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Article

The Variation of G and Λ in Cosmology

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Abstract: The idea of varying constants of nature is very old, and has commanded a lot of attention since first mooted. The variation in the gravitational parameter G and cosmological parameter Λ is still an active area of research. Since the idea of a varying G was introduced by Dirac almost a century ago, there are even theories that have variable G such as the Brans–Dicke theory and the scale covariant theory. Both these theories also have a varying Λ in their full generalisations. A varying Λ was also introduced around the same time as that of varying G. It is interesting to note that a possible solution to the cosmological constant problem can be realised from a dynamic Λ. In this work, we focus on a varying Λ and G framework. In almost all studies in the simplest framework of variables Λ and G, it is found that one of them has to increase with time. However, observations and theoretical considerations indicate that both Λ and G should decrease with time. In this paper, we propose a solution to this problem, finding theories in which both Λ and G decrease with time.

Keywords: variable Λ; variable G; conservation of energy momentum

1. Introduction

The idea of varying constants of nature, such as the fine-structure constant α, the speed of light c, Newton's gravitational constant G, Boltzmann's constant k_B, Planck's constant $\hbar$, and Fermi's constant G_F, is not new. There are many reasons to believe that these constants should vary [1,2].

- From quantum theory, string theory, and other similar points of view, there are strong reasons for believing in more than three spatial dimensions. Hence, the constants from these higher dimensions need not be constant as viewed from our three-dimensional point of view. Any change that is slow in the size of higher dimensions can be detected by changes in the "constants" in our 3-dimensional space.
- Symmetry-breaking processes that are spontaneous in the very early universe introduce irreducibly random elements as far as the values of the constants of nature.
- The outcome of a theory of quantum gravity is expected to be probabilistic, whose probability distributions for observables may not be very sharply peaked for all possibilities. So, the gravitation "constant", G, or $\dot{G}$ may vary.
- At present, we do not know why any of the constants of nature have the values that they do. Also, we have not been able to predict the value of any dimensionless constant before it has been measured.
- The measured values of the possible changes in the values of the constants of nature are usually weak. Sometimes, they are made out to sound strong by chosen parametrisations.
- Using publicly available data, Li et al. [3] were able to construct samples from 40 spectra of galaxies that emit Lyman α lines and 46 from QSOs in the redshift range $1.09 < z < 3.73$. Having used two methods, they were able to calculate $\alpha(z)$ by measuring the wavelengths of two components of the spin-orbit doublet. By analysing the spectra obtained, they found a change in α of $(-3 \pm 6) \times 10^{-5}$, as compared to the laboratory value. Many scientists suspect a bias in the observational measurements of the data, or in the laboratory calculations, and are presently re-analysing this matter.

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- Varying G and Λ can cause changes in the dynamics of the universe, thereby changing perturbations and growth of the perturbations on a large scale [4,5]. The apparent simplicity of the ΛCDM model belies the intricate challenges associated with the cosmological constant, including the cosmological constant problem and the coincidence problem. Consequently, alternative explanations have been sought. Their findings highlighted the differences and potential advantages of considering time-variable parameters in the cosmological model which can impact the formation of large-scale structures in the universe. While further research and observations are needed to validate their findings, valuable insights were gleaned to our understanding of the cosmos beyond the standard cosmological paradigm.

The literature on varying cosmological and gravitational constants is very vast, and we only focus on some of the key papers. The references therein provide additional reading matter. It seems that Dirac [6,7] first proposed the idea of varying G. The electromagnetic force between an electron and a proton is around 10^{40} times higher than the gravitational force. Furthermore, the radius of the observed universe is about 10^{40} times greater than the radius of the electron. In addition, $N_b = (c^3/(m_p G H_0)) \sim 10^{80}$, where N_b is the number of baryons in the universe and H_0 is the present value of the Hubble parameter. Dirac noticed the similarity between these numbers, and felt that this could not be a coincidence. Since $H_0 \sim t^{-1}$ is not constant, he proposed that Newton's gravitational constant G is not really a constant, but varies as $G \sim t^{-1}$. According to Dirac's choice, e, c, and m_e turn out to be constants that are fixed, but this choice is not unique to yield $G \sim t^{-1}$. One could make other choices to obtain the same result [8,9]. It is important to note that observations of the change in G yield only $\dot{G}/G \leq 10^{-2} H_0$ and not any stronger constraints [1].

