{\frac{A_1 t}{n} x} \tag{III.25}$$

so that $f(t) = 16\pi A_1 G(1)$. Now Laplace transform $G(x)$:

$$L(s) = \int_0^{\infty} e^{-sx} G(x)\, dx = \sum_{n=1}^{\infty} \frac{C_n}{s(n-A_1 t/s)} \quad . \tag{III.26}$$

Next define

$$L(s,y) = \sum_{n=1}^{\infty} \frac{C_n}{s(n-A_1 t/s)} y^{n-A_1 t/s} \quad . \tag{III.27}$$

Then $L(s) = L(s,1)$ and

$$s \frac{\partial}{\partial y} L(s,y) = \sum_{n=1}^{\infty} C_n y^{n-A_1 t/s-1} \equiv y^{-1-A_1 t/s} F(y). \tag{III.24}$$

Now $F(y)$ is easy to evaluate; we have $F(y) = \sum_{n=1}^{\infty} C_n y^n$ and, using (III.23), we see that

$$(F(y))^2 - F(y) + C_1 y = 0 \tag{III.25}$$

or

$$F(y) = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1 - 4 C_1 y} \quad . \tag{III.26}$$

Since $F(0) = 0$, we must take the negative root. We can now integrate the differential equation (III.24), and we find

$$L(s) = L(s,1) = \frac{1}{2s} \int_0^1 dy' \, (y')^{-1-At/s} \, (1 - \sqrt{1-4\,C_1\,y'}). \quad \text{(III.27)}$$

Finally, we must invert the Laplace transform to get G(x):

$$G(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(x)\, e^{sx}\, dx \; ; \quad \text{(III.28)}$$

then

$$f(t) = G(1) = 8\pi\, A_1 \int_0^1 \frac{dy'}{y'} \, (1 - \sqrt{1 - 4\,C_1\,y'})\, J_0(2\sqrt{A_1\, t \log y'}) \, . \quad \text{(III.29)}$$

This solves our problem. We have evaluated f(t), and hence $d\sigma/dt$, for all t. In particular, at $t = 0$, Eq. (III.29) yields the result that

$$\sigma_T = f(0) = 16\pi\, A_1 \left(1 - \sqrt{1 - 4\,C_1} + \tfrac{1}{2} \log C_1 \, \frac{1+\sqrt{1 - 4\,C_1}}{1-\sqrt{1 - 4\,C_1}} \right) . \quad \text{(III.30)}$$

We also have, of course, $\sigma_T(\text{inelastic}) = B_1 = 16\pi\, A_1\, C_1$. Hence,

$$\frac{\sigma_T(\text{elastic})}{\sigma_T} = 1 - \frac{\sigma_T(\text{inelastic})}{\sigma_T}$$

$$= 1 - \frac{C_1}{1 - \sqrt{1 - 4\,C_1} + \tfrac{1}{2} \log C_1 \, \dfrac{1 + \sqrt{1 - 4\,C_1}}{1 - \sqrt{1 - 4\,C_1}}} \, . \quad \text{(III.31)}$$

The right-hand side takes on its maximum value at $C_1 = \frac{1}{4}$ (note, in fact, that the entire solution fails if $C_1 > \frac{1}{4}$; we shall see the physical reasons for this later), and this value is 0.2. Thus we have the result that $\sigma_T(\text{elastic})/\sigma_T \le 0.2$.[5] This is contradicted by the pp data; however, we already know that this model cannot explain the experiments because it has no sign change in $f_0(t)$.

Another way to see the disagreement with experiment is to note that the data suggests, at the very smallest t, a falloff in $d\sigma/dt$ like exp (10t). Thus we expect $A_1 \sim 5/(\text{BeV})^2$. However, because of our restriction that $C_1 < \frac{1}{4}$, we then get too small a value of f(0), and hence of $(d\sigma/dt)_{t=0}$, to fit the experimental value of 90 mb/$(\text{BeV})^2$ (or, in more sensible units, of $225/(\text{BeV})^4$).

A final remark of interest in the Van Hove model is that the asymptotic form of f(t) for large t comes out, from (III.29), to be

$$f(t) \underset{t \to \infty}{\longrightarrow} 16\pi \frac{\sqrt{\log \frac{1}{4} C_1}}{(-t)} e^{-2\sqrt{-A_1 t \log \frac{1}{4} C_1}} . \qquad (III.32)$$

This behavior $f(t) \sim 1/t\, e^{-\sqrt{-t}}$ is a faster falloff than seems indicated by the data, which looks more like $f(t) \sim t^{-4}$. (In fact, a model due to Chou and Yang[4] suggests that $f(t) \sim (F(t))^2$ where $F(t)$ is the proton electromagnetic form factor. We will return to this in more detail later.) However, as we have already indicated, at large it is unlikely that the experiment is as yet in the asymptotic region, so just what f(t) is for large t is not really known.

To summarize, the Van Hove model, while it does not fit the data, is a useful illustration of how Eq. (III.12) could be used, provided that one is given an input $f_0(t)$ from someplace else. Our next task, then, should be to actually calculate an accurate $f_0(t)$ from inelastic intermediate states.

IV. Partial Waves

For many purposes, it is more convenient to look at the partial wave amplitude rather than the entire T(s,t). We had, we recall, defined the partial wave amplitude by

$$T(s,\ell) = -16\pi \frac{E}{p} \left(\frac{\eta_\ell(s)\, e^{2i\delta_\ell(s)} - 1}{2i} \right) . \qquad (IV.1)$$

Unitarity for partial waves reads

$$\mathrm{Im}\, T(s,\ell) = -\frac{1}{16\pi} \frac{p}{E} T(s,\ell)^2 - 16\pi \frac{E}{p} \left(\frac{1 - \eta_\ell^2(s)}{4} \right) . \qquad (IV.2)$$

Hence,

$$(\mathrm{Im}\, T(s,\ell))_{\text{inelastic}} = -16\pi \frac{E}{p} \left(\frac{1 - \eta_\ell^2(s)}{4} \right) . \qquad (IV.3)$$

Let us now invoke our assumption that $\mathrm{Im}\, T(s,t) = -s\, f(t) + 0(\mathrm{Im}\, \alpha(t))$ and $(\mathrm{Im}\, T(s,t))_{\text{inelastic}} = -s\, f_0(t) + 0(\mathrm{Im}\, \alpha(t))$. Then

$$16\pi \frac{E}{p} \cdot \frac{1 - \eta_\ell^2(s)}{4} = \frac{s}{2} \int_{-1}^{1} d(\cos\theta)\, P_\ell(\cos\theta)\, f_0(t) . \qquad (IV.4)$$

Equation (IV.4) permits us to calculate the inelasticity for any $f_o(t)$: explicitly,

$$\eta_\ell^2(s) = 1 - \frac{1}{4\pi}\int_{-s}^{0} dt\, f_o(t)\, P_\ell(1+2t/s) \quad . \qquad (IV.5)$$

Note that if $\eta_\ell(s) \to 1$ as $s \to \infty$ for fixed ℓ, we must have

$$\int_{-\infty}^{0} dt\, f_o(t) = 0$$

which demonstrates, among other things, that $f_o(t)$ would have to change sign. Experimentally, it is unclear if this equation holds or not, since $f_o(t)$ is not known for large (negative) t. Thus we do not know what experiment suggests for $\eta_\ell(\infty)$.

In the Van Hove model, we had $f_o = B_1 e^{A_1 t}$, which yields, for large s,

$$\eta_\ell(s) = \sqrt{1 - \frac{B_1}{4\pi}\sqrt{\frac{\pi s}{A_1}}\, e^{-A_1 s/2}\, I_{\ell+\frac{1}{2}}(A_1 s/2)} \quad , \qquad (IV.6)$$

and if we let $s \to \infty$, we have

$$\eta_\ell(s) \to \sqrt{1 - B_1/4\pi A_1} = \sqrt{1 - 4\, C_1} \quad . \qquad (IV.7)$$

Thus, in order that $\eta_\ell(s)$ remain real (which is obviously necessary physically) we must have $C_1 < \frac{1}{4}$ in the Van Hove model. This is the physical origin of the restriction we found earlier. If the restriction is satisfied, the η_ℓ approaches some constant between 0 and 1 asymptotically.

The principal value of the partial wave approach may lie in the fact that it permits us to make finite s corrections to the asymptotic differential cross sections. In the models we are discussing, we expect $\delta_\ell(s) \to 0$ as $s \to \infty$, so that

$$T(s,\ell) \to -16\pi \frac{\eta_\ell(s) - 1}{2i} \quad . \qquad (IV.8)$$

Thus

$$T(s,t) \to 16\pi\, i \sum_\ell (2\ell+1)\, P_\ell(\cos\theta)(\eta_\ell(s)-1) = -is\, f(t) \qquad (IV.9)$$

and therefore for large s

$$\eta_\ell(s) = 1 - \frac{1}{8\pi} \int_{-s}^{0} dt\, P_\ell(1 + 2t/s)\, f(t) \quad . \tag{IV.10}$$

(This equation is, of course, exactly the same as (IV.5), in view of the relation (III.12) between f(t) and $f_o(t)$.) Thus, if we knew the true asymptotic $d\sigma/dt$, and hence f(t), we could calculate $\eta_\ell(s)$.

Now the phase shift itself satisfies a dispersion relation. Again, (for s much bigger than masses) we have[11]

$$\delta_\ell(s) = -\frac{\sqrt{s}}{2\pi} \oint_{s_{\text{inelastic}}}^{\infty} \frac{\log \eta_\ell(s')}{\sqrt{s'}\,(s'-s)}\, ds' + \frac{\sqrt{s}}{\pi} \int_{\text{LHC}} \frac{\text{Im}\, \delta_\ell(s')}{\sqrt{s'}\,(s'-s)}\, ds' . \tag{IV.11}$$

It is plausible to believe that the leading behavior of $\delta_\ell(s)$ can be calculated from this relation, using the (known) asymptotic η_ℓ's. These phase shifts, together with η_ℓ, can then be inserted into Eq. (IV.1), and the partial wave expansion summed, to get a corrected T(s,t). This T(s,t), of course, approaches $-is\, f(t)$ as $s \to \infty$, but for finite s deviates from it, and the deviation is larger, for larger t. Thus we may be able to compare the corrected T to experiment at finite s, and thereby check that a guessed asymptotic f(t) is correct.

To make this idea more specific, let us write, for finite but large s,

$$T(s,t) = \bar{T}(s,t) + \Delta T(s,t) \tag{IV.12}$$

where $\bar{T}(s,t) = -is\, f(t)$ is the purely imaginary and truly asymptotic amplitude. Corresponding to this, let us write

$$T(s,\ell) = \bar{T}(s,\ell) + \Delta T(s,\ell) \tag{IV.13}$$

and

$$\eta_\ell(s) = \bar{\eta}_\ell(s) + \Delta\eta_\ell(s) \tag{IV.14}$$

where

$$\bar{T}(s,\ell) = -16\pi \frac{\bar{\eta}_\ell(s) - 1}{2i} \quad . \tag{IV.15}$$

Next, from (IV.1), we evidently can deduce that

$$\eta_\ell(s) \cos 2\,\delta_\ell(s) = 1 + \frac{1}{16\pi} \int_{-1}^{1} dx\, P_\ell(x)\, \text{Im}\, T(s,t) \tag{IV.16}$$

and that

$$\eta_\ell(s) \sin 2\,\delta_\ell(s) = -\frac{1}{16\pi}\int_{-1}^{1} dx\, P_\ell(x)\, \mathrm{Re}\, T(s,t). \qquad (IV.17)$$

Now, for large but finite s, we expect the phase shift δ_ℓ to be small. Thus (IV.16) and (IV.17) become, to first order in δ,

$$\eta_\ell(s) = 1 + \frac{1}{8\pi s}\int_{-s}^{0} dt\, P_\ell(1+2t/s)\, \mathrm{Im}\, T(s,t) \qquad (IV.18)$$

and

$$\eta_\ell(s)\,\delta_\ell(s) = -\frac{1}{16\pi s}\int_{-s}^{0} dt\, P_\ell(1+2t/s)\, \mathrm{Re}\, \Delta T(s,t). \qquad (IV.19)$$

But $\mathrm{Im}\, T(s,t) \approx \mathrm{Im}\, \bar{T}(s,t) = -s\, f(t)$, so that

$$\eta_\ell(s) \approx \bar{\eta}_\ell(s) = 1 - \frac{1}{8\pi}\int_{-s}^{0} dt\, P_\ell(1+2t/s)\, f(t) \qquad (IV.20)$$

and $\Delta\eta/\bar{\eta} \sim \mathrm{Im}\,\Delta T/\mathrm{Im}\,\bar{T} << 1$.

We may next invoke Eq. (IV.11). Let us write $\delta_\ell(s) = \bar{\delta}_\ell(s) + \Delta\delta_\ell(s)$, where

$$\bar{\delta}_\ell(s) = -\frac{\sqrt{s}}{2\pi}\, \mathrm{P}\!\!\int_{s_{inelastic}}^{\infty} \frac{\log \bar{\eta}_\ell(s')}{\sqrt{s'}\,(s'-s)}\, ds' \quad . \qquad (IV.21)$$

Then

$$\Delta\delta_\ell(s) = -\frac{\sqrt{s}}{2\pi}\, \mathrm{P}\!\!\int_{s_{inelastic}}^{\infty} \frac{ds'}{\sqrt{s'}\,(s'-s)}\, \frac{\Delta\eta_\ell(s')}{\bar{\eta}_\ell(s')}$$

$$+\frac{\sqrt{s}}{\pi}\int_{LHC} \frac{\mathrm{Im}\,\delta_\ell(s')}{\sqrt{s'}\,(s'-s)}\, ds' \quad . \qquad (IV.22)$$

Certainly the first term in (IV.22) is much less than the integral in (IV.21); if this were the full story, we would conclude directly that $\Delta\delta/\bar{\delta} \sim \Delta\eta/\bar{\eta} << 1$. However, the left-hand cut contribution to the phase shift muddies the story somewhat. We may hope that, since s is so large and positive, while s′ in the LHC integral is always negative, that the LHC integral is relatively unimportant. Thus we may hope that $\Delta\delta/\bar{\delta} << 1$, so that in Eq. (IV.19) we may replace the left-hand side by $\eta_\ell(s)\,\bar{\delta}_\ell(s)$. Then Eq. (IV.19) may be inverted, to give us an expression for Re T(s,t):

$$\mathrm{Re}\,\Delta T(s,t) = -16\pi \sum_{\ell} (2\ell+1)\, P_\ell(x)\, \bar{\eta}_\ell(s)\, \bar{\delta}_\ell(s) \quad . \qquad (IV.23)$$

Hopefully, and provided that the LHC contribution may be neglected, this formula gives a reasonable approximation the real part of the amplitude, in the region of large s but not so large s that it has gone away entirely. We may remark, in closing, that for computational purposes, it may be convenient to replace (IV.23) by the impact parameter form

$$\mathrm{Re}\,\Delta T(s,t) = -8\pi s \int_0^\infty b\, db\, J_0(b\sqrt{-t})\, \bar{\eta}(b)\, \bar{\delta}(s,b) \qquad (IV.24)$$

where, as usual, $(2\ell+1) = \sqrt{s}\, b$. In any event, whether (IV.23) or (IV.24) is more convenient, the expression (IV.21) is to be used for $\bar{\delta}$.

V. Models for $f_0(t)$

We left the basic problem, the calculation of $f_0(t)$, at the end of Section III. In pictures, $f_0(t)$ is displayed in Fig. 7. Explicitly,

$$(\mathrm{Im}\, T(s,t))_{\mathrm{inelastic}} = -\tfrac{1}{2} \sum_{n=3}^{\infty} (2\pi)^4\, \delta^4(P_n - P)\, T_{i\to n}\, T^*_{f\to n} \qquad (V.1)$$

where the sum is over all states of three or more particles, and $T_{i\to n}$ is the amplitude for the intial two particles to go to an n-particle state, and likewise for $T_{f\to n}$. P is the total four-momentum of the scattering, so that $P^2 = s$. P_n is the four-momentum of the n-particle state.

Our fundamental assumption is that $T_{i\to n}$ is such that for large s, $(\mathrm{Im}\,T)_{\mathrm{inelastic}}$ becomes proportional to s (up to higher order terms in Im $\alpha(t)$), and $(\mathrm{Im}\,T)_{\mathrm{inelastic}} \to$ -is $f_0(t)$. Equation (V.1) can then, in principle, be used to calculate $f_0(t)$.

What is needed is a model for $T_{i\to n}$, that is, for the amplitude for two particles to become n particles. Various such models can be thought of, some of which are suggestive, but none which have, so far at least, led to a really convincing understanding of the experiments. A partial list, with a brief explanation, of various models follows.

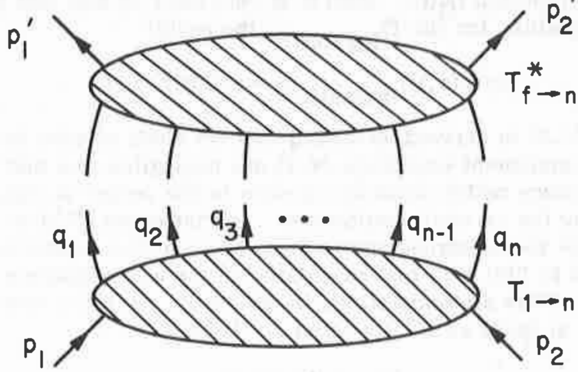


Fig. 7. Diagram representing contributions to $(\mathrm{Im}\, T(s,t))_{\mathrm{inelastic}}$

(i) Suppose that the scattering particles are composite, and for simplicity, suppose they are made of two constituent particles. (An illustration, if the scattering particles were mesons, might be quark-antiquark for the two constituents.) Then a simple picture for $(\mathrm{Im}\, T)_{\mathrm{inelastic}}$ is given in Fig. 8. Now suppose that the constituent-constituent scattering amplitude, at high energies, also has the characteristic form

$$T(s,t)_{CC} \to -is\, f_C(t) \tag{V.2}$$

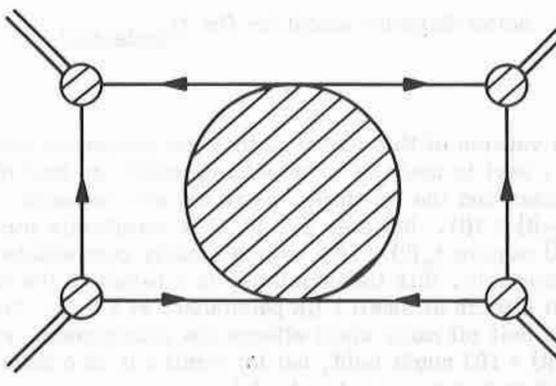


Fig. 8. Model for $(\mathrm{Im}\, T)_{\mathrm{inelastic}}$ in terms of some elementary constituent particles.

with some function $f_C(t)$. Then it is very easy to see that the diagram of Fig. 8 yields, for $(\text{Im } T)_{\text{inelastic}}$, the result

$$(\text{Im } T(s,t))_{\text{inelastic}} \to -s\ (F(t))^2\ f_C(t)\ . \qquad (V.3)$$

Equation (V.3) is derived assuming that off shell effects in the constituent-constituent amplitude (V.2) are negligible, so that the off shell amplitude which actually appears in the center of Fig. 8 can be replaced by the on shell amplitudes. In that event, F(t) in Eq. (V.3) is precisely the electromagnetic form factor of the scattering particle, normalized to F(0) = 1, and calculated in the approximation illustrated in Fig. 9. (This approximation, incidentally, yields a form factor falling off at least as t^{-2} for large t.[1]) Thus

$$f_0(t) = f_C(t)\ (F(t))^2\ . \qquad (V.4)$$

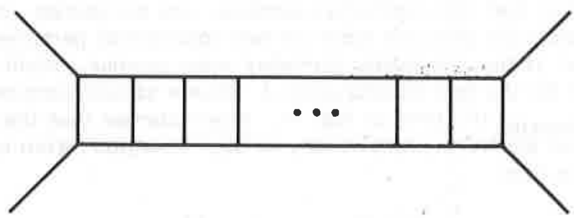


Fig. 9. Bethe-Salpeter model for $(\text{Im } T)_{\text{inelastic}}$.

Note that one version of this model is that the composite particle (if it is a meson, say) is made up of itself and itself, so that the constituent particles and the scattering particles are the same. Then we would have $f_C(t) = f(t)$. But then Eq. (V.4) is manifestly inconsistent since it would require $f_0(0) = f(0)$, which clearly contradicts Eq. (III.12). Presumably, this inconsistency is a result of the neglect of off mass shell effects at small t (in particular, at t = 0). For large t, it is plausible that off mass shell effects are unimportant, so that (V.4) with $f_C(t) = f(t)$ might hold, but for small t it is a priori unlikely to be true, and in fact turns out to be false.

In the absence of a theory from which $f_C(t)$ can be calculated, Eq. (V.4) and this entire composite model, is essentially useless. Perhaps only one remark is worth making, namely the following: If $f_C(t) \to$ const. $\equiv C$ as $t \to \infty$, and if $F(t) \sim t^{-n}$ for large t (n = 2 experimentally), then $f_0(t) \sim t^{-2n}$ for large t. But then the solution to Eq. (III.12) yields $f(t) \sim t^{-2n}$ at large t as well, since the integral in (III.12), with an input of $f(t) \sim t^{-2n}$, itself behaves asymptotically like t^{-2n}. Thus we have $f(t) \sim (F(t))^2$ for large t, a result first suggested by Chou and Yang,[2] and later elaborated on by Abarbanel, Drell, and Gilman,[3] among others.

(ii) A second type of model consists of guessing a form for $T_{2 \to n}$ and making use of Eq. (V.1) directly. Evidently, in order to be consistent with Eq. (III.12), and, indeed, with our entire outlook, our guess for $T_{2 \to n}$ must be such as to yield an $(\mathrm{Im}\ T)_{\mathrm{inelastic}}$ proportional to s (apart from higher order terms in $\mathrm{Im}\ \alpha_p(t'))$ at large s. Various models suggest themselves. For example, the multi-Regge model provides a form for $T_{2 \to n}$, as described at the end of Section II. However, various difficulties (which have yet to be cleared up) were mentioned there, which make its use in this context somewhat ambiguous. Perhaps a less deep, and more phenomenological, choice is to assume

$$T_{2\to n} = C \prod_{i=1}^{n} e^{-\alpha (p_\perp)_i^2} \qquad (V.5)$$

where $(p_\perp)_i$ is the transverse momentum of the i-th produced particle. Such a model is, roughly, consistent with present data.

(iii) A somewhat simplified version of model (ii) is simply to say $(\mathrm{Im}\ T)_{\mathrm{inelastic}}$ can be approximated by a sideways ladder, as shown in Fig. 9, and to calculate the ladder using the Bethe-Salpeter equation.

However, with regard to both models (ii) and (iii), (and indeed, for any models which try to calculate, or guess $T_{2 \to n}$), one may say that the crucial ingredient is a form for $T_{2 \to n}$ which can give an $(\mathrm{Im}\ T)_{\mathrm{inelastic}}$ which is linear in s. No (reasonable) such model has yet been made. In particular, (V.5) will certainly not give a linear behavior without dependence on $(p_\parallel)_i$, and, as was indicated earlier, the multi-Regge model for $T_{2 \to n}$ with a flat Pomeranchon input does not yield a linear dependence for $(\mathrm{Im}\ T)_{\mathrm{inelastic}}$ unless one adopts the ad hoc (but possibly true) rule that the Pomeranchon is exchanged only once.

We must therefore end on a somewhat inconclusive note: there exists, as yet, no plausible theory of diffraction scattering which starts with the basic assumptions of particle physics (that is, with unitarity, crossing, analyticity) and proceeds to a differential cross section. All that one can do is to use these general principles to pinpoint the production amplitude $T_{2 \to n}$ as the missing piece in the puzzle. Until a reliable theory of $T_{2 \to n}$, valid over most of phase space, is constructed, progress in understanding diffraction processes will be very limited.

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CAUSALITY IN ELECTROPRODUCTION AT HIGH ENERGY†

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I. Introduction

Electron scattering serves to produce a virtual photon of space-like four momentum which probes nucleon structure in a very clean way. Elastic scattering experiments have been carried out extensively, and we now have a fairly detailed knowledge of the nucleon form factors as functions of the virtual photon mass. In these experiments, the nucleon recoils elastically, the photon interacts with the nucleon material in a coherent manner, and these form factors are related, roughly, to the average shape of a nucleon. High energy inelastic electron-proton scattering experiments are now being performed although as yet we have only preliminary results. The inelastic total scattering cross section is described by two structure functions that depend on both the photon energy (ν) and the photon mass (k^2). In the inelastic process the photon interacts in an incoherent manner and it probes, roughly, the instantaneous construction of the proton rather than the average shape found in the elastic scattering experiments. The structure functions can be expressed in terms of the Fourier transform of the commutator of two electromagnetic current operators. The high energy behavior of the structure functions is therefore also correlated with the nature of this commutator at small space-time intervals. It is to this aspect of electroproduction that these lectures are addressed.

We shall describe the subject matter of these lectures with an annotated list. Sec. II reviews the kinematical description of the total inelastic electroproduction cross section and the relationship of the structure functions to a matrix element of current operators. The structure functions are the absorptive part of the forward, virtual

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photon scattering amplitude. Sec. III is devoted to the kinematics involved in the Regge pole analysis of this Compton amplitude, and it is shown that the Regge residue function must be singular if the vacuum trajectory is to contribute to the amplitude that describes real photon scattering. This must occur if the photoabsorption cross section is to approach a constant high energy limit. Such a singularity is obtained if the amplitude has a fixed pole in the angular momentum plane. Sec. IV reviews the experimental situation. The high energy limit of the photoabsorption cross section is seen to indeed be constant and, what is more striking, the high energy electroproduction cross section appears to be well described by a scaling limit of the structure functions, a limit in which $\nu \to \infty$ with k^2/ν fixed that involves a dimensionless function of the dimensionless parameter k^2/ν. The remainder of the lectures is devoted to investigating what applications causality, the condition that two current operators commute at space-like separation, may have in the understanding of these results.

The nature of causal representations is discussed in Sec. V without pretense to mathematical rigor but hopefully in a way that makes the structure of these representations clear. A causal representation, the Jost-Lehmann representation, is used to discuss the high energy behavior of the electroproduction structure functions in Sec. VI. With the assumption that the Jost-Lehmann weight functions are uniformly convergent, it is shown that the $k^2 \to \infty$ limit of the Regge asymptotic form is related to the small k^2/ν behavior of the scaling limit and, moreover, that this relation is in excellent accord with experiment. The Jost-Lehmann representation also provides a connection between the scaling limit and the behavior of current matrix elements on the light cone which suggests that the conformal group may have some role in the description of this limit. The relationship between equal-time commutators and the scaling limit is considered in Sec. VII. The validity of the scaling limit is shown to require that the spin-averaged nucleon matrix element of the commutator of two spatial current components at equal time must vanish. Finally, it is proven that if the corresponding commutator with one time derivative has a transverse structure then so does the scaling limit and, conversely, if the commutator has a longitudinal structure then so does the scaling limit.

The work described in Secs. VI and VII was carried out in collaboration with D. G. Boulware. I have also enjoyed conversations on some of this material with S. B. Treiman. These notes were written at the Aspen Center for Physics.

II. General Kinematics

The electromagnetic process e + N → e + (hadrons) is illustrated in Fig. 1. We denote the initial and final four momenta of the electron by q and q′, the initial momentum of the nucleon by p, and the total four momentum of the final hadronic state by P′. The four-momentum balance reads

$$P' + q' = p + q \; . \tag{II.1}$$

We label the initial and final spins of the electron by κ and κ', respectively, the spin of the initial nucleon by λ, and we use the symbol ζ' to represent all the variables of the final hadronic state other than its total form-momentum P′.

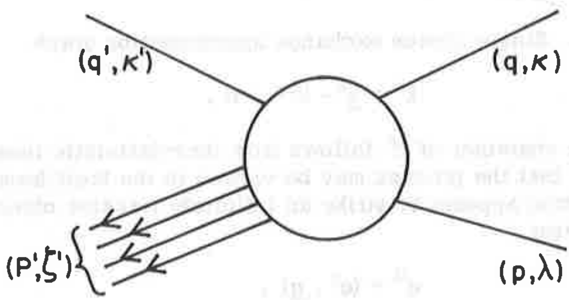

Fig. 1. Pictorial representation of the inelastic electron-proton scattering process.

Since electromagnetism is relatively weak, its effect can be treated in lowest order, and we need compute only the single photon exchange contribution[1] depicted in Fig. 2. In effect, the scattering of the electron serves only to produce a virtual photon of four momentum

$$\begin{aligned} k &= q - q' \\ &= P' - p \end{aligned} \tag{II.2}$$

which then probes the nucleon and excites it to some final hadronic state. The effective mass carried by the virtual photon (which is also the square of the four-momentum transfer imparted by the electron) is space-like

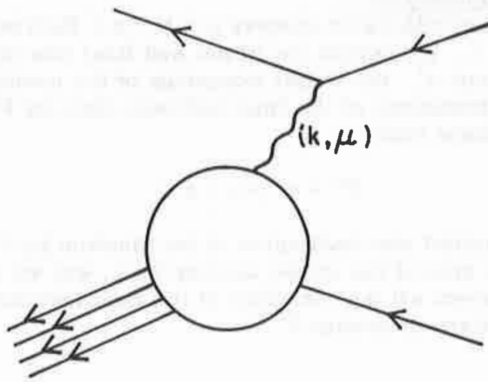

Fig. 2. Single photon exchange approximation graph.

$$k^2 = \underset{\sim}{k}^2 - k^{o2} > 0 \ . \qquad \text{(II.3)}$$

The space-like character of k^2 follows from its relativistic invariance and the remark that the process may be viewed in the Breit frame where the electron appears to strike an infinitely massive object. In this special frame

$$q^\mu = (q^o , \underset{\sim}{q}) \ ,$$

$$q'^\mu = (q^o , -\underset{\sim}{q}) ,$$

and

$$k^\mu = (0 , 2\underset{\sim}{q})$$

is manifestly space-like. In addition to the effective mass of the virtual photon, we shall use a relativistically invariant energy parameter

$$\nu = -pk \ . \qquad \text{(II.4)}$$

In the laboratory frame in which the initial nucleon is at rest

$$p^\mu = (m, \underset{\sim}{0}) \ ,$$

this parameter is essentially the energy of the virtual photon

$$\nu = mk^0 \quad ,$$

or, equivalently, it gives the energy loss of the scattered electron,

$$\nu = m(q^0 - q'^0).$$

In terms of these variables, the squared total mass of the final hadronic state is given by

$$s = -P'^2 = -(p + q - q')^2$$
$$= m^2 + 2\nu - k^2 \quad . \qquad \text{(II.5)}$$

If $s = m^2$ we have elastic electron-proton scattering which is, of course, a special case of the general inelastic process. For $s \geq (m + \mu_\pi)^2$ we have electropion production, and so forth. Thus

$$2\nu - k^2 \geq 0 \qquad \text{(II.6)}$$

with the equality holding only for elastic scattering while for inelastic scattering $2\nu - k^2$ exceeds $2m\mu_\pi + \mu^2_\pi$.

The transition matrix element in the single photon exchange approximation can be easily derived with the aid of the usual reduction technique. If an electron field is used to create one of the electron states and the reduction method applied to it, a matrix element involving the electron and photon fields is obtained. After expressing the photon field in terms of its hadronic current source j^μ one arrives at

$$\langle p'\zeta', q'\kappa' \text{ out} | p\lambda, q\kappa \text{ in}\rangle$$
$$= i(2\pi)^4 \, \delta(P' + q' - p - q)T, \qquad \text{(II.7)}$$

with

$$T = e^2 \bar{u}_{\kappa'}(q') \gamma_\mu u_\kappa(q) \, k^{-2} \, \langle P'\zeta' \text{ out} | j^\mu(0) | p\lambda\rangle \quad . \qquad \text{(II.8)}$$

Here we use an invariant normalization of states so that, for example,

$$\langle q'\kappa' | q\kappa\rangle = (2\pi)^3 \, 2q^0 \, \delta^{(3)}(q' - q) \, \delta_{\kappa'\kappa} \quad , \qquad \text{(II.9)}$$

and

$$\bar{u}_{\kappa'}(q)\gamma^{\mu}u_{\kappa}(q) = 2q^{\mu}\delta_{\kappa'\kappa} . \qquad \text{(II.10)}$$

The differential scattering cross section in the laboratory frame can now be calculated by the usual mnemonic method. The rate at which scattering occurs on a single target is the absolute square of the transition matrix element divided by the total elapsed time $2\pi\delta^{(1)}(0)$ and by the number of target particles which, according to the normalization convention (II.9), is $(2\pi)^3\delta^{(3)}(0)2m$. This rate divided by the flux of incident electrons is the differential cross section. Since $(2\pi)^3\delta^{(3)}(0)$ is associated with the volume of space, our normalization corresponds to an electron flux of $2q^0$ (electron velocity) $= 2|\underset{\sim}{q}|$. We shall calculate only the unpolarized cross section[2] so we average over initial helicities and sum over final helicities. Furthermore, we shall assume that only the final electron is detected, and so we sum over all final hadronic states. Accordingly, the cross section for scattering into some interval Δ of final electron momenta is given by

$$\Delta\sigma = \tfrac{1}{2}\Sigma_{\kappa\kappa'}\int_{\Delta}\frac{(d\underset{\sim}{q}')}{(2\pi)^3}\,\frac{1}{2q'^0}\,\tfrac{1}{2}\Sigma_{\lambda}\Sigma_{\zeta'}\int\frac{(d\underset{\sim}{P}')}{(2\pi)^3}\,\frac{1}{2P'^0}$$

$$(2\pi)^4\,\delta(P'+q'-p-q)|T|^2\,\frac{1}{2m2|\underset{\sim}{q}|} . \qquad \text{(II.11)}$$

The strong interaction part of the transition amplitude (II.8) enters into the cross section in the form

$$A^{\mu\nu} = \tfrac{1}{2}\Sigma_{\lambda}\Sigma_{\zeta'}\int\frac{(d\underset{\sim}{P}')}{(2\pi)^3}\,\frac{1}{2P'^0}\,(2\pi)^4\,\delta(P'-p-k)$$

$$\langle p\lambda|j^{\mu}(0)|P'\zeta'\ \text{out}\rangle\langle P'\zeta'\ \text{out}|j^{\nu}(0)|p\lambda\rangle . \qquad \text{(II.12)}$$

Here we have used the Hermitian property of the current to write

$$\langle P'\zeta'\ \text{out}|j^{\mu}|p\lambda\rangle^* = \langle p\lambda|j^{\mu}|P'\zeta'\ \text{out}\rangle . \qquad \text{(II.13)}$$

We should note that the structure tensor $A^{\mu\nu}$ is the absorptive part of the forward, virtual-photon Compton scattering amplitude. It can be expressed as the Fourier transform of a current matrix element. To this end, we make use of the representation

$$(2\pi)^4\,\delta(P'-p-k) = \int(dx)e^{i(P'-p-k)x} \qquad \text{(II.14)}$$

and of the energy-momentum operator P^{μ} in the form

$$e^{i(P'-p)x}\langle p\lambda|j^\mu(0)|P'\zeta'\ \text{out}\rangle = \langle p\lambda|e^{-iPx}j^\mu(0)e^{iPx}|P'\zeta'\ \text{out}\rangle$$
$$= \langle p\lambda|j^\mu(x)|P'\zeta'\ \text{out}\rangle\ . \tag{II.15}$$

In this way all reference to the particular final state is removed and the completeness of the final states

$$\Sigma_{\zeta'}\int\frac{(d\underline{P}')}{(2\pi)^3}\ \frac{1}{2P'^0}|P'\zeta'\ \text{out}\rangle\langle P'\zeta'\ \text{out}| = 1 \tag{II.16}$$

may be employed to give the simple result:

$$A^{\mu\nu} = \int(dx)e^{-ikx}\tfrac{1}{2}\Sigma_\lambda\langle p\lambda|j^\mu(x)j^\nu(0)|p\lambda\rangle\ . \tag{II.17}$$

This structure will be the basis for our later discussion of the theory of electroproduction. We note, incidentally, that the energy-momentum operator P^μ can be used to prove the translation invariance

$$\langle p\lambda|j^\mu(x)j^\nu(0)|p\lambda\rangle = \langle p\lambda|j^\mu(0)j^\nu(-x)|p\lambda\rangle\ . \tag{II.18}$$

Since we have used an invariant normalization of states, $A^{\mu\nu}$ is a Lorentz tensor and can be expanded in terms of $g^{\mu\nu}$, $p^\mu p^\nu$, $p^\mu k^\nu$, $k^\mu p^\nu$, and $k^\mu k^\nu$ with scalar coefficients that depend upon the two invariants that one can construct, ν and k^2. Not all of these scalars are independent however, for the current is conserved,

$$\partial_\mu j^\mu(x) = 0\ . \tag{II.19}$$

It follows from Eqs. (II.17) and (II.18) by partial integration that this requires

$$k_\mu A^{\mu\nu} = 0 = A^{\mu\nu}k_\nu\ . \tag{II.20}$$

We can combine $k^\mu k^\nu$ with $k^2 g^{\mu\nu}$ to immediately obtain one covariant that satisfies this constraint

$$^{(2)}t^{\mu\nu} = k^\mu k^\nu - k^2 g^{\mu\nu}\ . \tag{II.21b}$$

With $k^\mu k^\nu$ now eliminated from the list of tensor forms there remains only one combination that obeys the gauge invariance constraint (II.20):

$$
\begin{aligned}
{}^{(1)}t^{\mu\nu} &= p^\mu p^\nu k^2 - (p^\mu k^\nu + k^\mu p^\nu)pk + g^{\mu\nu}(pk)^2 \\
&= p^\mu p^\nu k^2 + (p^\mu k^\nu + k^\mu p^\nu)\nu + g^{\mu\nu}\nu^2 \ .
\end{aligned}
\tag{II.21a}
$$

Thus we have the decomposition

$$
A^{\mu\nu} = {}^{(1)}t^{\mu\nu} A_1(\nu, k^2) + {}^{(2)}t^{\mu\nu} A_2(\nu, k^2) \ . \tag{II.22}
$$

Covariants obtained from combinations of ${}^{(1)}t^{\mu\nu}$ and ${}^{(2)}t^{\mu\nu}$ with factors of ν or k^2 appearing as denominators should not be used, for they contain spurious kinematical singularities whose cancellation requires constraints between the corresponding scalar amplitudes. By construction, the covariants (II.21) are free of kinematical singularities and the structure functions $A_{1,2}$ are correspondingly free of such kinematical constraint.[3]

Since the tensor $A^{\mu\nu}$ was originally constructed in terms of a sum of squares (II.12) it is real, and, moreover it is positive-definite in the sense that

$$
a_\mu^* A^{\mu\nu} a_\nu \geq 0 \tag{II.23}
$$

for an arbitrary complex four-vector a_μ. The full content of this positivity condition can be obtained most easily if the vector a^μ is expanded as

$$
a^\mu = \alpha p^\mu + \beta k^\mu + \ell^\mu + m^\mu \ . \tag{II.24}
$$

Since p^μ is time-like, ℓ^μ and m^μ can be chosen to be two space-like vectors which are orthogonal to each other and to p^μ and k^μ. With this expansion, the positivity condition (II.23) becomes

$$
\begin{aligned}
&|\alpha|^2 (\nu^2 + k^2 m^2) [m^2 A_1 + A_2] \\
&+ (|\ell|^2 + |m|^2) [\nu^2 A_1 - k^2 A_2] \geq 0
\end{aligned}
\tag{II.25}
$$

and is therefore equivalent to the two constraints

$$
m^2 A_1 + A_2 \geq 0 , \tag{II.26a}
$$

and

$$
\nu^2 A_1 - k^2 A_2 \geq 0 \tag{II.26b}
$$

or,

$$A_1 \geq 0 \ , \tag{II.27a}$$

and

$$- m^2 A_1 \leq A_2 \leq (\nu^2/k^2) A_1 \ . \tag{II.27b}$$

To complete the cross section calculation, we need the electron tensor corresponding to the hadronic structure tensor $A^{\mu\nu}$. It is given by

$$a^{\mu\nu} = \tfrac{1}{2}\Sigma_{\kappa',\kappa}[\bar{u}_{\kappa'}(q')\,\gamma^{\mu} u_{\kappa}(q)]^{*}\bar{u}_{\kappa'}(q')\,\gamma^{\nu} u_{\kappa}(q). \tag{II.28}$$

Since we are interested only in collisions whose energies are several orders of magnitude larger than the electron mass, we incur essentially no error with the neglect of this mass, and we can use the zero mass projection

$$\Sigma_{\kappa} u_{\kappa}(q)\ \bar{u}_{\kappa}(q) = - \gamma q \tag{II.29}$$

with $q^2 = 0 = q'^2$ to get

$$\begin{aligned} a^{\mu\nu} &= \tfrac{1}{2} \operatorname{tr} \gamma q \gamma^{\mu} \gamma q' \gamma^{\nu} \\ &= 2(q^{\mu} q'^{\nu} + q'^{\mu} q^{\nu} - g^{\mu\nu} q' q). \end{aligned} \tag{II.30}$$

The differential cross section for a given momentum transfer and energy loss can now be written down. On using Eqs. (II.11), (II.8), (II.12), and (II.28), we get

$$\frac{d^2\sigma}{dk^2\,d\nu} = \int \frac{(dq')}{(2\pi)^3} \frac{1}{2q'^0}\,\delta(k^2 - (q'-q)^2)\ \delta(\nu + (q-q')p)$$
$$\frac{e^4}{(k^2)^2}\ a_{\mu\nu} A^{\mu\nu} \frac{1}{-4pq} \ , \tag{II.31}$$

where, since the electron mass is taken to vanish, the laboratory quantity $m|q|$ can be expressed as the invariant $-pq$. The formula (II.31) for the differential cross section exhibits it as a manifestly Lorentz invariant scalar. The phase space integral is easily done in the laboratory frame and the result can readily be written in an invariant form,

$$\int \frac{(dq')}{(2\pi)^3}\,\frac{1}{2q'^0}\,\delta(k^2-(q'-q)^2)\,\delta(\nu+(q'-q)p) = \frac{1}{(4\pi)^2}\,\frac{1}{(-qp)} \quad \text{(II.32)}$$

A little algebra now gives

$$\frac{d^2\sigma}{dk^2 d\nu} = \frac{\alpha^2}{k^2}\,\frac{1}{(qp)^2}\,\{[(q'p)(qp) - \tfrac{1}{2}m^2k^2 + \tfrac{1}{2}\nu^2]\,A_1(\nu,k^2)$$

$$-\tfrac{1}{2}k^2 A_2(\nu,k^2)\} \quad , \quad \text{(II.33)}$$

where $\alpha = (e^2/4\pi) \simeq 1/137$ is the fine structure constant.

It has become conventional to use a pair of structure functions[4] defined by

$$A^{\mu\nu} = (1/m^2)[p^\mu + k^\mu(\nu/k^2)][p^\nu + k^\nu(\nu/k^2)]\,4\pi m W_2(\nu,k^2)$$

$$+ [g^{\mu\nu} - k^\mu k^\nu(1/k^2)]\,4\pi m W_1(\nu,k^2) \quad \text{(II.34)}$$

which are not free of kinematical constraint. They are related to the kinematic singularity free amplitudes by

$$4\pi m W_2 = m^2 k^2 A_1 \quad , \quad \text{(II.35a)}$$

and

$$4\pi m W_1 = \nu^2 A_1 - k^2 A_2 \quad . \quad \text{(II.35b)}$$

These structure functions do have the advantage of putting the differential cross section in a simple form. If we write the electron initial and final laboratory energies as $\epsilon = q^0$, $\epsilon' = q'^0$, and use the laboratory angle defined by (remember that the electron mass is taken to vanish)

$$k^2 = -2q'q = 2\epsilon'\epsilon(1-\cos\theta) \quad , \quad \text{(II.36)}$$

and the corresponding solid angle

$$d\Omega = 2\pi\,|d\cos\theta| = (\pi/\epsilon'\epsilon)\,dk^2 \quad , \quad \text{(II.37)}$$

we have

$$\frac{d^2\sigma}{d\Omega d\epsilon'} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{MOTT}} \left\{W_2(\nu,k^2) + 2\tan^2\frac{\theta}{2}W_1(\nu,k^2)\right\}, \quad \text{(II.38)}$$

in which

$$\left(\frac{d\sigma}{d\Omega}\right)_{MOTT} = \frac{\alpha^2 \cos^2\theta/2}{4\epsilon^2 \sin^4\theta/2} \tag{II.39}$$

is the scattering cross section for a high energy electron on an infinitely heavy, spin zero, structureless target.

We have already noted that, in the single photon exchange approximation, the electroproduction process is equivalent to the absorption of a virtual photon. In the limit $k^2 \to 0$ this exchanged photon becomes real and hence the electroproduction cross section must become proportional to the photoabsorption cross section. This relationship can be made explicit if we compute the total cross section

$$\gamma + N \to \text{(hadrons)}$$

for the photon absorption process in terms of the structure tensor $A^{\mu\nu}$. Since this tensor is gauge invariant, the average over the photon polarizations is tantamount to its contraction with $\frac{1}{2}g_{\mu\nu}$ and thus, in the same manner in which the electroproduction cross section was calculated, we get

$$\begin{aligned}\sigma_\gamma(\nu) &= e^2 \tfrac{1}{2} g_{\mu\nu} A^{\mu\nu}\Big]_{k^2=0} \frac{1}{-4pk} \\ &= \pi\alpha\nu A_1(\nu, 0) \ .\end{aligned} \tag{II.40}$$

This limit leads to a third parameterization[5] of the structure functions which involves a decomposition of $A^{\mu\nu}$ into spatially longitudinal and transverse parts in the laboratory frame. Since the structure tensor is gauge invariant [Eq. (II.20)], its longitudinal piece is proportional to a time component and thus we may equally well speak of a scalar part rather than a longitudinal part. Now, if $\underset{\sim}{\epsilon}$ is a transverse vector,

$$\underset{\sim}{\epsilon} \cdot \underset{\sim}{k} = 0 \ , \tag{II.41}$$

we have, in the laboratory frame,

$$\underset{\sim}{\epsilon}^*_k A^{k\ell} \underset{\sim}{\epsilon}_\ell = |\underset{\sim}{\epsilon}|^2 \{\nu^2 A_1 - k^2 A_2\} \ , \tag{II.42a}$$

while

$$\underset{\sim}{k}_k A^{k\ell} \underset{\sim}{k}_\ell = \underset{\sim}{k}^2 \{\nu^2 A_1 + (\nu^2/m^2) A_2\} \ . \tag{II.42b}$$

It is conventional to write Eq. (II.42a) as

$$\sigma_T(\nu,k^2) = (\pi\alpha/\bar{\nu})[\nu^2 A_1(\nu,k^2) - k^2 A_2(\nu,k^2)] \qquad \text{(II.43a)}$$

where $\bar{\nu}$ is the energy that a real photon would have to produce the same missing mass [cf. Eq. (II.5)]

$$\bar{\nu} = \nu - \tfrac{1}{2}k^2 \qquad \text{(II.44)}$$

This transverse cross section reduces to the photoabsorption cross section when the virtual photon mass vanishes,

$$k^2 \to 0: \ \sigma_T(\nu,k^2) \to \sigma_\gamma(\nu) \ . \qquad \text{(II.45a)}$$

It is also conventional to write Eq. (II.42b) as

$$\sigma_s(\nu,k^2) = (\pi\alpha/\bar{\nu})\, m^2 k^2 [A_1(\nu,k^2) + (1/m^2) A_2(\nu,k^2)] \ , \qquad \text{(II.43b)}$$

which obeys

$$k^2 \to 0: \ \sigma_s(\nu,k^2) \to 0 \ . \qquad \text{(II.45b)}$$

Note that, according to Eq. (II.26), the transverse and scalar cross sections, σ_T and σ_s, are independent, positive quantities. They are related to the structure functions W_1 and W_2 by

$$4\pi m W_1 = (\bar{\nu}/\pi\alpha)\sigma_T, \qquad \text{(II.46a)}$$

and

$$4\pi m W_2 = \frac{\bar{\nu}}{\pi\alpha}\,\frac{m^2 k^2}{\nu^2 + m^2 k^2}(\sigma_s + \sigma_T) . \qquad \text{(II.46b)}$$

III. Regge Kinematics

We shall outline the kinematics involved in obtaining the Regge asymptotic behavior of the electroproduction structure functions. We begin by very briefly reviewing the Regge analysis of the scattering amplitude $T(\nu,t)$ of spinless particles. We denote the initial and final momenta of these particles by p,k and p',k' with

$$p' + k' = p + k \ , \qquad \text{(III.1)}$$

and use the variables

$$\nu = -\tfrac{1}{4}(k'+k)(p'+p) \quad , \tag{III.2}$$

$$t = -(p'-p)^2 \quad . \tag{III.3}$$

We shall assume that $p^2 = p'^2$ and $k^2 = k'^2$ so that even at $t = 0$ the cosine of the unphysical scattering angle in the t-channel obeys

$$\nu \to \infty: \ -\cos\theta_t \ \alpha \ \nu \quad . \tag{III.4}$$

The basic idea in the Regge method is to make use of the angular momentum decomposition of the scattering amplitude in the t-channel, and to use the Watson-Sommerfeld transformation to replace the partial wave sum by a contour integral in the angular momentum plane. In order to achieve good convergence of this integral, it is necessary to introduce amplitudes of definite "signature" with

$$T(\nu,t) = T^{+1}(\cos\theta_t,t) + T^{+1}(-\cos\theta_t,t)$$

$$+ T^{-1}(\cos\theta_t,t) - T^{-1}(-\cos\theta_t,t) \tag{III.5}$$

and

$$T^{\eta}(\cos\theta_t,t) = \Sigma_J \, t^{\eta J}(t) \, P_J(\cos\theta_t) \quad . \tag{III.6}$$

The Watson-Sommerfeld transformation gives

$$T^{\eta}(\cos\theta_t,t) = \oint dJ \frac{t^{\eta J}(t)}{2i \sin\pi J} P_J(-\cos\theta_t) , \tag{III.7}$$

and the integration contour, which originally encloses the positive integers, is opened up and the Regge poles, the poles of $t^{\eta J}(t)$ in J are encircled. The leading Regge pole, the pole at $J = \alpha(t)$ which lies furthest to the right in the angular momentum plane, gives a contribution of the form

$$T^{\eta\alpha}(\cos\theta_t,t) = \frac{-\bar{\beta}(t)}{\sin\pi\alpha(t)} P_{\alpha(t)}(-\cos\theta_t) \tag{III.8}$$

and, since

$$z \to \infty: \ P_\alpha(z) \ \alpha \ z^\alpha \tag{III.9}$$

this leading trajectory dominates the asymptotic behavior in ν of the scattering amplitude. [The presence of branch cuts in the angular momentum plane could invalidate this argument. However, we shall use the Regge analysis only at t = 0 where the leading pole should dominate.] In terms of the original amplitude we have

$$\nu \to \infty: \quad T(\nu,t) = -\beta(t)\, \nu^{\alpha(t)} \frac{1 + \eta e^{-\pi i \alpha(t)}}{\sin \pi \alpha(t)} \tag{III.10}$$

in which $\eta = \pm 1$ is the signature of the leading trajectory.

