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Two-body corrections to the g factors of the bound muon and nucleus in light muonic atoms

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Abstract. A nonrelativistic (NR) theory of recoil corrections to the magnetic moments of bound particles is revisited. A number of contributions can be described within an NR theory with the help of various potentials. We study those potential-type contributions for two-body atomic systems. We have developed an approach, that allows us to find the g factor for an electron or muon in a two-body bound system for an arbitrary electrostatic interaction together with the m/M recoil corrections, as well as the binding corrections to the g factor of the nucleus. We focus our attention on light muonic two-body atoms, where the recoil effects are enhanced. Both mentioned kinds of contributions have been previously known only for the pure Coulomb effects. We have applied the here-obtained master equations to a few particular cases of perturbations of the Coulomb potential. In particular, the results on the recoil corrections to the finite-nuclear-size (FNS) and Uehling-potential contributions to the g factor of the bound muon are obtained. The Uehling-potential and FNS contributions to the g factor of the bound nucleus have been found as well together with the related recoil corrections. We have generalized the results for the case of the g factor of a bound muon in a three-body atomic system consisting of an electron, a muon, and a spinless nucleus.

1 Introduction

The actual values of the g factors of the bound particles differ from their free values due to the binding effects. The leading binding correction to the g factor of the light orbiting particle in a hydrogen-like atom is a relativistic one [1]. The leading binding correction to the nuclear magnetic moment is recognized as the electron shielding (screening) correction.

Magnetic properties of various atomic systems, including the exotic ones, such as the magnetic moments or the g factors of their constituents, were studied in a variety of experiments. In particular, there are a number of experimental results on the bound magnetic moment of the [negative] muon available [2–7]. In many experiments the muonic atoms contain a number of particles: a nucleus, a muon at a low orbit, and a number of electrons. There are two types of effects which affect the muon g factor. One of them is due to a two-body system of a muon, sitting at a low orbit close to the nucleus, and the nucleus itself. The effects of binding a muon to a nucleus are of a clear two-body nature. The other group of effects is due to the electron(s) screening of the muon magnetic moment and involves more particles.

The compact muon–nucleus two-body system plays a role of a compound nucleus for the electron(s), which weakly penetrates it. The simplest muonic atom where the electron screening takes place is a system of an electron, a muon, and a nucleus. The most simple of them are the ones with a spinless nucleus. In such a case the magnetic moment and spin of the compound nucleus is the muon one and the screening of the magnetic moment of the compound nucleus is a two-body problem of an electron and a compound nucleus. Such a hierarchy technique for the study of neutral muonic helium has been successfully applied to the hyperfine structure [8–12] and the Lamb shift [13,14] of the electron energy levels.

In this paper we study both two-body and three-body muonic atoms. The theory of the bound g factors has a long history and involves many corrections to the g factors. The leading bound contributions are the Coulomb ones. They can be found, considering particles bound by the Coulomb potential in the presence of magnetic field. We can modify the potential adding various corrections to the Coulomb field. Here we study "potential-type" contributions, i.e., the contributions induced by those additions to the Coulomb potential. An interaction of two bound particles can be described by a potential in two cases. One of them is a relativistic case for a light particle within the external field approximation, while the other is a potential

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at the nonrelativistic approximation for two particles, which can be described by a one-body Schrödinger equation with the reduced mass.

We rely on a nonrelativistic expansion and consider the latter case. In the case of a potential-type contribution it may be possible to express the corrections to the g factor in terms of the corrections to the energy. In particular, relatively recently two relations between the contributions to the g factor of a bound muon (electron) and the energy were discovered for states of a two-body atom bound by an arbitrary electrostatic central potential [15,16].

We are to generalize those expressions on the recoil effects and consequently to obtain the results for two classes of the contributions, which have not been studied. One is for recoil corrections to the g factor of a muon, bound by an arbitrary potential, and the other is for the contributions to the g factor of the nucleus, both non-recoil and recoil, also bound with an arbitrary potential. The results are obtained below in a closed analytic form.

2 The g factors in a two-body muonic atom

The theory of the g factor of the bound muon and bound nucleus in a light two-body muonic atom is somewhat different from that for ordinary [electronic] atoms. One of the differences is the importance of the recoil corrections since $m_\mu/m_p \simeq 1/9$. The effects of the electronic vacuum polarization and the nuclear finite size (FNS) are enhanced, which makes another important difference.

