Adelic Eisenstein series on SL_n

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Abstract

In their seminal 1997 paper [1], Green and Gutperle studied modular forms, in particular non-holomorphic Eisenstein series, in the context of the low energy effective action of type IIB string theory. They demonstrated how these functions provide a window into the non-perturbative sector of four graviton scattering where the scattering amplitudes are encoded in the Fourier coefficients of the Eisenstein series. Their formalism was subsequently generalized to toroidal compactifications of type IIB theory which leads to Eisenstein series on the Cremmer-Julia series of Lie Groups and hints towards a class of functions which generalize the notion of automorphic forms.

In this thesis, we provide some background on studying automorphic forms in this context and lay out the necessary mathematical framework for finding their Fourier expansions with an emphasis on SL_n. A central part of this framework are the p-adic number fields $\mathbb{Q}_p$ which are completions of $\mathbb{Q}$ using inequivalent norms but nevertheless on the same footing as $\mathbb{R}$. In the context of Eisenstein series, an important decomposition is the so called Iwasawa decomposition working on the level of the group. With an interest in calculating Fourier coefficients explicitly, we derive closed formulae for the Iwasawa-decomposition of the groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Q}_p)$.

Drawing inspiration from SL_5 (which is the Cremmer-Julia group in seven dimensions), we present a formalism developed by OA, Gustafsson, Kleinschmidt, Liu and Persson which relates Fourier coefficients of automorphic forms attached to the minimal- and next-to-minimal automorphic representations of SL_n over maximal parabolic subgroups to their Whittaker functions. Using the Iwasawa formulae discussed above and a "reduction formula" due to Fleig, Kleinschmidt and Persson we give examples showing how one can obtain explicit results for these Fourier coefficients.

Keywords: Automorphic forms, Eisenstein series, p-adic numbers, adeles, Iwasawa decomposition, string amplitudes, non-perturbative

Zusammenfassung

In ihrem bahnbrechenden Artikel [1] aus dem Jahre 1997 haben Green und Gutperle Modulformen, insbesondere nicht-holomorphe Eisensteinreihen, im Zusammenhang mit der Niedrigenergiewirkung der typ IIB Stringtheorie untersucht. Sie haben damit gezeigt wie diese Funktionen ein Fenster in den nicht-störungstheoretischen Teil der Vier-Graviton-Streuung – bei der die Streuungsamplituden in den Fourierkoeffizienten der Eisensteinreihe kodiert sind – eröffnen. Der dabei entwickelte Formalismus wurde anschließend für toroidiale Kompaktifikationen der Typ IIB Theorie verallgemeinert, was zu Eisensteinreihen auf den Cremmer-Julia Reihen von Lie Gruppen führt und auf eine Klasse von Funktionen hindeutet, welche die Idee der automorphen Formen verallgemeinert.

In dieser Arbeit geben wir zunächst den nötigen Hintergrund zu automorphen Formen und legen das notwendige mathematische Rahmenwerk um deren Fourierreihe mit Schwerpunkt auf SL_n zu finden dar. Ein zentraler Teil dieses Rahmenwerks sind die p-adischen Zahlenkörper $\mathbb{Q}_p$, welche Körpererweiterungen von $\mathbb{Q}$ sind, die inäquivalente Normen nutzen, aber dennoch zu $\mathbb{R}$ gleichberechtigt sind. Im Zusammenhang mit den Eisensteinreihen ist die sogenannte Iwasawa Zerlegung eine wichtige Zerlegung welche auf dem Niveau der Gruppen agiert. Mit Blick auf die explizite Berechnung von Fourierkoeffizienten leiten wir einen geschlossenen Ausdruck für die Iwasawa Zerlegung der Gruppen $\mathrm{SL}_n(\mathbb{R})$ und $\mathrm{SL}_n(\mathbb{Q}_p)$ her.

Mit Inspiration von SL_5 (welche die Cremmer-Julia Gruppe in sieben Dimensionen ist) präsentieren wir einen Formalismus, welcher von OA, Gustafsson, Kleinschmidt, Liu und Persson entwickelt wurde, der die Fourierkoeffizienten von automorphen Formen in der minimalen- und nebst-zu-minimalen automorphen Darstellung von SL_n über maximalen parabolischen Untergruppen zu deren Whittakerfunktionen in Verbindung bringt. Unter Verwendung der obengenannten Iwasawa Zerlegung und einer "Reduktionsformel" von Fleig, Kleinschmidt und Persson geben wir Beispiele welche illustrieren wie man explizite Ausdrücke für diese Fourierkoeffizienten erhalten kann.

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To Annalisa Carlsson

⋆ 15 August 1919, † 30 October 2016

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1 Introduction

The aim of this thesis is to introduce the reader to certain techniques for the calculation of Fourier coefficients of a special class of functions defined on Lie groups, called *automorphic forms.* One branch of mathematics where calculations like these are studied is number theory. A special class of automorphic forms are so called *modular forms* which have played an important role in many areas of physics and mathematics. Loosely speaking, modular forms are functions defined on $\mathrm{SL}_2(\mathbb{R})$ which transform "nicely" under $\mathrm{SL}_2(\mathbb{Z})$. Modular forms have played an important role in many areas of physics, including conformal field theory and string theory. As of the late 90s, it has been known that certain types of modular forms (and their generalizations to automorphic forms) called *Eisenstein series* appear in string theory in such a way that they encode valuable physical information. In this thesis, we will study what kind of physical information these functions carry and how one goes about to uncover it.

We will make references to the following papers and proceedings:

- **Paper I** [2]: O. Ahlén, *Global Iwasawa-decomposition of* $\mathrm{SL}(n, \mathbb{A}_{\mathbb{Q}})$, submitted to International Journal of Number Theory

- **Paper II** [3]: O. Ahlén, H. P. A. Gustafsson, A. Kleinschmidt, B. Liu and D. Persson, *Fourier coefficients attached to small automorphic representations of* $\mathrm{SL}_n(\mathbb{A})$, submitted to Journal of Number Theory

- **Proceedings I** [4]: O. Ahlén, *Instantons in string theory*, published in AIP Conference Proceedings

1.1 Searching for a theory of quantum gravity

Arguably the longest standing problem in modern physics is to reconcile the two rather different physical theories of quantum mechanics which describes physics at the microscopic level and general relativity which gives provides a description of gravity, into one framework. Such a framework would be a theory of *quantum gravity.* The difficulty is easy to understand since these two theories are formulated in fundamentally different ways and deal with fundamentally different concepts. Central concepts in quantum mechanics include operators acting on states in Hilbert spaces and the quantization of energy, giving rise to such results as the Heisenberg uncertainty principle. General relativity on the other hand is a geometric theory about the curvature of spacetime, elegantly captured in the language of differential geometry. The source of this curvature is energy itself, made precise by the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{1.1}$$

This is a system of coupled partial differential equations where the left hand side speaks about the curvature of spacetime and the right hand side about the energy contents of the universe.

Taking into account the effects of special relativity for quantum mechanics gives rise to quantum field theory (QFT) which together with the concept of gauge symmetries has become an enormously successful framework to describe particle physics as well as condensed matter physics. The cutting edge theory of particle physics is a particular QFT modestly called the Standard Model and with the discovery of the Higgs boson in 2012 [5, 6] has been fully experimentally verified. What's more is that no observations have yet been made here on earth that contradict the Standard Model and general relativity even at the most powerful particle accelerator in the world, which to this date is the Large Hadron Collider (LHC), probing physics at 14 TeV. Since all known forms of matter and the their fundamental forces have thus been quantized, it is natural to believe that gravity itself is quantised as well, especially since Eq. (1.1) describes gravity in terms of the energy contents of the universe. Trying to put gravity into a quantum field theory however is incredibly difficult, much so since the resulting theory is not renormalizable. On a more conceptual level, thinking about gravity (the curvature of spacetime) as a quantum entity which hence undergoes quantum fluctuations leads to various puzzles such as that two points in spacetime might fluctuate between being and not being in causal contact.

The standard model describes three fundamental forces: the electromagnetic force, the weak force and the strong force. Compared to these three forces, gravity is incredibly weak and since the source of gravity is mass (or energy) itself, one concludes that the energy densities reached at the LHC are simply too small to show any traces of quantum gravity. Said differently, up to the energy levels probed at the LHC, gravity on the quantum level can be neglected and we can lead happy and predictive lives using only the Standard Model for microscopic physics and general relativity for macroscopic physics without the need of mixing them so long as we do not construct machines more powerful than the LHC. Only in situations in nature where the energy densities are large enough that gravity becomes a force comparable in strength to the other three forces would a theory of quantum gravity be required in order to understand what we observe. Unfortunately, those situations include some of the most interesting physical systems, for example systems where a spacetime singularity is present such as that inside a black hole or the Big Bang singularity.

Taking Newton's constant G, the speed of light c and Planck's constant $\hbar$, one can form the so called Planck length $l_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\mathrm{m}$ which could be assumed to set the length scale at which effects of quantum gravity come into play. This is far out of reach for current experiments with the LHC probing roughly $10^{-19}\,\mathrm{m}$. There has therefore been no experimental guidance to tell us which features a theory of quantum gravity should or should not possess. Instead, the approaches to quantum gravity have largely proceeded by extending and generalizing concepts in theoretical physics that have proven successful in the past. One example is the generalization of symmetries to so called supersymmetries, where in order for a theory to possess local supersymmetry it must necessarily also include gravity, making it a so called a theory of *supergravity*. See [7] for a great resource on this topic. Among the various supergravity theories that one can construct, the four dimensional so called $\mathcal{N} = 8$ theory has received much attention in the last few years, as a team of theorists have been able to conclude that this theory is finite up to four loops [8], meaning that the theory stays predictive since the quantum corrections that $\mathcal{N} = 8$ imposes on general relativity do not diverge which would have been a problem since the theory is not renormalizable. The same team is currently finishing calculations at five loops and if this trend continues it gives evidence for that $\mathcal{N} = 8$ could be the first known finite theory of quantum gravity in four dimensions.

Another theoretical approach that was investigated was to replace the point particle which has no size (mathematically a zero dimensional entity) in its relativistic treatment with an object that does have a size of dimension length (mathematically a one dimensional entity). Such an object is a string and the relativistic treatment of strings gave rise to *string theory*. Over the years, string theory evolved into a fully fledged framework in which general relativity and quantum mechanics can naturally be unified and is today one of the most prominent theories of quantum gravity. It is important to point out that string theory is but one of several candidate theories of quantum gravity but will be the one of interest in this thesis.

1.2 The evolution of string theory

The beginning ideas of what would later become string theory came about in the 1960s as a theoretical model to understand the strong interaction. In particle colliders, an abundance of strongly interacting particles (or hadrons) with seemingly large spins J were observed. It was observed that in plotting the spin versus the mass squared m^2, the particles fell on straight lines called Regge trajectories according to $m^2 = J/\alpha'$ with proportionality constant $\alpha' \sim 1\,\text{GeV}^{-2}$. This relationship was observed up to $J = 11/2$ and there was no reason to believe that it would stop there. The abundance of strongly interacting particles was in stark contrast with the weak and electromagnetic interactions where only a small number of weakly- and electrically charged particles had been observed. The particles observed were so numerous that it did not seem plausible that they were fundamental. Furthermore, no known consistent quantum field theories of higher spin particles were known to exist so it seemed likely that the observed particles were resonances of some other physical object. By considering the high energy behavior of the scattering of four pions, Veneziano was able to write down his famous amplitude in 1968 [9]

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} \tag{1.2}$$

where Γ is the Euler gamma function and $\alpha(x) = \alpha(0) + \alpha' x$. This amplitude describes the scattering of four pions exchanging particles of integer spin J and with masses $M_J^2 = (n - \alpha(0))/\alpha'$. Furthermore it has the desirable property of duality, meaning $A(s,t) = A(t,s)$ and that $A(s,t)$ can be written either as a sum over poles in the s-channel or equivalently as a sum over poles in the t-channel.

In spite of the favorable properties of the Veneziano amplitudes, there was difficulty in explaining the parton-like behavior of strong interactions that had been observed in experiments of deep inelastic scattering. As a model for the interactions between the partons, it was proposed that they are connected by small strings. This relativistic model of strings is what became string theory as we know it today. It was one of the big triumphs that this theory provided a way to derive the Veneziano amplitude Eq. (1.2) from first principles. The theory however came with some obvious drawbacks, namely the requirement of extra dimensions of spacetime, the existence of a tachyon as well as a massless spin-2 field in its spectrum. At roughly the same time, a competing quantum field theory called quantum chromodynamics was developed which offered a description of the strong interaction that full agreed with the experimental observations and string theory was abandoned as a theory of strong interactions.

The attention was instead turned towards string theory as a possible theory of quantum gravity. The strings of string theory can either be open or closed and quantization of the theory amounts to that the excitations (or vibrations) of the strings are quantized. The massless spin-2 field corresponds to an excitation of the closed string and was interpreted as the metric of general relativity, while other excitations correspond to other fields (or particles). It is in this way that string theory naturally unifies general relativity with quantum mechanics. In what became known as the first superstring revolution it turned out that after imposing supersymmetry, string theory (or rather superstring theory) turned into a theory of quantum gravity free from anomalies, without tachyons, with the presence of fermions and with amplitudes that are finite at each loop order. With the inclusion of supersymmetry, the number of required extra dimensions is different and the dimensionality of spacetime changed from 26 for the bosonic theory to 10 for superstring theory.

Supersymmetry in 10 dimensions allows for $\mathcal{N} = 1$ or $\mathcal{N} = 2$. Maximal supersymmetry, or $\mathcal{N} = 2$, can be realized either by letting the chiralities of the supersymmetries associated with the left and right movers be opposite or be aligned and the corresponding superstring theories are called type IIA and type IIB respectively (II stands for $\mathcal{N} = 2$) which are theories of oriented closed strings. Furthermore, by modding out the left-right symmetry of type IIB yet another superstring theory called type I can be constructed (I stands for $\mathcal{N} = 1$). Yet another possibility is to apply the 26 dimensional formalism of bosonic string theory to the left movers and the 10 dimensional superstring formalism to right movers. In doing so, one finds that the theory requires a local gauge symmetry of either SO_{32} or $\mathrm{E}_8 \times \mathrm{E}_8$. These theories are called the heterotic string theories and describe closed strings. There are thus five different superstring theories, all formulated in 10 dimensions: type I, SO_{32}-heterotic and $\mathrm{E}_8 \times \mathrm{E}_8$-heterotic all with $\mathcal{N} = 1$ supersymmetry, and type IIA and type IIB with $\mathcal{N} = 2$ supersymmetry. Only type I describes open strings. Each of the superstring theories have corresponding supergravity theories as their low-energy limits.

In what become known as the second superstring revolution, many more insights about superstring theory were had. A set of discrete transformations called dualities were found that provide relations among these five string theories. It was also understood by Ed Witten that at strong coupling, type IIA theory and the $E_8 \times E_8$-theory grow an additional dimension and are thus described by a full (quantum) 11-dimensional theory dubbed M-theory. All together these relations provide a "web of dualities" in which all five string theories are related and in some sense unified under 11-dimensional M-theory whose low-energy limit is also a supergravity theory, namely the unique 11-dimensional (maximally supersymmetric) supergravity theory called $D = 11$. For the case of type IIB in particular, a duality called S-duality relates type IIB theory at strong coupling to itself at weak coupling, allowing perturbation theory to give insights into the same theory at strong coupling. This duality leads to the concept of automorphy and will be one of the main tools in this thesis.

It was furthermore understood that the one-dimensional fundamental string is but a special case of p-dimensional extended objects called p-branes with the fundamental string corresponding to $p = 1$. A special class of p-branes are so called Dp-branes which exist in the type I and type II theories. They have the property that they are objects on which fundamental strings can end (D stands for Dirichlet). It was realized by Joe Polchinsky that the Dp-branes are dynamical objects in their own right whose tension is proportional to $1/g_s$ where g_s is the string coupling constant. At small g_s they thus become infinitely heavy and inaccessible in perturbation theory. Their contributions to a scattering process are therefore called non-perturbative. There is no reason to expect g_s to be small so it is important to understand these non-perturbative effects for scattering processes. This thesis focuses on IIB theory and will discuss a technique to exploit the discrete symmetries in type IIB theory as a way to gain access to these non-perturbative effects. These non-perturbative effects are encoded in automorphic forms.

Good introductory books to String Theory include the works by Green, Schwarz and Witten [10, 11]. For a more modern treatment including D-branes and the second superstring revolution see [12, 13, 14]. For a less detailed modern overview, see [15].

1.3 Compactification

It can be argued to be a virtue that superstring theory and M-theory predict the dimensionality of space-time to be $D = 10$ and $D = 11$ respectively. This prediction for the critical dimension is a consequence of requiring closure of the Lorentz-algebra and it is encouraging that the dimensions come out as natural numbers greater than or equal to four. The fact that they are greater than four calls for some extra work to be done in order to make recourse with the four dimensional world we experience. The standard way to do this is through what is called *compactification*, or *Kaluza-Klein theory*. Since this topic will play a role in this thesis, it will be briefly explained here. For a good set of lecture notes on this topic, see [16] available on the author's personal web page. It is also briefly explained in [17].

The general idea of Kaluza-Klein theory is to formulate a D-dimensional theory on the product space $\mathbb{R}^{1,D-1-d} \times X^d$ where X^d is a d-dimensional compact internal manifold. By taking the size of of the internal manifold to be small, one gets an effective theory in $D - d$ dimensions. The example that was initially studied by Kaluza and Klein formulated five dimensional general relativity on the space $M^5 = \mathbb{R}^{1,3} \times S^1_L$ where S_L is a circle of radius L. Letting z denote the coordinate on the circle and x^μ ($\mu = 0, 1, 2, 3$) denote the coordinate on $\mathbb{R}^{1,3}$, periodicity lets us expand a scalar field $\hat{\phi}$ on M^5 (hats denote 5-dimensional quantities) as

$$\hat{\phi}(x, z) = \sum_{n \in \mathbb{Z}} \phi_n(x) \mathrm{e}^{\mathrm{i}nz/L}. \tag{1.3}$$

The 5-dimensional wave equation

$$\hat{\Box}\hat{\phi} = 0 \tag{1.4}$$

implies the equations

$$\Box\phi_n - \frac{n^2}{L^2}\phi_n = 0 \tag{1.5}$$

for each Fourier mode, $n \in \mathbb{Z}$. These are 4-dimensional wave equations for scalar fields ϕ_n of mass $|n|/L$. In the limit $L \to 0$, i.e. for a small internal manifold all fields expect the massless ϕ_0 become infinitely massive. Hence, at low-energies these fields decouple from the theory and physics can be described in terms of an effective action involving the massless fields. An equivalent viewpoint is that only at high energies does one have enough resolution to "see" the small compact direction of the S^1_L. At low energies one instead obtains an effective four-dimensional description.

In a similar way as to the scalar field, the 5-dimensional metric tensor $\hat{g}_{MN}(x, z)$ decomposes into a 4-dimensional metric $\hat{g}_{\mu\nu}(x)$, a 4-vector $\hat{g}_{\mu z}(x)$ and a scalar field $\hat{g}_{zz}(x)$. It should be somewhat clear that upon compactification on a higher dimensional internal manifold whose degrees of freedom are labelled by indices $i, j, \ldots$ one will obtain additional scalar fields. Collectively they are called moduli and are typically grouped into the "dilatons" being the diagonal degrees of freedom in $\hat{g}_{ij}$ and the "axions" being the off diagonal degrees of freedom. An important mechanism is that the diffeomorphisms from the higher dimensional theory involving the compactified directions descend to gauge symmetries. In this way, Kaluza and Klein famously found that compactification of 5-dimensional gravity on a circle gives 4-dimensional Maxwell-Einstein theory with an additional scalar field coupled to the vector potential.

In this thesis, we will be concerned with toroidal compactifications of the maximally supersymmetric theories type IIB supergravity and type IIB string theory. It is known that from a phenomenological standpoint, toroidal compactifications are not feasible since they preserve all supersymmetries and the 4-dimensional universe we observe possesses less supersymmetry than that. Nevertheless, they are interesting from a theoretical standpoint since the equations of motion are constrained enough that it can tractable to investigate them.

2 Automorphic Forms and Fourier expansions

This chapter will give a formal introduction to the class of functions called automorphic forms which will be the key players of this thesis. In essence, an automorphic form is a function defined on a Lie group satisfying moderate growth conditions and certain prescribed differential equations. Furthermore, an automorphic form is invariant under translations of a discrete subgroup of its domain which should be thought of as a generalization of being periodic. As such, automorphic forms can be treated with Fourier theory and represented as a Fourier series. The Fourier coefficients of an automorphic form are of big interest to number theorists as they contain so called L-functions[1] of which the (analytic continuation of the) Riemann zeta function is the most well known example. In fact we will see how this particular L-function arises as a Fourier coefficient of an automorphic form. The Fourier coefficients of certain automorphic forms are also of big interest to physicists as they encode scattering amplitudes in string theory complete with non-perturbative contributions. This connection between automorphic forms and string theory will be the topic of Chapter 3. For an excellent and thorough review of the theory of automorphic forms with an emphasis on their role in physics, see the book [18].

2.1 Lie theory

In order to speak fluently about automorphic forms, we begin by revising some basic notions from the study of groups and their associated Lie algebras. A good resource on this topic is the book [19].

2.1.1 Lie algebras

Let $\mathfrak{g}(\mathbb{R})$ be the split real form of a finite dimensional simple Lie algebra from the Cartan-Killing classification. We can choose a maximal and abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ called a Cartan subalgebra. The dimension r of the Cartan subalgebra determines the rank of $\mathfrak{g}$. As the elements of the Cartan algebra have zero Lie brackets among themselves, the adjoint maps $\mathrm{ad}_h(\cdot) \equiv [h, \cdot]$ are simultaneously diagonalizable. The whole of $\mathfrak{g}$ must thus be spanned by elements which are simultaneous eigenvectors to the maps ad_h for all

[1] L stands for Robert Langlands, mathematician.

$h \in \mathfrak{h}$. The corresponding decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \tag{2.1}$$

is called the *root space decomposition* (relative to the chosen Cartan subalgebra $\mathfrak{h}$) where the sum goes over all non-zero generalized eigenvalues $\alpha : \mathfrak{h} \to \mathbb{R}$ called *roots* and each root space consists of all elements of $\mathfrak{g}$ with eigenvalue $\alpha(h)$ under the map ad_h,

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \quad \forall h \in \mathfrak{h}\}. \tag{2.2}$$

The roots α are linear functions on $\mathfrak{h}$ and are thus elements of the dual space, $\alpha \in \mathfrak{h}^\star$. Having picked a basis $H_1, \ldots, H_r$ of r basis vectors in $\mathfrak{h}$, one should think of each root α as an r-vector in the dual space with components $\alpha(H_i)$, $i = 1, \ldots, r$. The set of all non-zero roots is denoted Δ.

It is possible to define an inner product on $\mathfrak{g}$ through the so called *Killing form*

$$\kappa(x, y) \equiv \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) \quad \text{for} \quad x, y \in \mathfrak{g} \tag{2.3}$$

where $\circ$ denotes composition of maps and tr denotes the trace of a linear map. The linearity of the maps ad_x and ad_y makes κ a bilinear map. It was shown by Cartan that for the Lie algebras under consideration, the Killing form is non-degenerate on $\mathfrak{g}$ as well as non-degenerate on its restriction to $\mathfrak{h}$. As such, for each root $\alpha \in \mathfrak{h}^\star$ we can associate an element $H^\alpha \in \mathfrak{h}$ by requiring

$$\alpha(h) \propto \kappa(H^\alpha, h) \quad \forall h \in \mathfrak{h} \tag{2.4}$$

with some conveniently chosen proportionality constant. In this way, the Killing form extends to a non-degenerate metric on the dual space $\mathfrak{h}^\star$ as well.

At this point, the following statements may be proven rigorously (see [19] or [20] for a condensed version):

- The roots span all of $\mathfrak{h}^\star$.
- The root spaces $\mathfrak{g}_\alpha$ are all one dimensional and we let E^α denote a suitably normalized basis vector for each $\mathfrak{g}_\alpha$.
- The only multiples of a root α which are roots are $\pm\alpha$.
- There is a basis $\{H^1, \ldots, H^r\}$ of $\mathfrak{h}$ such that $\alpha(H^i) \in \mathbb{Z}$ for all roots α.
- It is possible to choose a system of *simple roots*

  $$\Pi = \{\alpha_1, \ldots, \alpha_r\} \tag{2.5}$$

  which is such that all roots can uniquely be written as linear combinations of the simple roots and each simple root cannot be expressed in terms of the other simple roots.

- For this choice of simple roots, each root can be written as a linear combination of the simple roots either with only positive integral coefficient (called a positive root) or with only negative integral coefficients (called a negative root).

The H^α defined above can be shown to be elements of $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and we can take their normalization to satisfy $\alpha(H_\alpha) = 2$. In doing so we get that the triple

$$(E^\alpha, H^\alpha, E^{-\alpha}) \in \mathfrak{g}_\alpha \times \mathfrak{h} \times \mathfrak{g}_{-\alpha} \tag{2.6}$$

obey the standard standard $\mathfrak{sl}_2$ relations

$$[H^\alpha, E^\alpha] = 2E^\alpha, \quad [H^\alpha, E^{-\alpha}] = -2E^{-\alpha}, \quad [E^\alpha, E^{-\alpha}] = H^\alpha \tag{2.7}$$

The generators E^α, H^α and $E^{-\alpha}$ are called *Chevalley generators* and specifically the E^α and $E^{-\alpha}$ are called raising- and lowering operators respectively. It is the way in which generators E^α and E^β of different roots α and β are intertwined that classifies the finite dimensional simple Lie algebras, and this information is encode in the so called *Cartan matrix A* which is can be represented pictorially by a *Dynkin diagram.* There is always a matter of choosing conventions and normalization when defining a Lie algebra so here we will simply state the conventions that are used later on in this thesis. The Killing form provides a non-degenerate inner product on $\mathfrak{g}$ and thus also on $\mathfrak{g}^\star$. Furthermore using the property of *invariance* (also called *compatibility* with the Lie bracket)

$$\kappa([x,y],z) = \kappa(x,[y,z]) \tag{2.8}$$

it can be extended to define a pairing between the Chevalley generators E^α as well. It is sometimes also convenient to introduce a basis of $\mathfrak{h}^\star$ dual to the simple roots, this basis is made up of the so called *fundamental weights* Λ_i $(i = 1, \ldots, r)$ which obey

$$\langle \Lambda_i, \alpha_j \rangle = \frac{1}{2}\langle \alpha_i, \alpha_i \rangle. \tag{2.9}$$

In this thesis we will be working with the Lie algebra $\mathfrak{sl}_n$ where all roots are of equal length,

$$\langle \alpha_i, \alpha_i \rangle = 2 \quad \forall i. \tag{2.10}$$

The space of positive/negative roots is denoted by $\Delta_\pm$ and are related by $\Delta_- = -\Delta_+$. We define the nilpotent subalgebras

$$\mathfrak{n} \equiv \mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta_+} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_- = \bigoplus_{\alpha\in\Delta_-} \mathfrak{g}_\alpha \tag{2.11}$$

The Lie algebra $\mathfrak{g}$ also has a compact subalgebra spanned by the combinations $E^\alpha - E^{-\alpha}$,

$$\mathfrak{k} \equiv \mathrm{span}_{\mathbb{R}}\{E^\alpha - E^{-\alpha} : \alpha \in \Delta_+\}. \tag{2.12}$$

This subalgebra is compact as the inner product on $\mathfrak{k}$ is negative definite:

$$\langle E_\alpha - E_{-\alpha} | E_\beta - E_{-\beta} \rangle = -2\delta_{\alpha,\beta}. \tag{2.13}$$

Another important concept for this thesis is the so called *Weyl group* $\mathcal{W}$ of a Lie algebra $\mathfrak{g}$. It is the subgroup of the isometry group of the root system consisting of the reflections through hyperplanes orthogonal to the simple roots. Having chosen a set of simple roots, this finite Coxeter group is generated by the fundamental reflections w_i which act on fundamental weights $\lambda \in \mathfrak{h}^\star$ according to

$$w_i(\lambda) = \lambda - 2\frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle}\alpha_i \tag{2.14}$$

As the Weyl group $\mathcal{W}$ is generated by the fundamental reflections, a general element of $\mathcal{W}$ is called a *Weyl word*. Each Weyl word can be written as a composition of the fundamental reflections in several ways and the smallest number of fundamental reflections required to represent a given Weyl word is called the height of the given word. The unique Weyl word of the largest height is called the *longest Weyl word* and denoted w_{long}. It has the property that it maps all simple roots to their negatives.

Lastly, an useful quantity is the so called Weyl vector

$$\rho = \frac{1}{2}\sum_{i=1}^{\operatorname{rank}\mathfrak{g}} \alpha_i = \sum_{i=1}^{\operatorname{rank}\mathfrak{g}} \Lambda_i. \tag{2.15}$$

A fundamental reflection w_i word acting on ρ gives

$$w_i(\rho) = \rho - \alpha_i. \tag{2.16}$$

2.1.2 Parabolic subalgebras

Given a Lie group, a particular class of subgroups which will be important in this thesis are the so called parabolic subgroups together with their associated Levi-decomposition.

Constructing a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ starts by picking a subset Σ of the simple roots Π of Eq. (2.5) (or equivalently picking a subset of nodes from the Dynkin diagram of $\mathfrak{g}$). By taking all possible Lie brackets of the corresponding Chevalley raising and lowering operators one generates a finite dimensional semisimple Lie algebra $\mathfrak{m}$ whose roots will be denoted by $\langle \Pi \rangle$. Adjoining all (remaining) positive roots of $\mathfrak{g}$, the parabolic subalgebra $\mathfrak{p}$ is defined as the direct sum over the root spaces touched so far together with the Cartan subalgebra:

$$\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{p})} \mathfrak{g}_\alpha \quad \text{where} \quad \Delta(\mathfrak{p}) = \Delta_+ \cup \langle \Sigma \rangle. \tag{2.17}$$

A parabolic subalgebra has a natural decomposition according to

$$\mathfrak{p} = \underbrace{\bigoplus_{\alpha\in\Delta_+\setminus\langle\Sigma\rangle_+}}_{\mathfrak{u}} \oplus \underbrace{\mathfrak{h} \oplus \bigoplus_{\alpha\in\langle\Sigma\rangle} \mathfrak{g}_\alpha}_{\mathfrak{l}} \tag{2.18}$$

called the *Levi decomposition*. The subalgebra $\mathfrak{u}$ is nilpotent and the subalgebra $\mathfrak{l}$ is reductive as it consists of the semisimple Lie algebra $\mathfrak{m}$ mentioned above together with additional semisimple generators from $\mathfrak{h}$. Separating $\mathfrak{m}$ from $\mathfrak{l}$ by writing $\mathfrak{m} = [\mathfrak{l}, \mathfrak{l}]$ leads to the so called *Langlands decomposition*

$$\mathfrak{p} = \mathfrak{u} \oplus \underbrace{\mathfrak{a}_P \oplus \mathfrak{m}}_{\mathfrak{l}}. \tag{2.19}$$

The special case of choosing all simple roots as the subset, $\Sigma = \Pi$, gives $\langle\Sigma\rangle = \Delta$ and thus $\mathfrak{u} = 0$ and $\mathfrak{p} = \mathfrak{l} = \mathfrak{m} = \mathfrak{g}$ and a trivial decomposition. The other extreme of choosing no simple roots, $\Sigma = \varnothing$, leads to $\langle\Sigma\rangle = \varnothing$ and thus $\mathfrak{u} = \mathfrak{n}$ from Eq. (2.11) and $\mathfrak{l} = \mathfrak{h}$. The resulting parabolic subalgebra

$$\mathfrak{b} \equiv \mathfrak{n} \oplus \mathfrak{h} \tag{2.20}$$

is called the *Borel subalgebra*. Adding the lowering operators $E^{-\alpha}$ to the Borel subalgebra by direct addition of the compact subalgebra $\mathfrak{k}$ gives the important *Iwasawa decomposition* of $\mathfrak{g}$

$$\mathfrak{g} \equiv \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{k} \tag{2.21}$$

which on the group level will play an important role in this work.

The special case when Σ contains all but one simple root is called a *maximal parabolic subalgebra*.

2.1.3 Interpretation for $\mathfrak{sl}_n$ and SL_n

Let's quickly go through how to think about the concepts above for the Lie algebra $\mathfrak{sl}_n$ in the fundamental representation (the set of all real traceless $n \times n$ matrices) and its associated Lie group $\mathrm{SL}_n(\mathbb{R})$ in the defining representation (the set of all real $n \times n$ matrices of unit determinant). We investigate the case $n = 4$ as it provides sufficient generality to exhibit the important features. The generalization to higher n is obvious.

The $(n-1 = 3)$-dimensional Cartan subalgebra is spanned by the Chevalley generators T_i

$$T_1 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & -1 & \\ & & & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}. \tag{2.22}$$

The three Chevalley generators for the simple roots are

$$E_{\alpha_1} = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad E_{\alpha_2} = \begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad E_{\alpha_3} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}. \tag{2.23}$$

There are three more raising operators, corresponding to the roots $\alpha_1+\alpha_2$, $\alpha_2+\alpha_3$ and $\alpha_1+\alpha_2+\alpha_3$. They are obtained by taking Lie brackets (in this case matrix commutators)

$$\begin{aligned} E_{\alpha_1+\alpha_2} &= [E_{\alpha_1}, E_{\alpha_2}] = \begin{pmatrix} 0 & & 1 & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad E_{\alpha_2+\alpha_3} = [E_{\alpha_2}, E_{\alpha_3}] = \begin{pmatrix} 0 & & & \\ & 0 & & 1 \\ & & 0 & \\ & & & 0 \end{pmatrix}, \\ E_{\alpha_1+\alpha_2+\alpha_3} &= [E_{\alpha_1}, [E_{\alpha_2}, E_{\alpha_3}]] = \begin{pmatrix} 0 & & & 1 \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}. \end{aligned} \tag{2.24}$$

The lowering operators $E_{-\alpha}$ are obtained similarly and also given by the matrix transpose $E_{-\alpha} = E_\alpha^\mathsf{T}$.

There are five (isomorphically distinct) possible parabolic subalgebras

$$\begin{aligned}
\Sigma = \{\alpha_1,\alpha_2,\alpha_3\}: \quad & \mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad 4 \\
\Sigma = \{\alpha_1,\alpha_2\}: \quad & \mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ & & & * \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} * & * & * & \\ * & * & * & \\ * & * & * & \\ & & & * \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0 & & & * \\ & 0 & & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \quad 31 \\
\Sigma = \{\alpha_1,\alpha_3\}: \quad & \mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0 & & * & * \\ & 0 & * & * \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad 2^2 \\
\Sigma = \{\alpha_1\}: \quad & \mathfrak{p} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} * & * & & \\ * & * & & \\ & & * & \\ & & & * \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0 & & * & * \\ & 0 & * & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \quad 21^2 \\
\Sigma = \varnothing: \quad & \mathfrak{p} = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0 & * & * & * \\ & 0 & * & * \\ & & 0 & * \\ & & & 0 \end{pmatrix} \quad 1^4.
\end{aligned} \tag{2.25}$$

The parabolic subgroup corresponding to $\Sigma = \{\alpha_2, \alpha_3\}$ is isomorphic to the one labelled 31 and those corresponding to $\Sigma = \{\alpha_2\}$ and $\Sigma = \{\alpha_3\}$ are isomorphic to the one labelled 21^2. The parabolic subalgebras are in one-to-one correspondence with the partitions of n as indicated on the right, in this case $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. This fact holds in general for $\mathfrak{sl}_n$ [21]. The parabolic subalgebras corresponding to 31 and 2^2 are maximal. For general n, a maximal parabolic subalgebra taxes the form

$$\mathfrak{p} = \begin{pmatrix} *_{m\times m} & *_{m\times(n-m)} \\ & *_{(n-m)\times(n-m)} \end{pmatrix}, \quad \mathfrak{l} = \begin{pmatrix} *_{m\times m} & \\ & *_{(n-m)\times(n-m)} \end{pmatrix}, \quad \mathfrak{u} = \begin{pmatrix} 0_{m\times m} & *_{m\times(n-m)} \\ & 0_{(n-m)\times(n-m)} \end{pmatrix} \tag{2.26}$$

where $m = 1, \ldots, n-1$, corresponding to the $n-1$ simple roots.

There are corresponding notions on the group level for all concepts listed above. They are given by the exponential of the corresponding generators or subalgebras where heuristically, the only difference is that there are ones on the diagonal instead of zeroes. The exponential $U = e^{\mathfrak{u}}$ of the unipotent subalgebra $\mathfrak{u}$ of some parabolic subalgebra $\mathfrak{p}$ is called the *unipotent radical* of the corresponding parabolic subgroup $P = e^{\mathfrak{p}}$ and the Levi-decompositions on the group level carries through

$$P = e^{\mathfrak{p}} = e^{\mathfrak{u}} e^{\mathfrak{l}} = UL. \tag{2.27}$$

A maximal parabolic subgroup is the exponentiation of a maximal parabolic subalgebra and is thus labelled by a simple root β. We write P_β for the maximal parabolic subgroup given by deleting the node corresponding to β from the Dynkin diagram. Note that the unipotent radical of a maximal parabolic subgroup of SL_n is abelian.

2.1.4 Iwasawa decomposition for $\mathrm{SL}_n(\mathbb{R})$

The Iwasawa decomposition exists on the group level for an arbitrary number field F and states[2]

$$G(F) = N(F)A(F)K(G(F)) \tag{2.28}$$

The unipotent subgroup $N = \mathrm{e}^{\mathfrak{n}}$ is the unipotent radical of the Borel subgroup $B = \mathrm{e}^{\mathfrak{b}}$ and as such it is also called the *maximal unipotent*. The abelian subgroup $A = \mathrm{e}^{\mathfrak{h}}$ is the exponential of the Cartan subalgebra and is called the *Cartan torus*. Lastly, the group $K(G(F))$ is called the *maximal compact subgroup* of $G(F)$. For $F = \mathbb{R}$ and $\mathfrak{g}$ being the split real form of A_n, it is given as the exponential of the compact subalgebra $\mathfrak{k}(\mathbb{R})$ of Eq. (2.12) so that we have

$$K(\mathrm{SL}_n(\mathbb{R})) = \mathrm{SO}_n(\mathbb{R}). \tag{2.29}$$

as can be seen on the level of the Lie algebra since we then have $\mathfrak{k}(\mathbb{R}) = \mathfrak{so}_n(\mathbb{R})$, the set of real traceless antisymmetric matrices, which is indeed the Lie algebra of $\mathrm{SO}_n(\mathbb{R})$. On the group level, we parametrize the Iwasawa decomposition in the following way (taking $\mathrm{SL}_4(\mathbb{R})$ as an example)

$$\mathrm{SL}_4(\mathbb{R}) \ni g = nak = \underbrace{\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & x_{1,4} \\ & 1 & x_{2,3} & x_{2,4} \\ & & 1 & x_{3,4} \\ & & & 1 \end{pmatrix}}_{n\in N} \underbrace{\begin{pmatrix} y_1 & & & \\ & y_2/y_1 & & \\ & & y_3/y_2 & \\ & & & 1/y_4 \end{pmatrix}}_{a\in A} \underbrace{k}_{\in K} \tag{2.30}$$

where $y_i > 0$ for all i. The generalization to $\mathrm{SL}_n(\mathbb{R})$ is obvious. Parametrized in the above way, the matrix elements $x_{i,j}$ are called *axions* and the y_i are called *dilatons*. For $F = \mathbb{R}$, this decomposition is unique.

Remark 2.1.
The nomenclature of axions and dilatons here stems from physics. Recall that a metric is a symmetric 2-tensor (or matrix) and from any element $g \in \mathrm{SL}_n$ one can form the symmetric matrix

$$gg^\intercal = nakk^\intercal a^\intercal n^\intercal = na^2 n^\intercal \tag{2.31}$$

which in certain physical scenarios can be understood as a metric. As mentioned in Section 1.3, the scalar fields arising as the diagonal degrees of freedom (here represented by the matrix a^2) from the compactified part of a metric are called dilatons and the off diagonal ones (represented by n) are called axions.

[2] Equivalently, one may also take $G = KAN$ as the Iwasawa decomposition. On the level of the Lie algebra the decomposition Eq. (2.21) is trivial while on the level of the group, using $G = NAK$ or $G = KAN$ is a matter of convention.

Later on, we will need explicit formulae for the axions and dilatons of an arbitrary matrix $M \in \mathrm{SL}_n(\mathbb{R})$. These were found in **Paper I**. The central result is as follows

Theorem 2.2 (Iwasawa decomposition for $\mathrm{SL}_n(\mathbb{R})$)
A matrix $M \in \mathrm{SL}_n(\mathbb{R})$ may be written uniquely as[3]

$$M = NAK \tag{2.32}$$

where $N \in \mathrm{SL}_n(\mathbb{R})$ is unit upper triangular, $A \in \mathrm{SL}_n(\mathbb{R})$ is diagonal with positive entries and $K \in \mathrm{SO}_n$. Furthermore, denoting the row-vectors in M by V_i, $i \in \{1, \ldots, n\}$, and parametrizing N and A as

$$N_{ij} = \begin{cases} 1, & i = j \\ x_{ij}, & i < j \\ 0, & i > j \end{cases} \quad \textit{and} \quad A_{ij} = \frac{y_i}{y_{i-1}}\delta_{ij} \quad \textit{with} \quad y_0 \equiv y_n \equiv 1, \tag{2.33}$$

we have that the axions and dilatons are given by

$$x_{\mu\nu} = y_{\nu-1}^2 \epsilon\left(V_\mu, V_{\nu+1}, \ldots, V_n; V_\nu, V_{\nu+1}, \ldots, V_n\right), \quad \mu < \nu, \quad \textit{and} \tag{2.34}$$

$$y_\mu^{-2} = \epsilon\left(V_{\mu+1}, \ldots, V_n; V_{\mu+1}, \ldots, V_n\right) \tag{2.35}$$

where ϵ denotes the totally antisymmetric product

$$\epsilon\left(A_1, \ldots, A_m; B_1, \ldots, B_m\right) = \delta^{i_1 \text{—} i_m}_{a_1 \text{—} a_m} \left(A_1\right)^{a_1} \ldots \left(V_m\right)^{a_m} \left(B_1\right)_{i_1} \ldots \left(B_m\right)_{i_m} \tag{2.36}$$

where the A's and B's are n-vectors and

$$\delta^{i_1 \text{—} i_m}_{a_1 \text{—} a_m} = m! \delta^{i_1}_{[a_1} \ldots \delta^{i_m}_{a_m]} = \frac{1}{(n-m)!} \epsilon_{a_1 \text{—} a_m \alpha_{m+1} \text{—} \alpha_n} \epsilon^{i_1 \text{—} i_m \alpha_{m+1} \text{—} \alpha_n} \tag{2.37}$$

denotes the generalized Kronecker delta.

The procedure of writing a real matrix M in Iwasawa form is tantamount to Gram-Schmidt orthogonalization of the n row-vectors in M for which there are recursive formulae. The orthogonal matrix K consists of n orthonormal row-vectors and the unit upper triangular matrix N together with the normalization in A then specifies the appropriate linear combinations of these row-vectors to build the row-vectors in M. Oftentimes in the literature, people denote the product of A and N as R and speak about the QR-decomposition[4].

A very quick way to arrive at the non-recursive formulae Eq. (2.34) and Eq. (2.35) given above is by means of the UL-decomposition as done in [22]. The argument goes like this: Write $MM^\mathsf{T} = NA^2N^\mathsf{T}$. The right hand side is then a UL-decomposition of MM^T and the matrix elements of A^2 and N must then be given by Eq. (4.82) and Eq. (4.83)

[3] There also exist the decomposition $G = KAN$ which works completely analogously. Which one is used in a physical setting is a matter of convention.

[4] This is in the case $M = KAN$, where $K = Q$ and $AN = R$.

respectively. As a complement to this method, we give a proof for the formulae above which doesn't rely on the UL-decomposition.

Proof. We have the equality

$$MM^{\mathsf{T}} = NA^2N^{\mathsf{T}}. \tag{2.38}$$

To illustrate the idea behind the proof, we write out the right hand side explicitly for the case $n = 4$

$$\begin{pmatrix} y_1^2 + \frac{x_{12}^2 y_2^2}{y_1^2} + \frac{x_{14}^2}{y_3^2} + \frac{x_{13}^2 y_3^2}{y_2^2} & \frac{x_{12} y_2^2}{y_1^2} + \frac{x_{14} x_{24}}{y_3^2} + \frac{x_{13} x_{23} y_3^2}{y_2^2} & \frac{x_{14} x_{34}}{y_3^2} + \frac{x_{13} y_3^2}{y_2^2} & \frac{x_{14}}{y_3^2} \\ \frac{x_{12} y_2^2}{y_1^2} + \frac{x_{14} x_{24}}{y_3^2} + \frac{x_{13} x_{23} y_3^2}{y_2^2} & \frac{y_2^2}{y_1^2} + \frac{x_{24}^2}{y_3^2} + \frac{x_{23}^2 y_3^2}{y_2^2} & \frac{x_{24} x_{34}}{y_3^2} + \frac{x_{23} y_3^2}{y_2^2} & \frac{x_{24}}{y_3^2} \\ \frac{x_{14} x_{34}}{y_3^2} + \frac{x_{13} y_3^2}{y_2^2} & \frac{x_{24} x_{34}}{y_3^2} + \frac{x_{23} y_3^2}{y_2^2} & \frac{x_{34}^2}{y_3^2} + \frac{y_3^2}{y_2^2} & \frac{x_{34}}{y_3^2} \\ \frac{x_{14}}{y_3^2} & \frac{x_{24}}{y_3^2} & \frac{x_{34}}{y_3^2} & \frac{1}{y_3^2} \end{pmatrix} \tag{2.39}$$

Starting from the $(4,4)$ entry and working "backwards", i.e. proceeding as $(4,4) \to (3,4) \to (3,3) \to (2,4) \to \dots$, we notice that each equation is solvable in terms of variables that have previously been determined. The (μ,μ)-equation allows for determination of $y_{\mu-1}$ and the (μ,ν) $(\mu < \nu)$ allows for determination of $x_{\mu\nu}$ all in terms of known variables. We now carry this out for the general case.

