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ANALYSIS OF THE PANOFSKY-WENZEL THEOREM IN PILLBOX CAVITIES WITH A BEAM PIPE

L. M. Wroe, A. Latina, S. Stapnes, W. Wuensch, CERN, Geneva, Switzerland
R. J. Apsimon[1], M. Southerby[1], Lancaster University, Lancaster, United Kingdom
[1]also at Cockcroft Institute, Warrington, United Kingdom

Abstract

In this paper, we derive the multipolar form of the change in transverse and longitudinal momenta of an ultra-relativistic charged particle that traverses a harmonic TM_{mn0} mode in a pillbox cavity with a beam pipe. The relevant equations are first formalised before presenting results from the numerical integration of RF cavity field maps. In particular, we show that the radial dependence of the change in transverse and longitudinal momenta through a TM_{mn0} mode has polynomial, and not Bessel, dependence.

INTRODUCTION

In 1956, Panofsky and Wenzel derived their eponymous theorem stating that the change in transverse momentum $\Delta p_\perp$ of a rigid particle of charge e and ultra-relativistic velocity $\vec{v} = c\hat{z}$ which traverses an RF cavity supporting a mode l of resonant angular frequency ω_l and longitudinal electric field E_z is [1,2]

$$\Delta p_\perp(r,\theta) = -i\frac{e}{\omega_l}\int_{-L/2}^{L/2} \nabla_\perp E_z(\vec{r},t)dz. \quad (1)$$

Here L is a longitudinal length that is sufficiently longer than the RF cavity such that electromagnetic fields decay to a negligeble value, $E_z(r,\theta,|z|>L) \sim 0$, and the integral limits in Eq. 1 can be replaced with $L \to \infty$.

In the case of the ultra-relativistic particle having coordinate $z = 0$ at $t = 0$ such that $t = z/c$ and the harmonic RF cavity mode having phase ψ at $t = 0$ such that $E_z(\vec{r},t) = E_z(\vec{r})e^{i(\omega_l t+\psi)}$, Panofsky-Wenzel's Eq. 1 becomes

$$\Delta p_\perp(r,\theta) = -i\frac{e}{\omega_l}\int_{-\infty}^{\infty} e^{i(\omega_l z/c+\psi)}\nabla_\perp E_z(\vec{r})dz. \quad (2)$$

Thus the change in transverse momentum is simple to calculate if the longitudinal electric field is known completely.

A completely general representation of the spatial component of the longitudinal electric field for a standing-wave mode l in an RF cavity is derived in Ref. [3] as

$$E_z(\vec{r}) = \int_{-\infty}^{\infty} dk\frac{e^{ikz}}{\sqrt{2\pi}}\sum_{m=0}^{\infty}\tilde{g}_m(k)R_m(\kappa_l r)\cos(m\theta-\phi_m)$$
$$= \sum_{m=0}^{\infty} E_{z,m}(r,z)\cos(m\theta-\phi_m). \quad (3)$$

In this formalism, $\tilde{g}_m(k)$ has the same units as the electric field and represents the strength of the multipole of order m and ϕ_m is the orientation of the multipole of order m ($\phi_m = 0$ is normal and $\phi_m = \pi/2$ skew). Furthermore,

$$R_m(\kappa_l r) = \begin{cases} J_m(\kappa_l r), & k < k_l; \\ I_m(\kappa_l r), & \text{otherwise}, \end{cases} \quad (4)$$

where $\kappa_l^2 = |k^2 - k_l^2|$ and $k_l = \omega_l/c$ is the wavenumber of the mode.