We must now extend this analysis to the nucleon spin averaged, virtual-photon Compton amplitude $T^{\mu\nu}$ whose absorptive (imaginary) part at t = 0 is the structure function $A^{\mu\nu}$. We can write

$$T^{\mu\nu} = \left(g^{\mu\alpha} - \frac{k'^{\mu}k'^{\alpha}}{k'^2}\right)\tfrac{1}{2}(p'+p)_\alpha \left(g^{\nu\beta} - \frac{k^{\nu}k^{\beta}}{k^2}\right)\tfrac{1}{2}(p'+p)_\beta T_2(\nu,t,k'^2,k^2)$$
$$+ \left(g^{\mu\nu} - \frac{k^{\mu}k'^{\nu}}{kk'}\right) T_1(\nu,t,k'^2,k^2) + \ldots , \tag{III.11}$$

where the omitted terms vanish at t = 0, and where the absorptive parts of $T_{1,2}$ at t = 0 are the structure functions $W_{1,2}$:

$$W_{1,2}(\nu,k^2) = \mathrm{Im}\, T_{1,2}(\nu,0,k^2,k^2) \ . \tag{III.12}$$

We have chosen the covariants such that gauge invariance is generally obeyed

$$k'_\mu T^{\mu\nu} = 0 = T^{\mu\nu} k_\nu \ . \tag{III.13}$$

They have kinematic singularities in k'^2, k^2 and t, but these are irrelevant to our present discussion.

The crucial aspect of the Regge analysis is an angular momentum decomposition in the crossed t-channel. Such a decomposition is obtained by examining the behavior of the amplitude under rotations of the photon variables while the (spin-averaged) nucleon variables are held fixed or, since rotating all the particles leaves the amplitude invariant, by examining the behavior of the amplitude under rotations of the nucleon pair with the photon variables held fixed. In our case, it is easier to consider the response of the amplitude to nucleon rotations in which the photon variables are kept fixed.

The behavior of T_1 under nucleon rotations is trivial to obtain, for its covariant involves only photon variables which are not altered by such a rotation. Hence, we immediately obtain the angular momentum expansion of the signatured amplitude

$$T_1{}^{\eta}(\cos\theta_t, t) = \Sigma_J t_1{}^{\eta J}(t) P_J(\cos\theta_t) \, , \tag{III.14}$$

and the asymptotic behavior given in Eq. (III.10). The vacuum trajectory $\alpha_P(t)$ which has positive signature, $\eta = +1$, should be the leading trajectory here, and, with $\alpha_P(0) = 1$, we find that the structure function W_1 behaves as

$$\nu \to \infty: \; W_1(\nu, k^2) = w_1(k^2)\nu \, . \tag{III.15}$$

Such a simple analysis cannot be applied to the other invariant T_2, for it is associated with a covariant that involves nucleon momenta as well as photon variables and this covariant is not fixed during the nucleon pair rotation. This difficulty can, however, be circumvented by the ruse[6]

$$\tfrac{1}{2}(p'+p)_\alpha = -2 \frac{\partial}{\partial k'^\alpha} \nu = -2 \frac{\partial}{\partial k^\alpha} \nu \, , \tag{III.16}$$

which enables one to write

$$\left(g^{\mu\alpha} - \frac{k'^\mu k'^\alpha}{k'^2}\right) \tfrac{1}{2}(p'+p)_\alpha \left(g^{\nu\beta} - \frac{k^\nu k^\beta}{k^2}\right) \tfrac{1}{2}(p'+p)_\beta T_2(\nu, t, k'^2, k^2)$$

$$= \left(g^{\mu\alpha} - \frac{k'^\mu k'^\alpha}{k'^2}\right)\left(g^{\mu\beta} - \frac{k^\nu k^\beta}{k^2}\right) 4 \frac{\partial}{\partial k'^\alpha} \frac{\partial}{\partial k^\beta} \; \tilde{T}_2(\nu, t, k'^2, k^2) \, . \tag{III.17}$$

The transverse projections annihilate k'^α and k^β so that the derivatives with respect to the masses k'^2 and k^2 do not contribute. We have defined $t = -(p'-p)^2$ and it is independent of k' and k'. Hence

$$T_2(\nu, t, k'^2, k^2) = \frac{\partial^2}{\partial \nu^2} \tilde{T}_2(\nu, t, k'^2, k^2) \, . \tag{III.18}$$

The covariant associated with $\tilde{T}_2$ now involves photon quantities that remain constant during the nucleon pair rotation and thus this new invariant has a simple angular momentum decomposition and the Regge asymptotic behavior given in Eq. (III.10). We take two derivatives of this formula to get

$$\nu \to \infty: \; T_2 = -\beta(t)\alpha(t)\,[\alpha(t)-1]\,\nu^{\alpha(t)-2} \frac{1+\eta e^{-\pi i \alpha(t)}}{\sin \pi\alpha(t)} \, . \tag{III.19}$$

We now encounter another difficulty: the leading trajectory should be the vacuum trajectory $\alpha_P(t)$ with positive signature, $\eta = +1$, but $\alpha_P(0) = 1$, and the factor $[\alpha_P(t)-1]$ apparently uncouples this trajectory at $t = 0$. This decoupling is related to the impossibility of coupling two photons of the same polarization to a vector particle at zero momentum transfer. However, it is possible[7] for the residue function to be singular at $t = 0$ such that the vacuum trajectory does contribute. This is permissible because we are considering the Compton amplitude only to lowest order in electromagnetism, and thus there is no bound on the partial wave amplitude. Such a singular behavior of the residue is obtained if the partial wave amplitude $\tilde{t}_2^{\eta J}$ has a multiplicative fixed pole at $J = 1$ of the form

$$\tilde{t}_2^{+1J} = \frac{\bar{\beta}(t)}{[J-1][J-\alpha_P(t)]} + \ldots \; . \tag{III.20}$$

The pole at $J = 1$ does not produce a pole in the partial wave expansion

$$T_2^{\eta} = \Sigma_J \tilde{t}_2^{\eta J}(t) \frac{\partial^2}{\partial \nu^2} P_J(\cos\theta_t) \tag{III.21}$$

since the ν-derivatives annhilate $P_1(\cos\theta_t)$. Thus the Watson-Sommerfeld transformation can be carried out with the result

$$\nu \to \infty: \; T_2 = -\bar{\beta}(t)\alpha_P(t)\nu^{\alpha_P(t)-2} \frac{1+e^{-\pi i \alpha_P(t)}}{\sin\pi\alpha_P(t)} \; , \tag{III.22}$$

and

$$\nu \to \infty: \; W_2(\nu, k^2) = w_2(k^2)\nu^{-1} \; . \tag{III.23}$$

In terms of the amplitudes that are free of kinematic singularity, we have

$$\nu \to \infty: \; A_2(\nu, k^2) = \beta_2(k^2)(\nu/m^4) \quad , \tag{III.24}$$

and, assuming that the $[\alpha_P(t)-1]$ zero is indeed cancelled by a singular residue,

$$\nu \to \infty: \; A_1(\nu, k^2) = \beta_1(k^2)(1/\nu m^2) \; . \tag{III.25}$$

We have scaled these formulas with appropriate powers of the nucleon mass to make the residue functions $\beta_{1,2}$ dimensionless. It follows from Eq. (II.40) that the total photoabsorption cross section has the constant limit

$$\nu \to \infty: \sigma_\gamma(\nu) = (\pi\alpha/m^2)\,\beta_1(0), \qquad \text{(III.26)}$$

This cross section would vanish asymptotically if we had not chosen a singular residue to enforce the vacuum trajectory contribution.

IV. Experiment

The experimental situation with regard to the high energy total cross section for the absorption of photons on protons is relatively clear. The measured[8)] cross section is displayed in Fig. 3 along with a plot of the function $[100 + 60\nu^{-\frac{1}{2}}]$. The $\nu^{-\frac{1}{2}}$ form of the correction to the constant asymptotic limit accounts approximately for lower Regge trajectory contributions such as the P' and A_2 which have $\alpha(0) \simeq \frac{1}{2}$. Thus it appears that the photoabsorption cross section does indeed become a constant in the high energy limit,

$$\sigma_\gamma(\nu{=}\infty) = 100 \times 10^{-30}\ \text{cm}^2 \quad (\sim 10\%)\ , \qquad \text{(IV.1)}$$

and that the vacuum Regge trajectory with $\alpha_P(0) = 1$ does contribute. In terms of the parameterization (III.26), the experimental value of the dimensionless residue is given by

$$\beta_1(0) = 10 \quad (\sim 10\%)\quad . \qquad \text{(IV.2)}$$

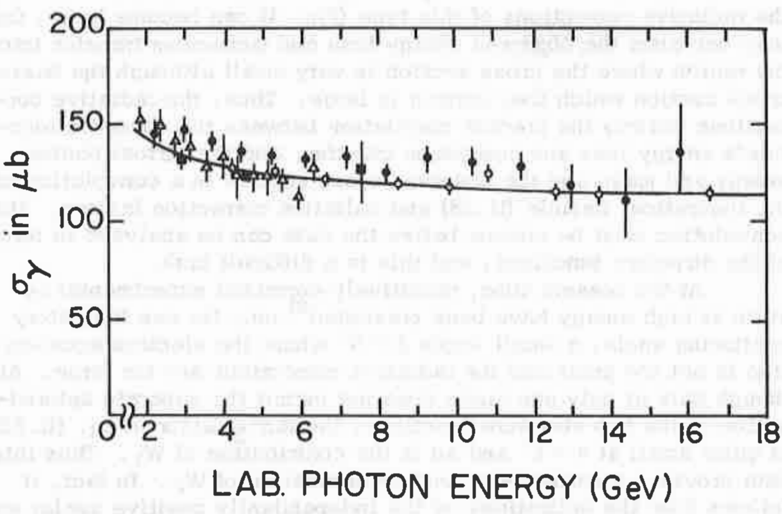

Fig. 3. Slightly cavalier representation of the total photoabsorption cross section at high energy.[8)]

The experimental situation in high energy electroproduction is not so transparent. We have found that this cross section can be described as

$$\frac{d^2\sigma}{d\Omega d\epsilon'} = \left(\frac{d\sigma}{d\Omega}\right)_{MOTT} \left\{W_2(\nu, k^2) + 2\tan^2\frac{\theta}{2} W_1(\nu, k^2)\right\} \quad . \qquad (II.38)$$

In principle, this cross section can be measured at a fixed energy loss ν and at a fixed momentum transfer k^2 but at various laboratory scattering angles θ and the ratio

$$\left(\frac{d^2\sigma}{d\Omega d\epsilon'}\right) \Big/ \left(\frac{d\sigma}{d\Omega}\right)_{MOTT}$$

can be plotted as a function of $2\tan^2\frac{\theta}{2}$. This ratio should appear as a straight line with a slope given by $W_1(\nu, k^2)$ and an intercept at $\theta = 0$ given by $W_2(\nu, k^2)$. In practice, the experimental analysis is not so simple because radiative corrections, the corrections due to photon emission, can be substantial. Since the electron is very light, it undergoes by far the greatest acceleration during the collision, and it is the principal source of the radiation. If the cross section decreases rapidly with increasing energy loss and momentum transfer, the radiative corrections of this type (Fig. 4) can become large, for they can alter the observed energy loss and momentum transfer into the region where the cross section is very small although the basic cross section which they correct is large. Thus, the radiative corrections destroy the precise correlation between the observed electron's energy loss and momentum transfer, and the virtual photon energy and mass, and the observed cross section in a convolution of the theoretical formula (II.38) and radiative correction factors. This convolution must be undone before the data can be analysed in terms of the structure functions, and this is a difficult task.

At the present time, radiatively corrected experimental results at high energy have been presented[9] only for one laboratory scattering angle, a small angle $\theta = 6^\circ$ where the electron acceleration is not too great and the radiative corrections are not large. Although data at only one angle does not permit the separate determination of the two structure functions, the $\tan^2\frac{\theta}{2}$ factor in Eq. (II.38) is quite small at $\theta = 6^\circ$ and so is the contribution of W_1. Thus this data provides a moderately good determination of W_2. In fact, it follows from the definitions of the independently positive scalar and transverse cross sections [Eqs. (II.46)] that W_1 is bounded by W_2 according to

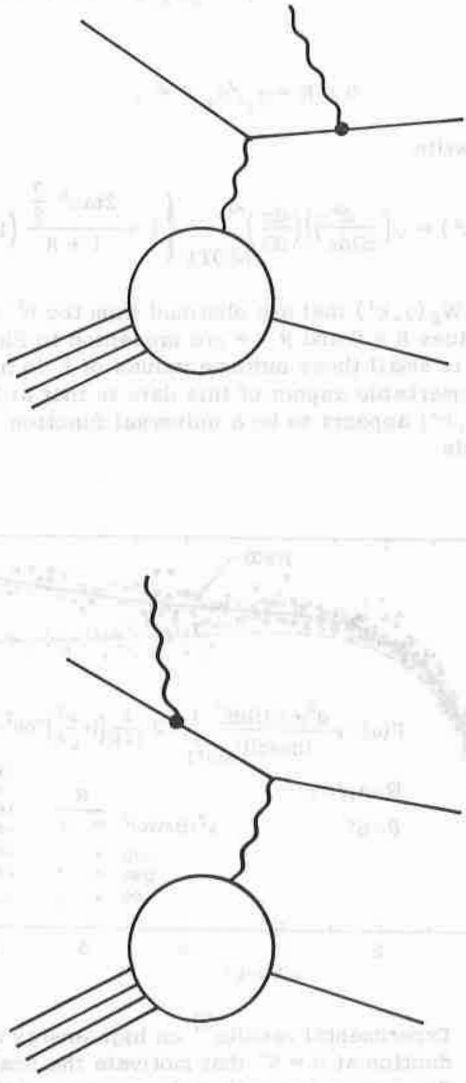

Fig. 4. Dominant photon emission graphs.

$$W_1(\nu, k^2) = (1 + R)^{-1} \left(1 + \frac{\nu^2}{m^2 k^2}\right) W_2(\nu, k^2) \, , \tag{IV.3}$$

in which

$$0 \le R = \sigma_S/\sigma_T < \infty \, . \tag{IV.4}$$

Hence we can write

$$\nu W_2(\nu, k^2) = \nu\left(\frac{d^2\sigma}{d\Omega d\epsilon'}\right)\left(\frac{d\sigma}{d\Omega}\right)^{-1}_{\text{MOTT}} \left\{1 + \frac{2\tan^2\frac{\theta}{2}}{1+R}\left(1 + \frac{\nu^2}{m^2 k^2}\right)\right\}^{-1} . \tag{IV.5}$$

The values of $\nu W_2(\nu, k^2)$ that are obtained from the 6° data for the two extreme values R = 0 and R = ∞ are presented in Fig. 5. As we noted, since θ is small these extreme values of R do not change W_2 greatly. The remarkable aspect of this data is that at large ν the quantity $\nu W_2(\nu, k^2)$ appears to be a universal function of the dimensionless variable

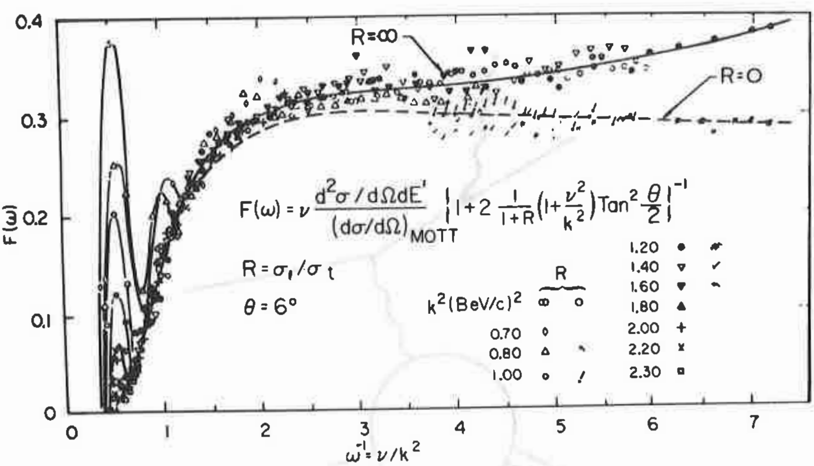

Fig. 5. Experimental results[9] on high energy electroproduction at θ = 6° that motivate the scaling law. The two curves follow the average of the data on νW_2 when the two extreme values R = 0 and R = ∞ are used in formula (IV.5).

$$\omega = k^2/\nu. \qquad \text{(IV.6)}$$

That is, it appears that $\nu W_2(\nu, k^2)$ takes on the same value for a range of ν and k^2 so long as the ratio k^2/ν is held fixed.

This result is consistent with the existence of a scaling limit $\nu \to \infty$:

$$W_1(\nu, k^2) = u_1(\omega) \ , \qquad \text{(IV.7a)}$$

$$W_2(\nu, k^2) = \frac{u_2(\omega)}{\nu} \ , \qquad \text{(IV.7b)}$$

or, in terms of the structure functions free of kinematic singularity,

$$A_1(\nu, k^2) = \frac{a_1(\omega)}{\nu^2} \ , \qquad \text{(IV.8a)}$$

$$A_2(\nu, k^2) = \frac{a_2(\omega)}{\nu} \ . \qquad \text{(IV.8b)}$$

We shall discuss this limit in Sec. VI. If $a_2(\omega)$ vanishes so does the high energy limit of the ratio R and $u_2(\omega)$, or equivalently $\omega a_1(\omega)$, is seen from Fig. 5 to be very nearly a constant for $\omega < \frac{1}{2}$ with the value

$$\omega < \tfrac{1}{2} : \omega a_1(\omega) = 4.0 \ (\sim 15\%) \qquad \text{(IV.9)}$$

We shall show in Sec. VI that the behavior of the scaling limit functions $a_{1,2}(\omega)$ near $\omega = 0$ is controlled by the value of the leading Regge trajectory $\alpha(0)$ and, in particular, that $\omega a_1(\omega)$ becomes constant for small ω with $\alpha(0) = 1$. We have just seen that the experimental data supports this limit.

V. Causal Commutators

We have found that the structure tensor may be expressed as

$$A^{\mu\nu} = \int (dx) e^{-ikx} \tfrac{1}{2}\Sigma_\lambda \langle p\lambda | j^\mu(x) j^\nu(0) | p\lambda \rangle \ , \qquad \text{(II.17)}$$

where k^0 or, equivalently, ν is positive. Now, by repeating the discussion that led to this result, it is easy to verify that for $\nu > 0$:

$$\int (dx) e^{-ikx} \tfrac{1}{2}\Sigma_\lambda \ \langle p\lambda | j^\nu(0) j^\mu(x) | p\lambda \rangle = 0 \ , \qquad \text{(V.1)}$$

for in this case one obtains $\delta(P'+k-p)$ whose argument cannot vanish. This follows from the stability of the nucleon, as is easily seen in the laboratory frame where the nucleon is at rest. In this frame $p^0 = m$

while, since the nucleon is stable, an intermediate state energy P'^0 must be at least as large as the nucleon mass m and $P'^0 - p^0 + k^0$ cannot vanish. Accordingly, we may express the structure tensor as the Fourier transform of a current commutator, $\nu > 0$:

$$A^{\mu\nu} = -i \int (dx) e^{-ikx} C^{\mu\nu}(x) \quad , \tag{V.2}$$

with

$$C^{\mu\nu}(x) = \tfrac{1}{2}\Sigma_\lambda \langle p\lambda | i[j^\mu(x), j^\nu(0)] | p\lambda \rangle . \tag{V.3}$$

The tensor decomposition (II.22) becomes

$$C^{\mu\nu}(x) = [p^\mu p^\nu(-\partial^2) + (p^\mu \partial^\mu + \partial^\mu p^\nu) p\partial - g^{\mu\nu}(p\partial)^2] C_1(x)$$

$$- [\partial^\mu \partial^\nu - g^{\mu\nu}\partial^2] C_2(x)$$

$$= C^{\nu\mu}(x) \quad , \tag{V.4}$$

and we have

$$\nu > 0: \; A_{1,2}(\nu, k^2) = -iC_{1,2}(k) \tag{V.5}$$

with

$$C_{1,2}(k) = \int (dx) e^{-ikx} C_{1,2}(x) . \tag{V.6}$$

Translation invariance,

$$\langle p\lambda | j^\mu(x) j^\nu(0) | p\lambda \rangle = \langle p\lambda | j^\mu(0) j^\nu(-x) | p\lambda \rangle \tag{II.18}$$

and the symmetry in the tensor indices exhibited in Eq. (V.4), give

$$C^{\mu\nu}(x) = -C^{\nu\mu}(-x) = -C^{\mu\nu}(-x) \quad . \tag{V.7}$$

Thus the invariant commutator functions are odd

$$C_{1,2}(-x) = -C_{1,2}(x) \quad . \tag{V.8}$$

We may now incorporate the physical postulate of causality, the requirement that two current operators commute at space-like coordinate separation:

$$x^2 > 0: [j^\mu(x), j^\nu(0)] = 0 \,, \qquad (V.9)$$

or

$$x^2 > 0: \; C^{\mu\nu}(x) = 0 \;. \qquad (V.10)$$

This condition is clearly satisfied if the invariant functions are causal,

$$x^2 > 0: \; C_{1,2}(x) = 0 \;. \qquad (V.11)$$

If the various components of Eq. (V.4) are examined in the nucleon rest frame, it can be shown,[10] conversely, that the invariant functions $C_{1,2}$ must be causal if the tensor $C^{\mu\nu}$ is to be causal. Hence, the vanishing of $C_{1,2}(x)$ at space-like separation (V.10) is a necessary and sufficient condition for causality.

We turn now to outline in a very heuristic manner how representations that exploit the causality information may be constructed. An odd, causal function may be written as

$$C(x) = \epsilon(x^0) \int_0^\infty da^2 \, \delta(x^2 + a^2) C(a^2, x) \;, \qquad (V.12)$$

in which $C(a^2, x)$ may be taken to be a function of a single invariant formed from x^2 and $(px)^2$. There are two natural choices for this invariant: $(px)^2/m^2$ which reduces to the square of the time t^2 in the nucleon rest frame, or, alternatively, $x^2 + (px)^2/m^2$ which reduces to the square of the spatial distance r^2 in the nucleon rest frame. The former choice leads to the Deser-Gilbert-Sudarshan[11] representation while the latter gives the Jost-Lehmann[12] representation. These representations contain information about the mass spectrum of the intermediate states as well as being causal. This information can be incorporated if $\epsilon(x^0)\delta(x^2 + a^2)$ is replaced by the vacuum commutator function

$$\Delta(x, m^2) = 2\pi i \int \frac{(dk)}{(2\pi)^4} \, e^{ikx} \, \epsilon(k^0) \, \delta(k^2 + m^2) \;, \qquad (V.13)$$

which is causal

$$x^2 > 0: \; \Delta(x, m^2) = 0 \;. \qquad (V.14)$$

We shall also need the relation

$$\Delta(x, m^2) = \epsilon(x^0) \Delta^{(1)}(x^2, m^2) \;. \qquad (V.15)$$

In order to accomplish this replacement we note that since $\Delta^{(1)}(x^2, m^2)$ is causal, the Fourier transform of the commutator function

$$\epsilon(k^0)\,\delta(k^2+m^2) = \frac{1}{2\pi i}\int (dx) e^{-ikx}\,\Delta(x, m^2) \qquad (V.16)$$

may be expressed as

$$\epsilon(k^0)\delta(k^2+m^2) = \frac{1}{2\pi i}\int (x) e^{-ikx}\int_0^\infty db^2\,\epsilon(x^0)\delta(x^2+b^2)\,\Delta^{(1)}(b^2, m^2)\,. \qquad (V.17)$$

Now employing the Fourier transform (V.16) we get

$$\epsilon(k^0)\delta(k^2+m^2) = \frac{1}{2\pi i}\int (dx) e^{-ikx}\int_0^\infty db^2\,\frac{1}{2\pi i}\int (dx') e^{-ixx'}$$
$$\Delta(x', b^2)\,\Delta^{(1)}(b^2, m^2)\,, \qquad (V.18)$$

and, upon interchanging integrals and replacing the variable k by x, we arrive at the lemma:

$$\epsilon(x^0)\delta(x^2+a^2) = (2\pi)^2\int_0^\infty db^2\,\Delta(x, b^2)\,\Delta^{(1)}(b^2, a^2)\,. \qquad (V.19)$$

Accordingly, again interchanging integrals, we may write Eq. (V.12) in the form

$$C(x) = \int_0^\infty db^2\,\Delta(x, b^2)\,D(b^2, x)\,, \qquad (V.20)$$

with

$$D(b^2, x) = (2\pi)^2\int_0^\infty da^2\,\Delta^{(1)}(b^2, a^2)\,C(a^2, x)\,. \qquad (V.21)$$

The Deser-Gilbert-Sudarshan representation is obtained by using the variable $(px)^2$:

$$C(x) = \int_0^\infty db^2\,\Delta(x, b^2)\,X(b^2, (px)^2)\,. \qquad (V.22)$$

We may write

$$X(b^2, (px)^2) = \int \frac{d\beta}{2\pi}\, e^{-i\beta px}\,\chi(\sigma, \beta^2)\,, \qquad (V.23)$$

in which $\sigma = b^2 - \beta^2 m^2$ to get

$$C(x) = \int d\sigma \int \frac{d\beta}{2\pi} \chi(\sigma, \beta^2) e^{-i\beta px} \Delta(x;\sigma+\beta^2 m^2) , \qquad (V.24)$$

and, taking the Fourier transform we obtain

$$C(k) = i\int d\sigma \int d\beta \chi(\sigma, \beta^2) \epsilon(\nu+\beta m^2) \delta(k^2 - 2\beta\nu + \sigma) . \qquad (V.25)$$

We have yet to impose the conditions implied by the mass spectrum of the intermediate states. We have seen that if $\nu > 0$ only one ordering in the commutator contributes, and we obtain the structure functions which have support only for $2\nu - k^2 \geq 0$ [Eq. (II.6)]. On the other hand, if $\nu < 0$ only the other ordering contributes with $-2\nu - k^2 \geq 0$. Hence, we must have generally

$$2|\nu| - k^2 \geq 0 , \qquad (V.26)$$

if $C(k)$ is nonvanishing. It is easy to obtain conditions on the domain of the parameters σ and β that are sufficient to ensure the spectral condition (V.26). This domain is also necessary for the validity of the spectral condition, but we will not prove that this is so. First we note that if $\nu = 0$, the δ-function in Eq. (V.25) becomes $\delta(k^2 + \sigma)$ and we guarantee that $C(k)$ is nonvanishing only for $-k^2 > 0$ by demanding that

$$0 \leq \sigma < \infty . \qquad (V.27)$$

The spectral condition is now satisfied for arbitrary values of ν if we require that

$$-1 \leq \beta \leq 1 . \qquad (V.28)$$

The scalar invariant $C(x)$ is equivalent to a matrix element of a scalar field commutator,

$$C(x) = \langle p | i[\varphi(x), \varphi(0)] | p \rangle . \qquad (V.29)$$

As we have remarked, the spectral conditions allow the two orderings of the commutator to be separated: in the nucleon rest frame one ordering contains only positive frequencies, the other only negative frequencies. Since the representation (V.24) involves a vacuum commutator function with masses larger than $|\beta| m$, its energy components dominate in the rest frame, $|k^0| > |\beta| p^0 = |\beta| m$, and the separation into the two orderings is clear. We have

$$A(x) = \langle p|\varphi(x)\,\varphi(0)|p\rangle$$

$$= \int d\sigma \int \frac{d\beta}{2\pi} \chi(\sigma,\beta^2) e^{-i\beta px} \Delta^{(+)}(x;\, \sigma+\beta^2 m^2) \ , \qquad (V.30)$$

with

$$\Delta^{(+)}(x, m^2) = \int \frac{(dk)}{(2\pi)^3} e^{ikx} \theta(k^0)\, \delta(k^2+m^2) \ . \qquad (V.31)$$

The Fourier transform of this representation gives, of course, the representation for the structure functions $A_{1,2}(\nu, k^2)$. The time-ordered product may now be constructed in terms of the separate orderings

$$T(x) = \langle p|\, iT(\varphi(x)\,\varphi(0))|p\rangle$$

$$= \theta(x^0)\langle p|i\varphi(x)\,\varphi(0)|p\rangle + \theta(-x^0)\langle p|i\varphi(0)\,\varphi(x)|p\rangle \qquad (V.32)$$

and we obtain

$$T(x) = \int d\sigma \int \frac{d\beta}{2\pi} \chi(\sigma,\beta^2)\, e^{-i\beta px} \Delta_+(x;\, \sigma+\beta^2 m^2) \ , \qquad (V.33)$$

with

$$\Delta_+(x;m^2) = \theta(x^0)\, i\Delta^{(+)}(x;m^2) + \theta(-x^0)\, i\Delta^{(+)}(-x;m^2)$$

$$= \int \frac{(dk)}{(2\pi)^4} e^{ikx} \frac{1}{k^2+m^2-i\epsilon} \ . \qquad (V.34)$$

A similar representation holds for the scalar invariants associated with the time-ordered product of the current operator. However, to obtain such a representation for these scalar invariants, the step functions $\theta(x^0)$ and $\theta(-x^0)$ must be commuted through the tensor covariants exhibited in Eq. (V.4), a process that generally leads to noncovariant contact or "seagull" terms multiplying $\delta(x-x')$. These noncovariant terms can be cancelled by a suitable definition[13] of a covariant time-ordered product with the result that it has the form of Lorentz covariants operating on scalar functions with the representation (V.33). In terms of the Fourier transform

$$T(\nu, k^2) = \int (dx)\, e^{-ikx}\, T(x) \ , \qquad (V.35)$$

we have

$$T(\nu, k^2) = \int d\sigma \int \frac{\chi(\sigma, \beta^2)}{k^2 - 2\beta\nu + \sigma - i\epsilon} \quad . \tag{V.36}$$

Despite the pleasant cast of the Deser-Gilbert-Sudarshan representation and the fact that it has been shown to be true to all orders in perturbation theory by Nakanishi,[14] we shall use instead the Jost-Lehmann representation which, in addition to having been established with abstract rigor, is more convenient for our purposes. It is

$$C(x) = \int_0^\infty ds\, \Psi(s, \zeta^2)\, \Delta(x;s) \quad , \tag{V.37}$$

with

$$\zeta^2 = x^2 + (px)^2/m^2 \quad . \tag{V.38}$$

We shall work almost entirely in the nucleon rest frame where $\zeta^2 = r^2$ is the spatial coordinate separation. This entails no loss of generality, however, for the result in an arbitrary frame is immediately obtained with the replacement

$$\underset{\sim}{r} \rightarrow \underset{\sim}{\zeta}$$

where

$$\underset{\sim}{\zeta} = \underset{\sim}{r} + \underset{\sim}{p}\left[\frac{\underset{\sim}{p}\cdot\underset{\sim}{r}}{m(p^0+m)} + \frac{t}{m}\right] , \tag{V.39}$$

for

$$\underset{\sim}{\zeta}^2 = x^2 + (px)^2/m^2 \quad . \tag{V.40}$$

As before, the spectral condition allows a separation of the two orderings in the commutator. If we write

$$\Psi(s, r^2) = \int \frac{(d\underset{\sim}{u})}{(2\pi)^3}\, e^{i\underset{\sim}{u}\cdot\underset{\sim}{r}}\, \psi(s, u^2) \quad , \tag{V.41}$$

then, in the nucleon rest frame, we get the structure function representation

$$A_{1,2}(\nu, k^2) = \int_0^\infty ds \int \frac{(d\underset{\sim}{u})}{(2\pi)^3}\, \psi_{1,2}(s, u^2)$$

$$2\pi\theta(\nu)\delta(k^2 - 2\underset{\sim}{u}\cdot\underset{\sim}{k} + u^2 + s) \quad . \tag{V.42}$$

The spectral conditions turn out to require that the weight functions vanish if the parameter u exceeds the nucleon mass,

$$|u| > m:\ \psi_{1,2}(s,u^2) = 0\ . \tag{V.43}$$

We note, for completeness, that the time-ordered product has the rest frame representation

$$T(\nu,k^2) = \int_0^\infty ds \int \frac{(d\underset{\sim}{u})}{(2\pi)^3}\ \frac{\psi(s,u^2)}{k^2 - 2\underset{\sim}{u}\cdot\underset{\sim}{k} + u^2 + s - i\epsilon}\ . \tag{V.44}$$

VI. Asymptotic Behavior

We consider first the behavior of the structure functions in the limit $\nu \to \infty$ with the ratio

$$\omega = k^2/\nu \tag{VI.1}$$

held fixed. Note that since the structure functions are nonvanishing only when $2\nu - k^2 \geq 0$ [Eq. (II.6)], the parameter ω lies in the range

$$0 \leq \omega \leq 2\ . \tag{VI.2}$$

We shall assume, following Bjorken,[15] that the complete structure tensor (II.22) remains finite in this limit in which the covariants ${}^{(1)}t^{\mu\nu}$ and ${}^{(2)}t^{\mu\nu}$ diverge as ν^2 and ν, respectively. Hence, we require that, with ω fixed,

$$\nu \to \infty:\ A_1(\nu,k^2) = \frac{a_1(\omega)}{\nu^2}\ , \tag{VI.3a}$$

$$\nu \to \infty:\ A_2(\nu,k^2) = \frac{a_2(\omega)}{\nu}\ . \tag{VI.3b}$$

We have already remarked [Sec. IV] that there is some experimental evidence in support of the existence of the limit (VI.3a), but there is yet none in support of the limit (VI.3b). The functions $a_{1,2}(\omega)$ are dimensionless functions of the dimensionless parameter ω. Thus, the existence of this limit implies in some sense that nature becomes scale invariant at high energies. Note that the positivity condition (II.27) requires that these functions obey the inequalities

$$a_1(\omega) \geq 0, \tag{VI.4a}$$

and

$$0 \le a_2(\omega) \le \omega^{-1} a_1(\omega) \quad . \tag{VI.4b}$$

These scaling limits can be related to the Jost-Lehmann weights $\psi_{1,2}$. If the angular integral in Eq. (V.42) is done, one gets

$$A_{1,2}(\nu, k^2) = \frac{1}{4\pi|\underset{\sim}{k}|} \int_0^\infty ds \int_0^m udu\psi_{1,2}(s, u^2)\theta(2u|\underset{\sim}{k}| - k^2 - u^2 - s) \quad , \tag{VI.5}$$

where, in the rest frame,

$$|\underset{\sim}{k}| = [k^2 + \nu^2/m^2]^{\frac{1}{2}} \quad . \tag{VI.6}$$

We shall make the basic assumption that the weights $\psi_{1,2}(s, u^2)$ decrease rapidly at large s, uniformly in u. Thus, since the variation of u is bounded, we can neglect both u^2 and s in the θ-function in Eq. (VI.5), and obtain

$$\nu \to \infty\colon\ A_{1,2}(\nu, k^2) = \frac{m}{4\pi\nu} \int_0^\infty ds \int_0^m udu\psi_{1,2}(s, u^2)\theta(2u - \omega m) \quad . \tag{VI.7}$$

This gives the limit (VI.3b) with the identification

$$a_2(\omega) = \frac{m}{4\pi} \int_0^\infty ds \int_{\frac{1}{2}m\omega}^m udu\psi_2(s, u^2) \quad . \tag{VI.8}$$

On the other hand, the requirement that $A_1(\nu, k^2)$ vanish more rapidly than $1/\nu$ demands that the corresponding integral involving ψ_1 must vanish for all ω or, on taking the derivative with respect to ω, we have that

$$\varphi_1(s, u^2) = \int_0^s ds' \psi_1(s', u^2) \tag{VI.9}$$

must obey

$$s \to \infty\colon\ \varphi_1(s, u^2) \to 0 . \tag{VI.10}$$

Hence, we can write

$$\psi_1(s, u^2) = \frac{\partial}{\partial s} \varphi_1(s, u^2) \tag{VI.11}$$

in the formula (VI.5) and integrate by parts to obtain the general result

$$A_1(\nu, k^2) = \frac{1}{4\pi|\underset{\sim}{k}|} \int_0^\infty ds \int_0^m udu\varphi_1(s, u^2)\delta(2u|\underset{\sim}{k}| - k^2 - u^2 - s) . \tag{VI.12}$$

This gives the desired $1/\nu^2$ limiting behavior with

$$a_1(\omega) = \left(\frac{m^2}{8\pi}\right)\left(\frac{\omega m}{2}\right)\int_0^\infty ds\varphi_1\left(s, \left(\frac{\omega m}{2}\right)^2\right) . \qquad \text{(VI.13)}$$

It is interesting to compare[16)] this scaling limit with the $\nu \to \infty$ limit in which k^2 is held fixed, the "Regge" limit,

$$k^2 \text{ fixed}, \quad |\underset{\sim}{k}| \to \nu/m \to \infty . \qquad \text{(VI.14)}$$

We consider first the limit of the structure function A_1. It follows from Eq. (VI.16) and our basic assumption that the Jost-Lehmann weight decreases rapidly at large s uniformly in u, that this limit probes the small u behavior of the weight $\varphi_1(s, u^2)$ and that A_1 will vanish at least as rapidly as ν^{-2} unless this behavior is singular. Since A_1 should approach $\nu^{\alpha-2}$, with $\alpha = 1$ for the leading vacuum trajectory, the weight $\varphi_1(s, u^2)$ must, in fact, be singular at $u = 0$. We obtain the Regge limit if we write

$$\varphi_1(s, u^2) = \frac{1}{u^{1+\alpha}}\sigma_1(s) + \bar{\varphi}_1(s, u^2) \qquad \text{(VI.15)}$$

with $\bar{\varphi}_1(s, u)$ regular at $u = 0$. Indeed the singular term gives, in view of the general formula (VI.12),

$$A_1^{(\alpha)}(\nu, k^2) = \frac{1}{4\pi|k|}\int_0^m \frac{du}{u^\alpha}\int_0^\infty ds\sigma_1(s)\delta(2u|k| - k^2 - s) , \qquad \text{(VI.16)}$$

where we have omitted the u^2 term in the δ-function since it affects neither the Regge nor the scaling limits. We can do the u-integral to get

$$A_1^{(\alpha)}(\nu, k^2) = \left(\frac{1}{8\pi}\right)\left[k^2 + \frac{\nu^2}{m^2}\right]^{\frac{1}{2}\alpha - 1}\int_0^\infty ds\sigma_1(s)\left[\frac{2}{k^2+s}\right]^\alpha , \qquad \text{(VI.17)}$$

and the Regge limit

$$\nu \to \infty: \; A_1(\nu, k^2) = m^{-4}\beta_1(k^2)(\nu/m^2)^{\alpha-2} , \qquad \text{(VI.18)}$$

with

$$\beta_1(k^2) = \left(\frac{m^2}{8\pi}\right)\int_0^\infty ds\sigma_1(s)\left[\frac{m}{k^2+s}\right]^\alpha . \qquad \text{(VI.19)}$$

On the other hand, the integral (VI.17) has a scaling limit (assuming that $\sigma(s)$ vanishes sufficiently rapidly at infinity) and gives a contribution

$$\nu \to \infty,\ \omega \text{ fixed:}\ A_1^{(\alpha)}(\nu, k^2) = \frac{a_1^{(\alpha)}(\omega)}{\nu^2}, \tag{VI.20}$$

with

$$a_1^{(\alpha)}(\omega) = \left(\frac{m^2}{8\pi}\right)\left(\frac{2}{m\omega}\right)^{\alpha} \int_0^{\infty} ds\sigma_1(s) \ . \tag{VI.21}$$

Note that this contribution behaves as $\omega^{-\alpha}$, while the regular part of the weight, $\bar{\varphi}_1(s, u^2)$, gives, according to Eq. (VI.13), a function that vanishes as ω near $\omega = 0$. We have thus found that the physical condition of causality, as conveyed by the Jost-Lehmann representation, implies a connection between the Regge limit and the $\omega \to 0$ behavior of the scaling limit. Indeed, if we compare Eqs. (VI.21) and (VI.19), we find that

$$a_1^{(\alpha)}(\omega) = \omega^{-\alpha} \lim_{k^2 \to \infty} (k^2/m^2)^{\alpha} \beta_1(k^2), \tag{VI.22}$$

and, as we have just remarked,

$$\omega \to 0:\ a_1(\omega) \to a_1^{(\alpha)}(\omega) \ . \tag{VI.23}$$

The conclusion is that the Regge residue $\beta_1(k^2)$ must have a large k^2 limit which is correlated with the value of the trajectory α such that the Regge asymptotic behavior (VI.18) is consistent with the scaling limit,[17] and, moreover, it is the Regge limit which controls the small ω dependence of the scaling function $a_1(\omega)$. Thus, the limit $\nu \to \infty$ with k^2 fixed gives the Regge form (VI.18), and then the limit $k^2 \to \infty$ may be taken to get the small $\omega = k^2/\nu$ behavior of the scaling function.

For the nucleon structure function, the leading trajectory should be the vacuum trajectory with $\alpha_P(0) = 1$, which gives

$$\omega \to 0:\ a_1(\omega) = \frac{a_1^P}{\omega}, \tag{VI.24}$$

with

$$a_1^P = \lim_{k^2 \to \infty} (k^2/m^2)\,\beta_1(k^2). \tag{VI.25}$$

This result is in beautiful accord with an experiment which indicates that (assuming the A_2 contribution to be small) $\omega a_1(\omega)$ is very nearly constant throughout the range $\omega < \frac{1}{2}$, with the value

$$\omega \leq \tfrac{1}{2}: \ \omega a_1(\omega) = 4.0 = a_1{}^P \qquad \text{(VI.26)}$$

within an error of 15% or so. Unfortunately, we only know the experimental value of the Regge residue at $k^2 = 0$ from the photoabsorption experiments which give

$$\beta_1(0) = 10 \qquad \text{(VI.27)}$$

within about a 10% error. If we make a very naive approximation in which the integral representation (VI.19) for the Regge residue is dominated at, say the ρ mass, $s = m_\rho^2$, we have

$$\beta_1(k^2) = \beta_1(0)\,\frac{m_\rho^2}{k^2 + m_\rho^2}\ , \qquad \text{(VI.28)}$$

and

$$a_1{}^P = \beta_1(0)(m_\rho^2/m^2) = 6.6\ , \qquad \text{(VI.29)}$$

which is about twice the correct experimental value. This should not be disturbing in the least, for the spectral weight in Eq. (VI.19) need not be positive.