The leading contributions in the first order for all these three effects (namely, due to the recoil, FNS, and electronic vacuum polarization (*aka* Uehling potential)) have been known for some time.

The leading recoil effects in the case of a pure Coulomb potential have been studied in [17–20]. The result for the g factor of a muon (or an electron) and a nucleus up to the third order in a combination of three small parameters $(\alpha, Z\alpha, m/M)$ is

$$g_\mu(ns) = g_\mu^{(0)} \left\{ 1 - \frac{1}{3} \left[\left(1 - \frac{3}{2}\frac{m}{M} \right) \right. \right.$$
$$\left. \left. - a_\mu \left(\frac{3}{2} - \frac{5}{2}\frac{m}{M} \right) \right] \frac{(Z\alpha)^2}{n^2} \right\}, \quad (1)$$

$$g_N(ns) = g_N^{(0)} \left\{ 1 - \frac{1}{3} \left[\left(1 - \frac{2m}{M} \right) \right. \right.$$
$$\left. \left. + \frac{1}{2(1+\kappa_N)}\frac{m}{M} \right] \frac{\alpha(Z\alpha)}{n^2} \right\}, \quad (2)$$

where we neglect the $(m/M)^2$ terms (see [21–23] for details). Note, $g_\mu^{(0)} = 2(1 + a_\mu)$ and $g_N^{(0)} = 2(1 + \kappa_N)$; a_μ, κ_N are the anomalous magnetic moments of the muon and nucleus; m is the mass of the orbiting particle, which is a muon in the muonic atoms, and M stands for the nuclear mass.

The leading FNS correction to the g factor of a bound muon (in a muonic atom) (or a bound electron in an ordinary hydrogen-like atom) in the leading non-relativistic

approximation is known in a closed form [24]

$$\Delta g_\mu(ns) = \frac{8}{3n^3} (Z\alpha)^4 m^2 R_N^2, \quad (3)$$

where R_N is the rms nuclear charge radius. The Uehling-potential correction (i.e., the correction due to the electronic vacuum polarization) to the bound muon g factor in two-body muonic atoms is known in closed analytic form [25,26] as well (see below).

An important group of the corrections to the bound g factors can be considered by using various potentials. That is not a general situation. For example, the muon (electron) self-energy contribution cannot be described in this way (see, e.g., [27]). The non-potential-type contributions will be considered elsewhere [28].

Still, the FNS effects as well as the effects of the vacuum polarization, can be successfully considered as the potential-type ones. We are to consider the recoil effects especially for muonic atoms, where the contributions of those types are of special importance, and so we take the advantage of the "potential" approach to the description of the corrections. Within such an approach some functional relations are simplified. In particular, the results above for the leading FNS and the leading Uehling contributions (in the non-recoil limit) could be easily derived using a relation [15] between the contributions to the bound muon (electron) g factor and to the energy of a hydrogen-like bound state of interest, which reads

$$\frac{\Delta g_i(ns)}{2} = \frac{2}{3}\frac{\partial E_i(ns)}{\partial m}, \quad (4)$$

where subscript i "marks" the effects of interest (FNS, Uehling correction, etc.). The relation is valid for an electron or muon bound by an arbitrary local electrostatic potential. We remind that here and throughout the paper m is the mass of the orbiting particle and never a reduced mass.

The relation is a relativistic one, but its use in a non-relativistic approximation allows us to simplify the required expressions for the energy. The Uehling correction to the energy in a muonic atom has been known in the close analytic form both in the non-relativistic [29–35] (see below) and relativistic case [34,35].

The mentioned relativistic relation (4) was obtained in the external-field approximation [15], while here we study the recoil corrections to the leading FNS and Uehling terms for the bound muon. We also derive the related corrections to the g factor of a bound nucleus in a two-body muonic atom. For the derivation of the appropriate nonrelativistic relations between the contributions to the g factor and to the energy we note that very often the leading correction due to a certain kind of effects can be presented in terms of the average of a very few operators over the wave functions. In particular, all the corrections of interest (including the recoil effects) can be expressed for the ns bound states in terms of one average, namely, $\langle \mathbf{p}^2 \rangle$, as it has been achieved in [19,20].

