HAMILTON-JACOBI EQUATIONS IN $SU(2,2)$ HOMOGENEOUS SPACES [1]

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Abstract: A study of separation of variables for the Hamilton-Jacobi equation in an homogeneous space based in $SU(2,2)$ is presented. Some considerations are discussed about the potentials appearing when symmetry reduction to $O(2,2)$ is done.

1 Introduction

Let consider the group $SU(2,2)$ acting in the manifold $\mathbb{C}^4$ as matrix multiplication; it preserves the hermitian form $F(x,y) = -\overline{x}_{-1}y_{-1} - \overline{x}_0 y_0 + \overline{x}_1 y_1 + \overline{x}_2 y_2$. Taking the orbit $M = \{y \in \mathbb{C}^4 / F(y,y) = -1\}$, the group $U(1)$ acts freely on M ($y \in M \rightarrow e^{i\theta} y \in M$). Let $\mathcal{M}$ be the space of orbits in M under $U(1)$. The action of $SU(2,2)$ on M commutes with the $U(1)$-action, then it is possible to define an action of $SU(2,2)$ on $\mathcal{M}$. In this way $\mathcal{M} \approx SU(2,2)/SU(2,1) \times U(1)$.

The action of $SU(2,2)$ on $\mathcal{M}$ gives a representation for the infinitesimal generators of the group by the fundamental fields

$$X_a = \frac{d}{dt}(f(\exp(-ta)y))|_{t=0} + \text{complex conjugate}$$

with $a \in su(2,2)$, $f \in C^\infty(\mathbb{C}^4)$.

Introducing in $\mathcal{M}$ affine coordinates $z_\mu = y_\mu / y_{-1}$, $\mu = 0,1,2$ the metric in $\mathcal{M}$ is given now by

$$ds^2 = \frac{(1 - z^\mu \overline{z}_\mu)(dz^\nu d\overline{z}_\nu) + (z^\mu d\overline{z}_\mu)(\overline{z}^\nu dz_\nu)}{(1 - z^\mu \overline{z}_\mu)^2}$$

With $z^\mu \overline{z}_\mu = -|z_o|^2 + |z_1|^2 + |z_2|^2$. This metric is a noncompact version ot the Fubini-Study metric [1].

The hamiltonian defined by $H = g^{ij}\overline{p}_i p_j$ take in our case the following form:

$$\begin{aligned} H \;=\; & -(1 + |z^0|^2 - |z^1|^2 - |z^2|^2) \\ & \{(|z^0|^2 + 1)|p_0|^2 + (|z^1|^2 - 1)|p_1|^2 + (|z^2|^2 - 1)|p^2|^2 + \overline{z}^0 z^1 \overline{p}_0 p_1 \\ & + z^0 \overline{z}^1 p_0 \overline{p}_1 + \overline{z}^0 z^2 \overline{p}_0 p_2 + z^0 \overline{z}^2 p_0 \overline{p}_2 + \overline{z}^1 z^2 \overline{p}_1 p_2 + z^1 \overline{z}^2 p_1 \overline{p}_2\} \end{aligned}$$

2 Separable systems

As is well known, separable systems of coordinates in homogeneous spaces can be associated with complete sets of commuting second order operators living in the enveloping algebra of the Lie algebra of the corresponding symmetry group of the equation. Because

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of this relation, a complete classification of the maximal abelian subalgebras (MASA's) of the symmetry Lie algebra is necessary.

As we have shown in a recent paper [2], $su(2,2)$ has 12 MASA's. One of them is four-dimensional and the others three-dimensional. Here we will consider systems involving three ignorable variables (i.e. variables not appearing in the metric). These variables are associated to a MASA, in the following way: if $\{Y_1, Y_2, Y_3\}$ is a basis for a MASA (written as first order differential operators) the equations $Y_i = \partial_{x_i}$, $i = 1,2,3$ allow to compute the ignorable variable x_i, $i = 1,2,3$. Note that, even if there exists a four-dimensional MASA, it gives rise only to two ignorable variables.