We turn now to the Regge asymptotic behavior of the other structure function, $A_2 \sim \nu^\alpha$. It follows from the Jost-Lehmann representation (VI.5) that, if the weight $\psi_2(s,u^2)$ decreases rapidly at large s, uniformly in u, then this weight must behave as $u^{-3-\alpha}$ near $u = 0$. But the very existence of the representation requires that the weight be integrable in $u^2 du$ at $u = 0$ which is apparently violated if $\psi_2 \sim u^{-3-\alpha}$. This dilemma is circumvented by the realization that the weights need not be ordinary functions but can be distributions. Thus, we can write

$$\psi_2(s,u^2) = \sigma_2(s)\nabla_u^2\{|u|^{-1-\alpha}\theta(m-|u|)\} + \bar{\psi}_2(s,u^2)\ , \qquad \text{(VI.30)}$$

in which $\bar{\psi}_2(s,u^2)$ is regular at $u = 0$, and the Laplacian with respect to u is to be treated in the usual distribution theory sense: it is to be integrated by parts. If the singular contribution is inserted into the Jost-Lehmann representation in its original form (V.42), several integrations by parts performed, and the angular integral done, one gets:

$$A_2^{(\alpha)}(\nu,k^2) = \frac{1}{4\pi|\underset{\sim}{k}|}\int_0^m \frac{du}{u^\alpha}\int_0^\infty ds\left\{4\left(\frac{\nu^2}{m^2}-s\right)\frac{d^2\sigma_2(s)}{d^2s}\right.$$

$$\left. - 6\sigma_2(s)\right\}\theta(2u|\underset{\sim}{k}|-k^2-u^2-s) \quad . \tag{VI.31}$$

The leading contribution to both limits comes from the term involving ν^2/m^2, and integrating by parts puts this contribution in the form

$$A_2^{(\alpha)}(\nu,k^2) \simeq \frac{\nu^2}{\pi m^2|\underset{\sim}{k}|}\int_0^m \frac{du}{u^\alpha}\left\{-\theta(2u|\underset{\sim}{k}|-k^2-u^2)\left.\frac{d\sigma_2(s)}{ds}\right]_{s=0}\right.$$

$$\left. + \int_0^\infty ds\,\frac{d\sigma_2(s)}{ds}\,\delta(2u|\underset{\sim}{k}|-k^2-u^2-s)\right\} \tag{VI.32}$$

which shows that the Regge limit is obtained only if we require that

$$\left.\frac{d\sigma_2(s)}{ds}\right]_{s=0} = 0 \quad . \tag{VI.33}$$

In this case we obtain:

$$A_2^{(\alpha)}(\nu,k^2) \simeq \frac{\nu^2}{2\pi m^2|\underset{\sim}{k}|^2}\,|\underset{\sim}{k}|^\alpha \int_0^\infty ds\,\frac{d\sigma_2(s)}{ds}\left[\frac{2}{k^2+s}\right]^\alpha \quad . \tag{VI.34}$$

In the Regge limit we have

$$\nu\to\infty:\ A_2(\nu,k^2) = \frac{1}{m^2}\,\beta_2(k^2)(\nu/m^2)^\alpha , \tag{VI.35}$$

with

$$\beta_2(k^2) = \frac{m^2}{2\pi}\int_0^\infty ds\,\frac{d\sigma_2(s)}{ds}\left[\frac{2m}{k^2+s}\right]^\alpha , \tag{VI.36}$$

while in the scaling limit

$$\nu\to\infty,\ \omega \text{ fixed: } A_2^{(\alpha)}(\nu,k^2) = \frac{a_2^{(\alpha)}(\omega)}{\nu} , \tag{VI.37}$$

with

$$a_2^{(\alpha)}(\omega) = \frac{1}{2\pi}\cdot\frac{1}{\omega}\left(\frac{2}{\omega m}\right)^\alpha \alpha\int_0^\infty ds\,\sigma_2(s) \quad . \tag{VI.38}$$

This singular contribution to the scaling limit behaves as $\omega^{-1-\alpha}$ while the regular contribution gives a constant term near $\omega = 0$ so we again have

$$\omega \to 0: \; a_2(\omega) \to a_2^{(\alpha)}(\omega) \; . \tag{VI.39}$$

And again we find that the large k^2 behavior of the Regge residue is correlated with the value of the trajectory such that the Regge limit is consistent with the scaling limit and gives the small ω form of this limit:

$$a_2^{(\alpha)}(\omega) = \omega^{-1-\alpha} \lim_{k^2 \to \infty} (k^2/m^2)^{1+\alpha} \beta_2(k^2) . \tag{VI.40}$$

We can gain some understanding of the nature of the scaling limit if we write the Jost-Lehmann representation for the structure functions in configuration space

$$A_{1,2}(x) = \int_0^\infty ds \, \Psi_{1,2}\left(s, x^2 + \frac{(px)^2}{m^2}\right) \Delta^{(+)}(x;s) \; . \tag{VI.41}$$

Near the light cone

$$x^2 \to 0: \; \Delta^{(+)}(x;s) \simeq \frac{1}{4\pi^2} \left\{ \frac{1}{x^2} + \frac{s}{4} \ln(sx^2) \right\} . \tag{VI.42}$$

Hence

$$\int_0^\infty ds \, \Psi_2\left(s, \frac{(px)^2}{m^2}\right) = \lim_{x^2 \to 0} \; 4\pi^2 x^2 A_2(x) \; , \tag{VI.43}$$

which expresses essentially the Fourier transform of the scaling function $a_2(\omega)$ [Eqs. (V.41) and (VI.8)] in terms of the singular behavior of the structure function $A_2(x)$ on the light cone. Since

$$\Psi_1(s, \zeta^2) = \frac{\partial}{\partial s} \Phi_1(s, \zeta^2) \; , \tag{VI.44}$$

we find that

$$\int_0^\infty ds \, \Phi_1\left(s, \frac{(px)^2}{m^2}\right) = -\lim_{x^2 \to \infty} \left[\frac{16\pi^2}{\ln(m^2 x^2)} \right] A_1(x) , \tag{VI.45}$$

which expresses the Fourier transform of the scaling function $a_1(\omega)$ in terms of the light cone behavior of the structure function $A_1(x)$. We find that the scaling functions $a_{1,2}(\omega)$ are not only dimensionless functions of a dimensionless parameter, but that they are determined by the behavior of the structure functions on the light cone. This suggests that the conformal group may play some role here, for it contains various transformations which change the scale of coordinates but which leave the light cone invariant.

The scaling functions can also be related to "almost equal-time commutators" at infinite momentum, as suggested by Bjorken.[15] We consider the infinite nucleon momentum limit of the spatial current commutator

$$\lim_{p^o \to \infty} \frac{1}{\pi} \int_0^\infty dt \sin(\omega p^o t) \int (d\underset{\sim}{r})\, C^{k\ell}(x)$$

$$= \lim_{p^o \to \infty} \frac{1}{\pi} \int_0^\infty dt \sin(\omega p^o t) \int (d\underset{\sim}{r})\, \omega^2 p^{o2}$$

$$\left\{\left[p^{o2}\, \delta^{k\ell} - p^k p^\ell\right] C_1(x) + \delta^{k\ell}\, C_2(x)\right\}, \qquad \text{(VI.46)}$$

where we have introduced the decomposition (V.4) and integrated the time derivatives by parts using the Jost-Lehmann result that $C_{1,2}(x)$ vanish at t = 0. The limit can be calculated if we use the Jost-Lehmann representation

$$C(x) = \int_0^\infty ds\, \Psi(s, \zeta^2)\, \Delta(x;s), \qquad \text{(V.37)}$$

with

$$\Psi(s,\zeta^2) = \int \frac{(d\underset{\sim}{u})}{(2\pi)^3}\, e^{i\underset{\sim}{u}\cdot\underset{\sim}{\zeta}}\, \psi(s,u^2)\,, \qquad \text{(V.41)}$$

and, in a general frame,

$$\underset{\sim}{\zeta} = \underset{\sim}{r} + \underset{\sim}{p}\left[\frac{\underset{\sim}{p}\cdot\underset{\sim}{r}}{m(p^o+m)} + \frac{t}{m}\right]. \qquad \text{(V.39)}$$

It is also convenient to do the k^o integral in (V.13) and use

$$\Delta(x;s) = \int \frac{(d\underset{\sim}{k})}{(2\pi)^3}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, \frac{\sin(k^2+s)^{\frac{1}{2}} t}{(k^2+s)^{\frac{1}{2}}}. \qquad \text{(V.13')}$$

The limit can then be performed and, bearing in mind the connection between the scaling functions and the Jost-Lehmann weights, Eqs. (VI.8) and (VI.13), one finds

$$\lim_{p^o \to \infty} \frac{1}{\pi} \int_0^\infty dt \sin(\omega p^o t) \int (d\underline{r})\, C^{k\ell}(x)$$

$$= \left[\hat{p}^k \hat{p}^\ell - \delta^{k\ell} \right] a_1(\omega) + \delta^{k\ell} \omega a_2(\omega) \ . \qquad \text{(VI.47)}$$

VII. Equal-Time Commutators and High Energy Behavior

We have found that the scaling limit is related to the behavior of a current function on the light cone. We turn now to investigate what information can be obtained from the behavior of this function at the tip of the light cone or, equivalently, from the nature of current commutators at equal time. The relationship between the scaling limit and equal-time current commutators is obtained if we recall that

$$C^{\mu\nu}(x) = \tfrac{1}{2}\Sigma_\lambda \langle p\lambda | i[j^\mu(x), j^\nu(0)] | p\lambda \rangle$$

$$= [p^\mu p^\nu(-\partial^2) + (p^\mu \partial^\nu + \partial^\mu p^\nu) p\partial - g^{\mu\nu}(p\partial)^2] C_1(x)$$

$$+ [g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu] C_2(x) \ , \qquad \text{(VII.1)}$$

with

$$C_{1,2}(x) = \int_0^\infty ds \Psi_{1,2}\left(s, x^2 + \frac{(px)^2}{m^2}\right) \Delta(x;s) \ , \qquad \text{(VII.2)}$$

and if we use the relations

$$\Delta(x;s)]_{t=0} = 0 \ , \qquad \text{(VII.3a)}$$

$$\partial_o \Delta(x;s)]_{t=0} = \delta(\underline{r}) \ , \qquad \text{(VII.3b)}$$

and

$$\partial^2 \Delta(x;s) = s\Delta(x;s) \ . \qquad \text{(VII.3c)}$$

The simplest case is the equal-time limit

$$\tfrac{1}{2}\Sigma_\lambda \langle p\lambda | i[j^\mu(x), j^\nu(0)]_{t=0} | p\lambda \rangle$$

$$= \{ [p^\mu \underline{\nabla}^\nu + \underline{\nabla}^\mu p^\nu - 2\delta^{\mu\nu}(\underline{p}\cdot\underline{\nabla})] p^o$$

$$- [\underline{p}^\mu n^\nu + n^\mu \underline{p}^\nu](\underline{p}\cdot\underline{\nabla}) \} \delta(\underline{r}) \int_0^\infty ds\, \Psi_1(s,0)$$

$$+ [\underline{\nabla}^\mu n^\nu + n^\mu \underline{\nabla}^\nu]\, \delta(\underline{r}) \int_0^\infty ds\, \Psi_2(s,0) \ . \qquad \text{(VII.4)}$$

Here, in order to achieve a compact form, we use the notation

$$\underline{\nabla}^{\mu} = (0,\underline{\nabla}) \;,$$

$$\underline{p}^{\mu} = (0,\underline{p}) \;,$$

$$n^{\mu} = (1,0) \;,$$

$$\delta^{\mu\nu} = g^{\mu\nu} + n^{\mu} n^{\nu} \;. \qquad \text{(VII.5)}$$

The first combination, with a coefficient

$$\int_0^\infty ds\, \Psi_1(s,0) = \int \frac{(d\underline{u})}{(2\pi)^3} \int_0^\infty ds\, \psi_1(s,u^2) \;, \qquad \text{(VII.6)}$$

gives a contribution both to the commutator of the spatial currents, $[j^k, j^\ell]$ and to an operator "Schwinger term," the time-space commutator $[j^o, j^k]$. This "Schwinger term" has a rather complicated vector structure [it is not simply $\nabla^k \delta(\underline{r})$] and its coefficient is, of course, related to that of the spatial commutator $[j^k, j^\ell]$. The second combination, with a coefficient

$$\int ds\, \Psi_2(s,0) \;,$$

produces only a "Schwinger term" $[j^o, j^k]$ with the simple vector structure $\nabla^k \delta(\underline{r})$.

In view of the discussion of the previous section [c.f. Eq. (VI.8)], we may do the angular integral and write the coefficient of the first combination as

$$\int_0^\infty ds\, \Psi_1(s,0) = \frac{1}{2\pi^2} \int_0^m du \int_u^m u' du' \int_0^\infty ds\, \psi_1(s,u'^2)$$

$$= \left(\frac{2m}{\pi}\right) \int_0^m du \lim_{\nu\to\infty} [\nu A_1(\nu,k^2)] \;, \qquad \text{(VII.7)}$$

in which

$$k^2 = \omega\nu \;,$$

with

$$\tfrac{1}{2} m\omega = u \;. \qquad \text{(VII.8)}$$

The scaling law asserts that $A_1(\nu,k^2)$ behave as $1/\nu^2$ in this limit and hence that this contribution to the equal-time commutator vanish. That is, the validity of the scaling law requires that the nucleon (spin-averaged) matrix element of the equal-time, space-space current commutator vanish and that the related "Schwinger term" with the complicated vector structure also vanish. The converse statement is true as well, for A_1 is positive semi-definite. Thus, if either the equal-time, space-space commutator vanishes or if the related "complicated Schwinger term" vanishes, then the limit of $\nu A_1(\nu,k^2)$ must vanish and the scaling law for A_1 must hold.

There now remains only the "simple Schwinger term" of the form $\nabla^k\delta(\underline{r})$ with a coefficient, following the previous discussion that led to Eq. (VII.7), given by

$$\int_0^\infty ds\,\Psi_2(s,0) = \left(\frac{2m}{\pi}\right)\int_0^m du a_2(\omega) \quad , \qquad \text{(VII.9)}$$

in which $\frac{1}{2}m\omega = u$. The function $a_2(\omega)$ is also positive semi-definite, and it thus appears that the vanishing of this "Schwinger term" requires that $a_2(\omega)$ vanish identically and vice versa. This is wrong because Eq. (VII.9) is wrong. The error lies in a formula used in the derivation of Eq. (VII,7)

$$\int \frac{(d\underline{u})}{(2\pi)^3}\psi(s,u^2) = \frac{1}{2\pi^2}\int_0^m du \int_u^m u'du'\;\psi(s,u'^2) \qquad \text{(VII.10)}$$

which does not hold if $\psi(s,u^2)$ is a singular distribution at $u = 0$. This is the case with $\psi_2(s,u^2)$ where we have seen [Eq. (VI.30)] that it has such a singularity[17a] of the form $\nabla_u^2 u^{-1-\alpha}$. This fact, unfortunately, casts some aspersion on the character of Eq. (VII.7). However, if we assume that $A_1(\nu,k^2)$ has a well-behaved Regge behavior, then the weight $\psi_1(s,u^2)$ does not have such a bad singularity at $u = 0$ and Eq. (VII.7) does hold as well as the discussion of the preceding paragraph.

The commutator involving a time derivative of the current is also directly related to the scaling limit. Because of current conservation and the translation invariance of the diagonal matrix element, the only independent equal-time commutator is $[\partial_0 j^k, j^\ell]$. It is straightforward to express this commutator in terms of integrals over the weights $\psi_{1,2}$, and one obtains, assuming that A_1 satisfies the scaling law so that Eq. (VII.7) vanishes, an expression of the form

$$\tfrac{1}{2}\Sigma_\lambda \langle p\lambda | i[\partial_o j^k(x),\ j^\ell(0)]_{t=0} | p\lambda\rangle$$

$$= \delta^{k\ell}\delta(r)A + \nabla^k\nabla^\ell\,\delta(r)B$$

$$+ \delta(r)(p^k p^\ell - p^2\delta^{k\ell})C_T + \delta(r)p^k p^\ell C_S\ , \qquad \text{(VII.11)}$$

in which the coefficients A, B, C_T and C_S are numbers that are independent of the nucleon momentum p. The weights $\psi_{1,2}$ occur in the coefficients C_T and C_S in a way that is directly related to the scaling limits (VI.8) and (VI.13) and these coefficients can be written as

$$C_T = \frac{1}{\pi}\int_o^2 d\omega[\omega a_1(\omega) - \omega^2 a_2(\omega)]\ , \qquad \text{(VII.12a)}$$

and

$$C_S = \frac{1}{\pi}\int_o^2 d\omega\omega^2 a_2(\omega)\ . \qquad \text{(VII.12b)}$$

Here there is no difficulty with the small u singularities of the Jost-Lehmann weights.

The structure of the commutator (VII.11) can be compared[18) with that arising from simple models of the current operator. If the current is composed of a bilinear combination of spin ½ fields as in a quark model with a neutral vector field interaction, then it follows from simply the structure of the commutator that it contains the nucleon momentum in the transverse combination $(p^k p^\ell - p^2\delta^{k\ell})$ or that

$$C_S = 0 \quad \text{(quark model)} \quad . \qquad \text{(VII.13b)}$$

On the other hand, if the current is constructed in terms of a spin one field, as in the algebra of fields model, then the nucleon momentum enters only in the longitudinal form $p^k p^\ell$ and

$$C_T = 0 \quad \text{(field algebra)} \quad . \qquad \text{(VII.13a)}$$

Now, the integrands that enter into the definition of C_T and C_S, (Eqs. (VII.12)), are proportional to the scaling limit of the transverse and scalar cross sections defined in Eqs. (II.42) and are positive semi-definite. Therefore, the vanishing of the integral requires the vanishing of the integrand, and we reach the important conclusion that if the current operator is composed of fundamental spin ½ fields, then the scaling limit of the structure function $A_2(\nu, k^2)$ vanishes, or

$$a_2(\omega) = 0 \quad \text{(quark model)} . \tag{VII.14b}$$

Conversely, if the current operator is constructed from a spin one field, then the scaling limit of the transverse cross section $\sigma_T(\nu, k^2)$ mush vanish, or

$$a_1(\omega) - \omega a_2(\omega) = 0 \quad \text{(field algebra)} \quad . \tag{VII.14a}$$

Unfortunately, the calculation of the model commutators depends upon the naive manipulation of bilinear operator products at a common space-time point, products that are not well defined. Therefore, the validity of these results is open to question. It has been shown[19] that they do not, in fact, hold in perturbation calculations in some models. However, since the perturbation calculations diverge at high energy and require renormalization to make them finite, it is not clear that they are a reliable guide to high energy behavior. There is also difficulty[20] in obtaining the scaling law itself in perturbation calculations; here additional logarithmic terms appear.

The electromagnetic mass of the nucleon can be expressed in terms of the Jost-Lehmann weights and its parts that may be divergent can be related to the nonvanishing of certain equal-time commutators. This problem has been discussed at some length in the literature.[21]

References

1. Although the contribution of double (hard) photon exchange is indeed small, of relative order $\alpha \simeq 1/137$, infrared photon corrections can be substantial as we shall indicate in Sec. IV. We shall proceed, however, with the optimistic position that such infrared effects have been correctly removed from the data so that we need analyze only the interesting strong-interaction structure of the electroproduction process.
2. There is nothing to be learned from high energy inelastic electron scattering experiments in which the polarizations of the incident and scattered electrons are measured. If these spins are both measured, the spin averaging in Eq. (II.28) is not done, and in the Breit frame where the analysis is clear, the tensor $a^{\mu\nu}$ is replaced by

$$a^{\mu\nu}_{\kappa',\kappa} = [\bar{u}_{\kappa'}(-\underline{q})\,\gamma^{(\mu} u_{\kappa}(\underline{q})]^* [\bar{u}_{\kappa'}(-\underline{q})\,\gamma^{\nu)} u_{\kappa}(\underline{q})] \; .$$

Here, as the notation indicates, we have symmetrized the indices $\mu\nu$ since only the symmetrical part survives the contraction with the symmetrical nucleon structure tensor $A^{\mu\nu}$. At high energy the electron mass can be neglected, and a simple calculation shows that, in the Breit frame,

$$a^{oo}_{\kappa',\kappa} = 0 ,$$

and

$$a^{k\ell}_{\kappa',\kappa} = 4(q^2\delta^{k\ell} - \underline{q}^k\underline{q}^\ell)\,\delta_{\kappa',\kappa\pm\frac{1}{2}} ,$$

or

$$a^{\mu\nu}_{\kappa',\kappa} = 2(q^\mu q'^\nu + q'^\mu q^\nu - g^{\mu\nu}q'q)\delta_{\kappa',\kappa\pm\frac{1}{2}} .$$

Hence, even if both the incident and scattered electron spins are measured, the electron current tensor $a^{\mu\nu}_{\kappa',\kappa}$ remains proportional to its spin averaged form (II.30), and no additional information is obtained. This situation is changed if the electron is replaced by a muon whose mass m_μ is not entirely negligible, for in this case

$$a^{oo}_{\kappa',\kappa} = 4m^2_\mu\delta_{\kappa',\kappa}$$

does not vanish, and the two structure functions are separately determined, in principle, by a spin analysis of both the incident and scattered muon at a fixed angle.

3. The forward scattering Compton amplitude can also be written in the form of Eq. (II.22). Note that both ${}^{(1)}t^{\mu\nu}$ and ${}^{(2)}t^{\mu\nu}$ vanish as $k^\alpha k^\beta$ in the limit $k \to 0$, while, except for the nucleon pole contribution, the Compton invariants are regular in this limit. Hence, the forward Compton amplitude approaches its nucleon pole contribution as $k \to 0$, with an error of order k^2. This is an example of a general class of low energy theorems that follow from gauge invariance and the absence of spurious, kinematic singularities.

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$$\frac{1}{2}\sum_{\lambda} \langle p\lambda | i \left[j^{o}(x), j^{k}(o) \right]_{t=0} | p\lambda \rangle$$
$$= \nabla^{k} \delta(r) \; (m^{2}/\pi) \int_{o}^{2} d\omega \; \bar{a}_{2}(\omega),$$

where $\bar{a}_2(\omega)$ is the scaling limit function with the Regge contributions of the form (VI.38) removed; it need not be positive.

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ELECTRON-POSITRON ANNIHILATION INTO HADRONS†

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This paper is a revised and enlarged version of notes previously prepared for the lectures delivered, last winter, at:

1. VIII Internationale Universitätswochen für Kernphysik der Universität Graz, Schladming, Austria -- February 24-March 8, 1969.
2. IVe Rencontre de Moriond sur les Interactions Electromagnétiques, Verbier, Switzerland -- March 11-March 21, 1969.

Special stress has been laid on some theoretical problems like the π-meson electromagnetic form factor and the data reviewed correspond to the latest experimental information available to us. A detailed study of electromagnetic mixing problems as those occurring between the ρ^o and ω has also been added.

SECTION A: The One-Photon Exchange Approximation

I. Structure of the Cross Section

1°) We are interested in the annihilation process

$$e^+ + e^- \Rightarrow f$$

where f is an arbitrary final state compatible with the usual conservation laws of electric, baryonic, leptonic charges ($Q = 0$; $B = 0$; $L_e = 0$; $L_\mu = 0$). Using the kinematics as indicated on Fig. 1

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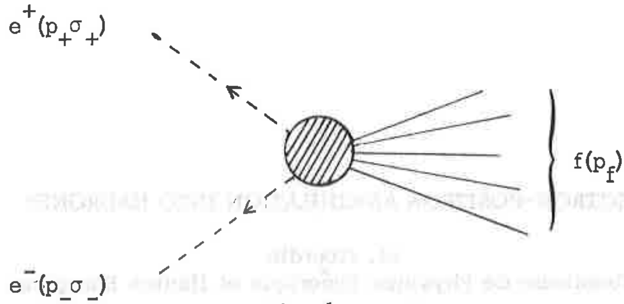

Fig. 1

the differential cross section is given by

$$d\sigma(e^+e^- \Rightarrow f) = \frac{1}{[(p_+p_-)^2 - m^4]^{\frac{1}{2}}} \left(\frac{m^2}{4} \sum_{\sigma_+\sigma_-} \right)$$

$$\sum_{\text{pol.f}} (2\pi)^4 \delta_4 (p_+ + p_- - p_f) |\langle f|T|e^+e^-\rangle|^2 \, d\rho_f$$

where m is the electron mass.

The final state density is written as

$$d\rho_f = \prod_{a \in f} \frac{N_a}{(2\pi)^3} \frac{d_3 p_a}{E_a} .$$

The normalization factor N_a is $\frac{1}{2}$ for bosons and the mass m_a for fermions; E_a and p_a are the energy and momentum of the particle a.

2°) Let us now assume that the electron-positron annihilation proceeds via the one-photon exchange. The transition matrix element is then factorized into the product of two matrix elements of the electromagnetic current

$$\langle f|T|e^+e^-\rangle = -\frac{e^2}{s} \langle f|J^{em}_\mu(0)|0\rangle \, \bar{V}_{\sigma_+}(p_+) \gamma^\mu u_{\sigma_-}(p_-) \qquad (A.1)$$

where $u_{\sigma_-}(p_-)$ is the free Dirac spinor for the electron and $V_{\sigma_+}(p_+)$ the free Dirac spinor for the positron.

The electric charge e is normalized so that $\alpha = e^2/4\pi = 1/137$ and the invariant quantity s is defined in our metric by

$$s = -(p_+ + p_-)^2 = -p_f^2$$

e.g. s is the square of the total energy of the centre-of-mass system. It is straightforward to perform the summation over the electron-positron polarizations

$$\frac{m^2}{4}\sum_{\sigma_+\sigma_-}\left[\bar{v}_{\sigma_+}(p_+)\gamma^\mu u_{\sigma_-}(p_-)\right]\left[\bar{v}_{\sigma_+}(p_+)\gamma^\nu u_{\sigma_-}(p_-)\right]^* = \tfrac{1}{4}\left(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu + \frac{s}{2}g^{\mu\nu}\right) \quad (A.2)$$

and the differential cross section takes the form

$$d\sigma(e^+e^- \Rightarrow f) = \frac{1}{[(p_+p_-)^2 - m^4]^{\frac{1}{2}}}\frac{e^4}{s^2}\tfrac{1}{4}\left(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu + \frac{s}{2}g^{\mu\nu}\right)\{f\}_{\mu\nu} \quad (A.3)$$

where the final-state tensor $\{f\}_{\mu\nu}$ is defined by

$$\{f\}_{\mu\nu} = \sum_{\text{pol.}f}(2\pi)^4\delta_4(p_+ + p_- - p_f)\,\langle f|J_\mu^{em}(0)|0\rangle\langle 0|J_\nu^{em}(0)|f\rangle\, d\rho_f\,. \quad (A.4)$$

3°) Let us now work in the electron-positron centre-of-mass system. We define

$$p_+ = (\vec{p},\, p_o) \qquad\qquad p_- = (-\vec{p},\, p_o)$$

$$\frac{s}{4} = p_o^2 = \vec{p}^2 + m^2\,.$$

The electron-positron tensor (A.2) has only space components because of the conservation of the electromagnetic current

$$p_+^\mu p_-^\nu + p_+^\nu p_-^\mu + \frac{s}{2}g^{\mu\nu} \Rightarrow \frac{s}{2}\left(\delta_{mn} - \frac{p_m p_n}{p_o^2}\right)$$

and Eq. (A.3) becomes

$$d\sigma(e^+e^- \Rightarrow f)_{CM} = \frac{4\pi^2\alpha^2}{s^{3/2}(s-4m^2)^{\frac{1}{2}}}\left(\delta_{mn} - \frac{p_m p_n}{p_o^2}\right)\{f\}_{mn} \quad (A.5)$$

The electron mass m can be neglected compared to the electron energy in almost all the applications and Eq. (A.5) reduces to

$$d\sigma(e^+e^- \Rightarrow f)_{CM} = \frac{4\pi^2\alpha^2}{s^2}\left(\delta_{mn} - \frac{p_m p_n}{\vec{p}^2}\right)\{f\}_{mn} . \qquad (A.6)$$

The total cross section $\sigma_{tot}(e^+e^- \Rightarrow f)$ is obtained integrating over all the angular variables

$$\left(\delta_{mn} - \frac{p_m p_n}{\vec{p}^2}\right)\{f\}_{mn} \Rightarrow \frac{2}{3}\,\mathrm{Tr}\,\{f\}$$

and the final expression for $\sigma_{tot}(e^+e^- \Rightarrow f)$ is simply

$$\sigma_{tot}(e^+e^- \Rightarrow f) = \frac{8\pi^2\alpha^2}{3}\,\frac{1}{s^2}\,\mathrm{Tr}\,\{f\} . \qquad (A.7)$$

Going back to the definition (A.4) of the final-state tensor $\{f\}_{\mu\nu}$, its trace Tr $\{f\}$ is given in an invariant way by

$$\mathrm{Tr}\{f\} = S_f (2\pi)^4 \delta_4(p_+ + p_- - p_f)\, \langle f|J_\mu^{em}(0)|0\rangle\langle 0|J_\nu^{em}(0)|f\rangle\, g^{\mu\nu} \qquad (A.8)$$

where the symbol S_f means

a) a summation over the polarization of the final-state particles
b) a phase-space integration

$$S_f \equiv \sum_{pol.f} \int d\rho_f .$$

II. The Final State $\bar{M}M$ Where M Is a Spinless Meson

1°) We restrict ourselves to a final two-body state $f \equiv \bar{M}M$ where M is a spin zero meson. For instance $M = \pi^+, K^+, K^o$ etc.

Let us first study the structure of the M meson electromagnetic vertex

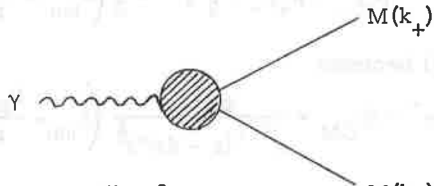

Fig. 2

Using the Lorentz covariance and the electromagnetic current conservation we simply have

$$\langle \bar{M}M | J_\mu^{em}(0) | 0 \rangle = (k_+ - k_-)_\mu F_M(s) . \qquad (A.9)$$

The invariant function $F_M(s)$ is, by definition, the M meson electromagnetic form factor and the normalization has been chosen so that $F_M(0)$ is the electric charge of the M meson in unit e.

2°) The tensor $\{\bar{M}M\}_{\mu\nu}$ is simply defined by Eqs. (A.4) and (A.9)

$$\{\bar{M}M\}_{\mu\nu} = \frac{1}{(4\pi)^2} |F_M(s)|^2 (k_+ - k_-)_\mu (k_+ - k_-)_\nu \left\{ \frac{d_3 k_+}{k_{+o}} \frac{d_3 k_-}{k_{-o}} \times \delta_4 (p_+ + p_- - k_+ - k_-) \right\} .$$

In the centre-of-mass system, the energy momentum variables are the following

$$k_+ = (\vec{k}, k_o) \qquad k_- = (-\vec{k}, k_o)$$

$$\frac{s}{4} = k_o^2 = \vec{k}^2 + m_M^2$$

where m_M is the M meson mass.

The phase-space density is simply written as

$$\frac{d_3 k_+}{k_{+o}} \frac{d_3 k_-}{k_{-o}} \delta_4 (p_+ + p_- - k_+ - k_-) \Rightarrow \frac{k}{\sqrt{s}} d\Omega_k$$

from which we deduce

$$\{\bar{M}M\}_{mn} = \frac{1}{\pi} \frac{d\Omega_k}{4\pi} \frac{k}{\sqrt{s}} |F_M(s)|^2 k_m k_n .$$

The differential cross section for the annihilation $e^+e^- \Rightarrow \bar{M}M$ takes the form

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{M}M)}{d\Omega} CM = \frac{\alpha^2}{8} \frac{1}{s} \frac{\left(1 - \frac{4m_M^2}{s}\right)^{3/2}}{\left(1 - \frac{4m^2}{s}\right)^{\frac{1}{2}}} \left[1 - \left(1 - \frac{4m^2}{s}\right) Z^2\right] |F_M(s)|^2$$

where Z is the cosine of the CM angle $(\vec{p}.\vec{k} = |p||k|Z)$.

Neglecting the electron mass m we obtain

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{M}M)_{CM}}{d\Omega} = \frac{\alpha^2}{8}\,\frac{1}{s}\left(1 - \frac{4m_M^2}{s}\right)^{3/2}(1-Z^2)\,|F_M(s)|^2 . \quad (A.10)$$

3°) The total cross section is computed by integrating the differential cross section (A.10)

$$\sigma_{tot}(e^+e^- \Rightarrow \bar{M}M) = \frac{\pi\alpha^2}{3}\,\frac{1}{s}\left(1 - \frac{4m_M^2}{s}\right)^{3/2}|F_M(s)|^2 . \quad (A.11)$$

An equivalent way to obtain the equality (A.11) is to calculate $Tr\{\bar{M}M\}$ and to use Eq. (A.7). We simply have

$$Tr\{\bar{M}M\} = \frac{1}{8\pi}\, s\left(1 - \frac{4m_M^2}{s}\right)^{3/2}|F_M(s)|^2 .$$

III. The Final State $\bar{N}N$ Where N Is a Spin J Particle

1°) We now study a more general final two-body state $f \equiv \bar{N}N$ where N is a spin J particle.

The tensor $\{\bar{N}N\}_{\mu\nu}$ is constructed from the matrix elements of the electromagnetic current between the vacuum and the $\bar{N}N$ state. After summation over the N and $\bar{N}$ polarizations, the only possible structure of $\{\bar{N}N\}_{\mu\nu}$ giving a nonvanishing contribution to the differential cross section is

$$\{NN\}_{\mu\nu} = \frac{2J+1}{(4\pi)^2}\{A(s)(k_+-k_-)_\nu + \tfrac{1}{2}(s-4m_N^2)B(s)g_{\mu\nu}\}$$
$$\times\left\{\frac{d_3k_+}{k_{+o}}\,\frac{d_3k_-}{k_{-o}}\,\delta_4(p_++p_--k_+-k_-)\right\} .$$

Using the centre-of-mass variables introduced in the previous section

$$\{\bar{N}N\}_{mn} = \frac{1}{\pi}\,\frac{d\Omega_k}{4\pi}\,\frac{k}{\sqrt{s}}\left\{A(s)k_m k_n + \tfrac{1}{2}B(s)k^2\delta_{mn}\right\}(2J+1) .$$

The differential cross section for the annihilation $e^+e^- \Rightarrow \bar{N}N$ takes the form

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{N}N)_{CM}}{d\Omega} = \frac{(2J+1)\alpha^2}{8}\,\frac{1}{3}\left(1 - \frac{4m_M^2}{s}\right)^{3/2}\left[A(s)(1-Z^2)+B(s)\right] . (A.12)$$

As a consequence of the one-photon exchange approximation the differential cross section for the electron-positron annihilation into a $\bar{N}N$ system is a linear function of Z^2 in the centre-of-mass system.

2°) Using the Lorentz covariance, the parity conservation and the time-reversal invariance, the electromagnetic vertex of the N particle depends on 2J+1 invariant functions called form factors

$$\langle NN | J_\mu^{em}(0) | 0 \rangle = \sum_{\ell=0}^{\ell=2J} I_\mu^\ell \, F_N^\ell(s)$$

where the I_μ^ℓ span a basis of covariants.

In a convenient basis the (2J+1) form factors $F_N^\ell(s)$ can be normalized at s = 0 to the (2J+1) static moments of the N particle. We then define the physical form factors and they are alternatively of the electric and magnetic type.

It is now a simple matter of algebra to relate the invariant functions A(s) and B(s) to the physical form factors $F_N^\ell(s)$. The result can be written in the form

$$A(s) = E(s) - \frac{s}{4m_N^2} M(s) \qquad B(s) - \frac{s}{2m_N^2} M(s) \qquad \text{(A.13)}$$

where E(s) and M(s) are sums of terms $|F_N^\ell(s)|^2$ of the electric type for E(s) (ℓ even) and of the magnetic type for M(s) (ℓ odd).

The normalization of E(s) and M(s) at s = 0 is simply

$$E(0) = q_N^2 \qquad\qquad M(0) = \frac{J+1}{3J} \mu_N^2$$

where q_N is the electric charge in unit e and μ_N the dipole magnetic moment in unit $e/2m_N$ of the particle N.

Inserting the decomposition (A.13) into Eq. (A.12) we obtain the general expression for the differential cross section

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{N}N)_{CM}}{d\Omega} = \frac{(2J+1)\alpha^2}{8} \frac{1}{s} \left(1 - \frac{4m_N^2}{s}\right)^{3/2} \times \left[(1 - Z^2)\, E(s) + (1 + Z^2) \frac{s}{4m_N^2} M(s) \right].$$

By measuring the angular distribution we can only separate the electric and magnetic contributions. The knowledge of the N and $\bar{N}$ polarizations is needed in order to obtain information about the individual form factors.

The total cross section is then given by

$$\sigma_{tot}(e^+e^- \Rightarrow \bar{N}N) = \frac{(2J+1)\pi\alpha^2}{3}\,\frac{1}{s}\left(1 - \frac{4m_N^2}{s}\right)^{3/2}\left[E(s) + \frac{s}{2m_N^2}\,M(s)\right]. \qquad \text{(A.15)}$$

3°) The case J = 0 has been considered in part II and we have only one electromagnetic form factor normalized to the electric charge. The case J = 1 will be interesting when the available incident energies will allow the production of vector meson pairs $\bar{V}V$ (like $\rho^+\rho^-$, $K^*\bar{K}^*$) or axial vector meson pairs.

We have two electric form factors, the charge form factor $F_V^0(s)$ and the quadrupole form factor $F_V^2(s)$ and one magnetic form factor $F_V^1(s)$. The corresponding expressions for E(s) and M(s) are simply

$$E(s) = |F_V^0(s)|^2 + \frac{s^2}{18m_V^2}|F_V^2(s)|^2 \qquad M(s) = \tfrac{2}{3}|F_V^1(s)|^2 .$$

The differential cross section for the reaction $e^+e^- \Rightarrow \bar{V}V$ is given by

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{V}V)}{d\Omega}\,\text{CM} = \frac{3\alpha^2}{8s}\left(1 - \frac{4m_V^2}{s}\right)^{3/2}\left\{(1-Z^2)\left[|F_V^0(s)|^2 + \frac{s^2}{18m_V^2}|F_V^2(s)|^2\right] + (1+Z^2)\frac{s}{6m_V^2}|F_V^1(s)|^2\right\}. \qquad \text{(A.16)}$$

and for the total cross section we obtain

$$\sigma_{tot}(e^+e^- \Rightarrow \bar{V}V) = \frac{\alpha^2\pi}{s}\left(1 - \frac{4m_V^2}{s}\right)^{3/2}\left\{|F_V^0(s)|^2 + \frac{s}{3m_V^2}|F_V^1(s)|^2 + \frac{s^2}{18m_V^2}|F_V^2(s)|^2\right\}. \qquad \text{(A.17)}$$

4°) Let us now consider the case $J = \frac{1}{2}$ corresponding for instance to nucleons and hyperons. We have one electric form factor and one magnetic form factor easily related to the form factors $F_1(s)$ and $F_2(s)$ defined in the usual Dirac basis by

$$\langle \bar{N}N | J_\mu^{em}(0) | 0\rangle = i\,\bar{u}(k_+)\left[\gamma_\mu F_1(s) + \frac{i}{4m_N}[\gamma_\mu,\gamma_\nu](k_+-k_-)^\nu F_2(s)\right]V(k_-)\,.$$

The normalization of F_1 and F_2 at $s = 0$ is given by

$$F_1(0) = q_N \qquad F_2(0) = \varkappa_N$$

where q_N is the electric charge in unit e and $\varkappa_N$ the anomalous magnetic moment in unit $e/2m_N$ of the particle N.

The differential cross section is given by

$$\frac{d\sigma(e^+e^- \Rightarrow \bar{N}N)_{CM}}{d\Omega} = \frac{\alpha^2 m_N^2}{s^2}\left(1 - \frac{4m_N^2}{s}\right)^{\frac{1}{2}}\left\{(1-Z^2)|F_1(s)|^2 + \frac{s}{4m_N^2}F_2(s)|^2 + (1+Z^2)\frac{s}{4m_N^2}|F_1(s)+F_2(s)|^2\right\} \tag{A.18}$$

and for the total cross section we obtain

$$\sigma_{tot}(e^+e^- \Rightarrow \bar{N}N) = \frac{8\pi\alpha^2 m_N^2}{3s^2}\left(1 - \frac{4m_N^2}{s}\right)^{\frac{1}{2}}\left\{|F_1(s) + \frac{s}{4m_N^2}F_2(s)|^2 + \frac{s^2}{2m_N^2}|F_1(s) + F_2(s)|^2\right\}\,. \tag{A.19}$$

IV. The Final State $P^o\gamma$ Where P^o Is a Pseudoscalar Meson

1°) We now study the final state $f \equiv P^o\gamma$ where P^o is a pseudoscalar meson of mass m_o as for instance $P_o = \pi^o, \eta, \eta'$.

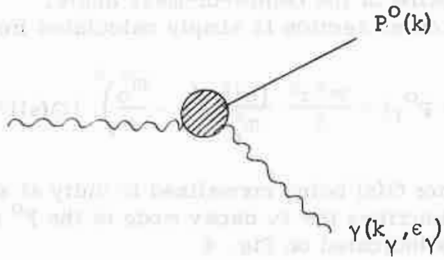

Fig. 3

From Lorentz covariance and parity conservation the matrix element of the electromagnetic current between the vacuum and the $P^o\gamma$ state has the general structure

$$\langle P^o\gamma | J_\mu^{em}(0) | 0\rangle = e\, g\, \frac{G(s)}{m_o} \epsilon_{\mu\nu\rho\sigma} k^\nu k_\gamma^\rho \epsilon_\gamma^\sigma$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the completely skew symmetric Rici tensor. The form factor G(s) has been normalized to unity at s = 0. In the centre-of-mass system we have

$$\langle P^o\gamma | J_m^{em}(0) | 0\rangle = e\, g\, \frac{G(s)}{m_c} \sqrt{s}\, \epsilon_{mpq} k^p \epsilon_\gamma^q .$$

After summation over the two transverse polarization states of the photon we obtain the space components of the tensor $\{P^o\gamma\}$ in the centre-of-mass sytem

$$\{P^o\gamma\}_{mn} = \frac{d\Omega_k}{4\pi} \frac{\alpha}{m_o^2} k^3 \sqrt{s} |g|^2 |G(s)|^2 \left[\delta_{mn} - \frac{k_m k_n}{k^2}\right]$$

and the trace of the tensor $\{P^o\gamma\}$

$$\mathrm{Tr}\ \{P^o\gamma\} = 2\alpha \frac{k^3\sqrt{s}}{m_o^2} |g|^2 |G(s)|^2 . \qquad \text{(A.21)}$$

2°) Combining Eqs. (A.6) and (A.20) we compute the differential cross section in the centre-of-mass system

$$\frac{d\sigma(e^+e^- \Rightarrow P^o\gamma)_{CM}}{d\Omega} = \frac{\pi\alpha^3}{8} \frac{|g|^2}{m_o^2} \left(1 - \frac{m_o^2}{s}\right)^3 (1+Z^2)\ |G(s)|^2 \qquad \text{(A.22)}$$

where Z is the cosine of the centre-of-mass angle.

The total cross section is simply calculated from Eqs. (A.7) and (A.21)

$$\sigma_{tot}(e^+e^- \Rightarrow P^o\gamma) = \frac{2\pi^2\alpha^3}{3} \frac{|g|^2}{m_o^2} \left(1 - \frac{m_o^2}{s}\right)^3 |G(s)|^2 . \qquad \text{(A.23)}$$

3°) The form factor G(s) being normalized to unity at s = 0, the coupling constant g describes the 2γ decay mode of the P^o meson. With the kinematics as indicated on Fig. 4

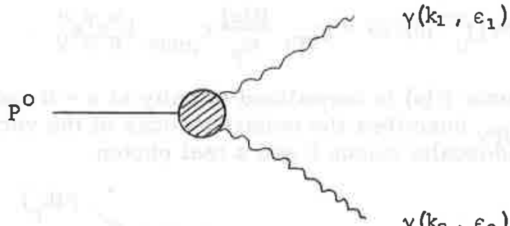

Fig. 4

the transition matrix element for the $P^o \Rightarrow 2\gamma$ is written as

$$\langle 2\gamma | T | P^o \rangle = e^2 \, g \frac{1}{m_o} \varepsilon_{\mu\nu\rho\sigma} \, k_1^\mu \, \varepsilon_1^\nu \, k_2^\rho \, \varepsilon_2^\sigma$$

It is then straightforward to compute the radiative decay width for the P^o meson, taking into account by a factor $\frac{1}{2}$ the Bose statistics satisfied by the two photons in the final state

$$\Gamma(P^o \Rightarrow 2\gamma) = \frac{\pi\alpha^2}{4} \, |g|^2 \, m_o \, . \qquad (A.24)$$

Taking into account this expression of the coupling constant g we can write the differential cross section (A.22) and the total cross section (A.23) in the equivalent form

$$\frac{d\sigma(e^+e^- \Rightarrow P^o\gamma)_{CM}}{d\Omega} = \frac{\alpha}{2} \, \frac{\Gamma(P^o \Rightarrow 2\gamma)}{m^3} \left(1 - \frac{m_o^2}{s}\right)^3 (1 + Z^2) \, |G(s)|^2$$

$$\sigma_{tot}(e^+e^- \Rightarrow P^o\gamma) = \frac{8\pi\alpha}{3} \, \frac{\Gamma(P^o \Rightarrow 2\gamma)}{m_o^3} \left(1 - \frac{m_o^2}{s}\right)^3 \, |G(s)|^2 \, .$$

4°) The previous calculation is easily extended to a final state $f \equiv PV$ where P is a pseudoscalar meson of mass m_P and V a vector meson of mass m_V.

Let us first define the various matrix elements entering the calculation

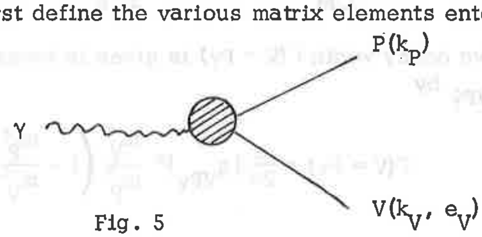

Fig. 5

$$\langle PV | J_\mu^{em}(0) | 0 \rangle = g_{VP\gamma} \frac{H(s)}{m_P} \varepsilon_{\mu\nu\rho\sigma} k_P^\nu k_V^\rho e_V^\sigma \quad .$$

The form factor H(s) is normalized to unity at s = 0 and the coupling constant $g_{VP\gamma}$ describes the radiative decay of the vector meson V into a pseudoscalar meson P and a real photon

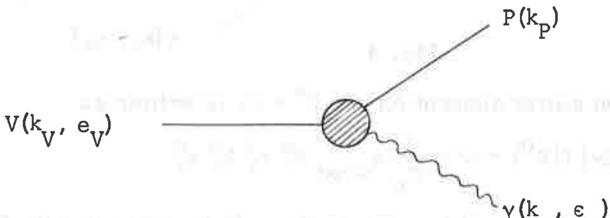

Fig. 6

$$\langle P\gamma | T | V \rangle = e\, g_{VP\gamma} \frac{1}{m_P} \varepsilon_{\mu\nu\rho\sigma} k_\gamma^\mu \varepsilon_\gamma^\nu k_V^\rho e_V^\sigma \quad .$$

The differential cross section for the process $e^+e^- \Rightarrow PV$ is given, in the centre-of-mass system, by

$$\frac{d\sigma(e^+e^- \Rightarrow PV)_{CM}}{d\Omega} = \frac{\alpha^2}{4} \frac{|g_{VP\gamma}|^2}{m_P^2} \left(\frac{k_{CM}}{\sqrt{s}}\right)^3 (1+Z^2)|H(s)|^2$$

and for the total cross section we obtain

$$\sigma_{tot}(e^+e^- \Rightarrow PV) = \frac{4\pi\alpha^2}{3} \frac{|g_{VP\gamma}|^2}{m_P^2} \left(\frac{k_{CM}}{\sqrt{s}}\right)^3 |H(s)|^2 .$$

The centre-of-mass momentum k_{CM} is related to masses by

$$k_{CM} = \frac{[s-(m_V+m_P)^2]^{\frac{1}{2}}[s-(m_V-m_P)^2]}{2\sqrt{s}} \quad .$$

The radiative decay width $\Gamma(V \Rightarrow P\gamma)$ is given in terms of the coupling constant $g_{VP\gamma}$ by

$$\Gamma(V \Rightarrow P\gamma) = \frac{\alpha}{24} |g_{VP\gamma}|^2 \frac{m_V^3}{m_P^2} \left(1 - \frac{m_P^2}{m_V^2}\right)^3 .$$

V. Three Pseudoscalar Meson Final State

1°) Let us now consider a final state with three pseudoscalar mesons $f \equiv P_1 P_2 P_3$ as for instance $\pi^+ \pi^- \pi^0$, $K \bar{K} \pi$

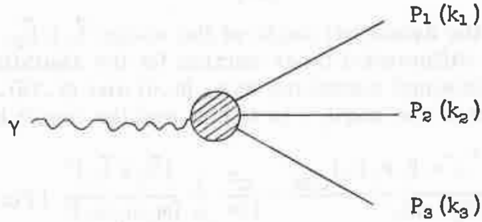

Fig. 7

Due to Lorentz covariance and to parity conservation the matrix element of the electromagnetic current between the vacuum and the $P_1 P_2 P_3$ state has the simple structure

$$\langle P_1 P_2 P_3 | J_\mu^{em}(0) | 0 \rangle = \varepsilon_{\mu\nu\rho\sigma} \left(\frac{k_1}{m_1}\right)^\nu \left(\frac{k_2}{m_2}\right)^\rho \left(\frac{k_3}{m_3}\right)^\sigma F(s_1, s_2, s_3) \ . \qquad \text{(A.25)}$$

The invariants s_1, s_2, s_3 are defined by

$$s_j = -k_j \cdot P \qquad P = k_1 + k_2 + k_3$$

with the relation

$$s_1 + s_2 + s_3 = s \ .$$

In the centre-of-mass system the invariants s_j are simply related to the energy of the meson P_j

$$s_j = \sqrt{s}\, E_j \qquad \sqrt{s} = E_1 + E_2 + E_3$$

and the three particle form factor $F(s_1, s_2, s_3)$ can be equivalently considered as a function of the variables s, E_1 and E_2

$$F(s_1, s_2, s_3) \Rightarrow F(s; E_1, E_2)$$

2°) In the centre-of-mass system the space components of the tensor $\{P_1 P_2 P_3\}$ are given by

$$\{P_1P_2P_3\}_{mn} = \frac{s}{(m_1 m_2 m_3)^2} |F(s; E_1, E_2)|^2 (\vec{k}_1 \times \vec{k}_2)_m (\vec{k}_1 \times \vec{k}_2)_n$$

$$\times \frac{1}{(4\pi)^3} dE_1 \, dE_2 \, d\cos\theta \tag{A.26}$$

where θ is the azimuthal angle of the vector $\vec{k}_1 \times \vec{k}_2$.