The value of $\langle \mathbf{p}^2 \rangle$ is an average over the wave function of a state of interest, while the wave function is a solution of a one-body equation with the reduced mass and an

arbitrary central electrostatic potential, which is usually the Coulomb one affected by corrections of various kinds. The advantage of the nonrelativistic approximation for the recoil corrections to the wave functions is that they are covered by the introduction of the reduced mass m_r.

The derivations of the pure Coulomb's relations in (1) and (2) are lengthy and cumbersome [17–20]. The essence of the derivation is a disentanglement of a motion of the center of mass of the two-body system and its relative motion. A consideration of pure Coulomb field offers certain simplifications and a compatible derivation with an arbitrary potential would be somewhat more complicated. Two former derivations [19,20] dealt in part with arbitrary potential, but the detailed description is not present there. Nevertheless, the agreement of intermediate formulas in [19,20], allows us to consider their intermediate results as well established and utilize them as a starting point of our derivation. Since the separation of two motions has already been done [19,20], we consider in this paper only relative motion of bound particles in the rest frame of the two-body atom.

The related expressions for the bound g factors of the orbiting particle (a muon) and the nucleus in an ns state with the required accuracy are of the form [19] (cf. [20])

$$g_\mu(\text{bound}) = g_\mu^{(0)} \left\{ 1 - \frac{1}{3} \left[1 - \frac{3a_\mu}{2(1+a_\mu)} \right. \right.$$
$$\left. \left. + \frac{1}{2(1+a_\mu)} \frac{m}{M} \right] \frac{\langle \mathbf{p}^2 \rangle}{m^2} \right\}, \quad (5)$$

$$g_N(\text{bound}) = g_N^{(0)} \left\{ 1 - \frac{1}{3Z} \right.$$
$$\left. \times \left[1 + \frac{1}{2(1+\kappa_N)} \frac{m}{M} \right] \frac{\langle \mathbf{p}^2 \rangle}{m^2} \right\}. \quad (6)$$

We emphasize that the equations have been applied in [19] for Coulomb-bound systems, but derived there for a more general case. They are valid for an arbitrary central electrostatic interaction and we take advantage of that to generalize the results previously obtained in the non-recoil limit (see, e.g., (4)). The result is consistent with an expression for the operators of the magnetic moment found in [20].

To apply expressions (5) and (6) we intend to use the non-relativistic relations discussed in [16] in the context of the bound g factor problem, which, in particular, has allowed us to find a non-relativistic version of (4). The derivation in [16] is valid for an electron or a muon, bound by an arbitrary local electrostatic potential $V(r)$, i.e., in the external-field approximation. However, it may be used for an NR two-body system bound by an electrostatic interaction. The required relation from [16] now takes the form

$$\langle \mathbf{p}^2 \rangle = -2m_r^2 \frac{\partial E}{\partial m_r} \simeq -2m^2 \left(1 - \frac{2m}{M} \right) \frac{\partial E}{\partial m_r}, \quad (7)$$

where energy E is the eigenvalue of the nonrelativistic Hamiltonian with reduced mass m_r and we neglect here the $(m/M)^2$ terms. The potential $V(r)$ is assumed to be mass independent, which is valid for a number of actual

perturbations of the Coulomb potential. The relation is somewhat similar to the one related to the virial theorem. Combining (5), (6), and (7), we immediately arrive at

$$g_\mu(\text{bound}) = g_\mu^{(0)} \left\{ 1 + \frac{2}{3} \left[1 - \frac{3a_\mu}{2(1+a_\mu)} \right. \right. \quad (8)$$
$$\left. \left. + \frac{1}{2(1+a_\mu)} \frac{m}{M} \right] \left(1 - \frac{2m}{M} \right) \frac{\partial E}{\partial m_r} \right\},$$

$$g_N(\text{bound}) = g_N^{(0)} \left\{ 1 + \frac{2}{3Z} \right.$$
$$\left. \times \left[1 + \frac{1}{2(1+\kappa_N)} \frac{m}{M} \right] \left(1 - \frac{2m}{M} \right) \frac{\partial E}{\partial m_r} \right\}. \quad (9)$$

Those two identities express the g factors of the bound spin particles in the s state in terms of the energy of the state of interest. They are valid for the effects related to the potentials for the effective one-body non-relativistic equation with the reduced mass and a central electrostatic potential. For the two-body atoms the leading binding potential is indeed the Coulomb one (for a point-like nucleus), which is perturbed by various effects.