Matrix elements of the left- and right hand sides of Eq. (2.38) evaluate to

$$\left(MM^{\mathsf{T}}\right)_{\mu\nu} = V_\mu \cdot V_\nu = (V_\mu)_A \, (V_\nu)^I \, \delta_I^A = \epsilon\,(V_\mu; V_\nu) \tag{2.40}$$

and (assuming $\mu < \nu$)

$$\begin{aligned} \left(NA^2N^{\mathsf{T}}\right)_{\mu\nu} &= \sum_{r=1}^{n}\sum_{s=1}^{n} N_{\mu r}\left(A^2\right)_{rs}\left(N^{\mathsf{T}}\right)_{s\nu} = \sum_{r=1}^{n}\sum_{s=1}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}\delta_{rs}N_{\nu s} = \\ &= \sum_{r=1}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}N_{\nu r} = \sum_{r=\nu}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}N_{\nu r} \\ &= x_{\mu\nu}\frac{y_\nu^2}{y_{\nu-1}^2} + \sum_{r=\nu+1}^{n} x_{\mu r}\frac{y_r^2}{y_{r-1}^2}x_{\nu r} \end{aligned} \tag{2.41}$$

respectively. Solving for $x_{\mu\nu}$ gives

$$x_{\mu\nu} = \frac{y_{\nu-1}^2}{y_\nu^2}\left(\epsilon\,(V_\mu; V_\nu) - \sum_{r=\nu+1}^{n}\frac{y_r^2}{y_{r-1}^2}x_{\mu r}x_{\nu r}\right). \tag{2.42}$$

We assume that all y_ρ for $\rho \geq \mu$, and $x_{\rho\sigma}$ for $\rho > \mu$, and $x_{\mu\sigma}$ for $\sigma > \nu$ have been found, and are of the form in Eqs. (2.34) and (2.35). The sum telescopes through the identity

$$\frac{\epsilon\left(V_\mu, V_{r+1}, \ldots, V_n; V_\nu, V_{r+1}, \ldots, V_n\right)}{\epsilon\left(V_{r+1}, \ldots, V_n; V_{r+1}, \ldots, V_n\right)} - \frac{y_r^2}{y_{r-1}^2} x_{\mu r} x_{\nu r} = \frac{\epsilon\left(V_\mu, V_r, \ldots, V_n; V_\nu, V_r, \ldots, V_n\right)}{\epsilon\left(V_r, \ldots, V_n; V_r, \ldots, V_n\right)} \tag{2.43}$$

which is proven in Appendix E. Applying Eq. (2.43) to Eq. (2.42) term by term starting with $r = n$ allows one to step down through the sum and obtain

$$\begin{aligned} x_{\mu\nu} &= \frac{y_{\nu-1}^2}{y_\nu^2} \frac{\epsilon\left(V_\mu, V_{\nu+1}, \ldots, V_n; V_\nu, V_{\nu+1}, \ldots, V_n\right)}{\epsilon\left(V_{\nu+1}, \ldots, V_n; V_{\nu+1}, \ldots, V_n\right)} = \\ &= y_{\nu-1}^2 \epsilon\left(V_\mu, V_{\nu+1}, \ldots, V_n; V_\nu, V_{\nu+1}, \ldots, V_n\right) \end{aligned} \tag{2.44}$$

which is exactly Eq. (2.34).

The dilaton $y_{\mu-1}$ is found through the (μ,μ)-equation

$$\begin{aligned} \left(NA^2N^\mathsf{T}\right)_{\mu\mu} &= \sum_{r=1}^{n}\sum_{s=1}^{n} N_{\mu r}\left(A^2\right)_{rs}\left(N^\mathsf{T}\right)_{s\mu} = \sum_{r=1}^{n}\sum_{s=1}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}\delta_{rs}N_{\mu s} = \\ &= \sum_{r=1}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}N_{\mu r} = \sum_{r=\mu}^{n} N_{\mu r}\frac{y_r^2}{y_{r-1}^2}N_{\mu r} \\ &= \frac{y_\mu^2}{y_{\mu-1}^2} + \sum_{r=\mu+1}^{n} x_{\mu r}^2 \frac{y_r^2}{y_{r-1}^2}. \end{aligned} \tag{2.45}$$

Solving for $y_{\mu-1}^{-2}$ gives

$$y_{\mu-1}^{-2} = \frac{1}{y_\mu^2}\left(\epsilon\left(V_\mu; V_\mu\right) - \sum_{r=\mu+1}^{n} x_{\mu r}^2 \frac{y_r^2}{y_{r-1}^2}\right). \tag{2.46}$$

We assume again that all "lower" variables are given of the form of Eqs. (2.34) and (2.35). The sum then telescopes through Eq. (2.43) with $\nu = \mu$ in precisely the same way as above. The result is

$$\begin{aligned} y_{\mu-1}^{-2} &= \frac{1}{y_\mu^2}\frac{\epsilon\left(V_\mu, V_{\mu+1}, \ldots, V_n; V_\mu, V_{\mu+1}, \ldots, V_n\right)}{\epsilon\left(V_{\mu+1}, \ldots, V_n; V_{\mu+1}, \ldots, V_n\right)} = \\ &= \epsilon\left(V_\mu, V_{\mu+1}, \ldots, V_n; V_\mu, V_{\mu+1}, \ldots, V_n\right) \end{aligned} \tag{2.47}$$

which is exactly Eq. (2.35). □

Remark 2.3.
The matrix K is given by solving equation Eq. (2.32) for K.

2.2 Automorphic Forms

Having defined the relevant concepts from Lie theory, we proceed by introducing the important class of functions called automorphic forms. An important subclass of automorphic forms are the so called Eisenstein series which will play a special role in this thesis.

Very colloquially speaking, an automorphic form is a complex valued function φ defined on a Lie group $G(\mathbb{R})$ satisfying three special properties:

1. Transform nicely under the discrete subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$: $\varphi(\gamma g) = j(\gamma)\varphi(g)$ for $\gamma \in G(\mathbb{Z})$. The complex valued function j here is called the *factor of automorphy*. This *functional relation* effectively cuts down the domain of φ to the left coset $G(\mathbb{Z})\backslash G(\mathbb{R})$.

2. Satisfy eigenvalue differential equations: The Lie algebra of $G(\mathbb{R})$ acts on φ as differential operators. We ask that φ is an eigenfunction under certain G-invariant differential operators such as the quadratic Casimir: $(\Delta_{G(\mathbb{Z})\backslash G(\mathbb{R})} - \mu)\varphi = 0$ for some $\mu \in \mathbb{C}$.

3. Satisfy moderate growth condition: The automorphic form may grow at most polynomially as we approach the boundaries on its domain.

The special class of automorphic forms on $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ are called *modular forms*.

For the purposes of this thesis, we are only interested in automorphic forms which are invariant under the discrete subgroup $G(\mathbb{Z})$, i.e. the factor of automorphy is trivial $j = 1$. Furthermore, we will only study automorphic forms which are also invariant under the right action of the maximal compact subgroup $K \subset G$. Thus we take the following as our definition of an automorphic form:

Definition 2.4 (Automorphic form)
An ***automorphic form*** *φ is a complex valued function $\varphi : G(\mathbb{R}) \to \mathbb{C}$ defined on the Lie group $G(\mathbb{R})$ satisfying*

1. *Left $G(\mathbb{Z})$-invariance ("automorphy"): $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in G(\mathbb{Z})$.*

2. *Right K-invariance ("sphericity"): $\varphi(gk) = \varphi(g)$ for all $k \in K$*

3. *Satisfy eigenvalue differential equations under action of the ring of G-invariant differential operators, such as the quadratic Casimir.*

4. *Satisfy moderate the growth condition that for any norm $||\cdot||$ on $G(\mathbb{R})$ we have $|\varphi(g)| \leq C||g||^n$ for some constant $C \in \mathbb{C}$ and $n \in \mathbb{N}$.*

Remark 2.5.
The property

$$\varphi(\gamma g k) = \varphi(g) \quad \text{for all} \quad \gamma \in G(\mathbb{Z}) \quad \text{and} \quad k \in K \tag{2.48}$$

implies that φ is really a function on a *double coset space*, $\varphi : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$.

We will now discuss the special class of automorphic forms called Eisenstein series which are the objects of interest for applications to string theory. Constructing an Eisenstein series is fairly straightforward. We looking to construct a function E which is left-invariant under $G(\mathbb{Z})$ and right invariant under K. Start by choosing some function $\chi : G(\mathbb{R}) \to \mathbb{C}$. Next, observe that one way of obtaining a function E which is manifestly left-invariant under $G(\mathbb{Z})$ is to define $E(g)$ as the average of $\chi(g)$ over the whole subgroup $G(\mathbb{Z})$. It could be the case that χ is stabilized by some subgroup $\Gamma_\infty \subset G(\mathbb{Z})$ so that should better be taken into account and we are looking at something like $E(g) = \sum_{\gamma\in\Gamma_\infty\backslash G(\mathbb{Z})} \chi(\gamma g)$. By also choosing χ to be right invariant under K, we get $E : G(\mathbb{Z})\backslash G(\mathbb{R})/K \to \mathbb{C}$. Finally, an Eisenstein series is obtained by furthermore choosing χ to be a *character* (see definition 2.16) on a parabolic subgroup $P(\mathbb{R})$ of $G(\mathbb{R})$. Such characters are labelled by weight vectors λ. As there are many different parabolic subgroups, we can construct many different Eisenstein series. Starting with the minimal parabolic subgroup, the Borel subgroup $B(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R}) \subset G(\mathbb{R})$, leads to the notion of the minimal parabolic Eisenstein series. We introduce the logarithm map $H : G(\mathbb{R}) \to \mathfrak{h}$ obeying $H(g) = H(nak) = H(a)$ such that $g \in Ne^{H(g)}K$.

Definition 2.6 (Minimal parabolic Eisenstein series)
The ***minimal parabolic Eisenstein series*** *for G is the coset sum*

$$E^G_\lambda(g) \equiv \sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} \mathrm{e}^{\langle\lambda+\rho|H(\gamma g)\rangle} \tag{2.49}$$

where $\lambda \in \mathfrak{h}\otimes\mathbb{C}$ and the Weyl vector ρ is included for a convenient normalization.

This sum is absolutely convergent as long as λ lies in the so called Godement range, meaning that the real part of $\langle\alpha|\lambda\rangle$ should be sufficiently large for all positive roots:

$$\mathrm{Re}\langle\alpha|\lambda\rangle > \langle\rho|\alpha\rangle \quad \text{for all} \quad \alpha\in\Pi_+. \tag{2.50}$$

The power functions[5] $\mathrm{e}^{\langle\lambda+\rho|H(\gamma g)\rangle}$ and hence also E^G_λ itself are eigenfunctions under the Laplace-Beltrami operator [23],

$$\Delta_{G/K}E^G_\lambda = \frac{1}{2}(\langle\lambda|\lambda\rangle - \langle\rho|\rho\rangle). \tag{2.51}$$

In fact, E^G_λ are eigenfunctions under the full ring of G-invariant differential operators, as required for the Eisenstein series E^G_λ to be an automorphic form. Using this fact, Langlands was able to show that E^G_λ can be meromorphically continued to an automorphic function on all of $G(\mathbb{Z})\backslash G(\mathbb{R})$ [24]

The name "minimal" in definition 2.6 refers to the role of the Borel subgroup $B = NA$ of G, also called the minimal parabolic subgroup. Another class of Eisenstein series are the so called maximal parabolic Eisenstein series, in which the Borel group is replaced

[5]The terms $\mathrm{e}^{\langle\lambda+\rho|H(\gamma g)\rangle}$ appearing in Eq. (2.49) are power behaved in the Cartan coordinates y_α, since $a \in A$ is parametrized as $= \mathrm{e}^{\sum_\alpha \ln(y_\alpha)H_\alpha}$.

with a maximal parabolic subgroup P_β due to an enlarged invariance in the character. This invariance is present if $\langle \lambda + \rho | H(g) \rangle$ is unchanged when g is left multiplied by any element of $P_\beta(\mathbb{Z})$, or equivalently when $\lambda + \rho$ is orthogonal to every simple root other than β. We then have that λ is restricted to lie on a line in $\mathfrak{h}^\star \times \mathbb{C}$. This case is most easily parametrized in terms of the fundamental weights by a single complex parameter s as

$$\lambda = 2s\Lambda_\beta - \rho. \tag{2.52}$$

Definition 2.7 (Maximal parabolic Eisenstein series)
Given a maximal parabolic subgroup P_β defined by a simple root β, the ***maximal parabolic Eisenstein series*** *for P_β is the coset sum*

$$E^G_{\beta;s}(g) \equiv \sum_{\gamma \in P_\beta(\mathbb{Z}) \backslash G(\mathbb{Z})} \mathrm{e}^{2s\langle \Lambda_\beta | H(\gamma g) \rangle}. \tag{2.53}$$

Remark 2.8.
There is a more general notion of maximal parabolic Eisenstein series in which an automorphic function ϕ on the Levi component L_β of $P_\beta = U_\beta L_\beta$ is included in the sum,

$$\sum_{\gamma \in P(\mathbb{Z}) \backslash G(\mathbb{Z})} \mathrm{e}^{2s\langle \Lambda_\beta | H(\gamma g) \rangle} \phi(\gamma g). \tag{2.54}$$

The maximal parabolic Eisenstein series defined above in definition 2.7 is of this form with $\phi \equiv 1$. It can be shown that a minimal parabolic Eisenstein series, Eq. (2.49), can be brought to this form and for λ of the form Eq. (2.52) we then get $\phi = 1$ and the notions of minimal- and maximal parabolic Eisenstein series coincide [25].

Remark 2.9.
The notion of Eisenstein series also exists for parabolic subgroups other than the minimal one and maximal ones, although they will not be of importance in this thesis. See [18] for a discussion.

For each Weyl word w, Eisenstein series satisfy the functional relation

$$E^G_\lambda(g) = M(w, \lambda) E^G_{w\lambda}(g) \tag{2.55}$$

where the function

$$M(w, \lambda) = \prod_{\substack{\alpha > 0 \\ w\alpha < 0}} \frac{\xi(\langle \lambda | \alpha \rangle)}{\xi(\langle \lambda | \alpha \rangle + 1)} \tag{2.56}$$

is called the *intertwiner* and the function

$$\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) \tag{2.57}$$

is called the *completed Riemann zeta function*. The Riemann zeta function ζ is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \operatorname{Re} s > 1 \tag{2.58}$$

and in his 1859 paper, Bernhard Riemann famously found the analytic continuation of ζ to the whole complex plane except at $s = 0$ and $s = 1$ where it has simple poles. This analytic continuation of $\zeta(s)$ exploits the functional relation

$$\xi(s) = \xi(1 - s). \tag{2.59}$$

The intertwiner, the Riemann zeta function and the completed Riemann zeta function will play important roles later in this thesis and a context for how these functions appear will be given in Chapter 4.

Example 2.10

Let's look at the easiest example of an Eisenstein series, namely for $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$. The only possible parabolic subgroup is all of G itself, with Levi subgroup equal to the Borel subgroup, thus there is only one type of Eisenstein series of the types discussed above that we can write down. We parametrize a group element as

$$\mathrm{SL}_2(\mathbb{R}) \ni g = nak = \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} k. \tag{2.60}$$

The square root for the SL_2-case is a matter of convention, higher rank groups are still parametrized as in Eq. (2.30). There is now only one simple root α and one fundamental weight Λ which equals the Weyl vector, thus we have

$$\lambda = 2s\Lambda - \rho \tag{2.61}$$

and

$$e^{\langle 2s\Lambda | H(g)\rangle} = y^s. \tag{2.62}$$

An element

$$l = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \tag{2.63}$$

of the arithmetic subgroup $\mathrm{SL}_2(\mathbb{Z})$ must satisfy $ad - bc = 1$. We can view this constraint as requiring $\gcd(c, d) = 1$ and Bézout's lemma then states that there exists a one-parameter family of solutions parametrized by $m \in \mathbb{Z}$ of the form

$$\left(\begin{smallmatrix} a_0 + mc & b_0 + md \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a_0 & b_0 \\ c & d \end{smallmatrix}\right) \tag{2.64}$$

where (a_0, b_0) is some solution whose existence is guaranteed by the lemma. Since the arithmetic Borel group is

$$B(\mathbb{Z}) = \left\{\left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right) : m \in \mathbb{Z}\right\}, \tag{2.65}$$

summing over $B(\mathbb{Z})\backslash G(\mathbb{Z})$ is achieved by summer over c and d coprime. To figure out the summand, we need an expression for the Cartan-coordinate y of the matrix lg. Using the closed formulae for the axions and dilatons of a general matrix from Section 2.1.4, we get

$$lg = lnak = \begin{pmatrix} a\sqrt{y} & \frac{b+ax}{\sqrt{y}} \\ c\sqrt{y} & \frac{d+cx}{\sqrt{y}} \end{pmatrix} k = \left(\begin{smallmatrix} 1 & x' \\ & 1 \end{smallmatrix}\right)\begin{pmatrix} \sqrt{y'} & \\ & 1/\sqrt{y'} \end{pmatrix} k' \quad \text{where} \quad y'^{-1} = c^2 y + \frac{(d+cx)^2}{y}. \tag{2.66}$$

Notice that we can write y' in term of a complex variable $z = x + iy$ as

$$y' = \frac{y}{|cz+d|^2} = \operatorname{Im} \underbrace{\frac{az+b}{cz+d}}_{z'} . \tag{2.67}$$

In fact, using the formulae for Iwasawa decomposition in Section 2.1.4, one can also verify that $x' = \operatorname{Re} z'$. The appearance of the complex coordinate z is due to the isomorphism $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \mathbb{H}$ with the Poincaré upper half plane $\mathbb{H}$ on which the *modular group* $SL_2(\mathbb{Z})$ acts as a Möbius transformation. This link is a good tool to gain an intuitive understanding of Eisenstein series but will not be available for the higher rank cases which will be analyzed later in this thesis.

We can now write

$$E_s^{SL_2}(g) = \sum_{\gcd(c,d)=1} \frac{y^s}{|c+dz|^{2s}} . \tag{2.68}$$

This function is referred to as a *non-holomorphic Eisenstein series* and was the first example of how automorphic forms encode perturbative as well as non-perturbative information about four graviton scattering in ten dimensional type IIB string theory as realized by Green and Gutperle in 1997 [1]. This link will be explained in Chapter 3.

Remark 2.11.
For the case $G(\mathbb{R}) = SL_2(\mathbb{R})$, i.e. the study of modular forms, there are many more interesting types of Eisenstein series, for instance the class of *holomorphic* modular forms. These functions do not however generalize to all higher rank groups and will therefore not play a role in this thesis.

2.3 Fourier expansions

The $G(\mathbb{Z})$-invariance of automorphic forms should be understood as a generalization of periodicity. As such, automorphic forms lend themselves to being analyzed using Fourier theory and expressed as a Fourier series. Consider an automorphic form $\varphi : SL_3(\mathbb{Z})\backslash SL_3(\mathbb{R})/SO_3(\mathbb{R}) \to \mathbb{C}$. In general, φ depends on five real variables, three axions x_1, x_2, x_3 which parametrize the maximal unipotent N and two dilatons y_1, y_2 which parametrize the Cartan torus A. Due to the left $G(\mathbb{Z})$-invariance, we have

$$\varphi(g) = \varphi\Bigg(\underbrace{\begin{pmatrix} 1 & n_1 & n_2 \\ & 1 & n_3 \\ & & 1 \end{pmatrix}}_{l} \underbrace{\underbrace{\begin{pmatrix} 1 & x_1 & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix}}_{n} \underbrace{\begin{pmatrix} y_1 & & \\ & y_2/y_1 & \\ & & 1/y_2 \end{pmatrix}}_{a} k}_{g} \Bigg) = \varphi\left(\begin{pmatrix} 1 & n_1+x_1 & n_2+x_2+n_1x_3 \\ & 1 & n_3+x_3 \\ & & 1 \end{pmatrix} ak \right) \tag{2.69}$$

for a matrix $l \in N(\mathbb{Z})$. We see that φ is periodic in the variable x_3 in the sense that for integer shifts $x_3 \to n_3 + x_3$, the function is completely invariant. The same is true for the variable x_2 as long as $n_1 = 0$. For x_1, we can compensate a shift $x_1 \to n_1 + x_1$ by a shift in the variable x_2. As such, the automorphic form φ is periodic in all three axions x_1, x_2 and x_3, although the periodicities are intertwined in a nontrivial way. The precise way in which the periodicity of an automorphic form on a group $G(\mathbb{R})$ works depends on the group structure and is captured in the root structure of its Lie algebra.

It is nevertheless still possible to express an automorphic form as a Fourier series, i.e. as a sum over basis functions multiplied by Fourier coefficients. In this procedure, one has the freedom of choosing which periodic degrees of freedom one wishes to capture in terms of Fourier coefficients. For the automorphic form φ above, we may decide to expand over all three axions but we may also decide to only expand over say x_3 and leave x_1 and x_2 unexpanded. These choices correspond to choosing a parabolic subgroup P of G. In the Levi decomposition $P = UL$, the degrees of freedom in $U \subset N$ are the ones which then will be expanded as a Fourier series. Extra care must be taken whenever the unipotent U is non-abelian, as will be explained below. Some good references for Fourier expansions of automorphic forms and Eisenstein series in particular include [18, 26].

Definition 2.12 (Commutator subgroup)
*Given a Lie group G, the **commutator subgroup** is denoted $[G, G]$ and is defined as*

$$[G, G] = \left\{ghg^{-1}h^{-1} : g, h \in G\right\}. \tag{2.70}$$

Definition 2.13 (Derived series)
*Given a Lie group G and defining $G^{(0)} \equiv G$, the **derived series** $G^{(i)}$, $i \in \mathbb{N}_0$ is formed by iterated commutator subgroups*

$$G^{(i)} = [G^{(i-1)}, G^{(i-1)}]. \tag{2.71}$$

Definition 2.14 (Abelianization)
*The **abelianization** of a group G is defined by*

$$G/[G, G]. \tag{2.72}$$

Remark 2.15.
The abelianization of a group G is abelian (commutative), since for $g, h \in G/[G, G]$ we have

$$gh \sim hgg^{-1}h^{-1}gh = hg \tag{2.73}$$

as representatives for a coset in $G/[G, G]$.

Definition 2.16 (Character)
*A **character** ψ is a complex valued group homomorphism.*

Remark 2.17.
The domain of a character is the abelianization of the group, since

$$\psi(gh) = \psi(g)\psi(h) = \psi(h)\psi(g) = \psi(hg). \tag{2.74}$$

The first and last equalities use the homomorphism property and the remaining equality uses the fact that c-numbers commute.

Example 2.18

Consider $G = \mathrm{SL}_4(\mathbb{R})$. The derived subgroups of the maximal unipotent subgroup N are

$$N^{(0)} = \left\{ \begin{pmatrix} 1 & x_1 & x_3 & x_6 \\ & 1 & x_2 & x_4 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} \right\}, \quad N^{(1)} = \left\{ \begin{pmatrix} 1 & & x_3 & x_6 \\ & 1 & & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, \quad N^{(2)} = \left\{ \begin{pmatrix} 1 & & & x_6 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, \quad N^{(\geq 3)} = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}. \tag{2.75}$$

Their abelianizations are

$$\begin{aligned} N^{(0)}/[N^{(0)}, N^{(0)}] &= \left\{ \begin{pmatrix} 1 & x_1 & & \\ & 1 & x_2 & \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} \right\}, \quad N^{(1)}/[N^{(1)}, N^{(1)}] = \left\{ \begin{pmatrix} 1 & & x_3 & \\ & 1 & & x_4 \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\} \\ N^{(2)}/[N^{(2)}, N^{(2)}] &= \left\{ \begin{pmatrix} 1 & & & x_6 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}, \quad N^{(\geq 3)}/[N^{(\geq 3)}, N^{(\geq 3)}] = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\}. \end{aligned} \tag{2.76}$$

A character ψ on a derived subgroup $U^{(i)}$ of a unipotent U is determined by its restriction to the abelianization $U^{(i)}/[U^{(i)}, U^{(i)}]$ as was shown above and parametrized by real numbers m_α, one number for each "simple root" in $U^{(i)}/[U^{(i)}, U^{(i)}]$. This fact is a consequence of the homomorphism property for ψ. Comparing with example 2.18, a character on $N^{(1)}$ for example depends on the coordinates x_3 and x_4 (but not on x_6 and is parametrized by two numbers m_3 and m_4. More generally, letting $\Delta^{(i)}(\mathfrak{u})$ denote the (positive) roots of $U^{(i)}$, a character ψ is then given by

$$\psi\left(\exp\left(\sum_{\alpha \in \Delta^{(i)}(\mathfrak{u})} x_\alpha E_\alpha\right)\right) = \prod_{\alpha \in \Delta^{(i)}(\mathfrak{u}) \setminus \Delta^{(i+1)}(\mathfrak{u})} e^{2\pi i m_\alpha x_\alpha}. \tag{2.77}$$

The character is trivial on $U(\mathbb{Z})$ if and only if all m_α are integers. We use the following vocabulary for characters

Definition 2.19 (Degenerate, generic and unramified character, charges)
*We say that a character ψ is **generic** if all m_α that define it are non-zero. We say that ψ is **degenerate** if only some (but not all) m_α are non-zero. We say that ψ **maximally degenerate** if all but one m_α vanish. We call ψ **unramified** if all non-zero m_α equal one. See remark 2.23.*

*The numbers m_α will be referred to as **charges** since their interpretation in a string theory context are as instanton charges, see Chapter 3. We say that a character with $m_\alpha \neq 0$ is charged on the root α (see notation in Appendix A).*

Having discussed multiplicative characters we can proceed and define the notion of a Fourier coefficient.

Definition 2.20 (Fourier coefficient)
Given an automorphic form φ on a Lie group G, a parabolic subgroup P thereof with Levi decomposition $P = UL$ and a character ψ on the unipotent $U(\mathbb{R})$ which is trivial

on $U(\mathbb{Z})$, we define a ***Fourier coefficient*** F^U_ψ *by*

$$F^U_\psi(g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\overline{\psi(u)}\,\mathrm{d}u \tag{2.78}$$

where the measure du *is the Haar measure on U, normalized so that $U(\mathbb{Z})\backslash U(\mathbb{R})$ itself has unit volume. We suppress the dependence on the automorphic form φ for the Fourier coefficient $F^U_\psi(g)$.*

Definition 2.21 (Whittaker function)
The special case of a Fourier coefficient of an Eisenstein series E^G_λ on the maximal unipotent U of the Borel subgroup $B = UL$ is called a ***Whittaker function*** *and denoted*

$$W_\psi(g) = \int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^G_\lambda(ng)\overline{\psi(n)}\,\mathrm{d}n. \tag{2.79}$$

We suppress the dependence on the weight vector λ for the Whittaker function $W_\psi(g)$.

Remark 2.22.
We use the same vocabulary as in definition 2.19 also for Fourier coefficients. An example of an unramified generic Whittaker function on $\mathrm{SL}_4(\mathbb{R})$ is

$$\int\limits_{(\mathbb{Z}\backslash\mathbb{R})^6} \varphi\left(\begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} g\right) \mathrm{e}^{2\pi\mathrm{i}(x_1+x_2+x_3)}\,\mathrm{d}^6x. \tag{2.80}$$

An example of a (ramified) degenerate Whittaker function is

$$\int\limits_{(\mathbb{Z}\backslash\mathbb{R})^6} \varphi\left(\begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix} g\right) \mathrm{e}^{2\pi\mathrm{i}(m_1x_1+m_2x_2)}\,\mathrm{d}^6x. \tag{2.81}$$

Remark 2.23.
A Fourier coefficient $F^U_\psi(g)$ (or $W_\psi(g)$) for a spherical automorphic form $\varphi(gk) = \varphi(g)$ is determined by its restriction to L (or A) since

$$\begin{aligned} F^U_\psi(ulk) &= \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(u'ulk)\overline{\psi(u')}\,\mathrm{d}u' = \\ &= \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(u''l)\overline{\psi(u''u^{-1})}\,\mathrm{d}(u''u^{-1}) = \\ &= \psi(u)F^U_\psi(l) \end{aligned} \tag{2.82}$$

where we have used that the Haar measure is invariant under translations. The corresponding equation for Whittaker functions is

$$W_\psi(nak) = \psi(n) W_\psi(a) \tag{2.83}$$

By summing up the Fourier coefficients for all characters on $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$, we capture the dependency of the automorphic form φ for the variables on which the characters depend (x_1, x_2 and x_3 in example 2.18). Said differently, we capture the dependency of φ on the abelianization of U [26],

$$\int\limits_{U^{(1)}(\mathbb{Z})\backslash U^{(1)}(\mathbb{R})} \varphi(ug)\,\mathrm{d}u = \sum_{\psi\in\mathfrak{C}(U)} F^U_\psi(g) \tag{2.84}$$

where $\mathfrak{C}(U)$ denotes the group of characters on $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$. The effect of the integral in the left hand side is to average out the dependency of φ on what lies outside the abelianization of U, i.e. the variables which the Fourier coefficients F^U_ψ cannot capture (x_4, x_5 and x_6 in example 2.18). To capture the entire automorphic form φ, we need to include Fourier coefficients on the derived subgroups of U. The complete Fourier expansions thus looks like

$$\varphi(g) = \sum_{\psi\in\mathfrak{C}(U)} F^U_\psi(g) + \underbrace{\sideset{}{'}\sum_{\psi\in\mathfrak{C}(U^{(1)})} F^{U^{(1)}}_\psi(g) + \ldots}_{\text{Non-abelian Fourier coefficients}} \;. \tag{2.85}$$

One term in the first sum corresponds to the trivial character $\psi = 1$. We distinguish this from the remaining terms by referring to the Fourier coefficient $F^U_{\psi=1}(g)$ as the *constant term* and the remaining Fourier coefficients $F^{U^{(i)}}_{\psi\neq1}(g)$ as *non-constant terms*. Since the character $\psi = 1$ has already been accounted for, it should not be included again among the non-abelian Fourier coefficients which is indicated by the prime on the sum over $\mathfrak{C}(U^{(1)})$.

Remark 2.24.
The constant term is by no means a constant, rather it is a function only on the Levi subgroup L but constant with respect to the unipotent U in the Levi decomposition $P = UL$.

Remark 2.25.
Let's contrast this program for Fourier expansion to the well known case of a periodic function f of one real variable, $f(x+1) = f(x)$. We have $G(\mathbb{R}) = \mathbb{R}$ under addition. This case doesn't quite fall under what is discussed above, since for example $\mathbb{R}$ does not have any unipotent subgroups but the comparison can still be made. The characters on

$\mathbb{R}$ trivial on $\mathbb{Z}$ are $\psi_m(x) = e^{2\pi i m x}$ where $m \in \mathbb{Z}$. The Fourier coefficients are

$$F^{\mathbb{R}}_{\psi_m}(x) = \int\limits_{\mathbb{Z}\backslash\mathbb{R}} f(x+u) e^{-2\pi i m u}\, du = \underbrace{\int\limits_{\mathbb{Z}\backslash\mathbb{R}} f(u) e^{-2\pi i m u}\, du}_{c_m \in \mathbb{C}}\, e^{2\pi i m x}. \tag{2.86}$$

and the Fourier expansion of f reads

$$f(x) = \sum_{m\in\mathbb{Z}} c_m e^{2\pi i m x} \tag{2.87}$$

which should be well known. The difference in vocabulary to what readers may be used to is that we call the entire object $F^{\mathbb{R}}_{\psi_m}(x)$ a Fourier coefficient, including both the c-numbers c_m as well as the basis functions ψ_m.

Example 2.26

We consider the Fourier expansion of the non-holomorphic Eisenstein series $E^{SL_2}_s$ given in example 2.10. The only available (proper) parabolic subgroup of $SL_2(\mathbb{R})$ is the Borel group

$$B(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y & \\ & 1/y \end{smallmatrix}\right) : x, y \in \mathbb{R}, y > 0 \right\} \tag{2.88}$$

and hence the only Fourier coefficients to calculate are the Whittaker functions $W_m(g)$. Characters $\psi_m : N(\mathbb{Z})\backslash N(\mathbb{R}) \to \mathbb{R}$ on the unipotent $N(\mathbb{R})$ trivial on $N(\mathbb{Z})$ are labelled by integers m and are given by

$$\psi_m\left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\right) = e^{2\pi i m x}. \tag{2.89}$$

Parametrizing the group element g as in Eq. (2.60), we are faced with the integrals

$$W_m(g) = \int\limits_{N(\mathbb{Z})\backslash N(\mathbb{R})} E^{SL_2}_s(ng)\overline{\psi_m(n)}\, dn = \int\limits_{\mathbb{Z}\backslash\mathbb{R}} \sum_{\gcd(c,d)=1} \frac{y^s}{|c + d(x+t+iy)|^{2s}} e^{2\pi i m t}\, dt. \tag{2.90}$$

This integral is non-trivial to calculate so at this stage we will be satisfied with simply stating the answer and give a few remarks. One finds

$$W_0(g) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s} \tag{2.91}$$

$$W_m(g) = \frac{2y^{1/2}}{\xi(2s)} |m|^{s-1/2} \sigma_{1-2s}(m) K_{s-1/2}(2\pi|m|y) e^{2\pi i m x}, \quad m \neq 0 \tag{2.92}$$

where σ_{1-2s} denotes the *divisor sum* given in Eq. (C.13) and $K_{s-1/2}$ is the modified Bessel function of the second kind. Since the maximal unipotent N is abelian for SL_2, we have that the non-holomorphic Eisenstein series can be written

$$E^{SL_2}_s(g) = \sum_{m\in\mathbb{Z}} W_m(g). \tag{2.93}$$

Remark 2.27.

The SL_2-Whittaker function Eq. (2.90) is the simplest example of a Fourier coefficient of an automorphic form the way they are defined in this thesis. The integral itself is difficult to calculate and the generalizations of this expression to automorphic forms on higher rank groups will give even more complicated integrals. There are several different techniques for dealing with thees types of integrals and this thesis will focus on the so called *adelic* method which is explained in Chapter 4. In some sense, these SL_2-Whittaker functions will turn out to be the building block of Fourier coefficients (not necessarily Whittaker functions) for certain automorphic forms on $SL_n(\mathbb{R})$ for arbitrary n.

Remark 2.28.
One method for arriving at the expressions Eqs. (2.91) and (2.92) is by so called *Poisson resummation*. The simplest example of Poisson resummation concerns a periodic function of one real variable $f(x+1) = f(x)$. Poisson resummation then states that

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k) \tag{2.94}$$

where $\hat{f}$ is the Fourier transform of f. By rewriting Eq. (2.68) it is possible to massage the expression until it has the form $E_s^{SL_2}(g) = \sum_{m\in\mathbb{Z}} c_m(y)\mathrm{e}^{2\pi imx}$. After this algebraic exercise, one can simply read off the Whittaker functions Eqs. (2.91) and (2.92). This technique generalizes poorly to higher rank groups as it is not always easy to express an Eisenstein series E_λ^G as a lattice sum such that Poisson resummation can be used.

Remark 2.29.
The adelic method relies on the existence of number fields other than the reals $\mathbb{R}$, called the *p-adic numbers* $\mathbb{Q}_p$. One proceeds by enlarging the domain of the automorphic forms to the so called adeles which include both the real as well as the p-adic numbers. Operating over the adeles greatly facilitates the calculation of Fourier coefficients. From the Fourier expansion of this "enlarged" adelic automorphic form one can then "project out" the Fourier expansion for the automorphic form defined only over the reals. In this context, the various factors that make up Eq. (2.92) can be understood as contributions coming from the real and p-adic parts of the adelic Fourier coefficient. This technique will be explored in detail in Chapters 4 and 6.

3 Automorphic Forms in String Theory

This thesis will illuminate how the theory of automorphic forms can be used to calculate graviton scattering amplitudes in type IIB string theory and access the non-perturbative part of the spectrum. In this chapter we shall make the connection between type IIB string theory and automorphic forms. **Proceedings I** offers a condensed version of this chapter.

Speaking mathematically, a scattering amplitude in string theory is a function of the data of the asymptotic states, the dimensionful parameter α' (of dimension length^2) as well the scalar fields which are present in the theory, called the *moduli*. Much as the string coupling g_s is given as the vacuum expectation value $g_s = \mathrm{e}^{\langle\phi\rangle}$ of the dilaton ϕ, we only need to model how an amplitude depends on the vacuum expectation values of the moduli. The set of all possible vacuum expectation values of the moduli is called *moduli space* and is denoted $\mathcal{M}$. We will let $g \in \mathcal{M}$ denote a general element of the moduli space.

It is often fruitful to write a scattering amplitude as a series expansion in some of its arguments. When the energies of the scattering states are small (in units of α', which sets the energy scale of the theory), one useful expansion is the so called *low-energy expansion* or *α'-expansion*. One forms dimensionless quantities by combining powers of α' with powers of the momenta. The particular scattering process we will be investigating is that of four gravitons in $D = 10 - d$ dimensions. Let k_i and ϵ_i $(i = 1, 2, 3, 4)$ denote the momenta and polarizations of the gravitons respectively. We form the dimensionless Lorenz invariant *Mandelstam variables*

$$s = -\frac{\alpha'}{4}(k_1 + k_2)^2, \quad t = -\frac{\alpha'}{4}(k_1 + k_3)^2, \quad u = -\frac{\alpha'}{4}(k_1 + k_4)^2, \tag{3.1}$$

satisfying $s + t + u = 0$ due to momentum conservation. Any symmetric polynomial in s, t, u can then be written as a polynomial in

$$\sigma_2 = s^2 + t^2 + u^2 \quad \text{and} \quad \sigma_3 = s^3 + t^3 + u^3 \tag{3.2}$$

and since an amplitude should posses the property of duality in the sense discussed in Section 1.2 we can expand it as a power series in σ_2 and σ_3. Similar simplifications allow us to capture the polarization dependence in a quantity we shall denote $\mathcal{R}^4$ which simply denotes the contraction of four linearised Riemann tensors $\mathcal{R}$ with two copies of the standard rank-8 tensor t_8 [27]. The (analytic part of the) four-graviton amplitude can therefore be expressed as [28]

$$\mathcal{A}^{(D)}(s,t,u,\epsilon_i;g) = \left(\mathcal{E}^{(D)}_{(0,-1)}(g)\frac{1}{\sigma_3} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\mathcal{E}^{(D)}_{(p,q)}(g)\sigma_2^p\sigma_3^q\right)\mathcal{R}^4 \tag{3.3}$$

The non-polynomial σ_3^{-1}-term is special in that it constitutes the contribution from pure supergravity, where $\mathcal{E}^{(D)}_{(0,-1)} = 3$. The remaining $\mathcal{E}^{(D)}_{(p,q)}$ are to be found.

Note that the terms in the double sum have increasing powers of the dimensionful constant α' whose dimensions are compensated by powers of the momenta. In a low-energy effective theory, the momenta turn into derivatives and the corresponding low-energy effective action takes the form [29]

$$S = \int \mathrm{d}^D x\sqrt{-G}\left(R + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(\alpha')^{3+2p+3q}\mathcal{E}^{(D)}_{(p,q)}(g)\nabla^{4p+6q}R^4\right) \tag{3.4}$$

where R denotes the Riemann tensor and $\nabla^{4p+6q}R^4$ denotes $4p+6q$ covariant derivatives acting on the same contraction of four Riemann tensors as that of $\mathcal{R}^4$. This action is the purely gravitational sector of the low-energy effective action of type IIB string theory and the higher derivative corrections to the supergravity Lagrangian R come from massive string states [27].

Let us discuss what is known about the moduli space $\mathcal{M}$. Cremmer and Julia showed [30, 31] that a maximally supersymmetric classical theory of gravity with a symmetry group $G(\mathbb{R})$ has the moduli space $\mathcal{M} = G(\mathbb{R})/K(G(\mathbb{R}))$. Furthermore, the precise symmetry groups $G(\mathbb{R})$ of toroidal compactifications of 11-dimensional supergravity to D dimensions were found in [32] and are given in table 3.1, their Dynkin diagrams are shown in Fig. 3.1. The low-energy limit of type IIB string theory and its toroidal compactifications to D dimensions are the D-dimensional maximal ungauged supergravities whose classical symmetry groups are those stated in table 3.1. For the full string theory however, quantum effects break this symmetry such that only discrete symmetries remain. These discrete symmetries are called dualities. One specific group of dualities are the so called T-dualities which arise from compactification on a torus and effectively interchange the (quantized) momenta in the compactified directions with the winding numbers of the strings [17]. Hull and Townsend [33] showed that the T-dualities combine with S-duality (under which type IIB is self dual) in a non-trivial way into the so called U-duality group, which here coincides with the arithmetic subgroup $G(\mathbb{Z}) \subset G(\mathbb{R})$ of the classical symmetry group. Furthermore, U-duality invariance must hold at each order in α' in Eq. (3.4). This effectively cuts down the moduli space to the "quantum" moduli space

$$\mathcal{M}_{\text{quant}} = G(\mathbb{Z})\backslash G(\mathbb{R})/K(G(\mathbb{R})) \tag{3.5}$$

which is the domain of the functions $\mathcal{E}^{(D)}_{(p,q)}$ appearing in the low-energy effective action Eq. (3.4). An equivalent viewpoint is that $\mathcal{E}^{(D)}_{(p,q)}$ are $G(\mathbb{Z})$-invariant functions on

$D=10-d$	$G(\mathbb{Z}) \equiv E_{d+1}(\mathbb{Z})$	$G(\mathbb{R}) \equiv E_{d+1}(\mathbb{R})$	$K(G(\mathbb{R}))$
10	$\mathrm{SL}_2(\mathbb{Z})$	$\mathrm{SL}_2(\mathbb{R})$	SO_2
9	$\mathrm{SL}_2(\mathbb{Z}) \times \mathbb{Z}_2$	$\mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}^+$	SO_2
8	$\mathrm{SL}_3(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$	$\mathrm{SL}_3(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$	$\mathrm{SO}_3 \times \mathrm{SO}_2$
7	$\mathrm{SL}_5(\mathbb{Z})$	$\mathrm{SL}_5(\mathbb{R})$	SO_5
6	$\mathrm{Spin}_{5,5}(\mathbb{Z})$	$\mathrm{Spin}_{5,5}(\mathbb{R})$	$(\mathrm{Spin}_5 \times \mathrm{Spin}_5)/\mathbb{Z}_2$
5	$E_{6(6)}(\mathbb{Z})$	$E_{6(6)}(\mathbb{R})$	$\mathrm{USp}_8/\mathbb{Z}_2$
4	$E_{7(7)}(\mathbb{Z})$	$E_{7(7)}(\mathbb{R})$	$\mathrm{SU}_8/\mathbb{Z}_2$
3	$E_{8(8)}(\mathbb{Z})$	$E_{8(8)}(\mathbb{R})$	$\mathrm{Spin}_{16}/\mathbb{Z}_2$

Table 3.1: The table lists the classical symmetry groups $G(\mathbb{R})$ of maximal ungauged supergravity in $D = 10 - d$ dimensions along with the U-duality groups $G(\mathbb{Z})$ which are the discrete symmetries that survive after quantum effects are turned on in the corresponding type IIB string theory on a d-torus. These groups are sometimes called the Cremmer-Julia sequence of hidden symmetries and denoted E_{d+1}. The table also lists the maximal compact subgroups $K(G(\mathbb{R})) \subset G(\mathbb{R})$. This data specifies the quantum moduli space in Eq. (3.5).

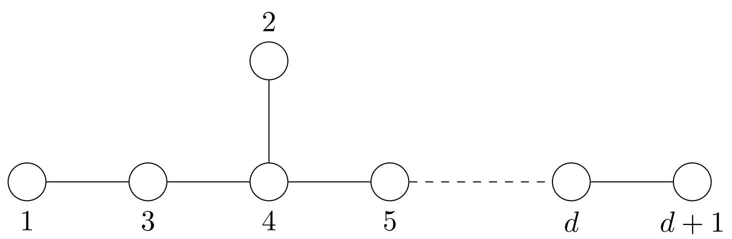

Figure 3.1: The Dynkin diagram of E_{d+1}, see also table 3.1. The nodes are added sequentially in specified order, also called Bourbaki labelling. The simple roots in the labeling will be called α_i, $i = 1, \ldots, d+1$.

$G(\mathbb{R})/K(G(\mathbb{R})$, and hence satisfy 1 and 2 in definition 2.4.