However, the early history of variable constants, and in particular, variable G, is not very clear, but there were several others apart from Dirac who were also involved in studying variable constants at around the same time. One of the earliest of these to study varying constants was Lord Kelvin and Tait [10], well over a century ago in 1874. This was some 30 years before Einstein came up with his special theory of relativity in 1905, in which c is an assumed constant. At the time of Kelvin, a varying speed of light was quite acceptable to the scientific community, as it played no special role in physics. However, after 1905, the situation changed completely, as expected. Now, it was not Dirac who had first conceived the idea of the so-called "Large Numbers Hypothesis (LNH)", which is usually attributed to his name. Hermann Weyl [11,12] in 1917 and 1919 speculated that the radius of the observed universe could be the the radius of some hypothetical particle whose ratio to the electron radius was of the order of 10^{42}. The coincidence was further developed by Eddington in 1931 [13], who related it to the estimated number of charged particles in the universe, which is around 10^{42}. Milne [14] came up with an idea of two systems of units, one for atomic purposes, and the other for gravitational, which were related by a logarithmic transformation. By requiring that the mass of the "universe" to be constant, he was able to derive a variation of G of the type $G \propto t$. However, it appears that Milne was motivated by his dislike of relativity rather than the LNH. The biologist Haldane [15,16] in 1935 took an interest in the theory of Milne, writing a few papers dealing with evolution of life. These authors proposed that biochemical activation energies could change on the τ timescale, yet look like constants as far as the t timescale was concerned. Hence, the universe does not evolve uniformly. Jordan [17,18] was also able to derive a variation of G as the inverse of time by a slightly different method. An interesting exposition of Jordan's cosmology is given by Dubois and Furza [19]. The variation $G \propto 1/t$ was shown to be unlikely by Teller [20] since it would lead to too quick a change in the temperature of the Earth, and life would not be able to exist. However, the ideas of Dirac led Jordan, Brans, and Dicke to develop the Brans–Dicke theory [21] in 1961, which would allow for the variation of G by the introduction of a suitable scalar field $\Phi(t)$, which had an evolution equation for $G(\Phi)$ in the theory.

Apart from that mentioned above, the motivation for varying G specifically is the following:

- Theoretically, many theories of gravity apart from general relativity and Newtonian gravity may be framed with a varying G [22–24].
- Experiments and observations have been used to set limits on $\dot{G}/G$ [25], including solar evolution, lunar occultations, and eclipses $(\sim 10^{-11}\ \text{yr}^{-1})$, paleontological evidence $(\sim 10^{-11}\ \text{yr}^{-1})$, white dwarf cooling and pulsations $(\sim 10^{-10}\ \text{yr}^{-1})$, neutron star masses and ages $(\sim 10^{-12}\ \text{yr}^{-1})$, star cluster evolution $(\sim 10^{-12}\ \text{yr}^{-1})$, big bang nucleosynthesis abundances $(\sim 10^{-12}\ \text{yr}^{-1})$, astroseismology $(\sim 10^{-12}\ \text{yr}^{-1})$, lunar laser ranging $(\sim 10^{-14}\ \text{yr}^{-1})$, evolution of planetary orbits $(\sim 10^{-14}\ \text{yr}^{-1})$, binary pulsars $(\sim 10^{-12}\ \text{yr}^{-1})$, high-resolution quasar spectra $(\sim 10^{-14}\ \text{yr}^{-1})$, gravitational wave observations of binary neutron stars $(\sim 10^{-8}\ \text{yr}^{-1})$, and supernovae. (See the list of references in [25]).
- An important point to note is that if one chooses a variation of G of the type $\dot{G}/G \leq 10^{-14}\ \text{yr}^{-1}$, then there will be no problems with the mass and size of galaxies, stars, and planets, as they will not be affected.
- It is interesting to note that a variation of fundamental constants can lead to the solution of the Hubble tension problem [26–28].

The ΛCDM model is currently the most favoured model for explaining the current acceleration of the universe, where Λ refers to the cosmological constant. Now, a major problem in cosmology is the cosmological constant problem [29,30]. Now, a dynamic cosmological parameter [31] can, inter alia, provide a solution to this problem. A Lagrangian description of variable Λ has been given by Poplawski [32]. The variation of the cosmological parameter seems to have first been considered by Bronstein [31,33] in 1933. Overduin and Cooperstock [34] have given a review of all the different forms of varying cosmological parameter Λ that have been considered in the literature as of 1998. A review of varying Λ and quintessence has been given by Kragh and Overduin [35].