Making use of the ignorable variables x_i, $i = 1,2,3$ we reduce the free hamiltonian in $\mathcal{M}$ to a new hamiltonian in a real $O(2,2)$ hyperboloid with a potential term. The work of finding the separable coordinate systems under $O(2,2)$ has been done by Kalnins and Miller[3]. There are at least six independent second order operators in the enveloping algebra of $su(2,2)$ commuting with the Hamiltonian and the generators of the fixed MASA. They provide a set of conserved quantities. Two of them can be chosen in involution with the hamiltonian and the squares of the generators of the MASA, determining a separable system and providing the complete integrability of the system, even more, superintegrability is also assured [4].

3 Explicit computation of the ignorable variables

Let us write the hamiltonian in the form:

$$H = cg^{\mu\nu} p_\mu \overline{p}_\nu, \qquad c \in \mathbb{R}$$

or, defining G and P_μ in a convenient way:

$$H = \frac{c}{2} P^T G P$$

$$G = \begin{bmatrix} 0 & g \\ g & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} p \\ \overline{p} \end{bmatrix}$$

Then, the following result holds:

The group generated by the MASA $\{Y_1, Y_2, Y_3\}$ and $Y_0 = iI$ with real parameters x_k can be used to construct a new system of real coordinates, $\{x_k, s_\mu,\ k = 0,1,2,3,\ \mu = -1,0,1,2\}$,such that, x_k are the ignorable variables, and s_μ belong to the real $O(2,2)$ hyperboloid: $s_{-1}^2 + s_1^2 - s_1^2 - s_2^2 = 1$.

If $B(x) = \exp(x_k Y_k)$, then the change of coordinates is given by:

$$y(x,s) = B(x)s$$

The hamiltonian can be written in these coordinates, and the result is:

$$H = \frac{c}{4} q_s^T g q_s + \frac{c}{4} q_x^T A^{-1} g (A^{-1})^+ q_x$$

That is, when we restrict ourselves to the real hyperboloid, we get the free hamiltonian in $O(2,2)$ plus a potential term depending of the specific choice of the MASA. The matrix A is part of the jacobian associated to this change of coordinates and can be easily computed. The quantities q are the conjugated momenta.

4 The compact Cartan subalgebra

In order to show how the method works, we present a detailed case, the corresponding to the compact Cartan subalgebra. This subalgebra is generated by the following 4×4 matrices of $su(2,2)$:

$$X_1 = \mathrm{diag}(i, -i, 0, 0), \quad X_2 = \mathrm{diag}(0, i, -i, 0), \quad X_3 = \mathrm{diag}(0, 0, i, -i)$$

The explicit form of the coordinate system is:

$$\begin{aligned} y_{-1} &= s_{-1} e^{i(x_0 + \frac{1}{2}x_1 - 2x_3)} \\ y_0 &= s_0 e^{i(x_0 - \frac{1}{2}x_1 - 2x_2)} \\ y_1 &= s_1 e^{i(x_0 - \frac{1}{2}x_1 + 2x_2)} \\ y_2 &= s_2 e^{i(x_0 + \frac{1}{2}x_1 + 2x_3)} \end{aligned}$$

The hamiltonian will be:

$$\begin{aligned} H &= \frac{c}{4}\{-q^2_{s_{-1}} - q^2_{s_0} + q^2_{s_1} + q^2_{s_2}\} + \\ & \frac{c}{64}\{-\frac{(q_0 - q_3 + 2q_1)^2}{s^2_{-1}} - \frac{(q_0 - q_2 - 2q_1)^2}{s_0^2} + \\ & \frac{(q_0 + q_2 - 2q_1)^2}{s_1^2} + \frac{(q_0 + q_3 + 2q_1)^2}{s_2^2}\} \end{aligned}$$

The separable coordinate systems in this case, are associated to six second order operators in involution, one of them the hamiltonian H. Three of them are the squares of the generators of the compact Cartan subalgebra, associated to the ignorable variables. The two remaining operators can be written as: $T = \sum_{i,j} a_{ij}\{X_i, X_j\}$, with $a_{ij} = a_{ji}$ and $i, j = 1, \ldots 15$, (X_i form a basis of $su(2,2)$) that is, T belongs to the enveloping algebra of $su(2,2)$. There are six independent operators of this kind:

$$T_1 = X_4^2 + X_5^2, \quad T_2 = X_6^2 + X_7^2, \quad T_3 = X_8^2 + X_9^2,$$
$$T_4 = X_{10}^2 + X_{11}^2, \quad T_5 = X_{12}^2 + X_{13}^2 \quad T_6 = X_{14}^2 + X_{15}^2$$