Among the string moduli is the string coupling g_s and by making the g_s-dependence of $\mathcal{E}^{(D)}_{(p,q)}$ explicit we can make recourse with string perturbation theory and extract the physical data that the automorphic functions $\mathcal{E}^{(D)}_{(p,q)}$ capture. String perturbation theory accesses the analytic dependence on g_s, i.e. the part of $\mathcal{E}^{(D)}_{(p,q)}$ which can be represented as a Taylor series in g_s. There is no reason however to expect the g_s dependence to be analytic and indeed it was argued by Schenker [34] that a genus expansion alone of scattering amplitudes in string theory is an asymptotic series and must be supplemented by non-analytic terms of the form e^{-1/g_s} to ensure convergence. This is indeed what we will find in $\mathcal{E}^{(D)}_{(p,q)}$. For now we shall emphasize that in the limit $g_s \to 0$ which corresponds to approaching a cusp in $\mathcal{M}_{\text{quant}}$, these non-perturbative effects vanish and we are left with the perturbative terms which satisfy the growth condition 4 of definition 2.4.

Next we turn to the question of whether the functions $\mathcal{E}^{(D)}_{(p,q)}$ satisfy differential equations which is point 3 in definition 2.4. Indeed, it was shown in [35] that for $D = 10$ the direct consequence of imposing supersymmetry on the action Eq. (3.4) is for $\mathcal{E}^{(10)}_{(0,0)}$ and $\mathcal{E}^{(10)}_{(1,0)}$ to obey eigenvalue equations under the Laplace-Beltrami operator Δ on the coset space $G(\mathbb{R})/K(G(\mathbb{R}))$ (the quadratic Casimir). Further work on the effects on supersymmetry in lower dimensions includes for example [28] and today there is an abundance of evidence supporting the fact that the functions $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$ and $\mathcal{E}^{(D)}_{(0,1)}$ satisfy the equations

$$\left(\Delta - \frac{3(11-D)(D-8)}{D-2}\right)\mathcal{E}^{(D)}_{(0,0)} = 6\pi\delta_{D,8} \tag{3.6}$$

$$\left(\Delta - \frac{5(12-D)(D-7)}{D-2}\right)\mathcal{E}^{(D)}_{(1,0)} = 40\zeta(2)\delta_{D,7} + 7\mathcal{E}^{(6)}_{(0,0)}\delta_{D,6} \tag{3.7}$$

$$\begin{aligned}\left(\Delta - \frac{6(14-D)(D-6)}{D-2}\right)\mathcal{E}^{(D)}_{(0,1)} = & -\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2 + 40\zeta(3)\delta_{D,6} \\ & + \frac{55}{3}\mathcal{E}^{(5)}_{(0,0)}\delta_{D,5} + \frac{85}{2\pi}\mathcal{E}^{(4)}_{(1,0)}\delta_{D,4}.\end{aligned} \tag{3.8}$$

In 10-dimensions, where $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, the quadratic Casimir is the only G-invariant differential operator. In lower dimensions there are however additional higher order G-invariant differential operators whose significance have been investigated in [36, 37, 38]. These findings show that the functions $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ satisfy 3 in definition 2.4. It should be noted that Eq. (3.8) is qualitatively different from Eqs. (3.6) and (3.7) in that it is sourced by the term $-\left(\mathcal{E}^{(D)}_{(0,0)}\right)^2$ in all dimensions. This is not tolerated by definition 2.4 and $\mathcal{E}^{(D)}_{(0,1)}$ therefore doesn't qualify as an automorphic form. It does however still possess the property of automorphy and for reasons to be outlined below seems to fall into a class of functions generalizing the concept of automorphic forms [39, 40]. Notably, the constraints from supersymmetry also set the corrections at orders $(\alpha')^1$ and $(\alpha')^2$ to zero.

The function $\mathcal{E}^{(D)}_{(0,1)}$ has been studied in various dimensions and expressions for it have been found as the integral over the so called Kawazumi-Zhang invariant [41, 42]. It would still be desirable to make the connection with Eisenstein series as close as possible and understand the function better in the context of automorphic forms.

3.1 R^4 in $D = 10$

The procedure of exploiting U-duality to uncover pieces of the low-energy effective action of type IIB string theory was first done in the 1997 seminal paper [1] by Green and Gutperle for $D = 10$. They treated the R^4 and $\nabla^4 R^4$ curvature corrections which work analogously. Here we will show how the R^4 correction $\mathcal{E}^{(10)}_{(0,0)}$ is found and how to extract the physical information it encodes.

In $D = 10$ we have $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$ and the classical moduli space is $\mathcal{M} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2$. U-duality requires left $\mathrm{SL}_2(\mathbb{Z})$-invariance and supersymmetry imposes

$$\left(\Delta - \frac{3}{4}\right)\mathcal{E}^{(10)}_{(0,0)} = 0. \tag{3.9}$$

This is exactly what is treated in example 2.10 with $s = 3/2$. In the context of string theory (or supergravity), the Poincaré upper half plane is now parametrized according to

$$\mathbb{H} \ni \tau = \tau_1 + i\tau_2 = \chi + \mathrm{i}\underbrace{\mathrm{e}^{-\phi}}_{g_s^{-1}} \tag{3.10}$$

where χ and ϕ are the two scalar fields (moduli) present in type IIB supergravity called the *axion* and *dilaton* respectively. The only lacking information to pin down a solution exactly are boundary conditions for the differential equation Eq. (3.9). We will comment on this a bit further down. The solution that Green and Gutperle found is

$$\mathcal{E}^{(10)}_{(0,0)}(\tau) = \sum_{(m,n)'\in\mathbb{Z}^2} \frac{\tau_2^{3/2}}{|m+n\tau_2|_\infty^3}. \tag{3.11}$$

The idea is now that since this function appears in an effective action, it should encode information about scattering amplitudes. In order to access this information, we need to write $\mathcal{E}^{(10)}_{(0,0)}$ in a form where we can interpret the results in this fashion. This form is the Fourier series representation

$$\mathcal{E}^{(10)}_{(0,0)}(\tau) = 2\zeta(3)\tau_2^{3/2} + 4\zeta(2)\tau_2^{-1/2} + 8\pi\sqrt{\tau_2}\sum_{N'\in\mathbb{Z}} \mu_{-2}(N)|N|_\infty K_1(2\pi|N|_\infty\tau_2)\mathrm{e}^{-2\pi\mathrm{i}N\tau_1} \tag{3.12}$$

as discussed in example 2.26. The power behaved terms in τ_2 (or the string coupling g_s) have a direct interpretation as the tree-level and one-loop contributions to the four graviton amplitude at third order in α'. These results are attainable by string perturbation theory and the fact that that the amplitudes thus found in $\mathcal{E}^{(10)}_{(0,0)}$, namely $2\zeta(3)$ and $4\zeta(2)$ respectively agree with perturbation theory is no accident but rather by construction. The infinite sum containing Bessel functions is exponentially suppressed in the weak coupling limit $\tau_2 \to \infty$ which corresponds to approaching the "cusp at infinity" in $\mathbb{H}$. It is by demanding agreement with known perturbative results that one obtains the proper boundary conditions for Eq. (3.9) so that the right multiple of $E^{\mathrm{SL}_2}_{3/2}$ is chosen for $\mathcal{E}^{(10)}_{(0,0)}$. It is noteworthy that the lack of additional perturbative terms provides a non-renormalization theorem stating that there will be no further contributions beyond 1-loops for the four graviton amplitude at third order in α'.

The infinite series has an interpretation as non-perturbative effects in type IIB string theory known as D-instantons as we shall see now. Instantons are solutions to the equations of motion with a finite value of the action. The first example of instantons in type IIB theory was found by Gibbons, Green and Perry in [43]. A brief outline of how these solutions are found and how their actions are computed can be found in **Proceedings I**. There are instantons as well as anti-instantons and their actions are

$$S_{\text{inst}}(\tau, mn) = -2\pi\mathrm{i}|mn|_\infty\tau \quad \text{and} \quad S_{\text{anti}}(\tau, mn) = 2\pi\mathrm{i}|mn|_\infty\overline{\tau}. \tag{3.13}$$

The actions are quantized in terms of a winding number m and n units of momentum. This picture comes from type IIA string theory on a circle S_R with radius R where the one-dimensional world volume of a D0-brane wraps around the circle. The momentum in the compact direction must also be quantized. T-duality maps Dp-branes in type IIA to D$(p-1)$-branes in type IIB on $S_{1/R}$. In the decompactification limit $R \to \infty$ one then obtains D(-1)-branes (also called D-instantons) of type IIB theory in $D = 10$ with quantized actions. See [43] for details.

Let us now see that the infinite series can be interpreted in terms of D-instantons. For large arguments, the Bessel functions are given by

$$K_1(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}(1 + \mathcal{O}(x^{-1})). \tag{3.14}$$

In the limit $\tau_2 \to \infty$, the infinite series in $\mathcal{E}^{(10)}_{(0,0)}$ becomes

$$\begin{aligned} &8\pi\sqrt{\tau_2}\sum_{N'\in\mathbb{Z}}\mu_{-2}(N)|N|_\infty K_1(2\pi|N|_\infty\tau_2)\mathrm{e}^{-2\pi\mathrm{i}N\tau_1} = \\ &= 4\pi\sum_{N'\in\mathbb{Z}}\sqrt{|N|_\infty}\mu_{-2}(N)\mathrm{e}^{2\pi(\mathrm{i}N\tau_1 - |N|_\infty\tau_2)}(1 + \mathcal{O}(\tau_2^{-1})) = \\ &= 4\pi\sum_{N=1}^{\infty}\sqrt{|N|_\infty}\mu_{-2}(N)\left(\mathrm{e}^{-S_{\text{inst}}(\tau,N)} + \mathrm{e}^{-S_{\text{anti}}(\tau,N)}\right)(1 + \mathcal{O}(\tau_2^{-1})). \end{aligned} \tag{3.15}$$

For each N, one obtains polynomials in the string coupling g_s weighted by the instanton and anti-instanton actions which has the direct interpretation as scattering in a background of D-instantons. The number theoretical factor $\mu_{-2}(N)$ has a special interpretation as well. It is called the instanton measure and takes into account the number of ways an integer N (also called the instanton charge) can be partitioned into a product mn of winding numbers and momenta.

3.2 $\nabla^6 R^4$ in $D = 10$

As commented above, the function $\mathcal{E}^{(10)}_{(0,1)}$ is not an automorphic form due to the source term its differential equation,

$$(\Delta - 12)\mathcal{E}^{(10)}_{(0,1)} = -(\mathcal{E}^{(10)}_{(0,0)})^2 \tag{3.16}$$

with $\mathcal{E}^{(10)}_{(0,0)}$ given in the section above. Since it is not an automorphic form, it will definitely not be an Eisenstein series. Green, Miller and Vanhove [40] were nevertheless able to solve the differential equation with the appropriate boundary conditions and find the Fourier expansion of $\mathcal{E}^{(10)}_{(0,0)}$. The constant term of the $\nabla^6 R^4$ interaction was found to be

$$\int_0^1 \mathcal{E}^{(10)}_{(0,1)}(\tau)\,\mathrm{d}\tau_2 = \frac{2\zeta(3)^2}{3}\tau_2^3 + \frac{4\zeta(2)\zeta(3)}{3}\tau_2 + \frac{4\zeta(4)}{\tau_2} + \frac{4\zeta(6)}{27\tau_2^3} + \dots \tag{3.17}$$

where the ellipsis consists of (non-perturbative) instanton–anti-instanton bound states where the net instanton charge is zero. That these terms appear in the constant term can be understood by making a Fourier ansatz in Eq. (3.16). The non-constant terms consist of instanton–anti-instanton pairs where the net instanton charge is non-zero.

In an appendix, the authors discussed expressing the solution as a Poincaré series

$$\mathcal{E}^{(10)}_{(0,1)}(\tau) = \sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})} \Phi(\gamma\cdot\tau). \tag{3.18}$$

The resulting function Φ is not a character on $G(\mathbb{R})$, which is no surprise since if that were the case, $\mathcal{E}^{(10)}_{(0,1)}$ would be an Eisenstein series. As has been observed by my co-supervisor Axel Kleinschmidt that the function Φ satisfies the "folded" equation

$$(\Delta - 12)\Phi(\tau) = -4\zeta(3)^2\tau_2^{3/2}E_{3/2}(\tau). \tag{3.19}$$

That this is true can be seen by applying the Poincaré sum $\sum_{\gamma\in B(\mathbb{Z})\backslash G(\mathbb{Z})}$ to both sides. The Laplacian Δ as well as the Eisenstein series $E_{3/2}$ are $G(\mathbb{Z})$-invariant. The factor $\tau_2^{3/2}$ then acts as the "seed" to generate the second factor of $E_{3/2}$ and the Poincaré sum of function Φ gives $\mathcal{E}^{(10)}_{(0,1)}$. The virtue of this observation is that Eq. (3.19) is easier to solve than Eq. (3.16). It was Kleinschmidt's suggestion that this approach might be fruitful in lower dimensions where the duality groups are of higher rank.

3.3 Lower dimensions

The takeaway message from the section above is that the correction functions $\mathcal{E}^{(D)}_{(p,q)}$ encode information about four graviton scattering and the actual scattering amplitudes are encoded in the Fourier coefficients of the functions with the constant term corresponding to perturbative effects and the non-constant terms corresponding to scattering in non-perturbative backgrounds. When stepping down in dimension by means of toroidal compactification, the moduli space $G(\mathbb{R})/K(G(\mathbb{R}))$ as well as the duality group $G(\mathbb{Z})$ grows. With the larger moduli space comes additional cusps in which we can consider the limiting behavior of $\mathcal{E}^{(D)}_{(p,q)}$. Furthermore, as was discussed in Section 2.3 there are now several possible ways (or "directions") in which one can construct a Fourier expansion, each corresponding to the choice of a parabolic subgroup of $G(\mathbb{R})$. It turns out that different Fourier expansions reveal different physical information.

There are three parabolic subgroups of special importance in string theory, each being a maximal parabolic subgroup. Being maximal parabolics, their Levi subgroups will all contain a GL_1-factor and it is the limiting behavior in regards to this degree of freedom that generalizes the weak coupling limit that was considered above. The three parabolic subgroups are labelled using the Bourbaki-labelling in Fig. 3.1:

- **String-perturbation limit** P_1: The maximal parabolic associated with node 1 in Fig. 3.1 is of D-type and the Levi factor is $L_1 = \mathrm{GL}_1 \times \mathrm{SO}_{d,d}$. The $\mathrm{SO}_{d,d}$ factor is associated with T-duality in type IIB theory in $10-d$ dimensions. The GL_1-factor controls the strength of the string interactions. The constant term in a Fourier expansion with respect to this parabolic corresponds to the perturbative terms of strings with worldsheets of different genera, as was seen in Section 3.1[6] while the non-perturbative terms correspond to the non-perturbative D-instantons.

- **M-theory limit** P_2: The Levi subgroup of P_2 is $L_2 = \mathrm{GL}_1 \times \mathrm{SL}_{d+1}$. The SL_{d+1}-factor is associated with a $d+1$-dimensional torus T^{d+1} (c.f. the modular group $\mathrm{SL}_2(\mathbb{Z})$ of the 2-torus T^2) and the GL_1-factor controls the volume of the T^{d+1}. The cusp in this limit corresponds to a large torus, effectively decompactifying the T^{d+1} and giving physics in $D = 10-d+d+1 = 11$ dimensions. The constant term gets its contributions from 11-dimensional supergravity on a $(d+1)$-dimensional torus and the non-perturbative states reflect non-perturbative states of M-theory.

- **Decompactification limit** P_{d+1}: Removing the "most recently added" node $d+1$ gives the Levi $L_{d+1} = \mathrm{GL}_1 \times E_d$ where the E_d is associated with the moduli space of type IIB theory on T^{d-1}. The GL_1 factor governs the size of the "last" circle and the cusp associated with this parabolic corresponds to this circle becoming large and decompactifying. The constant term corresponds to the automorphic forms which can be obtained from the $(D+1)$-dimensional theory. Taking repeated decompactification limits leads to a chain of relations letting one deduce all Eisenstein series from the 3-dimensional theory:

  $$\mathrm{E}_8 \supset \mathrm{E}_7 \supset \mathrm{E}_6 \supset \mathrm{Spin}_{5,5} \supset \mathrm{SL}_5 \supset \mathrm{SL}_3 \times \mathrm{SL}_2 \supset \mathrm{SL}_2. \tag{3.20}$$

  The non-constant terms correspond to black-hole state whose world-line wrap the large circle [29, 25].

It is the physical interest in Fourier expansions of Eisenstein series along these maximal parabolics that motivates the majority of the work in this thesis. There exists a formula for calculating constant terms known as Langlands' constant term formula (see theorem 4.39) and results can be found in [25]. No such formula exists however for the non-constant terms. In the remaining chapters, we are driven by the occurrence of SL_2 and SL_5 among the duality groups in table 3.1 to develop a formalism for calculating non-constant Fourier coefficients over maximal parabolics of a class of physically relevant automorphic forms on SL_n.

[6] Granted, in $D = 10$ there is not much of a choice of parabolics since the duality group $\mathrm{SL}_2(\mathbb{R})$ is of rank-1 and the only possible parabolic subgroup is maximal and equal to the Borel subgroup.

Another concept which plays an important role in understanding the Fourier expansions of the higher curvature corrections in type IIB theory is that of *automorphic representations*. This topic will be treated mathematically in Chapter 5 but first we shall see how it is to be understood form a physical perspective. As we shall see in Chapter 5, the arithmetic subgroup $L(\mathbb{Z})$ of the Levi carries a natural action on a Fourier coefficient F_ψ^U over a parabolic $P = UL$. An element $l \in L(\mathbb{Z})$ acts on the character ψ according to $l \cdot \psi(u) = \psi(lul^{-1})$ and thus partitions the character variety into so called character variety orbits such that Fourier coefficients that are attached to the same orbit are related. Having an interpretation for the non-constant Fourier coefficients F_ψ^U over the maximal parabolics P_1, P_2 and P_{d+1} as explained above in terms of physical objects (instantons and black-holes [18]) carrying charges corresponding to points on the character variety, the algebraic conditions specifying the character variety orbits into which the Fourier coefficients fall can be understood as physical constraints on the string states. The physical constraints governing the instantons are so called BPS-conditions which arise from the supersymmetry algebra of the theory.

The important point for this thesis is that the R^4, $\nabla^4 R^4$ and $\nabla^6 R^4$ corrections enjoy $\frac{1}{2}$-, $\frac{1}{4}$- and $\frac{1}{8}$-BPS protection respectively. The fraction indicates the smallest amount of supersymmetry that states entering into these interactions must preserve. From a mathematical point of view it means that for a given curvature correction function, say $\mathcal{E}^{(D)}_{(0,0)}$, Fourier coefficients which are too generic (or not degenerate enough) should vanish. In the case of the $\frac{1}{2}$-BPS term $\mathcal{E}^{(D)}_{(0,0)}$, only the maximally degenerate Fourier coefficients are non-vanishing. The corresponding mathematical notion is that of automorphic representations and we say that $\mathcal{E}^{(D)}_{(0,0)}$, $\mathcal{E}^{(D)}_{(1,0)}$nd $\mathcal{E}^{(D)}_{(0,1)}$ are attached to the minimal-, next-to-minimal- and next-to-next-to-minimal automorphic representations respectively of $G(\mathbb{R})$ [29].

3.4 $\nabla^6 R^4$ in $D = 7$

In this section, we will see how the technique of folding mentioned in Section 3.1 can be used to find a particular solution for $\mathcal{E}^{(7)}_{(0,1)}$ of the inhomogeneous Laplace equation Eq. (3.8). In $D = 7$, the U-duality group is $\mathrm{SL}_5(\mathbb{R})$ as shown in table 3.1 and the moduli space is $\mathrm{SL}_5(\mathbb{R})/\mathrm{SO}_5$. Equation (3.8) becomes

$$\left(\Delta - \frac{42}{5}\right)\mathcal{E}^{(7)}_{(0,1)} = -\left(\mathcal{E}^{(7)}_{(0,0)}\right)^2 \tag{3.21}$$

where the R^4-function is given by $\mathcal{E}^{(7)}_{(0,0)}(g) = 2\zeta(3)E^{\mathrm{SL}_5}_{\alpha_1,3/2}(g)$ and satisfies [28]

$$\left(\Delta + \frac{12}{5}\right)\mathcal{E}^{(7)}_{(0,0)} = 0. \tag{3.22}$$

We will analyze this equation over the mirabolic P_1, i.e. the maximal parabolic subgroup associated with node 1. The coset representative $g \in \mathrm{SL}_5(\mathbb{R})/\mathrm{SO}_5$ of the moduli space is parametrized by

$$g = ul = \left(\begin{smallmatrix} 1 & Q_1 \\ 0 & \mathbb{1} \end{smallmatrix}\right) \left(\begin{smallmatrix} r^{4/5} & 0 \\ 0 & r^{-1/5} e_4 \end{smallmatrix}\right) \tag{3.23}$$

where Q_1 is a four-component row vector and e_4 is a coset representative of $\mathrm{SL}_4(\mathbb{R})/\mathrm{SO}_4$. The Laplacian $\Delta = \Delta_{\mathrm{SL}_5}$ in these coordinates decomposes as [44]

$$\Delta_{\mathrm{SL}_5} = \frac{5}{8} r^2 \partial_r^2 - \frac{15}{8} r \partial_r + r^2 ||e_4^{-1} \partial_{Q_1}||^2 + \Delta_{\mathrm{SL}_4}. \tag{3.24}$$

We will make the assumption that $\mathcal{E}^{(7)}_{(0,1)}$ is a Poincaré series with respect to this maximal parabolic, i.e.

$$\mathcal{E}^{(7)}_{(0,1)}(g) = \sum_{\gamma \in P_1(\mathbb{Z}) \backslash \mathrm{SL}_5(\mathbb{Z})} \sigma(\gamma \cdot g). \tag{3.25}$$

Equation (3.21) then unfolds into

$$\left(\Delta - \frac{42}{5}\right) \sigma(g) = -4\zeta(3)^2 r^{12/5} E^{\mathrm{SL}_5}_{\alpha_1;3/2}(g). \tag{3.26}$$

Let us furthermore make the assumption that σ is left $P_1(\mathbb{Z})$-invariant so that it can be expressed as a Fourier series in the parabolic P_1,

$$\sigma(g) = \sum_{N_1 \in \mathbb{Z}^4} \sigma_{N_1}(r, e_4) e^{2\pi i Q_1 N_1} \tag{3.27}$$

with N_1 being column vectors of charges. Since the unipotent of a maximal parabolic subgroup of SL_n is abelian, this expression captures the whole of σ without need for non-abelian coefficients. The Eisenstein series in the right-hand side of (3.26) can also be written as a Fourier series over the unipotent as

$$E^{\mathrm{SL}_5}_{\alpha_1,3/2}(g) = \sum_{N_1 \in \mathbb{Z}} F_{N_1}(r, e_4) e^{2\pi i Q_1 N_1} \tag{3.28}$$

with

$$F_{N_1}(r, e_4) = \frac{2}{\zeta(3)} r^{7/5} \sigma_2(k) \frac{K_1(2\pi r ||e_4^{-1} N_1||)}{||e_4^{-1} N_1||} \quad \text{for } N_1 \neq 0 \text{ and} \tag{3.29}$$

$$F_0(r, e_4) = r^{12/5} + \frac{2\zeta(2)}{\zeta(3)} r^{2/5} E^{\mathrm{SL}_4}(2\Lambda_1 - \rho, e_4). \tag{3.30}$$

A technique for calculating Fourier coefficients of this kind, i.e. Fourier coefficients over a maximal parabolic for Eisenstein series on SL_n is described in Chapter 6. We obtain differential equations for the Fourier coefficients σ_{N_1} of σ.

3.4.1 Zero mode

For the zero mode σ_0 we have the equation

$$\left(\frac{5}{8}r^2\partial_r^2 - \frac{15}{8}r\partial_r + \Delta_{\mathrm{SL}_4} - \frac{42}{5}\right)\sigma_0(r, e_4) = \\ -4\zeta(3)^2 r^{24/5} - 8\zeta(2)\zeta(3)r^{14/5}E^{\mathrm{SL}_4}_{\alpha_1;3/2}(e_4). \tag{3.31}$$

Starting with the homogeneous equation and writing $\sigma_0^{(\mathrm{h})}(r, e_4) = \alpha(r)\beta(e_4)$, separation of variables gives

$$\frac{5}{8}r^2\alpha'' - \frac{15}{8}r\alpha' + \left(\gamma - \frac{42}{5}\right)\alpha = 0 \tag{3.32}$$

$$(\Delta_{\mathrm{SL}_4} - \gamma)\beta = 0. \tag{3.33}$$

The first of these is an Euler equation with solution

$$\alpha(r) = a_1 r^{2+2c} + a_2 r^{2-2c} \quad \text{where} \quad \gamma = \frac{5}{2}(c^2 - 1) + \frac{42}{5}. \tag{3.34}$$

The second equation is solved by an appropriate SL_4 Eisenstein series.

Looking for a particular solution of the form $c_1 r^{24/5} + c_2 r^{14/5} E^{\mathrm{SL}_4}_{\alpha_1;3/2}(e_4)$ we are lead to the particular solution

$$\sigma_0^{(\mathrm{p})}(r, e_4) = \frac{2\zeta(3)^2}{3}r^{24/5} + \frac{2\zeta(2)\zeta(3)}{3}r^{14/5}E^{\mathrm{SL}_4}_{\alpha_1;3/2}(e_4). \tag{3.35}$$

The full solution for the zero mode is now $\sigma_0 = \sigma_0^{(\mathrm{h})} + \sigma_0^{(\mathrm{p})}$.

3.4.2 Non-zero modes

For the non-zero modes σ_{N_1}, we have the equation

$$\underbrace{\left(\frac{5}{8}r^2\partial_r^2 - \frac{15}{8}r\partial_r - 4\pi^2 r^2 ||e_4^{-1}N_1||^2 + \Delta_{\mathrm{SL}_4} - \frac{42}{5}\right)}_{D}\sigma_{N_1} = \\ \underbrace{-16\pi\zeta(3)\sigma_2(k)}_{A_k}\, r^{19/5}\frac{K_1(2\pi r||e_4^{-1}N_1||)}{||e_4^{-1}N_1||}. \tag{3.36}$$

Using the convention

$$ds^2 = \frac{-1}{2}\,\mathrm{tr}\,\mathrm{d}M\mathrm{d}M^{-1} \tag{3.37}$$

for calculating the metric on a coset space $\mathcal{M}$ where M is the coset representative of $\mathcal{M}$, we calculate the Laplacian on SL_4/SO_4 and obtain (see Appendix A in [45])[7]

$$\Delta_{SL_4} = \frac{1}{2} g_{ik} g_{jl} \partial^{ij} \partial^{kl} - \frac{1}{8} \left(g_{ij} \partial^{ij} \right)^2 + \frac{5}{2} g_{ij} \partial^{ij} \tag{3.38}$$

where

$$\partial^{ij} \equiv \frac{\partial}{\partial g_{ij}} \quad \text{and} \quad g = e_4 e_4^{\mathsf{T}}. \tag{3.39}$$

Here we have recycled the letter g from denoting the coset representative on the SL_5-moduli space to the SL_4-moduli space. We introduce the notation (dropping the subscript on N_1)

$$u = ||e_4^{-1} N|| = \sqrt{N^{\mathsf{T}} g^{-1} N} = \sqrt{N_i g^{ij} N_j} \tag{3.40}$$

$$x = 2\pi r ||e^{-1} N|| = 2\pi r u \tag{3.41}$$

$$f_s^{\alpha\beta} = r^\alpha u^\beta K_s. \tag{3.42}$$

The following useful identities are easy to check (repeated indices are summed over)

$$g_{ij} \partial^{ij} u = -u \tag{3.43}$$

$$g_{ik} g_{jl} \partial^{ij} \partial^{kl} u = 9u \tag{3.44}$$

$$g_{ik} g_{jl} \left(\partial^{ij} u \right) \left(\partial^{kl} u \right) = u^2. \tag{3.45}$$

Furthermore to save space, the argument of all Bessel functions and their derivatives is omitted and they are always to be evaluated at x, so we have

$$K_s \equiv K_s(x) = K_s(2\pi r u) \quad \text{and} \quad f_s^{\alpha\beta} = r^\alpha u^\beta K_s(2\pi r u). \tag{3.46}$$

We will start by looking for a particular solution $\sigma_{N_1}^{(\mathrm{p})}$ to (3.36) and make an ansatz of the form

$$\sigma_{N_1}^{(\mathrm{p})} = A_k \sum_i B_i r^{\alpha_i} u^{\beta_i} K_{s_i} = A_k \sum_i B_i f_{s_i}^{\alpha_i \beta_i} \quad \text{with} \quad B_i \in \mathbb{R} \tag{3.47}$$

where the number of terms in the sum is to be determined. We hope to be able to tweak the values of B_i, α_i, β_i and s_i such that we find a particular solution. To this end, it is useful to see how the differential operator D acts on a term in (3.47). By using the

[7] The convention (3.37) differs by a factor of 2 from that used in [45]. Nevertheless, it has been checked that Eqs. (3.37) and (3.38) together produce the correct eigenvalues when acting on automorphic forms on SL_4

identities in Eqs. (3.43) to (3.45) one can derive the following relations

$$\begin{aligned} r^2\partial_r^2 f_s^{\alpha\beta} =& \alpha\left(\alpha-1\right) f_s^{\alpha\beta} + 2\alpha x f_s^{\prime\alpha\beta} + x^2 f_s^{\prime\prime\alpha\beta} && (3.48)\\ r\partial_r f_s^{\alpha\beta} =& \alpha f_s^{\alpha\beta} + x f_s^{\prime\alpha\beta} && (3.49)\\ g_{ik}g_{jl}\partial^{ij}\partial^{kl} f_s^{\alpha\beta} =& \left(\beta^2+8\beta\right) f_s^{\alpha\beta} + \left(2\beta+9\right) x f_s^{\prime\alpha\beta} + x^2 f_s^{\prime\prime\alpha\beta} && (3.50)\\ \left(g_{ij}\partial^{ij}\right)^2 f_s^{\alpha\beta} =& \beta^2 f_s^{\alpha\beta} + \left(2\beta+1\right) x f_s^{\prime\alpha\beta} + x^2 f_s^{\prime\prime\alpha\beta} && (3.51)\\ g_{ij}\partial^{ij} f_s^{\alpha\beta} =& -\beta f_s^{\alpha\beta} - x f_s^{\prime\alpha\beta}. && (3.52) \end{aligned}$$

The differential operator D can be expressed as

$$D = \frac{5}{8}r^2\partial_r^2 - \frac{15}{8}r - 4\pi^2 r^2 ||e^{-1}N||^2 + \frac{1}{2}g_{ik}g_{jl}\partial^{ij}\partial^{kl} - \frac{1}{8}\left(g_{ij}\partial^{ij}\right)^2 + \frac{5}{2}g_{ij}\partial^{ij} - \frac{42}{5}. \quad (3.53)$$

Acting on a term $f_s^{\alpha\beta}$ from Eq. (3.47) then gives

$$D f_s^{\alpha\beta} = x^2 f_s^{\prime\prime\alpha\beta} + x f_s^{\prime\alpha\beta}\frac{5\alpha+3\beta}{4} + f_s^{\alpha\beta}\left(\frac{-42}{5} + \frac{5\alpha\left(\alpha-4\right)}{8} + \frac{3\beta\left(\beta+4\right)}{8} - x^2\right). \quad (3.54)$$

We now use the modified Bessel equation

$$x^2 K_s^{\prime\prime}(x) + x K_s^{\prime}(x) - (x^2+s^2)K_s(x) = 0 \quad \text{or} \quad x^2 f_s^{\prime\prime\alpha\beta} + x f_s^{\prime\alpha\beta} - (x^2+s^2) f_s^{\alpha\beta} = 0 \quad (3.55)$$

to get rid of the second derivative. We get

$$\begin{aligned} & x^2 f_s^{\prime\prime\alpha\beta} + x f_s^{\prime\alpha\beta}\frac{5\alpha+3\beta}{4} + f_s^{\alpha\beta}\left(\frac{-42}{5} + \frac{5\alpha\left(\alpha-4\right)}{8} + \frac{3\beta\left(\beta+4\right)}{8} - x^2\right) \\ &= x f_s^{\prime\alpha\beta}\left(\frac{5\alpha+3\beta}{4} - 1\right) + f_s^{\alpha\beta}\left(\frac{-42}{5} + \frac{5\alpha\left(\alpha-4\right)}{8} + \frac{3\beta\left(\beta+4\right)}{8} + s^2\right). \end{aligned} \quad (3.56)$$

Let us now plug in the full ansatz Eq. (3.47) into the full differential equation, Eq. (3.36).

We get

$$D\left(A_k\sum_i B_i f_{s_i}^{\alpha_i\beta_i}\right) = A_k f_1^{19/5,-1} \quad \Leftrightarrow \tag{3.57}$$

$$\begin{aligned}\sum_i B_i\Bigg(& x f_{s_i}^{\prime\alpha_i\beta_i}\left[\frac{5\alpha_i+3\beta_i-4}{4}\right]+\\ & f_{s_i}^{\alpha_i\beta_i}\left[\frac{-42}{5}+\frac{5\alpha_i\left(\alpha_i-4\right)}{8}+\frac{3\beta_i(\beta_i+4)}{8}+s_i^2\right]\Bigg) = f_1^{19/5,-1} \quad \Leftrightarrow\end{aligned} \tag{3.58}$$

$$\begin{aligned}\sum_i B_i r^{\alpha_i-19/5}u^{\beta_i+1}\Bigg(& xK'_{s_i}\left[\frac{5\alpha_i+3\beta_i-4}{4}\right]+\\ & K_{s_i}\left[\frac{-42}{5}+\frac{5\alpha_i\left(\alpha_i-4\right)}{8}+\frac{3\beta_i\left(\beta_i+4\right)}{8}+s_i^2\right]\Bigg) = K_1 \quad \Leftrightarrow\end{aligned} \tag{3.59}$$

$$\begin{aligned}\sum_i B_i r^{\alpha_i-19/5}u^{\beta'_i}\Bigg(& xK'_{s_i}\left[\frac{5\alpha_i+3\beta'_i-7}{4}\right]+\\ & K_{s_i}\left[\frac{-381+25\alpha_i\left(\alpha_i-4\right)+15\beta'_i\left(\beta'_i+2\right)}{40}+s_i^2\right]\Bigg) = K_1 \quad \Leftrightarrow\end{aligned} \tag{3.60}$$

$$\begin{aligned}\sum_i B'_i r^{\alpha_i-\beta'_i-19/5}x^{\beta'_i}\Bigg(& xK'_{s_i}\left[\frac{5\alpha_i+3\beta'_i-7}{4}\right]+\\ & K_{s_i}\left[\frac{-381+25\alpha_i\left(\alpha_i-4\right)+15\beta'_i\left(\beta'_i+2\right)}{40}+s_i^2\right]\Bigg) = K_1.\end{aligned} \tag{3.61}$$

Fixing α_i to be $\alpha_i = \beta'_i + 19/5$ gives

$$\sum_i B'_i x^{\beta'_i}\left(xK'_{s_i}\left[3+2\beta'_i\right]+K_{s_i}\left[\left(\beta'_i-2\right)\left(\beta'_i+5\right)+s_i^2\right]\right) = K_1 \quad \Leftrightarrow \tag{3.62}$$

$$\begin{aligned}&\sum_i B'_i x^{\beta'_i}\left(K_{s_i}\left(s_i\left[3+2\beta'_i\right]+\left[\left(\beta'_i-2\right)\left(\beta'_i+5\right)+s_i^2\right]\right)+xK_{s_i+1}\left[-3-2\beta'_i\right]\right)\\ &\quad = K_1 \quad \Leftrightarrow\end{aligned} \tag{3.63}$$

$$\sum_i B'_i x^{\beta'_i}\left(K_{s_i}\left[\left(-2+s_i+\beta'_i\right)\left(5+s_i+\beta'_i\right)\right]+xK_{s_i+1}\left[-3-2\beta'_i\right]\right) = K_1 \quad \Leftrightarrow \tag{3.64}$$

$$\sum_i B''_i\Bigg(x^{\beta'_i}K_{s_i}\underbrace{\frac{\left(-2+s_i+\beta'_i\right)\left(5+s_i+\beta'_i\right)}{-3-2\beta'_i}}_{d\left(s_i,\beta'_i\right)}+x^{\beta'_i+1}K_{s_i+1}\Bigg) = K_1 \tag{3.65}$$

for new variables

$$\beta_i' = \beta_i + 1 = \alpha_i - 19/5 \tag{3.66}$$
$$B_i' = (2\pi)^{-\beta_i'} B_i \tag{3.67}$$
$$B_i'' = \left[-3 - 2\beta_i'\right] B_i' = \frac{-3 - 2\beta_i'}{(2\pi)^{\beta_i'}} B_i. \tag{3.68}$$

In going to Eq. (3.63) we used the Bessel relation

$$K_\nu'(z) = \frac{\nu}{z} K_\nu(z) - K_{\nu+1}(z). \tag{3.69}$$

The last step is valid as long as $3 + 2\beta_i' \neq 0$ which will be the case as long as we keep to $\beta_i' \in \mathbb{Z}$.

The name of the game is recursive elimination. We are free to keep adding terms to the sum $\sum_i$ in Eq. (3.47), and each time we do so we get two Bessel functions is the LHS of Eq. (3.65). The first thing to do is to add a term so that one of these Bessel functions cancels with the K_1 in the RHS. One is left with a stray term in the LHS which has to be cancelled by adding another term to the sum $\sum_i$. This process is repeated and a solution is found when the numerical coefficient d evaluates to zero and the recursion terminates. The procedure below illustrates this.

First, add a term

$$s_1 = 0, \quad \beta_1' = -1, \quad B_1'' = 1 \quad \Rightarrow \quad d(0, -1) = 12 \tag{3.70}$$

to get

$$x^{-1} K_0 12 + K_1 + \sum_i \cdots = K_1. \tag{3.71}$$

This successfully cancels the K_1 in the RHS. We must now cancel the new term $x^{-1}K_0 12$ in the LHS which we do by adding

$$s_2 = -1, \quad \beta_1' = -2, \quad B_2'' = -12 \quad \Rightarrow \quad d(-1, -2) = -10 \tag{3.72}$$

to get

$$x^{-1} K_0 12 + (-12) x^{-2} K_1 (-10) - 12 x^{-1} K_0 + \sum_i \cdots = 0. \tag{3.73}$$

Next, add

$$s_3 = -2, \quad \beta_3' = -3, \quad B_3'' = -120 \quad \Rightarrow \quad d(-2, -3) = 0 \tag{3.74}$$

to get

$$120x^{-2}K_{-1} + 0 + (-120)x^{-2}K_1 + \sum_i \cdots = 0 \tag{3.75}$$

at which point all terms have cancelled out and the recursion ends. We're done. The particular solution is hence given by

$$\sigma_N^{(\mathrm{p})} = A_k \sum_{i=1}^{3} B_i'' \left(d\left(s_i, \beta_i'\right) x^{\beta_i'} K_{s_i}' + x^{\beta_i'+1} K_{s_i+1} \right) \tag{3.76}$$

with

$$\begin{cases} s_1 = 0 \\ s_2 = -1 \\ s_3 = -2 \end{cases}, \quad \begin{cases} \beta_1' = -1 \\ \beta_2' = -2 \\ \beta_3' = -3 \end{cases}, \quad \begin{cases} B_1'' = 1 \\ B_2'' = -12 \\ B_3'' = -120 \end{cases} \Rightarrow \begin{cases} d(0,-1) = 12 \\ d(-1,-2) = -10 \\ d(-2,-3) = 0 \end{cases}. \tag{3.77}$$

Or in terms of the original variables α_i, β_i, s_i and B_i and the original ansatz Eq. (3.47)

$$\begin{aligned} \sigma_N^{(\mathrm{p})} &= A_k \left(\frac{-1}{2\pi} r^{14/5} u^{-2} K_0 + \frac{-12}{(2\pi)^2} r^{9/5} u^{-3} K_{-1} + \frac{-40}{(2\pi)^3} r^{4/5} u^{-4} K_{-2} \right) = \\ &= 32\pi^2 \zeta(3) \sigma_2(k) r^{24/5} \\ &\left(\frac{K_0\left(2\pi r ||e^{-1}N||\right)}{\left(2\pi ||e^{-1}N||\right)^2} + 12 \frac{K_1\left(2\pi r ||e^{-1}N||\right)}{\left(2\pi ||e^{-1}N||\right)^3} + 40 \frac{K_2\left(2\pi r ||e^{-1}N||\right)}{\left(2\pi ||e^{-1}N||\right)^4} \right) \end{aligned} \tag{3.78}$$

where we have used that the Bessel functions have even parity Eq. (3.83).

For the homogeneous solutions to Eq. (3.36), we investigate an individual term from the homogeneous version of (3.64)

$$B' x^{\beta'} \left(K_s \left[\left(-2 + s + \beta'\right)\left(5 + s + \beta'\right) \right] + x K_{s+1} \left[-3 - 2\beta' \right] \right) = 0. \tag{3.79}$$

Setting $\beta' = \frac{-3}{2}$ gives

$$B' x^{-3/2} \left(K_s \left[\left(\frac{-7}{2} + s \right) \left(\frac{7}{2} + s \right) \right] \right) = 0 \tag{3.80}$$

which is solved by setting $s = \pm 7/2$, both corresponding to the solution

$$\sigma_N^{(\mathrm{h})} = C r^{23/10} u^{-3/2} K_{7/2} = C' r^{19/5} \frac{K_{7/2}\left(2\pi r ||e^{-1}N||\right)}{\left(2\pi ||e^{-1}N||\right)^{3/2}} \tag{3.81}$$

for constants C and C'. The other linearly independent homogeneous solution is

$$D r^{23/10} u^{-3/2} I_{7/2} = D' r^{19/5} \frac{I_{7/2}\left(2\pi r ||e^{-1}N||\right)}{\left(2\pi ||e^{-1}N||\right)^{3/2}}. \tag{3.82}$$

Remark 3.1.
The solution Eq. (3.78) has been verified with Mathematica.

The modified Bessel functions of the second kind $K_\nu(z)$ have even parity

$$K_\nu(z) = K_{-\nu}(z) \tag{3.83}$$

and there is consequently also the relation

$$K'_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z} K_\nu(z) \tag{3.84}$$

which would have allowed us to step upwards in order during the recursion instead of downwards. I suspect that in this case, the other root $s_i + \beta'_i = 2$ in d would have terminated the recursion.

4 Adelic technology

Let me make an exception and start this chapter with a quote:

"God made the integers, all else is the work of man." – Leopold Kronecker[8]

This quote emphasizes the fact that the field of mathematics rests on a set of axioms. Starting with the integers (or the natural numbers), we can define additional sets of numbers and develop topics such as real- and complex analysis and differential geometry. An important but not widely known fact is that the "step" from rational numbers to real numbers is only one of many equally valid steps. Equivalently, there exists number fields other than the reals which are fundamentally different in their nature but on the same footing as the reals. These number fields are called the p-adic numbers $\mathbb{Q}_p$ and there is one such field for every prime number p. Having defined these new number fields, one can construct the adeles $\mathbb{A}$ as the direct product across this countably infinite list of number fields. As it turns out, the calculus over the adeles is especially powerful and will be exhibited in this chapter.

4.1 p-adic numbers

The reals $\mathbb{R}$ as well as the p-adics $\mathbb{Q}_p$ are completions of the rationals $\mathbb{Q}$ with respect to different norms. We start with a discussion of norms over the rationals. For a detailed reference on this, see [46].

4.1.1 Definitions

Definition 4.1 (Norm)
*A **norm** $|\cdot|$ over a field F is a map from F to the reals $\mathbb{R}$ satisfying*

- $|x| \geq 0$ — *Non-negativity* (4.1)
- $|x| = 0 \quad \Leftrightarrow \quad x = 0$ — *Positive definiteness* (4.2)
- $|xy| = |x||y|$ — *Multiplicativity* (4.3)
- $|x+y| \leq |x| + |y|$ — *Triangle inequality* (4.4)

for all $x, y \in F$.

Remark 4.2.
All norms satisfy $|1| = \left|\frac{x}{x}\right| = \frac{|x|}{|x|} = 1$.

[8]"*Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk.*" From an 1886 lecture of Kronecker, published in Weber 1891/92, 19.

Given a norm $|\cdot|_\star$ we can always construct another norm $|\cdot|_{\star\star} \equiv |\cdot|_\star^c$ as long as $0 < c < 1$. Eqs. (4.1) to (4.3) hold trivially and Eq. (4.4) holds since the function x^c is convex for $0 < c < 1$. We are therefore led to the following definition

Definition 4.3 (Equivalence of norms)
We say that a norm $|\cdot|_\star$ is ***equivalent*** *to another norm $|\cdot|_{\star\star}$ if there exists a real number $c > 0$ such that*

$$|x|_\star = |x|_{\star\star}^c \quad \text{for all } x. \tag{4.5}$$

Remark 4.4.
In terms of Cauchy sequences (see below), this definition is the same as the statement that two norms are equivalent so long as an arbitrary sequence is Cauchy with respect to one norm if and only if it is Cauchy with respect to the other.