The differential cross section for the annihilation $e^+e^- \Rightarrow P_1P_2P_3$ is obtained combining Eqs. (A.6) and (A.26). The integration with respect to the angle θ is trivial and the result is simply

$$\frac{d^2\sigma(e^+e^- \Rightarrow P_1P_2P_3)_{CM}}{dE_1 \, dE_2} = \frac{\alpha^2}{12\pi} \frac{1}{s} \frac{|\vec{k}_1 \times \vec{k}_2|^2}{(m_1 m_2 m_3)^2} |F(s; E_1, E_2)|^2 . \tag{A.27}$$

For the total cross section we have to integrate the expression (A.27) in a domain D(s) defined by the condition that k_1, k_2 and k_3 are sides of a triangle

$$\sigma_{tot}(e^+e^- \Rightarrow P_1P_2P_3) = \frac{\alpha^2}{12\pi s} \iint_{D(s)} dE_1 \, dE_2 \frac{|\vec{k}_1 \times \vec{k}_2|^2}{(m_1 m_2 m_3)^2} |F(s; E_1, E_2)|^2 . \tag{A.28}$$

VI. Discussion

1°) From the beginning of this chapter we have assumed that the annihilation of the electron-positron pair into hadrons proceeds via the exchange of a virtual time-like photon. What are the physical arguments to justify such an approximation?

First the two-photon exchange amplitude is expected to be reduced with respect to the one-photon exchange amplitude by a factor α. Secondly, the one-photon exchange approximation has been tested for space-like photons in various experiments.

a) angular distribution in elastic electron-proton and electron-nucleus scattering

b) angular distribution in inelastic electron-proton and electron-nucleus scattering

c) comparison of the elastic electron-proton and positron-proton cross sections

d) polarization of the recoil proton in elastic electron-proton scattering.

No evidence has been found for the presence of a measurable two-photon amplitude.

2^{o}) Let us consider now the actual case of time-like photons. In the one-photon exchange approximation the angular distribution for the e^+e^- annihilation into an arbitrary two-body final state is a linear function of Z^2 where Z is the cosine of the centre-of-mass system.

Moreover, in some particular cases, the angular distribution is predicted to be pure $(1 - Z^2)$ as for the $\pi^+\pi^-$, $K\bar{K}$ systems or pure $(1 + Z^2)$ as for the $\pi^o\gamma$, $\pi^o\omega$ systems.

It is certainly difficult to check carefully such a prediction. Nevertheless, one can, for instance, look for an asymmetry in the angular distribution with respect to a plane orthogonal to the incident direction. Such an asymmetry is obviously related to terms odd in Z.

3^{o}) Another way to detect the presence of a two-photon exchange contribution is to observe a final state which is an eigenstate of the charge conjugation operator with a positive eigenvalue

a) a 2γ state in pure electrodynamics

b) a $N^o N^o$ state where N^o is an eigenstate of the particle-antiparticle conjugation operator C like π^o, η, ρ^o, ω or φ. Because of the TCP invariance, the $N^o N^o$ state cannot be connected to one photon only and as an example the observation of the reaction $e^+e^- \Rightarrow \pi^o\pi^o$ is an unambiguous proof of the presence of a two-photon exchange amplitude.

Such a production can be enhanced by a strong final-state interaction as occurring for instance in the $\pi^o\pi^o$ system around the f^o resonance.

c) a $N^o M^o$ state where $N^o \neq M^o$ are both eigenstates of C with the same eigenvalue ($\pi^o\eta$, $\rho^o\omega$, $\rho^o\varphi$, $\omega\varphi$...). If $C_{N^o} = -1$ the particle M^o can be a photon ($\rho^o\gamma$, $\omega\gamma$, $\varphi\gamma$...). If C is conserved in the electromagnetic interactions, the $N^o M^o$ final state can be reached only via a two-photon exchange. But if C is not conserved the observation of a $N^o M^o$ system can be interpreted as a violation of C in the one-photon exchange amplitude.

SECTION B: The π-Meson Electromagnetic Form Factor

I. Measurement of the π-Meson Electromagnetic Form Factor

1^{o}) The π-meson electromagnetic vertex can be represented by the diagram of Fig. 1

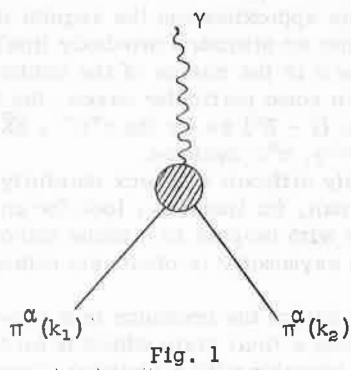

Fig. 1

The index $\alpha = +, 0, -$, indicates the charge state of the π^α meson. Using the Lorentz covariance and the electromagnetic current conservation, we obtain the following structure for the matrix element of the electromagnetic current between two one-π meson states

$$\langle \pi^\alpha | J_\mu^{em}(0) | \pi^\alpha \rangle = (k_1 + k_2)_\mu F_{\pi^\alpha}(s) \tag{B.1}$$

where $s = -(k_1 - k_2)^2$.

For π mesons on the mass shell, s is negative in Eq. (B.1). Using the hermiticity property of the electromagnetic current we easily check that the electromagnetic form factor $F_{\pi^\alpha}(s)$ is real in the space-like region $s \leq 0$.

$$F^*_{\pi^\alpha}(s) = F_{\pi^\alpha}(s) . \tag{B.2}$$

Applying now the TCP invariance we obtain

$$F_{\pi^\alpha}(s) = q^\alpha F_\pi(s)$$

where q^α is the electric charge of the meson π^α in unit e.

For a real photon, $s = 0$, the vertex function with the three particles on the mass shell reduces to the coupling constant which, in the present case is simply the electric charge q^α. It follows

$$F_\pi(0) = 1 . \tag{B.3}$$

The matrix element (B.1) is the analytic continuation of the matrix element of the electromagnetic current between the vacuum and a $\pi^+\pi^-$ state previously introduced in Section A. With the notation of Fig. 2

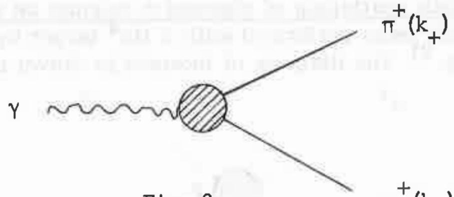

Fig. 2

we simply have

$$\langle \pi^+\pi^- | J_\mu^{em}(0) | 0 \rangle = (k_+ - k_-)_\mu \, F_\pi(s) \qquad \text{(B.4)}$$

where $s = -(k_+ + k_-)^2$.

For π mesons on the mass shell, s is positive in Eq. (B.4).

2^o) There exist, at least in principle, several ways to measure the π-meson electromagnetic form factor in the spacelike region $s < 0$. The available experimental information is an evaluation of the slope of $F_\pi(s)$ at $s = 0$. The convenient parameter used is the so-called root mean square radius defined by

$$r_\pi^2 = 6 \, F_\pi(0)$$

where the derivative is taken with respect to s.

a) Elastic scattering of charged π mesons on atomic electrons

The lowest order diagram is represented on Fig. 3

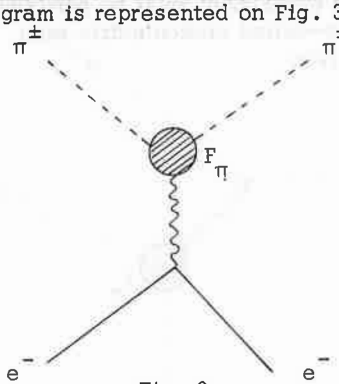

Fig. 3

and the experiment gives[1)]

$$r_\pi < 3 \text{ fermi.}$$

b) Coulomb scattering of charged π mesons on nucleus

The experiment has been performed with a He^4 target by comparing π^+ and π^- scattering.[2)] The diagram of interest is drawn on Fig. 4.

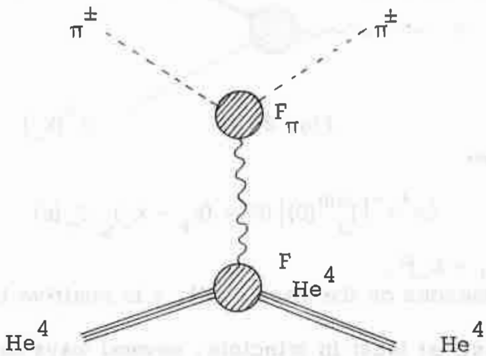

Fig. 4

The He^4 electromagnetic form factor is known from electron-He^4 elastic scattering experiments. The main difficulty in extracting $F_\pi(s)$ is an accurate determination of the nuclear effects. The result is

$$r_\pi < 0.9 \text{ fermi.}$$

c) Electroproduction of π^+ meson on proton

The electroproduction experiment must be performed in a kinematical situation where the so-called photoelectric term, represented on Fig. 5 plays an important role

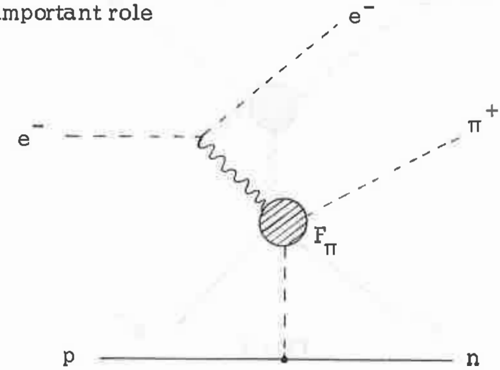

Fig. 5

Unfortunately the background is always important and for a large part, model dependent.

The result of two experiments is the following

$$r_{\pi} = (0.80 \pm 0.10) \text{ fermi}^{3)}$$

$$r_{\pi} = (0.86 \pm 0.14) \text{ fermi}.^{4)}$$

3°) In the timelike region the form factor $F_{\pi}(s)$ becomes complex above the $\pi^+\pi^-$ threshold $s_o = 4m_{\pi}^2$. The storage ring experiments as those recently performed in Novosibirsk and Orsay allow a direct measurement of $|F_{\pi}(s)|$ by looking at the electron-positron pair annihilation into a $\pi^+\pi^-$ system. The corresponding lowest-order diagram is represented on Fig. 6.

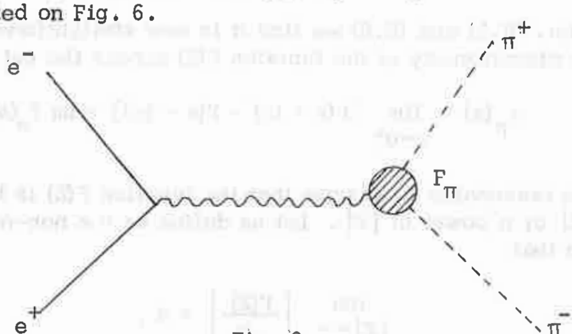

Fig. 6

II. Dispersion Relation

1°) We introduce the complex Z plane with a cut on the real positive axis starting from $s_o = 4m^2$ to $+\infty$ ($s = \text{Re } Z$)

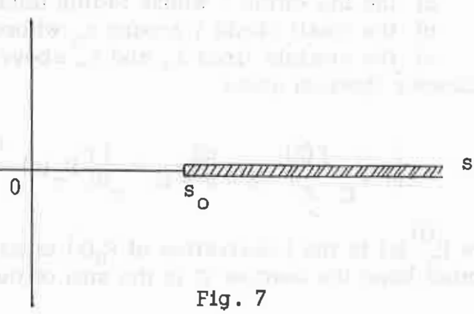

Fig. 7

We denote by $F(Z)$ an analytic function of Z in the complex cut plane and which coincides, on the real axis, with the π-meson electromagnetic form factor $F_\pi(s)$ following

$$F_\pi(s) = F(s) \quad \text{for } s < s_o$$

$$F_\pi(s) = \lim_{\varepsilon=0^+} F(s + i\varepsilon) \quad \text{for } s > s_o \ . \tag{B.5}$$

The reality condition (B.2) satisfied by $F_\pi(s)$ on the real negative axis becomes the Schwartz reflexion principle

$$F(Z^*) = F(Z)^* \quad . \tag{B.6}$$

Using Eqs. (B.5) and (B.6) we find it is now straightforward to compute the discontinuity of the function $F(Z)$ across the cut

$$\sigma_\pi(s) = \lim_{\varepsilon\to 0^+} [F(s + i\varepsilon) - F(s - i\varepsilon)] = \text{Im } F_\pi(s) \ . \tag{B.7}$$

2°) It is reasonable to assume that the function $F(Z)$ is bounded for large $|Z|$ by a power of $|Z|$. Let us define as n a non-negative integer such that

$$\lim_{|Z|\Rightarrow\infty} \left| \frac{F(Z)}{Z^n} \right| = 0 \ . \tag{B.8}$$

We now apply the Cauchy theorem to the function $F(Z)/Z^n$ which is meromorphic in the cut plane.

The contour C is shown in Fig. 8 and it can be divided into three parts

a) the big circle Γ whose radius tends to infinity
b) the small circle γ around s_o whose radius tends to zero
c) the straight lines L_+ and L_- above and below the cut.

The Cauchy theorem gives

$$\frac{1}{2i\pi} \int_C \frac{F(Z)}{Z^n} \frac{dZ}{Z - s - i\varepsilon} = \frac{1}{s^n}\left[F_\pi(s) - \sum_{j=0}^{j=n-1} s^j F_\pi^{(j)}(0)\right] \tag{B.9}$$

where $F_\pi^{(j)}(s)$ is the j derivative of $F_\pi(s)$ taken with respect to s. On the other hand the contour C is the sum of three contours

$$C = \Gamma + \gamma + (L_+ + L_-)$$

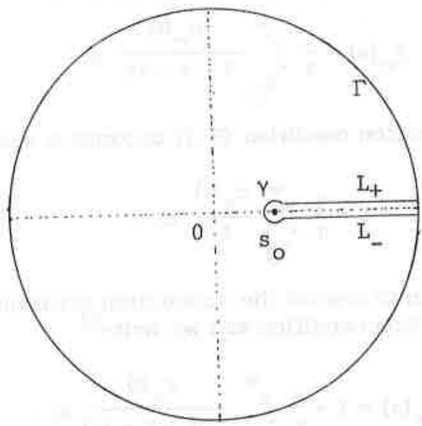

Fig. 8

and an equivalent expression can be obtained as the sum of three integrals along Γ, γ and $(L_+ + L_-)$

a) the contribution coming from Γ vanishes because of the condition

b) the contribution coming from γ vanishes if F(Z) is regular at $Z = s_o$; if not, we extract the singularity and we apply the same technique to the regular part

c) the contribution coming from $L_+ + L_-$ gives the dispersion integral after the use of equality (B.7), provided the integral converges

$$\frac{1}{2i\pi}\int_{L_+ + L_-} \frac{F(Z)}{Z^n} \frac{dZ}{Z - s - i\epsilon} = \frac{1}{\pi}\int_{s_o}^{\infty} \frac{\sigma_\pi(t)}{t^n(t - s - i\epsilon)} dt \, . \qquad (B.10)$$

The dispersion relation with n subtractions is finally obtained combining the equalities (B.9) and (B.10)

$$F_\pi(s) = \sum_{j=0}^{j=n-1} s^j F_\pi^{(j)}(0) + \frac{s^n}{\pi}\int_{s_o}^{\infty} \frac{\sigma_\pi(t)}{t^n(t - s - i\epsilon)} dt \, . \qquad (B.11)$$

If the function $F(Z)$ tends to zero at infinity in all the directions of the Z plane, we can write an unsubtracted dispersion relation

$$F_\pi(s) = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{\sigma_\pi(t)}{t - s - i\epsilon} dt \tag{B.12}$$

and the normalization condition (B.3) becomes a sum rule

$$1 = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{\sigma_\pi(t)}{t} dt \; .$$

If one subtraction is needed the subtraction constant is determined by the normalization condition and we write[5)]

$$F_\pi(s) = 1 + \frac{s}{\pi} \int_{s_o}^{\infty} \frac{\sigma_\pi(t)}{t(t - s - i\epsilon)} dt \; . \tag{B.13}$$

3°) In order to make useful the dispersion relation (B.11) we compute the spectral function $\sigma_\pi(s)$ using the unitarity property of the S matrix.

The spectral function $\sigma_\pi(s)$ is conveniently written as a sum of contributions due to intermediate states m

$$\sigma_\pi(s) = \sum_m \sigma_\pi^{(m)}(s)$$

and a straightforward calculation gives

$$\sigma_\pi^{(m)}(s) = S_n (2\pi)^4 \; \delta_4 (k_+ + k_- - k_m) \tfrac{1}{2} \; \langle m | \frac{(k_+ - k_-)^\mu J_\mu^{em}(0)}{(k_+ - k_-)^2} | 0 \rangle$$

$$\times \langle \pi^+ \pi | T | m \rangle^* \; . \tag{B.14}$$

The intermediate states $|m\rangle$ are restricted by the energy momentum, Dirac distribution and some other conservation laws. For instance all the possible intermediate states must have zero electric charge, zero baryonic charge and zero leptonic charges.

In the lowest-order approximation with respect to electromagnetic interactions the states m are strongly coupled to the $\pi^+\pi^-$ final state and have therefore strangeness $S = 0$ and total isotopic spin $I = 1$ (as a consequence of the generalized Pauli principle). It follows, in particular, that the G parity is conserved and there can be only even numbers of π mesons.

In the region $4m_\pi^2 \le s \le 16m_\pi^2$ only a $\pi^+\pi^-$ state can contribute and the so-called elasticity unitarity relation is diagramatically represented on Fig. 9.

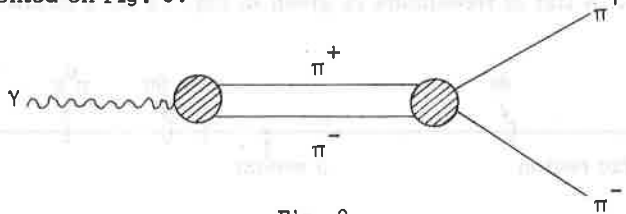

Fig. 9

Formula (B.14) reduces to

$$\sigma_\pi^{(2\pi)}(s) = \theta(s - s_o)\, F_\pi(s) \int \frac{d\Omega_n}{4\pi} Z \frac{k}{8\pi\sqrt{s}} \langle \pi^+\pi^- | T | \pi^+\pi^- \rangle^* \qquad (B.15)$$

when Z is the cosine of the centre-of-mass angle.

The amplitude for elastic $\pi^+\pi^-$ scattering in the total isotopic spin I = 1 can be expanded in Legendre polynomials of Z

$$\langle \pi^+\pi^- | T | \pi^+\pi^- \rangle = \frac{8\pi\sqrt{s}}{k} \sum_J (2J + 1)\, f_{1J}(s)\, P_J(s) \ .$$

From the elastic unitarity we have the constraint ($4m_\pi^2 \le s \le 16\, m_\pi^2$)

$$\mathrm{Im}\, f_{1J}(s) = |f_{1J}(s)|^2$$

or equivalently using the π-π phase shift

$$f_{1J}(s) = e^{i\delta_{1J}(s)} \sin \delta_{1J}(s) \ .$$

The angular integration in Eq. (B.15) extracts the P-wave term of the partial-wave expansion and the final result is simply

$$\sigma_\pi^{(2\pi)}(s) = F_\pi(s)\, f_{11}^*(s) \qquad (B.16)$$

or, using the phase-shift representation of $f_{11}(s)$:

$$\sigma_\pi^{(2\pi)}(s) = F_\pi(s)\, e^{-i\delta_{11}(s)} \sin \delta_{11}(s) \ . \qquad (B.17)$$

Equation (B.17) tells us that, in the region $4m_\pi^2 \leq s \leq 16m_\pi^2$, the phase of $F_\pi(s)$ is $\delta_{11}(s)$ modulo π.

4°) For $s \geq 16\ m_\pi^2$ other contributions can occur in the spectral function $\sigma_\pi(s)$. A list of thresholds is given in Fig. 10 for s below 1 GeV²

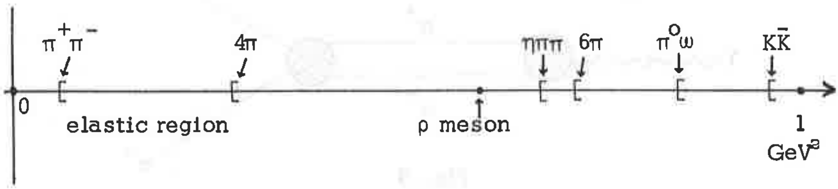

Fig. 10

Experimentally the ρ meson is a resonance in the J = 1, I = 1 partial wave of the π-π system which seems to be essentially elastic. The decay of the ρ meson into a 4π system is experimentally less than 1%.

It follows that, at least in the ρ-meson region, the 4π channel is not appreciably coupled to the 2π channel. Moreover, in the same region the phase shift $\delta_{11}(s)$ is always real. For these reasons the elastic unitarity relation

$$\sigma_\pi(s) = \sigma_\pi^{(2\pi)}(s)$$

is certainly valid for values of s above the 4π threshold and probably also in the ρ-meson region. We denote by $s_{inel.}$ the effective inelastic threshold and for $s_0 \leq s \leq s_{inel.}$ the phase of $F_\pi(s)$ is $\delta_{11}(s)$ modulo π.

III. The Phase Representation

1°) Let us recall the properties of the function F(Z) introduced in part II

a) F(Z) is analytic in the complex Z cut plane
b) $F(Z^*) = F(Z)^*$
c) F(Z) is bounded by a power of $|Z|$ as $|Z| \Rightarrow \infty$ in all direction.

We now assume a new condition on the function $\sigma_\pi(s)$ which is the discontinuity of F(Z) across the cut

d) $\sigma_\pi(s)$ is continuous and has only a finite number of zeros.

As a first consequence of these assumptions the function F(Z) has only a finite number of zeros.[6] From condition b) these zeros are distributed on the real axis, $s = a_j$ ($a_j < s_o$) or $s = b_k$ ($b_k \geq s_o$) or in pair of complex conjugate numbers in the Z plane: $Z = c_\ell$ and $Z = c_\ell^*$ as indicated on Fig. 11.

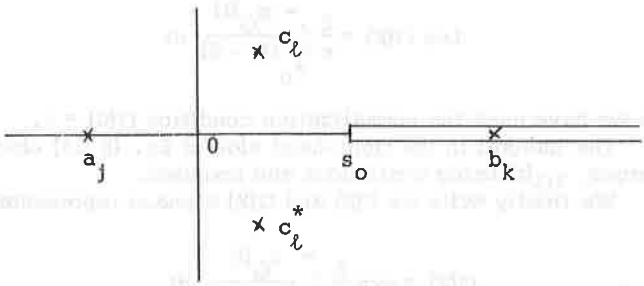

Fig. 11

Let us define the polynomial P(Z) by

$$P(Z) = \prod_{j,k,\ell}' \left(1 - \frac{Z}{a_j}\right)\left(1 - \frac{Z}{b_k}\right)\left(1 - \frac{Z}{c_\ell}\right)\left(1 - \frac{Z}{c_\ell^*}\right) .$$

Of course P(s) is real on the real axis and P(0) = 1.

The function G(Z) defined by

$$G(Z) = \frac{F(Z)}{P(Z)} \tag{B.18}$$

satisfies obviously the set of properties a), b), c), d) and has no zero in the complex Z plane. Moreover G(0) = 1.

Therefore, the function log G(Z) is also analytic in the cut plane and its discontinuity across the cut is given by

$$\varphi_G(s) = \frac{1}{2i} \lim_{\epsilon \Rightarrow 0^+} [\mathrm{Log}\, G(s+i\epsilon) - \mathrm{Log}\, G(s-i\epsilon)] = \arctan \frac{\mathrm{Im}\, F_\pi(s)}{\mathrm{Re}\, F_\pi(s)} . \tag{B.19}$$

From condition d), $\varphi_G(s)$ is continuous and bounded. By convention we choose $\varphi_G(s_o) = 0$ in what follows.

On the other hand, condition c) implies

$$\lim_{|Z| \Rightarrow \infty} \frac{\mathrm{Log}\, G(Z)}{Z} = 0 .$$

By applying the Cauchy theorem to the function Log G(Z)/Z with the contour C drawn in Fig. 8 we obtain the equality

$$\text{Log } G(Z) = \frac{Z}{\pi}\int_{s_o}^{\infty} \frac{\varphi_G(t)}{t(t-Z)}\, dt \qquad (B.20)$$

where we have used the normalization condition G(0) = 1.

The integral in the right-hand side of Eq. (B.20) obviously converges, $\varphi_G(t)$ being continuous and bounded.

We finally write for F(Z) and G(Z) a phase representation[7)]

$$G(Z) = \exp \frac{Z}{\pi}\int_{s_o}^{\infty} \frac{\varphi_G(t)}{t(t-Z)}\, dt \qquad (B.21)$$

$$F(Z) = P(Z) \exp \frac{Z}{\pi}\int_{s_o}^{s} \frac{\varphi_G(t)}{t(t-Z)}\, dt\,. \qquad (B.22)$$

2^o) Let us now study the asymptotic behaviour of G(Z) as a consequence of the phase representation (B.21).

By convention $\varphi_G(s) = 0$ and from assumption d), $\varphi_G(\infty)$ exists and is finite. Let us put

$$\varphi_G(\infty) = \pi N_G\,.$$

It is convenient to introduce an auxiliary function $\bar{\varphi}_G(s)$ which vanishes at infinity

$$\varphi_G(s) \equiv \bar{\varphi}_G(s) + \varphi_G(\infty)$$

and to define the integral

$$u_G(Z) = \frac{Z}{\pi}\int_{s_o}^{\infty} \frac{\bar{\varphi}_G(t)}{t(t-Z)}\, dt\,. \qquad (B.23)$$

A straightforward calculation gives

$$\text{Log } G(Z) = \text{Log}\left(\frac{s_o - Z}{s_o}\right)^{-N_G} + u_G(Z)\,. \qquad (B.24)$$

The asymptotic behaviour of $u_G(Z)$ and therefore of $G(Z)$ obviously depends on the precise high energy behaviour of the phase $\bar{\varphi}_G(s)$.

Let us assume for the moment that the integral

$$-\frac{1}{\pi}\int_{s_o}^{\infty}\frac{\bar{\varphi}_G(t)}{t}\,dt \tag{B.25}$$

converges and let us call its value ρ_∞.

It is straightforward to show that under reasonable conditions on the derivative of $\varphi_G(s)$ for large s

$$\lim_{|Z|\Rightarrow\infty} u_G(Z) = \rho_\infty \quad . \tag{B.26}$$

Using now Eq. (B.24) we obtain the asymptotic expression of $G(Z)$ for large Z

$$G(Z) \simeq e^{\rho_\infty}\left(1-\frac{Z}{s_o}\right)^{-N_G} \tag{B.27}$$

Such a situation occurs for instance when $\bar{\varphi}_G(s)$ behaves at infinity like $1/s^\epsilon$ with $\epsilon > 0$ or $1/\mathrm{Log}\, s)^\alpha$ with $\alpha > 1$.

If now the integral (B.25) diverges, Eq. (B.26) is no longer true and $u_G(Z)$ tends to infinity for large Z.

The more critical behaviour occurs when Z is in the cut because of the presence of principal value integral. Let us define

$$\rho_G(s) = \mathrm{Re}\, u_G(s+i\epsilon) = \frac{s}{\pi}\,\mathrm{PV}\int_{s_o}^{\infty}\frac{\bar{\varphi}_G(t)}{t(t-s)}\,dt \; .$$

It is straightforward to prove that

$$\lim_{s\Rightarrow+\infty}\frac{\rho_G(s)}{\mathrm{Log}\, s} = 0 .$$

Moreover, $\bar{\varphi}_G(s)$ having a finite number of zeros possesses asymptotically a definite sign, ϵ_G. We deduce

if $\epsilon_G = +1$ $\exp \rho_G(s) \Rightarrow 0$ less rapidly than any power of $1/s$

if $\epsilon_G = -1$ $\exp \rho_G(s) \Rightarrow \infty$ less rapidly than any power of s.

Let us consider, as an example, the case where $\bar{\varphi}_G(s)$ behaves for large positive s like π a/Log s. Using a theorem due to Frye and Warnock[8] on the asymptotic limit of principal value integrals, we have

$$\rho_G(s) = -\frac{1}{\pi}\int_{s_0}^{s} \frac{\bar{\varphi}_G(t)}{t} dt + 0\left[\frac{1}{\text{Log } s}\right] .$$

Therefore, for large positive s

$$\rho_G(s) \simeq -a \text{ Log}(\text{Log } s) \qquad \exp \rho_G(s) \simeq \frac{1}{(\text{Log } s)^a} . \tag{B.28}$$

3°) From Eq. (B.24) the function F(Z) can be written in the form

$$F(Z) = P(Z)\left(1 - \frac{Z}{s_0}\right)^{-N_G} \exp u_G(Z) \tag{B.29}$$

and we denote by R the degree of P(Z).

Using the results of the previous paragraph on the asymptotic behaviour of $u_G(Z)$ we easily check that F(Z) satisfies a dispersion relation of type (B.11) with n subtractions where the non-negative integer n is restricted by

$$n > R - N_G .$$

The equality $n = R - N_G$ can occur in the following particular situation

a) $\int \frac{\bar{\varphi}_G(t)}{t}$ dt diverges

b) $\epsilon_G = +1$, e.g. there exists a $T > s_0$ such that for $t > T$, $\bar{\varphi}_G(t) \geq 0$. (B.30)

The solution depends obviously of R arbitrary parameters one can choose as the zeros of P(Z).

Conversely, let us look for solutions of the dispersion relation (B.11) having the form (B.29). Now, n and N_G are fixed. In the general case, Eq. (B.11) has solutions if and only if $n + N_G > 0$ and the number of linearly independent solutions is then R + 1, where R is the maximum non-negative integer less than $n + N_G$.

In the particular case where the conditions (B.30) are fulfilled, even if $n + N_G = 0$, there exists a unique solution to Eq. (B.11).

As an illustration of the previous results, we consider the unsubtracted dispersion relation (B.12). In general, this equation has no solution if $N_G \leq 0$. However, if $N_G = 0$, the only possible solution must satisfy the particular conditions (B.30) as occurs, for instance, when the high-energy behaviour of $\bar{\varphi}_G(s)$ is given by

$$\bar{\varphi}_G(s) \simeq \frac{a\pi}{\mathrm{Log}\ s} \quad \text{with } a > 0 \ .$$

4°) From the unitarity relation (B.17), in the elastic region $s_o \leq s \leq 4s_o$ the phase $\varphi_F(s_o)$ of the form factor is equal modulo π to the phase $\delta_{11}(s)$ of the elastic π-π scattering amplitude $I = 1$, $J = 1$. Assuming $\delta_{11}(s_o) = 0$ we identify $\varphi_G(s)$ and $\delta_{11}(s)$ in this region.

As discussed in Section II, such an identification can be extended to a larger domain including in particular the ρ-meson region

$$\varphi_G(s) = \delta_{11}(s) \qquad \text{for } s_o \leq s \leq s_{inel.} \ .$$

More generally, we define a phase $\eta(s)$ which represents the contributions of states other than 2π to the unitarity relation

$$\varphi_G(s) \equiv \delta_{11}(s) + \eta(s) \ . \tag{B.31}$$

This phase $\eta(s)$ appears only for $s \geq s_{inel.}$ and we choose $\eta(s_{inel.}) = 0$. The Omnès function[9] associated to the π-π scattering amplitude $I = 1$, $J = 1$ is defined by

$$G_{11}(Z) = \exp \frac{Z}{\pi} \int_o^\infty \frac{\delta_{11}(t)}{t(t-Z)}\, dt \ .$$

As a consequence of equation (B.31) the contributions of the $\pi^+\pi^-$ intermediate state are explicitly exhibited following

$$G(Z) = G_{11}(Z) \exp \frac{Z}{\pi} \int_{s_{inel.}}^\infty \frac{\eta(t)}{t(t-Z)}\, dt \ .$$

Models can be used to construct $G_{11}(Z)$ and therefore to obtain approximate expressions for $G(Z)$ and $F(Z)$.

IV. The Modulus Representation

1°) We consider an analytic function $F(Z)$ with properties a), b), and c). In this section, condition d) is replaced by a weaker one d′).

The function F(Z) has only a finite number of zeros and its phase increases less rapidly, in modulus than $|Z|^{\frac{1}{2}}$ in all directions.

We can define a polynomial P(Z), of finite degree, having the same zeros as F(Z) and normalized so that P(0) = 1.

It follows that the function $G(Z) = P^{-1}(Z)\,F(Z)$ satisfies conditions a), b) and c). Moreover, it is normalized to unity at Z = 0 and has no zeros in the complex Z plane.

2^{o}) Equation (B.21) gives an expression of G(Z) in the cut plane in terms of the phase $\varphi_G(t)$. An equivalent expression of G(Z) can be obtained in terms of the modulus $|G_\pi(t)|$ assumed to be known on the cut $t \geq s_o$.[10),11)] It is the object of this section. Let us consider the function

$$\frac{\operatorname{Log} G(Z)}{(Z - s_o)^{\frac{1}{2}}} \;.$$

It is an analytic function of Z in the cut plane which tends to zero as $|Z| \Rightarrow \infty$ in all directions. We apply the Cauchy theorem to that function with the contour C of Fig. 8. The result is simply

$$\operatorname{Log} G(Z) = \frac{(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\operatorname{Log} |G_\pi(t)|}{(t - s_o)^{\frac{1}{2}}(t - Z)}\, dt \;. \tag{B.33}$$

The normalization condition G(0) = 1 gives a sum rule

$$\frac{2m_\pi}{\pi} \int_{s_o}^{\infty} \frac{\operatorname{Log}|G_\pi(t)|}{(t - s_o)^{\frac{1}{2}}\, t}\, dt = 0 \;. \tag{B.34}$$

Combining Eqs. (B.33) and (B.34) we can write a more convergent expression

$$\operatorname{Log} G(Z) = \frac{Z(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\operatorname{Log} |G_\pi(t)|}{(t - s_o)^{\frac{1}{2}}\, t(t - Z)}\, dt \;. \tag{B.35}$$

3^{o}) In particular the phase $\varphi_G(s)$ is deduced from Eq. (B.35) to be given by a principal value integral ($s \geq s_o$)

$$\varphi_G(s) = -\frac{s(s - s_o)^{\frac{1}{2}}}{\pi}\, PV \int_{s_o}^{\infty} \frac{\operatorname{Log} |G_\pi(t)|}{t(t - s_o)^{\frac{1}{2}}(t - s)}\, dt \;. \tag{B.36}$$

For $s_o \le s \le 4s_o$ the phase $\varphi_G(s)$ can be identified with the phase $\delta_{11}(s)$ of the P wave scattering amplitude for $\pi\pi$ scattering. Therefore, $\varphi_G(s)$ must behave like $(s - s_o)^{3/2}$ around the threshold point. Using the identity

$$\frac{(Z - s_o)^{\frac{1}{2}}}{(t - s_o)^{\frac{1}{2}}(t - Z)} \equiv \frac{d}{dt}\left(\mathrm{Log}\,\frac{(t - s_o)^{\frac{1}{2}} - (Z - s_o)^{\frac{1}{2}}}{(t - s_o)^{\frac{1}{2}} + (Z - s_o)^{\frac{1}{2}}}\right)$$

we perform an integration by parts in Eq. (B.33) and obtain an equivalent expression

$$\mathrm{Log}\,G(Z) = \mathrm{Log}\,G(s_o) - \frac{1}{i\pi}\int_{s_o}^{\infty} \mathrm{Log}\,\frac{(t - s_o)^{\frac{1}{2}} - (Z - s_o)^{\frac{1}{2}}}{(t - s_o)^{\frac{1}{2}} + (Z - s_o)^{\frac{1}{2}}}\,\frac{d}{dt}\mathrm{Log}|G_\pi(t)|dt \tag{B.37}$$

provided that $\mathrm{Log}\,|G_\pi(t)|$ is differentiable.

Let us make an expansion around $Z = s_o$ of the logarithmic term in the previous integral

$$\mathrm{Log}\,\frac{(t - s_o)^{\frac{1}{2}} - (Z - s_o)^{\frac{1}{2}}}{(t - s_o)^{\frac{1}{2}} + (Z - s_o)^{\frac{1}{2}}} \simeq -2\,\frac{(Z - s_o)^{\frac{1}{2}}}{(t - s_o)^{\frac{1}{2}}} + O[(Z - s_o)^{3/2}]\ .$$

Putting this in Eq. (B.37) we deduce an expansion of Log G(Z) around $Z \simeq s_o$

$$\mathrm{Log}\,G(Z) = \mathrm{Log}\,G(s_o) + \frac{2}{i\pi}(Z - s_o)^{\frac{1}{2}}\int_{s_o}^{\infty}\frac{dt}{(t - s_o)^{\frac{1}{2}}}\,\frac{d}{dt}\,\mathrm{Log}\,|G_\pi(t)| + {} + O[(Z - s_o)^{3/2}]$$

and the behaviour of $\varphi_G(s)$ around the threshold point implies the sum rule

$$\int_{s_o}^{\infty}\frac{dt}{(t - s_o)^{\frac{1}{2}}}\,\frac{d}{dt}\,\mathrm{Log}\,|G_\pi(t)| = 0\ . \tag{B.38}$$

The scattering length for the P-wave scattering amplitude is computed with an analogous method of integration by parts and the result is

$$\lim_{s \Rightarrow s_o} \frac{\delta_{11}(s)}{(s - s_o)^{3/2}} = \frac{4}{3\pi} \int_{s_o}^{\infty} \frac{dt}{(t - s_o)^{\frac{1}{2}}} \frac{d^2}{dt^2} \mathrm{Log}\,|G_\pi(t)|. \quad (B.39)$$

4°) The form factor in the spacelike region $s \leq 0$ is related to the modulus $|G_\pi(t)|$ in the timelike region by

$$G_\pi(s) = \exp\left[\frac{s(s_o - s)^{\frac{1}{2}}}{\pi} \int_{s_o}^{\infty} \frac{\mathrm{Log}\,|G_\pi(t)|}{t(t - s_o)^{\frac{1}{2}}(t - s)} dt\right]. \quad (B.40)$$

As an application of the previous expression the root mean square radius associated to the form factor $G_\pi(s)$ is represented by the highly convergent integral

$$r_G^2 = \frac{12 m_\pi}{\pi} \int_{s_o}^{\infty} \frac{\mathrm{Log}\,|G_\pi(t)|}{t^2 (t - s_o)^{\frac{1}{2}}} dt. \quad (B.41)$$

5°) Let us now study the high-energy behaviour for the phase $\varphi_G(s)$ in the timelike region ($s \Rightarrow + \infty$) and for the form factor $G_\pi(s)$ in the spacelike region ($s \Rightarrow - \infty$).

As a first remark, Eqs. (B.33) and (B.35) are equivalent if and only if the condition (B.34) is fulfilled. If not we define

$$x = \frac{1}{\pi} \int_{s_o}^{\infty} \frac{\mathrm{Log}\,|G_\pi(t)|}{(t - s_o)^{\frac{1}{2}} t} dt$$

and Eq. (B.35) is equivalently written as

$$\mathrm{Log}\,G(Z) = \frac{(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\mathrm{Log}\,|G_\pi(t)|}{(t - s_o)^{\frac{1}{2}}(t - Z)} dt + ix(Z - s_o)^{\frac{1}{2}}.$$

It is convenient to introduce a new function $\tilde{G}(Z)$ by

$$G(Z) = \tilde{G}(Z)\, e^{ix(Z - s_o)^{\frac{1}{2}}}.$$

On the cut $s \geq s_o$ we have the equality

$$\text{Log}\,|G_\pi(t)| = \text{Log}\,|\tilde{G}_\pi(t)|$$

and it follows that $\tilde{G}(Z)$ is a solution of an equation of the type (B.33)

$$\text{Log}\,\tilde{G}(Z) = \frac{(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\text{Log}\,|\tilde{G}_\pi(t)|}{(t - s_o)^{\frac{1}{2}}(t - Z)}\, dt$$

with the normalization condition $\tilde{G}(0) = e^{2xm_\pi}$.

We can therefore restrict our study to Eq. (B.33). The first proposition is the following: if there exists a finite number ν such that

$$\lim_{s \Rightarrow +\infty} \text{Log}\left\{\left(\frac{s}{s_o}\right)^{\nu \pm \epsilon} |\tilde{G}_\pi(s)|\right\} = \pm\infty \tag{B.42}$$

for all $\epsilon > 0$ then

$$\lim_{s \Rightarrow +\infty} \varphi_{\tilde{G}}(s) = \nu\pi \ . \tag{B.43}$$

Of course, from the existence of a polynomial bound for $|\tilde{G}(Z)|$ in the complex Z plane for large $|Z|$ (Eq. (B.8))

$$\lim_{|Z| \Rightarrow \infty} \left|\frac{\tilde{G}(Z)}{Z^n}\right| = 0$$

it follows

$$\nu > -n \ .$$

The quantity ν being defined by the condition (B.42) it is convenient to introduce a new analytic function $g(Z)$

$$g(Z) = \left(1 - \frac{Z}{s_o}\right)^{\nu} \tilde{G}(Z) \ . \tag{B.44}$$

Using the Cauchy theorem it is easy to prove the equality

$$\text{Log}\left(1 - \frac{Z}{s_o}\right) = \frac{(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\text{Log}\left(\frac{t}{s_o} - 1\right)}{(t - s_o)^{\frac{1}{2}}(t - Z)}\, dt$$

from which it follows, for g(Z), an equation analogous to (B.32)

$$\text{Log } g(Z) = \frac{(Z - s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\text{Log } |g_\pi(t)|}{(t - s_o)^{\frac{1}{2}} (t - Z)} dt \qquad (B.45)$$

with the normalization condition $g(0) = \exp(2x\, m_\pi)$.

We then are brought back to the case $\nu = 0$ in the previous proposition: if for all $\varepsilon > 0$

$$\lim_{s \Rightarrow +\infty} \text{Log} \left\{ \left(\frac{s}{s_o} \right)^{\pm\varepsilon} |g_\pi(s)| \right\} = \pm \infty \qquad (B.46)$$

then

$$\lim_{s \Rightarrow +\infty} \varphi_g(s) = 0 \ . \qquad (B.47)$$

The phase $\varphi_g(s)$ is normalized as usual to $\varphi_g(s_o) = 0$. From Eq. (B.45) we deduce its integral representation

$$\varphi_g(s) = -\frac{(s - s_o)^{\frac{1}{2}}}{\pi} \text{PV} \int_{s_o}^{\infty} \frac{\text{Log } |g_\pi(t)|}{(t - s_o)^{\frac{1}{2}} (t - s)} dt \ . \qquad (B.48)$$

From the restriction (B.46) on the high-energy behaviour of $|g_\pi(s)|$ in the timelike region it is straightforward to prove the result (B.47) as the limit, at infinite energy, of the principal value integral (B.48).

A more refined information about the high-energy behaviour of the phase $\varphi_g(s)$ is given by a second proposition

$$\lim_{s \Rightarrow +\infty} \frac{-\frac{1}{\pi} \int_{s_o}^{s} \frac{\varphi_g(t)}{t} dt}{\text{Log } |g_\pi(s)|} = 1 \ . \qquad (B.49)$$

The proof of Eq. (B.49) is obtained using the technique of the phase representation as explained in Sec. III.

Finally, as a consequence of the Phragmen-Lindelöf theorem, the high-energy behaviour of the form factor $g_\pi(s)$ in the spacelike region and in the timelike region are identical

$$\lim_{s \Rightarrow -\infty} \left| \frac{g_\pi(s)}{g_\pi(-s)} \right| = 1 \ . \qquad (B.50)$$

6°) We now consider the actual function $F(Z) \equiv P(Z)\, G(Z)$ and more specifically its logarithm.

Let us first apply the Cauchy theorem to the function

$$\frac{\operatorname{Log} P(Z)}{(Z - s_o)^{\frac{1}{2}}}$$

using an integration contour C_P excluding

a) the normal cut on the positive real axis $(s_o, +\infty)$ due to $(Z - s_o)^{\frac{1}{2}}$

b) the cuts associated to the zeros z_j of $P(Z)$ not located on the normal cut, e.g. of type a_j or c_ℓ, c_ℓ^*.

The discontinuity of $\operatorname{Log} P(Z)/(Z - s_o)^{\frac{1}{2}}$ across the normal cut is given by

$$\frac{1}{(s - s_o)^{\frac{1}{2}}} \lim_{\varepsilon \to 0^+} \tfrac{1}{2}[\operatorname{Log} P(s + i\varepsilon) + \operatorname{Log} P(s - i\varepsilon)] =$$

$$\frac{1}{(s - s_o)^{\frac{1}{2}}} [\operatorname{Log}|P(s)| + i\varphi_P(s_o)] .$$

Choosing $\varphi_P(0) = 0$, the phase $\varphi_P(s_o)$ is πN_{a^+} where N_{a^+} is the number of positive real zeros of $P(s)$ between 0 and s_o.

After a straightforward calculation of the contributions due to the other cuts of type b) we obtain the final result

$$\operatorname{Log} P(Z) = \frac{(Z-s_o)^{\frac{1}{2}}}{i\pi} \int_{s_o}^{\infty} \frac{\operatorname{Log}|P(t)|}{(t-s_o)^{\frac{1}{2}}(t-Z)} dt + i\varphi_P(s_o)$$

$$+ \sum_j \operatorname{Log} \frac{(z_j-s_o)^{\frac{1}{2}} - (Z-s_o)^{\frac{1}{2}}}{(z_j-s_o)^{\frac{1}{2}} + (Z-s_o)^{\frac{1}{2}}} . \qquad \text{(B.51)}$$

Combining Eqs. (B.33) and (B.51) we obtain a representation of $F(Z)$ in the cut plane in terms of the modulus $|F_\pi(t)|$ in the timelike region as measured in electron-positron annihilation experiments

$$\mathrm{Log}\,F(Z) = \frac{(Z-s_o)^{\frac{1}{2}}}{i\pi}\int_{s_o}^{\infty}\frac{\mathrm{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}(t-Z)}dt + i\varphi_P(s_o)$$

$$+\sum_j \mathrm{Log}\frac{(z_j-s_o)^{\frac{1}{2}} - (Z-s_o)^{\frac{1}{2}}}{(z_j-s_o)^{\frac{1}{2}} + (Z-s_o)^{\frac{1}{2}}} \quad . \qquad (B.52)$$

It is convenient to parametrize the zeros z_j in the following way

$$\left\| \begin{array}{ll} (s_o - a_j)^{\frac{1}{2}} = \alpha_j & \alpha_j > 0 \\ (c_\ell - s_o)^{\frac{1}{2}} = \beta_\ell + i\gamma_\ell & \beta_\ell > 0 \quad \gamma_\ell > 0 \\ (c_\ell^* - s_o)^{\frac{1}{2}} = -\beta_\ell + i\gamma_\ell & \end{array} \right.$$

Equation (B.52) becomes

$$\mathrm{Log}\,F(Z) = \frac{(Z-s_o)^{\frac{1}{2}}}{i\pi}\int_{s_o}^{\infty}\frac{\mathrm{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}(t-Z)}dt + i\varphi_P(s_o)$$

$$+\sum_j \mathrm{Log}\frac{\alpha_j + i(Z-s_o)^{\frac{1}{2}}}{\alpha_j - i(Z-s_o)^{\frac{1}{2}}} + \sum_j \mathrm{Log}\frac{\beta_\ell^2 + \gamma_\ell^2 + s_o - Z + 2i\,\gamma_\ell(Z-s_o)^{\frac{1}{2}}}{\beta_\ell^2 + \gamma_\ell^2 + s_o - Z - 2i\,\gamma_\ell(Z-s_o)^{\frac{1}{2}}} \, .$$

From the normalization condition $F(0) = 1$ we deduce a sum rule

$$\frac{2m_\pi}{\pi}\int_{s_o}^{\infty}\frac{\mathrm{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}t}dt = -i\varphi_P(s_o) + \sum_j \mathrm{Log}\frac{\alpha_j + 2m_\pi}{\alpha_j - 2m_\pi}$$

$$+\sum_\ell \mathrm{Log}\frac{\beta_\ell^2 + (\gamma_\ell + 2m_\pi)^2}{\beta_\ell^2 + (\gamma_\ell - 2m_\pi)^2} \, .$$

The first two terms in the right-hand side of the previous equality can be combined to give the simple result[12)]

$$\frac{2m_\pi}{\pi}\int_{s_o}^{\infty}\frac{\text{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}t}dt = \sum_j \text{Log}\left|\frac{\alpha_j+2m_\pi}{\alpha_j-2m_\pi}\right| + \sum_j \text{Log}\frac{\beta_\ell^2+(\gamma_\ell+2m_\pi)^2}{\beta_\ell^2+(\gamma_\ell-2m_\pi)^2} . \tag{B.53}$$

7°) The phase $\varphi_F(s)$ of the form factor $F_\pi(s)$ in the timelike region $s \geq s_o$ is deduced from Eq. (B.52) to be

$$\varphi_F(s) = \varphi_F(s_o) - \frac{(s-s_o)^{\frac{1}{2}}}{\pi}\text{PV}\int_{s_o}^{\infty}\frac{\text{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}(t-s)}dt$$

$$2\sum_j \text{Arc tan}\frac{(s-s_o)^{\frac{1}{2}}}{\alpha_j} + 2\sum_j \text{Arc tan}\frac{2\gamma_\ell(s-s_o)^{\frac{1}{2}}}{\beta_\ell^2+\gamma_\ell^2-(s-s_o)} . \tag{B.54}$$

Observing that, in Eq. (B.54) the angular contribution from each zero z_j is an increasing function of s starting from zero at $s = s_o$ and reaching $\pi/2$ when $s \Rightarrow +\infty$, we can obtain a lower bound for the phase $\varphi_F(s)$[12]

$$\varphi_F(s) - \varphi_F(s_o) \geq -\frac{(s-s_o)^{\frac{1}{2}}}{\pi}\text{PV}\int_{s_o}^{\infty}\frac{\text{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}(t-s)}dt . \tag{B.55}$$

From the unitarity relation (B.16) the phase difference $\varphi_F(s) - \varphi_F(s_o)$ is simply the phase shift $\delta_{11}(s)$ in the region $s_o \leq s \leq 4s_o$ and therefore must behave like $(s-s_o)^{3/2}$ around the threshold point $s = s_o$. Assuming that $F_\pi(s)$ does not have zeros on the cut, it is straightforward to generalize Eq. (B.38) and the result is simply[12]

$$\frac{1}{\pi}\int_{s_o}^{\infty}\frac{dt}{(t-s_o)^{\frac{1}{2}}}\frac{d}{dt}(\text{Log}|F_\pi(t)|) = \sum_j\frac{1}{\alpha_j} + 2\sum_\ell\frac{\gamma_\ell}{\beta_\ell^2+\gamma_\ell^2} . \tag{B.56}$$

The right-hand side of this equation is obviously positive and we obtain the following inequality

$$\frac{1}{\pi}\int_{s_o}^{\infty}\frac{dt}{(t-s_o)^{\frac{1}{2}}}\frac{d}{dt}\text{Log}|F_\pi(t)| \geq 0 \tag{B.57}$$

and the equal sign holds when and only when the form factor does not have zeros.