In this representation, $\tilde{g}_m(k)$ has no dependence on radius. Therefore, if the longitudinal field E_z is completely known on the surface of a cylinder of radius a, $g_m(k)$ can be explicitly calculated as

$$\tilde{g}_m(k) = \frac{1}{R_m(\kappa_l a)}\int_{-\infty}^{\infty}\frac{dz}{\sqrt{2\pi}}e^{-ikz}E_{z,m}(a,z). \quad (5)$$

By solving Eq. 5, the longitudinal electric field is then also calculable at any point in the cavity using Eq. 3 and, from this, so too is the change in transverse momentum by Panofsky-Wenzel's Eq. 2 which becomes

$$\Delta p_\perp = -i\frac{e}{\omega_l}\int_{-\infty}^{\infty}\nabla_\perp\Big(\int_{-\infty}^{\infty}\frac{dk}{\sqrt{2\pi}}e^{ikz}e^{i(k_l z+\psi)}$$
$$\sum_{m=0}^{\infty}\tilde{g}_m(k)R_m(\kappa_l r)\cos(m\theta-\phi_m)\Big)dz. \quad (6)$$

From here, we present analytical and numerical results from field map integrations using Mathematica [4] to derive the form of the change in transverse momentum through TM_{mn0} modes in RF cavities.

PILLBOX CAVITIES WITH BEAM PIPES

Figure 1 shows a pillbox cavity with a beam pipe and the relevant parameters used for this analysis: G is the cell length, R the cavity radius, a the beam pipe radius, and L the field map length.

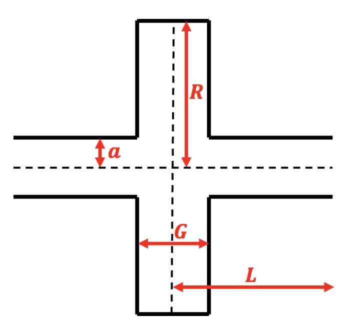

Figure 1: Cross-section of a pillbox cavity with beam pipe.

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The longitudinal electric field of the TM_{mnp} mode in the cavity shown in Fig. 1 is derived by assuming a fixed bore electric field of

$$E_z(a,\theta,z) = \begin{cases} E_G \cos(m\theta - \phi_m)\cos(k_p z), & |z| < G/2; \\ 0, & \text{otherwise}, \end{cases} \tag{7}$$

where $k_p = 2p\pi/G$.

Limiting this study to the normal $\phi_m = 0$, $p = 0$ modes and inserting Eq. 7 into Eq. 5 gives

$$\tilde{g}_m(k) = \frac{E_G}{R_m(\kappa_l a)} \int_{-G/2}^{G/2} \frac{dz}{\sqrt{2\pi}} e^{-ikz} = \frac{G E_G}{\sqrt{2\pi} R_m(\kappa_l a)} \frac{\sin(kG/2)}{kG/2}. \tag{8}$$

From Eq. 8, the longitudinal electric field throughout the cavity is determined using Eq. 3 as

$$E_z(\vec{r}) = \frac{G E_G}{\sqrt{2\pi} R_m(\kappa_l a)} \cos(m\theta) \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} \frac{\sin(kG/2)}{kG/2} R_m(\kappa_l r), \tag{9}$$

and, from here, the change in transverse momentum is calculable using the Panofsky-Wenzel theorem in Eq. 2 as

$$\Delta p_\perp(r,\theta) = -i \frac{e}{\omega_l} \frac{G E_G}{\sqrt{2\pi} R_m(\kappa_l a)} \cos(m\theta) \nabla_\perp \int_{-\infty}^{\infty} e^{i(k_l z + \psi)} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} e^{ikz} \frac{\sin(kG/2)}{kG/2} R_m(\kappa_l r) dz. \tag{10}$$

We also note explicitly that, under our assumptions, the change in longitudinal momentum is

$$\Delta p_z(r,\theta) = \frac{e}{c} \int_{-\infty}^{\infty} E_z(\vec{r},t) dz \rightarrow \Delta p_\perp = -i \frac{c}{\omega_l} \nabla_\perp p_z. \tag{11}$$

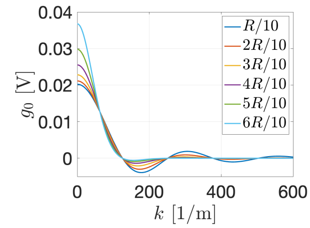
FIELD MAP INTEGRATIONS

Equations 8–10 are non-trivial to analyse directly[1] and so here we present an analysis of the 3 GHz TM_{010}, TM_{110}, and TM_{210} modes in pillbox cavities of length $G = c/2f \sim 50$ mm and with beam pipes of varying radii.