Some important norms on the rationals include the following

Definition 4.5 (Norms on $\mathbb{Q}$: trivial norm, infinity norm and p-adic norm)
The ***trivial norm*** *of a rational number $x \in \mathbb{Q}$ is defined by*

$$|x|_0 = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0. \end{cases} \tag{4.6}$$

The ***infinity norm*** *of a rational number $x \in \mathbb{Q}$ is defined by*

$$|x|_\infty = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases} \tag{4.7}$$

For a prime number p and a non-zero rational number $x \in \mathbb{Q}^\times$ expressed as $x = p^r \frac{m}{n}$ where m and n are integers with no factors of p and $r \in \mathbb{Z}$ is an integer (positive or negative) counting how many factors of p the number x carries (in the numerator or denominator), the ***p-adic norm*** *on $\mathbb{Q}$ is defined by*

$$\begin{aligned} |x|_p &= \left| p^r \frac{m}{n} \right|_p = p^{-r} \quad \textit{for} \quad x \neq 0 \quad \textit{and} \\ |0|_p &= 0 \end{aligned} \tag{4.8}$$

Remark 4.6.
Rational numbers which are close in the infinity norm may be p-adically far apart, for example

$$\begin{aligned} &\left| \frac{4}{500} - \frac{1}{500} \right|_\infty = \left| \frac{3}{2^2 5^3} \right|_\infty = \frac{3}{2^2 5^3} \quad \text{but} \\ &\left| \frac{3}{2^2 5^3} \right|_2 = 4, \quad \left| \frac{3}{2^2 5^3} \right|_3 = \frac{1}{3}, \quad \left| \frac{3}{2^2 5^3} \right|_5 = 125 \quad \text{and} \quad \left| \frac{3}{2^2 5^3} \right|_7 = 1. \end{aligned} \tag{4.9}$$

A powerful theorem due to Ostrowski now states that the norms thus defined exhaust all possible norms on $\mathbb{Q}$ up to equivalency.

Theorem 4.7 (Ostrowski 1916)
The only norms (up to equivalence) on $\mathbb{Q}$ are the trivial norm $|\cdot|_0$, the infinity norm $|\cdot|_\infty$ and the p-adic norms $|\cdot|_p$, one for each prime number p.

Proof. Consider a non-trivial norm $|\cdot|_\star$ on $\mathbb{Q}$. It is enough to investigate $|\cdot|_\star$ for natural numbers greater than one since an equivalence $|n|_\star = |n|^c_{\star\star}$ trivially holds for $n = 0$ and $n = 1$ and if it holds for $n \in \mathbb{N}_{\geq 2}$ it also holds for all rational numbers $q \in \mathbb{Q}$

$$|q|_\star = \left|\frac{a}{b}\right|_\star = \frac{|a|_\star}{|b|_\star} = \frac{|a|^c_\star}{|b|^c_{\star\star}} = \left|\frac{a}{b}\right|^c_{\star\star} = |q|^c_{\star\star}. \tag{4.10}$$

We study the two cases

I $\exists n \in \mathbb{N} : |n|_\star > 1$ (This will lead to the infinity norm)

II $\forall n \in \mathbb{N}, |n|_\star \leq 1$ (This will lead to the p-adic norms)

These two cases are exhaustive.

Case I: $\exists n \in \mathbb{N} : |n|_\star > 1$
Let $a, b, n \in \mathbb{N}$ such that $a, b > 1$. We can express b^n in base a as

$$b^n = \sum_{i=0}^{m} c_i a^i \quad \text{where} \quad c_i \in \{0, \ldots, a-1\} \quad \text{and} \quad m = \lfloor \log_a b^n \rfloor. \tag{4.11}$$

Taking the norm we find

$$|b|^n_\star = |b^n|_\star \leq a(m+1)\max\{|a|^m_\star, 1\} \leq a(n\log_a b + 1)\max\{|a|^{n\log_a b}_\star, 1\}. \tag{4.12}$$

The first inequality holds since for $i \in \{0, \ldots, m\}$ we have $|a^i|_\star = |a|^i_\star \leq \max\{|a|^m_\star, 1\}$ where the 1 gets chosen in case $|a|_\star < 1$. Furthermore the sum in Eq. (4.11) has at most a copies of a^i for each i of which there are $m+1$ possible values, and we can therefore bound it from above with $a(m+1)$ copies of an upper bound of $|a^i|_\star$ by the triangle inequality. Taking the n^{th} root and letting $n \to \infty$ gives

$$|b|_\star =\leq a^{1/n}(n\log_a b + 1)^{1/n}\max\{|a|^{\log_a b}_\star, 1\} \xrightarrow{n\to\infty} \max\{|a|^{\log_a b}_\star, 1\} \tag{4.13}$$

where we have used $n^{1/n} = \mathrm{e}^{(\log n)/n} \to \mathrm{e}^0 = 0$. Now choose b such that $|b|_\star > 1$ which is possible by the assumption of Case I. This yields

$$1 < |b|_\star \leq |a|^{\log_a b}_\star \quad \Rightarrow \quad |a|_\star > 1. \tag{4.14}$$

We also get

$$|b|_\star \leq |a|^{\log_a b}_\star = |a|^{\frac{\log b}{\log a}}_\star \quad \Leftrightarrow \quad \frac{\log|b|_\star}{\log b} \leq \frac{\log|a|_\star}{\log a}. \tag{4.15}$$

As a and b thus far were kept arbitrary, we could redo the above analysis with a and b interchanged and arrive at an inequality pointing the other way. This inequality is therefore really an equality an we have

$$\frac{\log |b|_\star}{\log b} = \frac{\log |a|_\star}{\log a} \quad \Leftrightarrow \quad \log |b|_\star = \log b \underbrace{\frac{\log |a|_\star}{\log a}}_{\lambda} \quad \Leftrightarrow \quad |b|_\star = b^\lambda = |b|_\infty^\lambda \tag{4.16}$$

and $|\cdot|_\star$ is hence equivalent to the infinity norm $|\cdot|_\infty$.

Case II: $\forall n \in \mathbb{N}, |n|_\star \leq 1$
The assumption that $|\cdot|_\star$ is non-trivial guarantees that

$$\exists n \in \mathbb{N} : |n|_\star < 1. \tag{4.17}$$

Denote the prime factorization of n by

$$n = \prod_p p^{e_p}. \tag{4.18}$$

The condition $|n|_\star < 1$ implies that $|q|_\star < 1$ for at least one prime q. We now claim that $|q|_\star < 1$ holds for one and only one prime q and the remaining primes obey $|p|_\star = 1, p \neq q$. This is proven by contradiction as follows.

Assume that there exists a prime $p \neq q$ such that

$$|p|_\star < 1 \quad \text{and} \quad |q|_\star < 1. \tag{4.19}$$

Let $e \in \mathbb{N}$ be such that

$$|p|_\star^e < \frac{1}{2} \quad \text{and} \quad |q|_\star^e < \frac{1}{2}. \tag{4.20}$$

Bézout's identity states that for nonzero integers a and b, there exists integers x and y such that

$$ax + by = \gcd(a, b). \tag{4.21}$$

We can therefore write

$$xp^e + yq^e = 1. \tag{4.22}$$

Taking the norm of this equation and using the triangle inequality as well as the assumption of Case II gives

$$|1|_\star = 1 \leq |x|_\star |p|_\star^e + |y|_\star |q|_\star^e \overset{!}{<} \frac{|x|_\star + |y|_\star}{2} \leq 1. \tag{4.23}$$

The exclamation mark points out the contradiction $1 < 1$. Hence exactly one prime q has $|q|_\star < 1$. Armed with this insight, let $\alpha = |q|_\star$ which gives $0 < \alpha < 1$ and define c by

$$\alpha = q^{-c} \quad \text{which gives} \quad c > 0. \tag{4.24}$$

We then have

$$|n|_\star = \left|\prod_p p^{e_p}\right|_\star = \prod_p |p|_\star^{e_p} = |q|_\star^{e_q} = \alpha^{e_q} = q^{-ce_q} = \left(q^{-e_q}\right)^c = |n|_q^c \tag{4.25}$$

and $|\cdot|_\star$ is hence equivalent to the q-adic norm $|\cdot|_q$. □

4.1.2 Construction of $\mathbb{Q}_p$

Recall that a Cauchy sequence is a sequence whose elements become arbitrarily close to one another. This notion of closeness relies on the existence of a norm. The real numbers can be constructed from the rationals by considering Cauchy sequences of rational numbers and defining irrational numbers, i.e. elements of $\mathbb{R} \setminus \mathbb{Q}$ as an equivalence class of Cauchy sequences. As an example, consider the sequence (in base 10)

$$3; \quad 3{,}1; \quad 3{,}14; \quad 3{,}141; \quad 3{,}1415; \quad 3{,}14159; \quad \ldots. \tag{4.26}$$

Each element here is a rational number but if the correct digits are added in each step and in particular added in such a way that they never repeat then under the infinity norm, this sequence will be convergent and can be defined as (a representative of) the real number π. Observe now that under the p-adic norms, a sequence like this has little hope of being convergent as $|q|_p$ depends on how the prime p sits in the rational q which can vary wildly with small changes in the decimal places of q. Still it is possible to find Cauchy sequences of rationals with respect to the p-adic norm. What would such a sequence of rational numbers look like? Start by considering a rational number q written in base p

$$q = \sum_{k=k_{\min}}^{k_{\max}} a_k p^k \in \mathbb{Q} \quad \text{where} \quad a_k \in \{0, \ldots, p-1\} \quad \text{and} \quad a_{k_{\min}} \neq 0. \tag{4.27}$$

The nature of the p-adic norm is such that large powers of p are p-adically small and adding terms $a_k p^k$ with larger and larger k will contribute smaller and smaller corrections to q which is equivalent to adding decimal digits in Eq. (4.26) for the real case. Completing $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$ therefore amounts to replacing the upper bound $k_{\max}$ by infinity. A general p-adic number x can thus be written

$$x = \sum_{k=k_{\min}}^{\infty} a_k p^k \in \mathbb{Q}_p \quad \text{where} \quad a_k \in \{0, \ldots, p-1\} \quad \text{and} \quad a_{k_{\min}} \neq 0. \tag{4.28}$$

This sum is still convergent under the p-adic norm, and the norm of x is in fact

$$|x|_p = \Bigg| p^{k_{\min}} \underbrace{\sum_{k=0}^{\infty} a_{k+k_{\min}} p^k}_{\text{No factors of } p} \Bigg|_p = p^{-k_{\min}}. \tag{4.29}$$

The sum carries no factors of p as it is an integer of the form $a_{k_{\min}} + p(\dots)$ where $0 \neq a_{k_{\min}} \neq p$. Trying to replace the lower limit $k_{\min}$ by negative infinity however would result in a divergent sum.

We define the set of p-adic integers $\mathbb{Z}_p$ as

$$\mathbb{Z}_p \equiv \{x \in \mathbb{Q}_p : |x|_p \leq 1\} \subset \mathbb{Q}_p. \tag{4.30}$$

and the p-adic units $\mathbb{Z}_p^\times$ as

$$\mathbb{Z}_p^\times \equiv \{x \in \mathbb{Q}_p : |x|_p = 1\} \subset \mathbb{Z}_p. \tag{4.31}$$

Note that the p-adic integers are compact in $\mathbb{Q}_p$. In the series representation Eq. (4.28), the p-adic integers correspond to the series which have $k_{\min} \geq 0$ while the p-adic units correspond to the series with $k_{\min} = 0$. The p-adic units are precisely those p-adic integers whose multiplicative inverses are still p-adic integers. Note lastly that in particular $\mathbb{Z} \subset \mathbb{Z}_p$ and for example $\frac{1}{2} \in \mathbb{Z}_3$ but $\frac{1}{2} \notin \mathbb{Z}_2$.

Proposition 4.8
For a rational number $x \in \mathbb{Q}$ we have that x is a p-adic integer for all p if and only if it is a proper integer

$$x \in \mathbb{Z}_p \quad \forall p \quad \Leftrightarrow \quad x \in \mathbb{Z}. \tag{4.32}$$

Proof. For a fixed p, the property $x \in \mathbb{Z}_p$ states that the prime number p appears with positive multiplicity in x, or equivalently that there are no net factors of p in the denominator of x. That this is true for all p means that x has unit denominator and hence $x \in \mathbb{Z}$. The implication in the other direction is trivial. □

4.1.3 Properties of $\mathbb{Q}_p$

The p-adic numbers $\mathbb{Q}_p$ behave very differently from the reals. This difference in behavior comes from the fact that the p-adics obey a stronger version of the triangle inequality Eq. (4.4), namely the so called *ultrametric property*

$$|x + y|_p \leq \max(|x|_p, |y|_p). \tag{4.33}$$

For rational $x = p^{r_1}\frac{m_1'}{n_1'}$ and $y = p^{r_2}\frac{m_2'}{n_2'}$ (the primes indicate no factors of p) we can see this by assuming $r_1 \leq r_2 \Leftrightarrow x \geq y$ (in the p-adic sense) without loss of generality and writing

$$\begin{aligned} |x+y|_p &= \left| p^{r_1}\frac{m_1'}{n_1'} + p^{r_2}\frac{m_2'}{n_2'} \right|_p = |p^{r_1} m_1' n_2' + p^{r_2} m_2' n_1'|_p = \\ &= |p^{r_1}(m_1' n_2' + p^{r_2 - r_1} m_2' n_1')|_p = \underbrace{p^{-r_1}}_{|x|_p} \underbrace{|m_1' n_2' + p^{r_2-r_1} m_2' n_1'|_p}_{\leq 1 \text{ since argument is in } \mathbb{N}} \leq |x|_p. \end{aligned} \tag{4.34}$$

For p-adic numbers

$$x = \sum_{k=k_{\min}}^{\infty} a_k p^k \quad \text{and} \quad y = \sum_{l=l_{\min}}^{\infty} b_l p^l \tag{4.35}$$

we can understand the ultrametric property by observing

$$x + y = \sum_{m=\min(k_{\min},l_{\min})}^{\infty} (a_m + b_m) p^m = \sum_{m=m_{\min}}^{\infty} c_m p^m \tag{4.36}$$

where we have defined $a_k = 0$ for $k < k_{\min}$ and similarly for b_l. The c's are defined by again requiring $c_m \in \{0, \ldots, p-1\}$. In the event that $a_m + b_m \geq p$, then a contribution of 1 carries over to c_{m+1} just like when adding real numbers together digit by digit[9]. It is now easy to see that

$$|x+y|_p \begin{cases} = |x|_p, & |x|_p > |y|_p \\ = |y|_p, & |x|_p < |y|_p \\ = |x|_p = |y|_p, & |x|_p = |y|_p \quad \text{and} \quad a_{k_{\min}} + b_{l_{\min}} \neq p \\ < |x|_p = |y|_p, & |x|_p = |y|_p \quad \text{and} \quad a_{k_{\min}} + b_{l_{\min}} = p \end{cases} \Bigg\} \leq \max(|x|_p, |y|_p). \tag{4.37}$$

A norm satisfying the ultrametric property (like the p-adic norms) is called "non-archimedean" while a norm which doesn't is called "archimedean". This name stems from the so called Archimedean axiom of geometry[10] which in the context of a number field K states that for $x, y \in K$ where $x < y$ there exists a natural number $n \in \mathbb{N}$ such that $\underbrace{x + \cdots + x}_{n \text{ times}} \equiv nx > y$. This is obviously not true for the p-adic norm as $|nx|_p = |n|_p |x|_p \leq |x|_p$ for all $n \in \mathbb{N}$.

Some examples of how geometry and algebra in the p-adic case differs from the real case include [47]:

- $\mathbb{Q}_p$ is not ordered. There is no notion of negative numbers in $\mathbb{Q}_p$.
- $x^2 + 1 = 0$ is solvable over $\mathbb{Q}_p$ for $p \equiv 1 \pmod 4$.
- Two balls in $\mathbb{Q}_p$ are either disjoint or one contained inside the other.
- Every point in a ball in $\mathbb{Q}_p$ is a center point.

Recall that the complex numbers $\mathbb{C}$ are the algebraic closure of $\mathbb{R}$, obtained as a field extension of degree 2 by adjoining a root to the polynomial equation $x^2 + 1 = 0$. The resulting number field $\mathbb{C}$ is both algebraically closed (fundamental theorem of algebra) and complete in the sense that all Cauchy sequences have limits in $\mathbb{C}$. For the p-adics however, the algebraic completion $\overline{\mathbb{Q}_p}$ involves an infinite sequence of field extensions

[9] Multiplication $(\sum_k a_k p^k)(\sum_l b_l p^k) = \sum_m c_m p^m$ works the same way.

[10] Book V, definition 4 of Euclid's Elements: Magnitudes are said to have a ratio to one another which can, when multiplied, exceed one another.

adjoining roots to polynomial equations of higher and higher degree. The resulting field $\overline{\mathbb{Q}_p}$ is not complete but upon completion using the norm on $\overline{\mathbb{Q}_p}$ induced from $|\cdot|_p$, one obtains a field $\mathbb{C}_p$ which is both algebraically closed and complete [46]. The fields $\mathbb{C}_p$ are much less understood than $\mathbb{C}$.

4.1.4 p-adic integration

As we are ultimately interested in performing Fourier analysis on functions with a p-adic domain, we will build up the necessary theory to understand the p-adic analog of a real expression like

$$\int_0^1 f(x)\mathrm{e}^{-2\pi \mathrm{i} m x}\,\mathrm{d}x. \tag{4.38}$$

This will require the notion of integration over $\mathbb{Q}_p$ as well as the notion of an *additive character*.

Integration over $\mathbb{Q}_p$ is defined with respect to the *additive measure* $\mathrm{d}x$, obeying

$$\mathrm{d}(x+a) = \mathrm{d}x \quad \text{and} \quad \mathrm{d}(ax) = |a|_p \mathrm{d}x. \tag{4.39}$$

Furthermore, this measure is normalized to unity over the p-adic integers

$$\int_{\mathbb{Z}_p} \mathrm{d}x = 1. \tag{4.40}$$

Since $\mathbb{Q}_p$ is unordered, we will never write down integrals with limits like the one in Eq. (4.38), rather the domain will be a prescribed subset of $\mathbb{Q}_p$ and the corresponding volume will be worked out by comparing it to the normalized volume of $\mathbb{Z}_p$. Consider for example the volume of the p-adic units $\mathbb{Z}_p^\times$, we have

$$\begin{aligned}\int\limits_{\mathbb{Z}_p^\times} \mathrm{d}x &= \int\limits_{\mathbb{Z}_p \smallsetminus (p\mathbb{Z}_p)} \mathrm{d}x = \int\limits_{\mathbb{Z}_p} \mathrm{d}x - \int\limits_{p\mathbb{Z}_p} \mathrm{d}x = \int\limits_{\mathbb{Z}_p} \mathrm{d}x - \int\limits_{\mathbb{Z}_p} \mathrm{d}(px) = \\ &= \int\limits_{\mathbb{Z}_p} \mathrm{d}x - \int\limits_{\mathbb{Z}_p} |p|_p\,\mathrm{d}x = 1 - p^{-1} = \frac{p-1}{p}.\end{aligned} \tag{4.41}$$

We can understand this intuitively by considering the general form of a p-adic integer

$$\sum_{k=0}^{\infty} a_k p^k \in \mathbb{Z}_p \quad \text{where} \quad a_k \in \{0, \dots, p-1\}. \tag{4.42}$$

The p-adic units are those p-adic integers which satisfy $a_0 \neq 0$ and the fraction $\frac{p-1}{p}$ simply corresponds to the $p-1$ possible remaining choices for the digit a_0.

Example 4.9

Let's look at an example of p-adic integration, namely integrating the function $|x|_p^s$ for $s \in \mathbb{C}$ over $\mathbb{Z}_p$. The derivation goes as follows

$$\begin{aligned}\int_{\mathbb{Z}_p} |x|_p^s \, \mathrm{d}x &= \sum_{k=0}^{\infty} \int_{p^k \mathbb{Z}_p^\times} |x|_p^s \, \mathrm{d}x = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^\times} \left|p^k x\right|_p^s \mathrm{d}(p^k x) = \sum_{k=0}^{\infty} \int_{\mathbb{Z}_p^\times} \left|p^k\right|_p^s |x|_p^s \left|p^k\right|_p \mathrm{d}x = \\ &= \sum_{k=0}^{\infty} p^{-ks-k} \int_{\mathbb{Z}_p^\times} \mathrm{d}x = \frac{1}{1-p^{-s-1}} \frac{p-1}{p}, \quad \operatorname{Re} s > -1.\end{aligned} \tag{4.43}$$

Now that the notion of integration has been introduced, let us look at the p-adic analog of the real expression $\mathrm{e}^{2\pi \mathrm{i} m x}$. We have the following definitions

Definition 4.10 (Fractional part of a p-adic number)
We define the ***fractional part*** $[x]_p$ *of a p-adic number* $x \in \mathbb{Q}_p$ *from its series representation*

$$x = \sum_{k=k_{\min}}^{\infty} a_k p^k = \underbrace{\sum_{k=k_{\min}}^{-1} a_k p^k}_{[x]_p} + \sum_{k=0}^{\infty} a_k p^k. \tag{4.44}$$

For p-adic integers $x \in \mathbb{Z}_p$ *we have* $[x]_p = 0$*. Notice that we always have* $[x]_p \in \mathbb{Q}$*.*

Remark 4.11.
Note that $x - [x]_p \in \mathbb{Z}_p$ for all p.

Proposition 4.12
For a rational number $x \in \mathbb{Q}$ *we have*

$$x - \sum_p [x]_p \in \mathbb{Z}. \tag{4.45}$$

Proof. For a fixed p, write

$$x - \sum_p [x]_p = x - [x]_p - \sum_{q \neq p} [x]_q. \tag{4.46}$$

By the triangle inequality for the p-adic norm, we have

$$\left|[x]_q\right|_p \leq \max\left\{|a_{k_{\min}} q^{k_{\min}}|_p, \ldots, |a_{-1} q^{-1}|_p\right\} = \max\left\{|a_{k_{\min}}|_p, \ldots, |a_{-1}|_p\right\} \leq 1 \tag{4.47}$$

since $a_k \in \mathbb{Z}$. This together with remark 4.11 shows that Eq. (4.46) is the sum of p-adic integers and is thus itself a p-adic integer. Since p is arbitrary, this property holds for all p and proposition 4.8 then gives that $x - \sum_p [x]_p$ must indeed be integer. □

Definition 4.13 (Additive character and unramified character)
An ***additive character*** $\psi_{u,p} : \mathbb{Q}_p \to S^1$ *on* $\mathbb{Q}_p$ *(*$p \leq \infty$*) is parametrized by a p-adic or*

real number $u \in \mathbb{Q}_p$ *and given by*

$$\psi_{u,p}(x) = \begin{cases} \mathrm{e}^{2\pi \mathrm{i} ux}, & p = \infty \\ \mathrm{e}^{-2\pi \mathrm{i}[ux]_p}, & p < \infty. \end{cases} \tag{4.48}$$

We say that a character is ***unramified*** *if* $u = 1$.

Remark 4.14.
Additive characters obey $\psi_{u,p}(x)\psi_{u,p}(y) = \psi_{u,p}(x+y)$ and $\psi_{u,p}(x)\psi_{v,p}(x) = \psi_{u+v,p}(x)$. For the real case, these identities are elementary while for the p-adic case they hold since

$$[a+b]_p = [a]_p + [b]_p - n \quad \text{for} \quad a, b \in \mathbb{Q}_p \quad \text{and} \quad n \in \mathbb{Z}. \tag{4.49}$$

The p-adic Fourier transform will involve integrating a function against the additive character $\psi_{u,p}$. Let's start by considering the integral of the character alone over $\mathbb{Z}_p$. We have

$$\int\limits_{\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[ux]_p}\, \mathrm{d}x = \gamma_p(u) \equiv \begin{cases} 1, & u \in \mathbb{Z}_p \\ 0, & u \neq \mathbb{Z}_p \end{cases} = \begin{cases} 1, & |u|_p \leq 1 \\ 0, & |u|_p > 1 \end{cases} \tag{4.50}$$

where we have introduced the function γ_p called the *p-adic gaussian* which will play a big role later in this thesis. To understand why this formula holds true, consider the two following cases.

(I) $u \in \mathbb{Z}_p$: We have $ux \in \mathbb{Z}_p$ and hence $[ux]_p = 0$ and the integral becomes $\int_{\mathbb{Z}_p} \mathrm{d}x = 1$.

(II) $u \notin \mathbb{Z}_p$: This case will yield zero as one is integrating a periodic function over one or more full periods. It is enough to keep track of how the first digit of ux integrates. Start by investigating the case $u \in p^{-1}\mathbb{Z}_p^\times$, parametrizing u as $u = p^{-1}n$ with $n \in \mathbb{N}$ fixed. We get

$$\int\limits_{\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[p^{-1}nx]_p}\, \mathrm{d}x = \int\limits_{\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[p^{-1}x]_p}\, \mathrm{d}x = \sum_{a=0}^{p-1} \mathrm{e}^{-2\pi \mathrm{i}a/p} \int\limits_{C_a} \mathrm{d}x = \frac{1 - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{p}p}}{1 - \mathrm{e}^{-\frac{2\pi \mathrm{i}}{p}}} \frac{1}{p} = 0 \tag{4.51}$$

where C_a is defined through $\mathbb{Z}_p = \cup_{a=0}^{p-1} C_a$, or all p-adic integers where the first digit a_0 in its series representation equals a. Each C_a has volume $1/p$ but the sum over exponentials averages to zero. The more general case $u = p^{-k}n$ with $n \in \mathbb{Z}_p^\times$ fixed and $n \in \mathbb{N}$ works analogously

$$\int\limits_{\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[p^{-k}nx]_p}\, \mathrm{d}x = \int\limits_{\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[p^{-k}x]_p}\, \mathrm{d}x \propto \sum_{a=0}^{p-1} \mathrm{e}^{-2\pi \mathrm{i}a/p^k} \int\limits_{p\mathbb{Z}_p} \mathrm{e}^{-2\pi \mathrm{i}[p^{-k}x]_p}\, \mathrm{d}x = 0 \tag{4.52}$$

where the proportionality sign is due to a volume factor.

4.2 Adeles

We have seen that $\mathbb{Q}$ has a countably infinite list of completions, labelled by the prime numbers p for the p-adics and ∞ for the reals. Ostrowski's theorem tells us that these completions are all on the same footing and gives no reason to pick out one completion over another. It is reasonable to think that something stands to be gained by keeping all these completions and performing analysis over all of them as an entity. This is exactly what is the adeles stand to do.

Definition 4.15 (The ring of adeles)
An ***adele*** *x is an infinite tuple $x = (x_\infty; x_2, x_3, x_5, \dots)$ where each $x_p \in \mathbb{Q}_p$ $(p \leq \infty)$. The space of all adeles $\mathbb{A}$ is defined as the infinite restricted (as indicated by the prime) direct product*

$$\mathbb{A} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p \tag{4.53}$$

where the restriction is such that almost all (meaning all but a finite number) of the components x_p of an adele must be p-adic integers. The norm of an adele, denoted as $|\cdot|$ (without ornaments) is given by

$$|x| = \prod_{p \leq \infty} |x_p|_p \tag{4.54}$$

and the restriction thus guarantees that this is finite.

Definition 4.16 (Global and local)
We will denote $\mathbb{Q}_\infty \equiv \mathbb{R}$. The word ***global*** *refers to mathematics taking place on the level of the adeles while the word* ***local*** *refers to each $\mathbb{Q}_p$ individually.*

Remark 4.17.
Two adeles x and y are added and multiplied point wise $x + y = (x_\infty + y_\infty; x_2 + y_2, \dots)$ and $xy = (x_\infty y_\infty; x_2 y_2, \dots)$. Note that not every adele has a multiplicative inverse, one example being the adeles x with $x_p = p$ $(p \leq \infty)$. This adele has $|x_p| = \frac{1}{p} \leq 1$ so that every $x_p \in \mathbb{Z}_p$ but the multiplicative inverses are all $x_p \neq \mathbb{Z}_p$. This is the reason why $\mathbb{A}$ is a ring and not a field.

As each $\mathbb{Q}_p$ for $p \leq \infty$ is a completion of $\mathbb{Q}$, we can embed the rationals into the adeles by writing

$$\mathbb{Q} \ni q \mapsto (q; q, q, \dots) \in \mathbb{A} \tag{4.55}$$

which is called *diagonal embedding.* This element is indeed an adele since the p-adic norms $|q|_p$ are only non-trivial for the finite number of primes that make up the rational q. A rational q embedded diagonally has unit norm in the adeles $|q| = 1$. This comes about since $|q|_p$ returns exactly p^{-a} where a is the multiplicity (positive or negative or zero) of the prime p as it appears in q. We therefore have

$$\prod_p |q|_p = \frac{1}{|q|_\infty} \quad \text{and hence} \quad |q| = \prod_{p \leq \infty} |q|_p = 1. \tag{4.56}$$

A mildly astonishing fact is that $\mathbb{Q}$ embedded in this way sits discretely inside $\mathbb{A}$! This is seen as follows: In order for a rational q (embedded diagonally) to be discrete in $\mathbb{A}$, we should be able to construct an open neighborhood around q which does not contain any other rationals. Such a neighborhood is

$$V = (q - 1/2, q + 1/2) \times \prod_p \mathbb{Z}_p. \tag{4.57}$$

This is open as $\mathbb{Z}_p$ is indeed an open ball

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\} = \{x \in \mathbb{Q}_p : |x|_p < a\} \quad \text{with} \quad 1 < a < p, \tag{4.58}$$

since the "next" larger value $|\cdot|_p$ takes after 1 is p. Note also that if $x \in \mathbb{Z}_p$ for all $p \leq \infty$, we must have $x \in \mathbb{Z}$ (and vice versa). Thus, a diagonally embedded rational $x \in V$ must also be an integer $x \in \mathbb{Z}$. The condition $x \in (q - 1/2, q + 1/2)$ then forces $x = q$ since there are no other integers in this interval. This shows that V is an open neighborhood of the rational q (embedded diagonally) which only contains q itself and no other rationals.

4.3 Adelic group theory

For a number field F, a linear algebraic group $G(F)$ $(p \leq \infty)$ is a subgroup of $\mathrm{GL}_n(F)$ defined by a set of polynomial conditions. The typical example is the group $\mathrm{SL}_n(F)$ which is the set of those $g \in \mathrm{GL}_n(F)$ satisfying the constraint $\det(g) = 1$. If the polynomial defining the algebraic group have coefficients in $\mathbb{Q}$, it makes sense talk about both the p-adic as well as the real algebraic groups $G(\mathbb{Q}_p)$ $(p \leq \infty)$ as subgroups of $\mathrm{GL}_n(\mathbb{Q}_p)$, since the p-adics as well as the reals contain $\mathbb{Q}$. Such is the case for the condition $\det(g) = 1$ and $\mathrm{SL}_n(\mathbb{Q}_p)$ $(p \leq \infty)$ thus denotes the set of $n \times n$-matrices of unit determinant with either p-adic or real entries.

An important difference for the p-adic group $G(\mathbb{Q}_p)$ is how the maximal compact subgroup $K(G(\mathbb{Q}_p))$ is defined. Recall that for a split real group $G(\mathbb{R})$, the maximal compact subgroup was generated by the maximal compact subalgebra $\mathfrak{k}$ of Eq. (2.12) and in particular for $\mathrm{SL}_n(\mathbb{R})$ we have $K(\mathrm{SL}_n(\mathbb{R})) = \mathrm{SO}_n(\mathbb{R})$. As the p-adic integers $\mathbb{Z}_p$ form a compact ring in $\mathbb{Q}_p$, it follows that the maximal compact subgroup of a p-adic group $G(\mathbb{Q}_p)$ equals its restriction to the p-adic integers,

$$K(G(\mathbb{Q}_p)) = G(\mathbb{Q}_p) \cap \mathrm{GL}_n(\mathbb{Q}_p) \equiv G(\mathbb{Z}_p), \quad p < \infty. \tag{4.59}$$

Thus, in the case $G(\mathbb{Q}_p) = \mathrm{SL}_n(\mathbb{Q}_p)$, the maximal compact subgroup is simply the group of (p-adic) integer matrices $K(\mathrm{SL}_n(\mathbb{Q}_p)) = \mathrm{SL}_n(\mathbb{Z}_p)$.

The adelic group $G(\mathbb{A})$ is then formed as the restricted product

$$G(\mathbb{A}) = G(\mathbb{R}) \times \underbrace{\prod_p{}' G(\mathbb{Q}_p)}_{G_f} \tag{4.60}$$

where the restriction now imposes that for an adelic group element

$$g = (g_\infty, g_2, g_3, g_5, \dots) \in G(\mathbb{A}), \tag{4.61}$$

almost all local group elements g_p must be lie in their maximal compact subgroup $G(\mathbb{Z}_p)$.

4.3.1 Iwasawa decomposition of $\mathrm{SL}_n(\mathbb{Q}_p)$

The Iwasawa decomposition for a p-adic group reads

$$G(\mathbb{Q}_p) = N(\mathbb{Q}_p)A(\mathbb{Q}_p)G(\mathbb{Z}_p). \tag{4.62}$$

Explicit results for the Iwasawa decomposition of p-adic matrices will be needed later on in this thesis and were worked out in **Paper I**. First of all unlike the real case, the Iwasawa decomposition of a p-adic matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$ is not unique. This can be seen simply by counting degrees of freedom. Considering Eq. (4.62) for $G(\mathbb{Q}_p) = \mathrm{SL}_n(\mathbb{Q}_p)$ gives

$$\begin{aligned} &\text{Number of degrees of freedom in LHS: } n^2 - 1 \\ &\text{Number of degrees of freedom in RHS: } \frac{n(n-1)}{2} + n - 1 + (n^2-1)'. \end{aligned} \tag{4.63}$$

The prime indicates that the $n^2 - 1$ degrees of freedom are for p-adic integers, but since the rational part of a p-adic number only has a finite number of digits (see Eq. (4.44)) while the integer part has has infinitely many, there is just as much freedom in a general p-adic number as in a p-adic integer. Hence for $\mathrm{SL}_n(\mathbb{Q}_p)$, we expect a $\left(\frac{n(n-1)}{2} + n - 1\right)$-parameter family of decompositions. The following theorem from **Paper I** shows that this is the case and offers an explicit parametrization for this family of decompositions.

Theorem 4.18 (Family of Iwasawa decompositions for $\mathrm{SL}_n(\mathbb{Q}_p)$)
Given an Iwasawa decomposition NAK of $M \in \mathrm{SL}_n(\mathbb{Q}_p)$, all other decompositions $N'A'K'$ can be found by writing $N' = NAXA^{-1}$, $A' = AY$ and $K' = (XY)^{-1}K$ and letting X range over all unit upper triangular matrices[11] in $\mathrm{SL}_n(\mathbb{Z}_p)$ and Y over all diagonal $n \times n$-matrices whose entries are p-adic units. Furthermore, each choice of X and Y yields a different decomposition.

Proof. Consider two Iwasawa decompositions of M

$$M = N_1A_1K_1 = N_2A_2K_2, \tag{4.64}$$

[11] Unit upper triangular refers to upper triangular with ones on the diagonal.

Without loss of generality, we may take N_2 to be of the form

$$N_2 = N_1 A_1 \nu A_1^{-1} \quad \text{and } A_2 \text{ to be of the form} \quad A_2 = A_1 \alpha \tag{4.65}$$

where ν is unit upper triangular and α is diagonal. Define the matrix

$$Z = K_1 K_2^{-1} = A_1^{-1} N_1^{-1} N_2 A_2 = \nu\alpha. \tag{4.66}$$

Note that $Z \in \mathrm{SL}_n(\mathbb{Z}_p)$ as it is the product of two $\mathrm{SL}_n(\mathbb{Z}_p)$-matrices and also that it is upper triangular. We have

$$1 = |1|_p = |\det Z|_p = |\det \nu\alpha|_p = |\det \alpha|_p = \prod_{i=1}^{n-1} |\alpha_i|_p \tag{4.67}$$

where α_i are the diagonal elements of α. Since α_i are also the diagonal elements of Z, we have $|\alpha_i|_p \leq 1$. The line above then implies $|\alpha_i|_p = 1$ for all i since if $|\alpha_i|_p < 1$ for some i we would necessarily need $|\alpha_i|_p > 1$ for other i in order for their product to equal unity. Hence α is of the form stated in the theorem. Since α consists of p-adic units, it preserves the p-adic norms of the matrix element of a matrix under multiplication. We therefore have

$$\nu\alpha \in \mathrm{SL}_n(\mathbb{Z}_p) \Leftrightarrow \nu \in \mathrm{SL}_n(\mathbb{Z}_p) \tag{4.68}$$

and hence ν is also of the form stated in the theorem. Solving Eq. (4.66) for K_2 gives

$$K_2 = (\nu\alpha)^{-1} K_1. \tag{4.69}$$

We have thus proven that any two Iwasawa decompositions are related as stated in the theorem. Furthermore, it is clear that since N and A are invertible, the maps

$$X \mapsto N' = NAXA^{-1} \quad \text{and} \quad Y \mapsto Y' = AY \tag{4.70}$$

are injective and therefore by varying X and Y, one generates every decomposition exactly once. □

Remark 4.19.
X and Y parametrize the $\left(\frac{n(n-1)}{2} + n - 1\right)$-family of decompositions respectively mentioned above.

An important corollary is that for a matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$ written in Iwasawa form, the p-adic norms of the dilatons $y_1, \ldots, y_{n-1}$ are unique.

Corollary 4.20
For a matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$, the norms of the dilatons are unique across all Iwasawa decompositions of M.

Proof. By theorem 4.18, the semisimple part of any Iwasawa decomposition of M is of the form AY where A is the semisimple part of some Iwasawa decomposition and Y is diagonal with entries in the p-adic units, hence the norms of the diagonal elements are unchanged under $A \mapsto AY$. Regarding the dilatons y_i in Eq. (2.30), we get that the norm of y_1 in unchanged, and hence also that of y_2 etcetera. □

For the application of p-adic numbers to the theory of automorphic forms in this thesis, the important pieces of information in a p-adic matrix will turn out to be these norms. **Paper I** thus also contains a closed formula for the p-adic norms of the dilatons of an arbitrary matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$. This formula and its proof relies on the notion of a matrix minor as well as a modified version of the LU-decomposition which I decide to call the strong UL-decomposition. The proof of this formula simultaneously defines an algorithm for computing a complete Iwasawa decomposition of a matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$. Together with theorem 4.18 one can then obtain a complete parametrization for the Iwasawa decompositions of a given p-adic matrix.

Definition 4.21 (Minor)
*Given an $m \times n$ matrix M, a **minor of order k***

$$M\left(\begin{smallmatrix} r_1 & \dots & r_k \\ c_1 & \dots & c_k \end{smallmatrix}\right) \tag{4.71}$$

is the determinant of the submatrix of M obtained by only picking the k rows $\{r_i\}$ and k columns $\{c_i\}$.

*If the rows and columns agree, i.e. $r_i = c_i$ for all $i \in \{1, \dots, k\}$, then the minor is called a **principal minor**.*

If the rows selected are the first k rows in order,

$$M\left(\begin{smallmatrix} 1 & \dots & k \\ c_1 & \dots & c_k \end{smallmatrix}\right), \tag{4.72}$$

*the minor is called a **leading minor** while if they are the last k rows in order,*

$$M\left(\begin{smallmatrix} m-k+1 & \dots & m \\ c_1 & \dots & c_k \end{smallmatrix}\right), \tag{4.73}$$

*we will call it an **anti-leading minor**.*

Hence, the minors

$$M\left(\begin{smallmatrix} 1 & \dots & k \\ 1 & \dots & k \end{smallmatrix}\right) \quad \text{and} \quad M\left(\begin{smallmatrix} m-k+1 & \dots & m \\ n-k+1 & \dots & n \end{smallmatrix}\right) \tag{4.74}$$

*are called the **leading principal minor** and the **anti-leading principal minor** of order k respectively.*

The empty minor is defined as

$$M(\,) \equiv 1. \tag{4.75}$$

Remark 4.22.
A minor is totally antisymmetric under permutations of the rows as well as the columns. Hence it vanishes if some r's coincide or some c's coincide.

Remark 4.23.

A minor can be expanded along a row or column according the formula

$$\begin{aligned} M\left(\begin{smallmatrix} r_1 & \dots & r_k \\ c_1 & \dots & c_k \end{smallmatrix}\right) &= \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} r_1 \\ c_a \end{smallmatrix}\right) M\left(\begin{smallmatrix} r_2 & \dots & r_a & r_{a+1} & \dots & r_k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) = \\ &= \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} r_a \\ c_1 \end{smallmatrix}\right) M\left(\begin{smallmatrix} r_1 & \dots & r_{a-1} & r_{a+1} & \dots & r_k \\ c_2 & \dots & c_a & c_{a+1} & \dots & c_k \end{smallmatrix}\right) \end{aligned} \tag{4.76}$$

called Laplace expansion.

Next we recall the famous LU-decomposition due to Alan Turing together with a useful lemma.

Lemma 4.24
Given a non-singular square matrix M of size n, there is a permutation matrix P such that the leading principal minors of MP are all non-zero.

Proof. See [48]. Below we prove a more powerful version of this lemma adapted for anti-leading minors. □

Thanks to this lemma, given a non-singular square matrix M one can always find a permutation matrix P such that LU-decomposition of MP is possible.

Theorem 4.25 (LU-decomposition)
Let F be a field. A matrix $M \in \mathrm{SL}(n, F)$ can be written as

$$\begin{aligned} M &= MPP^{-1} = LDUP^{-1} = \\ &= \begin{pmatrix} 1 & & & \\ l_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & \\ & \frac{y_2}{y_1} & & \\ & & \ddots & \\ & & & \frac{1}{y_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \dots & u_{n1} \\ & 1 & \dots & u_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} P^{-1} \end{aligned} \tag{4.77}$$

where D has unit determinant and its elements are given by

$$y_p = (MP)\begin{pmatrix} 1 & \dots & p \\ 1 & \dots & p \end{pmatrix}, \tag{4.78}$$

L is unit lower triangular with matrix elements

$$l_{ip} = (MP)\begin{pmatrix} 1 & \dots & p-1 & i \\ 1 & \dots & p-1 & p \end{pmatrix} \Big/ (MP)\begin{pmatrix} 1 & \dots & p \\ 1 & \dots & p \end{pmatrix} = (MP)\begin{pmatrix} 1 & \dots & p-1 & i \\ 1 & \dots & p-1 & p \end{pmatrix} \Big/ y_p \quad i \geq p, \tag{4.79}$$

U is unit upper triangular with matrix elements

$$u_{pi} = (MP)\begin{pmatrix} 1 & \dots & p-1 & p \\ 1 & \dots & p-1 & i \end{pmatrix} \Big/ (MP)\begin{pmatrix} 1 & \dots & p \\ 1 & \dots & p \end{pmatrix} = (MP)\begin{pmatrix} 1 & \dots & p-1 & p \\ 1 & \dots & p-1 & i \end{pmatrix} \Big/ y_p \quad p \leq i \tag{4.80}$$

and P is almost[12] *a permutation matrix.*

Proof. See [22]. □

For our purposes of Iwasawa decomposition, we will need a UL-decomposition rather than the conventional LU-decomposition. In this case, leading principal minors get replaced by anti-leading principal minors.[13] We begin by proving a more powerful version of lemma 4.24 (adapted for anti-leading minors) which lets us construct a UL-decomposition with the added benefit that one may first permute the columns of the matrix to be decomposed and freely choose which column should stand on the right so long as the bottom element in this column is nonzero. This added freedom in the UL-decomposition is what the adjective "strong" is referring to. The strong UL-decomposition will turn out to be useful in computing the Iwasawa decomposition of an arbitrary matrix in $\mathrm{SL}_n(\mathbb{Q}_p)$.

We begin by proving a more powerful version of lemma 4.24 (adapted for anti-leading minors) which lets us construct a UL-decomposition with the added property that one may choose freely the rightmost column in the matrix to be decomposed as long as the bottom element in this column is nonzero. This added freedom in the UL-decomposition is what the adjective "strong" is referring to. The strong UL-decomposition will turn out to be useful in computing the Iwasawa decomposition of an arbitrary matrix in $\mathrm{SL}_n(\mathbb{Q}_p)$.

Lemma 4.26
Given a non-singular square matrix M of size n, there is a permutation matrix Π_a such that the anti-leading principal minors of $M\Pi_a$ are all non-zero where Π_a moves column a to the rightmost position and we require the bottom element of column a to be non-zero.

Proof. The permutation of columns is realized by multiplication of a permutation matrix from the right. The proof works by induction and the cases $n = 1$ and $n = 2$ are obvious. Assume that the statement holds true for matrices of size up to and including $n-1$ and consider a matrix of size n. Start by permuting the columns of M so that column a is not in the leftmost position and consider the $(n-1) \times n$ matrix obtained by deleting the top row of the permuted matrix. This matrix contains $n-1$ linearly independent rows and hence also contains $n-1$ linearly independent columns where column a can be assumed to be one of them, since it is not a zero-column. Permute these columns to the rightmost $n-1$ positions and apply the induction hypothesis to the bottom right $(n-1) \times (n-1)$ block of the permuted matrix, which is allowed since this block is guaranteed to be a non-singular matrix. This establishes that the first $n-1$ anti-leading principal minors of $M\Pi$ are non-zero and the rightmost column of $M\Pi$ is column a of M, where Π denotes the resulting permutation matrix. The last anti-leading principal

[12] In the case that P is an odd permutation, we may replace it with for example $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} P$ to give it a positive determinant and thereby preserve the determinant condition on both sides.