The value of the integral (B.57) gives a measure of the importance of the zeros in the region around the threshold.

Finally, the scattering length for the P wave ππ scattering amplitude is computed from Eqs. (B.39) and (B.54)

$$\lim_{s\to s_o} \frac{\delta_{11}(s)}{(s-s_o)^{3/2}} = \frac{4}{3\pi}\int_{s_o}^{\infty} \frac{dt}{(t-s_o)^{\frac{1}{2}}} \frac{d^2}{dt^2} \mathrm{Log}|F_\pi(t)|$$

$$-\frac{2}{3}\sum_j \frac{1}{\alpha_j^3} - \frac{4}{3}\sum_j \frac{\gamma_\ell(\gamma_\ell^2 - 3\beta_\ell^2)}{(\beta_\ell^2+\gamma_\ell^2)^3} \quad . \tag{B.58}$$

8°) In the spacelike region $s \le 0$ the form factor $F_\pi(s)$ has the following representation

$$F_\pi(s) = \exp\left\{\frac{(s_o - s)^{\frac{1}{2}}}{\pi}\int_{s_o}^{\infty} \frac{\mathrm{Log}|F_\pi(t)|}{(t-s_o)^{\frac{1}{2}}(t-s)}dt\right\}$$

$$\times (-1)^{N_a+} \prod_j \frac{\alpha_j-(s_o-s)^{\frac{1}{2}}}{\alpha_j+(s_o-s)^{\frac{1}{2}}} \prod_\ell \frac{\beta_\ell^2+[\gamma_\ell-(s_o-s)^{\frac{1}{2}}]^2}{\beta_\ell^2+[\gamma_\ell+(s_o-s)^{\frac{1}{2}}]^2} \ . \tag{B.59}$$

The condition $F_\pi(0) = 1$ implies the relation (B.53). It can be observed that all the logarithm terms in Eq. (B.53) are positive and it follows immediately a second inequality

$$\frac{2m_\pi}{\pi}\int_{s_o}^{\infty} \frac{dt}{(t-s_o)^{\frac{1}{2}}} \frac{\mathrm{Log}|F_\pi(t)|}{t} \ge 0 \ . \tag{B.60}$$

Again the equal sign holds when and only when the form factor does not have zeros.

The value of the integral (B.60) gives a measure of the importance of the zeros in the region around the origin.

Using similar arguments we can obtain, in the region $s \le s_o$ an inequality which generalizes (B.60)[12]

$$|F_\pi(s)| \le \exp \frac{(s_o - s)^{\frac{1}{2}}}{\pi} \int_{s_o}^{\infty} \frac{\mathrm{Log}|F_\pi(t)|}{(t - s_o)^{\frac{1}{2}}(t - s)} dt \ . \qquad (B.61)$$

In other words, Eq. (B.61) gives an upper bound of the form factor $F_\pi(s)$ in the spacelike region in terms of the modulus of the form factor as measured in the timelike region.

V. Form Factors with an Exponential Decreasing in the Spacelike Region

1°) Let us begin with the following theorem of the theory of distributions.[13]

The function F(Z) has the properties a), b) and c) if and only if there exists a real valued tempered distribution S whose support is contained in $\{x | x \ge 0\}$ such that for all Z in the complex cut plane, we have the representation

$$F(Z) = \int_{-\infty}^{+\infty} S(x)\, e^{ix(Z-s_o)^{\frac{1}{2}}} dx \qquad (B.62)$$

where

$$0 < \arg\, (Z - s_o)^{\frac{1}{2}} < \pi \ .$$

In particular on the real axis, the form factor $F_\pi(s)$ is a tempered distribution given by

$$F_\pi(s) = \int_{-\infty}^{+\infty} S(x)\, e^{ix(s-s_o)^{\frac{1}{2}}} dx \qquad \text{if } s \ge s_o \qquad (B.63)$$

$$F_\pi(s) = \int_{-\infty}^{+\infty} S(x)\, e^{-x(s_o-s)^{\frac{1}{2}}} dx \qquad \text{if } s \le s_o \qquad (B.64)$$

We easily check that for spacelike values of s, $F_\pi(s)$ is real.

The normalization of S is obtained from the condition F(0) = 1

$$1 = \int_{-\infty}^{+\infty} S(x)\, e^{-2m_\pi x} dx \ . \qquad (B.65)$$

Conversely if the form factor $F_\pi(s)$ is the boundary value of an analytic function $F(Z)$ with properties a), b) and c), then $F_\pi(s)$ is a tempered distribution and using the theory of Fourier transform, we obtain

$$S(x) = \frac{1}{2\pi} \int_{s_o}^{\infty} \frac{ds}{(s - s_o)^{\frac{1}{2}}} \operatorname{Re} e^{ix(s-s_o)^{\frac{1}{2}}} F_\pi(s) . \qquad \text{(B.66)}$$

2°) We restrict ourselves, in the following, to form factors which decrease like $\exp(-a|s|^{\frac{1}{2}})$ when $s \Rightarrow -\infty$ in the spacelike region. Such a behaviour has been suggested by Wu and Yang[14] in the framework of a model for large angle scattering at high energy in strong interactions.

It can be proved[15] that the two following statements

α) $\limsup_{s \Rightarrow -\infty} |s|^{-\frac{1}{2}} \operatorname{Log}|F_\pi(s)| \leq -a$ with $a > 0$

β) the support of S is contained in $\{x | x \geq a\}$

are equivalent.

Therefore, if

$$\lim_{s \Rightarrow -\infty} |s|^{-\frac{1}{2}} \operatorname{Log}|F_\pi(s)| = -\infty$$

the support of S is empty. Thus S is zero and also $F(Z)$ assumed to fulfill conditions a), b) and c).

A lower bound for the decreasing of $F_\pi(s)$ in the spacelike region is precisely that considered above in α).

3°) The high-energy behaviour α) of $F_\pi(s)$ in the spacelike region implies some interesting properties of the discontinuity $\sigma_\pi(s)$ of $F(Z)$ across the cut.

a) The discontinuity $\sigma_\pi(s)$ cannot decrease as fast as $\exp(-\beta\, s^{\frac{1}{2}})$ as $s \Rightarrow +\infty$.[16]

b) The function $F(Z)$ and the derivatives of F satisfy generalized dispersion relations without subtractions

$$F(Z) = \lim_{\epsilon \Rightarrow 0^+} \frac{1}{\pi} \int_{s_o}^{\infty} e^{-\epsilon t} \frac{\sigma_\pi(t)}{t - Z} dt$$

$$F^{(j)}(Z) = \lim_{\epsilon \Rightarrow 0^+} \frac{j!}{\pi} \int_{s_o}^{\infty} e^{-\epsilon t} \frac{\sigma_\pi(t)}{(t - Z)^{j+1}} dt \quad . \qquad (B.67)$$

In other words, the function $F(Z)$ is uniquely determined by its discontinuity $\sigma_\pi(s)$ when α) is satisfied.[15),17)]

c) The discontinuity $\sigma_\pi(s)$ satisfies convergence sum rules[15)]

$$\lim_{\epsilon \Rightarrow 0^+} \int_{s_o}^{\infty} e^{-\epsilon t} t^N \sigma_\pi(t) dt = 0 \qquad (B.68)$$

for all non-negative integers N.

d) The discontinuity $\sigma_\pi(s)$ must have an infinite number of changes of signs. More precisely, let us denote by n(s) the number of zeros of $\sigma_\pi(s)$ in the interval T - S where T is fixed, then[15)]

$$\limsup_{s \Rightarrow +\infty} s^{-\frac{1}{2}} n(s) \geq \frac{a}{\pi} \quad . \qquad (B.69)$$

4°) From the unitarity relation (B.16) the phase difference $\varphi_F(s) - \varphi_F(s_o)$ is simply the phase shift $\delta_{11}(s)$ in the low-energy region and therefore must behave like $(s - s_o)^{3/2}$ around the threshold point $s = s_o$. Using the representation (B.63) this last condition is simply written as

$$\int_{-\infty}^{+\infty} x \, S(x) \, dx = 0 \; . \qquad (B.70)$$

The scattering length for the P-wave $\pi\pi$ scattering amplitude is then deduced from Eq. (B.63) to be

$$\lim_{s \Rightarrow s_o} \frac{\delta_{11}(s)}{(s - s_o)^{3/2}} = -\frac{1}{6} \frac{\int_{-\infty}^{+\infty} x^3 S(x) \, dx}{\int_{-\infty}^{+\infty} S(x) \, dx} \quad . \qquad (B.71)$$

5°) The root mean square radius is determined using an expansion of Eq. (B.64) around the origin s = 0. With the normalization condition (B.65) we obtain

$$\langle r_\pi^2 \rangle = \frac{3}{4m_\pi^2} \int_{-\infty}^{+\infty} x\, e^{-2m_\pi x}\, S(x)\, dx \ .$$

VI. Models

1°) The partial wave amplitude

$$h_{11}(s) = \frac{\sqrt{s}}{k^3}\, e^{i\delta_{11}(s)} \sin \delta_{11}(s) \quad s \geq s_0 \qquad \text{(B.72)}$$

is the limiting value of an analytic function H(Z) in the complex Z plane with two cuts

a) a right-hand cut on the positive real axis from s_0 to $+\infty$

b) a left-hand cut on the negative real axis from $-\infty$ to 0

Because of the threshold behaviour of the P-wave phase shift $\delta_{11}(s)$ the expression (B.72) is regular at the point $s = s_0$.

2°) Let us now write the function H(Z) as the ratio of two functions

$$H(Z) = \frac{N(Z)}{D(Z)}$$

where

N(Z) is an analytic function of Z except on the left-hand cut

D(Z) is an analytic function of Z except on the right-hand cut

The discontinuity of D(Z) on the right-hand cut is given by

$$\text{Im } D(s) = -\frac{k^3}{\sqrt{s}}\, N(s) = -\frac{1}{8}\, \frac{(s - s_0)^{3/2}}{s^{\frac{1}{2}}}\, N(s) \ .$$

The number of subtractions needed to write a dispersion relation for D(s) depends obviously on the high-energy behaviour of the left-hand cut contributions represented by the function N(s). For instance, if $\lim_{s \Rightarrow +\infty} N(s) = 0$ one subtraction can be sufficient and the subtraction point can be chosen at s = 0.

$$D(s) = D(0) - \frac{s}{8\pi} \int_{s_o}^{\infty} \frac{(t - s_o)^{3/2}}{t^{3/2}} \frac{N(t)}{t - s - i\epsilon} dt \tag{B.73}$$

but if $\lim_{s \Rightarrow +\infty} N(s) = \text{const.}$ two subtractions are necessary and choosing always $s = 0$ as the subtraction point

$$D(s) = D(0) + s\, D'(0) - \frac{s^2}{8\pi} \int_{s_o}^{\infty} \frac{(t - s_o)^{3/2}}{t^{3/2}} \frac{N(t)}{t - s - i\epsilon} . \tag{B.74}$$

Conversely the discontinuity of $N(Z)$ across the left-hand cut $(s < 0)$ is the product of $D(s)$ by the discontinuity of $H(Z)$ across the same left-hand cut.

3°) The N/D formalism will be used in the following to construct partial-wave amplitudes like (B.72) from specific assumptions concerning the left-hand cut contributions. Models for $\delta_{11}(s)$ are built in this way corresponding to particular forms of $N(s)$.

a) Frazer and Fulco[18] replace the left-hand cut by a pole on the real negative axis at $s = -s_1$. The function $N(s)$ is approximated on the right-hand cut by

$$N(s) = \frac{s_1}{s + s_1} .$$

It is possible to write for $D(s)$ a dispersion integral in the form (B.73) and to compute explicitly the dispersion integral. The result is for $D(s)$, an expression with two arbitrary parameters $D(0)$ and s_1.

b) Vaughn and Wali[19] approximate the left-hand cut by a double pole using, as numerator function

$$N(s) = \left(\frac{s_1}{s + s_1}\right)^2 .$$

In their paper, the dispersion integral is written in the form (B.74) with two subtractions at $s = 0$ in order to have a more rapid convergence of the dispersion integral. Choosing arbitrarily $D'(0) = 1$, Vaughn and Wali deduce an expression for $D(s)$ with two adjustable parameters $D(0)$ and s_1.

c) Gounaris and Sakurai[20] make the crudest but simplest assumption of a constant numerator function. The dispersion relation for $D(s)$ is used in the form (B.74) and again the result is, for $D(s)$, an explicit expression with two arbitrary parameters $D(0)$ and $D'(0)$.

In all the three models the two free constants are determined by the requirements to have, in the partial-wave amplitude, the ρ-meson resonance, at the mass m_ρ with the width Γ_ρ.

4°) In the Frazer-Fulco model the function H(Z) is found to be

$$H^{-1}(Z) = (Z - s_o)\, f(Z) + a + bZ \qquad (B.75)$$

where the function f(Z) is defined by

$$f(Z) = \frac{1}{8\pi}\left(\frac{Z - s_o}{Z}\right)^{\frac{1}{2}} \text{Log}\, \frac{(Z - s_o)^{\frac{1}{2}} + Z^{\frac{1}{2}}}{(Z - s_o)^{\frac{1}{2}} - Z^{\frac{1}{2}}} \ . \qquad (B.76)$$

The constants a and b are related to the parameters D(0) and s_1 by

$$a = D(0) + \frac{m_\pi^2}{\pi}$$

$$b = \frac{1}{s_1}\left[a - (s_o + s_1)\, f(-s_1)\right] \ . \qquad (B.77)$$

By construction $H^{-1}(-s_1) = 0$.

The Gounaris-Sakurai model gives, for H(Z), an expression identical to Eq. (B.75). The constant a is unchanged and the constant b is given by

$$b = D'(0) - \frac{1}{3\pi} \ .$$

We now consider more specifically the scattering amplitude $h_{11}(s)$ in the physical region $s \geq s_o$. Using Eqs. (B.72) and (B.75) we obtain for the phase shift $\delta_{11}(s)$ a so-called generalized effective range formula

$$\frac{k^3}{\sqrt{s}} \cot \delta_{11}(s) = k^2\, h(s) + a + bs \qquad (B.78)$$

where the function h(s) is deduced, from Eq. (B.76), to be

$$h(s) = \frac{2}{\pi} \frac{k}{\sqrt{s}} \text{Log}\, \frac{\sqrt{s} + 2k}{2m_\pi} \ .$$

With the convention $\delta_{11}(s_o) = 0$ the P-wave phase shift as given by the model (B.78) tends to zero at infinite energy with the following behaviour

$$\delta_{11}(s) \simeq \frac{\pi}{\mathrm{Log}\,\frac{s}{m_\pi^2}} \quad . \tag{B.79}$$

The existence of an elastic π-π resonance, the ρ meson, in the I = 1, J = 1 channel, determines the constants a and b by the two constraints on the phase shift $\delta_{11}(s)$

$$\cot \delta_{11}(m_\rho^2) = 0$$

$$\frac{d}{ds}\,\delta_{11}(m_\rho^2) = \frac{1}{m_\rho \Gamma_\rho} \quad .$$

The result is

$$a = m_\pi^2\ h(m_\rho^2) + m_\rho^2 \left[\frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} + k_\rho^2\ h'(m_\rho^2) \right]$$

$$b = -\tfrac{1}{4}\ h(m_\rho^2) - \left[\frac{k_\rho^3}{m_\rho^2 \Gamma_\rho} + k_\rho^2\ h'(m_\rho^2) \right]$$

where

$$k_\rho = \tfrac{1}{2}(m_\rho^2 - 4m_\pi^2)^{\frac{1}{2}} \quad .$$

The value of s_1 for which $H^{-1}(-s_1) = 0$, is found to be very large

$$s_1 \simeq 9.6\ 10^8\ m_\rho^2 \quad .$$

5°) We now construct the Omnès function $G_{11}(Z)$ defined in Eq. (B.32)

$$G_{11}(Z) = \exp \frac{Z}{\pi} \int_{s_0}^{\infty} \frac{\delta_{11}(t)}{t(t-Z)}\, dt \quad .$$

From the considerations of the second paragraph of this section we easily deduce that the function Q(Z) defined by

$$Q(Z) = G_{11}(Z)\ D(Z)$$

is an entire function of Z in the complex Z plane. The function $G_{11}(Z)$ is then related to the scattering amplitude H(Z) by

$$G_{11}(Z) = \frac{H(Z)}{H(0)} \frac{Q(Z)}{N(Z)} . \tag{B.80}$$

We assume H(Z) and N(Z) to be known; the function Q(Z) is then restricted by the following properties of $G_{11}(Z)$ previously discussed

a) no zeros in the complex Z plane

b) $G_{11}(0) = 1$

c) correct asymptotic behaviour as studied in Sec. III.

In the actual case where the phase shift $\delta_{11}(s)$ is described by the effective range formula (B.78) the only singularity of H(Z) we have to cancel is the pole $Z = -s_1$ so that

$$\frac{Q(Z)}{N(Z)} = 1 + \frac{Z}{s_1} .$$

The corresponding Omnès function is found to be

$$G_{11}(Z) = \left(1 + \frac{Z}{s_1}\right) \frac{a - \frac{m_\pi^2}{\pi}}{(Z - s_0)\, f(Z) + a + bZ} . \tag{B.81}$$

On the real axis in the timelike region above the cut $s > s_0$ the Omnès function is given by

$$\lim_{\epsilon \Rightarrow 0^+} G_{11}(s + i\epsilon) = \left(1 + \frac{s}{s_1}\right) \times$$

$$\frac{m_\rho^2\left(1 + d\frac{\Gamma_\rho}{m_\rho}\right)}{m_\rho^2 - s + \frac{m_\rho^2 \Gamma_\rho}{k_\rho^3}\{k^2[h(s) - h(m_\rho^2)] + (m_\rho^2 - s)k_\rho^2 h'(m_\rho^2)\} - i m_\rho \Gamma_\rho \left(\frac{k}{k_\rho}\right)^3 \frac{m_\rho}{\sqrt{s}}} \tag{B.82}$$

where the constant d is given by

$$d = \frac{3}{\pi} \frac{m_\pi^2}{k_\rho^2} \operatorname{Log} \frac{m_\rho + 2k_\rho}{2m_\pi} + \frac{m_\rho}{2\pi\, k_\rho} - \frac{m_\pi^2\, m_\rho}{\pi\, k_\rho^3} . \tag{B.83}$$

Equation (B.82) exhibits the following properties

a) Near the ρ-meson mass $s \simeq m_\rho^2$ the term in brackets in the denominator of Eq. (B.82) behaves like $(s - m_\rho^2)^2$ and therefore it can be ignored in this region. For practical computations $G_{11}(s)$ reduces to the usual P-wave form

$$G_{11}(s) \simeq \frac{m_\rho^2\left(1 + d\,\frac{\Gamma_\rho}{m_\rho}\right)}{m_\rho^2 - s + O[(m_\rho^2 - s)^2] - i\,m_\rho\Gamma_\rho\left(\frac{k}{k_\rho}\right)^3 \frac{m_\rho}{\sqrt{s}}} .$$

b) A correct normalization of $G_{11}(s)$ at $s = 0$ implies the presence in the numerator of $G_{11}(s)$ of the constant d giving a measure of nonzero width corrections. For instance at $s = m_\rho^2$ the value of $G_{11}(s)$ is given by

$$G_{11}(m_\rho^2) \simeq i\,\frac{m_\rho}{\Gamma_\rho}\left(1 + d\,\frac{\Gamma_\rho}{m_\rho}\right) .$$

The numerical value of d for the actual ρ-meson mass is close to $d \simeq 0.48$ and we obtain a 14% effect for $|G_{11}(m_\rho^2)|^2$.

c) As a common property of all the width energy dependent Breit-Wigner expressions the actual maximum of $|G_{11}(s)|^2$ is not at $s = m_\rho^2$ but is somewhat shifted towards the left.

d) Because of the very high value of s_1 the factor $(1 + s/s_1)$ in Eq. (B.82) can be disregarded in the domain of validity of the effective range expansion (B.78) e.g. for values of s between the threshold and 1 or 2 GeV².

VII. Experiments

1°) The modulus of the π-meson electromagnetic form factor has been measured in the ρ^0 resonance region by observing the reaction

$$e^+ + e^- \Rightarrow \pi^+ + \pi^- .$$

A systematic analysis of the experiments performed in Novosibirsk[21] and in Orsay[22] has been made by Roos and Pisut[23] using different parametrizations of the π-meson electromagnetic form factor.

The possibility of zeros of $F_\pi(s)$ has been disregarded and the form factor has been identified with the Omnès function $G_{11}(s)$ corresponding to the elastic unitarity relation.

The problem is then reduced to the construction of a phase shift $\delta_{11}(s)$ reproducing correctly the main features of the actual P-wave phase shift and in particular the resonance property of the scattering amplitude in the ρ-meson region. Some of these models have been discussed in detail in Sec. VI but the Roos and Pisut analysis covers a larger domain of possibilities for the energy dependence of the width.

As expected, the resonance parameters depend strongly on the differences in the formulae used for the fits but it seems difficult to choose clearly the best phenomenological form for $F_\pi(s)$, all the experimental data being concentrated in the ρ^0-meson region. Nevertheless an energy-dependent width as suggested by the P-wave character of the final-state interaction gives better results especially for the Orsay data.

2°) Let us now consider the data more quantitatively. The measurements cover a range of total energy $\sqrt{s}$ from 580 MeV to 1030 MeV approximatively by 30 MeV steps, with nine values for the Novosibirsk experiments and seven values for the Orsay experiments.

We first use a simple Breit-Wigner formula ignoring the normalization condition at s = 0

$$|F_\pi(s)|^2 = \frac{m_\rho^4}{(m_\rho^2 - s)^2 + m_\rho^2\Gamma_\rho^2}\,(F_\pi(0))^2 \ .$$

The result of this three-parameter fit is given in the following table

	Degree of freedom	χ^2	Mass (MeV)	Width (MeV)	$F_\pi(0)$
Novosibirsk	6	2.3	754 ± 9	105 ± 20	0.9 ± 0.11
Orsay	4	8.6	762 ± 6	117 ± 11	0.12 ± 0.08

Table 1

A second fit is made, using now the model proposed by Frazer and Fulco in 1959 and by Gounaris and Sakurai in 1968. Such a model based on the effective range expansion of the π-π phase shift $\delta_{11}(s)$ has been extensively discussed in Sec. VI and the explicit expression of the associated Omnès function is given in Eq. (B.82) with the practical form (B.83) in the ρ^0-meson region.

The results of this two-parameter fit are given in Table 2.

	Degree of freedom	χ^2	Mass (MeV)	Width (MeV)
Novosibirsk	7	4.5	768 ± 10	140 ± 14
Orsay	5	8.6	772 ± 6	113 ± 8

Table 2

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SECTION C. Currents and Spectral Representations

I. Conserved Currents

1°) We first consider a conserved current $J_\mu^\alpha(x)$ where α is the weight of the adjoint representation of an internal symmetry group like SU(2) or SU(3). It is always possible to choose α such that the current component $J_\mu^\alpha(x)$ is an hermitian operator

$$J_\mu^\alpha(x)^* = J_\mu^\alpha(x) \quad .$$

By assumption the current is divergenceless

$$\partial^\mu J_\mu^\alpha(x) = 0 \qquad (C.1)$$

where ∂^μ is a short notation for $\partial/\partial x_\mu$.

The space integral of the time component of the current is associated to a conserved quantity

$$Q^\alpha = \int J_0^\alpha(x)\, d_3 x$$

and Q^α is generally called a charge.

2°) A Källen-Lehmann[1),2)] representation can be written for the vacuum expectation value of the product of two current-components. The structure of such a representation is determined by the Lorentz covariance and the divergence condition (C.1)

$$\langle 0|\, J_\mu^\alpha(x)\, J_\nu^\beta(0)\,|0\rangle = \int_0^\infty \rho^{\alpha\beta}(m^2)\left[g_{\mu\nu} - \frac{1}{m^2}\partial_\mu \partial_\nu\right] i\,\Delta^+(x;m^2)\,dm^2 \quad . \qquad (C.2)$$

The invariant distribution $\Delta^+(x,m^2)$ is defined by its four dimensional Fourier transform

$$\Delta^{(\,)}(x;m^2) = \frac{1}{(2\pi)^4}\int e^{ik\cdot x}\, \Delta^{(\,)}(k;m^2)\, d_4 k$$

$$\Delta^{\pm}(k;m^2) = \mp\, 2i\pi\, \theta(\pm k_0)\, \delta(k^2 + m^2)$$

where θ is the usual step function and $\delta(k^2+m^2)$ the invariant Dirac distribution on the mass hyperboloid.

If the vacuum belongs to the scalar representation of the symmetry group the spectral function $\rho^{\alpha\beta}(m^2)$ can be written as

$$\rho^{\alpha\beta}(m^2) = \delta^{\alpha\beta}\,\rho^{\alpha\alpha}(m^2)\ .$$

Because of the hermitian character of the current the spectral function $\rho^{\alpha\beta}(m^2)$ satisfies

$$\rho^{\alpha\beta}(m^2)^* = \rho^{\beta\alpha}(m^2)\ .$$

In particular the diagonal elements $\rho^{\alpha\alpha}(m^2)$ are real functions.

3°) Analogous Källen-Lehmann representations can be written for the product of the two current-components in the reversed order

$$\langle 0|\, J_\nu^\beta(0)\, J_\mu^\alpha(x)\,|0\rangle = \int_0^\infty \rho^{\beta\alpha}(m^2)\left[g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right]\frac{1}{i}\,\Delta^-(x;m^2)\,dm^2$$

and for the commutator of two such current components

$$\langle 0|\,[J_\mu^\alpha(x),\, J_\nu^\alpha(0)]\,|0\rangle = \int_0^\infty \rho^{\alpha\alpha}(m^2)[\,g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu]\ i\,\Delta(x;m^2)\,dm^2 . \qquad (C.3)$$

The causal invariant distribution $\Delta \equiv \Delta^+ + \Delta^-$ is defined by

$$\Delta(k;m^2) = -\,2i\,\pi\,\varepsilon(k_o)\,\delta(k^2+m^2)$$

where $\varepsilon(k_o)$ is the discontinuous function sign of k_o.

4°) It is now convenient to take the Fourier transform of both sides of the Källen-Lehmann representation (C.2) in order to study some properties of the spectral function $\rho^{\alpha\beta}(m^2)$

$$\int e^{-ik\cdot x}\langle 0|\,J_\mu^\alpha(x)\,J_\nu^\beta(0)\,|0\rangle\,d_4x = 2\pi\,\theta(k_o)\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\rho^{\alpha\beta}(-k^2)\,. \qquad (C.4)$$

We introduce a complete set of intermediate states $|n\rangle$

$$\langle 0|\,J_\mu^\alpha(x)\,J_\nu^\beta(0)\,|0\rangle = \sum_n S_n\,\langle 0|\,J_\mu^\alpha(x)\,|n\rangle\,\langle n|\,J_\nu^\beta(0)\,|0\rangle\ .$$

Let us recall that the symbol S_n defined in Sec. A implies a summation over the polarization of the particles of the intermediate state $|n\rangle$ and a phase-space integration.

From space-time translation invariance

$$\langle 0 | J^{\alpha}_{\mu}(x) | n \rangle = e^{ip_n \cdot x} \langle 0 | J^{\alpha}_{\mu}(0) | n \rangle .$$

The x integration in the left-hand side of Eq. (C.4) is now easily performed

$$\sum_n S_n (2\pi)^4 \delta_4 (p_n - k) \langle 0 | J^{\alpha}_{\mu}(0) | n \rangle \langle n | J^{\beta}_{\nu}(0) | 0 \rangle = 2\pi\, \theta(k_o) \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right] \rho^{\alpha\beta}(-k^2) .$$

Contracting this with the metric tensor $g^{\mu\nu}$ we finally obtain

$$\rho^{\alpha\beta}(s) = \frac{1}{6\pi} \theta(k_o) \sum_n S_n (2\pi)^4 \delta_4 (p_n - k) \langle 0 | J^{\alpha}_{\mu}(0) | n \rangle \langle n | J^{\beta}_{\nu}(0) | 0 \rangle g^{\mu\nu} \qquad (C.5)$$

where $s = -k^2$.

It is now straightforward to prove that $\rho^{\alpha\alpha}(s)$ is a definite positive function using for instance the property for the matrix elements of a conserved current, to have only space components in the centre-of-mass system ($\vec{k} = 0$)

$$\rho^{\alpha\alpha}(s) = \frac{1}{6\pi} \theta(k_o) \sum_n S_n (2\pi)^4 \delta_4 (p_n - k) \sum_{\ell=1}^{\ell=3} |\langle 0 | J^{\alpha}_{\ell}(0) | n \rangle|^2 . \qquad (C.6)$$

II. Non-Conserved Current

1°) The current density $J^{\alpha}_{\mu}(x)$ is always an hermitian operator but its four divergence is different from zero. Therefore two spectral functions are needed to write Källen-Lehmann representations like (C.2) and (C.3)

$$\langle 0 | J^{\alpha}_{\mu}(x)\, J^{\beta}_{\nu}(0) | 0 \rangle = \int_0^\infty \{ \rho^{\alpha\beta}_{(1)}(m^2) \left[g_{\mu\nu} - \frac{1}{m^2} \partial_\mu \partial_\nu \right] - \rho^{\alpha\beta}_{(0)}(m^2) \partial_\mu \partial_\nu \} i\Delta^{+}(x; m^2)\, dm^2 \qquad (C.7)$$

$$\langle 0|[J^{\alpha}_{\mu}(x),\ J^{\alpha}_{\nu}(0)]|0\rangle = \int_0^{\infty} \{\rho^{\alpha\beta}_{(1)}(m^2)\left[g_{\mu\nu} - \frac{1}{m^2}\partial_{\mu}\partial_{\nu}\right]$$

$$- \rho^{\alpha\alpha}_{(0)}(m^2)\,\partial_{\mu}\partial_{\nu}\}\, i\Delta(x;m^2)\, dm^2 \ . \quad (C.8)$$

Of course when the current is conserved the spectral function $\rho^{\alpha\alpha}_{(0)}(m^2)$ vanishes identically.

2°) We use the same method as in Sec. I to study the spectral functions $\rho^{\alpha\beta}_{(1)}(m^2)$ and $\rho^{\alpha\beta}_{(0)}(m^2)$. The four-dimensional Fourier transform of the Källen-Lehmann representation (C.7) is simply

$$\int e^{-ik\cdot x}\langle 0|J^{\alpha}_{\mu}(x)\ J^{\beta}_{\nu}(0)|0\rangle d_4x = 2\pi\ \theta(k_o)\left\{\left[g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}\ \rho^{\alpha\beta}_{(1)}(-k^2)\right.\right.$$

$$\left.\left. + k_{\mu}k_{\nu}\ \rho^{\alpha\beta}_{(0)}(-k^2)\right]\right\} \ . \quad (C.9)$$

We introduce again a complete set of intermediate states $|n\rangle$ in the left-hand side of Eq. (C.9). We perform the x integration and obtain the general relation

$$2\pi\ \theta(k_o)\left\{\left[g_{\mu\nu} + \frac{k_{\mu}k_{\nu}}{s}\quad \rho^{\alpha\beta}_{(1)}(s) + k_{\mu}k_{\nu}\ \rho^{\alpha\beta}_{(0)}(s)\right]\right\}$$

$$= \sum_n S_n (2\pi)^4 \delta_4(p_n - k)\ \langle 0|J^{\alpha}_{\mu}(0)|n\rangle\langle n|J^{\beta}_{\nu}(0)|0\rangle$$

where as previously $s = -k^2$.

In the centre-of-mass system $\vec{k} = 0$ the tensor $g_{\mu\nu} - (k_{\mu}k_{\nu}/k^2)$ has only nonvanishing space-space components ($\mu \neq 0$, $\nu \neq 0$) whereas the tensor $k_{\mu}k_{\nu}$ has only one non-vanishing time-time component ($\mu = 0$, $\nu = 0$). We are then able to compute independently the two spectral functions $\rho^{\alpha\alpha}_{(1)}(s)$ and $\rho^{\alpha\alpha}_{(0)}(s)$

$$\rho^{\alpha\alpha}_{(1)}(s) = \frac{1}{6\pi}\,\theta(k_o)\sum_n S_n (2\pi)^4 \delta_4(p_n - k)\sum_{\ell=1}^{\ell=3} |\langle 0|J^{\alpha}_{\ell}(0)|n\rangle|^2 \quad (C.10)$$

$$\rho^{\alpha\alpha}_{(0)}(s) = \frac{1}{2\pi s}\,\theta(k_o)\sum_n S_n(2\pi)^4\,\delta_4(p_n - k)\,|\langle 0|J^\alpha_0(0)|n\rangle|^2 \qquad (C.11)$$

From Eq. (C.10) and (C.11) we conclude that the two spectral functions $\rho^{\alpha\alpha}_{(1)}(s)$ and $\rho^{\alpha\alpha}_{(0)}(s)$ are non-negative

$$\rho^{\alpha\alpha}_{(1)}(s) \geq 0 \qquad \rho^{\alpha\alpha}_{(0)}(s) \geq 0 \quad .$$

Of course, if one of the spectral functions is identically zero, the second one is definite positive.

III. Equal-Time Commutators

1°) The distribution $\Delta(x;m^2)$ satisfies the integral representation

$$\Delta(\vec{r},t;\, m^2) = -\frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\,\frac{\sin\omega_k t}{\omega_k}\,d_3 k \qquad (C.12)$$

where $x = (\vec{r},t)$ and $\omega_k = \sqrt{-\vec{k}^2 + m^2}$.

Equation (C.12) is used to prove the following properties of the distribution Δ and its derivatives at time $t = 0$

$$\Delta(\vec{r},0;\, m^2) = 0 \qquad \partial_j\,\Delta(\vec{r},0;\, m^2) = 0$$

$$\partial_o\Delta(\vec{r},0;m^2) = -\frac{\partial}{\partial t}\Delta(\vec{r},0;\, m^2) = \delta_3(\vec{r})$$

$$\partial_j\partial_k\,\Delta(\vec{r},0;\, m^2) = 0 \qquad \partial_o\partial_o\,\Delta(\vec{r},0;\, m^2) = 0 \qquad (C.13)$$

2°) Let us go back to the Källen-Lehmann representation of the vacuum expectation value of the commutator of two components of a conserved current (C.3)

$$\langle 0|[J^\alpha_\mu(x),\, J^\alpha_\nu(0)]|0\rangle = \int_0^\infty \rho^{\alpha\alpha}(m^2)\left[g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu\right] i\,\Delta(x;m^2)\,dm^2 \ .$$

As a consequence of relations (C.13), we easily obtain the vacuum expectation value of equal time commutators [3)-5)]

$$\langle 0|[J^\alpha_0(\vec{r},0),\, J^\alpha_0(0)]|0\rangle = 0 = \langle 0|[J^\alpha_k(\vec{r},0),\, J^\alpha_\ell(0)]|0\rangle \qquad (C.14)$$

$$\langle 0|[J_0^\alpha(\vec{r},0), J_\ell^\alpha(0)]|0\rangle = \frac{1}{i}\partial_\ell \delta_3(\vec{r}) \int_0^\infty \frac{\rho^{\alpha\alpha}(m^2)}{m^2} dm^2 \qquad (C.15)$$

3°) If now J^α is a nonconserved current, using the same method, we easily check that Eq. (C.14) remains true and Eq. (C.15) is simply replaced by[6)]

$$\langle 0|[J_0^\alpha(\vec{r},0), J_\ell^\alpha(0)]|0\rangle = \frac{1}{i}\partial_\ell \delta_3(\vec{r}) \int_0^\infty \frac{\rho_{(1)}^{\alpha\alpha}(m^2)}{m^2} + \rho_{(0)}^{\alpha\alpha}(m^2) \quad dm^2 . \qquad (C.16)$$

4°) The equal-time commutator of the time component with a space component of a current density has the minimal structure

$$[J_0^\alpha(\vec{r},0), J_\ell^\beta(0)] = c_\gamma^{\alpha\beta} J_\ell^\gamma(0) \delta_3(\vec{r}) + S^{\alpha\beta} \frac{1}{i}\partial_\ell \delta_3(\vec{r})$$

where $c_\gamma^{\alpha\beta}$ are the skew symmetric structure constants of the symmetry group.

The Schwinger term is defined as $S^{\alpha\beta}$. The vacuum expectation value of the Schwinger term--or the Schwinger term itself if it is a c-number--is given by

$$\langle 0|S^{\alpha\alpha}|0\rangle = C^\alpha$$

where the quantity C^α is computed from Eq. (C.15) for a conserved current

$$C^\alpha = \int_0^\infty \frac{\rho^{\alpha\alpha}(m^2)}{m^2} \quad dm^2 \qquad (C.17)$$

and from Eq. (C.16) for a nonconserved current

$$C^\alpha = \int_0^\infty \left[\frac{\rho_{(1)}^{\alpha\alpha}(m^2)}{m^2} + \rho_{(0)}^{\alpha\alpha}(m^2)\right] dm^2 \quad . \qquad (C.18)$$

In both cases, the positivity properties of the spectral functions force away C^α to be positive and therefore nonzero.

IV. Time Ordered Products

1°) The time ordered product of two current components is defined by

$$T(J_\mu^\alpha(x)\, J_\nu^\beta(0)) = \theta(x_o)\, J_\mu^\alpha(x)\, J_\nu^\beta(0) + \theta(-x_o)\, J_\nu^\beta(0)\, J_\mu^\alpha(x)$$

and we have to construct a Källen-Lehmann representation for the vacuum expectation value of the T product starting from the representation (C.7). We are then led to consider two distributions

a)
$$\theta(x_o)\,\Delta^+(x;m^2) - \theta(-x_o)\,\Delta^-(x;m^2) \equiv \Delta_F(x;m^2) \quad .$$

It can be easily checked that the Feynman distribution $\Delta_F(x;m^2)$ is solution of the Green equation associated to the Klein-Gordon equation and its four-dimensional Fourier transform is simply given by

$$\Delta_F(k;m^2) = \lim_{\epsilon \Rightarrow 0^+} \frac{-1}{k^2 + m^2 - i\epsilon} \quad . \qquad \text{(C.19)}$$

b)
$$\theta(x_o)\,\partial_\mu\partial_\nu\,\Delta^+(x;m^2) - \theta(-x_o)\,\partial_\mu\partial_\nu\,\Delta^-(x;m^2) \quad .$$

A straightforward calculation, using in particular the relations (C.13), gives the following equality between distributions

$$\theta(x_o)\partial_\mu\partial_\nu\,\Delta^+(x;m^2) = \theta(-x_o)\partial_\mu\partial_\nu\,\Delta^-(x;m^2) \equiv \partial_\mu\partial_\nu\,\Delta_F(x;m^2) + g_{\mu o}\, g_{\nu o}\,\delta_4(x).$$

We are now in a position to write the Källen-Lehmann representation[5)]

$$\langle 0|\, T(J_\mu^\alpha(x) J_\nu^\alpha(0))\, |0\rangle = \int_0^\infty \{\rho_{(1)}^{\alpha\alpha}(m^2)[g_{\mu\nu} - \frac{1}{m^2}\partial_\mu\partial_\nu] - \rho_{(0)}^{\alpha\alpha}(m^2)\,\partial_\mu\partial_\nu\}$$

$$\times\, i\Delta_F(x;m^2)\, dm^2 + \frac{1}{i}\, g_{\mu o}\, g_{\nu o}\,\delta_4(x) \int_0^\infty \left[\frac{\rho_{(1)}^{\alpha\alpha}(m^2)}{m^2} + \rho_{(0)}^{\alpha\alpha}(m^2)\right] dm^2 . \qquad \text{(C.20)}$$

2°) The time-ordered product of two current components is covariant except at the point x = 0 where the product of distributions introduces singularities.

The second term in the right-hand side of Eq. (C.20) exhibits clearly such a feature. Moreover, the noncovariant part of the time-ordered product is proportional to the Schwinger term previously computed in Eq. (C.18).

3°) The Fourier transform of the time-ordered product is usually defined by

$$\Delta^{\alpha}_{\mu\nu}(k) = i\int e^{-ik\cdot x}\langle 0|T(J^{\alpha}_{\mu}(x)\,J^{\alpha}_{\nu}(0))|0\rangle\, d_4x\ .$$

From Eqs. (C.19) and (C.20) the expression of $\Delta^{\alpha}_{\mu\nu}(k)$ is

$$\Delta^{\alpha}_{\mu\nu}(k) = \int_0^{\infty}\frac{dm^2}{k^2+m^2-i\epsilon}\left\{g_{\mu\nu}\,\rho^{\alpha\alpha}_{(1)}(m^2) + k_{\mu}k_{\nu}\left[\frac{\rho^{\alpha\alpha}_{(1)}(m^2)}{m^2} + \rho^{\alpha\alpha}_{(0)}(m^2)\right]\right\}$$

$$+\, g_{\mu o}\, g_{\nu o}\int_0^{\infty}\left[\frac{\rho^{\alpha\alpha}_{(1)}(m^2)}{m^2} + \rho^{\alpha\alpha}_{(0)}(m^2)\right] dm^2\ . \qquad \text{(C.21)}$$

V. Electromagnetic Current

1°) The electromagnetic current is a conserved vector current. The spectral function $\rho^{em}(s)$ is defined by an equation analogous to (C.5)

$$\rho^{em}(s) = \frac{1}{6\pi}\sum_n S_n\,(2\pi)^4\,\delta_4(p_n - k)\langle 0|J^{em}_{\mu}(0)|n\rangle\langle n|J^{em}_{\nu}(0)|0\rangle\, g^{\mu\nu}. \qquad \text{(C.22)}$$

Comparing with Eq. (A.8) we obtain

$$\rho^{em}(s) = \sum_n \rho^{em}_n(s)$$

$$\rho^{em}_n(s) = \frac{1}{6\pi}\,\mathrm{Tr}\,\{n\}\ .$$

The spectral function $\rho^{em}_n(s)$ is then related to the total cross section for e^+e^- annihilation into a final state n by using formula (A.7) in the one-photon exchange approximation

$$\rho^{em}_n(s) = \frac{s^2}{16\pi^3\alpha^2}\,\sigma_{tot}(e^+e^- \Rightarrow n)\ . \qquad \text{(C.23)}$$

2°) The electromagnetic current can be decomposed into an isovector component and an isoscalar component disregarding other possibilities

$$J^{em}_{\mu} = J^{I_3}_{\mu} + \tfrac{1}{2}\, J^{Y}_{\mu} \qquad \text{(C.24)}$$

$J_\mu^{I_3}$ is the third component of the isotopic spin current

J_μ^Y is the hypercharge current.

The equality (C.24) implies obviously the Gell-Mann-Nishijima relation for the charges: $Q = I_3 + \frac{1}{2}Y$ but the reverse is not true. In the framework of SU(3) symmetry it is convenient to use the weights associated to particles of the adjoint representation. The U-spin scalar electromagnetic current is written as

$$J_\mu^{em} + J_\mu^3 + \frac{1}{\sqrt{3}} J_\mu^8$$

J_μ^3 corresponds to an isovector particle (ρ^o meson)

J_μ^8 corresponds to an isoscalar particle (φ_8 meson) .

In the lowest order approximation with respect to electromagnetic interactions isospin invariance can be used and the corresponding relations for the spectral functions are

$$\rho^{em}(s) = \rho^{33}(s) + \frac{1}{3}\rho^{88}(s) \qquad \text{(C.25)}$$

with

$$\rho^{33}(s) = \frac{s^2}{16\pi^3\alpha^2}\sigma_{tot}(e^+e^- \Rightarrow I = 1)$$

$$\rho^{88}(s) = \frac{3s^2}{16\pi^3\alpha^2}\sigma_{tot}(e^+e^- \Rightarrow I = 0) \ . \qquad \text{(C.26)}$$

The first correction to Eq. (C.25) is given by interference terms

$$\rho^{em}(s) = \rho^{33}(s) + \frac{1}{3}\rho^{88}(s) + \frac{1}{\sqrt{3}}[\rho^{38}(s) + \rho^{83}(s)] \ .$$

3°) The Schwinger terms C^3 and C^8 or more precisely the vacuum expectation value of the Schwinger terms are immediately expressed as integrals over the total cross sections for e^+e^- annihilation into hadrons

$$C^3 = \frac{1}{16\pi^3\alpha^2}\int_{4m_\pi^2}^{\infty} s\,\sigma_{tot}(e^+e^- \Rightarrow I = 1)\,ds \qquad (C.27)$$

$$C^8 = \frac{3}{16\pi^3\alpha^2}\int_{9m_\pi^2}^{\infty} s\,\sigma_{tot}(e^+e^- \Rightarrow I = 0)\,ds \;. \qquad (C.28)$$

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SECTION D. Vacuum Polarization

I. Modification of the Photon Propagator

1°) The photon propagator is the vacuum expectation value of the time-ordered product of two components of the electromagnetic field $A_\mu(x)$. In the energy momentum space the distribution $D_{\mu\nu}(k)$ is defined by

$$D_{\mu\nu}(k) = i\int e^{-ik\cdot x}\,\langle 0|\,T(A_\mu(x)\,A_\nu(0))\,|0\rangle\,d_4x \;. \qquad (D.1)$$

In order to write a Källen-Lehmann representation for $D_{\mu\nu}(k)$ it is convenient to first consider the vacuum expectation value of the commutator of two components of $A_\mu(x)$

$$\langle 0|\,[A_\mu(x), A_\nu(0)]\,|0\rangle = \int_0^\infty \{\sigma_1(m^2)g_{\mu\nu} + \sigma_o(m^2)\partial_\mu\partial_\nu\}\, i\,\Delta(x;m^2)\,dm^2 \,. \qquad (D.2)$$

The two functions $\sigma_1(m^2)$ and $\sigma_o(m^2)$ can be related to the spectral function $\rho^{em}(m^2)$ defined in Eq. (C.22) using a Yang-Feldman equation for the interpolating field $A_\mu(x)$

$$A_\mu(x) = A_\mu^o(x) + e\int D_R(x-y)\,J_\mu^{em}(y)\,d_4y$$

where $A^o_\mu(x)$ is the free photon field and D_R the retarded Green distribution associated to the Klein-Gordon equation with mass zero. The result has been obtained by Källen[1])

$$\sigma_1(m^2) = -\delta(m^2) - e^2 \frac{\rho^{em}(m^2)}{m^4}$$

$$\sigma_o(m^2) = e^2 \frac{\rho^{em}(m^2)}{m^6} - \delta(m^2)\, e^2 \int_0^\infty \frac{\rho^{em}(t)}{t^3}\, dt\,. \quad (D.3)$$

We then use the techniques of Part IV of Sec. C. to deduce from Eqs. (D.2) and (D.3) the Källen-Lehmann representation for the photon propagator $D_{\mu\nu}(k)$,[2])

$$D_{\mu\nu}(k) = -\frac{g_{\mu\nu}}{k^2 - i\varepsilon} - \left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2 - i\varepsilon}\right] e^2 \int_0^\infty \frac{\rho^{em}(m^2)}{m^4} \frac{dm^2}{k^2 + m^2 - i\varepsilon}\,. \quad (D.4)$$

The first term in the right-hand side of Eq. (D.4) is the free field propagator and the second term is a gauge invariant correction.