In general relativity, Λ is a strict constant and associated with the vacuum energy density, $\rho(vac)$. However, in the anaFLRW universe, quantum field theory entails a cosmological constant Λ that is changing with time. This also means a vacuum density that is changing with time. Via quantum effects, Λ turns out to vary: $\Lambda \rightarrow \Lambda + \delta\Lambda$. The GR limit can be recovered smoothly. The connection of the varying Λ with quantum field theory can be motivated from semi-qualitative renormalisation group arguments [36]. However, an explicit quantum field theory calculation has appeared only very recently [37–41]. Also, recently, a running vacuum model was studied in the Brans–Dicke theory, and it was found that observations favour this model over the one with constant Λ [42].

The authors of [42] solved both the perturbation and background equations for the variable cosmological parameter model (BDRVM) and the vacuum energy density model (VED). Data from the latest SNIa + H(z) + BAO + LSS + CMB observations were used in the analysis. To ascertain whether these models perform better than the ΛCDM in general relativity, use was made of the criteria from the AIC and DIC statistics. The data from three different sources were used to test the two different models. It was found that both the two models can fit the observations, but the BDRVM one performed slightly better. Hence, a dynamical Λ performs better than the constant Λ, and the BDRVM model has the potential to solve the two tensions in general relativity. However, more investigations are necessary.

Hence, there is a strong motivation to study the variation of the gravitational parameter G and the cosmological parameter Λ in cosmology.

In this work, in Section 2, we firstly review the simplest formulation of a Lagrangian description of general relativity with variable G and Λ. We point out that it is not possible to have both decreasing Λ and G. A decreasing G is favoured for many reasons, inter alia, giving rise to late-time acceleration without the need for any exotic matter [43], as in the ΛCDM model. Then, in Sections 3 and 5, we examine two theories, viz. the scale covariant theory and $f(R,T)$ theory. These two theories naturally have variable Λ and G. We show that it is possible to have both decreasing Λ and G.

2. Lagrangian Formulation

Let us consider Einstein's field equations in suitable units with variable G and Λ

$$R_{ab} - \frac{1}{2}RG_{ab} + \Lambda(x^d)G_{ab} = G(x^d)T_{ab} \tag{1}$$

where the symbols have their usual meanings, but Λ and G are allowed to be variable. It should be noted that one has to have a variable Λ together with variable G in order for the usual conservation law to hold. This form of the field equations can be shown to arise from several different approaches. Firstly, the following Lagrangian [44].We assume that G and Λ are related, and that the action is

$$A = \int d^4x\mathcal{L} = \int d^4x\{\sqrt{-g}[R/G - V(G)] + L_m\} + A'... \tag{2}$$

where $V(G)$ is a function of G, G a variable, and L_m the matter Lagrangian. The reason for the introduction of A' is to take care of terms in the second derivatives of the metric. This enables simple equations in the variables G and Λ [45]. The Euler–Lagrange equations used here are

$$\frac{\partial\mathcal{L}}{\partial G} = \nabla_a \frac{\partial\mathcal{L}}{\partial(\partial_a G)} \tag{3}$$

This yields

$$V'(G) = \frac{R}{G^2} \tag{4}$$

Varying the action (2) with respect to g_{ab}, we obtain

$$R_{ab} - \frac{1}{2}Rg_{ab} = GTab - (\frac{1}{2}GV(G))g_{ab} \tag{5}$$

where the matter tensor T_{ab} arises from the matter Lagrangian L_m. If we now let

$$\frac{1}{2}GV(G) = \Lambda \tag{6}$$

then we finally obtain

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda gab = GT_{ab} \tag{7}$$

where G and Λ are variable.

From Equation (7), we can derive the field equations as

$$-3H^2 + \Lambda(t) = -G(t)\rho \tag{8}$$

$$-2\dot{H} - 3H^2 + \Lambda(t) = G(t)p \tag{9}$$

From the above two equations, we can derive a modified energy conservation equation with variable G and Λ, as follows:

$$\dot{\rho} + 3H(\rho + p) = -\rho\left(\frac{\dot{G}}{G}\right) - \frac{\dot{\Lambda}}{G}. \tag{10}$$

We notice that in this simplest formalism with variable G and Λ, using a Lagrangian formulation, we can still retain the usual energy conservation law in general relativity and obtain a separate equation for the variation of G and Λ, as follows:

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{11}$$

$$\rho\left(\frac{\dot{G}}{G}\right) + \frac{\dot{\Lambda}}{G} = 0. \tag{12}$$

Another Lagrangian derivation has been given by Lau [46] and Lau and Prokhovnik [47] in terms of a scalar formulation for Λ.