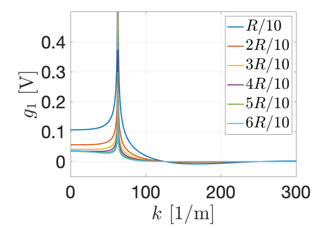
Figure 2 shows $\tilde{g}_0(k)$, $\tilde{g}_1(k)$, and $\tilde{g}_2(k)$ for the TM_{010}, TM_{110}, and TM_{210} modes for different beam pipe radii a. In the case of $m \geq 1$, we note that the function diverges at $k = k_l \sim 62.9\,\text{m}^{-1}$.

Figure 3 shows $E_z(r,0,z,t=0)$ and $E_z(r,0,z,t=z/c)$ when $\psi = 0$ in the TM_{010}, TM_{110}, and TM_{210} modes for a fixed beam pipe radius[2]. The peak field at $z = 0$ varies as $J_m(k_l r)$ and the leakage of the field into the beam pipe beyond $|z| > G/2c \sim 25.0$ mm is visible. For the oscillating case, this coincides with a change in sign of the electric field.

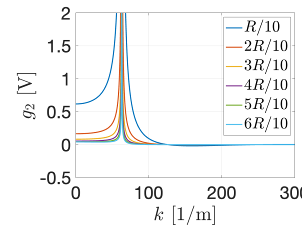
[1] Refs. [5,6] present a thorough treatment of the simpler, $a = 0$ regime.
[2] To achieve sufficient accuracy, 1000 integration steps were taken in the dz-integral up to 100 mm and 2000 in the dk-integral up to $5000k_l$.



(a) TM_{010}.

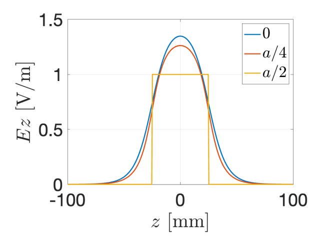


(b) TM_{110}.

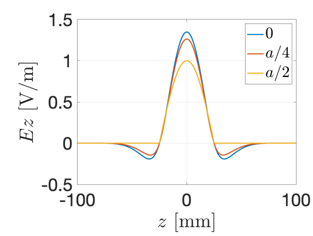


(c) TM_{210}.

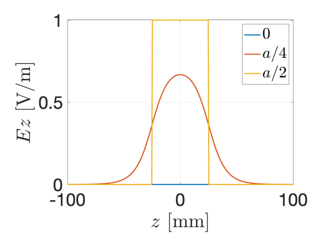
Figure 2: $\tilde{g}_0(k)$, $\tilde{g}_1(k)$, and $\tilde{g}_2(k)$ for the 3 GHz modes TM_{010} ($R \sim 38.3$ mm), TM_{110} ($R \sim 60.9$ mm), and TM_{210} ($R \sim 81.7$ mm) modes with $G = c/2f$ and $E_g = 1$ V/m. The legend denotes the beam pipe radius.



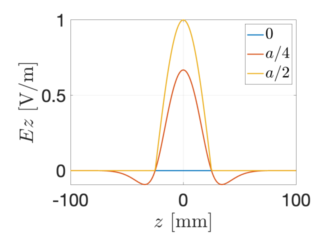
(a) Static TM_{010}.