2°) The modification of the photon propagator is measured by the function $\pi(s)$ defined by

$$\pi(s) = e^2 \int_0^\infty \frac{\rho^{em}(t)}{t^2} \frac{dt}{t - s - i\varepsilon}\,. \quad (D.5)$$

Equation (D.4) is equivalently written as

$$D_{\mu\nu}(k) = \frac{g_{\mu\nu}}{s + i\varepsilon}[1 - s\,\pi(s)] - \frac{k_\mu k_\nu}{s + i\varepsilon}\pi(s) \quad (D.6)$$

where $s = -k^2$.

The hadronic contributions to $\pi(s)$ are associated to the total cross section for the electron-positron annihilation into hadrons. From Eq. (C.23) we obtain

$$\pi^{(hadrons)}(s) = \frac{1}{\pi e^2} \int_0^\infty \frac{\sigma_{tot}(e^+e^- \Rightarrow \text{hadrons})}{t - s - i\varepsilon}\, dt\,. \quad (D.7)$$

3°) One of the cleanest ways to measure the hadronic modification to the photon propagator is to look at the reaction

$$e^+ + e^- \Rightarrow \mu^+ + \mu^- \ .$$

The total cross section of the previous reaction is simply the product of the usual uncorrected cross section as calculated in electrodynamics by a vacuum polarization factor

$$\sigma_{tot}(e^+e^- \Rightarrow \mu^+\mu^-) = \sigma_{elect.}(s)|1 - s\,\pi(s)|^2 \qquad (D.8)$$

where

$$\sigma_{elect.}(s) = \frac{4\pi\alpha^2}{3s}\left(1 - \frac{4m_\mu^2}{s}\right)^{\frac{1}{2}}\left(1 + \frac{2m_\mu^2}{s}\right) \ .$$

II. Charge Renormalization

1°) The photon propagator has been written in Sec. I in the general form

$$D_{\mu\nu}(k) = g_{\mu\nu}\,F(s) + k_\mu k_\nu\,G(s) \ .$$

The ratio of the bare electric charge e_o to the observed electric charge e is defined by[3]

$$\left(\frac{e_o}{e}\right)^2 = \frac{\lim\limits_{s\Rightarrow\infty} s\,F(s)}{\lim\limits_{s\Rightarrow 0} s\,F(s)} \ .$$

The function F(s) is related to $\pi(s)$ by the Eq. (D.6) and using the integral representation (D.5) we obtain[1]

$$\frac{\delta e_o^2}{e^2} = e^2 \int_0^\infty \frac{\rho^{em}(t)}{t^2}\,dt$$

where by definition

$$e_o^2 = e^2 + \delta\, e_o^2 \ .$$

2°) The hadronic contributions to the charge renormalization are written as integrals involving the total cross section for electron-positron annihilation into hadrons

$$\frac{\delta e_0^2}{e^2}(\text{hadrons}) = \frac{1}{\pi e^2}\int_0^\infty \sigma_{tot}(e^+e^- \Rightarrow \text{hadrons})\, dt \,. \quad (D.9)$$

III. Hadronic Contribution to the Muon Anomalous Magnetic Moment

1°) The general method to obtain the hadronic contributions to the muon anomalous magnetic moment $a_\mu = \frac{1}{2}(g_\mu - 2)$ due to vacuum polarization corrections is well known.[4),5)] The resulting expression corresponding to the class of Feynman diagrams shown on Fig. 1

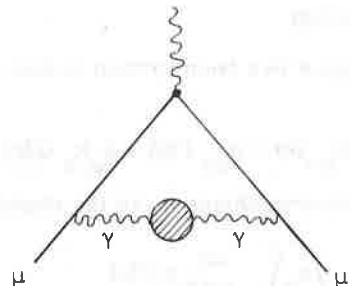

Fig. 1

has the following structure

$$a_\mu(\text{hadrons}) = \frac{1}{\pi e^2}\int_0^\infty \sigma_{tot}(e^+e^- \Rightarrow \text{hadrons})\, K_\mu^{(2)}(t)\, dt \quad (D.10)$$

where the weight function $K_\mu^{(2)}(t)$ is the second-order vertex function given by the integral representation

$$K_\mu^{(2)}(t) = \frac{\alpha}{\pi}\int_0^1 dx \frac{x^2(1-x)}{x^2 + \frac{t}{m_\mu^2}(1-x)}$$

where m_μ is the muon mass.

2°) The explicit form of $K_\mu^{(2)}(t)$ is known.[6),7)] Obviously

$$K_\mu^{(2)}(0) = \frac{\alpha}{\pi}$$

and for $t \geq 4m_\mu^2$ a convenient parametrization is the following

$$K_\mu^{(2)}(t) = \frac{\alpha}{\pi}\left\{\frac{y^2}{2}(2 - y^2) + (1 + y)^2(1 + y^2)\,\frac{\text{Log}(1 + y) - y + \frac{y^2}{2}}{y^2} + \frac{1 + y}{1 - y}\, y^2 \,\text{Log}\, y\right\}$$

where

$$y = \frac{1 - \left(1 - \frac{4m_\mu^2}{t}\right)^{\frac{1}{2}}}{1 + \left(1 - \frac{4m_\mu^2}{t}\right)^{\frac{1}{2}}} \quad .$$

For large t, $K_\mu^{(2)}(t)$ goes to zero as 1/t. It then appears that the high energy contributions to a_μ(hadrons) are depressed by the factor $K_\mu^{(2)}(t)$ and the integral (D.10) is dominated by the low values of t, in particular those values of t where the electron-positron annihilation cross sections have been recently measured.

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SECTION E. High-Energy Behaviour of Electron-Positron Annihilation Cross Sections

I. Spectral Representations

1°) We have written, in Secs. C and D, integrals involving the total cross section for electron-positron annihilation into hadrons.

a) The Schwinger term in Eq. (C.17)

$$C^{em}(\text{hadrons}) = \frac{1}{16\pi^3\alpha^2}\int_0^\infty s\,\sigma_{tot}(s)\,ds \quad . \qquad \text{(E.1)}$$

b) The charge renormalization in Eq. (D.9)

$$\frac{\delta e_o^2}{e^2}\,(\text{hadrons}) = \frac{1}{\pi e^2}\int_0^\infty \sigma_{tot}(s)\, ds \qquad (E.2)$$

c) μ meson anomalous magnetic moment in Eq. (D.10)

$$a_\mu(\text{hadrons}) = \frac{1}{\pi e^2}\int_0^\infty \sigma_{tot}(s)\, K_\mu^{(2)}(s)\, ds \qquad (E.3)$$

where $K_\mu^{(2)}(s)$ behaves like $1/s$ for large s.

We must now examine the important problem of the convergence of these integrals.

2^o) The quark model offers a possibility to evaluate the Schwinger terms and to study the high energy behaviour of the total cross section $\sigma_{tot}(e^+ + e^- \Rightarrow \text{hadrons})$.

It has been shown by Gribov, Ioffe and Pomerantchuk[1] that the Schwinger terms are infinite and that the expected high energy behaviour of the total cross section is

$$\lim_{s\Rightarrow\infty} s\,\sigma_{tot}(s) = \text{const.} \qquad (E.4)$$

e.g. the same type of behaviour as for the $e^+ + e^- \Rightarrow \mu^+ + \mu^-$ total cross section in pure electrodynamics with only one photon exchanged.

Therefore

(E.1) diverges linearly

(E.2) diverges logarithmically

(E.3) converges.

3^o) Nevertheless it is possible to construct models where the result (E.4) is incorrect.

In a simplified version of the gluon model, Hayot and Nieh[2] conclude that the constant in Eq. (E.4) must be zero.

In the algebra of field model of Kroll, Lee and Zumino[3] where the electromagnetic current is identified to a sum of massive vector meson fields

$$J_\mu^{em} = - \sum_{a=\rho,\omega,\varphi} \frac{m_a^2}{f_a^2}\, V_\mu^a$$

the Schwinger term is simply given by

$$C^{em} = \sum_{a=\rho,\omega,\varphi} \frac{m_a^2}{f_a^2}$$

and its finiteness implies, for the total cross sections, a very stringent high-energy behaviour

$$\lim_{s\Rightarrow\infty} s^2 \sigma_{tot}(s) = 0 \ . \tag{E.5}$$

4°) Recently, Sakurai[4] speculating about the possibility of a free field behaviour of the current suggests the highly convergent limit

$$\lim_{s\Rightarrow\infty} s^3 \sigma_{tot}(s) = 0 \ . \tag{E.6}$$

II. Form Factors

1°) Let us discuss now the asymptotic form of the cross section in a particular channel, for instance the $\pi^+\pi^-$ channel. From the results of Sec. A., we have, in the one-photon exchange approximation, the following high-energy behaviour

$$\sigma_{tot}(e^+ + e^- \Rightarrow \pi^+ + \pi^-) = \frac{\text{const}}{s} |F_\pi(s)|^2 \ .$$

Nothing is known about the behaviour of the π-meson electromagnetic form factor in the timelike region but we can imagine two possible situations

a) The Phragmèn-Lindelöf theorem works and the high-energy behaviour is the same in the timelike region and in the spacelike region. For instance such a situation can occur if a phase representation can be used for F_π (see Sec. B, parts II and III).

b) The Phragmèn-Lindelöf theorem does not work and the two high-energy behaviours are not related. For instance the form factor, in the spacelike region, decreases exponentially like $e^{-a|s|^{\frac{1}{2}}}$, the spectral function in the timelike region has a very complicated oscillating structure and we retain only polynomial bounds for the form factor in the timelike region (see Sec. B-IV).

2°) Of course, we do not know the high-energy behaviour of the π-meson electromagnetic form factor in the spacelike region. Let us look, as a guide, at the nucleon electromagnetic form factors. From high-energy experiments, they decrease rapidly for large $|s|$ at least like $|s|^{-1}$, probably like $|s|^{-2}$ and perhaps more rapidly like $|s|^{-3}$.

Assuming analogous behaviour in the timelike region for the π-meson electromagnetic form factor we easily check that the $\pi^+\pi^-$ contributions to integrals like (E.1), (E.2) and (E.3) will be finite.

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SECTION F: Decay of Vector Mesons Into A Lepton-Antilepton Pair

I. One Level Vector Meson Dominance Model

1°) We consider the annihilation process $e^+ + e^- \Rightarrow F$ with a threshold s_F. Let us try to formulate in a naive way the general consequences of the vector meson dominance model in this specific case.

If a vector meson $V(\rho, \omega, \varphi)$ is physically realizable as an intermediate unstable state ($m_V^2 > s_F$) and if V can decay strongly in the state F, then the reaction $e^+e^- \Rightarrow F$ is dominated in the neighbourhood of $s = m_V^2$ by the V meson contribution according to the chain

$$e^+ + e^- \Rightarrow V \Rightarrow F$$

and described by the diagram

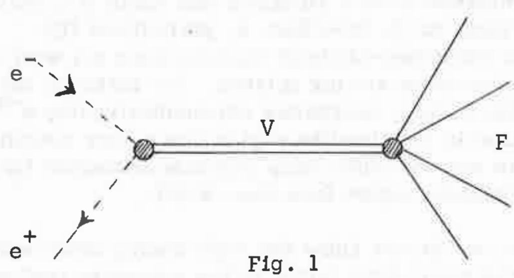

Fig. 1

2^{o}) At the V meson mass the total cross section is factorized as

$$\sigma_{tot}(e^{+} + e^{-} \Rightarrow V \Rightarrow F)_{s=m_V^2} = \frac{12\pi}{m_V^2} \frac{\Gamma(V \Rightarrow e^+e^-)}{\Gamma_V} \frac{\Gamma(V \Rightarrow F)}{\Gamma_V} . \quad (F.1)$$

Formula (F.1) is used to extract from experiment the partial decay widths $\Gamma(V \Rightarrow e^+e^-)$.

3^{o}) The ω meson and the φ meson can be strongly coupled to the same state, for instance the $\pi^+\pi^-\pi^o$ state. But all interferences between the ω-meson and the φ-meson contributions are always extremely small and can be neglected. More generally the V-meson contribution is important only in a range of energy approximately defined by $|\sqrt{s} - m_V| \lesssim \Gamma_V$ and it is clear in particular that the two domains

$$|\sqrt{s} - m_\omega| \lesssim \Gamma_\omega \quad \text{and} \quad |\sqrt{s} - m_\varphi| \lesssim \Gamma_\varphi$$

do not overlap.

4^{o}) The situation is a priori different for the ρ meson and the ω meson where the mass m_ω belongs to the range $|\sqrt{s} - m_\rho| \lesssim \Gamma_\rho$. If the final state is a pure hadronic state like $\pi^+\pi^-$, $\pi^+\pi^-\pi^{o}$ the total isotopic spin is well defined--using the additional information $J^{PC} = 1^{--}$ -- and therefore also the G parity. The ρ meson and the ω meson having opposite values of the G parity, the ρ-ω interference will occur only via electromagnetic interactions and formula (F.1) holds at the lowest order with respect to electromagnetic interactions.

5^{o}) If however the final state contains in addition to hadrons, photons or leptons, for instance a $\pi^o\gamma$ or a $\eta\gamma$ state, the G parity of the final state is no more defined, the ρ-meson and the ω-meson contributions occur on the same footing and the ρ-ω interference can be important.

The mixing effects are disregarded in what follows and will be studied in the next part.

II. The $\rho^{o} \Rightarrow e^+e^-$ Decay

1^{o}) The final state F is a $\pi^+\pi^-$ pair. At the total energy $s = m_\rho^2$ formula (F.1) is simply

$$\frac{\Gamma(\rho \Rightarrow e^+e^-)}{\Gamma_\rho} = \frac{m_\rho^2}{12\pi}\,\sigma_{tot}(e^+ + e^- \Rightarrow \pi^+ + \pi^-)_{s=m_\rho^2}\,. \quad (F.2)$$

2°) In the Orsay experiment the measured cross section at $s = m_\rho^2$ is[1)]

$$\sigma_{tot}(e^+e^- \Rightarrow \pi^+\pi^-)_{s=m_\rho^2} = (1.57 \pm 0.21)\ 10^{-30}\ cm^2.$$

The best fit of the π-meson electromagnetic form factor has given (see Sec. B)[1)]

$$m_\rho = (772 \pm 6)\ \text{MeV} \qquad \Gamma_\rho = (113 \pm 8)\ \text{MeV}$$

from which we deduce, using Eq. (F.2)

$$\frac{\Gamma(\rho \Rightarrow e^+e^-)}{\Gamma_\rho} = (6.37 \pm 0.85)\ 10^{-5}$$

and

$$\Gamma(\rho \Rightarrow e^+e^-) = (7.20 \pm 0.92)\ \text{keV}\,. \quad (F.3)$$

III. The $\omega \Rightarrow e^+e^-$ Decay

1°) The final state F is $\pi^+\pi^-\pi^o$. At the total energy $s = m_\omega^2$ formula (F.1) becomes

$$\frac{\Gamma(\omega \Rightarrow e^+e^-)}{\Gamma_\omega} = \frac{\Gamma_\pi}{\Gamma(\omega \Rightarrow \pi^+\pi^-\pi^o)}\,\frac{m_\omega^2}{12\pi}\,\sigma_{tot}(e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^o)_{s=m_\omega^2}\,. \quad (F.4)$$

2°) In the Orsay experiment the ω mass is fixed to $m_\omega = 783$ MeV and the ω total width is found to be[1)]

$$\Gamma_\omega = (16.2 \pm 3.2)\ \text{MeV}$$

e.g. larger than the world data average value of[2)]

$$\Gamma_\omega = (12.2 \pm 1.3)\ \text{MeV}$$

Including the possiblity of non-resonating background the Orsay people choose the width as given by the average of the world data and the measured cross section at $s = m_\omega^2$ is[1]

$$\sigma_{tot}(e^+e^- \Rightarrow 3\pi)_{s=m_\omega^2} = (1.65 \pm 0.31)\ 10^{-30}\ \text{cm}^2 .$$

3°) The branching ratio $\Gamma(\omega \Rightarrow 3\pi)/\Gamma_\omega$ is given by the world average value[2]

$$\frac{\Gamma(\omega \Rightarrow 3\pi)}{\Gamma_\omega} = 0.907 \pm 0.010 .$$

Using Eq. (F.4) we obtain

$$\frac{\Gamma(\omega \Rightarrow e^+e^-)}{\Gamma_\omega} = (7.7 \pm 1.4)\ 10^{-5}$$

and

$$\Gamma(\omega \Rightarrow e^+e^-) = (0.94 \pm 0.18)\ \text{keV} . \qquad \text{(F.5)}$$

IV. The $\varphi \Rightarrow e^+e^-$ Decay

1°) The final state F can be K^+K^-, $K^o\bar{K}^o$ or $\pi^+\pi^-\pi^o$. At the total energy $s = m_\varphi^2$, formula (F.1) becomes

$$\frac{\Gamma(\varphi \Rightarrow e^+e^-)}{\Gamma_\varphi} = \frac{\Gamma_\varphi}{\Gamma(\varphi \Rightarrow F)} \frac{m_\varphi^2}{12\pi} \sigma_{tot}(e^+ + e^- \Rightarrow F)_{s=m_\varphi^2} . \qquad \text{(F.6)}$$

2°) In the Orsay experiment the $K^o\bar{K}^o$ and the $\pi^+\pi^-\pi^o$ modes of the φ meson were detected and identified by looking at a $\pi^+\pi^-$ pair in various kinematical situations. An experiment for the K^+K^- mode is in progress.

The φ-meson mass has been taken at its world data average value $m_\varphi = 1019.3$ MeV and the φ-meson width has been measured by this experiment[1]

$$\Gamma_\varphi = (4.2 \pm 0.9)\ \text{MeV} .$$

The cross section at $s = m_\varphi^2$ for the $K^o\bar{K}^o$ final state is[1]

$$\sigma_{tot}(e^+e^- \Rightarrow K^o\bar{K}^o)_{s=m_\varphi^2} = (1.71 \pm 0.28)\ 10^{-30}\ \text{cm}^2 .$$

3°) The comparison of the total cross sections for $e^+ + e^- \Rightarrow K^o + \bar{K}^o$ and for $e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^o$ at $s = m_\varphi^2$ gives, in the vector meson dominance model, the ratio of the partial decay widths of the φ meson into $K^o\bar{K}^o$ and $\pi^+\pi^-\pi^o$. The branching ratio obtained in the Orsay experiment is[1)]

$$\frac{\Gamma(\varphi \Rightarrow \pi^+\pi^-\pi^o)}{\Gamma(\varphi \Rightarrow K^o\bar{K}^o)} = 0.667 \pm 0.157 \qquad \text{(F.7)}$$

in disagreement with the value 0.354 deduced from other experiments.[2)]

4°) On the other hand, assuming isotopic spin invariance for the decay amplitudes $\varphi \Rightarrow K^o\bar{K}^o$ and $\varphi \Rightarrow K^+K^-$ and taking into account the phase-space corrections due to the $K^o - K^+$ mass difference and the electromagnetic corrections we find a theoretical prediction for the ratio of the partial-decay widths for $\varphi \Rightarrow K^o\bar{K}^o$ and $\varphi \Rightarrow K^+K^-$ [3)]

$$\frac{\Gamma(\varphi \Rightarrow K^+K^-)}{\Gamma(\varphi \Rightarrow K^o\bar{K}^o)}\text{th} \simeq 1.60 \qquad \text{(F.8)}$$

again in disagreement with the average value 1.21 deduced from actual experiments.

5°) Assuming that the other decay modes of the φ meson are small, one can deduce from Eqs. (F.7) and (F.8) the three branching ratios for the main decay modes

$$\begin{array}{ll} \varphi \Rightarrow K^+K^- & (49 \pm 2.5)\% \\ \varphi \Rightarrow K^oK^o & (30.6 \pm 1.5)\% \\ \varphi \Rightarrow \pi^+\pi^-\pi^o & (20.4 \pm 4.0)\% \end{array}$$

Using Eq. (F.6) we obtain

$$\frac{\Gamma(\varphi \Rightarrow e^+e^-)}{\Gamma_\varphi} = (3.98 \pm 0.62)\, 10^{-4}$$

and

$$\Gamma(\varphi \Rightarrow e^+e^-) = (1.67 \pm 0.25) \text{ keV} . \qquad \text{(F.9)}$$

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SECTION G. Formalism for the Vector Meson Dominance Model Including Mixing

I. Mixing Problem

1°) The precise problem of interest is the ρ-ω mixing due to electromagnetic interactions but such a problem can be formulated in quite a more general way, as follows.

We use a Hamiltonian language and we divide the total Hamiltonian H into two parts

$$H = H_o + H_I .$$

Let us first consider the Hamiltonian H_o as the dominant part. We assume that there exists, in some sense, discrete states $|m\rangle$ stable or unstable, related to H_o and having the same quantum numbers as far as the total Hamiltonian H is concerned. These states $|m\rangle$ are distinguished, in H_o, by a conservation law associated to a particular invariance property, L, of H_o as for instance:

SU(3) invariance	isotopic spin invariance I
hypercharge Y	PC conservation

Therefore, in H_o, the states $|m\rangle$ are mutually orthogonal and in particular their relative phases are not observable. The states $|m\rangle$ span a subspace $\mathcal{E}$ of the Hilbert space of the physical states associated to H_o.

2°) Now what happens when the perturbation H_I is introduced? First the conservation law L is no more valid and we have a violation of the previous invariance. It follows that the states $|m\rangle$ can now mix and the physical states correspond to well defined superpositions of the $|m\rangle$'s.

However the mixing effects will be important only for those states such that their mass differences are small compared to their masses. We give some examples of mixing in Table I.

H_o	H_I	Particles (H_o)	Quantum Numbers (H)	Particles (H)	Invariance (H_o)
H_{strong} (SU_3 invariant)	$H_{medium\ strong}$	φ_8 ω_1	$J^{PG} = 1^{--}$	φ_o ω_o	SU(3)
H_{strong}	$H_{electromagnetic}$	ρ_o ω_o Σ^o Λ^o Y_1^* Y_o^*	$J^P = 1^-$ $J^P = \frac{1}{2}^+$ $J^P = \frac{3}{2}^-$	ρ ω Σ^o_{phys} Λ_{phys} Y^*_{1phys} Y^*_{0phys}	I
$H_{strong} + H_{em}$	H_{weak}	K^o $\bar{K}^o$	$J = 0$	K_S K_L	Y
$H_{st} + H_{em} + H_w$ (PC=+1)	H_w (PC=-1)	K_1^o K_2^o	$J = 0$	K_S K_L	PC

Table I

II. General Method

1°) There exist essentially two approaches of the mixing of almost degenerate particles. The first one is based on the Wigner-Weisskopf[1] perturbation theory and is essentially a time-dependent treatment of unstable particles. It has been for instance successfully applied to the $K^o\bar{K}^o$ problem by Lee, Oehme and Yang.[2] Analogous techniques have been used by Bernstein and Feinberg[3] for the ω-ρ electromagnetic mixing but some difficulties are due to the unstable character of the states before mixing and to a possible variation of the ρ width with the energy.

2°) The second method is based on the properties of the propagator matrix and has been proposed by Jacob and Sachs.[4] The unstable particles are now considered as resonances in a scattering problem. An application of this method has been made by Sachs[5] to the $K^o\bar{K}^o$ mixing and by Harte and Sachs[6] to the problem of the neutral vector meson mixing.

3°) We use the propagator method because we are first interested in the scattering aspect. Moreover, the time-dependent formalism has no direct relation with experiment for particles decaying strongly with a lifetime of the order 10^{-21} - 10^{-22}s and clearly the time distribution of the decay products cannot be reached with the actual experimental techniques.

4°) The formalism is presented here for vector mesons but it can be obviously adapted to particles having different spins.

Let us write the vector meson propagator in the form

$$\Delta_{\mu\nu}(p) = g_{\mu\nu} F(s) + p_\mu p_\nu G(s)$$

where $p^2 + s = 0$. In all the applications, we consider the vector meson is coupled to a conserved vector current so that only the part F(s) will contribute.

The function F(s) is the boundary value of an analytic function F(Z); it is convenient to represent it in the form[4]

$$F(Z) = [W(Z) - Z\,I]^{-1} . \qquad (G.1)$$

In the space $\mathcal{E}$, the function W(Z) is a matrix and I is the unit operator.

The physical particles are associated to the poles of the propagator located in the second sheet near the physical region. The real part and the imaginary one of such a pole Z_a are related to the mass and the width of the physical particle and the Z_a's are solutions of the equation

$$\det [W(Z_a) - Z_a I] = 0 \quad . \tag{G.2}$$

We assume the existence of ℓ such poles if ℓ is the dimension of the space $\mathcal{E}$.

5°) The practical way to resolve Eq. (G.2) is to diagonalize the matrix W(Z) defining eigenstates $|a(Z)>$, $|b(Z)>$, ... which are associated to the physical particles at the points Z_a, Z_b, ... Of course, such a mixing is energy dependent but we have a control about such a dependence.

6°) The precise form of W(Z) is arbitrary at the beginning. The unitarity of the S matrix will impose restrictions in a vector meson dominance model.

III. Normal Form of the Propagator

1°) We start with an orthonormal basis $|m\rangle$ defined by the unperturbed Hamiltonian H_0 in the finite dimensional subspace $\mathcal{E}$ (in practice the dimension of $\mathcal{E}$ will be 2 or 3). The characteristic relations are

$$\langle m|n\rangle = \delta_{mb} \qquad I = \sum_m |m\rangle\langle m| \tag{G.3}$$

where I is the projector on the subspace $\mathcal{E}$, e.g. the unit operator in $\mathcal{E}$. In the absence of the perturbation H_I the matrix function W(Z) has a diagonal representation in the previous basis.

Introducing now the perturbation H_I we first have a slight modification of the diagonal matrix elements $\langle m|W(Z)|m\rangle$ and secondly the appearance of nondiagonal matrix elements $\langle m|W(Z)|n\rangle$ proportional to H_I.

2°) Let us now assume that for any given Z, at least in the neighbourhood of the physical region, W(Z) can be brought in a diagonal form by a linear transformation in $\mathcal{E}$ represented by a complex regular matrix C(Z).

The right eigenvectors $|a(Z)\rangle$ are defined by the homogeneous equation

$$[W(Z) - W_a(Z)\, I]\ |a(Z)\rangle = 0 \qquad (G.4)$$

where $W_a(Z)$ is the corresponding eigenvalue function. The linear transformation is then written

$$|a(Z)\rangle = \sum_m C_{am}(Z)\,|m\rangle \ . \qquad (G.5)$$

The hermitian conjugate vectors $\langle a(Z)|$ are not in general left eigenvectors of $W(Z)$ because the matrix $C(Z)$ is not unitary. These left eigenvectors $\langle \tilde{a}(Z)|$ defined by the homogeneous equation

$$\langle \tilde{a}(Z)|\,[W(Z) - W_a(Z)\, I] = 0 \qquad (G.6)$$

are related to the original basis $\langle m|$ by the inverse linear transformation $C^{-1}(Z)$

$$\langle \tilde{a}(Z)| = \sum_m \langle m|\, C^{-1}_{ma}(Z) \qquad (G.7)$$

whereas the vectors $\langle a(Z)|$ are related to the original basis by the transformation $C^*(Z)$.

3^o) The bases $|a(Z)\rangle$ and $|\tilde{a}(Z)\rangle$ are not orthogonal bases and we can briefly sketch some of the most useful properties. Using the matrix $D(Z)$ defined by

$$D(Z) = C(Z)\, C^*(Z)$$

we easily obtain from the definitions (G.5) and (G.7) the following relations:

$$\langle b(Z)|a(Z)\rangle = D_{ab}(Z) \qquad \langle \tilde{b}(Z)|\tilde{a}(Z)\rangle = D^{-1}_{ab}(Z)$$

$$\langle \tilde{b}(Z)|a(Z)\rangle = \delta_{ab} = \langle b(Z)|\tilde{a}(Z)\rangle \ . \qquad (G.8)$$

In this way the operator I can be decomposed into

$$I = \sum_{a,b} D^{-1}_{ba}(Z)\,|a(Z)\rangle\langle b(Z)| \qquad I = \sum_{a,b} D_{ba}(Z)\,|\tilde{a}(Z)\rangle\langle\tilde{b}(Z)|$$

$$I = \sum_{a} |a(Z)\rangle\langle\tilde{a}(Z)| \qquad I = \sum_{a} |\tilde{a}(Z)\rangle\langle a(Z)| \quad . \tag{G.9}$$

Let us remark that the operators $|a(Z)\rangle\langle\tilde{a}(Z)|$ are idempotent, orthogonal two by two but not self-adjoint and therefore they are not projectors.

4°) Let us consider the interesting case of two dimensions. In order to simplify the notation we forget for the moment the Z dependence.

$$C = \begin{vmatrix} p & -q \\ r & s \end{vmatrix} \quad .$$

We are free to use the normalization conditions $\langle a|a\rangle = \langle b|b\rangle = 1$ or

$$|p|^2 + |q|^2 = 1 \qquad |r|^2 + |s|^2 = 1 \tag{G.10}$$

so that the matrix D is simply written as

$$D = \begin{vmatrix} 1 & p\bar{r} - q\bar{s} \\ \bar{p}r - \bar{q}s & 1 \end{vmatrix} \quad .$$

We are also free to make a choice of phases: p and s real, the relative phase between vectors of the original basis being arbitrary before mixing. Therefore we have two independent complex mixing parameters

$$\varepsilon_1 = \frac{q}{p} \qquad \varepsilon_2 = \frac{r}{s} \quad . \tag{G.11}$$

In the orthonormal basis W(Z) is represented by a 2 x 2 matrix

$$W_o = \begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix} \quad .$$

The diagonal form

$$W_D = \begin{vmatrix} W_a & 0 \\ 0 & W_D \end{vmatrix}$$

is obtained after applying the linear transformation C

$$W_D = C\, W_o\, C^{-1} \quad .$$

We can then compute the two mixing parameters ϵ_1, ϵ_2 and the two eigenvalues W_a, W_b in terms of the W_{ij}'s

$$\epsilon_2 = -\frac{W_{12}}{W_{11} - W_{22}} \; \frac{2}{1 + \left[1 + \dfrac{4W_{12}W_{21}}{(W_{11} - W_{22})^2}\right]^{\frac{1}{2}}}$$

$$\epsilon_1 = -\frac{W_{21}}{W_{11} - W_{22}} \; \frac{2}{1 + \left[1 + \dfrac{4W_{12}W_{21}}{(W_{11} - W_{22})^2}\right]^{\frac{1}{2}}}$$

$$W_a = \tfrac{1}{2}(W_{11} + W_{22}) + \tfrac{1}{2}(W_{11} - W_{22}) \left[1 + \frac{4W_{12}W_{21}}{(W_{11} - W_{22})^2}\right]^{\frac{1}{2}}$$

$$W_b = \tfrac{1}{2}(W_{11} + W_{22}) - \tfrac{1}{2}(W_{11} - W_{22}) \left[1 + \frac{4W_{12}W_{21}}{(W_{11} - W_{22})^2}\right]^{\frac{1}{2}} \quad . \qquad \text{(G.12)}$$

5^o) If time-reversal invariance holds, the matrix W(Z) is symmetric in the original basis[6)]

$$\langle m | W(Z) | n \rangle = \langle n | W(Z) | m \rangle$$

and we obtai n interesting constraints on the linear transformation C(Z). For instance, in the two dimensional case we have the two relations $r = q$ $s = p$ (see Eqs. (G.12))

$$C = \begin{vmatrix} p & -q \\ q & p \end{vmatrix} \qquad \text{(G.13)}$$

and the mixing is described by only one mixing parameter

$$\epsilon_1 = \epsilon_2 = \epsilon \qquad \text{(G.14)}$$

As a second consequence, the hermitian product $\langle b|a\rangle$ becomes purely imaginary

$$\langle b|a\rangle = \bar{q}p - \bar{p}q = -2i(1 + |\epsilon|^2)\ \mathrm{Im}\ \epsilon\ . \tag{G.15}$$

IV. T Matrix Amplitude and Vector Meson Dominance Model

1°) Decay Amplitude

We consider the decay amplitude for a vector meson V of energy momentum p and polarization λ into a final state F. From Lorentz covariance we have

$$\langle F|T|V_m(\lambda)\rangle = a^m_\mu(F)\ e^\mu(p,\lambda) \tag{G.16}$$

where the index m refers to the type of vector meson.

The polarization four vector $e_\mu(p,\lambda)$ is submitted to the supplementary condition

$$p^\mu\ e_\mu(p,\lambda) = 0\ .$$

The four vector $a^m_\mu(F)$ is then orthogonal to p_μ and can be expanded on the basis of the polarization vectors

$$a^m_\mu(F) = \sum_\lambda a^m_\lambda(F)\ e^*_\mu(p,\lambda) \tag{G.17}$$

and the amplitude $a^m_\lambda(F)$ is just the amplitude we start with

$$\langle F|T|V_m(\lambda)\rangle = a^m_\lambda(F)\ . \tag{G.18}$$

The amplitudes $a^m_\lambda(F)$ can be considered as the components of a vector in the three dimensional space of the polarizations.

2°) Decay width

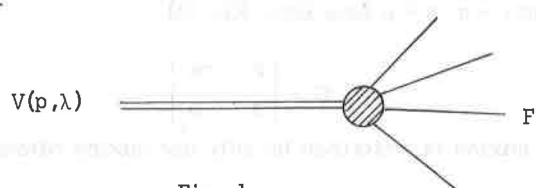

Fig. 1

We denote by p_α the energy momentum four vector of a particle α of the final state F. The total energy momentum is defined as

$$p_F = \sum_{\alpha \in F} p_\alpha \; .$$

The summation operation S_F introduced in Sec. A has the explicit form

$$S_f = \prod_{\alpha \in F} \frac{N_\alpha}{(2\pi)^3} \sum_{\text{pol } F} \int \prod_{\alpha \in F} \frac{d_3 p_\alpha}{E_\alpha}$$

where the N_α's are the normalization constants.

The partial width for the decay $V_m \Rightarrow F$ is easily computed to be

$$\Gamma(V_m \Rightarrow F) = \frac{1}{m_V} \frac{1}{6} \sum_\lambda S_F (2\pi)^4 \delta_4 (p_F - p) |a_\lambda^m(F)|^2 \; . \qquad (G.19)$$

3°) Scattering Amplitude

We now study the scattering amplitude from a state I to a state F, in the vector meson dominance model, when both I and F are possible decaying states of the vector meson V.

The T matrix amplitude corresponding to the diagram of Fig. 2

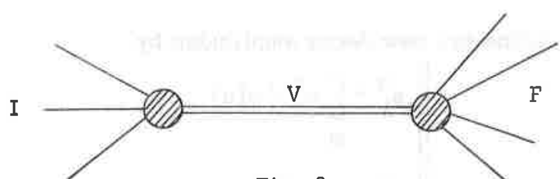

Fig. 2

is given by

$$\langle F|T|I\rangle = \sum_{m,n} a_\mu^m(F) \langle m|\Delta^{\mu\nu}(p)|n\rangle a_\nu^n(I)^* \; .$$

We assume the off-mass-shell decay amplitudes a_μ to be orthogonal to p_μ at least for one of the two states I or F. Such a property is true if the currents associated to vector mesons are conserved. Therefore only the $g^{\mu\nu}$ part of the propagator contributes

$$\langle F|T|I\rangle = \sum_{m,n} a_\mu^m(F)\,\langle m|\frac{g^{\mu\nu}}{W(s)-sI}|n\rangle\, a_\nu^n(I)^* . \qquad (G.20)$$

We then can expand the decay amplitudes a(F) and a(I) following equations (G.17) and using the orthonormality property of the polarization vectors

$$e_\mu^*(p,\lambda)\, e^\mu(p,\lambda') = \delta_{\lambda\lambda'} ,$$

we write the transition matrix element in the form

$$\langle F|T|I\rangle = \sum_{m,n}\sum_\lambda a_\lambda^m(F)\,\langle m|\frac{1}{W(s)-sI}|n\rangle\, a_\lambda^n(I)^* . \qquad (G.21)$$

Let us now introduce the eigenvectors of W and use the completeness relation (G.9)

$$I = \sum_a |a\rangle\langle\tilde{a}|$$

$$\langle m|\frac{1}{W(s)-sI}|n\rangle = \sum_a \langle m|a\rangle\,\frac{1}{W_a(s)-s}\,\langle\tilde{a}|m\rangle .$$

We then define two new decay amplitudes by

$$a_\lambda^a = \sum_m a_\lambda^m\,\langle m|a\rangle$$

$$a_\lambda^{\tilde{a}} = \sum_m a_\lambda^m\,\langle m|\tilde{a}\rangle .$$

We finally obtain a third expression for the transition matrix element

$$\langle F|T|I\rangle = \sum_a\sum_\lambda a_\lambda^a(F)\,\frac{1}{W_a(s)-s}\,a_\lambda^{\tilde{a}}(I)^* . \qquad (G.22)$$

4°) Resonant cross section

We consider the process A + B ⇒ F dominated by the vector mesons V in the direct channel. We identify the transition amplitude with its resonant part as given in Eq. (G.22) by the vector meson dominance model.

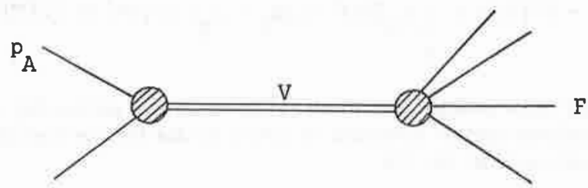

Fig. 3

The square of the total energy in the centre-of-mass system is called as usual s

$$s = -(p_A + p_B)^2 = -p_F^2 \quad .$$

The centre-of-mass momentum in the initial state is given by

$$K_{AB}(s) = \tfrac{1}{2}\left\{\frac{\left[s-(m_A+m_B)^2\right]\left[s-(m_A-m_B)^2\right]}{s}\right\}^{\frac{1}{2}}$$

and the invariant effective phase space is defined by

$$\prod_{\alpha\in F}\frac{d_3 p_\alpha}{E_\alpha} \equiv d_4 p_F \, dL_F \quad .$$

The differential cross section for the process A + B ⇒ (V) ⇒ F is then computed to be

$$\frac{d\sigma(A+B\Rightarrow V\Rightarrow F)}{dL_F} = \frac{1}{\sqrt{s}}\,\frac{1}{K_{AB}(s)}\,\frac{N_A N_B}{(2s_A+1)(2s_B+1)}$$
$$\sum_{\substack{\text{pol}\\ F}} (2\pi)^4 \prod_{\alpha\in F}\frac{N_\alpha}{(2\pi)^3}\sum_{\substack{\text{pol}\\ F}}\Big|\sum_{a,\lambda}\frac{a_\lambda^{a}(F)\,a_\lambda^{\tilde a}(AB)^*}{W_a(s)-s}\Big|^2 \quad . \qquad \text{(G.23)}$$

V. Unitarity Constraints

1°) The unitarity property of the S matrix implies for the transition matrix element the following relation

$$\langle F|T - T^*|I\rangle = i \sum_n S_n (2\pi)^4 \, \delta_4 (p_F - p_n) \langle F|T^*|n\rangle \langle n|T|i\rangle \, . \quad (G.24)$$

We apply this general unitarity relation to the particular case where the transition matrix element is given by its vector meson dominance model expression (G.21)

$$\langle F|T|I\rangle = \sum_{m,n} \sum_{\lambda} a_\lambda^m(F) \, a_\lambda^n(I)^* \langle m| \frac{1}{W(s) - sI} |n\rangle$$

$$\langle F|T^*|I\rangle = \sum_{m,n} \sum_{\lambda} a_\lambda^m(F) \, a_\lambda^n(I)^* \langle m| \frac{1}{W^*(s) - sI} |n\rangle \, .$$

2°) Let us first compute the left-hand side of the unitarity relation

$$\langle F|T - T^*|I\rangle = \sum_{m,n} \sum_{\lambda} a_\lambda^m(F) \, a_\lambda^n(I)^* \langle m| \frac{1}{W(s) - sI} - \frac{1}{W^*(s) - sI} |n\rangle \, .$$

Consider the matrix identity

$$[W(s)-sI]^{-1} - [W^*(s)-sI]^{-1} \equiv [W^*(s)-sI]^{-1}[W^*(s)-W(s)][W(s)-sI]^{-1}$$

and use the two equivalent decompositions of the identity (G.9)

$$I = \sum_b |\tilde{b}\rangle\langle b| \qquad\qquad I = \sum_a |a\rangle\langle\tilde{a}| \quad .$$

Taking into account the eigenvalue equations (G.4) and (G.6) we obtain

$$[W(s)-sI]^{-1} - [W^*(s)-s]^{-1} = \sum_{a,b} |b\rangle\langle a| \frac{\langle b|a\rangle \, [\tilde{W}_b(s) - W_a(s)]}{[\tilde{W}_b(s) - s][W_a(s) - s]}$$

and finally after summation over the indices m and n

$$\langle F|T - T^*|I\rangle = \sum_{a,b} \sum_{\lambda} a_\lambda^{\tilde{b}}(F)\, a_\lambda^{\tilde{a}}(I)^* \frac{\langle b|a\rangle\,[\bar{W}_b(s) - W_a(s)]}{[\bar{W}_b(s) - s][W_a(s) - s]} . \quad (G.25)$$

3°) Let us now compute the right-hand side of the unitarity relation retaining only, by consistency, intermediate states n of the type I or F.

$$RHS = i\sum_n S_n (2\pi)^4\, \delta_4(p_F - p_n) \sum_{\substack{m,n\\p,q}} \sum_{\substack{\lambda\\\mu}} a_\lambda^m(F)\, a_\lambda^{p*}(n)\, a_\mu^q(n)\, a_\mu^n(I)^* \quad \times$$

$$\langle m|\frac{1}{W^*(s) - sI}|p\rangle\langle q|\frac{1}{W(s) - sI}|n\rangle \quad .$$

Again we use the two forms of the relation (G.9)

$$RHS = i\sum_n S_n (2\pi)^4\, \delta_4(p_F - p_n) \sum_{a,b} \sum_{\lambda,\mu} a_\lambda^{\tilde{b}}(F)\, a_\mu^{\tilde{a}}(I)^*\, a_\lambda^b(n)^*\, a_\mu^a(n) \quad \times$$

$$\frac{1}{[\bar{W}_b(s) - s][W_a(s) - s]} \quad . \quad (G.26)$$

4°) The two expressions (G.25) and (G.26) must be equal for all initial states I and all final states F. It follows immediately the unitarity constraints on the vector meson propagator W(s)

$$\frac{1}{2i}[\bar{W}_b(s) - W_a(s)]\langle b(s)|a(s)\rangle = \frac{1}{6}\sum_n S_n (2\pi)^4 \delta_4(p_n - p) \sum_\lambda a_\lambda^b(n)^*\, a_\lambda^a(n) \quad . \quad (G.27)$$

For the diagonal elements where the normalization $\langle a|a\rangle = 1$ has been chosen, formula (G.27) reduces to

$$- \operatorname{Im} W_a(s) = \sigma_a(s) = \sum_n \sigma_a^n(s) \quad (G.28)$$

with

$$\sigma_a^n(s) = \frac{1}{6} S_n (2\pi)^4\, \delta_4(p_n - p) \sum_\lambda |a_\lambda^a(n)|^2 \quad .$$

Comparing this with the expression (G.19) for the partial decay width $\Gamma(a \Rightarrow n)$ we deduce

$$\Gamma(V_a \Rightarrow n) = \frac{1}{m_a} \sigma_a^n(m_a^2) \quad . \tag{G.29}$$

For the nondiagonal elements $a \neq b$ we obtain a Bell-Steinberger[7)] type relation.

5°) Equivalent constraints can be obtained proceeding in the original orthogonal. They are simply written as

$$\frac{1}{2i} \langle p | W^*(s) - W(s) | q \rangle = \frac{1}{6} \sum_n S_n (2\pi)^4 \delta_4 (p_n - p) \sum_\lambda a_\lambda^p(n)^* \, a_\lambda^q(n) \quad . \tag{G.30}$$

It is then convenient to decompose the matrix W(s) into its hermitian and skew hermitian parts $W \equiv R - i\,\Sigma$ where both R and Σ are hermitian matrices. The unitarity constraints (G.30) determine the matrix Σ

$$\langle p_- | \Sigma | q \rangle = \frac{1}{6} \sum_n S_n (2\pi)^4 \; \delta_4 (p_n - p) \sum_\lambda \; a_\lambda^p(n)^* \, a_\lambda^q(n) \quad .$$

VI. Mass and Width Parameters

1°) Let us split the function $W_a(s)$ for s real in the physical region, into its real and imaginary parts

$$W_a(s) = \rho_a(s) = i\, \sigma_a(s) \quad .$$

The imaginary part $\sigma_a(s)$ is determined by the unitarity condition (G.28) but the real part $\rho_a(s)$ is free up to now.

2°) In order to obtain information about $\rho_a(s)$ we have to express the existence of a pole for the propagator located in the second sheet at a point Z_a defined by the condition (G.2) which reduces here to

$$W_a(Z_a) = Z_a \quad . \tag{G.31}$$

We define $Z_a = x_a - i\, y_a$ both x_a and y_a being real and in order to be physically acceptance the pole Z_a must satisfy the two conditions

$$x_a, y_a > 0 \qquad \frac{y_a}{x_a} \ll 1 .$$

The physical interpretation of Z_a in terms of mass and width of the resonance is naturally given by

$$Z_a = \left(m_a - i\frac{\Gamma_a}{2}\right)^2$$

or

$$x_a = m_a^2\left[1 - \tfrac{1}{4}\left(\frac{\Gamma_a}{m_a}\right)^2\right] \qquad y_a = m_a \Gamma_a \quad . \tag{G.32}$$

3°) On the other hand, the mass and the width of the resonance are usually computed from the relations

$$m_a^2 = \rho_a(m_a^2) \qquad\qquad m_a \Gamma_a = \sigma_a(m_a^2) \tag{G.33}$$

and we want to relate the two definitions of m and Γ calculating Z_a from the conditions (G.33).