Since ρ is non-negative, it follows immediately from Equation (12) that if $\dot{G}$ is positive (negative), then $\dot{\Lambda}$ is negative (positive). So, a decreasing cosmological parameter implies an increasing gravitational parameter, and vice versa. Usually, a decreasing cosmological parameter is favoured, and this implies an increasing gravitational parameter. A variation of the type (12) seems to have been first considered by DerSarkissian [48], and since then, has been considered by many authors. A partial list of them is [49–56]. Further references can be found in these citations.

It is interesting to investigate what is the situation in other theories of gravity.

3. Scale Covariant Theory

Canuto et al. [57] came up with an alternative theory—the scale-covariant theory of gravitation. In this theory, there are two systems of units. One of them is gravitational units in which Einstein's field equations are valid, and the other is atomic units in which physical quantities are expressed in atomic units. A conformal transformation relates the two systems of units, as follows:

$$\bar{g}_{ab} = \phi^2 g_{ab}, \tag{13}$$

where indices a, b take their values $0, 1, 2, 3$, the bar indicates gravitational units, and unbarred quantities refer to atomic units. The scalar function ϕ satisfies $0 < \phi < \infty$. The action for the theory is

$$I = \int (A\phi^4 - \phi^2 R + 6\phi^a\phi_a + 2G(\phi)L)\sqrt{-g}d^4x, \tag{14}$$

where R is the Ricci scalar, L is the Lagrangian of the matter, and ϕ_a denotes the ordinary derivative. We note that the gravitational parameter $G(\phi)$ is no longer a constant, but a function of ϕ. Performing a variation of (14), we find the field equations for the scale covariant theory as [57]

$$R_{ab} - \frac{1}{2}Rg_{ab} + f_{ab}(\phi) + \Lambda(\phi)g_{ab} = GT_{ab}, \tag{15}$$

where

$$f_{ab}(\phi) = \frac{1}{\phi^2}\Big[2\phi\phi_{a;b} - 4\phi_a\phi_b - g_{ab}(2\phi\phi^d{}_{;d} - \phi^d\phi_d)\Big]. \tag{16}$$

Here, R_{ab} and T_{ab} stand for the Ricci tensor and energy-momentum tensor, respectively. Again, we notice that the cosmological parameter $\Lambda(\phi)$ is no longer a constant, but a function of the scalar ϕ. The scale-covariant theory involves a non-minimal coupling between the gauge function ϕ and the Ricci scalar R.

For a flat FLRW space-time, the metric is

$$ds^2 = -a^2(t)(dx^2 + dy^2 + dz^2) + dt^2. \tag{17}$$

where a is the scale factor. Also, the matter-stress tensor for a fluid that is perfect can be written as

$$T_{ab} = pg_{ab} + (\rho + p)u_a u_b, \tag{18}$$

where ρ, p, and u^a represent the energy density, pressure, and the four-velocity vector, respectively. In a co-moving coordinate system, $u^a u_a = -1$ and $u^a u_b = 0$. The field equations in the scale covariant theory for a flat FLRW space-time can be found by expanding the tensor Equations (15) and (16), as follows:

$$2\dot{H} + 3H^2 + 6H\frac{\dot{\phi}}{\phi} + 2\frac{\ddot{\phi}}{\phi} - \frac{\dot{\phi}^2}{\phi^2} = -8\pi G p + \Lambda, \tag{19}$$

$$3H^2 + 6H\frac{\dot{\phi}}{\phi} + 3\frac{\dot{\phi}^2}{\phi^2} = 8\pi G\rho + \Lambda, \tag{20}$$

The continuity equation, which is a consequence of the field Equations (19) and (20), is given by [58–60]

$$\dot{\rho} + 3H\left[\rho + p\left(1 + \frac{\dot{\phi}}{H\phi}\right)\right] = -\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) - \frac{\dot{\Lambda}}{G}. \tag{21}$$