The equations above are the main result of this paper. They are the master equations, which allow us to take advantage of the current theoretical situation when the theory of the energy levels is in general more advanced than the theory of the bound g factor. In a number of cases we know a contribution to the energy, while the related contribution to the g factor is unknown. In particular, the derived master equations cover two classes of new corrections. The upper equation (8) produces the results for the recoil corrections with an arbitrary effective potential, while the bottom one (9) delivers the results for an arbitrary potential for the non-recoil limit as well as for the recoil corrections. The original results for the specific corrections are summarized below for the g factors of a bound muon and bound nucleus in a two-body atoms and generalized in Section 4 for the case of the g factor of a muon in a three-body system (electron–muon–nucleus).

Prior to considering various specific contributions, let's mention that equation (8) reproduces two similar relations for the g factor of a bound muon, namely, the nonrelativistic limit of equation (4) previously obtained in [15] and the result for the contribution of the anomalous magnetic moment in the non-recoil limit for an arbitrary potential derived in [16]. None of the recoil contributions have been considered previously for an arbitrary potential. Neither the relations to the energy were derived for the g factor of a bound nucleus.

The master equation (8) for the arbitrary potential in the leading non-relativistic approximation reproduces the pure Coulomb result (see (1)) both for a particle with a pure Dirac magnetic moment and for a particle with a non-zero anomalous magnetic moment. Since the relation in (8) reproduces the known relations [15,16] for the non-recoil limit, it apparently reproduces the known external field approximation results for the Uehling contribution [25,26] (see below) and the leading FNS contribution [24] (see (3)) in the leading non-relativistic approximation.

However, the recoil corrections to them have not been known and we consider them below.

3 Recoil corrections to the g factors in a two-body muonic atom

Let's start with the recoil correction to the FNS contribution in a hydrogen-like muonic atom. The leading FNS contribution to the energy is well known as

$$\Delta E_{\text{FNS}}(ns) = \frac{2}{3n^3}(Z\alpha)^4 m_r^3 R_N^2, \tag{10}$$

which immediately allows us to find the FNS-recoil correction to the g factors of the bound muon and the bound nucleus in a two-body muonic atom

$$\Delta g_\mu(ns) = 4\left[1 - \frac{3m}{2M}\right]\frac{\Delta E_{\text{FNS}}(ns)}{m_r}, \tag{11}$$

where the non-recoil term in the theory of the g factor of a muon (see (3)) has been previously known [24]. (We neglect the radiative contributions to the FNS term in $g_\mu(ns)$ [16].) A part of the recoil correction is contained in $\Delta E_{\text{fns}}(ns)/m_r$, since the recoil effects are already incorporated in the leading FNS contribution (10) by using the reduced mass. The other recoil FNS corrections to the energy are of higher order in $Z\alpha$ or in $Z\alpha m R_N$ (see, e.g., [16]). (The recoil FNS contribution to the g factor of a bound muon of order $(Z\alpha)^4(mR_N)^2$ was previously studied in [36]. Our result disagrees with that of [36].)

The results for the Uehling correction in muonic atoms are more complicated. The correction to the energy is of the form

$$\Delta E_{\text{Ueh}}(ns) = \frac{\alpha}{\pi}\frac{(Z\alpha)^2 m_r}{n^2} \times F_{ns}^U\left(\frac{Z\alpha m_r}{nm_e}\right), \tag{12}$$

where in particular [29,30]

$$F_{1s}^U(y) = -\frac{1}{3}\left[\frac{2y^4 - y^2 - 4}{y^3}\frac{\ln\left(y + \sqrt{y^2 - 1}\right)}{\sqrt{y^2 - 1}} + \pi\frac{3y^2 + 4}{2y^3} - \frac{11y^2 + 12}{3y^2}\right]. \tag{13}$$

The function F^U for the other states can be found in [29, 30,32–34]. Similar to the case of the FNS contribution, the leading recoil effects are already included in (12) as long as we use the reduced mass appropriately.