We first expand $W_a(Z_a)$ around the point m_a^2 retaining only first-order terms

$$W_a(Z_a) \simeq W_a(m_a^2) + (Z_a - m_a^2)\, W_a'(m_a^2) \ .$$

Using the equality (G.31) for $W_a(Z_a)$ and the conditions (G.33) for $W_a(m_a^2)$ we obtain the approximate expression

$$Z_a = m_a^2 - i\, m_a \Gamma_a \frac{1}{1 - W_a'(m_a^2)} \ . \tag{G.34}$$

The function $\rho_a(s) - s$ is then expanded around the point $s = m_a^2$ following

$$\rho_a(s) - s = m_a^2 - s + (m_a^2 - s)^2\, \tau_a(s)$$

where $\tau_a(s)$ and its first derivative $\tau_a'(s)$ are assumed to be regular at the point $s = m_a^2$. The first derivative of $\rho_a(s)$ vanishes at $s = m_a^2$ and formula (G.34) becomes

$$Z_a = m_a^2 - i\, m_a \Gamma_a \frac{1}{1 + i\,\sigma_a'(m_a^2)} \qquad \text{(G.35)}$$

so that

$$x_a = m_a^2 \left[1 - \frac{\Gamma_a}{m_a} \frac{\sigma_a'(m_a^2)}{1 + \sigma_a'^2(m_a^2)} \right] \qquad y_a = m_a \Gamma_a \frac{1}{1 + \sigma_a'^2(m_a^2)} \;.$$

4°) Let us evaluate the correction factor in the case of the ρ meson. Using

$$m_\rho \simeq 770 \text{ MeV} \qquad \Gamma_\rho \simeq 110 \text{ MeV}$$

we obtain

$$x_\rho \simeq m_\rho^2 \times 0.965 \qquad y_\rho \simeq m_\rho \Gamma_\rho \times 0.969 \;.$$

VII. Factorization

1°) For the final states $F = \pi^+\pi^-, \pi^+\pi^-\pi^o, \pi^o\gamma, \eta\gamma$ etc... the matrix element $\langle F|T|V\rangle$ has only one Lorentz covariant because of the conservation of parity. Therefore the amplitude $a_\lambda^V(F)$ can be factorized into the product of a dynamical function $a_\lambda(F)$ by a coupling constant f_{VF}. In practice f_{VF} is assumed to be energy independent at least in the neighbourhood of the vector meson region.

$$a_\lambda^V(F) = f_{VF}\, a_\lambda(F) \;. \qquad \text{(G.36)}$$

The same property holds for a lepton-antilepton state in the one-photon exchange model.

2°) However, such a property is not general and there exist some rare decay modes like, for instance, $\pi^+\pi^-\gamma$ [8)] where the factorization is not possible.

3°) Let us now define the function

$$\gamma_F(s) = \frac{1}{6} S_F (2\pi)^4 \delta_4(p_F - p) \sum_\lambda |a_\lambda(F)|^2 \;. \qquad \text{(G.37)}$$

For those states where the factorization relation (G.36) is true the partial decay width is simply given by

$$\Gamma(V_a \Rightarrow F) = \frac{1}{m_a} |f_{AF}|^2 \gamma_F(m_a^2) \quad . \tag{G.38}$$

On the other hand, the unitarity relation (G.27) is strongly dominated by factorizable states and we can write

$$\frac{1}{2i} [\bar{W}_b(s) - W_a(s)] \langle b(s) | s(s) \rangle = \sum_F{}' \bar{f}_{bF} f_{aF} \gamma_F(s) \tag{G.39}$$

the sum $\sum_F{}'$ being extended to factorizable states only.

4°) We compute the total cross section for the process

$$A + B \Rightarrow (V) \Rightarrow F$$

where AB and F are two factorizable states.

The starting expression is the equation (G.23) and we sketch only the main points of the calculation:

a) It is convenient to introduce, as an intermediate step, the density matrices for vector meson decay

$$(\rho_n)_{\lambda\lambda'} = \frac{\sum_{\text{pol } n} a_\lambda(n)\, a_{\lambda'}(n)^*}{\sum_{\text{pol } n} \sum_\mu |a_\mu(n)|^2} \quad . \tag{G.40}$$

It is trivial to check the two characteristic properties

$$\rho_n^* = \rho_n \qquad \text{Tr}\, \rho_n = 1$$

and for vector meson the average value over angles gives

$$\langle \text{Tr}\, [\rho_I \rho_F] \rangle = \frac{1}{3} \quad .$$

b) The dynamical functions $\gamma(s)$ for the states F and AB are given from Eq. (G.37) by the explicit expressions

$$\left\| \begin{aligned} \gamma_F(s) &= \frac{1}{6}(2\pi)^4 \prod_{\alpha \in F} \frac{N_\pi}{(2\pi)^3} \int dL_F \sum_{\text{pol } F} \sum_{\lambda} |a_\lambda(F)|^2 \\ \gamma_{AB}(s) &= \frac{1}{6\pi} N_A N_B \frac{K_{AB}(s)}{\sqrt{s}} \sum_{\text{pol } AB} \sum_{\lambda'} |a_{\lambda'}(AB)|^2 \end{aligned} \right. .$$

Combining now all these results we obtain the final form

$$\sigma_{tot}(A + B \Rightarrow V \Rightarrow F) = \frac{12\pi}{(2s_A+1)(2s_B+1)} \frac{\gamma_{AB}(s)\,\gamma_F(s)}{K^2_{AB}(s)} \left| \sum_a \frac{f_{aF}\, f_{\tilde{a}AB}}{W_a(s) - s} \right|^2 . \tag{G.41}$$

VIII. Electron-Positron Annihilation

1°) Let the initial state be an electron-positron pair. The matrix element for the vector meson decay into a lepton-antilepton pair

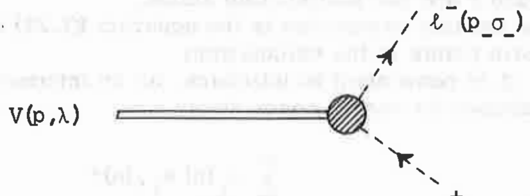

Fig. 4

has the general structure due to Lorentz covariance and parity conservation

$$\langle \ell^+\ell^- | T | V(\lambda) \rangle = e^2 \bar{u}_{\sigma_-}(p_-)[h_1^V(s)\gamma_\mu + i\, h_2^V(s)(p_+ - p_-)_\mu] v_{\sigma_+}(p_+) e^\mu(p,\lambda)$$

where h_1 and h_2 are two arbitrary form factors.

It is usual to compute h_1 and h_2 in the one-photon exchange approximation (algebra of field model)

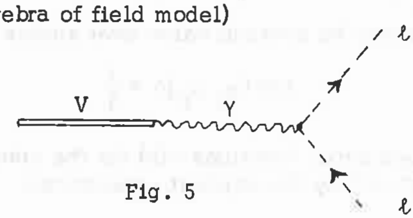

Fig. 5

and the result is

$$h_1^V(s) = -\frac{m_V^2}{f_V}\frac{1}{s} \qquad h_2(s) = 0 \quad .$$

In such a model only one invariant function $h_1^V(s)$ occurs and we can apply the factorization procedure. The reduced amplitude is written as

$$a_\lambda(\ell^+\ell^-) = -\frac{e^2}{s}\bar{u}_{\sigma_-}(p_-)\,\gamma_\mu\, v_{\sigma_+}(p_+)\, e^\mu(p,\lambda)$$

and the function $\gamma_{\ell^+\ell^-}(s)$ is easily calculated after summation over the lepton, antilepton and vector meson polarizations

$$\gamma_{\ell^+\ell^-}(s) = \frac{4\pi\,\alpha^2}{3}\,\frac{1}{s}\left(1 + \frac{2m^2}{s}\right)\left(1 - \frac{4m^2}{s}\right)^{\frac{1}{2}}$$

where m is the lepton mass. We always have $m^2/s \ll 1$ and up to terms m^4/s^2, $\gamma_{\ell^+\ell^-}(s)$ turns out to be independent of the lepton mass

$$\gamma_{\ell^+\ell^-}(s) = \frac{4\pi\,\alpha^2}{3}\,\frac{1}{s} \quad . \tag{G.42}$$

The radiative decay width $\Gamma(V_a \Rightarrow \ell^+\ell^-)$ is immediately computed to be

$$\Gamma(V_a \Rightarrow \ell^+\ell^-) = \frac{4\pi\,\alpha^2}{3}\,\frac{m_a}{|f_a|^2} \tag{G.43}$$

Using now the experimental result quoted in Section F for these radiative decays, we deduce from formula (G.43) the corresponding values for the coupling constants

$$\frac{|f_\rho|^2}{4\pi} = 1.90 \pm 0.25 \qquad \frac{|f_\omega|^2}{4\pi} = 14.8 \pm 2.8 \qquad \frac{|f_\varphi|^2}{4\pi} = 11.0 \pm 1.6 \quad .$$

Finally, we compute the total cross section for electron-positron annihilation into a final state F in the vector meson dominance VMD model. We have $s_A = s_B = \frac{1}{2}$ and, neglecting again the electron mass, the C.M. momentum is simply $K_{ee}(s) = \sqrt{s}/2$. Combining Eqs. (G.41) and (G.42) the result is

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow F) = \frac{(4\pi\alpha)^2}{s^2}\,\gamma_F(s)\Big|\sum_a m_a^2 \frac{f_{AF}}{f_a}\frac{1}{W_a(s)-s}\Big|^2 . \quad (G.44)$$

3°) Final state $\pi^+\pi^-$

The matrix element for the decay of a vector meson V in a $\pi^+\pi^-$ pair

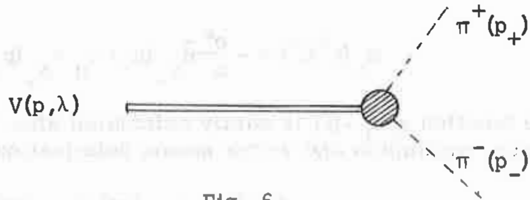

Fig. 6

has the general structure due to Lorentz covariance and parity conservation

$$\langle \pi^+\pi^- | T | V(\lambda)\rangle = f_{V\pi\pi}\,(p_+ - p_-)_\mu\, e^\mu(p,\lambda) .$$

The dynamical amplitude $a_\lambda(\pi^+\pi^-)$ is simply written

$$a_\lambda(\pi^+\pi^-) = (p_+ - p_-)_\mu\, e^\mu(p,\lambda) .$$

The summation over the vector meson polarizations is straightforward

$$\sum_\lambda |a_\lambda(\pi^+\pi^-)|^2 = (p_+ - p_-)^2 = s - 4m_\pi^2 = 4K_{\pi\pi}^2(s) .$$

The function $\gamma_{\pi\pi}(s)$ is given by

$$\gamma_{\pi\pi}(s) = \frac{1}{6\pi}\,\frac{K_{\pi\pi}^3(s)}{\sqrt{s}} \quad (G.45)$$

and the expression of the decay width is simply

$$\Gamma(V \Rightarrow \pi^+\pi^-) = \frac{2}{3}\,\frac{|f_{V\pi\pi}|^2}{4\pi}\,\frac{K_{\pi\pi}^3(m_V^2)}{m_V^2} . \quad (G.46)$$

Combining equations (G.44) and (G.45) we obtain the total cross section for the process $e^+ + e^- \Rightarrow \pi^+ + \pi^-$ in the VMD model approximation

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \pi^+ + \pi^-) = \frac{\pi\alpha^2}{3}\,\frac{1}{s}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \left|\sum_a m_a^2 \,\frac{f_{a\pi\pi}}{f_{\tilde{a}}}\,\frac{1}{W_a(s) - s}\right|^2 . \tag{G.47}$$

It is interesting to compare Eq. (G.47) with the general structure of the $e^+ + e^- \Rightarrow \pi^+ + \pi^-$ total cross section due to the one-photon exchange approximation and given in Eq. (A.11) of Section A. We then obtain the VMD model of the π form factor including electromagnetic mixing effects

$$F_\pi(s) = \sum_a m_a^2 \,\frac{f_{a\pi\pi}}{f_{\tilde{a}}}\,\frac{1}{W_a(s) - s} \quad . \tag{G.48}$$

Of course the sum $\sum_a$ is restricted here to $a = \rho$ and ω.

4°) Final state $\pi^+\pi^-\pi^o$

The matrix element for the decay of a vector meson V into $\pi^+\pi^-\pi^o$

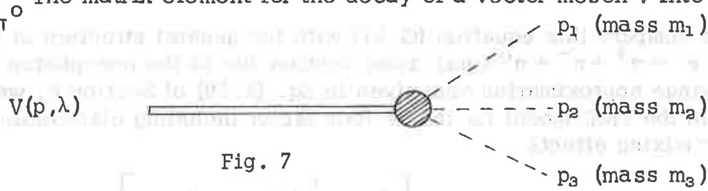

Fig. 7

has the general structure due to Lorentz covariance and parity conservation

$$\langle \pi^+\pi^-\pi^o | T | V(\lambda)\rangle = f_{V3\pi}\,\epsilon_{\mu\nu\rho\sigma}\, e^\mu(p,\lambda)\left(\frac{p_1}{m_1}\right)^\nu \left(\frac{p_2}{m_2}\right)^\rho \left(\frac{p_3}{m_3}\right)^\sigma \Phi(s;\, E_1\, E_2)$$

where the C.M. energy variables are defined by $E_j = -p \cdot p_j / \sqrt{s}$.

The dynamical amplitude $a_\lambda(3\pi)$ is simply written as

$$a_\lambda(3\pi) = \epsilon_{\mu\nu\rho\sigma}\, e^\mu(p,\lambda)\left(\frac{p_1}{m_1}\right)^\nu \left(\frac{p_2}{m_2}\right)^\rho \left(\frac{p_3}{m_3}\right)^\sigma \Phi(s;\, E_1\, E_2)$$

and the calculation of the function $\gamma_{3\pi}(s)$ is straightforward

$$\gamma_{3\pi}(s) = \frac{1}{3}\,\frac{1}{(4\pi)^3}\,\frac{s}{(m_1 m_2 m_3)^2} \iint_{D(s)} |\Phi(s;\, E_1 E_2)|^2\, |\vec{p}_1 \times \vec{p}_2|^2\, dE_1\, dE_2 \tag{G.49}$$

where the domain of integration D(s) is defined by the condition that p_1, p_2, p_3 are sides of a triangle.

The 3π decay width of a vector meson V takes then the form

$$\Gamma(V \Rightarrow 3\pi) = \frac{1}{3}\,\frac{|f_{V3\pi}|^2}{(4\pi)^3}\,\frac{m_V}{(m_1 m_2 m_3)^2} \iint_{D(m_V^2)} |\Phi(m_V^2; E_1 E_2)|^2 \times |\vec{p}_1 \times \vec{p}_2|^2 \; dE_1\, dE_2 \quad . \tag{G.50}$$

Combining now Eqs. (G.44) and (G.49) we compute the total cross section for the process $e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^0$ in the VMD model approximation

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \pi^+ + \pi^- + \pi^0) = \frac{\alpha^2}{12\pi s}\left\{ \iint_{D(s)} |\Phi(s;\, E_1 E_2)|^2 \times |\vec{p}_1 \times \vec{p}_2|^2 \; dE_1\, dE_2 \right\} \left| \sum_a m_a^2 \, \frac{f_{a3\pi}}{f_{\tilde{a}}} \, \frac{1}{W_a(s) - s} \right|^2 \quad . \tag{G.51}$$

If we compare this equation (G.51) with the general structure of the $e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^0$ total cross section due to the one-photon exchange approximation and given in Eq. (A.28) of Section A, we obtain the VMD model for the 3π form factor including electromagnetic mixing effects

$$F_{3\pi}(s;\, E_1 E_2) = \Phi(s;\, E_1 E_2)\left[\sum_a m_a^2 \, \frac{f_{a3\pi}}{f_{\tilde{a}}} \, \frac{1}{W_a(s) - s} \right] .$$

In the ω-ρ region, the sum $\sum_a$ has two terms, the ω term and the ρ term with the electromagnetic mixing.

In the φ region, the dominant contribution is due to the φ but we can have a small contribution due to the ρ.

5°) Final state $K\bar{K}$

The calculations are identical to those made in paragraph 3 for the $\pi^+\pi^-$ system. We have here two possible states depending on the charge of the K mesons, K^+K^- and $K^0\bar{K}^0$, noted as $K_r \bar{K}_r$ with $r = +$ or 0. In particular, the dynamical function $\gamma_{K\bar{K}}(s)$ is given by

$$\gamma_{K_r \bar{K}_r}(s) = \frac{1}{6\pi}\,\frac{K^3_{K_r \bar{K}_r}(s)}{\sqrt{s}} \tag{G.52}$$

and the partial decay width for the decay $V \Rightarrow K\bar{K}$ is simply

$$\Gamma(V \Rightarrow K_r\bar{K}_r) = \frac{2}{3}\,\frac{|f_{VK_r\bar{K}_r}|^2}{4\pi}\,\frac{K^3_{K_r\bar{K}_r}(m^2_V)}{m^2_V} \quad . \tag{G.53}$$

Let us remark the magnitude of the phase space corrections due to the $K^+ - K^o$ mass difference

$$\frac{K^3_{K^+K^-}(m^2_\varphi)}{K^3_{K^o\bar{K}^o}(m^2_\varphi)} \simeq 1.54$$

and the electromagnetic corrections to the SU(2) invariance of the coupling constants have been estimated

$$\left|\frac{f_{\varphi K^+K^-}}{f_{\varphi K^o\bar{K}^o}}\right|^2 \simeq 1.04 \quad .$$

Combining Eqs. (G.44) and (G.52) we obtain the total cross section for the process $e^+ + e^- \Rightarrow K + \bar{K}$ in the VMD model approximation

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow K_r + \bar{K}_r) = \frac{\pi\alpha^2}{3}\,\frac{1}{s}\left(1 - \frac{4m^2_{K_r}}{s}\right)^{3/2} \left|\sum_a m^2_a \frac{f_{aK_r\bar{K}_r}}{f_{\tilde{a}}}\,\frac{1}{W_a(s)-s}\right|^2 . \tag{G.54}$$

Comparing Eq. (G.54) with the general structure of the $e^+ + e^- \Rightarrow K + \bar{K}$ total cross section due to the one-photon exchange approximation and given in Eq. (A.11) of Section A, we obtain the VMD model for the K meson form factors

$$F_{K_r}(s) = \sum_a m^2_a \frac{f_{aK_r\bar{K}_r}}{f_{\tilde{a}}}\,\frac{1}{W_a(s) - s} \quad . \tag{G.55}$$

Of course in the φ region the sum $\sum_a$ is strongly dominated by the φ-meson contribution but nevertheless we can have a small contamination of the ρ because of the relatively large ρ-meson width.

6^o) Final state $\pi^o\gamma$

The matrix element for the decay of a vector meson V into a $\pi^o\gamma$ state

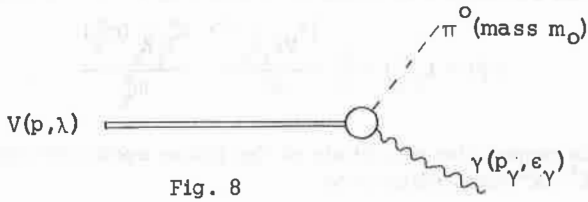

Fig. 8

has the general structure due to Lorentz covariance and parity conservation

$$\langle \pi^o\gamma | T | V(\lambda)\rangle = \frac{e}{m_o} f_{V\pi^o\gamma}\, \epsilon_{\mu\nu\rho\sigma}\, p^\mu\, e^\nu(p,\lambda)\, p_\gamma^\rho\, \epsilon_\gamma^\sigma \quad .$$

The amplitude $a_\lambda(\pi^o\gamma)$ is simply written

$$a_\lambda(\pi^o\gamma) = \frac{e}{m_o}\, \epsilon_{\mu\nu\rho\sigma}\, p^\mu\, e^\nu(p,\lambda)\, p_\gamma^\rho\, \epsilon_\gamma^\sigma \quad .$$

The summation over the photon and the vector meson polarizations is straightforward

$$\sum_{\text{pol. }\gamma} \sum_\lambda |a_\lambda(\pi^o\gamma)|^2 = \frac{e^2}{m_o^2}\, 2s\, K^2_{\pi^o\gamma}(s) = \frac{e^2}{2m_o^2}(s - m_o^2)^2 \quad .$$

The function $\gamma_{\pi^o\gamma}(s)$ is then given by

$$\gamma_{\pi^o\gamma}(s) = \frac{\alpha}{24}\, \frac{(s - m_o^2)^3}{m_o^2\, s} \tag{G.56}$$

and the expression of the decay width $V \Rightarrow \pi^o\gamma$ is simply deduced from (G.56) to be

$$\Gamma(V \Rightarrow \pi^o\gamma) = \frac{\alpha}{24}\, |f_{V\pi^o\gamma}|^2\, \frac{m_V^3}{m_o^2}\left(1 - \frac{m_o^2}{m_V^2}\right)^3 \quad . \tag{G.57}$$

We combine Eqs. (G.44) and (G.57) to obtain the total cross section for the process $e^+ + e^- \Rightarrow \pi^o + \gamma$ in the VMD model approximation

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \pi^o + \gamma) = \frac{2\pi^2\alpha^3}{3}\, \frac{1}{m_o^2}\left(1 - \frac{m_o^2}{s}\right)^3 \left|\sum_a m_a^2\, \frac{f_{a\pi^o\gamma}}{f_{\tilde{a}}}\, \frac{1}{W_a(s) - s}\right|^2 . \tag{G.58}$$

The comparison of Eqs. (G.58) and (A.23) gives the VMD model expression of the $\pi^o\gamma$ form factor

$$g\, G_{\pi^o\gamma}(s) = \sum_a m_a^2 \frac{f_{a\pi^o\gamma}}{f_{\tilde{a}}} \frac{1}{W_a(s) - s} . \qquad (G.59)$$

The previous sum $\sum_a$ is extended to the three vector mesons ρ, ω and φ. As a last remark, the results corresponding to the process $e^+ + e^- \Rightarrow \eta + \gamma$ are immediately obtained if m_o is the η mass and if the coupling constant $f_{a\pi^o\gamma}$ is replaced by $f_{a\eta\gamma}$.

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SECTION H: The ω-ρ Mixing

I. Experiments

1°) Let us begin with the Orsay experiment[1] on $e^+ + e^- \Rightarrow \pi^+ + \pi^+$ where six points in the energy range of the ω meson have been measured for the total cross section in addition to the seven points already quoted in Table 2 of Section B.

The theoretical formula needed to analyze the experimental data has been given in Eq. (G.47) of the previous section

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \pi^+ + \pi^-) = \frac{\pi\alpha^2}{3}\frac{1}{s}\left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} \left|\sum_a m_a^2 \frac{f_{a\pi\pi}}{f_{\tilde{a}}} \frac{1}{W_a(s) - s}\right|^2 \qquad (H.1)$$

where the sum $\sum_a$ in Eq. (H.1) is extended to the ρ-meson and ω-meson contributions.

The diagonal terms of the W(x) matrix can be reasonably approximated in this range of energy by

$$W_\rho(s) = m_\rho^2 - i\, m_\rho \Gamma_\rho \left[\frac{K_{\pi\pi}(s)}{K_{\pi\pi}(m_\rho^2)}\right]^3 \frac{m_\rho}{\sqrt{s}}$$

$$W_\omega(s) = m_\omega^2 - i\, m_\omega \Gamma_\omega \quad .$$

Each contribution depends on the mass m_a, on the width Γ_a and on the ratio of coupling constants $f_{a\pi\pi}/\tilde{f}_a$.

The ρ contribution has been represented by a π-meson electromagnetic form factor as suggested by Gounaris and Sakurai.[2] The equivalent formulation here is simply to correct the universality for the ρ coupling by a width dependent factor[3),4)]

$$\frac{f_{\rho\pi\pi}}{\tilde{f}_\rho} \simeq 1 + d\,\frac{\Gamma_\rho}{m_\rho} \qquad \text{with } d \simeq 0.48 \quad .$$

For the ω contribution, the mass m_ω and the width Γ_ω have been taken to their world average values and we have only an unknown complex parameter $f_{\omega\pi\pi}/\tilde{f}_\omega$. Its modulus, combined with the value of $|\tilde{f}_\omega|$ as deduced from a previous measurement on $e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^0$ will give the decay width for the mode $\omega \Rightarrow \pi^+ + \pi^-$ using formula (G.46). Its phase and more precisely the phase difference between $f_{\omega\pi\pi}/\tilde{f}_\omega$ and $f_{\rho\pi\pi}/\tilde{f}_\rho$ is called $\Phi_{2\pi}$.

We now consider the three following fits of the Orsay data:

(I) 7 points excluding the ω region
Fit with the ρ contribution: 2 free parameters m_ρ and Γ_ρ

(II) 13 points: assuming no ω-ρ interference
Fit with the ρ contribution: 2 free parameters

(III) 13 points: assuming no ω-ρ interference
Fit with the ρ and ω contributions: 4 free parameters

Fit	Degree of freedom	χ^2	m_ρ (MeV)	Γ_ρ (MeV)	$\Gamma^{\frac{1}{2}}(\omega \Rightarrow \pi^+\pi^-)$ (MeV$^{\frac{1}{2}}$)	$\Phi_{2\pi}$
I	5	8.6	772 ± 6	113 ± 8	-	-
II	11	18.29	768 ± 5	119.5 ± 3.3	0	-
III	9	8.66	773.5 ± 5.4	110.7 ± 5.3	0.63 ± 0.23	(196 ± 28)°

Table 1

2°) For comparison we quote now four series of results concerning the $\omega \Rightarrow \pi^+ + \pi^-$ mode

a) Compilation made by Walker et al[5)]

$$\frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\omega \Rightarrow 3\pi)} = \left(1.8 \begin{array}{l} +1.2 \\ -0.6 \end{array}\right) 10^{-2}$$

b) Compilation made by Lütjens and Steinberger[6)]

$$\frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\omega \Rightarrow 3\pi)} < 0.8 \times 10^{-2}$$

c) Experiment on $\pi^+ p \Rightarrow \pi^+ p \pi^+ \pi^-$ by Alff-Steinberger et al[7)]

$$\frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\omega \Rightarrow 3\pi)} \lesssim 2 \times 10^{-2}$$

d) Experiment on $K^- p \Rightarrow \Lambda \pi^+ \pi^-$ by Flatté et al[8)]

After subtraction of the dominant process $K^- p \Rightarrow Y_1^*(1385)\,\pi$, there remained 3887 events which were analyzed with two extreme assumptions

α)-complete cohrence in the ρ and ω production:

Fit made with $m_\rho = 750$ MeV, $\Gamma_\rho = 100$ MeV, $m_\omega = 782$ MeV, $\Gamma_\omega = 9$ MeV

$$\frac{\Gamma^{\frac{1}{2}}(\omega \Rightarrow 2\pi)}{\Gamma^{\frac{1}{2}}(\omega \Rightarrow 3\pi)} = 0.17 \pm 0.03 \quad .$$

β)-complete incoherence in the ρ and ω production:

Fit made with m_ρ free and found to be m_ρ = (740 ± 7)MeV

$$\frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\omega \Rightarrow 3\pi)} = (8.2 \pm 2)\ 10^{-2} \quad .$$

II. Model

1°) The natural order of magnitude expected for an electromagnetic amplitude $\omega \Rightarrow \pi^+ + \pi^-$ as compared with the amplitude $\rho \Rightarrow \pi^+ + \pi^-$ is obviously the fine structure constant α.

For instance, consider the one-photon exchange model for such a decay

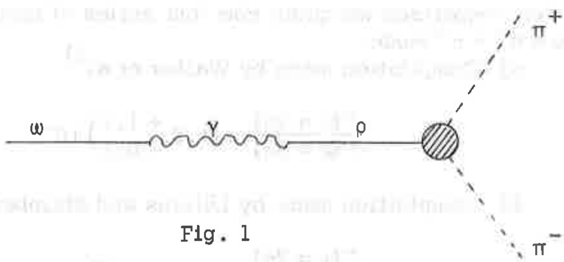

Fig. 1

A straightforward calculation gives

$$\left|\frac{f_{\omega 2\pi}}{f_{\rho 2\pi}}\right| \simeq 1.3\ \alpha \qquad \text{phase}\left(\frac{f_{\omega 2\pi}}{f_{\rho 2\pi}}\right) \simeq -75^\circ$$

leading to a decay width for $\omega \Rightarrow \pi^+ + \pi^-$ of

$$\Gamma(\omega \Rightarrow \pi^+ + \pi^-) \simeq 10\ \text{keV} \quad . \tag{H.2}$$

The Orsay measurement

$$\Gamma(\omega \Rightarrow \pi^+ + \pi^-) = \left(400 \begin{array}{l} +340 \\ -240 \end{array}\right)\ \text{keV} \tag{H.3}$$

looks considerably larger and perhaps the dominant effect is concentrated in the nondiagonal elements of the propagator matrix W_{12} and W_{21} or, equivalently, in the mixing parameters ϵ_1 and ϵ_2 which can be larger than α because of the small ω-ρ mass difference, a point already emphasized by Glashow.[9]

2°) As a first consequence of the model, the direct transitions

$$\omega_o \Rightarrow 2\pi \qquad\qquad \rho_o \Rightarrow 3\pi$$

can be neglected with respect to the ones induced by the mixing. Fortunately the experimental data like (H.3) are small enough to justify a first order calculation with respect to the mixing parameters ϵ_1 and ϵ_2. We make such an approximation from now

a) Final state $\pi^+\pi^-$ $\qquad f_{\omega_o\pi\pi} \simeq 0$

$$f_{\rho\pi\pi} \simeq p\, f_{\rho_o\pi\pi} \qquad\qquad f_{\omega\pi\pi} \simeq r\, f_{\rho_o\pi\pi} \, .$$

Therefore for the 2π decay mode of ρ and ω we have

$$\frac{\Gamma(\omega \Rightarrow \pi^+\pi^-)}{\Gamma(\rho \Rightarrow \pi^+\pi^-)} \simeq \left|\frac{r}{p}\right|^2 = |\epsilon_2|^2 \, . \tag{H.4}$$

From the Orsay result

$$|\epsilon_2| = 0.06 \pm 0.02 \tag{H.5}$$

b) Final state $\pi^+\pi^-\pi^o$ $\qquad f_{\rho_o 3\pi} \simeq 0$

$$f_{\rho 3\pi} \simeq -q\, f_{\omega_o 3\pi} \qquad\qquad f_{\omega 3\pi} \simeq s\, f_{\omega_o 3\pi} \, .$$

Therefore, for the 3π decay mode of ρ and ω we have

$$\frac{\Gamma(\rho \Rightarrow 3\pi)}{\Gamma(\omega \Rightarrow 3\pi)} \simeq \left|\frac{q}{s}\right|^2 = |\epsilon_1|^2 \, . \tag{H.6}$$

If time reversal invariance holds, $\epsilon_1 = \epsilon_2$ and we have the relation

$$\frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\rho \Rightarrow 2\pi)} = \frac{\Gamma(\rho \Rightarrow 3\pi)}{\Gamma(\omega \Rightarrow 3\pi)} \, . \tag{H.7}$$

3°) The unitarity constraints are written as (Eq. (G.26))

$$\left\| \begin{array}{ll} - \text{Im } W_a(s) = \sum_F |f_{AF}|^2 \gamma_F(s) & a = \rho, \omega \\ \frac{1}{2i}[\bar{W}_\omega(s) - W_\rho(s)] \langle \omega(s) | \rho(s) \rangle = \sum_F \bar{f}_{\omega F} f_{\rho F} \gamma_F(s) . & \end{array} \right.$$

The 2π and 3π contributions are easily calculated to be in the model

$$\text{for } 2\pi: \qquad p\bar{r} \, |f_{\rho_o 2\pi}|^2 \gamma_{2\pi}(s)$$

$$\text{for } 3\pi: \qquad -q\bar{s} \, |f_{\omega_o 3\pi}|^2 \gamma_{3\pi}(s)$$

What can we do for the other contributions ? Let us examine in some detail the $\pi^o\gamma$ contribution. The model cannot make predictions because both amplitudes $\rho_o \Rightarrow \pi^o\gamma$ and $\omega_o \Rightarrow \pi^o\gamma$ are of electromagnetic nature. We simply have

$$\bar{f}_{\omega\pi^o\gamma} f_{\rho\pi^o\gamma} = -q\bar{s}|f_{\omega_o\pi^o\gamma}|^2 + p\bar{r}|f_{\rho_o\pi^o\gamma}|^2 + p\bar{s} f_{\rho_o\pi^o\gamma} \bar{f}_{\omega_o\pi^o\gamma} - q\overline{r f_{\rho_o\pi^o\gamma}} f_{\omega_o\pi^o\gamma} . \tag{H.8}$$

Experimentally, the partial decay width $\Gamma(\omega \Rightarrow \pi^o\gamma)$ is of the order 1.2 MeV and for the partial decay width $\Gamma(\rho \Rightarrow \pi^o\gamma)$ we only know an upper limit of 0.4 MeV. It follows that the $\pi^o\gamma$ contributions to the unitarity relation are only small corrections and it is sufficient to retain only the first two terms in Eq. (H.8).

We then obtain

$$\frac{1}{2i}[\bar{W}_\omega - W_\rho](p\bar{r} - q\bar{s}) \simeq - p\bar{r} \, \text{Im } W_{\rho_o} + q\bar{s} \, \text{Im } W_{\omega_o} \tag{H.9}$$

and using the trace condition

$$W_\omega + W_\rho = W_{\omega_o} + W_{\rho_o}$$

Eq. (H.9) is equivalently written in the more convenient form

$$\frac{p\bar{r} - q\bar{s}}{p\bar{r} + q\bar{s}} = \frac{\bar{\epsilon}_2 - \epsilon_1}{\bar{\epsilon}_2 + \epsilon_1} = \frac{1}{i} \frac{\mathrm{Im}\, W_{\rho_o} - \mathrm{Im}\, W_{\omega_o}}{\mathrm{Re}\, W_{\omega} - \mathrm{Re}\, W_{\rho}} \quad . \tag{H.10}$$

The right-hand side of Eq. (H.10) being purely imaginary, we immediately deduce the relation

$$|\epsilon_1| = |\epsilon_2| \quad . \tag{H.11}$$

In particular the equality (H.7) holds in the model independently of the time reversal invariance.

On the other hand putting

$$\epsilon_1 = |\epsilon| \exp i\varphi_1 \qquad\qquad \epsilon_2 = |\epsilon| \exp i\varphi_2$$

the two phases φ_1 and φ_2 are related by Eq. (H.10)

$$\tan \frac{\varphi_1 + \varphi_2}{2} = \frac{\mathrm{Im}\, W_{\rho_o} - \mathrm{Im}\, W_{\omega_o}}{\mathrm{Re}\, W_{\omega} - \mathrm{Im}\, W_{\rho}} \quad .$$

At lowest order, $W_{\rho_o} = W_\rho$, $W_{\omega_o} = W_\omega$, (see Eqs. (G.12)) and we make the numerical calculation with

$$\left\| \begin{array}{l} W_\rho = m_\rho^2 - i\, m_\rho \Gamma_\rho(s) \\ \\ W_\omega = m_\omega^2 - i\, m_\omega \Gamma_\omega \end{array} \right. \quad .$$

The result is

$$\varphi_1 + \varphi_2 = (202 \pm 12)^\circ \tag{H.12}$$

where the error is essentially due to the uncertainty on the ρ-meson mass taken as (772 ± 6) MeV. The variation with the energy of the phase in the range $m_\rho \leq \sqrt{s} \leq m_\omega$ is less than 1°.

The additional prediction of time reversal invariance is simply

$$\varphi_1 = \varphi_2 = (101 \pm 6)^\circ \quad (\text{modulus } \pi) \; . \tag{H.13}$$

4°) Let us emphasize the close analogy between such a model for the ω-ρ interference and the superweak model for neutral K-meson decay. A correspondence can be made in the following way

$$\rho_o \leftrightarrow K_1^o \qquad \omega_o \leftrightarrow K_2^o$$

$$\rho \leftrightarrow K_S \qquad \omega \leftrightarrow K_L$$

$$\omega_o \not\to 2\pi \qquad K_2^o \not\to 2\pi$$

$$\rho_o \not\to 3\pi \qquad K_1^o \not\to 3\pi$$

III. Lepton-Antilepton Decay Mode

1°) The quantities measured in electron-positron annihilation experiments are the coupling constants $f_{\tilde{\rho}}$ and $f_{\tilde{\omega}}$. The problem is now to extract information about the decay coupling constants f_ρ and f_ω and the bare coupling constant f_{ρ_o} and f_{ω_o}.

2°) Decay coupling constants

Using Eqs. (G.5) and (G.7) we express the coupling constants $f_{\tilde{\rho}}$, $f_{\tilde{\omega}}$, f_ρ and f_ω in terms of the bare coupling constants f_{ρ_o} and f_{ω_o}

$$\frac{\bar{f}_{\tilde{\rho}}}{f_\rho} = (p\,s + q\,r)\,\frac{\dfrac{p}{f_{\rho_o}} - \dfrac{q}{f_{\omega_o}}}{\dfrac{s}{f_{\rho_o}} - \dfrac{r}{f_{\omega_o}}} \qquad \frac{\bar{f}_{\tilde{\omega}}}{f_\omega} = (p\,s + q\,r)\,\frac{\dfrac{p}{f_{\rho_o}} + \dfrac{s}{f_{\omega_o}}}{\dfrac{q}{f_{\rho_o}} + \dfrac{p}{f_{\omega_o}}}\,. \tag{H.14}$$

If time reversal invariance holds we have

$$r = q \qquad\qquad s = p$$

and the two coupling constants f_{ρ_o} and f_{ω_o} can be chosen both real. It follows from Eq. (H.14) the trivial relation

$$\frac{\bar{f}_{\tilde{\rho}}}{f_\rho} = \frac{\bar{f}_{\tilde{\omega}}}{f_\omega} = p^2 + q^2 = 1 + 0(\epsilon^2) \tag{H.15}$$

in terms of partial widths, time reversal invariance implies

$$\frac{\Gamma(\tilde{a} \Rightarrow e^+e^-)}{\Gamma(a \Rightarrow e^+e^-)} = \frac{|f_a|^2}{|f_{\tilde{a}}|^2}\;\frac{\langle\tilde{a}|\tilde{a}\rangle}{\langle a|a\rangle} = 1 \qquad a = \rho,\omega\,. \tag{H.16}$$

If time-reversal invariance is not valid, we have for the ratio of the coupling constants $\bar{f}_{\tilde{a}}/f_a$ a first order correction in ϵ proportional to a time-reversal violation parameter.

3°) Bare coupling constants

The relations between the decay coupling constants f_ρ, f_ω and the bare coupling constants f_{ρ_o}, f_{ω_o} are given by Eqs. (G.5)

$$\frac{f_{\rho_o}}{f_\rho} = p - q\,\frac{f_{\rho_o}}{f_{\omega_o}} \qquad \frac{f_{\omega_o}}{f_\omega} = s + r\,\frac{f_{\omega_o}}{f_{\rho_o}} \quad . \qquad \text{(H.17)}$$

We define the deviation from unity of these ratios in the following way

$$\frac{f_{\rho_o}}{f_\rho} = (1 + \delta_\rho)\, e^{i\varphi_\rho} \qquad \frac{f_{\omega_o}}{f_\omega} = (1 + \delta_\omega)\, e^{i\varphi_\omega} \quad . \qquad \text{(H.18)}$$

In the lowest order with respect to the mixing parameters, we simply have

$$\delta_\rho \simeq -\operatorname{Re}\epsilon_1 \frac{f_{\rho_o}}{f_{\omega_o}} \qquad \delta_\omega \simeq \operatorname{Re}\epsilon_2 \frac{f_{\omega_o}}{f_{\rho_o}}$$

$$\varphi_\rho \simeq -\operatorname{Im}\epsilon_1 \frac{f_{\rho_o}}{f_{\omega_o}} \qquad \varphi_\omega \simeq \operatorname{Im}\epsilon_2 \frac{f_{\omega_o}}{f_{\rho_o}} \quad . \qquad \text{(H.19)}$$

The mixing effects are always very small for the ρ meson but they are an order of magnitude larger for the ω meson.

If time-reversal invariance holds, $\epsilon_1 = \epsilon_2 = |\epsilon| \exp i\varphi$ and we can always define the ρ_o and ω_o states so that f_{ρ_o} and f_{ω_o} are both real and positive. From the experimental data we have

$$\frac{f_{\rho_o}}{f_{\omega_o}} = 0.36 \pm 0.04 \qquad \frac{f_{\omega_o}}{f_{\rho_o}} = 2.79 \pm 0.33 \quad .$$

Let us now make a numerical estimate using $|\epsilon| \simeq 0.06$ and φ as predicted by the model of Part II: $\varphi = 101°$ or $-79°$. The results of computations including second order terms in $|\epsilon|$ are given in Table 2

	$\varphi = 101^\circ$	$\varphi = -79^\circ$
δ_ρ	+0.003	-0.006
φ_ρ	-1.2°	$+1.2^\circ$
δ_ω	-0.022	+0.045
φ_ω	$+9.7^\circ$	-9°

Table 2

For the radiative widths we simply have

$$\frac{\Gamma(\rho_o \Rightarrow e^+e^-)}{\Gamma(\rho \Rightarrow e^+e^-)} = 1 - 2\delta_\rho \qquad \frac{\Gamma(\omega_o \Rightarrow e^+e^-)}{\Gamma(\omega \Rightarrow e^+e^-)} = 1 - 2\delta_\omega \quad .$$

The corrections due to mixing are far below the experimental uncertainties.

IV. Hadronic Decay Modes

1°) An interference effect between the ρ-meson and the ω-meson contributions has been observed in the process $e^+ + e^- \Rightarrow \pi^+ + \pi^-$. The experiment has been reported in Part I and we only comment about the interpretation of the phase $\Phi_{2\pi}$ experimentally measured and defined by

$$\Phi_{2\pi} = \text{phase}\left(\frac{f_{\omega 2\pi}}{f_{\rho 2\pi}} \; \frac{\bar{f}_{\tilde{\rho}}}{\bar{f}_{\tilde{\omega}}}\right) \quad .$$

Using time-reversal invariance and the notations of Part III we have

$$\Phi_{2\pi} = \varphi_\omega - \varphi_\rho + \text{phase}\left(\frac{f_{\omega 2\pi}}{f_{\rho 2\pi}}\right) \quad .$$

In the framework of the model proposed in Part II we simply have $f_{\omega 2\pi}/f_{\rho 2\pi} = \varepsilon$ so that

$$\Phi_{2\pi} = \varphi_\omega - \varphi_\rho + \varphi \quad . \tag{H.20}$$

Using now $|\epsilon| = 0.06$ to estimate φ_ω and φ_ρ the theoretical predictions are

$$\Phi_{2\pi} = 112^\circ \quad \text{or} \quad \Phi_{2\pi} = -89^\circ$$

with errors of order 6° to 10° because of the uncertainties on the ρ mass. The value found in the Orsay experiment $\Phi_{2\pi}^{\exp} = (-164 \pm 28)^\circ$ disagrees with both predictions. Nevertheless, we think that experiment can accommodate the theoretical value $\Phi_{2\pi} = -89^\circ$ (associated to $\varphi = -79^\circ$).

2°) It will be very interesting to detect the interference between the ρ-meson and the ω-meson contributions in the process $e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^0$. Crudely speaking the roles of the ω and the ρ mesons are exchanged and the magnitude of the interference effects into

$$e^+ + e^- \Rightarrow \pi^+ + \pi^- \qquad \text{and} \qquad e^+ + e^- \Rightarrow \pi^+ + \pi^- + \pi^0$$

are related by a factor of the order

$$\left(\frac{f_{\omega_0}}{f_{\rho_0}}\right)^2 \left(\frac{\Gamma_\omega}{\Gamma_\rho}\right)^2 \simeq 0.11 \quad .$$

Therefore the measurement of the ω-ρ interference in the 3π case will be an order of magnitude harder than in the 2π case assuming comparable statistics.

Let us recall the predictions of the model. First for the decay rates (Eq. (H.7))

$$\frac{\Gamma(\rho \Rightarrow 3\pi)}{\Gamma(\omega \Rightarrow 3\pi)} = \frac{\Gamma(\omega \Rightarrow 2\pi)}{\Gamma(\rho \Rightarrow 2\pi)} \quad .$$

Secondly for the phases (Eq. (H.12))

$$\Phi_{2\pi} + \Phi_{3\pi} = (22 \pm 12)^\circ \tag{H.21}$$

where the phase $\Phi_{3\pi}$ is defined as

$$\Phi_{3\pi} = \text{phase}\left(\frac{f_{\rho 3\pi}}{f_{\omega 3\pi}} \; \frac{\tilde{f}_\omega}{\tilde{f}_\rho}\right) .$$

Both relations are independent of the validity of time-reversal invariance.

3°) We now assume to be measured, in magnitude and phase, the two electromagnetic coupling constants $f_{\omega 2\pi}$ and $f_{\rho 3\pi}$. A straightforward application of Eqs. (G.5) using time-reversal invariance gives

$$\frac{f_{\omega 2\pi}}{f_{\rho_o 2\pi}} = q + p\,\frac{f_{\omega_o 2\pi}}{f_{\rho_o 2\pi}} \qquad \frac{f_{\rho 3\pi}}{f_{\omega_o 3\pi}} = -q + p\,\frac{f_{\rho_o 3\pi}}{f_{\omega_o 3\pi}} \; .$$

Eliminating q, we obtain

$$\frac{f_{\omega 2\pi}}{f_{\rho_o 2\pi}} + \frac{f_{\rho 3\pi}}{f_{\omega_o 3\pi}} = \left(\frac{f_{\omega_o 2\pi}}{f_{\rho_o 2\pi}} + \frac{f_{\rho_o 3\pi}}{f_{\omega_o 3\pi}}\right) p \; . \qquad \text{(H.22)}$$

In the model of Part II, the right-hand side has been neglected. A priori it must be of order α, e.g. small with respect to the mixing parameter ϵ. Experiment will test such an assumption via Eq. (H.22).

V. The $\pi^o\gamma$ Decay Mode

1°) At lowest order in ϵ and assuming time-reversal invariance, the decay coupling constants $f_{\rho\pi^o\gamma}$ and $f_{\omega\pi^o\gamma}$ are related to the bare coupling constant by

$$\left\| \begin{aligned} f_{\rho\pi^o\gamma} &= f_{\rho_o\pi^o\gamma} - \epsilon\, f_{\omega_o\pi^o\gamma} \\ f_{\omega\pi^o\gamma} &= f_{\omega_o\pi^o\gamma} + \epsilon\, f_{\rho_o\pi^o\gamma} \end{aligned} \right. \; . \qquad \text{(H.23)}$$

The two transitions $\rho \Rightarrow \pi^o + \gamma$ and $\omega \Rightarrow \pi^o + \gamma$ are both of electromagnetic nature and they can be, a priori, of the same order of magnitude.