4. Models in SCT with Decreasing Parameters

The modified energy conservation Equation (21) can be split up in the following way:

$$\dot{\rho} + 3H(\rho + p) + 3p\frac{\dot{\phi}}{\phi} = 0 \tag{22}$$

$$\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) + \frac{\dot{\Lambda}}{G} = 0 \tag{23}$$

The reason for splitting up the modified conservation equation in this is the following. The term $3p\frac{\dot{\phi}}{\phi}$ involves the term p, and hence it needs to be added to the term $3Hp$. Also, we note that when $\phi = 1$ in Equation (22), then we recover general relativity. Secondly, putting $\phi = 1$ in Equation (23), we obtain the same equation as Equation (11) [48]. We can write Equation (23) in the following way:

$$\frac{\dot{\Lambda}}{G} = -\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) \tag{24}$$

From this equation, it is possible to deduce that there is a large class of solutions that permit both decreasing Λ and G. The variable Λ is positive [61] and expected to decrease with time for many reasons, including a possible solution to the cosmological constant problem [62]. Therefore, since G is positive, the left side of Equation (24) is negative. This means that the right side must also be negative, i.e.,

$$-\rho\left(\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi}\right) < 0 \tag{25}$$

or, since the energy density ρ is assumed positive, we must have

$$\frac{\dot{G}}{G} + \frac{\dot{\phi}}{\phi} > 0 \tag{26}$$

or,

$$\frac{\dot{G}}{G} > -\frac{\dot{\phi}}{\phi} \tag{27}$$

For decreasing G, we must have $\dot{G}/G < 0$. Unfortunately, in the scale covariant theory, the specific choice of the gauge function ϕ is not dictated to by the theory itself, but has to be put in by hand. Then, one has to check how well this fits in with observational constraints. Canuto et al. [57,63] have resricted ϕ as follows:

$$\phi \sim t^n, \ -1 \leq n \leq +1 \tag{28}$$

Then

$$\frac{\dot{\phi}}{\phi} = \frac{n}{t} \tag{29}$$

Let us choose $n = 1$, and for G let us choose

$$G = \frac{1}{logt} \tag{30}$$

Then, $\dot{G}/G < 0$, and we find that the requirement (27) is satisfied.

Hence, we have demonstrated that there are solutions in the scale covariant theory for which both the cosmological parameter Λ and gravitational parameter G decrease with time. This is not possible in the equivalent formulation in general relativity.

5. $f(R, T)$ Theory of Gravity

Another very interesting modified gravity theory is $f(R, T)$ gravity [64], which has recently garnered the attention of researchers. In this theory, the Lagrangian comprises arbitrary functions of the the trace (T) of the energy-momentum tensor, and the Ricci scalar (R). The variation of the energy-momentum tensor with respect to the metric leads to a source term, i.e., $f(L_m)$. Hence, a different set of field equations is obtained for different choices of L_m. If the action of $f(R, T)$ gravity, then the field equations are obtained by [64] as follows:

$$S = \int \sqrt{-g}\left(\frac{1}{G}f(R, T) + L_m\right)d^4x \tag{31}$$

where L_m is the matter Lagrangian density and $f(R, T)$ is an arbitrary function of the trace (T) of the energy momentum tensor T_{ab} and the Ricci scalar (R). The field equations of the theory are obtained in the standard way by the usual variation. Consider a perfect fluid given by (18), and matter Lagrangian density $L_m = -p$. In order to analyse whether we could obtain both Λ and G decreasing, we choose a relatively simple form of $f(R, T)$, viz., $f(R, T) = R + 2\eta T$, where η is a constant which indicates the departure from general relativity. For the above choices, the action (31) leads to the following field equations:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = (G + 2\eta)T_{ab} + \eta(\rho - p)g_{ab} \tag{32}$$

For $\eta = 0$, we obtain the field Equations (7).