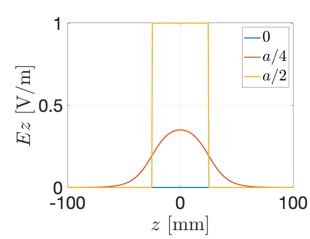
(b) Oscillating TM_{010}.



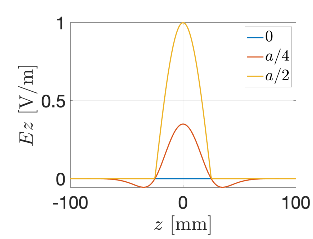
(c) Static TM_{110}.



(d) Oscillating TM_{110}.



(e) Static TM_{210}.



(f) Oscillating TM_{210}.

Figure 3: $E_z(r,0,z,t=0)$ and $E_z(r,0,z,t=z/c)$ along the different radii stated in the legend for the 3 GHz TM_{010} ($R \sim 38.3$ mm), TM_{110} ($R \sim 60.9$ mm), and TM_{210} ($R \sim 81.7$ mm) modes with $G = c/2f$ and $a = R/2$.

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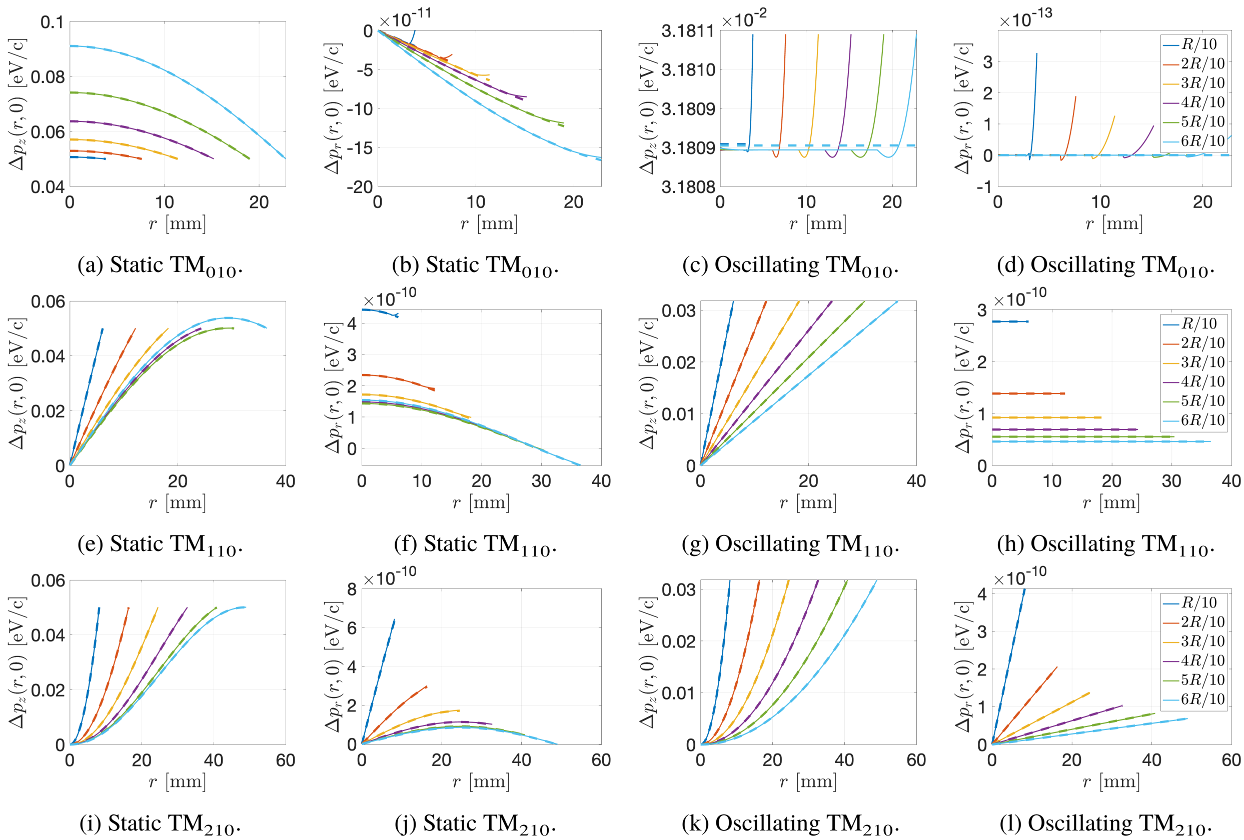