Experimentally the radiative decay mode $\omega \Rightarrow \pi^o + \gamma$ has been measured and the result is[10]

$$\Gamma(\omega \Rightarrow \pi^o + \gamma) = (1.16 \pm 0.20)\ \text{MeV} \; .$$

Using the relation (G.57) between the coupling constant $f_{\omega\pi^o\gamma}$ and the partial width $\Gamma(\omega \Rightarrow \pi^o\gamma)$ we find

$$|f_{\omega\pi^0\gamma}| = 0.40 \pm 0.04 \quad . \tag{H.24}$$

For the radiative decay mode $\rho \Rightarrow \pi^0 + \gamma$ we only have an upper bound[11)]

$$\Gamma(\rho \Rightarrow \pi^0 + \gamma) < 0.4 \text{ MeV} \quad .$$

We then deduce an inequality

$$\left| \frac{f_{\rho\pi^0\gamma}}{f_{\omega\pi^0\gamma}} \right| < 0.6 \quad .$$

2°) We have proposed, with Cremmer,[12)] a model where $f_{\rho\pi^0\gamma}$ turns out to be small. This model is based on an extrapolation at zero energy s = 0 of the VMD model approximation of the electromagnetic form factor $G_{\pi^0\gamma}(s)$ computed in Eq. (G.59)

$$g\, G_{\pi^0\gamma}(s) = \sum_a m_a^2 \frac{f_{a\pi^0\gamma}}{f_{\tilde{a}}} \frac{1}{W_a(s) - s} \quad .$$

Time-reversal invariance being assumed, we obtain the two basic relations

$$\tfrac{1}{2}\, g = \frac{f_{\rho\pi^0\gamma}}{f_{\rho\pi\pi}} \qquad \text{for the isovector part} \tag{H.25}$$

$$\tfrac{1}{2}\, g = \frac{f_{\omega\pi^0\gamma}}{f_\omega} + \frac{f_{\varphi\pi^0\gamma}}{f_\varphi} \quad \text{for the isoscalar part} \; . \tag{H.26}$$

The constant g characterizes the $\pi^0 \Rightarrow 2\gamma$ decay (see Eq. (A.24))

$$\Gamma(\pi^0 \Rightarrow 2\gamma) = \frac{\pi\alpha^2}{4} m_0 |g|^2$$

and using now the experimental result

$$\Gamma(\pi^0 \Rightarrow 2\gamma) = (7.2 \pm 1.2) \text{ eV}$$

we obtain

$$\tfrac{1}{2}|g| = (1.79 \pm 0.15)\, 10^{-2} \; .$$

3°) Let us first study the isovector transition. From the ρ-meson width as measured in the Orsay experiment, $\Gamma_\rho = (111 \pm 6)$ MeV we deduce the value of the coupling constant $f_{\rho\pi\pi}$ (Eq. (G.46))

$$\frac{f^2_{\rho\pi\pi}}{4\pi} = 2.13 \pm 0.13 \qquad f_{\rho\pi\pi} = 5.17 \pm 0.15 \quad .$$

From Eq. (H.25) we compute the coupling constant $f_{\rho\pi^o\gamma}$

$$|f_{\rho\pi^o\gamma}| = (9.25 \pm 0.85)\, 10^{-2} \quad .$$

The corresponding radiative decay width $\Gamma(\rho \Rightarrow \pi^o\gamma)$ is then predicted to be (Eq. (G.57))

$$\Gamma(\rho \Rightarrow \pi^o\gamma) = (59 \pm 10) \text{ keV} \tag{H.28}$$

and the ratio of the two radiative coupling constants is

$$\left| \frac{f_{\rho\pi^o\gamma}}{f_{\omega\pi^o\gamma}} \right| = 0.23 \pm 0.03 \quad . \tag{H.29}$$

4°) The photon-vector meson coupling constants are computed from the Orsay experiments and we have

$$|f_\omega| = 13.63 \pm 1.30 \qquad\qquad |f_\varphi| = 11.8 \pm 0.9 \quad . \tag{H.30}$$

Combining now the two experimental results (H.24) and (H.30) we obtain

$$\left| \frac{f_{\omega\pi^o\gamma}}{f_\omega} \right| = (2.92 \pm 0.42)\, 10^{-2} \quad .$$

Such a value has to be compared with the experimental value of $\frac{1}{2}|g|$ as given in Eq. (H.27). In order to satisfy the condition (H.26) we need a small φ-meson contribution in the $\pi^o \Rightarrow 2\gamma$ decay amplitude. Of course, the ω-meson contribution is the dominant one but the smallness of the $\pi^o \Rightarrow 2\gamma$ width is due, in this model, to a partial cancellation between the ω-meson and the φ-meson contributions of opposite signs. As an estimate for the φ contribution we find

$$\left| \frac{f_{\omega\pi^o\gamma}}{f_\varphi} \right| \simeq (1.1 \pm 0.6)\, 10^{-2} \quad .$$

We then compute a range of possible values for the radiative decay width of the φ meson

$$\Gamma(\varphi \Rightarrow \pi^{o} + \gamma) \simeq (50 - 700)\ \text{keV} .$$

The most recent experiment made in DESY[13] gives the upper limit

$$\Gamma(\varphi \Rightarrow \pi^{o}\gamma) < 18\ \text{keV}$$

but more experimental information is needed to confirm such a value.

5°) Let us go back to the ω-ρ interference problem. If the order of magnitude (H.29) obtained for the ratio $f_{\rho\pi^o\gamma}/f_{\omega\pi^o\gamma}$ is correct we can replace Eqs. (H.23) by

$$\left\| \begin{array}{l} f_{\rho\pi^o\gamma} = f_{\rho_o\pi^o\gamma} - \varepsilon\, f_{\omega_o\pi^o\gamma} \\ f_{\omega\pi^o\gamma} \simeq f_{\omega_o\pi^o\gamma} \end{array} \right.$$

and the contribution due to ε in $f_{\rho\pi^o\gamma}$ remains a correction and cannot explain, by itself, the complete $\rho \Rightarrow \pi^o + \gamma$ transition. We then have, for the process $e^+ + e^- \Rightarrow \pi^o + \gamma$, in the ω-ρ region a dominant contribution due to the ω-meson amplitude and a ω-ρ interference which looks to be constructive from Eqs. (H.25) and (H.26) and which is dominated by the term $f_{\rho_o\pi^o\gamma}\, f_{\omega_o\pi^o\gamma}$.

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SECTION I: Vector Meson Dominance Model and Spectral Representations

I. Sum Rules

1°) The spectral representation of the Fourier transform of the vacuum expectation value of the time ordered product of two components of a conserved current $J^{\alpha}_{\mu}(x)$ has been obtained in Eq. (C.21) of Section C

$$\Delta^{\alpha}_{\mu\nu}(k) = g_{\mu\nu}\int_0^{\infty}\frac{\rho^{\alpha\alpha}(m^2)}{k^2+m^2-i\varepsilon}\,dm^2 + k_{\mu}k_{\nu}\int_0^{\infty}\frac{\rho^{\alpha\alpha}(m^2)}{m^2(k^2+m^2-i\varepsilon)}\,dm^2$$

$$+\, g_{\mu o}\, g_{\nu o}\int_0^{\infty}\frac{\rho^{\alpha\alpha}(m^2)}{m^2}\,dm^2 \quad . \qquad (I.1)$$

We consider more specifically the electromagnetic current J^{em}_{μ} and its isovector and isoscalar parts

$$J^{em}_{\mu} = J^{3}_{\mu} + \frac{1}{\sqrt{3}}\, J^{8}_{\mu}$$

and we want to derive consequences of an asymptotic SU(3) symmetry.

2°) The assumption made by Das, Mathur and Okubo[1] is the following: the SU(3) symmetry becomes exact for the distribution $\Delta^{\alpha}_{\mu\nu}(k)$ in the limit $k_{\mu} \Rightarrow \infty$. We then derive

a) a convergence relation

$$\lim_{k_{\mu}\Rightarrow\infty}\left[\Delta^{3}_{\mu\nu}(k) - \Delta^{8}_{\mu\nu}(k)\right] = 0 \quad .$$

The spectral function integral associated to the $k_{\mu}k_{\nu}$ term in Eq. (I.1) must vanish and we obtain a first sum rule

$$\int_0^{\infty}\frac{[\rho^{33}(m^2) - \rho^{88}(m^2)]}{m^2}\,dm^2 = 0 \qquad W_1$$

which is a first Weinberg type sum rule[2] one can also interpret as the equality of two Schwinger terms (see Eq. (C.17))

$$C^3 - C^8 = 0 \qquad \text{(I.2)}$$

b) a superconvergence relation.

Moreover, we can also impose the vanishing of the spectral function integral associated to the $g_{\mu\nu}$ term in Eq. (I.1) and we obtain a second second sum rule

$$\int_0^\infty [\rho^{33}(m^2) - \rho^{88}(m^2)]\, dm^2 = 0 \qquad W_2$$

which is a second Weinberg type sum rule[2] requiring to be convergent a more rapid decreasing of the spectral function at high energy than in the previous case. Therefore, W_2 is highly questionable.

3°) A different form of the SU(3) asymptotic symmetry can be assumed fixing k^2 to be zero.[3] We then deduce

a) a convergence relation

$$\lim_{k_\mu \Rightarrow \infty} [\Delta^3_{\mu\nu}(k) - \Delta^8_{\mu\nu}(k)]_{k^2=0} = 0 \ .$$

The spectral function integral associated to the $k_\mu k_\nu$ term in Eq. (I.1) written at $k^2 = 0$ must vanish and we obtain a new sum rule

$$\int_0^\infty \frac{[\rho^{33}(m^2) - \rho^{88}(m^2)]}{m^4}\, dm^2 = 0 \qquad W_0$$

which can be interpreted as the equality of the isoscalar and isovector hadronic contributions to the charge renormalization.[4]

b) a superconvergence relation.

We obviously obtain the sum rule W_1 .

II. Vector Meson Dominance Model

1°) The spectral functions ρ^{em}, ρ^{33} and ρ^{88} have been related in Section C to total cross sections for electron-positron annihilation into hadrons (Eqs. (C.23) and (C.26))

$$\rho^{em}(s) = \frac{s^2}{16\pi^3\alpha^2}\ \sigma_{tot}(e^+ + e^- \Rightarrow \text{hadrons})$$

$$\rho^{33}(s) = \frac{s^2}{16\pi^3\alpha^2}\ \sigma_{tot}(e^+ + e^- \Rightarrow I = 1)$$

$$\rho^{88}(s) = \frac{3s^2}{16\pi^3\alpha^2}\ \sigma_{tot}(e^+ + e^- \Rightarrow I = 0)\ . \qquad (I.3)$$

2°) The total cross section for the process $e^+ + e^- \Rightarrow F$ is written in the vector meson dominance model as (Eq. (G.44))

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow F) = \frac{(4\pi\alpha)^2}{s^2}\gamma_F(s)\left|\sum_a m_a^2 \frac{f_{aF}}{f_{\tilde{a}}} \frac{1}{W_a(s) - s}\right|^2 . \qquad (I.4)$$

In this section we neglect all the interferences between vector meson contributions ($\rho^{38} = \rho^{83} = 0$) and Eq. (I.4) is replaced by

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow F) \simeq \frac{(4\pi\alpha)^2}{s^2}\gamma_F(s) \sum_a \left| m_a^2 \frac{f_{aF}}{f_{\tilde{a}}} \frac{1}{W_a(s) - s}\right|^2 . \qquad (I.5)$$

We sum over all possible final states F

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \text{hadrons}) \simeq \frac{12\pi}{s} \sum_a \sigma_a^{e^+e^-}(s)\, \sigma_a(s) \frac{1}{|W_a(s) - s|^2} \qquad (I.6)$$

where the functions $\sigma_a^F(s)$

$$\sigma_a^F(s) = |f_{aF}|^2\ \gamma_F(s) \qquad\qquad \sigma_a(s) = \sum_F \sigma_a^F(s)$$

have been previously introduced in Sec. G.

3°) The total cross sections play the role of spectral functions in the integral representations we are considering. It is then convenient, in a first calculation, to make a narrow width approximation.

The basic formula will be a definition of the Dirac distribution as a limit of a sequence of functions

$$\lim_{\varepsilon \Rightarrow 0^+} \frac{\varepsilon}{|X - i\varepsilon|^2} = \pi\, \delta(X) \quad . \qquad \text{(I.7)}$$

Substituting

$$X = m_a^2 - s \qquad\qquad \varepsilon = m_a \Gamma_a = \sigma_a (m_a^2)$$

we obtain

$$\sigma_{tot}(e^+ + e^- \Rightarrow V \Rightarrow \text{hadrons}) \simeq 12\pi^2 \sum_a \frac{\Gamma(a \Rightarrow e^+e^-)}{m_a} \delta(s - m_a^2) \ . \qquad \text{(I.8)}$$

4°) In the narrow-width approximation of the vector meson dominance model, the sum rules W_0 , W_1 and W_2 become[1),4)]

$$m_\rho^{-1}\Gamma(\rho \Rightarrow e^+e^-) - 3[m_\omega^{-1}\Gamma(\omega \Rightarrow e^+e^-) + m_\varphi^{-1}\Gamma(\varphi \Rightarrow e^+e^-)] = 0 \qquad V_0$$

$$m_\rho\, \Gamma(\rho \Rightarrow e^+e^-) - 3[m_\omega\, \Gamma(\omega \Rightarrow e^+e^-) + m_\varphi\, \Gamma(\varphi \Rightarrow e^+e^-)] = 0 \qquad V_1$$

$$m_\rho^3\, \Gamma(\rho \Rightarrow e^+e^-) - 3[m_\omega^3\, \Gamma(\omega \Rightarrow e^+e^-) + m_\varphi^3\, \Gamma(\varphi \Rightarrow e^+e^-)] = 0 \qquad V_2$$

Inserting the experimental data from Orsay we obtain

$$+ 0.51 \pm 0.75 = 0 \qquad V_0$$

$$- 0.75 \pm 0.80 = 0 \qquad V_1$$

$$- 1.58 \pm 0.70 = 0 \qquad V_2$$

5°) From the mass inequalities between vector mesons

$$m_\rho < m_\omega < m_\varphi$$

it follows that the three sum rules V_0 , V_1 , V_2 are not compatible. If one of them turns out to be exact, we must introduce in the two other ones a correction factor associated to a particular breaking of the SU(3) symmetry.

For instance, consider V_0 to be correct. Such a correction factor has been proposed by Sugawara[5)] who replaces the Das-Mathur-Okubo sum rule V_1 by

$$m_\rho \Gamma(\rho \Rightarrow e^+e^-) - 3[m_\omega \Gamma(\omega \Rightarrow e^+e^-) + m_\varphi \Gamma(\varphi \Rightarrow e^+e^-)] \frac{3m_\rho^2}{4m_{K^*}^2 - m_\rho^2} = 0 \qquad S$$

With the Orsay data we obtain

$$+0.28 \pm 0.65 = 0 \qquad \underline{S}$$

Let us also remark that if the electromagnetic current is only a U spin scalar and contains a component belonging to the scalar representation of SU(3), the sum rules V_0, V_1, V_2 become inequalities only

$$(\rho) - 3[(\omega) + (\varphi)] \leq 0 \ .$$

6°) For the ρ-meson contribution the narrow-width approximation must be corrected and we proceed in the following way. We assume the total cross section $\sigma_{tot}(e^+ + e^- \Rightarrow \Gamma = 1)$ to be dominated by the $\pi^+\pi^-$ contribution given by

$$\sigma_{tot}(e^+ + e^- \Rightarrow \pi^+ + \pi^-) = \frac{\pi \alpha^2}{3} \frac{1}{s} \left(1 - \frac{4m_\pi^2}{s}\right)^{3/2} |F_\pi(s)|^2 \qquad (I.9)$$

and for $F_\pi(s)$ we take the model proposed by Gounaris and Sakurai[6) (see Eq. (B,82)). We have used such a form in the entire range of integration whereas it has been tested only in the ρ-meson region. We must keep in mind that such an approximation is very doubtful especially in a calculation where the high-energy region plays an important role (Schwinger terms).

III. Schwinger Terms

1°) The hadronic contributions to the Schwinger term associated to the electromagnetic current are given, in the narrow-width approximation of the VMD model by

$$C^{em}(\text{hadrons}) = \frac{3}{4\pi \alpha^2} \sum_a m_a \Gamma(a \Rightarrow e^+e^-) = \sum_a \frac{m_a^2}{|f_a|^2} \ . \qquad (I.10)$$

2°) The ρ-meson contribution to Eq. (I.10) is identified with the isovector Schwinger term C^3. From experiment

$$C^3 = (2.49 \pm 0.32) 10^{-2} \text{ GeV}^2 \qquad (I.11)$$

A more sophisticated calculation of C^3, using a numerical integration gives a -7% effect

$$C^3 = (2.31 \pm 0.17)\, 10^{-2}\ \mathrm{GeV}^2 . \qquad \text{(I.12)}$$

3°) The isoscalar part of C^{em} is the sum of the ω and φ contributions. From experiment

$$\omega \text{ contribution} = (0.33 \pm 0.06)\, 10^{-2}\ \mathrm{GeV}^2$$

$$\varphi \text{ contribution} = (0.75 \pm 0.10)\, 10^{-2}\ \mathrm{GeV}^2 .$$

The Schwinger term C^8 is three times the isoscalar contribution so that

$$C^8 = (3.24 \pm 0.48)\, 10^{-2}\ \mathrm{GeV}^2 . \qquad \text{(I.13)}$$

4°) The electromagnetic Schwinger term C^{em}(hadrons) is given, in that model, by the sum of the three vector meson contributions. We obtain

$$C^{em}(\text{hadrons}) = (3.57 \pm 0.48)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.11)}$$

$$C^{em}(\text{hadrons}) = (3.39 \pm 0.39)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.12)}$$

5°) The Das-Mathur-Okubo sum rule V_1 gives

$$C^3 - C^8 = (-0.75 \pm 0.80)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.11)}$$

$$C^3 - C^8 = (-0.93 \pm 0.75)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.12)}$$

6°) The Sugawara sum rule S gives

$$C^3 - \frac{3m_\rho^2}{4m_{K^*}^2 - m_\rho^2}\, C^8 = (+0.28 \pm 0.65)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.11)}$$

$$C^8 - \frac{3m_\rho^2}{4m_{K^*}^2 - m_\rho^2}\, C^8 = (+0.10 \pm 0.50)\, 10^{-2}\ \mathrm{GeV}^2 \quad \text{with} \quad \text{(I.12)}$$

IV. Charge Renormalization

1°) The hadronic contributions to the charge renormalization are given in the narrow-width approximation of the VMD model combining Eqs. (D.9) and (I.8)

$$\frac{\delta e_o^2}{e^2} = \frac{3}{\alpha} \sum_a \frac{\Gamma(a \Rightarrow e^+e^-)}{m_a} = \alpha \sum_a \frac{4\pi}{|f_a|^2} \quad . \qquad \text{(I.14)}$$

2°) The ρ, ω and φ contributions are computed using the experimental Orsay data

ρ contribution $(0.526 \pm 0.062)\,\alpha$

ω contribution $(0.067 \pm 0.012)\,\alpha$

φ contribution $(0.091 \pm 0.013)\,\alpha$

so that we find

$$\frac{\delta\, e_o^2}{e^2} = (3.65 \pm 0.52)\; 10^{-3} \quad . \qquad \text{(I.15)}$$

3°) For the sum rule V_0, we obtain

$$\frac{4\pi}{|f_\rho|^2} - 3\left[\frac{4\pi}{|f_\omega|^2} + \frac{4\pi}{|f_\varphi|^2}\right] = 0.051 \pm 0.075 \quad .$$

V. Muon Anomalous Magnetic Moment [7)]

1°) The hadronic contributions to the muon anomalous magnetic moment are given, in the narrow width approximation of the VMD model combining Eq. (D.10) and (I.8)

$$a_\mu(\text{hadrons}) = \frac{3}{\alpha} \sum_a \frac{\Gamma(a \Rightarrow e^+e^-)}{m_a} K_\mu^{(2)}(m_a^2) = \alpha \sum_a \frac{4\pi}{|f_a|^2} K_\mu^{(2)}(m_a^2) \,. \qquad \text{(I.16)}$$

2°) For the ρ-meson contribution, experiments give

$$a_\mu(\rho) = (5.0 \pm 0.3)\; 10^{-8} \quad . \qquad \text{(I.17)}$$

We also have evaluated the integral (D.10) using the $\pi^+\pi^-$ cross section as given in Eq. (I.9). The numerical integration leads to the result

$$a_\mu(\pi^+\pi^-) = (5.4 \pm 0.3)\, 10^{-8} \qquad \text{(I.18)}$$

which is 8% larger than the narrow-width estimate (I.17).

3°) For the isoscalar contributions we have

$$a_\mu(\omega) = (0.61 \pm 0.11)\, 10^{-8}$$

$$a_\mu(\varphi) = (0.50 \pm 0.07)\, 10^{-8} \ .$$

4°) Combining now isoscalar and isovector contributions we obtain

$$a_\mu(\text{hadrons}) = (6.5 \pm 0.5)\, 10^{-8} \ .$$

The theoretical prediction including 2nd, 4th, 6th order calculations is now given by[8]

$$a_\mu = (116587 \pm 2)\, 10^{-8} \ ,$$

the last experimental value is[9]

$$a_\mu = (116616 \pm 31)\, 10^{-8} \ .$$

Therefore

$$a_\mu(\text{th}) - a_\mu(\text{exp}) = (-29 \pm 34)\, 10^{-8} \ .$$

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SPECTRAL FUNCTION SUM RULES
FROM IDENTITIES OF THE JACOBI TYPE†‡

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I. Introduction

During the past two years there has been much interest in applications, extensions and derivations of the Weinberg[1)] spectral function sum rules.[2)-8)] It has been noted[2)] soon after Weinberg's original derivation of the SU(2) × SU(2) sum rules, that the Jacobi identity may be used to derive the first sum rule for any Lie algebra from the assumption that the Schwinger terms in the commutators ($x_0 = y_0$) $[J_0^{\alpha}(x), J_k^{\beta}(y)]$ are c numbers. Here, $J_\mu^{\alpha}(x)$ denotes the currents of the Lie algebra considered. Jackiw[3)] has also used the Jacobi identity to derive the second Weinberg sum rule for the SU(2) ⊗ SU(2) currents.

Among the extensions of the Weinberg sum rules, Rothleitner[4)] has derived a sum rule for baryon spectral functions. The main assumption of Ref. 4 concerns the commutator of the time component of the axial current $A_0^{a}(x)$ (a = 1,2,3) and the nucleon field $\bar{\psi}(y)$ at equal times. The assumed[9)] commutator reads

$$\left[A_0^{a}(x), \bar{\psi}(y)\right] = -r_A \bar{\psi}(x)\gamma_5\tau^{a}\delta(\underset{\sim}{x}-\underset{\sim}{y}) + (\text{possible } (\Delta I=\tfrac{3}{2})\text{-terms}). \tag{1}$$

The absolute value of the constant r_A may be determined from current algebra.[13)] From this commutator, using the techniques of Ref. 1, Rothleitner derived a sum rule for baryon spectral functions. The same sum rule (in the approximation of one particle intermediate states) has been derived by M. Sugawara[12)] from his set of self consistency conditions, derived in Refs. 12 from the $\underset{\sim}{x}$-integrated Eq. (1), and additional assumptions. These conditions agree with experiment and thus support Eq. (1).

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In the present note we first present (for $SU(2)\otimes SU(2)$) a simplified version of the proof of the first Weinberg sum rule, described in Ref. 2. Assuming charge-current commutators we will see that the first Weinberg sum rule holds if and only if either ($x_o = y_o = z_o$)

$$\left\langle \left[Q_5{}^a(x_o),\, [A_k{}^b(y),\, V_o{}^c(z)] \right] \right\rangle_o = 0 \tag{2}$$

or

$$\left\langle \left[Q_5{}^a(x_o),\, [V_k{}^b(y),\, A_o{}^c(z)] \right] \right\rangle_o = 0\ . \tag{3}$$

It turns out that these expressions are equal and proportional to $\partial_k \delta(\underline{y}-\underline{z})$, i.e. only the Schwinger term in $\left[J_k{}^b(y),\, J_o^{\prime c}(z) \right]$ could possibly contribute.

Next we assume in addition that the divergence of the axial vector current commutes at equal times with the space components of the currents. From this we derive conditions equivalent to the second Weinberg sum rule. One of these conditions reads

$$\left\langle \left[Q_5{}^a(x_o), \left[\int d^3y A_k{}^b(y), \frac{\partial}{\partial x^o} V_i^c(x) - \frac{\partial}{\partial x^i} V_o{}^c(x) \right] \right] \right\rangle_o = 0 \tag{4}$$

and is equivalent to one of the assumptions made in Ref. 3 to derive the second sum rule. In Ref. 3, the consequences of current conservation which we need here are also assumed. In our treatment, however, we need not make the additional assumptions of this reference. We then do not gain the additional information on the commutator

$$\left[\dot{J}_i{}^c(x) - \frac{\partial}{\partial x^i} J_o{}^c(x),\, J_k^{\prime k}(y) \right]$$

obtained by Jackiw. The main result--that Eq. (4) implies the second sum rule--remains however also under our weaker assumptions.

These considerations serve as an introduction and illustration to the main purpose of the present talk: to present a condition[8] for the Rothleitner-Sugawara sum rule derived from the following identity of the Jacobi type

$$\left[[a,b],c \right]_+ + \left[[b,c]_+, a \right] = \left[[c,a],b \right]_+ . \tag{5}$$

To this purpose we will have to assume that the divergence of the axial current and the nucleon field commute at equal times[11] and that

$$[Q_5{}^a(y_o),\ \bar\psi(y)] = -r_A\bar\psi(y)\gamma_5\tau^a$$
$$+ \text{possible } (\Delta I = \tfrac{3}{2})\text{-terms}. \qquad (6)$$

Our assumption, Eq. (6), is more general than the assumption of Eq. (1) and allows for arbitrary Schwinger terms.[16] This way we obtain that the Rothleitner-Sugawara sum rule is equivalent to $(x_o = y_o = z_o)$

$$\left\langle\left[Q_5{}^a(x_o),\ \left[\int d^3y\ \dot\psi(y),\ \bar\psi(z)\right]\right]\right\rangle_o = 0. \qquad (7)$$

For additional sum rules similar conditions are derived, assuming that the axial current is conserved.

To illustrate possible applications, we assume the first two sum rules to hold. The relations obtained predict the existence of a P_{11} ($m > 1470$ MeV) nucleon resonance (the $P_{11}(1750)$?) from the existence of the four nucleon resonances $P_{11}(940)$, $P_{11}(1470)$, $S_{11}(1550)$ and $S_{11}(1710)$.

II. The Weinberg Sum Rules

To fix notation, we explicitly write the spectral representation as

$$i\left\langle\left[J_\mu{}^a(x),\ J_\nu{}^b(y)\right]\right\rangle_o = \delta^{ab}\int dm^2\left[\rho_I{}^{(J)}(m^2)g_{\mu\nu}+\rho_{II}{}^{(J)}(m^2)\frac{\partial^2}{\partial x^\mu \partial x^\nu}\right]\Delta(x-y;m^2). \qquad (8)$$

Here, the spectral functions fulfill the conditions

$$0 \le \rho_I{}^{(J)}(m^2) \le m^2\rho_{II}{}^{(J)}(m^2) \ . \qquad (9)$$

The second equality sign holds if and only if $J_\mu{}^a(x)$ is a conserved current. In this case, there is only one independent spectral function $\rho^{(J)}(m^2)$, which we define to be

$$\rho^{(J)}(m^2) = \rho_I{}^{(J)}(m^2) = m^2\rho_{II}{}^{(J)}(m^2) \ . \qquad (10)$$

The original Weinberg sum rules for $SU(2)\otimes SU(2)$ assume the currents to be conserved. Then, the first sum rule reads

$$\int m^{-2}\left[\rho^{(V)}(m^2) - \rho^{(A)}(m^2)\right]dm^2 = 0 \qquad (11)$$

and the second one reads

$$\int \left[\rho^{(V)}(m^2) - \rho^{(A)}(m^2) \right] dm^2 = 0 \ . \tag{12}$$

If zero mass states are absent, infinite many sum rules for spectral functions of nonconserved currents reduce to Eq. (11) or (12) in the limit of current conservation. These sum rules differ by the scalar or pseudoscalar contributions[17] and it is easy to see that only

$$W_{II} \equiv \int \left[\rho_{II}^{(V)}(m^2) - \rho_{II}^{(A)}(m^2) \right] dm^2 = 0 \tag{13}$$

reduces to Weinberg's first sum rule, for which the pion is treated as Goldstone boson.[18] Due to the smallness of the mass of the only spin zero state assumed to contribute (the pion) all the generalisations of the second sum rule are equivalent to[18]

$$W_{II} \equiv \int \left[\rho_{I}^{(V)}(m^2) - \rho_{I}^{(A)}(m^2) \right] dm^2 = 0 \quad . \tag{14}$$

In our treatment, Eqs. (13) and (14) will show up. Note that in our notation W_{II} and W_I represent (in this order!) the first or second Weinberg sum rule, respectively.

In order to derive conditions for the first Weinberg sum rule, we introduce as our first assumption:

A1. The commutators of the SU(2) ⊗ SU(2) charges $Q_1^a(x_o)$ and $Q_5^a(x_o)$ with the currents $V_\mu^a(x)$ and $A_\mu^a(x)$ are of standard current algebra form.

Note that nothing is assumed about commutators of space components with space components and that arbitrary Schwinger terms are allowed in all the commutators.

We will sometimes make explicit the contributions from possible violations of the Jacobi identity and define I(A,B,C) by

$$I(A,B,C) = \sum_{\text{cycl.}(A,B,C)} \left\langle \left[A,[B,C] \right] \right\rangle_o \ . \tag{15}$$

Under Assumption A1 we now have ($x_o = y_o = z_o$, not summed over i)

$$I(Q_5^{\,a}(x_o),\, V_\mu^{\,b}(y),\, A_o^{\,c}(z)) - \left\langle\left[Q_5^{\,a}(x_o),\left[V_\mu^{\,b}(y),A_o^{\,c}(z)\right]\right]\right\rangle_o$$

$$= i\epsilon^{abc}\left\{\left\langle\left[A_o^{\,i}(z),A_\mu^{\,i}(y)\right]\right\rangle_o - \left\langle\left[V_o^{\,i}(z),V_\mu^{\,i}(y)\right]\right\rangle_o\right\}$$

$$= \begin{cases} 0 & \text{for } \mu = 0 \\ i\epsilon^{abc}\, W_{II}\, \dfrac{\partial}{\partial z^k}\,\delta(\underline{y}-\underline{z}) & \text{for } \mu = k\ . \end{cases} \qquad (16)$$

To obtain the last line, we have used the spectral representation, Eq. (8), and have performed the equal time limit under the integral, using

$$-\frac{\partial}{\partial x_o}\,\Delta(x;m^2)\Big|_{x_o=0} = \delta(\underline{x}) \quad . \qquad (17)$$

Now, assuming $I = 0$, Eq. (16) shows that $\left\langle\left[Q_5^{\,a}(x_o),\left[V_k^{\,b}(y),A_o^{\,c}(z)\right]\right]\right\rangle_o$ is proportional to $\partial_k\delta(\underline{y}-\underline{z})$ and thus at most the first order Schwinger term of $\left[A_o^{\,c}(z),V_k^{\,b}(y)\right]$ may survive in Eq. (16). The sum rule Eq. (13) is thus equivalent to

$$\left\langle\left[Q_5^{\,a}(x_o),\left[V_k^{\,b}(y),\, A_o^{\,c}(z)\right]\right]\right\rangle_o = 0 \qquad (18)$$

or to

$$\int (z-y)_k \left\langle\left[Q_5^{\,a}(x_o),\left[V_k^{\,b}(y),\, A_o^{\,c}(z)\right]\right]\right\rangle_o = 0\ . \qquad (19)$$

If we had performed the above manipulations starting from $I_o(Q_5^{\,a}(x_o),\, A_\mu^{\,b}(y),\, V_o^{\,c}(z))$ the result would have been

$$I_o(Q_5^{\,a}(x_o),A_\mu^{\,b}(y),V_o^{\,c}(z)) - \left\langle\left[Q_5^{\,a}(x_o),\left[A_\mu^{\,b}(y),V_o^{\,c}(z)\right]\right]\right\rangle_o$$

$$= \begin{cases} 0 & \text{for } \mu = 0 \\ -i\epsilon^{abc}\, W_{II}\, \dfrac{\partial}{\partial z^k}\,\delta(\underline{y}-\underline{z}) & \text{for } \mu = k\ . \end{cases} \qquad (20)$$

The conclusions which follow are analogous to the ones above.

In order to deal with the second Weinberg sum rule we state two additional assumptions which we will use alternatively. These are:[19)]

A2. It is for $x_o = y_o$

$$\left[\int d^3x\,\partial^\mu A_\mu{}^a(x),\; A_k{}^b(y)\right] \tag{21}$$

and

A3. It is for $x_o = y_o$

$$\left[\int d^3x\,\partial^\mu A_\mu{}^a(x),\; V_k{}^b(y)\right] = 0\;. \tag{22}$$

There are two situations in which the above assumptions evidently hold. If the pion is treated as a Goldstone boson and the axial current is conserved, Eqs. (21) and (22) hold trivially. They are canonical rules in a model for which PCAC holds with a canonical pion field and with canonical vector or axial vector field proportional to the vector or axial vector current, respectively (see also the "Note added").

We first derive from A2 and charge-current commutators the following equal time commutator

$$\left[Q_5{}^a(y_o),\dot{A}_k{}^b(y)\right] = \frac{\partial}{\partial y^o}\left[Q_5{}^a(y_o),A_k{}^b(y)\right] - \left[\dot{Q}_5{}^a(y_o),A_k{}^b(y)\right]$$

$$= i\epsilon^{abd}\,\dot{V}_k{}^d(y) - \left[\int d^3x\,\partial^\mu A_\mu{}^a(x),A_k{}^b(y)\right] = i\epsilon^{abd}\,\dot{V}_k{}^d(y)\;. \tag{23}$$

From this, using charge-current commutators again, we get

$$I(Q_5{}^a(x_o),V_\mu{}^b(y),\dot{A}_k{}^c(z)) - \left\langle\left[Q_5{}^a(x_o),\left[V_\mu{}^b(y),\dot{A}_k{}^c(z)\right]\right]\right\rangle_o$$

$$= i\epsilon^{abc}\left\{\left\langle\left[\dot{A}_k{}^i(z),A_\mu{}^i(y)\right]\right\rangle_o - \left\langle\left[\dot{V}_k{}^i(z),V_\mu{}^i(y)\right]\right\rangle_o\right\}$$

$$= -i\epsilon^{abc}\int dm^2\left\{\left[\rho_I{}^{(V)}(m^2) - \rho_I{}^{(A)}(m^2)\right]g_{\mu k} + \right.$$

$$\left. + \left[\rho_{II}{}^{(V)}(m^2) - \rho_{II}{}^{(A)}(m^2)\right]\frac{\partial^2}{\partial y^\mu \partial y^k}\right\}\frac{\partial}{\partial z^o}\Delta(y-z;m^2)$$

$$= \begin{cases} 0 & \text{for } \mu = 0 \\ i\epsilon^{abc}\left[W_I g_{k\ell} + W_{II}\dfrac{\partial^2}{\partial y^\ell \partial y^k}\right]\delta(\underline{y}-\underline{z}) & \text{for } \mu = \ell\;. \end{cases} \tag{24}$$

The coefficient of the second order Schwinger term vanishes if the first Weinberg sum rule holds. We next use Eq. (20) to subtract this term out. Specializing to $\mu = \ell$ in Eq. (24) and differentiating Eq. (16) with respect to y^ℓ we obtain, after subtraction,

$$I(Q_5{}^a(x_o), V_\ell{}^b(y), \dot{A}_k{}^c(z)) - I(Q_5{}^a(x_o), V_\ell{}^b(y), \frac{\partial}{\partial z^k} A_o{}^c(z))$$
$$- \left\langle \left[Q_5{}^a(x_o), \left[V_\ell{}^b(y), \dot{A}_k{}^c(z) - \frac{\partial}{\partial z^k} A_o{}^c(z) \right] \right] \right\rangle_o$$
$$= i\epsilon^{abc} W_I \, g_{k\ell} \, \delta(\underline{y} - \underline{z}) \qquad . \tag{25}$$

Note that the above expression contains a non-Schwinger term only.

We next assume the Jacobi identities to hold and get, after integrating over $\underline{y}$

$$- \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y V_\ell{}^b(y), \dot{A}_k{}^c(z) - \frac{\partial}{\partial z^k} A_o{}^c(z) \right] \right] \right\rangle_o =$$
$$= i\epsilon^{abc} W_I \, g_{\ell k} \quad . \tag{26}$$

Obviously, if the assumption A2 were not made, the term

$$+ \left\langle \left[\int d^3y V_\ell{}^b(y), \left[\int d^3x \, \partial^\nu A_\nu{}^a(x), A_k{}^c(z) \right] \right] \right\rangle_o \tag{28}$$

would appear on the left hand side of Eq. (26).

If we had performed the same manipulations as above, starting from $I(Q_5{}^a(x_o), A_\ell{}^b(y), V_k{}^c(z))$ and the assumption A3 instead of A2, analogous results would have followed. Especially, we would have gotten instead of Eq. (26)

$$- \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y \, A_\ell{}^b(y), \dot{V}_k{}^c(z) - \frac{\partial}{\partial z^k} V_o{}^c(z) \right] \right] \right\rangle_o =$$
$$= i\epsilon^{abc} W_I \, g_{\ell k} \qquad . \tag{29}$$

The following two sets of equations will summarize the results:

$$i\epsilon^{abc} W_I g_{k\ell} = \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y \, V_\ell{}^b(y), \dot{A}_k{}^c(z) - \frac{\partial}{\partial z^k} A_o{}^c(z) \right] \right] \right\rangle_o$$
$$= \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y \, V_\ell{}^b(y), \dot{A}_k(z) \right] \right] \right\rangle_o \tag{30}$$

and

$$i\epsilon^{abc} W_{II} g_{k\ell} = \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y A_\ell{}^b(y), \dot{V}_k{}^c(z) - \frac{\partial}{\partial z^k} V_o{}^c(z) \right]\right] \right\rangle$$

$$= \left\langle \left[Q_5{}^a(x_o), \left[\int d^3y A_\ell{}^b(y), \dot{V}_k{}^c(z) \right]\right] \right\rangle_o \qquad (31)$$

The Eqs. (30) are derived under assumption A1 and A2; the Eqs. (31) under assumptions A1 and A3. Note again that the condition Eq. (4) is included in Eqs. (31). Note that also the Eqs., in which the $\underset{\sim}{y}$ integration is replaced by an $\underset{\sim}{z}$ integration in Eqs. (30) and (31), hold.

III. The Baryon Spectral Function Sum Rules

We write the spectral representation for the vacuum expectation value of the anticommutator of the baryon field as

$$\langle [\psi(x), \bar{\psi}(y)]_+ \rangle_o = i \int dm^2 \{ F_+^2(m^2)(ij^\mu \frac{\partial}{\partial x^\mu} + m) +$$

$$+ F_-^2(m^2)(ij^\mu \frac{\partial}{\partial x^\mu} - m)\} \Delta(x-y; m^2). \qquad (32)$$

The $F_\pm^2(m^2)$ are positive spectral functions and represent the contributions from the $I = \frac{1}{2}$, $J = \frac{1}{2}^\pm$ baryon spectrum respectively. We will assume the commutator Eq. (6) throughout this section. It can be seen that the $(\Delta I = \frac{3}{2})$-terms would not contribute and thus we simplify our derivation by assuming that they are absent. We assume in addition:[22]

A4. It is

$$\left[\int d^3x \, \partial^\mu A_\mu{}^a(x), \psi(y) \right] = 0 . \qquad (33)$$

Eq. (33) would evidently follow from current conservation and is one of the canonical rules, if $\bar{\psi}(y)$ and the pion field, defined by PCAC, are canonical fields. A derivation completely analogous to the one in Eq. (23) now gives us from (6) and (33), as in Ref. 11,

$$\left[Q_5{}^a(x_o), \dot{\psi}(y) \right] = -\gamma_5 \tau^a r_A \dot{\psi}(y). \qquad (34)$$

We write next the Jacobi type identity, Eq. (5), for $a = Q_5{}^a(x_o)$, $b = \dot{\psi}(y)$, and $c = \bar{\psi}(z)$. This gives[8),24)] us

$$\left[Q_5{}^a(x_o), \left[\dot\psi(y), \bar\psi(z)\right]_\pm\right]$$

$$= \left[\left[Q_5(x_o), \dot\psi(y)\right], \bar\psi(z)\right]_+ + \left[\left[Q_5(x_o), \bar\psi(z)\right], \dot\psi(y)\right]_+$$

$$= -r_A\left[\gamma_5\tau^a\dot\psi(y), \bar\psi(z)\right]_+ - r_A\left[\dot\psi(y), \bar\psi(z)\gamma_5\tau^a\right]_+ . \qquad (35)$$

We take vacuum expectation values and use the spectral representation. Because of the presence of j_5 in Eq. (35) no term proportional to j can contribute. Thus we get

$$\left\langle\left[Q_5{}^a(y_o), \left[\dot\psi(y), \bar\psi(z)\right]_+\right]\right\rangle_o$$

$$= 2r_A i\tau^a\gamma_5 \int dm^2\, m(F_+^2(m^2) - F_-^2(m^2))\, \delta(\underline{y} - \underline{z}) \quad . \qquad (36)$$

Before we discuss this result, we derive additional rules like (36) from assuming that the axial current is conserved. This is, we assume[23)]

<u>A5</u>. It is

$$\partial^\mu A_\mu{}^a(x) = 0 \quad . \qquad (37)$$

Then we derive

$$\left[Q_5{}^a(x_o), \frac{\partial^n}{\partial y_o^n}\psi(y)\right] = -\gamma_5\tau^a r_A \frac{\partial^n}{\partial y_o^n}\psi(y) \qquad (38)$$

for any integer n. Using this relation, the identity Eq. (5) with $a = Q_5{}^a(x_o)$, $b = (\frac{\partial}{\partial y_o})^{2n-1}\psi(y)$, and $c = \bar\psi(z)$ reads[24),25)]

$$\left\langle\left[Q_5{}^a(x_o), \left[\left(\frac{\partial}{\partial y_o}\right)^{2n-1}\psi(y), \bar\psi(z)\right]_+\right]\right\rangle_o$$

$$= -r_A\gamma_5\tau^a\left\langle\left[\left(\frac{\partial}{\partial y_o}\right)^{2n-1}\psi(y), \bar\psi(z)\right]_+\right\rangle_o - \left\langle\left[\left(\frac{\partial}{\partial y_o}\right)^{2n-1}\psi(y), \bar\psi(z)\right]_+\right\rangle r_A\gamma_5\tau^a$$

$$= -2i\tau^a\gamma_5 r_A \int dm^2\, m\left[F_+^2(m^2) - F_-^2(m^2)\right]\left(\frac{\partial^2}{\partial \underline{y}\partial \underline{y}} - m^2\right)^{n-1} \delta(\underline{y} - \underline{z}) \quad . \qquad (39)$$

We discuss this relation for n = 1,2 only. The generalisations will be obvious. For n = 1, Eq. (39) is identical to Eqs. (35)

and (36). For n = 2, there is only a non-Schwinger term contained in the Eq. (39). The Schwinger term is multiplied by the expression Eq. (36) and the non-Schwinger term is multiplied by

$$S_3 = \int dm^2\, m^3 (F_+^2(m^2) - F_-^2(m^2)). \qquad (40)$$

Thus, if A5 holds, $\langle [Q_5^a(x_o), [\dot{\psi}(y), \bar{\psi}(z)]]\rangle_o$ vanishes if and only if S_3 vanishes. If A4 holds, the vanishing of $\langle [Q_5^a(x_o), [\dot{\psi}(y), \bar{\psi}(z)]_+]\rangle_o$ is equivalent with

$$S_1 = \int dm^2\; m(F_+^2(m^2) - F_-^2(m^2)) = 0 . \qquad (41)$$

If A5 holds in addition, Eq. (41) is also equivalent with the vanishing of the Schwinger term in Eq. (39).

In the one particle intermediate states approximation, Eq. (41) and (40) read

$$\sum_{i=1}^{R} m_i\, \epsilon_i\, F_i^2 = 0 \qquad (42)$$

and

$$\sum_{i=1}^{R} m_i^3\, \epsilon_i\, F_i^2 = 0 \quad , \qquad (43)$$

respectively. Here, we have enumerated the nucleon resonances by i = 1,...,R and ϵ_i denotes the parity of the respective resonance. Evidently, either of the Eqs. (42) or (43) can hold only if baryons of opposite parities actually exist.

The sum rules Eqs. (41), (42) have been derived by J. Rothleitner[4] and M. Sugawara,[12] respectively. The experimental success of the considerations of Refs. 12 strongly supports Eq. (42) and thus shows that to the approximation to which Eq. (33) holds, the expression $\langle [Q_5^a(x_o), [\dot{\psi}(y), \bar{\psi}(z)]]\rangle_o$ should vanish.

Finally, to illustrate possible applications, let us assume that both Eqs. (42) and (43) hold. Restrictions will follow from the positivity of the F_i^2's. Enumerating the nucleon resonances $N_1 = P_{11}(940)$, $N_2 = P_{11}(1466)$, $N_3 = S_{11}(1548)$ and $N_4 = S_{11}(1709)$ by $N_1, \ldots N_4$, we write Eqs. (42) and (43) as

$$m_1 F_1^2 + m_2 F_2^2 + \sum_{i=5}^{R} \epsilon_i m_i F_i^2 = m_3 F_3^2 + m_4 F_4^2 \qquad (45)$$

and

$$m_1{}^3F_1{}^2 + m_2{}^3F_2{}^2 + \sum_{i=5}^{R} \epsilon_i m_i{}^3 F_i{}^2 = m_3{}^3F_3{}^2 + m_4{}^3F_4{}^2 \quad . \tag{46}$$

Multiplying (45) by $m_2{}^2$ and subtracting the Eq. (46) from the result we get

$$m_1(m_2{}^2 - m_1{}^2)F_1{}^2 + \sum_{i=5}^{R} \epsilon_i m_i (m_2{}^2 - m_i{}^2)F_i{}^2$$

$$= m_3(m_2{}^2 - m_3{}^2)F_3{}^2 + m_4(m_2{}^2 - m_4{}^2)F_4{}^2 \quad . \tag{47}$$

The right hand side is not positive and the first term on the left hand side is not negative. Thus, unless all the $F_i{}^2$'s vanish, at least one term in the sum is negative. Giving the number 5 to it, we have

$$\epsilon_5(m_2{}^2 - m_5{}^2) < 0 \ . \tag{48}$$

As it seems unlikely that a still undiscovered nucleon resonance with a mass smaller than m_2 exists, we have the prediction

$$\epsilon_5 = +1, \quad m_5 > m_2 \ . \tag{49}$$

This agrees with the existence of the $P_{11}(1750)$ nucleon resonance.

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Note added: It should be pointed out that the assumptions Eqs. (21) and (22) follow from the current-field identities

$$g_A a_\mu{}^a(x) = f_\pi \partial_\mu \varphi^a(x) + A_\mu{}^a(x) \ ,$$

$$g_A v_\mu{}^a(x) = V_\mu{}^a(x) \ ,$$

and

$$f_\pi m_\pi{}^2 \varphi^a(x) = \partial^\mu A_\mu{}^a(x) \quad .$$

Here, $V_\mu^{\,a}(x)$, $a_\mu^{\,a}(x)$, and $\varphi^a(x)$ denote fields which are proportional to canonical vector, axial vector, and pion fields. It can be seen that--unless the pion is free--the interaction must contain in this case derivatives of the pion field (H. Genz and J. Katz, On Current-Field Identities, Purdue University preprint). If the interaction Lagr Lagrangian does not contain derivatives of the vector or axial vector fields, then also

$$\dot{V}_k^{\,a}(x) - \partial_k V_o^{\,a}(x) = \dot{v}_k^{\,a}(x) - \partial_k v_o^{\,a}(x)$$

or

$$\dot{A}_k^{\,a}(x) - \partial_k A_o^{\,a}(x) = \dot{a}_k^{\,a}(x) - \partial_k a_o^{\,a}(x)$$

belong to the canonical variables (of course, if $V_\mu^{\,a}(x)$ or $A_\mu^{\,a}(x)$ are themselves proportional to canonical fields, the same conclusions hold). Thus, the left hand side of Eq. (29) or Eq. (26), respectively, would vanish in this case and this would show the validity of the second Weinberg sum rule. However, as recently shown, in case of canonical realizations of current-field identities the interaction Lagrangian L_I also contains derivatives of the spin one fields and thus Eq. (29) [or Eq. (26)] provides a test for the validity of this sum rule, namely

$$i\epsilon^{abc} W_I = -\left\langle \left[Q_5^{\,a}(x_o), \left[\int d^3y\, A_\ell^{\,b}(y), \frac{\partial L_I}{\partial \dot{V}_k(z)} \right]\right] \right\rangle_o .$$

References and Footnotes

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9. This commutator has been assumed and discussed by a number of authors.[8),10)-14)] See Ref. 15 for another proposal.
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16. If the $(\Delta I = \frac{3}{2})$-terms are absent, it has been shown[14)] that for fermions fields introduced into a field theory of currents Schwinger terms are present in Eq. (1).
17. Only $\rho_{II}^{(J)}(m^2)$ contains such contributions.
18. For the nonconserved ones of the SU(3) ⊗ SU(3) currents, the other generalisations may be of interest.
19. To derive the main results, Eq. (30) and (31), it would be sufficient to have, instead of A2 and A3, the weaker assumptions

$$\left\langle\left[\int d^3y\, V_\ell{}^b(y), \left[\int d^3x\, \partial^\mu A_\mu{}^a(x), A_k{}^c(z)\right]\right]\right\rangle_0 = 0$$

or

$$\left\langle\left[\int d^3y\, A_\ell{}^b(y), \left[\int d^3x\, \partial^\mu A_\mu{}^a(x), V_k{}^c(z)\right]\right]\right\rangle_0 = 0,$$

respectively.
20. In the minimal algebra model of Bjorken and Brandt[21)] the Jacobi identity which is relevant here does in fact not hold. However, $I_0 = 0$ also in this model and the Weinberg sum rules do hold.
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22. To derive Eq. (36) it would be sufficient to assume

$$\left\langle\left[\bar{\psi}(z), \left[\int d^3x\, \partial^\mu A_\mu{}^a(x), \psi(y)\right]\right]\right\rangle_0 = 0.$$

23. The results would also follow from some weaker, however complicated, conditions.
24. Contributions from a possible violation of the identity, Eq. (6), might easily be put in.
25. With $b = (\frac{\partial}{\partial y_0})^{2n} \psi(y)$ an identity evolves.
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