The commutation table for the operators T_i is the following:

[,]	T_1	T_2	T_3	T_4	T_5	T_6
T_1		A	B	$-A$	$-B$	0
T_2			C	$-A$	0	C
T_3				0	$-B$	$-C$
T_4					D	D
T_5						$-D$
T_6						

where A, B, C, D are linearly independent third order operators. We can choose for instance Q_3 and Q_4, given by $Q_3 = T_3 - (X_1 + X_2 + X_3)^2$ and $Q_4 = T_4 - X_2^2$ which are the casimirs of two commuting $su(1,1)$ subalgebras of $su(2,2)$ (subgroup type coordinates). Thus, we have a set of six second order operators in involution.

Now, in order to reduce the problem to $O(2,2)$ we put $x_0 = x_1 = x_2 = x_3 = 0$ and obtain $H = \frac{c}{4}\{-q^2_{s_{-1}} - q^2_{s_0} + q^2_{s_1} + q^2_{s_2}\} + V(s, q_x)$, $Q_3 = X_8^2$,$Q_4 = X_{10}^2$ because $X_1 = X_2 = X_3 = X_5 = X_7 = X_9 = X_{11} = X_{13} = X_{15} = 0$. The non-vanishing generators of $su(2,2)$ are six and are a basis for an $o(2,2)$ algebra:

$$
\begin{aligned}
X_4 = I_{-1,0}, \quad X_6 = I_{-1,1}, \quad X_8 = I_{-1,2}, \\
X_{10} = I_{0,1}, \quad X_{12} = I_{0,2} \quad X_{14} = I_{1,2}
\end{aligned}
$$

and then $Q_3 = I^2_{-1,1}$ and $Q_4 = I^2_{0,1}$. The operators $I_{-1,1}$ and $I_{0,1}$ generate a MASA of $o(2,2)$. Thus they are related with two ignorable variables. According to the work of Kalnins and Miller[3] the corresponding $O(2,2)$ coordinate system is:

$$
\begin{aligned}
s_{-1} &= \cosh u_1 \cosh u_2 \\
s_0 &= \sinh u_1 \sinh u_3 \\
s_1 &= \sinh u_1 \cosh u_3 \\
s_2 &= \cosh u_1 \sinh u_2
\end{aligned}
$$

Finally, the coordinate system for $SU(2,2)$ in affine coordinates is:

$$
\begin{aligned}
z_0 &= \tanh u_1 \frac{\sinh u_3}{\cosh u_2} e^{-i(x_1+2x_2-2x_3)} \\
z_1 &= \tanh u_1 \frac{\cosh u_3}{\cosh u_2} e^{-i(x_1-2x_2-2x_3)} \\
z_2 &= \tanh u_2 e^{4ix_3}
\end{aligned}
$$

5 Conclusions

This communication is part of a more complete study of the problem of separation of variables for the Hamilton-Jacobi equation in the space $SU(2,2)/SU(2,1) \times U(1)$. It should be viewed in the context of different research programs, the classification of maximal abelian subalgebras of the classical Lie algebras[2], the problem of separation of variables [3] and the construction of integrable and superintegrable hamiltonians [4]. The complete results will appear in a forthcoming paper.

References

[1] Kobayashi, S., Nomizu, K. *Foundations of Differential Geometry*, Interscience, New York, (1969).

[2] del Olmo, M. A., Rodríguez, M.A., Winternitz, P. and Zassenhaus, H. *Maximal Abelian Subalgebras of Pseudounitary Lie Algebras*, Linear Algebra and Applications, (1990).

[3] Kalnins, E.G., Miller, W.,Proc.Roy.Soc. Edinburgh, **79A**, 227, (1977)

[4] Evans,N.W. *Super-Integrability in Classical Mechanics*, Phys. Rev, (1990)