(a) Static TM_{010}. (b) Static TM_{010}. (c) Oscillating TM_{010}. (d) Oscillating TM_{010}.

(e) Static TM_{110}. (f) Static TM_{110}. (g) Oscillating TM_{110}. (h) Oscillating TM_{110}.

(i) Static TM_{210}. (j) Static TM_{210}. (k) Oscillating TM_{210}. (l) Oscillating TM_{210}.

Figure 4: $\Delta p_z(r,0)$ and $\Delta p_x(r,0)$ up to the beam pipe radius in 3 GHz TM_{010} ($R \sim 38.3$ mm), TM_{110} ($R \sim 60.9$ mm), and TM_{210} ($R \sim 81.7$ mm) modes. The different lines denote the momentum change for the different beam pipe radii a stated in the legend. The dashed lines are fits to a Bessel function (static plots) or polynomial (oscillating plots).

Figure 4 shows $\Delta p_z(r,0)$ and $\Delta p_x(r,0)$ for static and oscillating TM_{010} (top), TM_{110} (middle), and TM_{210} (bottom) modes for a range of beam pipe radii. For the static cases, the change in longitudinal momentum of the TM_{m10} mode varies as $J_m(k_l r)$ and the corresponding change in transverse momentum as $J_{m-1}(k_l r)$. For the oscillating cases, however, the longitudinal momentum of the TM_{m10} mode varies as r^m and the corresponding change in transverse momentum as r^{m-1} (the divergence to the polynomial fit seen approximately 5 mm from the beam pipe, particularly in Fig. 4c, is numerical error). That is, Figs. 4c and 4d show the change in longitudinal and transverse momentum in the time-varying TM_{010} mode is constant and zero respectively for all horizontal offsets, and Figs. 4g and 4h show they are linear and constant for the TM_{110} mode, Figs. 4k and 4l show they are quadratic and linear for the TM_{210} mode.

Including the time variation of the RF cavity field therefore leads to the change in longitudinal and transverse momentum varying as a polynomial with the radius and not with a Bessel. Generalising this conclusion, the change in longitudinal and transverse momenta of an ultra-relativistic charged particle traversing a TM_{m10} mode excited in a pillbox cavity with a beam pipe vary as $\Delta p_z \propto C_{m-1} r^m$ and $\Delta p_x \propto C_m r^{m-1}$, where the polynomial coefficient is $C_N \propto 1/a^N$.

This conclusion recovers the well-known result that for a TM_{010} mode, the energy gain (i.e. longitudinal momentum change) is constant and the magnetic focusing (i.e. transverse momentum change) is zero regardless of any radial offset [7].

CONCLUSION

The presented numerical field map integration results show that the radial dependence of the change in longitudinal and transverse momenta of a rigid, ultra-relativistic particle traversing a harmonic TM_{mn0} mode in a pillbox cavity with a beam pipe vary as r^m and r^{m-1} respectively.

The method and results presented here can be utilised in designing azimuthally modulated RF cavities for precision control over the multipolar components in their modes. This is beneficial for many applications including the removal of unwanted higher-order multipolar components introduced by incorporating power and/or higher-order mode couplers, or the addition of wanted higher-order multipolar components to create a bespoke electromagnetic field for a tailored application.

This polynomial dependence breaks down as the assumptions of a non-rigid and/or non-relativistic beam are relaxed. Such a regime also offers an interesting avenue for exploration in the design of RF cavities in the injection stages.

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