Applying relation (8) we find for the Uehling-recoil correction to the g factors of the bound muon (cf. [25,26])

$$\Delta g_\mu(ns) = \frac{4}{3}\frac{\alpha}{\pi}\frac{(Z\alpha)^2}{n^2}\left[1 - \frac{3m}{2M}\right]G_{ns}(Z\alpha m_r/m_e), \tag{14}$$

where

$$G_{ns}(y) = \left[F_{ns}^U(y) + y\frac{\partial}{\partial y}F_{ns}^U(y)\right],$$

and the radiative corrections to the Uehling term. The explicit result for a muon in the ground state reads

$$G_{1s}(y) = -\frac{1}{3}\left[\frac{2y^6 - 3y^4 + 12y^2 - 8}{y^3(y^2 - 1)}\frac{\ln\left(y + \sqrt{y^2 - 1}\right)}{\sqrt{y^2 - 1}} - \frac{4\pi}{y^3} - \frac{5y^4 - 20y^2 + 24}{3y^2(y^2 - 1)}\right]. \tag{15}$$

Let's now consider the contributions to the g factor of a bound nucleus based on equation (9). The theory of the g factor of a bound nucleus is less advanced than that of a bound muon (electron). The master equation (9) allows us to reproduce the known results for the Coulomb potential (see (1)). A number of corrections to the g factor of a bound muon (electron) are known in the non-recoil limit, however none of the related corrections are known for a bound nucleus. In the above we have considered the leading FNS contribution and the Uehling contributions to the g factor of the bound muon. Based on equation (9) we generalize the results on the case for the nucleus g factor.

The leading FNS contribution for the g factor of a bound nucleus is found to be (cf. (11))

$$\Delta g_N(ns) = \frac{2}{Z}g_N^{(0)}\left[1 - \left(2 - \frac{1}{2(1 + \kappa_N)}\right)\frac{m}{M}\right] \times \frac{\Delta E_{\text{fns}}(ns)}{m_r}, \tag{16}$$

where $g_N^{(0)}$ is the value of the free nucleus g factor and $\Delta E_{\text{fns}}(ns)$ is defined in (10). The result for the Uehling correction, following (9), reads (cf. (14))

$$\Delta g_N(ns) = \frac{2}{3}\frac{\alpha}{\pi}\frac{\alpha(Z\alpha)}{n^2}g_N^{(0)}\left[1 - \left(2 - \frac{1}{2(1 + \kappa_N)}\right)\frac{m}{M}\right] \times G_{ns}(Z\alpha m_r/m_e). \tag{17}$$

4 Results for a three-body muonic atom

Often muonic atoms also contain some electrons and the muonic magnetic moment is shielded by them. Actually, the only existing results on the g factor of a bound muon [2–7] are from the study of such atoms. In the muonic atoms with electrons the muon is usually at its ground state. The muon–nucleus bound system is a compact one and weakly affected by the electrons. The results for two-body muonic atoms, obtained above, could be used in calculations for the compact muon–nucleus system, which plays the role of a compound nucleus for the electrons. To study the electron shielding correction we should know the properties of that compound nucleus.

We discuss the simplest muonic atom where the effects of electronic screening are presented. That is a three-body atom consisting of an electron, a muon, and a spinless nucleus. Such a simple exotic system (namely, the muonic helium) was actually produced [37] and the related magnetic moments were measured [38], but with a rather low accuracy. We investigate such systems because of their

theoretical simplicity rather than because of the existing experimental data. The theoretical result should give a clear understanding of the order of magnitude of contributions of different effects, which may help to develop a theory for more complicated muonic atoms.

At the first stage we consider a two-body muonic ion and deduce the magnetic moment of a bound muon, while at the second stage we are to consider a hydrogen-like electronic atom with a compound nucleus in order to obtain a value of its nuclear magnetic moment (cf. [11–13]). The situation is greatly different from ordinary atoms with two electrons, which being identical particles require a different approach. For the sake of simplicity we consider the muon in the ground state (which is the most realistic scenario for experimental observations) and therefore the spin of the compound nucleus is 1/2. Collecting all the corrections to the g factor of a bound muon at the ground state up to the third order of magnitude in either combination of α, $Z\alpha$, and m/M, discussed above, we find the g factor of a muon in the two-body subsystem

$$g_\mu^{(2)} = g_\mu^{(0)} \times \left\{ 1 - \frac{(Z\alpha)^2}{3} \left[1 - \frac{3}{2}\frac{m_\mu}{M} + \frac{3(Z+1)}{2} \left(\frac{m_\mu}{M} \right)^2 \right. \right.$$
$$\left. - \frac{3}{2}\frac{\alpha}{2\pi} \left(1 - \frac{5}{3}\frac{m_\mu}{M} \right) - \frac{2\alpha}{\pi} \left(1 - \frac{3}{2}\frac{m_\mu}{M} \right) \right.$$
$$\left. \left. \times G_{1s} \left(\frac{Z\alpha m_r}{m_e} \right) \right] \right\}. \qquad (18)$$