Then, the FLRW metric (17) in conjunction with the field Equation (32) lead to the Raychaudhuri-type equation

$$-2\dot{H} - 3H^2 + \frac{k}{S^2} + \Lambda = -\eta G\rho + (1 + \eta)Gp \tag{33}$$

and the Friedmann equation

$$-3H^2 + \Lambda = -(1 + 3\eta)G\rho + \eta Gp \tag{34}$$

We note that for $\eta = 0$, we obtain the Equation (7). By differentiating Equation (34) and substituting into Equation (33), we are able to derive an energy conservation type equation in $f(R, T)$ gravity

$$\dot{\rho} + 3H(\rho + p) = -\left(\frac{\dot{G}}{G}\right) - \frac{\dot{\Lambda}}{G} - \frac{3\eta}{G}\dot{\rho} - \frac{6\eta}{G}\rho H - \frac{6\eta}{G}pH + \frac{\eta}{G}\dot{p}. \tag{35}$$

Again, for $\eta = 0$, we see that we obtain the same equation as Equation (11). To have the minimum departure from general relativity, we assume the usual energy conservation law by splitting the modified energy conservation law (35), as follows:

$$\dot{\rho} + 3H(\rho + p) = 0 \tag{36}$$

and the following equation, which includes the evolution of G and Λ:

$$-\left(\frac{\dot{G}}{G}\right)\rho - \frac{\dot{\Lambda}}{G} - \frac{3\eta}{G}\dot{\rho} - \frac{6\eta}{G}\rho H - \frac{6\eta}{G}pH + \frac{\eta}{G}\dot{p} = 0 \tag{37}$$

This equation can be written as

$$-\left(\frac{\dot{G}}{G}\right)\rho - \frac{\dot{\Lambda}}{G} - \frac{2\eta}{G}[\dot{\rho} + 3H(\rho + p)] - \frac{\eta}{G}\dot{\rho} + \frac{\eta}{G}\dot{p} = 0 \tag{38}$$

By Equation (36), we finally obtain

$$\dot{\Lambda} = -\dot{G}\rho - \eta\dot{\rho} + \eta\dot{p} \tag{39}$$

Hence, here we can have both G and Λ decreasing by choosing $\dot{G}\rho > -\eta\dot{\rho} + \eta\dot{p}$. In fact, we have the possibility of all nine combinations of decreasing Λ, constant Λ and increasing Λ with decreasing G, constant G, and even increasing G. Hence, we have demonstrated that in $f(R,T)$ theory of gravity, it is possible to have both G and Λ decreasing.

6. Results

We have analysed cosmological models in which the gravitational parameter G and the cosmological parameter Λ are allowed to be variable in the simplest formalism, as considered by many authors. In general relativity, one parameter has to be increasing and the other decreasing. The situation in several modified theories of gravity is shown to be more flexible, and we can have both parameters decreasing, which is preferred by observations.

We comment on splitting the full energy conservation law (35) into Equations (36) and (37). Firstly, we wish to regain the usual energy conservation law as in general relativity in the appropriate limit. Secondly, it enables us to obtain one more equation if we are seeking a full solution to the equations. Several authors have used this splitting, such as [65–67] and references therein.

7. Discussion

The idea of variable constants of nature is not new and goes back over a century. In this work, we focus on variable cosmological and gravitational parameters. We firstly consider the simplest extension of general relativity allowing for variable cosmological and gravitational parameters. Such a similar formalism can also arise in special forms of other theories of gravity such as the Brans–Dicke theory (or the scalar tensor theory based on it), the scale covariant theory and $f(R,T)$ gravity. In general relativity, one parameter can increase, but the other has to decrease. The preferred variation of these parameters is a decrease in both. We find that in the scale covariant theory and $f(R,T)$ gravity, we can have both parameters decreasing. Harko and Mak [68] also found a similar result with particle creation in general relativity with variables G and Λ. We also noted that a variation of fundamental constants of nature can solve the familiar Hubble tension problem. A variation of Λ provides a better fit to the data than does the standard ΛCDM model [69,70]. Sola et al. [71] carried out a very detailed analysis of several models with a running vacuum parameter (basically a varying Λ), and found a better fit to observations than the ΛCDM model. The transition redshift was found to be approximately the same as that of the ΛCDM model. Perico et al. [72] investigated a number of models in which the variation of Λ is expressed in a power series of the H. The proposed class of such models provides the following:

- A possible solution to the graceful exit problem;
- A new solution to inflation;
- The radiation and matter eras;
- At the present time, dark energy that evolves slowly;
- The present acceleration of the universe;
- A de Sitter stage as the end state.

The late-time acceleration is similar to the concordance ΛCDM model. Another pleasing feature of these models is that they allow for a smooth change between all the eras of the evolution, i.e., the initial inflation era, the radiation- and matter-dominated era, and final late-time accelerated expansion. As far as the future is concerned, better observations will be able to constrain the fundamental constants and their variation to better accuracy.

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