[13] Switching between LU and UL introduces no complications for proofs, which work analogously for the two. Which one is required depends on which of the conventions KAN or NAK is used.

minor is simply $\det(M\Pi)$ which is non-zero since $M\Pi$ is non-singular. Peano's axiom of induction now establishes the lemma. □

Theorem 4.27 (Strong UL-decomposition)
Let F be a field. A matrix $M \in \mathrm{SL}(n, F)$ with $M_{na} \neq 0$ can be written as

$$M = M\Pi_a\Pi_a^{-1} = V\Delta\Lambda\Pi_a^{-1} =$$
$$= \begin{pmatrix} 1 & v_{12} & \dots & v_{n1} \\ & 1 & \dots & v_{2n} \\ & & \ddots & \vdots \\ & & & 1 \end{pmatrix} \begin{pmatrix} \eta_1 & & & \\ & \frac{\eta_2}{\eta_1} & & \\ & & \ddots & \\ & & & \frac{1}{\eta_{n-1}} \end{pmatrix} \begin{pmatrix} 1 & & & \\ \lambda_{21} & 1 & & \\ \vdots & \vdots & \ddots & \\ \lambda_{n1} & \lambda_{n2} & \dots & 1 \end{pmatrix} \Pi_a^{-1} \tag{4.81}$$

where Δ has unit determinant and its elements are given by

$$\eta_p = \left((M\Pi_a) \left(\begin{smallmatrix} p+1 & \dots & n \\ p+1 & \dots & n \end{smallmatrix} \right) \right)^{-1}, \tag{4.82}$$

V is unit upper triangular with matrix elements

$$v_{pi} = (M\Pi_a) \left(\begin{smallmatrix} p & i+1 & \dots & n \\ i & i+1 & \dots & n \end{smallmatrix} \right) \big/ (M\Pi_a) \left(\begin{smallmatrix} i & \dots & n \\ i & \dots & n \end{smallmatrix} \right) = (M\Pi_a) \left(\begin{smallmatrix} p & i+1 & \dots & n \\ i & i+1 & \dots & n \end{smallmatrix} \right) \eta_{i-1} \quad p \leq i, \tag{4.83}$$

Λ is unit lower triangular with matrix elements

$$\lambda_{ip} = (M\Pi_a) \left(\begin{smallmatrix} i & i+1 & \dots & n \\ p & i+1 & \dots & n \end{smallmatrix} \right) \big/ (M\Pi_a) \left(\begin{smallmatrix} i & \dots & n \\ i & \dots & n \end{smallmatrix} \right) = (M\Pi_a) \left(\begin{smallmatrix} i & i+1 & \dots & n \\ p & i+1 & \dots & n \end{smallmatrix} \right) \eta_{i-1} \quad i \geq p \tag{4.84}$$

and Π_a is a permutation matrix which moves column a to the rightmost position, subject to the caveat explained in footnote 12.

Proof. Denote

$$W = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} = W^{-1}. \tag{4.85}$$

Note that conjugating a matrix by W "rotates" it by 180°. Write the matrix $WM\Pi_aW^{-1}$ using the LU-decomposition (here we get $P = \mathbb{1}$ thanks to the action of Π_a)

$$WM\Pi_aW^{-1} = LDU. \tag{4.86}$$

Solve for M and write it as

$$M = \underbrace{W^{-1}LW}_{V}\underbrace{W^{-1}DW}_{\Delta}\underbrace{W^{-1}UW}_{\Lambda}\Pi_a^{-1}. \tag{4.87}$$

The formula for the matrix elements of V, Δ and Λ follow from theorem 4.25. Alternatively, the formulae may be proven from first principles using the same technique as for the UL-decomposition in [22]. □

Remark 4.28.
All minors in the formulae above contain column a of the original matrix M.

Together with the lemmas in Appendix B, we now have what we need to derive a closed formula for the p-adic norms of the dilatons of an arbitrary matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$. This is the main result of **Paper I**.

Theorem 4.29 (Norms of the dilatons of an $\mathrm{SL}_n(\mathbb{Q}_p)$-matrix)
The norms of the dilatons in the Iwasawa decomposition of a matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$ are given by

$$|y_{n-k}|_p = \left(\max_{\sigma \in \Theta_k^n} \left\{ \left| M \begin{pmatrix} n-k+1 & \dots & n \\ \sigma(1) & \dots & \sigma(k) \end{pmatrix} \right|_p \right\} \right)^{-1} \quad \textit{where} \quad k \in \{1, \dots, n-1\} \tag{4.88}$$

and Θ_k^n denotes the set of all ordered subsets of $\{1, \dots, n\}$ of order k.[14]

Alternatively in terms of the generalized Plücker coordinates ρ_k,

$$|y_{n-k}|_p = \left(\max_{x \in \rho_k} \left\{ |x|_p \right\} \right)^{-1} \quad \textit{where} \quad k \in \{1, \dots, n-1\}. \tag{4.89}$$

Proof. The proof works by induction. Suppose that the formula holds up to and including $\mathrm{SL}_n(\mathbb{Q}_p)$ and consider $\mathrm{SL}_n(\mathbb{Q}_p)$.

Restrict to the case $|M_{na}|_p \geq |M_{ni}|_p$ for $i \in \{1, \dots, n\}$, i.e. the element with the largest p-adic norm sits in column a. If there is no unique such element, any one of the largest elements on the bottom row of M may play the role of M_{na}. Note that $M_{na} \neq 0$ since otherwise the bottom row would be a zero-row, rendering M singular. Performing a strong UL-decomposition on M where we move column a to the rightmost place gives

$$M = V\Delta\Lambda\Pi_a^{-1} = V\Delta \begin{pmatrix} 1 & & & & \\ \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ 1 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 1 & n \end{smallmatrix}\right)\eta_{n-2} & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 2 & n \end{smallmatrix}\right)\eta_{n-2} & \dots & 1 & \\ \tilde{M}\left(\begin{smallmatrix} n \\ 1 \end{smallmatrix}\right)\eta_{n-1} & \tilde{M}\left(\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right)\eta_{n-1} & \dots & \tilde{M}\left(\begin{smallmatrix} n \\ n-1 \end{smallmatrix}\right)\eta_{n-1} & 1 \end{pmatrix} \Pi_a^{-1} \tag{4.90}$$

where we have denoted $\tilde{M} = M\Pi_a$. Note that $\Pi_a^{-1} \in \mathrm{SL}_n(\mathbb{Z}_p)$. Note furthermore that the bottom row in the matrix Λ is simply a permutation of the bottom row of M divided by $\tilde{\eta}_{n-1}^{-1} = \tilde{M}\left(\begin{smallmatrix} n \\ n \end{smallmatrix}\right) = M_{na}$. Because M_{na} is assumed to have the largest p-adic norm, we get that every element in the bottom row of Λ is a p-adic integer. Thanks to the unit

[14] In words: The norm of the dilaton y_{n-k} is the inverse of the norm of the largest anti-leading principal minor of order k. The formula produces the desired result $y_0 = y_n = 1$ for $k = n$ and $k = 0$ respectively.

lower triangular form, the bottom row easily factorizes out on the right and we are left with

$$M = V\Delta \begin{pmatrix} 1 & & & & \\ \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ 1 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 & 1 & & & \\ \vdots & \vdots & \ddots & & \\ \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 1 & n \end{smallmatrix}\right)\eta_{n-2} & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 2 & n \end{smallmatrix}\right)\eta_{n-2} & \dots & 1 & \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} R\Pi_a^{-1} \tag{4.91}$$

where $R \in \mathrm{SL}_n(\mathbb{Z}_p)$ contains the bottom row of Λ. The block diagonal form implies that there will be no further contributions to the dilaton y_{n-1} and its norm is now fixed at

$$|y_{n-1}|_p = |\eta_{n-1}|_p = \left|\frac{1}{\tilde{M}\left(\begin{smallmatrix} n \\ n \end{smallmatrix}\right)}\right|_p = |M\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right)|_p^{-1} = \left(\max_{\sigma\in\Theta_1^n}\left\{\left|M\left(\begin{smallmatrix} n \\ \sigma(1) \end{smallmatrix}\right)\right|_p\right\}\right)^{-1}, \tag{4.92}$$

again using the fact that the element M_{na} has the largest p-adic norm. This expression is of the form Eq. (4.88). Putting $n = 2$ here proves the base case $\mathrm{SL}_2(\mathbb{Q}_p)$. Next we treat the dilatons $y_1, \dots, y_{n-2}$.

Note that we can write

$$\begin{aligned} &\begin{pmatrix} 1 & & & \\ \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ 1 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 & 1 & & \\ \vdots & \vdots & \ddots & \\ \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 1 & n \end{smallmatrix}\right)\eta_{n-2} & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 2 & n \end{smallmatrix}\right)\eta_{n-2} & \dots & 1 \end{pmatrix} = \\ &= \begin{pmatrix} \tilde{M}\left(\begin{smallmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{smallmatrix}\right)\eta_0 & \tilde{M}\left(\begin{smallmatrix} 1 & 2 & \dots & n \\ 2 & 2 & \dots & n \end{smallmatrix}\right)\eta_0 & \dots & \tilde{M}\left(\begin{smallmatrix} 1 & 2 & \dots & n \\ n-1 & 2 & \dots & n \end{smallmatrix}\right)\eta_0 \\ \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ 1 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 & \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ 2 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 & \dots & \tilde{M}\left(\begin{smallmatrix} 2 & 3 & \dots & n \\ n-1 & 3 & \dots & n \end{smallmatrix}\right)\eta_1 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 1 & n \end{smallmatrix}\right)\eta_{n-2} & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ 2 & n \end{smallmatrix}\right)\eta_{n-2} & \dots & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ n-1 & n \end{smallmatrix}\right)\eta_{n-2} \end{pmatrix}. \end{aligned} \tag{4.93}$$

We apply the induction hypothesis to this $(n-1)\times(n-1)$-diagonal block of Eq. (4.91) and compute its contribution to the norm of the dilaton $y_{n-(k+1)}$[15] for $k \in \{1, \dots n-1\}$. The formula Eq. (4.88) implies that we need to compute minors like

$$\begin{vmatrix} \tilde{M}\left(\begin{smallmatrix} n-k & n-k+1 & \dots & n \\ \tilde{\sigma}(1) & n-k+1 & \dots & n \end{smallmatrix}\right)\eta_{n-k-1} & \tilde{M}\left(\begin{smallmatrix} n-k & n-k+1 & \dots & n \\ \tilde{\sigma}(2) & n-k+1 & \dots & n \end{smallmatrix}\right)\eta_{n-k-1} & \dots & \tilde{M}\left(\begin{smallmatrix} n-k & n-k+1 & \dots & n \\ \tilde{\sigma}(k) & n-k+1 & \dots & n \end{smallmatrix}\right)\eta_{n-k-1} \\ \tilde{M}\left(\begin{smallmatrix} n-k-1 & n-k-2 & \dots & n \\ \tilde{\sigma}(1) & n-k-2 & \dots & n \end{smallmatrix}\right)\eta_{n-k-2} & \tilde{M}\left(\begin{smallmatrix} n-k-1 & n-k-2 & \dots & n \\ \tilde{\sigma}(2) & n-k-2 & \dots & n \end{smallmatrix}\right)\eta_{n-k-2} & \dots & \tilde{M}\left(\begin{smallmatrix} n-k-1 & n-k-2 & \dots & n \\ \tilde{\sigma}(k) & n-k-2 & \dots & n \end{smallmatrix}\right)\eta_{n-k-2} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ \tilde{\sigma}(1) & n \end{smallmatrix}\right)\eta_{n-2} & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ \tilde{\sigma}(2) & n \end{smallmatrix}\right)\eta_{n-2} & \dots & \tilde{M}\left(\begin{smallmatrix} n-1 & n \\ \tilde{\sigma}(k) & n \end{smallmatrix}\right)\eta_{n-2} \end{vmatrix} \tag{4.94}$$

[15] We choose to consider $y_{n-(k+1)} = y_{n-1-k}$ so that we get nice expressions in what follows. Later we send k to $k-1$ to compare with Eq. (4.88).

where $\tilde{\sigma} \in \Theta_k^{n-1}$. Using lemma B.4, this evaluates to

$$\begin{aligned} &\tilde{M}\begin{pmatrix} n-k & \dots & n-1 & n \\ \tilde{\sigma}(1) & \dots & \tilde{\sigma}(k) & n \end{pmatrix} \tilde{M}\begin{pmatrix} n-k+1 & \dots & n \\ n-k+1 & \dots & n \end{pmatrix} \dots \tilde{M}\begin{pmatrix} n-1 & n \\ n-1 & n \end{pmatrix} \eta_{n-k-1} \dots \eta_{n-2} = \\ &= \tilde{M}\begin{pmatrix} n-k & \dots & n-1 & n \\ \tilde{\sigma}(1) & \dots & \tilde{\sigma}(k) & n \end{pmatrix} \tilde{M}\begin{pmatrix} n-k & \dots & n \\ n-k & \dots & n \end{pmatrix} \end{aligned} . \tag{4.95}$$

The dilaton $y_{n-(k+1)}$ already has a contribution $\eta_{n-(k+1)} = \left(\tilde{M}\begin{pmatrix} n-k & \dots & n \\ n-k & \dots & n \end{pmatrix}\right)^{-1}$ from the matrix Δ. Multiplying these contributions together and considering all subsets $\tilde{\sigma} \in \Theta_k^{n-1}$ gives the final answer for the norm of the dilaton $y_{n-(k+1)}$ in the case $|M_{na}|_p \geq |M_{ni}|_p$ as

$$\left|y_{n-(k+1)}\right|_p = \left(\max_{\tilde{\sigma} \in \Theta_k^{n-1}} \left\{ \left|\tilde{M}\begin{pmatrix} n-k & \dots & n-1 & n \\ \tilde{\sigma}(1) & \dots & \tilde{\sigma}(k) & n \end{pmatrix}\right|_p \right\}\right)^{-1} . \tag{4.96}$$

Note that the exact form of the permutation matrix Π_a becomes irrelevant, since the expression above anyway considers all order k subsets of the columns 1 through $n-1$ in $\tilde{M}$. Note furthermore that any minus signs present in Π_a as discussed in footnote 12 make no difference since the minors sit inside a norm.

We can think of the order k-subsets $\tilde{\sigma} \in \Theta_k^{n-1}$ as order $k+1$ subsets $\sigma \in \Theta_{k+1}^n$ restricted such that $\sigma(k+1) = n$. Writing the equation above in terms of M and replacing k by $k-1$ gives

$$|y_{n-k}|_p = \left(\max_{\sigma \in \Theta'^n_k} \left\{ \left|M\begin{pmatrix} n-k+1 & \dots & n \\ \sigma(1) & \dots & \sigma(k) \end{pmatrix}\right|_p \right\}\right)^{-1} \tag{4.97}$$

where the prime on Θ'^n_k indicates that all subsets must contain a. To finalize the proof, we must show that Eq. (4.88) reduces to Eq. (4.97) in the case $|M_{na}|_p \geq |M_{ni}|_p$. This is equivalent to showing that for every anti-leading minor which doesn't include column a, one can find an anti-leading minor which does include column a and whose p-adic norm is at least as large as the original minor. To see that this is true, pick an anti-leading minor of order k which doesn't contain column a and consider its norm[16]

$$\left|M\begin{pmatrix} n-k+1 & \dots & n \\ \sigma_a(1) & \dots & \sigma_a(k) \end{pmatrix}\right|_p = \left|\sum_{i=1}^{k} (-1)^i M\begin{pmatrix} n \\ \sigma_a(i) \end{pmatrix} M\begin{pmatrix} n-k+1 & \dots & n-k+i-1 & n-k+i & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(i-1) & \sigma_a(i+1) & \dots & \sigma_a(k) \end{pmatrix}\right|_p \tag{4.98}$$

where $\sigma_a \in \Theta_k^n$ and $a \notin \sigma_a$. Define $(-1)^k u_i$ as the minor under consideration but with $\sigma_a(i)$ replaced by a. This minor is then already present in Eq. (4.97). Laplace expansion

[16] By operating inside of a norm, we don't need to keep track of the overall sign.

gives

$$
\begin{aligned}
&(-1)^k u_i \equiv M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+i-1 & n-k+i & n-k+i+1 & \dots & n \\ \sigma_a(1) & \dots & \sigma_a(i-1) & a & \sigma_a(i+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) = \\
&= \sum_{j=1}^{i-1}(-1)^{k+j} M\left(\begin{smallmatrix} n \\ \sigma_a(j) \end{smallmatrix}\right) M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+j-1 & n-k+j & \dots & n-k+i-2 & n-k+i-1 & n-k+i & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(j-1) & \sigma_a(j+1) & \dots & \sigma_a(i-1) & a & \sigma_a(i+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) \\
&+(-1)^{k+i} M\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right) M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+i-1 & n-k+i & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(i-1) & \sigma_a(i+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) \\
&+ \sum_{j=i+1}^{k}(-1)^{k+j} M\left(\begin{smallmatrix} n \\ \sigma_a(j) \end{smallmatrix}\right) M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+i-1 & n-k+i & n-k+i+1 & \dots & n-k+j-1 & n-k+j & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(i-1) & a & \sigma_a(i+1) & \dots & \sigma_a(j-1) & \sigma_a(j+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) .
\end{aligned}
\tag{4.99}
$$

We now show that the norm of the minor under consideration Eq. (4.98) can be expressed in terms of the norm of a sum of u's as

$$
\left| M\left(\begin{smallmatrix} n-k+1 & \dots & n \\ \sigma_a(1) & \dots & \sigma_a(k) \end{smallmatrix}\right)\right|_p = \left| \frac{1}{M\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right)} \sum_{i=1}^{k} M\left(\begin{smallmatrix} n \\ \sigma_a(i) \end{smallmatrix}\right) u_i \right|_p . \tag{4.100}
$$

By making recourse with Eq. (4.98), it is easy to see that the terms in the right hand side of Eq. (4.98) come from the second line in the right hand side of Eq. (4.99). To see that the remaining terms cancel, consider the term inside the sum in the right hand side of Eq. (4.100) with $i = I$ and $j = J$ where $J < I$ (coming from the first line of the right hand side of Eq. (4.99))

$$
\begin{aligned}
&M\left(\begin{smallmatrix} n \\ \sigma_a(I) \end{smallmatrix}\right) (-1)^J M\left(\begin{smallmatrix} n \\ \sigma_a(J) \end{smallmatrix}\right) \\
&M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+J-1 & n-k+J & \dots & n-k+I-2 & n-k+I-1 & n-k+I & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(J-1) & \sigma_a(J+1) & \dots & \sigma_a(I-1) & a & \sigma_a(I+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) .
\end{aligned}
\tag{4.101}
$$

This cancels with the term $i = J$ and $j = I$ (coming from the third line of the right hand side of Eq. (4.99)) as is seen by writing

$$
\begin{aligned}
&M\left(\begin{smallmatrix} n \\ \sigma_a(J) \end{smallmatrix}\right) (-1)^I M\left(\begin{smallmatrix} n \\ \sigma_a(I) \end{smallmatrix}\right) \\
&M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+J-1 & n-k+J & n-k+J+1 & \dots & n-k+I-1 & n-k+I & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(J-1) & a & \sigma_a(J+1) & \dots & \sigma_a(I-1) & \sigma_a(I+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) = \\
&= M\left(\begin{smallmatrix} n \\ \sigma_a(J) \end{smallmatrix}\right) (-1)^{-J+1} M\left(\begin{smallmatrix} n \\ \sigma_a(I) \end{smallmatrix}\right) \\
&M\left(\begin{smallmatrix} n-k+1 & \dots & n-k+J-1 & n-k+J & \dots & n-k+I-2 & n-k+I-1 & n-k+I & \dots & n-1 \\ \sigma_a(1) & \dots & \sigma_a(J-1) & \sigma_a(J+1) & \dots & \sigma_a(I-1) & a & \sigma_a(I+1) & \dots & \sigma_a(k) \end{smallmatrix}\right) .
\end{aligned}
\tag{4.102}
$$

This establishes Eq. (4.100). The ultrametric property of the p-adic norm now gives that the norm of the minor under consideration Eq. (4.100) is dominated by the norm of the largest u

$$\left|\sum_{i=1}^{k} \frac{M\left(\begin{smallmatrix} n \\ \sigma_a(i) \end{smallmatrix}\right)}{M\left(\begin{smallmatrix} n \\ a \end{smallmatrix}\right)} u_i\right|_p \leq \max\left\{\left|\frac{M_{n,\sigma_a(1)}}{M_{na}} u_1\right|_p, \dots, \left|\frac{M_{n,\sigma_a(k)}}{M_{na}} u_k\right|_p\right\} \leq \\ \leq \max\left\{|u_1|_p, \dots, |u_k|_p\right\} \tag{4.103}$$

where we have used $|M_{na}|_p \geq |M_{ni}|_p$. Since the u's are all present in Eq. (4.97), we can safely include the minor under consideration as its would never be picked over the u's. This argument holds for all anti-leading minors of order k and we may therefore include them all in the right hand side of Eq. (4.97), successfully reproducing Eq. (4.88). Peano's axiom of induction now establishes the theorem. □

Remark 4.30.
The proof shows that in the case $|M_{na}|_p \geq |M_{ni}|_p$, the expression Eq. (4.88) reduces to Eq. (4.97).

Remark 4.31.
The method of using strong UL-decomposition to reduce the problem of Iwasawa decomposition of an $n \times n$-matrix to that of an $(n-1) \times (n-1)$-matrix can be iterated and defines an algorithm for a complete Iwasawa decomposition $M = NAK$ of any matrix $M \in \mathrm{SL}_n(\mathbb{Q}_p)$. All other Iwasawa decompositions can then be found by using theorem 4.18. Using this method to derive a general formula for the matrix elements of N and A (and hence K) given M seems feasible. Such a formula is expected to express also the axions in terms of a conditional clause like max. In this endeavour, it would probably be convenient to use theorem 4.18 to impose some standard normalization on the axions and dilatons, for instance one may normalize the dilatons to be just pure powers of p.

4.4 Adelization

The idea behind adelization is to enlarge the domain of a function from the reals to the adeles. Starting with a function $f_\mathbb{R} : \mathbb{R} \to \mathbb{C}$ defined over the reals, consider a family $f_p : \mathbb{Q}_p \to \mathbb{C}$ of functions each with domain $\mathbb{Q}_p$. One can then construct the global function $f_\mathbb{A}$ as the Euler product

$$f_\mathbb{A}(x) = f_\mathbb{R}(x_\infty) \prod_{p<\infty} f_p(x_p) \quad \text{for an adele} \quad x \in \mathbb{A}. \tag{4.104}$$

The global function is defined over $\mathbb{A}$ and if the functions f_p are such that $f_p(1) = 1$, then one can recover the real function $f_\mathbb{R}(x_\infty)$ by evaluating

$$f_\mathbb{R}(x_\infty) = f_\mathbb{A}(x_\infty; 1, 1, 1, \dots). \tag{4.105}$$

We say that $f_\mathbb{A}$ is an *adelization* of $f_\mathbb{R}$.

We furthermore have the notion of local additive characters over the adeles. Using the characters of definition 4.13 we can construct a character $\psi_u : \mathbb{A} \to U(1)$ parametrized by an adele u by the Euler product

$$\psi_u(x) = \prod_{p\leq\infty} \psi_{u,p}(x_p) = \mathrm{e}^{2\pi\mathrm{i}u_\infty x_\infty} \prod_p \mathrm{e}^{-2\pi\mathrm{i}[u_p x_p]_p} = \mathrm{e}^{2\pi\mathrm{i}\left(u_\infty x_\infty - \sum_p [u_p x_p]_p\right)}. \tag{4.106}$$

We will also use the notation $\psi_u(x) = \mathrm{e}^{2\pi\mathrm{i}ux}$. Note that for $u = m \in \mathbb{Q}$ diagonally embedded in $\mathbb{A}$, we have for a rational number $r \in \mathbb{Q}$

$$\psi_m(x + r) = \psi_m(x)\psi_m(r) = \psi_m(x) \tag{4.107}$$

where we have used proposition 4.12 for

$$\psi_m(r) = \mathrm{e}^{2\pi\mathrm{i}\left(mr - \sum_p [mr]_p\right)} = 1. \tag{4.108}$$

The character $\psi_m : \mathbb{Q}\backslash\mathbb{A} \to U(1)$ is thus trivial on $\mathbb{Q}$.

4.4.1 Strong Approximation

As mentioned in remark 2.29, one method for computing (real) Fourier coefficients of a (real) automorphic form $\varphi : G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ is using the adelic framework. This method works by enlarging the domain of φ from the real group $G(\mathbb{R})$ to the adelic group $G(\mathbb{A})$ in such a way that one can "project out" the desired results for real case. This process is called adelization and it rests on a central result called strong approximation which lets one replace the double quotient space $G(\mathbb{Z})\backslash G(\mathbb{R})/K(\mathbb{R})$ with the adelic quotient space $G(\mathbb{Q})\backslash G(\mathbb{R})/K(\mathbb{R})$. The advantage is that left $G(\mathbb{Z})$-invariance is replaced with left $G(\mathbb{Q})$-invariance and since $\mathbb{Q}$ is a field while $\mathbb{Z}$ is merely a ring, it unlocks group theoretical theorems which rely on a Lie group G being defined over a field.

To describe strong approximation, we start with some notation. We already have

$$G_f = \prod_p{}' G(\mathbb{Q}_p). \tag{4.109}$$

Furthermore, let

$$K(G(\mathbb{A})) \equiv K_\mathbb{A} = \underbrace{K(G(\mathbb{R}))}_{K_\infty} \times \underbrace{\prod_p G(\mathbb{Z}_p)}_{K_f} \tag{4.110}$$

denote the maximal compact subgroup of the adelic group $G(\mathbb{A})$. The Iwasawa decomposition on the adelic level then states $G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K_\mathbb{A}$ and makes use of the local decompositions described in Sections 2.1.4 and 4.3.1.

Strong approximation now states that we have the isomorphism

$$G(\mathbb{Z})\backslash G(\mathbb{R}) \cong G(\mathbb{Q})\backslash G(\mathbb{A})/K_f. \tag{4.111}$$

A very useful consequence of this isomorphism is that any function $\phi_{\mathbb{R}}$ on $G(\mathbb{Z})\backslash G(\mathbb{R})$ can be lifted to a function $\phi_{\mathbb{A}}$ on the adelization $G(\mathbb{Q})\backslash G(\mathbb{A})/K_f$. Since the automorphic forms we will be dealing with are right-invariance under the maximal compact subgroup $K(G(\mathbb{R}))$ to begin with, our lift will take place from $G(\mathbb{Z})\backslash G(\mathbb{R})/K(G(\mathbb{R}))$ to $G(\mathbb{Q})\backslash G(\mathbb{A})/K_{\mathbb{A}}$. In this fashion, left $G(\mathbb{Z})$-invariance gets replaced with left $G(\mathbb{Q})$-invariance.

Remark 4.32.
There exists a more general statement of strong approximation involving left Γ-invariant functions where Γ not necessarily $G(\mathbb{Z})$ but possibly some other discrete subgroup of $G(\mathbb{R})$. For the more general statement and a proof thereof, see [18].

Remark 4.33.
Applying the result Eq. (4.111) to the group $G(\mathbb{R}) = \mathbb{R}^\times$ under multiplication gives

$$\mathbb{A}^\times = \mathbb{Q}^\times \times \underbrace{\mathbb{R}^+ \times \prod_p \mathbb{Z}_p^\times}_{J} \tag{4.112}$$

where $\mathbb{A}^\times$ denotes the set of adeles with multiplicative inverses (also called the *ideles*). Equivalently we have that J is a fundamental domain for $\mathbb{A}^\times$,

$$\mathbb{A}^\times = \bigsqcup_{k\in\mathbb{Q}^\times} kJ. \tag{4.113}$$

Remark 4.34.
In oral presentations, I have suggested to my audience as a rule thumb that "$\mathbb{Q}$ *is to* $\mathbb{A}$ *as* $\mathbb{Z}$ *is to* $\mathbb{R}$". For readers who are new to the topic of adeles and strong approximation, this can be a useful proverb to keep in mind.

4.4.2 Adelization of Eisenstein Series and Whittaker functions

The adelization of Eisenstein series is straightforward. Recall that an Eisenstein series is formed by defining a character $\chi_{\lambda,\infty} : G(\mathbb{R}) \to \mathbb{C}$ parametrized by a weight vector (or complex linear functional) $\lambda \in \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}(\mathbb{R})^\star = \mathfrak{h}_{\mathbb{C}}^\star$ and averaging this character over a discrete subgroup of $G(\mathbb{R})$. The adelization involves extending the domain of this character to all of $G(\mathbb{A})$ and averaging over a discrete subgroup of $G(\mathbb{A})$, meaning a subgroup of $G(\mathbb{Q})$.

There are obviously several ways of extending the character χ_λ to $G(\mathbb{A})$ leading to different adelizations of an Eisenstein series. We require an adelic character

$$\chi_\lambda(g) = \chi_{\lambda,\infty}(g_\infty) \prod_{p<\infty} \chi_p(g_p) \quad \text{for} \quad g \in G(\mathbb{A}) \tag{4.114}$$

such that

$$\chi_\lambda((g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)) = \chi_{\lambda,\infty}(g_\infty). \tag{4.115}$$

The requirement that χ_λ be a character on $G(\mathbb{A})$ translates to that each χ_p should be characters on $G(\mathbb{Q}_p)$. As such, they already satisfy $\chi_p(\mathbb{1}) = 1$ and Eq. (4.115) is satisfied. There is a one-to-one correspondence between characters on $G(\mathbb{Q}_p)$ ($p \leq \infty$) and the set of complex linear functionals. The adelization that is right for our purposes is the one where the same root λ is chosen at each local place, $\chi_p = \chi_{\lambda,p}$ and

$$\chi_\lambda(g) = \prod_{p\leq\infty} \chi_{\lambda,p}(g_p). \tag{4.116}$$

Parametrizing an element $a \in A(\mathbb{A})$ of the adelic Cartan torus as

$$a = \exp\left(\sum_{\alpha\in\Pi} y_\alpha H_\alpha\right) \quad \text{where} \quad H_\alpha \in \mathfrak{h}(\mathbb{R}) \quad \text{and} \quad y_\alpha \in \mathbb{A} \tag{4.117}$$

we define the logarithm map $H : G(\mathbb{A}) \to \mathfrak{h}(\mathbb{R})$ using the global Iwasawa decomposition $G(\mathbb{A}) = N(\mathbb{A})A(\mathbb{A})K_{\mathbb{A}}$ as

$$H(g) = H(nak) = H(a) = \log|a| \equiv \sum_{\alpha\in\Pi} |y_\alpha| H_\alpha. \tag{4.118}$$

The adelization χ_λ of the real character $\chi_{\lambda,\infty}$ is then

$$\chi_\lambda(g) = \mathrm{e}^{\langle\lambda+\rho|H(g)\rangle} = |a^{\lambda+\rho}|. \tag{4.119}$$

This character is left-invariant under $B(\mathbb{Q}) = N(\mathbb{Q})A(\mathbb{Q})$ (diagonally embedded). This is seen as follows. For an element $\tilde{n}\tilde{a}$ of $B(\mathbb{Q})$ where $\tilde{n} \in N(\mathbb{Q})$ and $\tilde{a} \in A(\mathbb{Q})$, we have

$$H(\tilde{n}\tilde{a}nak) = H(\tilde{a}na) = H(\underbrace{\tilde{a}n\tilde{a}^{-1}}_{\in N(\mathbb{A})}\tilde{a}a) = H(\tilde{a}a). \tag{4.120}$$

Since $\tilde{a} \in A(\mathbb{Q})$ is diagonally embedded, the dilatons $\tilde{y}_\alpha$ of $\tilde{a}$ multiply the dilatons y_α in a at each local place, meaning $y_\alpha \to \tilde{y}_\alpha y_\alpha$. Since $\tilde{y}_\alpha$ is a diagonally embedded rational, Eq. (4.56) gives $|\tilde{a}_\alpha a_\alpha| = |a_\alpha|$ and hence $H(\tilde{a}a) = H(a)$ showing that χ_λ is left $B(\mathbb{Q})$-invariant. As such, the proper expression for an adelic (minimal) Eisenstein series is

$$E_\lambda^{G(\mathbb{A})}(g) = \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \chi_\lambda(\gamma g). \tag{4.121}$$

We recover the real (minimal) Eisenstein series $E_\lambda^G(g_\infty)$ of Eq. (2.49) by writing

$$E_\lambda^{G(\mathbb{R})}(g_\infty) = E_\lambda^{G(\mathbb{A})}((g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)). \tag{4.122}$$

This fact is due to the isomorphism $B(\mathbb{Q})\backslash G(\mathbb{Q}) \cong B(\mathbb{Z})\backslash G(\mathbb{Z})$, meaning that the sum in the adelic Eisenstein series Eq. (4.121) contains "as many terms" as the real Eisenstein series Eq. (2.49) and choosing to evaluate it at $g = (g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)$ gives back the real case. The isomorphism is proven for $G = \mathrm{SL}_2$ in [18] and follows generally from the work by Steinberg [49] using the Bruhat decomposition.

Remark 4.35.
The discussion above was for minimal parabolic Eisenstein series. There is an analogous treatment for the maximal parabolic Eisenstein series of Eq. (2.53).

Following the adelization of Eisenstein series, the Fourier expansion also undergoes an adelic lift. Recall that a real automorphic form is left invariant (read periodic) under the integer group $G(\mathbb{Z})$ and Fourier expansion over a parabolic subgroup $P = UL$ is done with characters ψ on the unipotent $U(\mathbb{R})$ trivial on $U(\mathbb{Z})$. The character $\psi \in \mathrm{Hom}(U(\mathbb{Z})\backslash U(\mathbb{R}), U(1))$ being a homomorphism onto the circle $U(1)$ is in turn labelled by some integer *winding numbers* m_i and given as the product of exponentials $\mathrm{e}^{2\pi \mathrm{i} m_i x_i}$. The Fourier expansion involves summing over all characters which translates to a sum over integers, $\sum_\psi \sim \sum_{m_i \in \mathbb{Z}}$.

For adelic automorphic forms there is an analogous treatment. The adelic lift of an automorphic form is defined on $G(\mathbb{A})$ and left invariant under the rational group $G(\mathbb{Q})$. As such, the Fourier expansion will be with respect to characters on $U(\mathbb{A})$ trivial on the rationally embedded $U(\mathbb{Q})$. Such a character $\psi \in \mathrm{Hom}(U(\mathbb{Q})\backslash U(\mathbb{A}), U(1))$ is labelled rational winding numbers. The Fourier expansion will then be given as sums over rational numbers $\sum_\psi \sim \sum_{q_i \in \mathbb{Q}}$ as alluded to in remark 4.33.

A Fourier coefficient of an adelic automorphic form φ with respect to a character χ takes the form (compare with definition 2.20)

$$F_\psi^{U(\mathbb{A})}(u) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\,\mathrm{d}u. \tag{4.123}$$

The Fourier expansion takes the form

$$\varphi(g) = \sum_{\psi \in \mathfrak{c}(U(\mathbb{A}))} F_\psi^{U(\mathbb{A})}(g) + \dots \tag{4.124}$$

where $\mathfrak{c}(U(\mathbb{A}))$ denotes the group of characters on $U(\mathbb{A})$ trivial on $U(\mathbb{Z})$ and the ellipsis stands for possible non-abelian Fourier coefficients. We emphasise that the sum $\sum_{\psi \in \mathfrak{c}(U(\mathbb{A}))}$ is tantamount to a sum over rational numbers.

Lastly, the adelic Fourier coefficients $F_\psi^{U(\mathbb{A})}$ themselves can be projected down onto the real Fourier coefficients just as the automorphic forms can

$$F_\psi^{U(\mathbb{A})}((g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)) = F_\psi^{U(\mathbb{R})}(g_\infty), \tag{4.125}$$

which we will make use of in Chapter 6. See [18] for a proof of this.

Remark 4.36 (Single valuedness of the Whittaker function).
Since we have seen that the p-adic Iwasawa decomposition is not unique, one may worry about whether or not an adelic Whittaker function $W_\psi(g)$ evaluated at $g = nak$ is

uniquely defined or whether it depends upon the chosen Iwasawa decomposition $g_p = n_p a_p k_p$ at the p-adic places for each g_p. Fortunately, this is not the case for SL_n as we describe here.

Working at a specific prime p, consider the (spherical) p-adic Whittaker function $W_{\psi,p}$ (henceforth we will be dropping subscripts p for the variables in this remark in the interest of legibility). We can find a criterion for all $a \in A(\mathbb{Q}_p)$ for which $W_{\psi,p}(a)$ may be non-vanishing. Taking $n \in N(\mathbb{Z}_p)$, we also have $n \in K(G(\mathbb{Q}_p)) = G(\mathbb{Z}_p)$ and by sphericity we can write

$$W_{\psi,p}(a) = W_{\psi,p}(an) = W_{\psi,p}(ana^{-1}a) = \psi(ana^{-1})W_{\psi,p}(a) \tag{4.126}$$

since $ana^{-1} \in N$. We get that $W_{\psi,p}(a)$ may only be non-vanishing if $\psi(ana^{-1}) = 1$ for all $n \in N(\mathbb{Z}_p)$.

From theorem 4.18, we have that the general form of the p-adic Iwasawa decomposition of a p-adic matrix $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ is

$$g = \underbrace{naxa^{-1}}_{\in N(\mathbb{Q}_p)} \underbrace{ay}_{\in A(\mathbb{Q}_p)} \underbrace{(xy)^{-1}k}_{\in G(\mathbb{Z}_p)} \tag{4.127}$$

where $g = nak$ is some Iwasawa decomposition and $x \in N(\mathbb{Z}_p)$ and $y \in A(\mathbb{Z}_p^\times)$. Now consider the general expression

$$\begin{aligned} W_{\psi,p}(g) = W_{\psi,p}(naxa^{-1}ay(xy)^{-1}k) = \psi(naxa^{-1})W_{\psi,p}(ay) = \\ = \psi(n)\psi(axa^{-1})W_{\psi,p}(ay). \end{aligned} \tag{4.128}$$

The ambiguity due to y is resolved since the norms of the p-adic dilatons in ay are the same as those of a and the argument ay enters as the argument of the adelic Eisenstein-series which only cares about these norms. We therefore have

$$W_m(g) = \psi_m(n)\psi_m(axa^{-1})W_m(a). \tag{4.129}$$

Since $x \in N(\mathbb{Z}_p)$, Eq. (4.126) gives that $\psi(axa^{-1}) = 1$ whenever $W_{\psi,p}(a)$ is non-vanishing and we are left with

$$W_{\psi,p}(g) = \psi(n)W_{\psi,p}(a). \tag{4.130}$$

We learn that we can evaluate a p-adic Whittaker function for whichever decomposition we please.

Remark 4.37 (Contributions from p-adic axions).
In later chapters, we will have a reason to "project down" an adelic Fourier coefficient of the form $F_\psi^{U(\mathbb{A})}(lg)$ where l is a rational and diagonally embedded matrix. Looking at an expression like

$$F_\psi^{U(\mathbb{A})}(l(g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)), \tag{4.131}$$

one might wonder whether ψ evaluated for the p-adic unipotents in the p-adic Iwasawa decomposition of l at the local places gives a non-trivial overall phase or if the fact that l is rational triggers an effect similar to Eq. (4.56) which conspires to make the p-adic contributions to the overall phase trivial. We will see that this is not the case by considering the example from SL_2 where we do get a phase.

We are interested in understanding the overall phase coming from the p-adic axions for the $SL_2(\mathbb{A})$ Whittaker function in the expression

$$W_m^{\mathbb{A}}(l(g_\infty, \mathbb{1}, \mathbb{1}, \dots)) \tag{4.132}$$

where the matrix l is rational and diagonally embedded. The phase in question is of the form

$$\phi = \prod_{p<\infty} \psi_{m,p}(N_p(l)) \tag{4.133}$$

where $N_p : G(\mathbb{Q}_p) \to N(\mathbb{Q}_p)$ picks out the unipotent part of the p-adic Iwasawa decomposition of its argument. We will be working at a fixed prime and thus drop subscripts of p for simplicity. Inequalities are evaluated using the p-adic norm. A convenient p-adic Iwasawa decomposition of an arbitrary rational matrix $l \in SL_2(\mathbb{Q})$ is

$$l = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & b/d \\ & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & \\ & d \end{pmatrix} \begin{pmatrix} 1 & \\ c/d & 1 \end{pmatrix}, & d \geq c \\ \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & a/c \\ & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & \\ & c \end{pmatrix} \begin{pmatrix} 1 & \\ -d/c & 1 \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, & c > d. \end{cases} \tag{4.134}$$

Here we have set $a = \frac{1+bc}{d}$ (we are content with investigating the case $d \neq 0$). Denoting the decomposition by

$$l = nak = \begin{pmatrix} 1 & \nu \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} k, \tag{4.135}$$

the dilaton α and axion ν become

$$\alpha = \begin{cases} d^{-1}, & d \geq c \\ c^{-1}, & c > d \end{cases} \quad \text{and} \quad \nu = \begin{cases} b/d, & d \geq c \\ a/c = 1/(dc) + b/d, & c > d. \end{cases} \tag{4.136}$$

The non-vanishing condition (4.126) gives

$$1 = \psi_m \left(\begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & z \\ & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & \\ & \alpha \end{pmatrix} \right) = e^{-2\pi i [m\alpha^2 z]_p} \tag{4.137}$$

for all $z \in \mathbb{Z}_p$. In particular we get $m\alpha^2 \in \mathbb{Z}_p$ and for the case at hand

$$\begin{cases} md^{-2} \in \mathbb{Z}_p, & d \geq c \\ mc^{-2} \in \mathbb{Z}_p, & c > d \end{cases} \tag{4.138}$$

The two cases can be unified in the following way. Start with the case $d \geq c$ and observe that

$$md^{-2} \in \mathbb{Z}_p \quad \Leftrightarrow \quad mc^{-2} \underbrace{\frac{c^2}{d^2}}_{\in \mathbb{Z}_p} \in \mathbb{Z}_p \quad \Rightarrow \quad mc^{-2} \in \mathbb{Z}_p. \tag{4.139}$$

The case $c > d$ is treated analogously and the non-vanishing condition can thus be stated as

$$md^{-2} \in \mathbb{Z}_p \quad \text{and} \quad mc^{-2} \in \mathbb{Z}_p. \tag{4.140}$$

For fixed l, let Ω denote the set of primes such that $c > d$. The phase ϕ becomes

$$\phi = \prod_{p \notin \Omega} \mathrm{e}^{-2\pi \mathrm{i}[mb/d]_p} \prod_{p \in \Omega} \mathrm{e}^{-2\pi \mathrm{i}[ma/c]_p}. \tag{4.141}$$

We will now see that dividing the Euler product into the two cases $d \geq c$ and $c > d$ with the set Ω is unnecessary and one can proceed as if $d \geq c$ for all p and still obtain the same result. Note that in the case $p \in \Omega \Leftrightarrow c > d$, we have

$$ma/c = \frac{m}{dc} + m\frac{b}{d} = \underbrace{\frac{m}{d^2}}_{\in \mathbb{Z}_p} \underbrace{\frac{d}{c}}_{\in \mathbb{Z}_p} + m\frac{b}{d}. \tag{4.142}$$

Since the first term is a p-adic integer, it will not influence the p-rational part $[ma/c]_p$. We therefore have

$$\phi = \prod_{p \notin \Omega} \mathrm{e}^{-2\pi \mathrm{i}[mb/d]_p} \prod_{p \in \Omega} \mathrm{e}^{-2\pi \mathrm{i}[mb/d]_p} = \prod_{p < \infty} \mathrm{e}^{-2\pi \mathrm{i}[mb/d]_p}. \tag{4.143}$$

Using proposition 4.12 which states that

$$x \in \mathbb{Q} \quad \Rightarrow \quad x - \sum_{p < \infty} [x]_p \in \mathbb{Z} \tag{4.144}$$

we arrive at

$$\phi = \mathrm{e}^{-2\pi \mathrm{i} mb/d}. \tag{4.145}$$

This phase is in general non-trivial. Note also that using theorem 2.2 we calculate

$$lg_\infty = ln_\infty a_\infty k_\infty = \begin{pmatrix} 1 & x'_\infty \\ & 1 \end{pmatrix} a'_\infty k'_\infty \tag{4.146}$$

where $x'_\infty = \frac{(cx_\infty + d)(b(cx_\infty + d) + x_\infty) + c(bc+1)}{d(c^2 + (cx_\infty + d)^2)}$ and x_∞ is the axion in n_∞. The overall phase in Eq. (4.132) is then

$$\mathrm{e}^{2\pi \mathrm{i}(x'_\infty - b/d)} \tag{4.147}$$

which is not even trivial for $x_\infty = 0$.