The magnetic moment of the compound nucleus is essentially the muon one and it differs considerably from the related Dirac value for a "nuclear" particle with charge $+(Z-1)e$ and mass $M + m_\mu$. An important nuclear characteristic for the calculation of various corrections is the anomalous magnetic moment of such a compound nucleus η, which is defined by the relation

$$(1 + \eta)\frac{(Z-1)\,e}{M + m_\mu} = -(1 + a_\mu + b_\mu)\frac{e}{m_\mu},$$

where b_μ is the correction due to the muon–nucleus binding effects. Various effects, discussed above, contribute to b_μ. At low Z the bound term ($b_\mu \simeq -(Z\alpha)^2/3$) is much smaller than the free one ($a_\mu \simeq \alpha/2\pi$), but for higher Z they may be comparable.

In contrast to a muon, with a small value of the anomalous magnetic moment ($\sim 10^{-3}$), and a proton, which has a value comparable with unity ($\kappa_p \simeq 2.9$), the anomalous magnetic moment of the compound nucleus has a value which is parametrically enhanced ($\sim -M/(Zm_\mu)$). The equation for the g factor of the [compound] nucleus (see, e.g., (2) and (9)) deals with the factor

$$\frac{1}{1 + \eta} \simeq -\frac{(Z-1)\,m_\mu}{M + m_\mu}(1 - a_\mu - b_\mu), \qquad (19)$$

where we neglect the higher order effects since this factor is involved only in the calculation of small corrections.

We consider a three-body system, electron–muon–nucleus, with a muon in the ground state ($1s$) and an electron in the ns state. The electron shielding in a two body

with an orbiting electron and the nucleus with charge $Z - 1$, mass $M + m_\mu$ and anomalous magnetic moment η (as found in (19)) is presented by the factor (cf. (2))

$$\frac{g_\mu^{(3)}}{g_\mu^{(2)}} = 1 - \frac{(Z-1)\alpha^2}{3n^2} \left[1 - \frac{2m_e}{M + m_\mu} - \frac{Z-1}{2}\frac{m_e m_\mu}{M^2} \right]. \qquad (20)$$

The presented corrections are up to the third order in all the parameters ($\alpha, Z\alpha, m_e/m_\mu, m_e/M$), but m_μ/M, for which we give an additional term of the expansion. The last term in the factor in (20) (with $Z - 1$) is the contribution due to the anomalous magnetic moment of the compound nucleus (cf. (19)). Expressions (18) and (20) include only the potential-type contributions. For the non-potential-type ones, see [28].

The bound electron g factor in such an atomic system is covered in (1) by using the g factor of the orbiting particle (which is a muon in (1), but is an electron now) and with the nuclear charge $Z - 1$ instead of Z, etc.:

$$\frac{g_e(ns)}{g_e^{(0)}} = 1 - \frac{1}{3} \left[1 - \frac{3}{2}\frac{m_e}{M + m_\mu} - \frac{3a_e}{2} \right] \frac{((Z-1)\alpha)^2}{n^2}.$$

The g factor of a bound electron is known in principle with a higher accuracy than presented in (1) (see, e.g. [27]). Such a universal theory for the g factor of a bound electron in a hydrogen-like atom presents the most important generic contributions. The only additional enhanced contribution to the bound electron g factor in the electron–muon–nucleus bound system is due to the large nuclear size of the compound nucleus (see, e.g., [13,39]). The related contribution with the muon in the $1s$ state is (cf. (3))

$$\Delta g_e(ns) = -\frac{8(Z-1)^3\alpha^2}{Z^2 n^3} \left(\frac{m_e}{m_\mu} \right)^2, \qquad (21)$$

which is a correction of the fourth order. It is the electron in the ns state, while the muon is at its $1s$ state. A similar FNS correction exists also for the g factor of the muon

$$\Delta g_\mu(ns) = -\frac{8(Z-1)^2\alpha^2}{Z^2 n^3} \left(\frac{m_e}{m_\mu} \right)^2. \qquad (22)$$