4.5 Important formulae

The adelic framework allows for the derivation of several very useful formulae for the analysis of automorphic forms. It does so in part by replacing left $G(\mathbb{Z})$-invariance with left $G(\mathbb{Q})$-invariance which unlocks group theoretical theorems and techniques which are valid for groups defined over fields. One important such theorem is the so called *Bruhat* *decomposition* which paves the way for computing the Fourier expansion of the $\mathrm{SL}_2(\mathbb{R})$-Eisenstein series $E_s^{\mathrm{SL}_2(\mathbb{R})}(g)$ of Eq. (2.68) of example 2.10. The Bruhat decomposition furthermore underpins the proof of the famous *Langlands constant term formula.*

Theorem 4.38 (Bruhat decomposition)
An algebraic group $G(\mathbb{Q})$ over the rationals decomposes into a disjoint union according to

$$G(\mathbb{Q}) = \bigcup_{w \in \mathcal{W}} B(\mathbb{Q}) w B(\mathbb{Q}). \tag{4.148}$$

4.5.1 Fourier expansion of the SL_2 Eisenstein series

It was mentioned in remarks 2.28 and 2.29 that the Fourier coefficients Eq. (2.91) can be calculated in several ways and that using the adelic framework is the preferred way for this thesis. Here we will take a look at how the derivation of the SL_2-Whittaker function from example 2.26 takes place in the adelic framework since it is a good demonstration of the virtue of working over the adeles.

We consider the Eisenstein series

$$E_s^{G(\mathbb{R})}(g) = \sum_{\gamma \in B(\mathbb{Z}) \backslash G(\mathbb{Z})} \chi_s(\gamma g) \tag{4.149}$$

of example 2.10 where $G = \mathrm{SL}_2$ and $\chi_s(g) = \mathrm{e}^{2s\langle \Lambda | H(g) \rangle}$. We have $\chi_s = \prod_{p \leq \infty} \chi_{s,p}$ and

$$\chi_{s,\infty}\left(\begin{pmatrix} 1 & n_\infty \\ & 1 \end{pmatrix} \begin{pmatrix} y_\infty & 0 \\ 0 & y_\infty^{-1} \end{pmatrix} k_\infty \right) = |y_\infty^2|_\infty^s = y_\infty^{2s} \tag{4.150}$$

for the real place and

$$\chi_{s,p}\left(\begin{pmatrix} 1 & n_p \\ & 1 \end{pmatrix} \begin{pmatrix} y_p & 0 \\ 0 & y_p^{-1} \end{pmatrix} k_\infty \right) = |y_p^2|_p^s \tag{4.151}$$

for the p-adic places. We are interested in the integrals

$$W_m^{\mathbb{R}}(g_\infty) = \int\limits_{N(\mathbb{Z}) \backslash N(\mathbb{R})} E_s^{G(\mathbb{R})}(n_\infty g_\infty) \overline{\psi_{m,\infty}(n_\infty)} \, \mathrm{d}n_\infty \tag{4.152}$$

of Eq. (2.90) which will separate into the constant term $W_0^{\mathbb{R}}$ for the character $\psi_{0,\infty} = 1$ and the non-constant terms $W_m^{\mathbb{R}}$ for the characters $\psi_{m,\infty}(n_\infty) = \mathrm{e}^{2\pi i m x_\infty}$ with $n_\infty = \begin{pmatrix} 1 & x_\infty \\ & 1 \end{pmatrix}$ and $m \in \mathbb{Z} \setminus \{0\}$. Passing to the adelic framework means to analyze the adelic lift $E_s^{G(\mathbb{A})}$ of the real Eisenstein series and its adelic Whittaker functions

$$W_m^{\mathbb{A}}(g) = \int\limits_{N(\mathbb{Q}) \backslash N(\mathbb{A})} E_\lambda^{G(\mathbb{A})}(ng) \overline{\psi_m(n)} \, \mathrm{d}n \tag{4.153}$$

where ψ_m is constructed as an Euler-product using the additive characters introduced in definition 4.13. The desired real Whittaker functions are then obtained by "projecting out"

$$W_m^{\mathbb{R}}(g_\infty) = W_m^{\mathbb{A}}((g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)). \tag{4.154}$$

The idea for calculating $W_\psi^{\mathbb{A}}$ in the adelic framework is to use the Bruhat decomposition according to

$$\begin{aligned} W_\psi^{\mathbb{A}}(g) &= \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E(ng)\overline{\psi(n)}\,\mathrm{d}n = \\ &= \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma ng)\overline{\psi(n)}\,\mathrm{d}n = \\ &= \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})} \sum_{\delta\in(\gamma^{-1}B(\mathbb{Q})\gamma\cap B(\mathbb{Q}))\backslash B(\mathbb{Q})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi(\gamma\delta ng)\overline{\psi(n)}\,\mathrm{d}n = \\ &= \sum_{\gamma\in B(\mathbb{Q})\backslash G(\mathbb{Q})/B(\mathbb{Q})} \int\limits_{(\gamma^{-1}B(\mathbb{Q})\gamma\cap N(\mathbb{Q}))\backslash N(\mathbb{A})} \chi(\gamma ng)\overline{\psi(n)}\,\mathrm{d}n = \\ &= \sum_{w\in\mathcal{W}} \int\limits_{(w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q}))\backslash N(\mathbb{A})} \chi(wng)\overline{\psi(n)}\,\mathrm{d}n. \end{aligned} \tag{4.155}$$

The Whittaker function is now given as a sum over the Weyl group of integrals over the character χ instead of the automorphic form E. Note that in going to line four, we are making a change of variables $\delta n \to n$ and using the fact that ψ is trivial on $G(\mathbb{Q})$ (and not just $G(\mathbb{Z})$ which would have been the case if we were not operating over the adeles). The analysis now splits into analyzing the two cases $\psi = 1$ and $\psi \neq 1$. We will outline the derivation for $\psi \neq 1$.

The Weyl-group of SL_2 consists of the two elements

$$\mathbb{1} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \quad \text{and} \quad w_{\text{long}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.156}$$

called the trivial Weyl-word and the longest Weyl-word. We also have

$$\psi_m(n) = \mathrm{e}^{2\pi\mathrm{i}mx_\infty} \prod_p \mathrm{e}^{-2\pi\mathrm{i}[mx_p]_p}. \tag{4.157}$$

The trivial Weyl word leads us to the integral

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_s(ng)\overline{\psi_m(n)}\,\mathrm{d}n. \tag{4.158}$$

Noting that $\chi_s(ng) = \chi_s(g)$ is independent of the integration variable n, we are left with integrating the character $\psi_m(n)$ over a full period and hence the integral vanishes.

The non-trivial Weyl-word gives the integral

$$\int\limits_{N(\mathbb{A})} \chi_s(w_{\text{long}}na)\overline{\psi_m(n)}\,\mathrm{d}n. \tag{4.159}$$

Note now that the domain has been unfolded to include the full $N(\mathbb{A})$ because of the action of w_{long}. The integral factorizes into the local places and when evaluated at $g = (g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)$ it can be written

$$\begin{aligned}&\int\limits_{\mathbb{R}} \chi_s\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&t_\infty\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&x_\infty\\&1\end{smallmatrix}\right)\begin{pmatrix}y_\infty^{1/2}&\\&y_\infty^{-1/2}\end{pmatrix}\right)\mathrm{e}^{-2\pi\mathrm{i}mt_\infty}\,\mathrm{d}t_\infty\\ \prod_p &\int\limits_{\mathbb{Q}_p} \chi_s\left(\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&t_p\\&1\end{smallmatrix}\right)\right)\mathrm{e}^{2\pi\mathrm{i}[mt_p]_p}\,\mathrm{d}t_p.\end{aligned} \tag{4.160}$$

To evaluate these integrals, we need to write the arguments of χ_s in Iwasawa form, or at least calculate the corresponding norms[17] of the Cartan coordinates y'_p ($p \le \infty$) of the corresponding arguments. Eq. (2.35) gives

$$\begin{aligned}&\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&t_\infty\\&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&x_\infty\\&1\end{smallmatrix}\right)\begin{pmatrix}y_\infty^{1/2}&\\&y_\infty^{-1/2}\end{pmatrix} = n'_\infty\begin{pmatrix}y'_\infty&0\\&y'^{-1}_\infty\end{pmatrix}k'_\infty\\ &\text{where}\quad (y'_\infty)^2 = y_\infty^{-1}\left(1+(t_\infty+x_\infty)^2 y_\infty^{-2}\right)^{-1}\end{aligned} \tag{4.161}$$

and the real integral hence becomes

$$\begin{aligned}&\int\limits_{\mathbb{R}} y_\infty^{-s}\left(1+(t_\infty+x_\infty)^2y_\infty^{-2}\right)^{-s}\mathrm{e}^{-2\pi\mathrm{i}mt_\infty}\,\mathrm{d}t_\infty =\\ &= \mathrm{e}^{2\pi\mathrm{i}mx_\infty}y_\infty^{-s+1}\int\limits_{\mathbb{R}}\left(1+t_\infty^2\right)^{-s}\mathrm{e}^{-2\pi\mathrm{i}my_\infty t_\infty}\,\mathrm{d}t_\infty =.\\ &= \mathrm{e}^{2\pi\mathrm{i}mx_\infty}\frac{2\pi^s}{\Gamma(s)}y_\infty^{1/2}|m|_\infty^{s-1/2}K_{s-1/2}(2\pi|m|_\infty y_\infty)\end{aligned} \tag{4.162}$$

where we have used the formula

$$\int\limits_{\mathbb{R}}(1+u^2)^{-s}e^{-2\pi\mathrm{i}mu}\,\mathrm{d}u = \frac{2\pi^s}{\Gamma(s)}|m|_\infty^{s-1/2}K_{s-1/2}(2\pi|m|_\infty) \tag{4.163}$$

for the inverse Fourier transform of the Bessel function.

[17] Recall that the p-adic Iwasawa decomposition is not unique but the p-adic norms of the Cartan coordinates are, and since χ_p only depends on these norms, $\chi_p(g_p)$ is always well defined.

Furthermore, theorem 4.29 gives

$$\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t_p \\ & 1 \end{smallmatrix}\right) = n'_p \begin{pmatrix} y'_p & 0 \\ & y_p'^{-1} \end{pmatrix} k'_p \quad \text{where} \quad |y'_p| = \max(1, |t_p|_p)^{-1} \tag{4.164}$$

and the p-adic integrals become

$$\int\limits_{\mathbb{Q}_p} \max(1, |t_p|_p)^{-2s} e^{2\pi i [m t_p]_p} \, dn_p. \tag{4.165}$$

This integral can be evaluated using techniques similar to those demonstrated in Section 4.1.4 (see [18] for a derivation) and gives

$$\int\limits_{\mathbb{Q}_p} \max(1, |t_p|_p)^{-2s} e^{2\pi i [m t_p]_p} \, dn_p = \gamma_p(m)(1 - p^{-2s}) \frac{1 - p^{-2s+1}|m|_p^{2s-1}}{1 - p^{-2s+1}}. \tag{4.166}$$

The full SL_2-Whittaker function evaluated at $(g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)$ (and thus the real SL_2-Whittaker function) is then

$$\begin{aligned} W_m^{\mathbb{R}}(g_\infty) &= W_m^{\mathbb{A}}((g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)) = \\ &= e^{2\pi i m x_\infty} \frac{2\pi^s}{\Gamma(s)} y_\infty^{1/2} |m|_\infty^{s-1/2} K_{s-1/2}(2\pi |m|_\infty y_\infty) \\ &\quad \prod_p \gamma_p(m)(1 - p^{-2s}) \frac{1 - p^{-2s+1}|m|_p^{2s-1}}{1 - p^{-2s+1}}. \end{aligned} \tag{4.167}$$

The p-adic Whittaker functions Eq. (4.166) is made up of three important factors. Firstly, the Euler product over the p-adic gaussians γ_p ensures that $m \in \mathbb{Z}$. Secondly, the Euler product over the factors $1 - p^{-2s}$ gives a factor $\zeta(2s)^{-1}$ which combines with $(\Gamma(s)\pi^{-s})^{-1}$ of the archimedean Whittaker function Eq. (4.162) to give a factor $\xi(2s)^{-1}$. Lastly, the Euler product over $\frac{1-p^{-2s+1}|m|_p^{2s-1}}{1-p^{-2s+1}}$ gives rise to the divisor sum $\sigma_{1-2s}(m)$. See Appendix C for the details on Euler products. With these observations we get

$$W_m^{\mathbb{R}}(g_\infty) = e^{2\pi i m x_\infty} \frac{2}{\xi(2s)} y_\infty^{1/2} |m|_\infty^{s-1/2} K_{s-1/2}(2\pi |m|_\infty y_\infty) \sigma_{1-2s}(m) \tag{4.168}$$

for integer m and zero for non-integer rational m. This result is in agreement with Eq. (2.92).

Lastly we shall mention that the Fourier expansion of the adelic SL_2-Eisenstein series has the form

$$E_s^{SL_2(\mathbb{A})}(g) = \sum_{m \in \mathbb{Q}} W_m^{\mathbb{A}}(g) \tag{4.169}$$

since the characters on $SL_2(\mathbb{A})$ trivial on $SL_2(\mathbb{Q})$ are parametrized by rational numbers and summing over them all is therefore accomplished by summer over the rationals.

When projecting down we get

$$E_s^{\mathrm{SL}_2(\mathbb{A})}((g_\infty;\mathbb{1},\mathbb{1},\mathbb{1},\dots)) = E_s^{\mathrm{SL}_2(\mathbb{R})}(g_\infty) = \sum_{m\in\mathbb{Q}} W_m^{\mathbb{A}}((g_\infty;\mathbb{1},\mathbb{1},\mathbb{1},\dots)) = \sum_{m\in\mathbb{Z}} W_m^{\mathbb{R}}(g_\infty). \tag{4.170}$$

The sum over the rationals collapses to a sum over the integers thanks to the p-adic gaussian $\gamma_p(m)$. For higher rank groups and for Fourier coefficients over larger parabolic subgroups there can be examples where the argument of a p-adic gaussian is a more complicated expression involving several charges m_i such that a rational sum collapses to an integer sum with additional constraints describing an intricate lattice.

What remains to be done now is to the repeat the above analysis for $m = 0$ to reproduce the constant term of Eq. (2.91). Rather than spending time on the special case of SL_2, we will state a famous closed formula for the constant term of any adelic Eisenstein series.

4.5.2 Langlands' constant term formula

The Langlands' constant term formula is a staple of the analytical power that the adelic framework offers. This formula will not be needed in this thesis but is an interesting result to state nevertheless. Regarding the adelic Eisenstein series Eq. (4.121), we have the following result

Theorem 4.39 (Langlands' constant term formula)
The constant term of $E_\lambda^{G(\mathbb{A})}$ with respect to the maximal unipotent N is given by

$$\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} E_\lambda^{G(\mathbb{A})}(ng)\,\mathrm{d}n = \sum_{w\in\mathcal{W}} a^{w\lambda+\rho} M(w,\lambda) \tag{4.171}$$

where $g = nak$ and

$$M(w,\lambda) = \prod_{\substack{\alpha>0\\ w\alpha<0}} \frac{\xi(\langle\lambda|\alpha\rangle)}{\xi(1+\langle\lambda|\alpha\rangle)} \tag{4.172}$$

is the intertwiner of Eq. (2.56).

The proof of this theorem makes recourse with the Bruhat decomposition as demonstrated in the section above. The integral again splits into an Euler product with the archimedean place contributing the right factors of π and Γ-functions to combine with the non-archimedean contribution of the productand $\frac{1}{1-p^{-s}}$ to construct the completed Riemann zeta functions ξ. A full proof of this theorem can be found in [18].

Remark 4.40.
There also exists a constant term formula for the expansion over a (minimal) unipotent U belonging to a maximal parabolic subgroup (with Langlands decomposition) $P = UA_PL$. The summand on the right hand side then also contain Eisenstein series on L and the sum runs over a quotient of the Weyl group of G.

Remark 4.41.
For local generic unramified p-adic Whittaker functions

$$\int_{N(\mathbb{Z}_p)\backslash N(\mathbb{Q}_p)} E_\lambda^{G(\mathbb{A})}\overline{\psi(n)}\,\mathrm{d}n \quad \psi \text{ generic and unramified,} \tag{4.173}$$

there exists a closed formula called the Casselman-Shalika formula similar to Langlands' constant term formula but no such formula is known for the archimedean place $p = \infty$.

Remark 4.42.
Langland's constant term formula cannot be applied for calculating the constant term of the $\nabla^6 R^4$-function $\mathcal{E}_{(0,1)}^{(D)}$ of Chapter 3. This is due to the fact that $\mathcal{E}_{(0,1)}^{(D)}$ is not strictly speaking an automorphic form, because of the non-linear source term in Eq. (3.8). Another way of seeing this is that in an ansatz like Eq. (3.18) or Eq. (3.25), Φ and σ are not characters. In order to fix the integration constant on the homogeneous solution of Eq. (3.8), one needs a way of calculating the constant term of $\mathcal{E}_{(0,1)}^{(D)}$ to match up with known results from string perturbation theory. It is an open question whether or not there exists a convenient closed formula analogous to theorem 4.39 and answering this question will most likely require a better understanding of how the $\frac{1}{8}$-BPS protected function $\mathcal{E}_{(0,1)}^{(D)}$ fits into the representation theory of $G(\mathbb{A})$.

4.5.3 Reduction formula

A very useful formula is the so called reduction formula of [50]. It is an adelic formula which relates a degenerate Whittaker function evaluated on $A(\mathbb{A}) \subset G(\mathbb{A})$ to a generic Whittaker function on a subgroup $G'(\mathbb{A}) \subset G(\mathbb{A})$ evaluated at the identity.

Theorem 4.43 (Reduction formula)
We have the adelic equation

$$W_\psi(a) = \sum_{w_c w_0' \in \mathcal{C}_\psi} a^{(w_c w_0')^{-1}\lambda+\rho} M(w_c^{-1},\lambda) W'_{\psi^a}(w_c^{-1}\lambda, \mathbb{1}) \tag{4.174}$$

where λ denotes the weight vector of the underlying Eisenstein series and ψ^a denotes the "twisted character" $\psi^a(n) = \psi(ana^{-1})$ of Appendix C. The character ψ should be degenerate, meaning that it is charged only on a subset of the simple roots (or nodes in the Dynkin diagram) of $G(\mathbb{A})$. By G' we denote the subgroup defined by removing the uncharged nodes and W' denotes a Whittaker function on G'. Furthermore, w_0' denotes the longest Weyl word in the Weyl group of G' and w_c is thus the summation variable. Lastly, the set $\mathcal{C}_\psi$ is defined by

$$\mathcal{C}_\psi = \left\{ w \in \mathcal{W} : w\Pi' < 0 \right\} \tag{4.175}$$

where Π' is the set of simple roots of G' and $\mathcal{W}$ is the Weyl group of G. This set turns out to be especially simple to parametrize using the so called "orbit method" by writing

$w_c w_0' \in \mathcal{C}_\psi$ as explained in [51].

Remark 4.44.
The reduction formula implies that a maximally degenerate Whittaker function on a group $G(\mathbb{A})$ can be expressed in terms of Whittaker functions on $\mathrm{SL}_2(\mathbb{A})$ and that a Whittaker function on G charged on only two commuting roots can be expressed in terms of Whittaker functions on $\mathrm{SL}_2(\mathbb{A}) \times \mathrm{SL}_2(\mathbb{A})$ which in turns are products of two $\mathrm{SL}_2(\mathbb{A})$ Whittaker functions.

Since Whittaker functions are determined by their restriction to the Cartan torus A as explained in remark 2.23, the fact that the reduction formula only expresses the a-dependence in $g = nak$ is no shortcoming.

4.6 Applications to physics

This section is unrelated from the rest of this thesis but could be of some general interest.

p-adic numbers and adeles are commonplace in many branches of mathematics, most notably number theory. One example of a direct application is through the so called "Hasse principle", or the "local-global principle" wherein one can establish the existence (or lack thereof) of rational solutions to certain polynomial equations by analyzing the same equation over all completions $\mathbb{Q}_p$ ($p \leq \infty$) of the rationals. The intuition behind this is that is that given a polynomial P with rational coefficients, a rational root of P is also a p-adic and a real root since $\mathbb{Q} \subset \mathbb{Q}_p$ ($p \leq \infty$). It could therefore be promising to analyze P over each $\mathbb{Q}_p$ as if it's the case that roots cannot be found for each $p \leq \infty$ one can conclude that no rational roots can exist. The Hasse principle goes further and investigates when the existence of p-adic and real roots implies the existence of rational (or "global") roots. The same principle can be applied for other number fields or even rings, like the ring of integers. A central example of when the local-global principle does apply (meaning that the existence of roots in $\mathbb{Q}_p$ for $p \leq \infty$ implies the existence of roots in $\mathbb{Q}$) is for non-degenerate quadratic forms over $\mathbb{Q}$, a result known as the Hasse-Minkowski theorem. Details can be found in [52].

In other areas of science, p-adic numbers have been applied to topics such as computer science, quantum mechanics (via a theory of p-adic probability theory [53]) and most notably for this thesis, p-adic string theory. Some people argue that the archimedean axiom is a physical axiom about measurements [54, 55], stating that two scales can always be compared. As the argument goes, physics at the Planck scale might be very different than physics at the length scales we have thus far been able to model and probe using real numbers and continuous and archimedean geometry, and at this scale the physics might be non-archimedean. In the context of string theory, this postulate has been investigated in a number of publications [55, 56, 57, 58, 59] with the biggest surge in research in the late 1980s.

A particularly celebrated result was that of Freund and Witten [57] where they constructed p-adic counterparts to the Veneziano amplitude and found a particularly simple expression for their Euler product. The starting point is the (real) tree level scattering amplitude $A_\infty^{(N)}(k_1,\dots,k_N)$ of N open string tachyons with momenta k_i which after SL_2 gauge fixing takes the form (suppressing the momentum dependence)

$$A_\infty^{(N)} = \int\limits_{(\mathbb{Q}_\infty)^{N-3}} \mathrm{d}x_2 \dots \mathrm{d}x_{N-2} \prod_{i=2}^{N-2} |x_i|_\infty^{k_1\cdot k_i} |1-x_j|_\infty^{k_{N-1}\cdot k_i} \prod_{2\le i<j\le N-2} |x_i - x_j|_\infty^{k_i\cdot k_j}. \quad (4.176)$$

In constructing a p-adic analogue to this, one need to decide on which archimedean quantities are to become non-archimedean. Possible avenues to consider are to let amplitudes be valued in $\mathbb{Q}_p$, to let spacetime be a manifold with charts valued in $\mathbb{Q}_p$ or let the string action be valued in $\mathbb{Q}_p$. For the generalization in [58] these possibilities are left untouched and what is changed is to make the boundary of the string worldsheet p-adic which leads to the p-adic analog of Eq. (4.176)

$$A_p^{(N)} = \int\limits_{(\mathbb{Q}_p)^{N-3}} \mathrm{d}x_2 \dots \mathrm{d}x_{N-2} \prod_{i=2}^{N-2} |x_i|_p^{k_1\cdot k_i} |1-x_j|_p^{k_{N-1}\cdot k_i} \prod_{2\le i<j\le N-2} |x_i - x_j|_p^{k_i\cdot k_j}. \quad (4.177)$$

Restricting to the case $N = 4$, both Eq. (4.176) and Eq. (4.177) can be expressed in terms of the real and p-adic Gelfand–Graev–Tate gamma functions (the p-adic or real Fourier transform of $|\cdot|_p^{s-1}$, $p \le \infty$)

$$\Gamma_p(s) = \int_{\mathbb{Q}_p} |x|_p^s \psi_{1,p}(x) \frac{\mathrm{d}x}{|x|_p} = \begin{cases} 2\Gamma(s)\cos\frac{\pi s}{2}, & p = \infty \\ \frac{1-p^{s-1}}{1-p^{-s}}, & p < \infty \end{cases} \quad (4.178)$$

where $\Gamma(s)$ is the usual Euler gamma function. The 4-point amplitudes for all $p \le \infty$ can then be written

$$A_p^{(4)} = \prod_{x=s,t,u} \Gamma_p(-\alpha(x)) \quad \text{where } \alpha \text{ is the Regge trajectory} \quad \alpha(x) = 1 + \frac{x}{2} \quad (4.179)$$

for Mandelstam variables s, t and u. Using identities such as

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{and} \quad \Gamma(z) = \pi^{-1/2} 2^{z-1} \Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) \quad \text{and Eq. (C.15)} \quad (4.180)$$

the authors were able to establish the remarkably simple formula

$$\prod_{p\le\infty} A_p^{(4)} = 1. \quad (4.181)$$

This result sparked great interest in possible p-adic analogues of string theory. For $N > 4$, the integrals Eqs. (4.176) and (4.177) have been solved but do not combine into a nice Euler product in this way. The same problem for the tree level scattering of closed strings also doesn't have any known nice adelic properties.

More recently there has been some interesting results regarding p-adic numbers in the context of the AdS/CFT correspondence [60, 61]. Gubser et. al. succeeded in constructing a bulk space whose boundary is $\mathbb{Q}_p$, or a field extension $\mathbb{Q}_p^n$ thereof for higher dimensional physics. This bulk space is a version of the so called "Bruhat-Tits tree" and is discrete in nature. The authors consider a scalar field ϕ with the action

$$S = \sum_{\langle ab \rangle} \frac{1}{2}(\phi_a - \phi_b)^2 + \sum_a \left(\frac{1}{2} m_p^2 \phi_a^2 - J_a \phi_a \right)^2 \tag{4.182}$$

where the sum goes over lattice sites in the bulk and $\langle ab \rangle$ denotes nearest neighbors. Here ϕ_a is the value of the scalar field at site a. The equation of motion for ϕ reads

$$\left(\Box_a - m_p\right)^2 \phi_a = J_a \tag{4.183}$$

where $\Box_a$ is the p-adic (discrete) analog of the Laplace-Beltrami operator and acts on the variable a. Searching for a Green's function G to Eq. (4.183) satisfying

$$(\Box_a + m_p^2) G(a,b) = \delta_{a,b} \tag{4.184}$$

gives

$$G(a,b) = \frac{\zeta_p(2\Delta)}{p^\Delta} p^{-\Delta d(a,b)} \tag{4.185}$$

where

$$\zeta_p(s) = \frac{1}{1 - p^{-s}} \tag{4.186}$$

is the p-productand in the Euler product representation Eq. (C.15) of the Riemann zeta function. Furthermore, Δ satisfies $\zeta_p(\Delta - n)\zeta_p(-\Delta) m_p^2 = -1$ and $d(a,b)$ denotes the distance between sites a and b in the bulk. G is the bulk-to-bulk propagator. By taking the limit $b \to x$ where the bulk variable a is at the boundary we get the bulk-to-boundary propagator

$$K(z_0, z; x) = \frac{\zeta_p(2\Delta)}{\zeta_p(2\Delta - n)} \frac{|z_0|_p^\Delta}{||(z_0, z - x)||_p^{2\Delta}} \tag{4.187}$$

where the bulk point a has been written as the vector (z_0, z) (see [60] for details) and $|| \cdot ||_p$ denotes the p-adic vector norm obeying

$$||(x_1, \ldots, x_n)||_p = \max(|x_1|_p, \ldots, |x_1|_p) \quad \text{for} \quad (x_1, \ldots, x_n) \in \mathbb{Q}_p^n. \tag{4.188}$$

These results should be compared with the well known archimedean (i.e. "normal" or Euclidean AdS/CFT) where one has [62]

$$K(z_0, \boldsymbol{z}; \boldsymbol{x}) = \frac{\zeta_\infty(2\Delta)}{\zeta_\infty(2\Delta - n)} \frac{z_0^\Delta}{||(z_0, \boldsymbol{z} - \boldsymbol{x})||_\infty^2 \Delta} \tag{4.189}$$

where

$$\zeta_\infty(s) = \pi^{-s/2}\Gamma(s/2) \tag{4.190}$$

which should be recognized as the archimedean factor in the Euler product for the completed Riemann zeta function Eq. (C.16). The completed Riemann zeta function obeys the particularly nice functional equation

$$\xi(s) = \xi(1-s) \tag{4.191}$$

and as such, taking the Euler product over all bulk-to-boundary propagators (meaning over the p-adic AdS/CFT as well as Euclidean AdS/CFT) gives a propagator of an adelic spacetime with especially nice properties. These observations may count as evidence that the ultimately correct description of nature should be adelic (global) rather than just real or p-adic (local).

5 Automorphic representations and nilpotent orbits

Let us switch to the real numbers for a moment and consider the space $\mathcal{A}(G(\mathbb{Z})\backslash G(\mathbb{R}))$ (henceforth abbreviated as $\mathcal{A}_\mathbb{R}$) of automorphic forms on the group $G(\mathbb{R})$. Given an element $\varphi \in \mathcal{A}_\mathbb{R}$, there is evidently a natural group action π of $G(\mathbb{R})$ on φ given by

$$[\pi(h)\varphi](g) \equiv \varphi(gh), \quad g, h \in G(\mathbb{R}) \tag{5.1}$$

called the *right regular action* [18] . Additionally, the Lie algebra $\mathfrak{g}_\mathbb{C}$, and even the full universal enveloping algebra $\mathcal{U}(\mathfrak{g}(\mathbb{C}))$ (henceforth abbreviated as $\mathcal{U}$) acts on φ as differential operators according to

$$(D_X\varphi)(g) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\,\varphi\left(g e^{tX}\right)\Big|_{t=0}, \quad X \in \mathcal{U}(\mathfrak{g}(\mathbb{C})). \tag{5.3}$$

This formula can be compared with the definition of a tangent vector in differential geometry.

Whenever there is a group action, one may ask the question of how the corresponding space decomposes into irreducible representations. As the centre $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$ commutes with $G(\mathbb{R})$, it is natural to organize the irreducible components of $\mathcal{A}_\mathbb{R}$ with respect to their eigenvalues under the differential operators in $\mathcal{Z}(\mathfrak{g}_\mathbb{C})$. This corresponds to point 3 in definition 2.4. The decomposition of $\mathcal{A}_\mathbb{R}$ with respect to the right regular action, compatible with the action of $\mathcal{U}$ is what leads to the notion of automorphic representations. There is quite a lot of detail which goes into this topic and comprehensive treatment is beyond the scope of this thesis. Here we will summarize the key points and present the relevant results to be used in Chapter 6. For a fuller treatment, see [18, 63, 64].

It turns out that the space $\mathcal{A}_\mathbb{R}$ carries yet another action under the so called *Hecke operators* which also should be taken into account. The Hecke operators are generated by operators $T_p : \mathcal{A}_\mathbb{R} \to \mathcal{A}_\mathbb{R}$ labelled by the prime numbers p and the spectrum of T_p for each p should also be taken into account in the decomposition of $\mathcal{A}_\mathbb{R}$. However, the action of the Hecke operators on an automorphic form $\varphi \in \mathcal{A}_\mathbb{R}$ cannot be realized in terms of the right (or left) action of $G(\mathbb{R})$. There is however a convenient way to take

[18] This action is indeed a group homomorphism:

$$[\pi(h_1)\pi(h_2)\varphi](g) = [\pi(h_1)[\pi(h_2)\varphi]](g) = [\pi(h_1)\varphi](gh_2) = \varphi(gh_1h_2) = [\pi(h_1h_2)\varphi](g). \tag{5.2}$$

There is also the left regular action $[\pi_L(h)\varphi](g) = \varphi(h^{-1}g)$ where the inverse is required in order for π_L to be a homomorphism. For the study of automorphic representations, these two actions are equivalent.

the action of Hecke operators into account for the representation theory of $G(\mathbb{R})$ which involves passing to the adelic framework. Studying the right action of $G(\mathbb{A})$ on the space $\mathcal{A}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ (henceforth abbreviated as just $\mathcal{A}$) of automorphic forms on $G(\mathbb{A})$, one finds that the eigenvalues of T_p parametrise the irreducible representations of the right regular action of the local groups $G(\mathbb{Q}_p)$. From the adelic perspective, the Hecke algebra plays the same role at the local places $G(\mathbb{Q}_p)$ as the universal enveloping algebra $\mathcal{U}$ does at the archimedean place and by studying the adelic automorphic forms $\mathcal{A}$ under the right action of the adelic group $G(\mathbb{A})$, the Hecke algebra is automatically taken into account.

An automorphic representation (π, V) is thus defined as an irreducible constituent in the decomposition of $\mathcal{A}$ under the action π of $(\mathfrak{g}, K_\infty)\times G_f$ where K_∞[19] and G_f act on an automorphic form φ with the right regular action and $\mathfrak{g}$ acts compatibly[20] as differential operators. This notion of automorphic representations is enough for the purposes of this thesis. A more precise and technical definition can be found in [18].

One component of the irreducible submodules of the space $\mathcal{A}$ is the so called *principle series*. Without going into too much detail, they are the Poincaré sums[21] of functions

$$f \in I(\lambda) \equiv \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi \equiv \{f : G(\mathbb{A}) \to \mathbb{C} | f(bg) = \chi(b)f(g), b \in B(\mathbb{A})\} \tag{5.4}$$

which "transform according to the character $\chi : B \to \mathbb{C}$". These spaces come in a continuum and are parametrized by the weight λ which defines the character $\chi : B \to \mathbb{C}$. The (adelic analog of the) minimal Eisenstein-series Eq. (2.49) arise from $\chi \in \mathrm{Ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ itself. Furthermore, one can also think about the space $\mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})}\chi$ for non-minimal parabolics. Bear in mind however that the minimal- and maximal parabolic Eisenstein series are two sides of the same coin as discussed in remark 2.8.

Definition 5.1 (Gelfand-Kirillov dimension)
*Even though the principal series representations $I(\lambda)$ are infinite-dimensional, one can still ascribe the notion of a size or dimensionality to them. This is the so called **Gelfand-Kirillov dimension**, denoted* GKdim. *Loosely speaking, the Gelfand-Kirillov dimension of $I(\lambda)$ is defined as the smallest number of variables required to realize the functions in $I(\lambda)$.*

Example 5.2

[19] One restricts to $K_\infty \subset G(\mathbb{R})$ so that the compactness of K_∞ produces finite-dimensional representations in the decomposition of $\mathcal{A}$.

[20] The action of elements $k \in K_\infty$ and $X \in \mathfrak{g}$ on an automorphic form $\phi \in \mathcal{A}$ do not commute but rather obey $D_X\pi(k)\varphi = \pi(k)D_{k^{-1}Xk}\varphi$. This non-commutativity must be taken into account and the proper space that does this is a so called $(\mathfrak{g}, K_\infty)$-module.

[21] A Poincaré sum of a function f is an averaging across a discrete subgroup $\Gamma \subset G$ in order to construct another function which is automorphic on Γ. The Eisenstein series are an example of this.

The maximal parabolic subgroup $P_i \subset G$ of $G = \mathrm{SL}_n$ associated with node i has the form

$$P_i = \begin{pmatrix} \mathrm{GL}_i & U_i \\ 0 & \mathrm{GL}_{n-i} \end{pmatrix} \quad \text{together with a unit determinant condition on the GL-pieces.} \tag{5.5}$$

The representation space $\mathrm{Ind}_{P_i(\mathbb{A})}^{G(\mathbb{A})} \chi$ is *induced* from $\chi : P_i \to \mathbb{C}$ in the sense that the behavior of a function $\mathrm{Ind}_{P_i(\mathbb{A})}^{G(\mathbb{A})} \chi \ni f : G(\mathbb{A}) \to \mathbb{C}$ on the subgroup $P_i \in G$ is governed by χ but the degrees of freedom living in $P_i \backslash G$ are unspecified. We therefore have

$$\mathrm{GKdim}\, \mathrm{Ind}_{P_i(\mathbb{A})}^{G(\mathbb{A})} \chi = \dim P_i(\mathbb{A}) \backslash G(\mathbb{A}) = \dim U_i = i(n-i). \tag{5.6}$$

With this very brief introduction to automorphic representations, let us investigate their meaning for Fourier coefficients of automorphic forms. We will now stay in the adelic framework for the remainder of this thesis.

5.1 Nilpotent orbits

Consider a Fourier coefficient

$$F_\psi^U(g) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\, \mathrm{d}u \tag{5.7}$$

over a parabolic subgroup $P = UL$. Under the left-action element $\gamma \in L(\mathbb{Q})$ in the rational (i.e. discrete) Levi subgroup, we have the following

$$\begin{aligned} F_\psi^U(lg) &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ulg)\overline{\psi(u)}\, \mathrm{d}u = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(l^{-1}ug)\overline{\psi(u)}\, \mathrm{d}u = \\ &= \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(g)\overline{\psi(lul^{-1})}\, \mathrm{d}u = F_{\psi^\gamma}^U(g) \end{aligned} \tag{5.8}$$

where we have used left $L(\mathbb{Q})$-invariance for φ and the invariance of the Haar-measure[22] under the change of variables $u \to lul^{-1}$. We have also defined the "twisted character" (see Appendix C)

$$\psi^l(u) \equiv \psi(lul^{-1}). \tag{5.9}$$

This result shows that Fourier coefficients fall into orbits under the action of the discrete Levi subgroup $L(\mathbb{Q})$ and in the full Fourier expansion Eq. (2.85) of an automorphic form φ we therefore expect to be able to organize the Fourier coefficients into these orbits. We will now see that these orbits are none other than the so called *nilpotent orbits* of the Lie-algebra $\mathfrak{u}$ of the unipotent U.

[22] The proof of this involves using the property $|q| = 1$ for a rational number $q \in \mathbb{Q}$ embedded diagonally into $\mathbb{A}$, as discussed in Eq. (4.56)

Focusing for the moment on abelian unipotents (which includes unipotents of maximal parabolic subgroups), it was argued in Eq. (2.77) that a character ψ evaluated on a group element $u = \exp(X)$ where $X = \sum_{\alpha\in\Delta(\mathfrak{u})} x_\alpha E_\alpha \in \mathfrak{u}$ is the associated Lie algebra element, takes the form

$$\psi(\exp(X)) = \prod_{\alpha\in\Delta(\mathfrak{u})} \mathrm{e}^{2\pi i m_\alpha x_\alpha} \equiv \mathbf{e}(\omega(X)) \tag{5.10}$$

where $\omega \in \mathfrak{u}^\star$ is an element of the dual space of the Lie algebra $\mathfrak{u}$, defined by the charges m_α. Given the non-degeneracy of the Killing form, the dual space $\mathfrak{u}^\star$ can be identified with $\mathfrak{u}$ itself and we have

$$\psi(\exp(X)) = \mathbf{e}(\kappa(C, X)) \tag{5.11}$$

for some Lie algebra element $C \in \mathfrak{u}$. The character ψ is trivial on $U(\mathbb{Q})$ if and only if all charges m_α are rational numbers. A Fourier expansion involves summing over all possible such characters and thus summing over all possible rational matrices $C \in \mathfrak{u}(\mathbb{Q})$. The space $\mathfrak{u}(\mathbb{Q})$ is therefore called the *character variety* and it is this space that is acted upon in Eq. (5.8) and Eq. (5.9) by the rational Levi $L(\mathbb{Q})$ according to

$$\psi^l(\exp(X)) = \psi(l\exp(X)l^{-1}) = \psi(\exp(lXl^{-1})) = \mathbf{e}(\kappa(C, lXl^{-1})) = \mathbf{e}(\kappa(l^{-1}Cl, X)). \tag{5.12}$$

Under the action of $L(\mathbb{Q})$, the character variety $\mathfrak{u}(\mathbb{Q})$ splits into so called *character variety orbits* which will turn out to be nilpotent orbits.

Further down, we will cite results about the vanishing of certain Fourier coefficients of an automorphic form φ depending on the precise automorphic representation that φ is attached to and which orbit in the character variety the given Fourier coefficients fall into. We will make statements along the lines of *"φ is attached to a small automorphic representation, therefore this Fourier coefficient which falls into a large orbit must vanish"*. The vanishing of a Fourier coefficient is governed by whether or not it belongs to the so called *wavefront set* (to be made precise in Section 5.2) of an automorphic representation. This is a property which must be governed by the group G itself and not depend on the particular parabolic $P = LU$ chosen for the Fourier expansion. Because of the inclusions $\mathfrak{u}(\mathbb{Q}) \subset \mathfrak{g}(\mathbb{Q})$ and $L(\mathbb{Q}) \subset G(\mathbb{Q})$ one studies the more general problem of nilpotent orbits of a group G on its own Lie algebra $\mathfrak{g}$. Given knowledge about the wavefront set of a given automorphic representation, it is then necessary to know how the L-orbits embed into the G-orbits in order to deduce whether or not a Fourier coefficient F^U_ψ vanishes. This has been computed in [26].

5.1.1 Nilpotent orbits for $\mathfrak{sl}_n$

Consider a nilpotent matrix X. Nilpotency means that $X^k = 0$ for a sufficiently large integer k. Upon conjugation with another matrix g, taking $X \mapsto gXg^{-1}$, the nilpotency property is retained, since $(gXg^{-1})^k = gX^kg^{-1} = 0$. Given a group G with associated Lie algebra $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{k}$ (see Eq. (2.21)), it therefore makes sense to speak about nilpotent orbits of the nilpotent radical $\mathfrak{n}$ under the adjoint action of G.

The nilpotent orbits of $\mathfrak{g}$ under the action of $G(\mathbb{C})$ have been classified, see [21] for a good reference on this. For $\mathfrak{g} = \mathfrak{sl}_n$ the classification is very conveniently organized into partitions of the number n. This comes from the fact that a nilpotent matrix $X \in \mathfrak{sl}_n$ has a Jordan normal form with blocks sizes given by a partition $[d_1, \dots, d_n]$. In particular, this means that the set of nilpotent orbits is finite.

We will take a closer look at the case $n = 4$ and make comments about how the concepts generalize to arbitrary n. Keep in mind the discussion about parabolic subalgebras in Section 2.1.3. There are five partitions of the natural number four: $4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1$. These partitions will also be denoted 4, 31, 2^2, 21^2 and 1^4 respectively. The five nilpotent orbits are denoted $\mathcal{O}_4$, $\mathcal{O}_{31}$, $\mathcal{O}_{2^2}$ and $\mathcal{O}_{1^4}$ and have representatives

$$\begin{aligned} X_4 &= \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}, \quad X_{31} = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{pmatrix}, \quad X_{2^2} = \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \\ X_{21^2} &= \begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \quad \text{and} \quad X_{1^4} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \end{aligned}. \tag{5.13}$$

Considering that Weyl reflections are included in the adjoint action of SL_4, it is easy to see that a given Whittaker function is associated with precisely one of these four nilpotent orbits. In particular, a generic Whittaker function is associated with the so called principal orbit $\mathcal{O}_{\text{prin}} = \mathcal{O}_4$ and a maximally degenerate Whittaker function is associated with the so called minimal orbit $\mathcal{O}_{\text{min}} = \mathcal{O}_{21^2}$. The constant term is associated with the trivial orbit $\{0\} = \mathcal{O}_{1^4}$ (see below).

A more rigorous construction of the nilpotent orbits rests heavily on the so called Jacobson-Morozow theorem, stating that for every nilpotent element $X \in \mathfrak{g}$, one can find a triple (X, Y, H) satisfying the $\mathfrak{sl}_2$-commutation relations

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{5.14}$$

An important result is then that the only possible eigenvalues of the simple roots of $\mathfrak{g}$ under the action of H are 0, 1 and 2. This is sometimes illustrated in a so called weighted Dynkin diagram. The generator H then offers a grading

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \bigoplus_{i=1}^{2} (\mathfrak{g}(i) \oplus \mathfrak{g}(-i)) \tag{5.15}$$

of the Lie algebra $\mathfrak{g}$ and the dimension of a nilpotent orbit $\mathcal{O}$ is given by

$$\dim \mathcal{O} = \dim \mathfrak{g} - \dim \mathfrak{g}(0) - \dim \mathfrak{g}(1). \tag{5.16}$$

This is lemma 4.1.3 in [21]. The more general statement is

$$\dim \mathcal{O}_X = \dim \mathfrak{g} - \dim \mathfrak{g}_X \tag{5.17}$$

for a nilpotent orbit with generator $X \in \mathfrak{g}$. Where $\mathfrak{g}_X = \{Y \in \mathfrak{g} : [X, Y] = 0\}$ denotes the *centralizer of X in $\mathfrak{g}$*.

There exists a notion of an ordering among the nilpotent orbits of a group, namely $\mathcal{O} < \mathcal{O}'$ if and only if $\overline{\mathcal{O}} < \overline{\mathcal{O}'}$ where the closure $\mathcal{O}$ of an orbit is with respect to the Zariski-topology. This ordering allows one to arrange the nilpotent orbits in a Hasse diagram. For a general group G with Lie algebra $\mathfrak{g}$, this Hasse diagram can have quite some structure in it, see for example [65].

There are a few distinguished orbits however which are possessed by all G, the largest one being called the *principal* or *regular* orbit $\mathcal{O}_{\text{prin}}$ of dimension $\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g}$. The weighted Dynkin diagram of $\mathcal{O}_{\text{prin}}$ has every node labelled 2. All nilpotent orbits are contained in the closure of the principal orbit, $\mathcal{O} \subseteq \overline{\mathcal{O}_{\text{prin}}}$. Furthermore every $\mathfrak{g}$ also has an orbit called the *subregular* orbit $\mathcal{O}_{\text{subreg}}$ sitting directly below $\mathcal{O}_{\text{prin}}$ in the Hasse diagram and with $\dim\mathcal{O}_{\text{subreg}} = \dim\mathfrak{g} - \operatorname{rank}\mathfrak{g} - 2$. Additionally, every $\mathfrak{g}$ also contains the trivial orbit $\{0\}$ which also obeys $\{0\} \subset \mathcal{O}$ for every non-trivial orbit $\mathcal{O}$. Lastly, just above the trivial orbit in the Hasse diagram sits the *minimal* orbit $\mathcal{O}_{\text{min}}$ obeying $\mathcal{O}_{\text{min}} \subseteq \overline{\mathcal{O}}$ for every non-trivial orbit $\mathcal{O}$.

In general, there is a rich structure of nilpotent orbits between the subregular and minimal orbits but for $\mathrm{A}_n = \mathfrak{sl}_n$ the ordering of the orbits can be easily read off from the partitions that label them, they all have different dimensions and the Hasse diagram is just a straight line. Given two orbits $\mathcal{O}_{[p_1,\ldots,p_n]}$ and $\mathcal{O}_{[p'_1,\ldots,p'_n]}$ of $\mathfrak{sl}_n$, we have $\mathcal{O}_{[p_1,\ldots,p_n]} > \mathcal{O}_{[p'_1,\ldots,p'_n]}$ if and only if there exists a $k \in \mathbb{N}$ such that $p_k > p'_k$ and $p_i \geq p'_i$ for $i = 1, \ldots, k-1$. For this reason, in the case of A-type algebras of rank at least 3 (and in fact also for D- and E-type[23]) there is in particular also a well defined *next-to-minimal orbit* $\mathcal{O}_{\text{ntm}}$.

For $\mathfrak{sl}_n$, the regular orbit is given by the partition $[n]$, the subregular is given by $[n-1, 1]$, the next-to-minimal is given by $[2^2, 1^{n-4}]$, the minimal is given by $[2, 1^{n-2}]$ and the trivial by $[1^n]$.

5.2 Wavefront sets and small representations

Consider a Fourier coefficient F^U_ψ of an automorphic form φ and assume that $F^U_\psi \equiv 0$. The Fourier coefficient in question belongs to some nilpotent orbit as explained above. Transforming ψ with the elements $l \in L(\mathbb{Q})$ of the discrete Levi, we can map ψ to all other characters $\psi' = \psi^l$ belonging to the same nilpotent orbit. We get $0 \equiv F^U_\psi(lg) = F^U_{\psi'}(g)$ and thus: if a nilpotent orbit has associated with it a Fourier coefficient which vanishes, all Fourier coefficients associated with that nilpotent orbit must vanish. Conversely, so long as a nilpotent orbit has associated with it a non-vanishing Fourier coefficient, all Fourier coefficients associated with nilpotent orbit must be non-vanishing. The vanishing properties of Fourier coefficients turns out to be a property of the automorphic representation π to which ϕ is attached. This leads to the notion of the wavefront set $\mathrm{WF}(\pi)$ of an automorphic representation.

Definition 5.3 (Wavefront set)
*Let π be an automorphic representation of $G(\mathbb{A})$. The **wavefront set** $\mathrm{WF}(\pi)$ of π is*

[23] "I've got life. You're always on my mind. You gotta believe in something alright."

given by

$$\mathrm{WF}(\pi) = \bigcup_i \overline{\mathcal{O}_i} \tag{5.18}$$

where the union runs over all nilpotent orbits associated with non-vanishing Fourier coefficients.

Remark 5.4.
It should be mentioned that the notion of a global wavefront set, i.e. a wavefront set for an automorphic representation of $G(\mathbb{A})$ is a more delicate concept than the wavefront set of representation of $G(\mathbb{Q}_p)$. It is conceivable that the wavefront set $\mathrm{WF}(\pi)$ of a local representation π_p of a global automorphic representation $\pi = \otimes_{p\leq\infty}\pi_p$ varies across p, although this is expected not to happen for Eisenstein series [66]. The non-archimedean case $p < \infty$ was initially studied by Mœglin-Waldspurger [67] and the archimedean case $p = \infty$ by Matumoto [68]. The connection between nilpotent orbits and Fourier coefficient in the global setting was studied further by Ginzburg in [66] and Miller-Sahi in [26].

Remark 5.5.
It is known that the nilpotent orbits contributing to the union Eq. (5.18) are so called "special"[24] [66]. For $\mathfrak{sl}_n$ however, every orbit is special so this statement becomes trivial.

Given a nilpotent orbit $\mathcal{O}_\pi$ of dimension $2n$, one can construct a function space with Gelfand-Kirillov dimension n associated with the automorphic representation π using Kirillov's so called "orbit method". This function space then has $\mathrm{WF}(\pi) = \overline{\mathcal{O}_\pi}$. An automorphic representation with a particularly small Gelfand-Kirillov dimension therefore has a particularly small wavefront set, meaning that many of its Fourier coefficients vanish. This leads to the notion of small representations and in particular the minimal and (where applicable) next-to-minimal representations.

Definition 5.6 (Minimal representation)
An automorphic representation π of a split real Lie group G satisfying

$$\mathrm{WF}(\pi) = \overline{\mathcal{O}_{\mathrm{min}}} \tag{5.19}$$

is called a ***minimal representation*** *of G. The orbit $\mathcal{O}_{\mathrm{min}}$ is uniquely defined for every every such G.*

Definition 5.7 (Next-to-minimal representation)
An automorphic representation π of a split real Lie group G of ADE-type satisfying

$$\mathrm{WF}(\pi) = \overline{\mathcal{O}_{\mathrm{ntm}}} \tag{5.20}$$

is called a ***next-to-minimal representation*** *of G. The orbit $\mathcal{O}_{\mathrm{ntm}}$ is uniquely defined for the ADE groups.*

[24] See section 6.3 of [21] for a definition of special.

Example 5.8

Consider the SL_5 Eisenstein series $E^{\mathrm{SL}_5(\mathbb{A})}_{[0,s,0,0]}$. The notation implies that the defining weight is $\lambda = 2s\Lambda_2 - \rho$. As as explained in remark 2.8, this is really a maximal parabolic Eisenstein series for the parabolic P_2 and is attached to the induced representation $\mathrm{Ind}^{G(\mathbb{A})}_{P_2(\mathbb{A})}\chi$ where $\chi(g) = \mathrm{e}^{\langle\lambda+\rho|H(g)\rangle} = \mathrm{e}^{\langle 2s\Lambda_2|H(g)\rangle}$. example 5.2 gives that the Gelfand-Kirillov dimension of this representation is

$$\mathrm{GKdim}\,\mathrm{Ind}^{G(\mathbb{A})}_{P_2(\mathbb{A})}\chi = 2\cdot 3 = 6. \tag{5.21}$$

To figure out which representation $E^{\mathrm{SL}_5(\mathbb{A})}_{[0,2,0,0]}$ belongs to, we search for a nilpotent orbit of SL_5 with dimension $2\cdot 6 = 12$. There are 7 orbits in total with the order

$$\mathcal{O}_{[5]} > \mathcal{O}_{[41]} > \mathcal{O}_{[32]} > \mathcal{O}_{[31^2]} > \mathcal{O}_{[2^21]} > \mathcal{O}_{[21^3]} > \mathcal{O}_{[1^5]}. \tag{5.22}$$

The correct orbit turns out to be $\mathcal{O}_{[2^21]}$, the next-to-minimal orbit. Let us construct this orbit to see that it indeed has the correct dimension.

We are searching for a Jacobson-Morozow triple (X, Y, H) for $\mathcal{O}_{[2^21]}$ and the partition $[2^2, 1]$ leads us to write down

$$H = \begin{pmatrix} 1 &&&& \\ & -1 &&& \\ && 1 && \\ &&& -1 & \\ &&&& 0 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & 1 &&& \\ & 0 &&& \\ && 0 & 1 & \\ &&& 0 & \\ &&&& 0 \end{pmatrix}. \tag{5.23}$$

The semisimple element H can be brought to the form

$$\tilde{H} = \begin{pmatrix} 1 &&&& \\ & 1 &&& \\ && 0 && \\ &&& -1 & \\ &&&& -1 \end{pmatrix} \tag{5.24}$$

using Weyl reflections. This form lets us read off the weighted Dynkin diagram as 0—1—1—0. The simple roots and their negatives then have weights in the H-grading according to

$$\begin{pmatrix} 0 & 0 &&& \\ 0 & 0 & 1 && \\ & -1 & 0 & 1 & \\ && -1 & 0 & 0 \\ &&& 0 & 0 \end{pmatrix}. \tag{5.25}$$

By taking commutators we can fill in the matrix and find the weights of all Chevalley generators,

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 \\ -1 & -1 & 0 & 1 & 1 \\ -2 & -2 & -1 & 0 & 0 \\ -2 & -2 & -1 & 0 & 0 \end{pmatrix}. \tag{5.26}$$

We find that $\dim\mathfrak{g}(0) = 9 - 1 = 8$ (due to tracelessness) and $\dim\mathfrak{g}(1) = 4$. Using Eq. (5.16) we get $\dim\mathcal{O}_{[2^2,1]} = \dim\mathfrak{g} - \dim\mathfrak{g}(0) - \dim\mathfrak{g}(1) = 5^2 - 1 - 8 - 4 = 12$.

This establishes that $E^{\mathrm{SL}_5(\mathbb{A})}_{[0,s,0,0]}$ is attached to an automorphic representation with wavefront set given by $\overline{\mathcal{O}_{\mathrm{ntm}}}$ which is the next-to-minimal representation of SL_5. We can therefore deduce that a Fourier coefficient associated with the orbit $\mathcal{O}_{[31^2]}$ (or higher), for example

$$\int\limits_{(\mathbb{Q}\backslash\mathbb{A})^{10}} E^{\mathrm{SL}_5(\mathbb{A})}_{[0,s,0,0]}\left(\begin{pmatrix} 1 & x_1 & x_5 & x_8 & x_{10} \\ & 1 & x_2 & x_6 & x_9 \\ && 1 & x_3 & x_7 \\ &&& 1 & x_4 \\ &&&& 1 \end{pmatrix} g\right) \mathbf{e}(m_1x_1 + m_2x_2)\,\mathrm{d}^{10}x. \tag{5.27}$$

should vanish. We also expect a Fourier coefficient associated with $\mathcal{O}_{[2^21]}$, for example

$$\int\limits_{(\mathbb{Q}\backslash\mathbb{A})^{10}} E^{\mathrm{SL}_5(\mathbb{A})}_{[0,s,0,0]}\left(\begin{pmatrix} 1 & x_1 & x_5 & x_8 & x_{10} \\ & 1 & x_2 & x_6 & x_9 \\ && 1 & x_3 & x_7 \\ &&& 1 & x_4 \\ &&&& 1 \end{pmatrix} g\right) \mathbf{e}(m_1x_1 + m_3x_3)\,\mathrm{d}^{10}x \tag{5.28}$$

should be non-vanishing, as well as the maximally degenerate coefficients associated with the minimal orbit $\mathcal{O}_{[21^3]}$.

5.3 Application: Fourier coefficient in minimal unipotent of SL_3 minimal Eisenstein series

Here we will give an example of how the adelic framework and automorphic representation theory enters into explicit calculations. Let's consider an Eisenstein series $E_\lambda^{\mathrm{SL}_3(\mathbb{A})}$ on SL_3 and try to calculate the following Fourier coefficient (we will write $E \equiv E_\lambda^{\mathrm{SL}_3(\mathbb{A})}$ for short)

$$F_{m_3'}^{U_{\min}(\mathbb{A})}(g) = \int\limits_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \mathbf{e}(m_3' x_3)\, \mathrm{d}^1 x \quad \text{with} \quad m_3' \neq 0. \tag{5.29}$$

The "min" here pertains to that the unipotent in question is the minimal unipotent. Let's first see how this expression can be related to Whittaker functions. Using the summation representation of the adelic Dirac delta function Eq. (A.2) we can insert another integration variable as follows

$$\begin{aligned} F_{m_3'}^{U_{\min}(\mathbb{A})}(g) &= \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \delta(x_1)\, \mathbf{e}(m_3' x_3)\, \mathrm{d}^2 x = \\ &= \sum_{m_1} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & \\ & & 1 \end{pmatrix} g\right) \mathbf{e}(m_1 x_1 + m_3' x_3)\, \mathrm{d}^2 x. \end{aligned} \tag{5.30}$$

With the change of variables $x_3 \to x_3 - \frac{m_1 x_1}{m_3'}$ we can eliminate the x_1-dependence in the exponential. This change of variables is simply a translation which doesn't change the integration domain (after the translation we are still integrating over a full period) and the rational coefficient $\frac{m_1}{m_3'}$ leaves the Haar-measure invariant. We find that we can factorize the resulting matrix in the argument of E as follows

$$\begin{pmatrix} 1 & x_1 & x_3 - \frac{m_1 x_1}{m_3'} \\ & 1 & \\ & & 1 \end{pmatrix} = l_1^{-1} \begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & \\ & & 1 \end{pmatrix} l_1 \quad \text{where} \quad l_1 = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \frac{-m_1}{m_3'} \end{pmatrix}. \tag{5.31}$$

The rational matrix l_1 describes both the translation of x_3 and also a Weyl reflection exchanging x_1 and x_3. We then have

$$F_{m_3'}^{U_{\min}(\mathbb{A})}(g) = \sum_{m_1} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(l_1^{-1} \begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & \\ & & 1 \end{pmatrix} l_1 g\right) \mathbf{e}(m_3' x_3)\, \mathrm{d}^2 x. \tag{5.32}$$

Next, we would like to use the automorphy (in this case left $\mathrm{SL}_3(\mathbb{Q})$-invariance) of E to cancel the l_1^{-1}. It might look bad that $\det l_1 = -1$ so $l_1 \notin \mathrm{SL}_3(\mathbb{Q})$. This is easily overcome by a change of variables, for example $x_1 \to -x_1$ which gives

$$\begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & \\ & & 1 \end{pmatrix} \to \begin{pmatrix} 1 & x_3 & -x_1 \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}\begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}. \tag{5.33}$$

The matrix $\mathrm{diag}(1, 1, -1)$ then combines with l_1 to give something with unit determinant. This technicality does not change the end result so it will be ignored here. Note that cancelling the rational matrix l_1^{-1} would have been impossible without the adelic framework, since the real Eisenstein series is only left $\mathrm{SL}_3(\mathbb{Z})$-invariant. We thus have

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_1} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(\begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & \\ & & 1 \end{pmatrix} l_1 g\right) \mathbf{e}(m_3' x_3)\, \mathrm{d}^2 x. \tag{5.34}$$

We proceed by introducing another integration variable by inserting another Dirac delta function

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_1, m_2} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^3} E\left(\begin{pmatrix} 1 & x_3 & x_1 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} l_1 g\right) \mathbf{e}(m_2 x_2 + m_3' x_3)\, \mathrm{d}^3 x. \tag{5.35}$$

Note that the unipotent is now "full" and we have reached the form of a Whittaker function

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_1, m_2} W_{m_3', m_2}(l_1 g). \tag{5.36}$$

This calculation shows how we started with a Fourier coefficient on a unipotent $U \subset N$ and related it to a sum of Whittaker functions. This is valuable in and of itself since Whittaker functions have been studied in more detail in the literature and more about their behavior is known.

Let us now see what happens if we choose the Eisenstein series E such that it is attached to the minimal representation of $\mathrm{SL}_3(\mathbb{A})$. A calculation analogous to example 5.8 shows that $\lambda = 2s\Lambda_1 - \rho$ puts $E^{\mathrm{SL}_3(\mathbb{A})}_\lambda$ in the minimal representation. Our knowledge about wavefront sets, namely that $\mathrm{WF}(\pi_{\min}) = \overline{\mathcal{O}_{\min}} = \overline{\mathcal{O}_{[2,1]}}$ for SL_3 tells us that the only non-vanishing Whittaker functions in Eq. (5.36) are the (maximally) degenerate ones $W_{m_3', 0}$. For an Eisenstein series in the minimal representation, we thus have

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_1} W_{m_3', 0}(l_1 g). \tag{5.37}$$

Next, we will use the reduction formula theorem 4.43 to evaluate these degenerate Whittaker functions. The formula is written here again for convenience,

$$W_\psi(a) = \sum_{w_c w_0' \in \mathcal{C}_\psi} a^{(w_c w_0')^{-1}\lambda + \rho} M(w_c^{-1}, \lambda) W'_{\psi^a}(w_c^{-1}\lambda, \mathbb{1}) \tag{5.38}$$

	w_c	$\langle w_c^{-1}\lambda + \rho \vert \alpha_1 \rangle$	$M(w_c^{-1}, \lambda)$	$(w_c w_0')^{-1}\lambda + \rho$
$*$	Id	$2s$	1	$[2-2s, 2s-1]$
	w_{12}	0	$\frac{\xi(2s-2)}{\xi(2s)}$	$[2, 2-2s]$

Table 5.1: Data for the reduction formula Eq. (6.19) to evaluate $W_{m_3',0}(a)$ on SL_3 with $\lambda = 2s\Lambda_1 - \rho$. The star indicates the Weyl word that contributes to the reduction formula.

We start by writing (bearing in mind that these are adelic quantities)

$$l_1 g = l_1 nak = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & \frac{-m_1}{m_3'} \end{pmatrix} \begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & \\ & y_2/y_1 & \\ & & 1/y_2 \end{pmatrix} k = \underbrace{\begin{pmatrix} 1 & x_1' & x_3' \\ & 1 & x_2' \\ & & 1 \end{pmatrix}}_{n'} \underbrace{\begin{pmatrix} y_1' & & \\ & y_2'/y_1' & \\ & & 1/y_2' \end{pmatrix}}_{a'} k' \tag{5.39}$$

which gives

$$W_{m_3',0}(l_1 g) = \mathrm{e}^{2\pi \mathrm{i} m_3 x_1'} W_{m_3',0}(a'). \tag{5.40}$$

The Weyl group of SL_3 has $3! = 6$ elements, denoted

$$\begin{aligned} \mathbb{1} &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad w_1 = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = w_0', \quad w_2 = \begin{pmatrix} 1 & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix} \\ w_{12} &= \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad w_{21} = \begin{pmatrix} & -1 & \\ & & -1 \\ 1 & & \end{pmatrix}, \quad w_{121} = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix} = w_{212}. \end{aligned} \tag{5.41}$$

The degenerate Whittaker function $W_{m_3',0}$ has the associated group $G' = \mathrm{SL}_2$ and the longest Weyl word of G' is the one denoted w_0' above. Among the six Weyl words, only the two words w_1 and w_{121} satisfy the property $w\Pi' < 0$, or in this case $w\alpha_1 < 0$. Written in the form $w = w_c w_0'$ we get that the sum over Weyl words in the reduction formula contains two terms, namely $w_c = \mathbb{1}$ and $w_c = w_{12}$. The next step is to calculate the quantity $(a')^{(w_c w_0')^{-1}\lambda + \rho}$ for $\lambda = 2s\Lambda_1 - \rho$ and $\rho = \Lambda_1 + \Lambda_2$. This quantity determines the y'-dependencies of $W_{m_3',0}(a')$. We will also need the numerical factor $M(w_c^{-1}, \lambda)$ given by Eq. (2.56). Finally, the generic Whittaker function $W'_{\psi^{a'}}$ is with respect to an Eisenstein series on G' with weight vector $w_c^{-1}\lambda$ projected onto the root space of G'. This projection is given by the inner product $\langle w_c^{-1}\lambda + \rho | \alpha_1 \rangle$. These calculations are straightforward algebraic exercises involving the Killing form. The results are shown in table 5.1.

From the data in the table, we get that the term coming from $w_c = \mathbb{1}$ is

$$
\begin{aligned}
&\left|y_1'\right|_\infty^{2-2s} \left|y_2'\right|_\infty^{2s-1} 1 \\
&\frac{2\pi^s}{\Gamma(s)} \left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty^{s-1/2} K_{s-1/2}\left(2\pi\left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty\right) \\
&\prod_p \gamma_p\left(\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right)\left(1-p^{-2s}\right)\frac{1-p^{-2s+1}\left|\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right|_p^{2s-1}}{1-p^{-2s+1}}.
\end{aligned}
\tag{5.42}
$$

The first line constitutes the factor $a^{(w_c w_0')^{-1}\lambda+\rho}$ and $M(w_c^{-1}, \lambda)$. The second and third lines are the adelic SL_2 Whittaker function evaluated at $\mathbb{1}$ with weight vector $w_c^{-1}\lambda-\rho$ for $w_c = \mathbb{1}$. This quantity was calculated in Section 4.5.1 and summarized in Appendix C. The fact that $\langle w_c^{-1}\lambda+\rho|\alpha_1\rangle = 2s$ for $w_c = \mathbb{1}$ means that we can take Eq. (C.27) at face value, setting $x = 0$ and $y = 1$ since W'_{ψ^a} is to be evaluated at unity. The notation $\psi^{a'}$ denotes the "twisted" character

$$
\psi^{a'}(n) = \psi(a'n(a')^{-1})
\tag{5.43}
$$

as discussed in Appendix C. As explained in there, the effect of the twisted character is to replace the charge m_3' by $m_3'\frac{(y_{1,p}')^2}{y_{0,p}'y_{2,p}'}$ at all local places $p \leq \infty$, where $y_{0,p}' = 1$. In a moment we will set $g = (g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)$ and solve for x_3', y_1' and y_2' to get an expression for the real version of the Fourier coefficient in Eq. (5.29) for the minimal SL_3 Eisenstein series, but before that we shall see that term coming from the Weyl word w_{12} in table 5.1 vanishes.

The inner product $w_c^{-1}\lambda - \rho = 0 = 2 \cdot 0$ for $w_c = w_{12}$ tells us to substitute $s \to 0$ in Eq. (C.27) and we are led to write down

$$
\begin{aligned}
&\left|y_1'\right|_\infty^{2} \left|y_2'\right|_\infty^{2-2s} \frac{\xi(2s-2)}{\xi(2s)} \\
&\frac{2\pi^0}{\Gamma(0)} \left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty^{-1/2} K_{-1/2}\left(2\pi\left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty\right) \\
&\prod_p \gamma_p\left(\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right)\left(1-p^{0}\right)\frac{1-p^{1}\left|\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right|_p^{1}}{1-p^{1}},
\end{aligned}
\tag{5.44}
$$

Note that the Euler product over $(1-p^{-2s})$ combines with $\frac{\pi^s}{\Gamma(s)}$ to give a factor of $\frac{1}{\xi(2s)}$ as explained in Appendix C. Recall that the Riemann zeta function $\zeta(s)$ (just like the completed Riemann zeta function $\xi(s)$) is analytic in the whole complex plane excepet in the points $s = 0$ and $s = 1$ where it exhibits simple poles. The above expression effectively contains the factor $\frac{1}{\xi(0)} = 0$ which makes it vanish. It is good to notice however that factors of $\xi(0)$ or $\xi(1)$ can be tolerated in the denominator if they are simultaneously compensated by such factors in the numerator coming from the intertwiner $M(w_c^{-1}, \lambda)$. This mechanism plays an important role in [50, 51].

Remark 5.9.
Note that we are calculating a Fourier coefficient in the minimal orbit for an automorphic representation belonging to the minimal representation and found that only one Weyl word contributes in the reduction formula. This is a general feature and similarly Fourier coefficients in the next-to-minimal orbit of automorphic representations in the next-to-minimal automorphic representation also have only one Weyl word contributing. This can be seen for example in tables 6.1 to 6.4.

We now know that we have the equality (on the level of adeles)

$$\begin{aligned} F_{m_3'}^{U_{\min}(\mathbb{A})}(g) = \sum_{m_1} \left|y_1'\right|_\infty^{2-2s} \left|y_2'\right|_\infty^{2s-1} \\ \frac{2\pi^s}{\Gamma(s)} \left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty^{s-1/2} K_{s-1/2}\left(2\pi\left|\frac{m_3'(y_{1,\infty}')^2}{y_{2,\infty}'}\right|_\infty\right) \\ \prod_p \gamma_p\left(\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right)(1-p^{-2s})\frac{1-p^{-2s+1}\left|\frac{m_3'(y_{1,p}')^2}{y_{2,p}'}\right|_p^{2s-1}}{1-p^{-2s+1}}. \end{aligned} \tag{5.45}$$

Let us now set $g = (g_\infty; \mathbb{1}, \mathbb{1}, \mathbb{1}, \dots)$. We must write the argument $l_1 g$ in Iwasawa form at all local places. Starting with the archimedean place $p = \infty$, using theorem 2.2 we have

$$\begin{aligned} &\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & \frac{m_1}{m_3'} & 1 \end{pmatrix}\begin{pmatrix} 1 & x_{1,\infty} & x_{3,\infty} \\ & 1 & x_{2,\infty} \\ & & 1 \end{pmatrix}\begin{pmatrix} y_{1,\infty} & & \\ & y_{2,\infty}/y_{1,\infty} & \\ & & 1/y_{2,\infty} \end{pmatrix} k_\infty = \\ &= \begin{pmatrix} 1 & x_{1,\infty}' & x_{3,\infty}' \\ & 1 & x_{2,\infty}' \\ & & 1 \end{pmatrix}\begin{pmatrix} y_{1,\infty}' & & \\ & y_{2,\infty}'/y_{1,\infty}' & \\ & & 1/y_{2,\infty}' \end{pmatrix} k_\infty' \end{aligned} \tag{5.46}$$

where

$$\begin{aligned} x_{1,\infty}' &= x_{3,\infty} + \frac{m_1}{m_3'} x_{1,\infty} - x_1 x_2 \\ y_{1,\infty}' &= y_{1,\infty} \\ y_{2,\infty}' &= \sqrt{\frac{1}{\left(\frac{m_3 x_{2,\infty} - m_1}{m_3 y_{2,\infty}}\right)^2 + \left(\frac{y_{2,\infty}}{y_{1,\infty}}\right)^2}} \end{aligned} \tag{5.47}$$

At the non-archimedean places $p < \infty$, theorem 4.29 gives

$$\begin{pmatrix} 1 & & \\ 0 & 1 & \\ 1 & \frac{-m_1}{m_3'} & 1 \end{pmatrix}\mathbb{1} = \begin{pmatrix} 1 & & \\ & 1 & x_{2,p}' \\ & & 1 \end{pmatrix}\begin{pmatrix} y_{1,p}' & & \\ & y_{2,p}'/y_{1,p}' & \\ & & 1/y_{2,p}' \end{pmatrix} k_p' \tag{5.48}$$

where

$$|y'_{1,p}|_p = 1 \quad \text{and} \quad |y'_{2,p}|_p^{-1} = \max\left(\left|\frac{m_1}{m'_3}\right|_p, 1\right). \tag{5.49}$$

Starting with the p-adic gaussian γ_p of Appendix C, we find that the argument has norm

$$\left|\frac{m'_3(y'_{1,p})^2}{y'_{2,p}}\right|_p = \max(|m_1|_p, |m'_3|_p) = |k|_p \quad \text{where} \quad k = \gcd(m_1, m'_3) \tag{5.50}$$

as shown in Eq. (C.10). One may worry about whether the greatest common denominator is well defined since in general m_1 and m'_3 are rational numbers. This is taken care of due to the p-adic gaussians

$$\prod_p \gamma_p\left(\frac{m'_3(y'_{1,p})^2}{y'_{2,p}}\right) = \prod_p \gamma_p(m_1)\gamma_p(m'_3) \tag{5.51}$$

as discussed around Eq. (C.20). This Euler-product restricts the m_1-sum to be a sum over integers rather than rationals and is also restricts m'_3 to be integer. The greatest common divisor in Eq. (5.50) is thus well defined. The rest of the expression reassembles in accordance with Eq. (C.30) and after simplifications we are left with

$$\begin{aligned} &F^{U_{\min}(\mathbb{R})}_{m'_3}(g_\infty) = \\ &\quad = \sum_{m_1=-\infty}^{\infty} \frac{2}{\xi(2s)} y_{1,\infty} y_{2,\infty}^{2s-1} e^{2\pi i\left((x_{3,\infty} - x_{1,\infty}x_{2,\infty})m'_3 + x_{1,\infty}m_1\right)} \\ &\qquad \left(|m'_3|_\infty \frac{y_{1,\infty}}{y_{2,\infty}\sqrt{y_{2,\infty}^4 + y_{1,\infty}^2\left(x_{2,\infty} - \frac{m_1}{m'_3}\right)^2}}\right)^{s-1/2} \\ &\qquad K_{s-1/2}\left(2\pi|m'_3|_\infty \frac{y_{1,\infty}\sqrt{y_{2,\infty}^4 + y_{1,\infty}^2\left(x_{2,\infty} - \frac{m_1}{m'_3}\right)^2}}{y_{2,\infty}}\right) \\ &\qquad \sigma_{1-2s}(k)\left(\frac{k}{m'_3}\right)^{2s-1} \end{aligned} \tag{5.52}$$

where we have used

$$|y_{2,p}|_p = \left|\frac{m'_3}{m'_3 y_{2,p}}\right|_p^{-1} = \left|\frac{m'_3}{k}\right|_p \tag{5.53}$$

as well as Eq. (C.5).

5.4 Application: Deriving Bessel identities

The virtue of the calculation above was that we were able to relate a Fourier coefficient over a non-maximal unipotent to Whittaker functions, for which we can use the reduction formula theorem 4.43 together with the explicit Iwasawa formulae theorems 2.2 and 4.29 to obtain explicit expressions. It is interesting to note that there is some freedom in how the Fourier coefficient gets related to Whittaker functions. Consider Eq. (5.30), rather than inserting the new integration variable at the $(1,2)$-place, we could have placed it at the $(2,3)$-place, according to

$$\begin{aligned} F^{U_{\min}(\mathbb{A})}_{m_3'}(g) &= \sum_{m_2} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(\begin{pmatrix} 1 & & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix} g\right) \mathbf{e}(m_2 x_2 + m_3' x_3)\, \mathrm{d}^2 x = \\ &= \sum_{m_2} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^2} E\left(\begin{pmatrix} 1 & & x_2 \\ & 1 & x_3 \\ & & 1 \end{pmatrix} l_2 g\right) \mathbf{e}(m_3' x_3)\, \mathrm{d}^2 x \quad \text{where} \quad l_2 = \begin{pmatrix} 0 & 1 & \\ 1 & \frac{m_2}{m_3'} & \\ & & 1 \end{pmatrix}. \end{aligned} \tag{5.54}$$

Going this route lets us relate the Fourier coefficient to Whittaker functions as

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_1, m_2} W_{m_1, m_3'}(l_2 g) \tag{5.55}$$

which for an Eisenstein series in the minimal representation gives

$$F^{U_{\min}(\mathbb{A})}_{m_3'}(g) = \sum_{m_2} W_{0, m_3'}(l_2 g). \tag{5.56}$$

Using the reduction formula and projecting down to the reals in the same way as demonstrated above then gives the expression

$$\begin{aligned} &F^{U_{\min}(\mathbb{R})}_{m_3'}(g_\infty) = \\ &\quad \sum_{m_2=-\infty}^{\infty} \frac{2}{\xi(2s)} y_{2,\infty}^{3-2s} e^{2\pi i\left(x_{3,\infty} m_3' + x_{2,\infty} m_2\right)} \\ &\qquad \left(|m_3'|_\infty \frac{y_{2,\infty}\sqrt{y_{1,\infty}^4 + y_{2,\infty}^2\left(x_{1,\infty} + \frac{m_2}{m_3'}\right)^2}}{y_{1,\infty}} \right)^{s-1} \\ &\qquad K_{s-1}\left(2\pi |m_3'|_\infty \frac{y_{2,\infty}\sqrt{y_{1,\infty}^4 + y_{2,\infty}^2\left(x_{1,\infty} + \frac{m_2}{m_3'}\right)^2}}{y_{1,\infty}} \right) \sigma_{2-2s}(k). \end{aligned} \tag{5.57}$$

Note that the order of the Bessel function has been shifted by 1/2. This is a general feature when moving charges around on the nodes of a Whittaker function.

Since Eq. (5.52) and Eq. (5.57) express the same quantity in different ways, we must have that the right hand sides are equal. This amounts to a non-trivial equality which most notably relates Bessel functions of shifted orders. After some cancellations and simplifications, this equality is most conveniently stated as

$$f(x_1, x_3, y_1, y_2, s, m) = f(-x_3, x_1, y_2, y_1, -(s + 1/2), m) \tag{5.58}$$

where the function f is defined as

$$\begin{aligned} f(x_1, x_3, y_1, y_2, s, m) &\equiv \\ &\equiv y_2^{1-2s} e^{\pi i x_1 x_3 m} \sum_{q\in\mathbb{Z}} e^{2\pi i x_3 q} \left(|m|_\infty y_1^{-1} y_2 \sqrt{y_1^4 + y_2^2(q/m + x_1)^2} \right)^s \\ &\qquad \sigma_{-2s}(k) K_s \left(2\pi |m|_\infty y_1^{-1} y_2 \sqrt{y_1^4 + y_2^2(q/m + x_1)^2} \right) \end{aligned} \tag{5.59}$$

and we have dropped the ∞-superscript on the real variables. The variables take values according to

$$x_1, x_3 \in \mathbb{R}, \quad y_1, y_2 \in \mathbb{R}_+, \quad s \in \mathbb{C}, \quad m \in \mathbb{Z} \quad \text{and} \quad k = \gcd(q, m). \tag{5.60}$$

The example outlined here is the simplest example of how the adelic formalism together with the knowledge about the wavefront set of automorphic representations can be used to derive interesting equalities. By considering larger groups, for example SL_5, a coefficient like

$$F^{U_{\min}(\mathbb{A})}_{m'_{10}}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} E\left(\begin{pmatrix} 1 & & & x_{10} & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix} g\right) \mathbf{e}(m'_{10} x_{10})\, d^1x \tag{5.61}$$

for an Eisenstein series in the minimal representation can be related to the Whittaker functions $W_{m'_{10},0,0,0}$, $W_{0,m'_{10},0,0}$, $W_{0,0,m'_{10},0}$ or $W_{0,0,0,m'_{10}}$ which would give equalities schematically of the form

$$\cdots \sum K_s \cdots = \cdots \sum K_{s+1/2} \cdots = \cdots \sum K_{s+1} \cdots = \cdots \sum K_{s+3/2} \cdots . \tag{5.62}$$

For an Eisenstein series in the next-to-minimal representation, the summand in each of these four expressions would involve a product of Bessel functions. These technology in this thesis is sufficient to derive these identities and those of SL_n for arbitrary n. It is an interesting question whether or not the identities produced by considering SL_n are contained within those produced from SL_{n+1}.

6 Formalism for SL_n

In this chapter, we present the main results of **Paper II**. Sections 5.3 and 5.4 provide a good frame of reference for understanding how these results were obtained and what possible additional results could be obtained. Before proceeding, it is worth to point out that the discussion about nilpotent orbits in Section 5.1.1 from [21] for SL_n, in particular parametrizing the orbits in terms of partitions of n was for the real case $\mathrm{SL}_n(\mathbb{R})$. The parametrization is slightly different for the rational case $\mathrm{SL}_n(\mathbb{Q})$ and is discussed in [69]. The main result is

Proposition 6.1 (Proposition 4, [69])
Let $\underline{p} = [p_1 p_2 \dots p_r]$ be an ordered partition of n, with $p_1 \geq p_2 \geq \dots \geq p_r$ and let $m = \gcd(\underline{p}) = \gcd(p_1, p_2, \dots, p_r)$. For $d \in \mathbb{Q}^\times$, define $D(d) = \mathrm{diag}(1, 1, \dots, 1, d)$ and let also $J_{\underline{p}}$ be the standard (lower triangular) Jordan matrix corresponding to $\underline{p}$: $J_{\underline{p}} = \mathrm{diag}(J_{[p_1]}, J_{[p_2]}, \dots, J_{[p_r]})$, where $J_{[p]}$ is a $p \times p$ matrix with non-zero elements only on the subdiagonal which are one, for example $J_{[3]} = \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$

1. *For each $d \in F^\times$, the matrix $D(d)J_{\underline{p}}$ is a representative of a rational nilpotent orbit of SL_n parametrized by $\underline{p}$, and conversely, every orbit parametrized by $\underline{p}$ has a representative of this form. We say that the rational orbit represented by $D(d)J_{\underline{p}}$ is parametrized by $(\underline{p}, d)$.*

2. *The $\mathrm{SL}_n(\mathbb{Q})$-orbits represented by $D(d)J_{\underline{p}}$ and $D(d')J_{\underline{p'}}$ coincide if and only if $\underline{p} = \underline{p'}$ and $d \equiv d'$ in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^m$.*

Since we are interested in Fourier coefficients over a maximal parabolic subgroup P_m, we need the $L_m(\mathbb{Q})$-orbits rather than the full $\mathrm{SL}_n(\mathbb{Q})$-orbits. As discussed in **Paper II**, the $L_m(\mathbb{Q})$-orbits are characterized by a partition $[2^r 1^{n-2r}]$ together with a number $d \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^k$. Here, r takes values $0 \leq r \leq \min(m, n-m)$ and $k = \gcd([2^r 1^{n-2r}])$. We let $y(Y_r(d))$ denote convenient representatives of these $L_m(\mathbb{Q})$-orbits. Note however that any Fourier coefficient associated with a nilpotent orbit where $r > 2$ lies outside the wavefront set for automorphic forms attached to the minimal or next-to-minimal representations of SL_n for $n \geq 5$. By restricting to such automorphic forms we have $k = 1$ and thus $d = 1$, and the only representatives we need here are[25]

$$\begin{aligned} y(Y_1) &= \begin{pmatrix} 0_m & Y_1 \\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{pmatrix} \quad \text{where} \quad Y_1 = \begin{pmatrix} 0_{(m-1)\times 1} & 0_{(m-1)\times(n-m-1)} \\ 1 & 0_{1\times(n-m-1)} \end{pmatrix} \quad \text{and} \\ y(Y_2) &= \begin{pmatrix} 0_m & Y_2 \\ 0_{(n-m)\times m} & 0_{(n-m)\times(n-m)} \end{pmatrix} \quad \text{where} \quad Y_2 = \begin{pmatrix} 0_{(m-2)\times 1} & 0_{(m-2)\times(n-m-2)} \\ \begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix} & 0_{1\times(n-m-2)} \end{pmatrix}. \end{aligned} \tag{6.1}$$

[25] For $d = 1$, we omit d when writing $Y_r(d)$.

The representatives above are used as data in specifying the multiplicative characters $\psi_{y(Y_r(d))}$ on the unipotent U_m. As an element of the character variety, the non-zero entries in $y(Y_r(d))$ label which matrix elements in the unipotent U_m are charged, for example for $n = 5$ we have

$$\psi_{y(Y_2)}\left(\begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}\right) = \mathbf{e}(x_2 + x_6) \tag{6.2}$$

See **Paper II** for a discussion on the general $L_m(\mathbb{Q})$-orbits $([2^r 1^{n-2r}], d)$ and the form of their representatives $y(Y_r(d))$.

We will have two arithmetic subgroups of $L_m(\mathbb{Q})$ acting on these representatives. Letting ψ_0 be a character on the maximal unipotent N and T be the diagonal elements of $\mathrm{SL}_n(\mathbb{Q})$, we let T_{ψ_0} denote the stabilizer of ψ_0 under the action $[h\circ\psi_0](n) = \psi_0(hnh^{-1})$ for $h \in T$.

The first arithmetic subgroup is

$$\Gamma_i(\psi_0) := \begin{cases} (\mathrm{SL}_{n-i}(\mathbb{Q}))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(\mathbb{Q}) & 1 \le i \le n-2 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_0} & i = n-1\,, \end{cases} \tag{6.3}$$

where $(\mathrm{SL}_{n-i}(\mathbb{Q}))_{\hat{Y}}$ is the stabilizer of $\hat{Y} = (1, 0, 0, \ldots, 0)^{\mathsf{T}} \in \mathrm{Mat}_{(n-i)\times 1}(\mathbb{Q})$ and consists of elements $\left(\begin{smallmatrix} 1 & \xi \\ 0 & h \end{smallmatrix}\right)$, with $h \in \mathrm{SL}_{n-i-1}(\mathbb{Q})$ and $\xi \in \mathrm{Mat}_{1\times(n-i-1)}(\mathbb{Q})$.

The second arithmetic subgroup is

$$\Lambda_j(\psi_0) := \begin{cases} (\mathrm{SL}_j(\mathbb{Q}))_{\hat{X}} \backslash \mathrm{SL}_j(\mathbb{Q}) & 2 \le j \le n-1 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_0} & j = 1\,, \end{cases} \tag{6.4}$$

where $(\mathrm{SL}_j(\mathbb{Q}))_{\hat{X}}$ is the stabilizer of $\hat{X} = (0, \ldots, 0, 1) \in \mathrm{Mat}_{1\times j}(\mathbb{Q})$ with respect to right multiplication. In Appendix D we derive representatives for each of these four cosets. Whenever $\psi_0 \equiv 1$ we write $\Gamma_i(1)$ as Γ_i and $\Lambda_j(1)$ as Λ_j.

Associated with the arithmetic subgroups Γ_i and Λ_j we also have the embeddings $\iota, \hat{\iota} : \mathrm{SL}_{n-i} \to \mathrm{SL}_n$ respectively for any $0 \le i \le n-1$. These embeddings are defined by

$$\iota(\gamma) = \begin{pmatrix} I_i & 0 \\ 0 & \gamma \end{pmatrix} \qquad \hat{\iota}(\gamma) = \begin{pmatrix} \gamma & 0 \\ 0 & I_i \end{pmatrix}\,, \tag{6.5}$$

and we have suppressed their dependence on i for brevity. Note that for $i = 0$, they are just the identity maps for SL_n.

The central result in **Paper II** is a formula expressing the unramified Fourier coefficients over a maximal parabolic subgroup for automorphic forms in the minimal or next-to-minimal representations of SL_n for $n \ge 5$. For results on $n = 3$ and $n = 4$, see [70]. The case $n = 2$ does not arise since the only parabolic subgroup is the Borel subgroup. The main result of **Paper II** reads as follows (the notation is explained below)

Theorem 6.2 (Main theorem of **Paper II**)
Let π be a minimal or next-to-minimal irreducible automorphic representation of $\mathrm{SL}_n(\mathbb{A})$*, and let r_π be* 1 *or* 2 *respectively (which denotes the maximal rank of the character matrix Y_r). Furthermore let $\varphi \in \pi$, P_m be the maximal parabolic subgroup associated with the simple root α_m and $U \equiv U_m$ and L_m be the unipotent radical and Levi subgroup respectively. Let ψ_U be a non-trivial character on U_m with Fourier coefficient*

$$\mathcal{F}_U(\varphi, \psi_U; g) = \int\limits_{U_m(\mathbb{Q})\backslash U_m(\mathbb{A})} \varphi(ug)\psi_U^{-1}(u)\, \mathrm{d}u.$$

Then, there exists an element $l \in L_m(\mathbb{Q})$ such that

$$\mathcal{F}_U(\varphi, \psi_U; g) = \mathcal{F}_U(\varphi, \psi_{y(Y_r(d))}; lg)$$

for the standard character $\psi_{y(Y_r(d))}$ described above.

Additionally, all $\mathcal{F}_U(\varphi, \psi_{y(Y_r(d))}; lg)$ for $r > r_\pi$ vanish identically. The remaining (non-constant) coefficients can be expressed in terms of Whittaker functions as follows:

(i) *If $\pi = \pi_{\mathrm{min}}$:*

$$\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\psi_{\alpha_m}^{-1}(n)\, \mathrm{d}n. \tag{6.6}$$

(ii) *If $\pi = \pi_{\mathrm{ntm}}$:*

$$\begin{aligned}\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = &\int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\psi_{\alpha_m}^{-1}(n)\, \mathrm{d}n\, + \\ &+ \sum_{j=1}^{m-2} \sum_{\gamma \in \Lambda_j(\psi_{\alpha_m})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(n\hat{\iota}(\gamma)g)\psi_{\alpha_j,\alpha_m}^{-1}(n)\, \mathrm{d}n\, + \\ &+ \sum_{i=m+2}^{n-1} \sum_{\gamma \in \Gamma_i(\psi_{\alpha_m})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(n\iota(\gamma)g)\psi_{\alpha_m,\alpha_i}^{-1}(n)\, \mathrm{d}n.\end{aligned} \tag{6.7}$$

(iii) *If $\pi = \pi_{\mathrm{ntm}}$:*

$$\mathcal{F}_U(\varphi, \psi_{y(Y_2)}; g) = \int\limits_{C(\mathbb{A})} \int\limits_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(n\omega cg)\psi_{\alpha_1,\alpha_3}^{-1}(n)\, dn\, dc\,, \tag{6.8}$$

where ω is the Weyl element mapping a torus element according to

$$(t_1, t_2, \ldots, t_n) \mapsto (t_{m-1}, t_{m+2}, t_m, t_{m+1}, t_1, t_2, \ldots, t_{m-2}, t_{m+3}, t_{m+4}, \ldots, t_n)\,,$$

and the subgroup C of U_m is given by

$$C = C_1 C_2 \quad \textit{where} \quad C_1 = X_{e_m - e_{m+2}} \prod_{i=1}^{m-2} X_{e_i - e_{m+2}} \quad \textit{and} \quad C_2 = \prod_{i=1}^{m-2} X_{e_i - e_{m+1}}. \tag{6.9}$$

Here, $X_{e_i - e_j}$ denotes the one dimensional abelian subgroup of SL_n with unit diagonal and one entry on row-i and column-j.