Above we consider a system of the electron–muon–nucleus as a hierarchy system. First we consider the muon–nucleus system as a two-body one and next we consider the electron bound by a compound nucleus. Such a hierarchy approach was efficiently used for the description of the $e\mu$ hyperfine structure in neutral muonic helium ($e-\mu-$helium nucleus) [8–12] and the Lamb shift of the electronic levels [13,14]. If the muon is in a low state such a consideration is as good as a standard consideration of, say, the deuterium atom. The latter can be treated as a hydrogenlike atom with a compound $p-n$ nucleus or as a three-body $e-p-n$ system. The fact that the [compound] nucleus is bound by the strong interactions in deuterium and by the electromagnetic interactions in muonic systems is irrelevant. It is important that in muonic low states the muon is bound to the nucleus much stronger than the electron(s) because of the much shorter distances (by a factor

$\sim m_e/m_\mu = 1/207$). All the essentially three-body effects can be considered perturbatively (see, e.g., a discussion in [13]).

The corrections to the bound g factors due to the finite size of the compound nucleus in (21) and (22) are in fact the leading corrections due to the three-body effects. The electron "recognizes" that the compound nucleus consists of a muon and a nucleus. The effects of this type should also involve the polarizability of the compound nucleus, but they are of a higher order in m_e/m_μ than the FNS effects (see, e.g., [13,14]).

5 Conclusions

Concluding, we have revisited the nonrelativistic theory of the g factors of the two-body atomic systems with a muon. A special attention has been paid to the recoil corrections to the g factor of a bound muon, which are important in the case of muonic atoms. We have found the recoil corrections to the finite-nuclear-size contribution and to the Uehling contribution to the g factor of a bound muon in a hydrogen-like muonic atom.

The obtained recoil results are not relevant to ordinary [electronic] atoms, where the recoil effects are two orders of magnitude smaller than for muonic ones. For the not very low Z, the numerical value of the correction is of relative order 1×10^{-4} (comparing with the leading FNS term) and can be neglected comparing with its uncertainty. In the case of lighter nuclei, such as with the α particle, the FNS contribution is smaller and the recoil correction to it is even less important.

We have evaluated all the third-order potential-type terms. Some fourth-order potential-type contributions have also been considered both for two-body and three-body atomic systems, namely those in which one of the small parameters is m_μ/M, which is the most important parameter at low Z.

In the case of medium Z the largest value of the small parameters belongs to $Z\alpha$. The higher order $Z\alpha$ contributions require a relativistic consideration. The largest of those relativistic contributions is the $(Z\alpha)^4$ term for g_μ [1] and $\alpha(Z\alpha)^3$ term for g_N [40–43]. They are known in the external-field approximation. The most important two-body correction is of order $(Z\alpha)^4 m_\mu/M$ [42,44,45]. Those results, namely, the relativistic corrections to g_μ [1] and g_N [40–43] in the external field approximation and the m_μ/M recoil term [45], are known exactly in $Z\alpha$. The Uehling correction (cf. (14) [25,26]) and the FNS term (cf. (11)), presented [15] in terms of the FNS energy [46] (the leading order in FNS and exactly in $Z\alpha$ (see discussion in [16])), are also known exactly.

The results for some contributions (such as for the self-energy one) in the external-field approximation are of the same form for g_μ in a muonic hydrogen-like atom and g_e in an ordinary hydrogen-like atom. The state of the art for g_e is discussed in [27] (see also [28]).

In muonic atoms certain higher order corrections to the energy are specific (in comparison with ordinary atoms) (see, e.g., [47]). Those contributions to the energy could be converted to contributions to g_μ using our nonrelativistic relation (8), with the inclusion of the recoil effects, or a similar one (4) [15] for relativistic corrections.

The actual experiments have been performed on rather neutral atoms with a muon and several electrons. The experimental accuracy has reached the level of a few parts in 10^5. For this reason we have considered the simplest of the systems with electron screening of the bound muon magnetic moment, namely, a three-body muonic atom, which consists of an electron, a muon, and a spinless nucleus. The experimental results for such atoms [2–4] have reached accuracy at the level of a few parts in 10^5 at medium and high values of Z. We will consider practical numerical applications elsewhere [28] together with the calculation of the non-potential-type contributions.

Author contribution statement

The authors performed all the work together.

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