Proof (sketch). The proof of theorem 6.2 in **Paper II** is split over a number of lemmas. To convey the idea of the proof we will look at the special case $n = 5$ and $m = 3$. Let us begin with $r = 1$ and consider the Fourier coefficient

$$\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^6} \varphi\left(\begin{pmatrix} 1 & & & u_8 & u_{10} \\ & 1 & & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} g\right) \overline{\mathbf{e}(u_3)}\, \mathrm{d}^6 u. \tag{6.10}$$

Note the shorthand $\psi_{\alpha_3}(n) = \mathbf{e}(n_3)$ for $n \in N$ which we will be using. The first step is to fill out the next column to the left of U_3. This is done by "multiplying by one" as demonstrated for example in Eq. (5.30). In this case, it involves introducing two additional integration variables u_2 and u_5 and thus summing over all pairs of rational charges for these matrix elements, equivalent to a sum over characters. We separate the term with zero charges (the trivial character) from the rest of the sum. Just like shown in Section 5.3, this remaining sum over rationals (different characters) can now be brought to a sum over a subgroup of $L_3(\mathbb{Q})$ with each term using a standard character and the argument of φ being translated by the rational matrices. It is a non-trivial fact that the correct arithmetic subgroup is $\Lambda_2(\psi_{\alpha_3})$. We obtain

$$\begin{aligned} \mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = &\sum_{\gamma_2 \in \Lambda_2(\psi_{\alpha_3})} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^8} \varphi\left(\begin{pmatrix} 1 & & u_5 & u_8 & u_{10} \\ & 1 & u_2 & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \hat{\iota}(\gamma_2) g\right) \overline{\mathbf{e}(u_2 + u_3)}\, \mathrm{d}^8 u \\ &+ \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^8} \varphi\left(\begin{pmatrix} 1 & & u_5 & u_8 & u_{10} \\ & 1 & u_2 & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} g\right) \overline{\mathbf{e}(u_3)}\, \mathrm{d}^8 u. \end{aligned} \tag{6.11}$$

This procedure is now repeated on both terms for the next column to the left to obtain

$$\begin{aligned} &\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = \\ &= \sum_{\substack{\gamma_2 \in \Lambda_2(\psi_{\alpha_3}) \\ \gamma_1 \in \Lambda_1(\psi_{\alpha_3})}} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^9} \varphi\left(\begin{pmatrix} 1 & u_1 & u_5 & u_8 & u_{10} \\ & 1 & u_2 & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \hat{\iota}(\gamma_1)\hat{\iota}(\gamma_2) g\right) \overline{\mathbf{e}(u_1 + u_2 + u_3)}\, \mathrm{d}^9 u \\ &+ \sum_{\gamma_2 \in \Lambda_2(\psi_{\alpha_3})} \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^9} \varphi\left(\begin{pmatrix} 1 & u_1 & u_5 & u_8 & u_{10} \\ & 1 & u_2 & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} \hat{\iota}(\gamma_2) g\right) \overline{\mathbf{e}(u_2 + u_3)}\, \mathrm{d}^9 u + \dots . \end{aligned} \tag{6.12}$$

The ellipsis denotes two additional terms coming from the second line in Eq. (6.11). The next steps are to sequentially expand along the empty rows. This works analogously with the only difference that the correct arithmetic subgroups of $L_m(\mathbb{Q})$ are now the Γ_i's. In this case we only have one row remaining and after expansion we have a full unipotent and have thus reached the form of a Whittaker function. We will get eight terms in total, they are

$$\begin{aligned}
\mathcal{F}_U(\varphi, \psi_{y(Y_1)}; g) = &\sum_{\substack{\gamma_1 \in \Lambda_1(\psi_{\alpha_3}) \\ \gamma_2 \in \Lambda_2(\psi_{\alpha_3}) \\ \gamma_4 \in \Gamma_4(\psi_{\alpha_3})}} W_{\alpha_1,\alpha_2,\alpha_3,\alpha_4}(\iota(\gamma_4)\hat{\iota}(\gamma_1)\hat{\iota}(\gamma_2)g) \\
&+ \sum_{\substack{\gamma_2 \in \Lambda_2(\psi_{\alpha_3}) \\ \gamma_4 \in \Gamma_4(\psi_{\alpha_3})}} W_{\alpha_2,\alpha_3,\alpha_4}(\iota(\gamma_4)\hat{\iota}(\gamma_2)g) \\
&+ \sum_{\substack{\gamma_1 \in \Lambda_1(\psi_{\alpha_3}) \\ \gamma_4 \in \Gamma_4(\psi_{\alpha_3})}} W_{\alpha_1,\alpha_3,\alpha_4}(\iota(\gamma_4)\hat{\iota}(\gamma_1)g) \\
&+ \sum_{\gamma_4 \in \Gamma_4(\psi_{\alpha_3})} W_{\alpha_3,\alpha_4}(\iota(\gamma_4)g) \\
&+ \sum_{\substack{\gamma_1 \in \Lambda_1(\psi_{\alpha_3}) \\ \gamma_2 \in \Lambda_2(\psi_{\alpha_3})}} W_{\alpha_1,\alpha_2,\alpha_3,}(\hat{\iota}(\gamma_1)\hat{\iota}(\gamma_2)g) \\
&+ \sum_{\gamma_2 \in \Lambda_2(\psi_{\alpha_3})} W_{\alpha_2,\alpha_3,}(\hat{\iota}(\gamma_2)g) \\
&+ \sum_{\gamma_1 \in \Lambda_1(\psi_{\alpha_3})} W_{\alpha_1,\alpha_3,}(\hat{\iota}(\gamma_1)g) \\
&+ W_{\alpha_3}(g).
\end{aligned} \tag{6.13}$$

For an automorphic form $\varphi_{\min}$ in the minimal representation, only maximally degenerate Whittaker functions are non-vanishing and we are left with

$$\mathcal{F}_U(\varphi_{\min}, \psi_{y(Y_1)}; g) = W_{\alpha_3}(g). \tag{6.14}$$

For an automorphic form φ_{ntm} in the next-to-minimal representation, we can tolerate doubly charged Whittaker functions as long as the charges sit on commuting roots, as discussed in Section 5.2 and we get

$$\mathcal{F}_U(\varphi_{\mathrm{ntm}}, \psi_{y(Y_1)}; g) = \sum_{\gamma_1 \in \Lambda_1(\psi_{\alpha_3})} W_{\alpha_1,\alpha_3,}(\hat{\iota}(\gamma_1)g) + W_{\alpha_3}(g). \tag{6.15}$$

For the rank-2 coefficient

$$\mathcal{F}_U(\varphi, \psi_{y(Y_2)}; g) = \int\limits_{(\mathbb{Q}\backslash\mathbb{A})^6} \varphi\left(\begin{pmatrix} 1 & & & u_8 & u_{10} \\ & 1 & & u_6 & u_9 \\ & & 1 & u_3 & u_7 \\ & & & 1 & \\ & & & & 1 \end{pmatrix} g\right) \overline{\mathbf{e}(u_3 + u_9)}\, \mathrm{d}^6 u. \tag{6.16}$$

the story is slightly different. In order to repeat the procedure above, we need the charges to sit on the upper off diagonal. It is slightly unclear what the most suitable nodes (simple roots) are and in **Paper II** we simply use the Weyl reflection ω to swap the charged rows into row 1 and 3, thereby landing the charged matrix elements on the simple roots α_1 and α_3. Following that we populate the integration matrix in a similar fashion as has already been demonstrated. □

6.1 Applications

As can be seen from table 3.1, the group SL_5 is the symmetry and duality group for type IIB theory in $D = 7$ and it is this fact that provided the initial motivation for developing the formalism above. In $D = 7$, one finds Eisenstein series in the minimal and next-to-minimal representations of SL_5 encoding perturbative and non-perturbative physics in their Fourier coefficients. We shall now see how the formalism above can be used to calculate these Fourier coefficients over the maximal parabolic P_4, associated with the M-theory limit. The calculation above has a lot in common with Sections 5.3 and 5.4 so we will be brief in some steps and refer the reader to these sections for clarification of the calculus.

6.1.1 Generalities

With applications to string theory in mind, we are interested in expressions of the form

$$F_\psi^{U(\mathbb{R})}(g) = \int\limits_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(ug)\overline{\psi(u)}\, \mathrm{d}u \tag{6.17}$$

where $U(\mathbb{R})$ is the unipotent of a maximal parabolic subgroup of $\mathrm{G}(\mathbb{R})$, ψ is a rank-1 or rank-2 character on $U(\mathbb{R})$ and φ is an automorphic form in the minimal- or next-to-minimal automorphic representations of $\mathrm{G}(\mathbb{R})$. Any such coefficient can be brought to a standard form using the action of the arithmetic Levi subgroup $L(\mathbb{Z})$. For this reason, we will restrict to $\psi = \psi_{y(Y_1)}$ for the rank-1 case and $\psi = \psi_{y(Y_2)}$ for the rank-2 case and demonstrate how to apply theorem 6.2. The techniques demonstrated here allow for the calculation of all such Fourier coefficients for automorphic forms in the minimal and next-to-minimal representations on SL_n.

In order to apply theorem 6.2, we first perform an adelic lift (see Section 4.3)

$$F_\psi^{U(\mathbb{R})}(g_\infty) = F_\psi^{U(\mathbb{A})}((g_\infty, \mathbb{1}, \mathbb{1}, \dots)) = \int\limits_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(u(g_\infty, \mathbb{1}, \mathbb{1}, \dots))\overline{\psi(u)}\, \mathrm{d}u. \tag{6.18}$$

The theorem now gives $F_\psi^{U(\mathbb{A})}$ in terms of adelic Whittaker functions. These Whittaker functions will then be evaluated using the adelic reduction formula theorem 4.43, repeated here for convenience

$$W_\psi(a) = \sum_{w_c w_0' \in \mathcal{C}_\psi} a^{(w_c w_0')^{-1}\lambda+\rho} M(w_c^{-1}, \lambda) W'_{\psi^a}(w_c^{-1}\lambda, 1). \tag{6.19}$$

The power of this formula lies in that it expresses a degenerate Whittaker function evaluated on the Cartan torus of a group $G(\mathbb{A})$ as a sum of generic Whittaker functions on a subgroup $G'(\mathbb{A})$. This subgroup $G'(\mathbb{A})$ is determined by deleting all nodes in the Dynkin diagram of $G(\mathbb{A})$ on which ψ is not charged. λ denotes the weight of the underlying Eisenstein series, w_0' denotes the longest Weyl word on G' and $\mathcal{C}_\psi$ denotes the set

$$\mathcal{C}_\psi = \{w \in \mathcal{W} \mid w\Pi' < 0\} \tag{6.20}$$

where Π' is the set of simple roots of G', hence w_c is the summation variable and corresponds to a specific representative of the quotient Weyl group $\mathcal{W}/\mathcal{W}'$. Lastly, ρ denotes the Weyl vector Eq. (2.15), M denotes the intertwiner Eq. (2.56) and ψ^a denotes the "twisted character" defined in Appendix C.

The evaluation of a real Fourier coefficient $F_\psi^{U(\mathbb{R})}$ over a maximal parabolic $P = UL$ of $G = \mathrm{SL}_n$ looks like

$$\begin{aligned}
&F_\psi^{U(\mathbb{R})}(g_\infty) = F_\psi^{U(\mathbb{A})}((g_\infty, \mathbb{1}_n, \mathbb{1}_n, \dots)) && \text{Adelic lift}\\
&= \sum_\psi \sum_{l\in\Lambda \text{ or } l\in\Gamma} W_\psi(l(g_\infty, \mathbb{1}_n, \mathbb{1}_n, \dots)) && \text{theorem 6.2}\\
&= \sum_\psi \sum_{l\in\Lambda \text{ or } l\in\Gamma} W_\psi((n_\infty a_\infty k_\infty, n_2 a_2 k_2, n_3 a_3 k_3, \dots)) && \text{Iwasawa decomposition}\\
&= \sum_\psi \left(\prod_{p\le\infty} \psi_p(n_p)\right) \sum_{l\in\Lambda \text{ or } l\in\Gamma} W_\psi((a_\infty, a_2, a_3, \dots)) && W_\psi(nak) = \psi(n)W_\psi(a)\\
&= \sum_\psi \psi_\infty(n_\infty) \sum_{l\in\Lambda \text{ or } l\in\Gamma} \sum_w a^{\cdots} M(\dots) W'_{\psi^a}(\dots, 1) && \text{Reduction formula Eq. (6.19).}
\end{aligned} \tag{6.21}$$

The fourth line extracts the unipotent n_p-dependence at each local place $p \le \infty$. In the fifth line we have used that only the archimedean unipotent n_∞ contributes. This is contrary to the general case as argued in remark 4.37. The reason that the p-adic unipotent matrices n_p of the p-adic Iwasawa decomposition of $l \in L(\mathbb{Q}) \subset L(\mathbb{Q}_p)$ above drop out in this case goes as follows. In using theorem 6.2, we will be faced with evaluating Whittaker functions such as

$$\begin{aligned}
&W_{\alpha_j,\alpha_m}(\hat\iota(\lambda_j)g) \quad \text{for} \quad j \le m-2 \quad \text{where} \quad \lambda_j \in \Lambda_j \quad \text{and}\\
&W_{\alpha_m,\alpha_i}(\iota(\gamma_i)g) \quad \text{for} \quad i \ge m+2 \quad \text{where} \quad \gamma_i \in \Gamma_i\,.
\end{aligned} \tag{6.22}$$

We have that the Levi elements γ_i and λ_j are embedded in SL_n as (see Eq. (6.5))

$$\hat{\imath}(\lambda_j) = \begin{pmatrix} \lambda_j & \\ & I_{n-j} \end{pmatrix} \quad \text{and} \quad \iota(\gamma_i) = \begin{pmatrix} I_i & \\ & \gamma_i \end{pmatrix} . \tag{6.23}$$

It is clear from the block-diagonal form that the unipotent n_p in the p-adic Iwasawa decomposition of $\hat{\imath}(\lambda_j)$ (and $\iota(\gamma_i)$) will feature the same block-diagonal form. Since W_{α_j,α_m} (and W_{α_m,α_i}) is only sensitive to the unipotent on rows j and $m \geq j+2 > j$ (on rows i and $m \leq i-2 \leq i$), the block diagonal structure of n_p implies $\psi_{\alpha_j,\alpha_m;p}(n_p) = 1$ (and $\psi_{\alpha_m,\alpha_i;p}(n_p) = 1$).

For a real matrix $g \in \mathrm{SL}_n(\mathbb{R})$, we will denote its Iwasawa decomposition

$$g = n_\infty a_\infty k_\infty = \begin{pmatrix} 1 & x_{12} & \cdots & \cdots & x_{1n} \\ & 1 & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 & & & & \\ & y_2/y_1 & & & \\ & & \ddots & & \\ & & & y_{n-1}/y_{n-2} & \\ & & & & 1/y_{n-1} \end{pmatrix} k_\infty . \tag{6.24}$$

Similarly, for a p-adic matrix $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ we denote it as

$$g = n_p a_p k_p = n_p \begin{pmatrix} \eta_{1,p} & & & & \\ & \eta_{2,p}/\eta_{1,p} & & & \\ & & \ddots & & \\ & & & \eta_{n-1,p}/\eta_{n-2,p} & \\ & & & & 1/\eta_{n-1,p} \end{pmatrix} k_p . \tag{6.25}$$

theorem 2.2 offers closed formulae for the x's and the y's and theorem 4.29 gives a closed formula for the p-adic norm $|\eta_{i,p}|_p$ of the η's.

In what follows, we shall make use of all formulae that are derived or stated in Appendices A.2 and C along with the notation explained in Appendix A.1. Furthermore, we abandon the Bourbaki labeling (for SL_5) of Fig. 3.1 and simply label the nodes of $\mathfrak{sl}_n$ as $\alpha_1, \ldots, \alpha_{n-1}$ from "left to right".

6.1.2 Example: Rank-1 coefficient of π_{min} on $P_{\alpha_4} \subset \mathrm{SL}_5$

Here, we will calculate the real rank-1 Fourier coefficient (6.17) with $\psi = \psi_{y(Y_1)}$ over the maximal parabolic

$$P_{\alpha_4} = \mathrm{GL}(4) \times \mathrm{GL}_1 \times U_{\alpha_4} \subset \mathrm{SL}_5 \quad \text{subject to} \quad \det(\mathrm{GL}(4) \times \mathrm{GL}_1) = 1, \tag{6.26}$$

for the maximal parabolic Eisenstein series $E^{\mathrm{SL}_5}_{\alpha_1,s}(g)$ which lies in the minimal automorphic representation of SL_5 as can be checked by a calculation similar to that of example 5.8. The unipotent radical of P_{α_4} is

$$U(\mathbb{R}) = U_{\alpha_4}(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & & & & * \\ & 1 & & & * \\ & & 1 & & * \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \right\} . \tag{6.27}$$

	w_c	$\langle w_c^{-1}\lambda+\rho\vert\alpha_4\rangle$	$M\left(w_c^{-1},\lambda\right)$	$\left(w_c w_0'\right)^{-1}\lambda+\rho$
	Id	0	1	...
	w_1	0	...	...
	w_{12}	0	...	...
*	w_{123}	$2\left(s-\frac{3}{2}\right)$	$\frac{\xi(2s-3)}{\xi(2s)}$	$[0,0,0,5-2s]$

Table 6.1: Data for the reduction formula (6.19) to evaluate $W_{\alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_1-\rho$. The star indicates the one and only row that contributes in the sum over Weyl words.

theorem 6.2 gives

$$F^{U(\mathbb{A})}_{\psi_{y(Y_1)}}(g) = W_{\alpha_4}(g). \tag{6.28}$$

The Whittaker function is found by the reduction formula with data given in table 6.1. In this case, there is no diagonally embedded rational matrix l, meaning that the argument at the non-archimedean places is $\mathbb{1}_5$ and hence we have $|\eta_{1,p}|_p = |\eta_{2,p}|_p = |\eta_{3,4}|_p = |\eta_{4,p}|_p = 1$. We get

$$\begin{aligned}
&W_{\alpha_4}(\lambda;(g_\infty,\mathbb{1}_5,\mathbb{1}_5,\dots)) = \\
&= \mathbf{e}(x_{45})\left(y_4^{5-2s}\frac{\xi(2s-3)}{\xi(2s)}\prod_{p<\infty}|\eta_{4,p}|_p^{5-2s}\right)B_{s-3/2}\left(\frac{y_4^2}{y_3},1\right) \\
&\times\prod_{p<\infty}\gamma_p\left(\frac{\eta_{4,p}^2}{\eta_{3,p}}\right)\left(1-p^{-2(s-3/2)}\right)\frac{1-p^{-2(s-3/2)+1}\left|\frac{\eta_{4,p}^2}{\eta_{3,p}}\right|_p^{2(s-3/2)-1}}{1-p^{-2(s-3/2)+1}} \\
&= \mathbf{e}(x_{45})y_4^{5-2s}\frac{1}{\xi(2s)}2\left|\frac{y_4^2}{y_3}\right|_\infty^{s-2}K_{s-2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \\
&= 2\,\mathbf{e}(x_{45})y_3^{2-s}y_4\frac{1}{\xi(2s)}K_{s-2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) = F^{U(\mathbb{R})}_{\psi_y(Y_1)}(g_\infty).
\end{aligned} \tag{6.29}$$

The x's and y's are the Iwasawa coordinates for the matrix g_∞ as in (6.24). The function B_s that appears is a more compact way of writing the archimedean SL_2 Whittaker function defined explicitly in Eq. (C.28).

Parametrizing g_∞ as

$$g_\infty = ue = \begin{pmatrix} I_4 & Q \\ 0 & 1\end{pmatrix}\begin{pmatrix} r^{-1/4}e_4 & 0 \\ 0 & r\end{pmatrix} \quad \text{where} \quad e_4\in\mathrm{SL}_4(\mathbb{R}), \tag{6.30}$$

we get in particular that

$$y_3 = r^{-3/4}||Ne_4|| \quad \text{and} \quad y_4 = r^{-1}\,, \tag{6.31}$$

where $N = \left(\begin{smallmatrix} 0 & 0 & 0 & 1 \end{smallmatrix}\right)$ so that Ne_4 is equal to the last row in e_4. This is obtained using the formula (2.35). We get in particular that

$$y_3^{2-s}y_4 = r^{2s-5}\left(r^{-5/4}||Ne_4||\right)^{s-2} \quad \text{and} \quad \frac{y_4^2}{y_3} = r^{-5/4}||Ne_4||. \tag{6.32}$$

The more general (real) ramified Fourier coefficient has the expression

$$\begin{aligned} &\int E^{\mathrm{SL}_5}_{\alpha_1,s}\left(\left(\begin{smallmatrix} 1 & & & & u_1 \\ & 1 & & & u_2 \\ & & 1 & & u_3 \\ & & & 1 & u_4 \\ & & & & 1 \end{smallmatrix}\right) g_\infty\right) \overline{\mathbf{e}(m_1u_1 + m_2u_2 + m_3u_3 + m_4'u_4)}\, \mathrm{d}^4u = \\ &= \mathbf{e}(NQ)r^{2s-5}\frac{2}{\xi(2s)}\sigma_{4-2s}(k)\left(r^{-5/4}||Ne_4||\right)^{s-2}K_{s-2}\left(2\pi r^{-5/4}||Ne_4||\right) \\ &= \mathbf{e}(NQ)r^{\frac{3s}{4}-\frac{5}{2}}\frac{2}{\xi(2s)}\frac{\sigma_{2s-4}(k)}{|k|_\infty^{\frac{s-2}{2}}}||\tilde{N}e_4||^{s-2}K_{s-2}\left(2\pi|k|_\infty r^{-5/4}||\tilde{N}e_4||\right) \end{aligned} \tag{6.33}$$

for integer m's while for non-integer rational m's it vanishes. Here g_∞ has been parametrized as above, $N = \left(\begin{smallmatrix} m_1 & m_2 & m_3 & m_4' \end{smallmatrix}\right) = k\tilde{N}$, $k = \gcd(N)$ and $m_4' \neq 0$. This expression can also be found by starting from the more general standard form $\psi_{y(kY_1)}$ of the Fourier coefficient. This corresponds to $N = \left(\begin{smallmatrix} 0 & 0 & 0 & k \end{smallmatrix}\right)$ and its $L(\mathbb{Z})$ orbit gives the general expression (6.33). Eq. (6.33) agrees with [29, Eq. (H.37)] where the Fourier coefficients were computed using Poisson resummation.

6.1.3 Example: Rank-1 coefficient of π_{ntm} on $P_{\alpha_4} \subset \mathrm{SL}_5$

Here, we will calculate the real rank-1[26] Fourier coefficient Eq. (6.17) with $\psi = \psi_{y(Y_1)}$ over the same parabolic P_{α_4} as above, for the maximal parabolic Eisenstein series $E^{\mathrm{SL}_5}_{\alpha_2,s}(g)$ which lies in the next-to-minimal automorphic representation of SL_5 as is verified in example 5.8.

[26] There is no rank-2 character for this parabolic.

	w_c	$\langle w_c^{-1}\lambda+\rho\vert\alpha_4\rangle$	$M\left(w_c^{-1},\lambda\right)$	$\left(w_c w_0'\right)^{-1}\lambda+\rho$
	Id	0	1	...
	w_2	0	...	...
	w_{21}	0	...	...
*	w_{23}	$2(s-1)$	$\frac{\xi(2s-2)}{\xi(2s)}$	$[2s-1,0,0,4-2s]$
*	w_{213}	$2(s-1)$	$\frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)}$	$[3-2s,2s-2,0,4-2s]$
*	w_{2132}	$2(s-1)$	$\frac{\xi(2s-3)\xi(2s-2)}{\xi(2s)\xi(2s-1)}$	$[0,4-2s,2s-3,4-2s]$
	w_{213243}	0	...	...

Table 6.2: Data for the reduction formula (6.19) to evaluate $W_{\alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_2-\rho$. The stars indicate which rows contribute in the sum over Weyl words.

theorem 6.2 gives

$$\begin{aligned}
&\mathcal{F}^{\mathbb{A}}(E(2s\Lambda_2-\rho),\psi_{y(Y_1)};g) = \\
&= W_{\alpha_4}(g) + \sum_{\lambda_1\in\Lambda_1(\psi_{\alpha_4})} W_{\alpha_1,\alpha_4}(\lambda_1 g) + \sum_{\lambda_2\in\Lambda_2} W_{\alpha_2,\alpha_4}(\lambda_2 g) \\
&= W_{\alpha_4}(g) \\
&\quad + \sum_{z'} W_{\alpha_1,\alpha_4}\Bigg(\underbrace{\begin{pmatrix} z &&&& \\ & 1 &&& \\ && 1/z && \\ &&& 1 & \\ &&&& 1\end{pmatrix}}_{l_z} g\Bigg) \\
&\quad + \sum_{x',y} W_{\alpha_2,\alpha_4}\Bigg(\underbrace{\begin{pmatrix} x^{-1} && \\ y & x & \\ && I_3\end{pmatrix}}_{l_{xy}} g\Bigg) \\
&\quad + \sum_{x'} W_{\alpha_2,\alpha_4}\Bigg(\underbrace{\begin{pmatrix} 0 & -x^{-1} & \\ x & 0 & \\ && I_3\end{pmatrix}}_{l_x} g\Bigg),
\end{aligned} \tag{6.34}$$

using the representatives derived in Appendix D.

The first Whittaker function is found by the reduction formula with the data of table 6.2. In this case, there is no diagonally embedded rational matrix l, or equivalently

	w_c	$\langle w_c^{-1}\lambda+\rho\vert\alpha_1\rangle$	$\langle w_c^{-1}\lambda+\rho\vert\alpha_4\rangle$	$M\left(w_c^{-1},\lambda\right)$	$(w_c w_0')^{-1}\lambda+\rho$
	Id	0	0	1	...
	w_2	$2\left(s-\frac{1}{2}\right)$	0	...	...
*	w_{23}	$2\left(s-\frac{1}{2}\right)$	$2(s-1)$	$\frac{\xi(2s-2)}{\xi(2s)}$	v
	w_{2132}	0	$2(s-1)$	...	...
	w_{213243}	0	0	...	...

Table 6.3: Data for the reduction formula (6.19) to evaluate $W_{\alpha_1,\alpha_4}(a)$ on SL_5 with $\lambda=2s\Lambda_2-\rho$. The stars indicate which Weyl words contribute to the reduction formula. We wrote $v=[3-2s,2s-2,0,4-2s]$ here to conserve space.

$l=I_5$, and hence we have $|\eta_{1,p}|_p=|\eta_{2,p}|_p=|\eta_{3,4}|_p=|\eta_{4,p}|_p=1$. We get

$$\begin{aligned}
&W_{\alpha_4}((g_\infty,\mathbb{1}_n,\mathbb{1}_n,\dots))=\\
&=\mathbf{e}(x_{45})B_{s-1}\left(\frac{y_4^2}{y_3},1\right)\Bigg(y_1^{2s-1}y_4^{4-2s}\frac{\xi\left(2s-2\right)}{\xi\left(2s\right)}\prod_{p<\infty}|\eta_{1,p}|_p^{2s-1}|\eta_{4,p}|_p^{4-2s}\\
&+y_1^{3-2s}y_2^{2s-2}y_4^{4-2s}\frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)}\prod_{p<\infty}|\eta_{1,p}|_p^{3-2s}|\eta_{2,p}|_p^{2s-2}|\eta_{4,p}|_p^{4-2s}+\\
&+y_2^{4-2s}y_3^{2s-3}y_4^{4-2s}\frac{\xi(2s-3)\xi(2s-2)}{\xi(2s)\xi(2s-1)}\prod_{p<\infty}|\eta_{2,p}|_p^{4-2s}|\eta_{3,p}|_p^{2s-3}|\eta_{4,p}|_p^{4-2s}\Bigg)\\
&\times\prod_{p<\infty}\gamma_p\left(\frac{\eta_{4,p}^2}{\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right)\frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{4,p}^2}{\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}}\\
&=\mathbf{e}(x_{45})\Bigg(y_1^{2s-1}y_4^{4-2s}\frac{1}{\xi\left(2s\right)}+y_1^{3-2s}y_2^{2s-2}y_4^{4-2s}\frac{\xi(2s-2)}{\xi(2s)\xi(2s-1)}\\
&+y_2^{4-2s}y_3^{2s-3}y_4^{4-2s}\frac{\xi(2s-3)}{\xi(2s)\xi(2s-1)}\Bigg)2\left|\frac{y_4^2}{y_3}\right|_\infty^{s-3/2}K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right)\\
&=2\,\mathbf{e}(x_{45})\Bigg(y_1^{2s-1}y_3^{3/2-s}y_4\frac{1}{\xi\left(2s\right)}+y_1^{3-2s}y_2^{2s-2}y_3^{3/2-s}y_4\frac{\xi(2s-2)}{\xi(2s)\xi(2s-1)}\\
&+y_2^{4-2s}y_3^{s-3/2}y_4\frac{\xi(2s-3)}{\xi(2s)\xi(2s-1)}\Bigg)K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right)\,.
\end{aligned}\tag{6.35}$$

The x's and y's are the Iwasawa coordinates for the matrix g_∞ as in (6.24).

The second Whittaker function is found by the reduction formula with the data given in table 6.3. The p-adic Iwasawa decomposition of l_z has

$$|\eta_{1,p}|_p=|\eta_{2,p}|_p=|z|_p\quad\text{and}\quad|\eta_{3,p}|_p=|\eta_{4,p}|_p=1\,.\tag{6.36}$$

w_c	$\langle w_c^{-1}\lambda+\rho\|\alpha_2\rangle$	$\langle w_c^{-1}\lambda+\rho\|\alpha_4\rangle$	$M\left(w_c^{-1},\lambda\right)$	$\left(w_c w_0'\right)^{-1}\lambda+\rho$
Id	$2s$	0	1	...
w_{21}	0	0	...	...
w_{23}	0	$2(s-1)$	...	...
$*$ w_{213}	$2(s-1)$	$2(s-1)$	$\frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)}$	v
w_{213243}	0	0	...	...

Table 6.4: Data for the reduction formula (6.19) to evaluate $W_{\alpha_2,\alpha_4}(a)$ on SL_5 with $\lambda = 2s\Lambda_2 - \rho$. The star indicates the Weyl word that contributes to the reduction formula. We wrote $v = [0, 4-2s, 2s-3, 4-2s]$ to save space.

We get

$$\begin{aligned}
&\sum_{z'} W_{\alpha_1,\alpha_4}((g_\infty, \mathbb{1}_n, \mathbb{1}_n, \dots)) = \\
&= \sum_{z'} \mathbf{e}(x_{12}+x_{45}) y_1^{3-2s} y_2^{2s-2} y_4^{4-2s} \frac{\xi(2s-2)}{\xi(2s)} \\
&\qquad B_{s-1/2}\left(\frac{y_1^2}{y_2},1\right) B_{s-1}\left(\frac{y_4^2}{y_3},1\right) \prod_{p<\infty} |\eta_{1,p}|_p^{3-2s} |\eta_{2,p}|_p^{2s-2} |\eta_{4,p}|_p^{4-2s} \\
&\qquad \prod_{p<\infty} \gamma_p\left(\frac{\eta_{1,p}^2}{\eta_{2,p}}\right)\left(1-p^{-2(s-1/2)}\right) \frac{1-p^{-2(s-1/2)+1}\left|\frac{\eta_{1,p}^2}{\eta_{2,p}}\right|_p^{2(s-1/2)-1}}{1-p^{-2(s-1/2)+1}} \times \\
&\qquad \times \prod_{p<\infty} \gamma_p\left(\frac{\eta_{4,p}^2}{\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right) \frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{4,p}^2}{\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}} \\
&= \sum_{z'\in\mathbb{Z}} \mathbf{e}(x_{12}+x_{45}) y_1^{3-2s} y_2^{2s-2} y_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} \prod_{p<\infty} |z|_p^1 \\
&\qquad 4\left|\frac{y_1^2}{y_2}\right|_\infty^{s-1} \left|\frac{y_4^2}{y_3}\right|_\infty^{s-3/2} K_{s-1}\left(2\pi\left|\frac{y_1^2}{y_2}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \\
&\qquad \sigma_{-2(s-1/2)+1}(z) \\
&= \sum_{z'\in\mathbb{Z}} 4\,\mathbf{e}(x_{12}+x_{45}) y_1 y_2^{s-1} y_3^{3/2-s} y_4 \frac{1}{\xi(2s)\xi(2s-1)} |z|_\infty^{-1} \\
&\qquad K_{s-1}\left(2\pi\left|\frac{y_1^2}{y_2}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \sigma_{2-2s}(z)\,.
\end{aligned} \tag{6.37}$$

The x's and y's are the Iwasawa coordinates for the matrix $l_z g_\infty$.

The third and fourth Whittaker functions are found by the reduction formula with the data from table 6.4. The p-adic Iwasawa decomposition of l_{xy} has

$$|\eta_{1,p}|_p^{-1} = \max\{|y|_p, |x|_p\} \quad \text{and} \quad |\eta_{2,p}|_p = |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1\,. \tag{6.38}$$

We get

$$\begin{aligned}
&\sum_{x',y} W_{\alpha_2,\alpha_4}(l_{xy}(g_\infty, \mathbb{1}_n, \mathbb{1}_n, \dots)) = \\
&= \sum_{x',y} \mathbf{e}(x_{23}+x_{45}) y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)} \\
&\qquad B_{s-1}\left(\frac{y_2^2}{y_1 y_3}, 1\right) B_{s-1}\left(\frac{y_4^2}{y_3}, 1\right) \prod_{p<\infty} |\eta_{2,p}|_p^{4-2s} |\eta_{3,p}|_p^{2s-3} |\eta_{4,p}|_p^{4-2s} \\
&\qquad \prod_{p<\infty} \gamma_p\left(\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right) \frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}} \times \\
&\qquad \times \prod_{p<\infty} \gamma_p\left(\frac{\eta_{4,p}^2}{\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right) \frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{4,p}^2}{\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}} \\
&= \sum_{x',y\in\mathbb{Z}} \mathbf{e}(x_{23}+x_{45}) y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} 4 \left|\frac{y_2^2}{y_1 y_3}\right|_\infty^{s-3/2} \left|\frac{y_4^2}{y_3}\right|_\infty^{s-3/2} \\
&\qquad K_{s-3/2}\left(2\pi\left|\frac{y_2^2}{y_1 y_3}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \sigma_{-2(s-1)+1}(k) \\
&= \sum_{x',y\in\mathbb{Z}} 4\,\mathbf{e}(x_{23}+x_{45}) y_1^{3/2-s} y_2^1 y_4^1 \frac{1}{\xi(2s)\xi(2s-1)} \\
&\qquad K_{s-3/2}\left(2\pi\left|\frac{y_2^2}{y_1 y_3}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \sigma_{3-2s}(k)\,,
\end{aligned} \tag{6.39}$$

where $k = \gcd(|y|, |x|)$. Here, the x's and y's are the Iwasawa coordinates for the matrix $l_{xy} g_\infty$.

The p-adic Iwasawa decomposition of l_x has

$$|\eta_{1,p}|_p^{-1} = \max\{|0|_p, |x|_p\} = |x|_p \quad \text{and} \quad |\eta_{2,p}|_p = |\eta_{3,p}|_p = |\eta_{4,p}|_p = 1\,. \tag{6.40}$$

We get

$$
\begin{aligned}
&\sum_{x'} W_{\alpha_2,\alpha_4}\left(\Lambda; l_x(g_\infty, \mathbb{1}_n, \mathbb{1}_n, \dots)\right) = \\
&= \sum_{x'} \mathbf{e}(x_{23}+x_{45}) y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{\xi(2s-2)^2}{\xi(2s)\xi(2s-1)} \\
&\qquad B_{s-1}\left(\frac{y_2^2}{y_1 y_3}, 1\right) B_{s-1}\left(\frac{y_4^2}{y_3}, 1\right) \prod_{p<\infty} |\eta_{2,p}|_p^{4-2s} |\eta_{3,p}|_p^{2s-3} |\eta_{4,p}|_p^{4-2s} \\
&\qquad \prod_{p<\infty} \gamma_p\left(\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right) \frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{2,p}^2}{\eta_{1,p}\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}} \\
&\qquad \prod_{p<\infty} \gamma_p\left(\frac{\eta_{4,p}^2}{\eta_{3,p}}\right)\left(1-p^{-2(s-1)}\right) \frac{1-p^{-2(s-1)+1}\left|\frac{\eta_{4,p}^2}{\eta_{3,p}}\right|_p^{2(s-1)-1}}{1-p^{-2(s-1)+1}} \\
&= \sum_{x'\in\mathbb{Z}} \mathbf{e}(x_{23}+x_{45}) y_2^{4-2s} y_3^{2s-3} y_4^{4-2s} \frac{1}{\xi(2s)\xi(2s-1)} 4 \left|\frac{y_2^2}{y_1 y_3}\right|_\infty^{s-3/2} \left|\frac{y_4^2}{y_3}\right|_\infty^{s-3/2} \\
&\qquad K_{s-3/2}\left(2\pi\left|\frac{y_2^2}{y_1 y_3}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \sigma_{-2(s-1)+1}(x) \\
&= \sum_{x'\in\mathbb{Z}} 4\,\mathbf{e}(x_{23}+x_{45}) y_1^{3/2-s} y_2^1 y_4^1 \frac{1}{\xi(2s)\xi(2s-1)} \\
&\qquad K_{s-3/2}\left(2\pi\left|\frac{y_2^2}{y_1 y_3}\right|_\infty\right) K_{s-3/2}\left(2\pi\left|\frac{y_4^2}{y_3}\right|_\infty\right) \sigma_{3-2s}(x).
\end{aligned}
\tag{6.41}
$$

The x's and y's are the Iwasawa coordinates for the matrix $l_x g_\infty$.

The complete Fourier coefficient $F^{U(\mathbb{R})}_{\psi_{y(Y_1)}}(g_\infty)$ is then given by the combination of Eqs. (6.35), (6.37), (6.39) and (6.41). This form differ from that presented in [29, Eq. (H.52)] where it is given as an integral over the product of two Bessel functions. The different forms should come as no surprise as we have already established the existence of non-trivial relationships involving sums of Bessel functions in Section 5.4. It is straightforward to derive relationships which also involve integrals over Bessel functions. It is reasonable to believe that the equality of $F^{U(\mathbb{R})}_{\psi_{y(Y_1)}}(g_\infty)$ as derived here and computed in [29, Eq. (H.52)] amounts to an identity of this sort, although that has not yet been confirmed.

7 Outlook

In this thesis, we have discussed how automorphic forms appear in the low energy effective action of type IIB string theory on tori and how physical information about graviton scattering can be extracted from their Fourier expansions. Furthermore, we have given a formalism for calculating the full Fourier expansion of Eisenstein series in the minimal- and next-to-minimal automorphic representations of SL_n over maximal parabolics. There are two immediately interesting avenues for generalizations of the results presented in this thesis.

7.1 Generalizing automorphic forms

The nomenclature we shall use is the following. A sum of the form

$$\sum_{\gamma \in \Gamma \backslash G(\mathbb{Z})} \tag{7.1}$$

is called a *Poincaré sum*. Starting with a function $\chi : G(\mathbb{R}) \to \mathbb{C}$, we can construct a new function $F : G(\mathbb{R}) \to \mathbb{C}$ by applying the Poincaré sum to χ,

$$F(g) = \sum_{\gamma \in \Gamma_\chi \backslash G(\mathbb{Z})} \chi(\gamma g) \tag{7.2}$$

where Γ_χ is the stabilizer of χ in $G(\mathbb{Z})$. The virtue of this is that the resulting function F is automorphic,

$$F(\gamma g) = F(g) \quad \text{for all} \quad \gamma \in G(\mathbb{Z}). \tag{7.3}$$

The special case in which χ is a character on G (meaning a complex valued homomorphism) carries the name Eisenstein series and are guaranteed to be automorphic forms in the sense of definition 2.4.

Eisenstein series appear as the first two non-trivial higher curvature corrections, the R^4 correction $\mathcal{E}^{(D)}_{(0,0)}$ and the $\nabla^4 R^4$ correction $\mathcal{E}^{(D)}_{(1,0)}$ at orders $(\alpha')^3$ and $(\alpha')^5$ respectively in the α'-expansion. The BPS-protection of these interactions translates into statements about automorphic representations. As such, one knows a priori what to expect regarding the Fourier coefficients of $\mathcal{E}^{(D)}_{(0,0)}$ and $\mathcal{E}^{(D)}_{(1,0)}$ and there is a rich set of tools at one's disposal including formulae like the Langland's constant term formula and the Casselman-Shalika formula. The Langland's constant term formula is especially important for calculating the constant term in the string perturbation limit (the perturbative terms) of these functions in order to match it up with the known results from string perturbation theory.

The next function is the $\nabla^6 R^4$ correction $\mathcal{E}^{(D)}_{(0,1)}$ at order $(\alpha')^6$. It is also automorphic as a consequence of U-duality and enjoys $\frac{1}{8}$-BPS protection which should have implications regarding the vanishing properties of its Fourier coefficients. Its status as an automorphic form however is spoiled by the non-linearity $(\mathcal{E}^{(D)}_{(0,0)})^2$ in the right hand side of Eq. (3.8). This non-linearity makes finding $\mathcal{E}^{(D)}_{(0,1)}$ much more difficult. Green, Miller and Vanhove found an explicit solution in $D = 10$ in [40]. In an appendix, they express this function in the form

$$\mathcal{E}^{(10)}_{(0,1)} = \sum_{\gamma \in B(\mathbb{Z})\backslash G(\mathbb{Z})} \Phi(\gamma g) \tag{7.4}$$

for $G = \mathrm{SL}_2$. The function χ here is not a character and $\mathcal{E}^{(10)}_{(0,1)}$ is thus not an Eisenstein series. It is furthermore not clear how $\mathcal{E}^{(D)}_{(0,1)}$ should be viewed from the point of view of representation theory and if the function Φ should possess any special properties.

In Sections 3.2 and 3.4 we see how the unfolding technique simplifies Eq. (3.8) in $D = 10$ and $D = 7$ and find particular solutions. Once particular solutions have been found, the next step is to calculate their constant terms. Langland's constant term formula no longer applies and in [40] the authors rely on algebraic techniques to calculate Fourier coefficients of $\mathcal{E}^{(10)}_{(0,1)}$. I investigated the feasibility of using the adelic framework for calculating the constant term of $\mathcal{E}^{(D)}_{(0,1)}$. The results are summarized below.

Letting $\Omega = x + iy$ denote an element of the Poincaré upper half plane, we write Ω instead of g. We also denote $\mathcal{E}^{(10)}_{(0,1)} = f$. In [40], the Fourier expansion of f is given as

$$f(x+iy) = \sum_m \widehat{f}_m(y) e^{2\pi i m x} \tag{7.5}$$

where the constant term is

$$\widehat{f}_0(y) = \frac{2}{3}\zeta(3)^2 y^3 + \frac{4}{3}\zeta(2)\zeta(3) y + 4\zeta(4) y^{-1} + \frac{4}{27}\zeta(6) y^{-3} + \sum_{m\neq 0} \widehat{f}^P_{m,-m}(y) \tag{7.6}$$

and $\widehat{f}^P_{m,-m}(y)$ are bilinear in Bessel K functions corresponding to chargeless instanton-anti instanton pairs which vanish in the weak coupling limit. We are trying to reproduce the power behaved terms.

Using unfolding as discussed in Section 3.2, a solution f can be found in the form of the Poincaré series

$$f(\Omega) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \chi(\gamma \cdot \Omega) \tag{7.7}$$

where the non-character χ is given as the Fourier series [40]

$$\chi(\Omega) = \sum_{m\in\mathbb{Z}} \underbrace{c_m(y) e^{2\pi i m x}}_{\chi_m(\Omega)} \tag{7.8}$$

where

$$c_0(y) = \underbrace{\frac{2}{3}\zeta(3)^2}_{A} y^3 + \underbrace{\frac{1}{9}\pi^2\zeta(3)}_{B} y \tag{7.9}$$

and

$$\begin{aligned} c_m(y) = 8\zeta(3)\sigma_{-2}(|m|)y\bigg[&\left(1 + \frac{40}{(2\pi|m|y)^2}\right)K_0(2\pi|m|y) \\ &+ \left(\frac{12}{2\pi|m|y} + \frac{80}{(2\pi|m|y)^3}\right)K_1(2\pi|m|y) \\ &+ \frac{-16\sqrt{2}}{3\sqrt{\pi}\,(2\pi|m|y)^{1/2}}K_{7/2}(2\pi|m|y)\bigg]. \end{aligned} \tag{7.10}$$

We would like to calculate the constant term

$$\widehat{f}_0(y) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} f(ng)\,\mathrm{d}n = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \chi(\gamma n g)\,\mathrm{d}n \tag{7.11}$$

in a similar way as as done for the SL_2-Eisenstein series in Section 4.5.1 by utilizing the adelic framework and the Bruhat decomposition. We adelize the function f by replacing

$$\chi(\Omega) = \chi(g_\infty) \to \chi_{\mathbb{A}}(g) \tag{7.12}$$

where $g = (g_\infty, g_2, \dots)$ and $\chi_{\mathbb{A}}$ is a function on $G(\mathbb{A})$ such that if we restrict ourselves to $G(\mathbb{R})$ by taking $g = (g_\infty, \mathbb{1}, \dots)$, we reproduce the real function χ described above, $\chi_{\mathbb{A}}((g_\infty, \mathbb{1}, \dots)) = \chi(g_\infty)$. Notice that χ is given as a sum $\chi = \sum_m \chi_m$ in (7.8). Each χ_m enter linearly into the expression (7.11) for the constant term of f, so we can treat each m separately. We therefore make the replacement

$$\chi_m(\Omega) \equiv \chi_{m,\mathbb{Q}_\infty}(\Omega) \to \chi_{m,\mathbb{A}}(g) = \prod_{p\leq\infty} \chi_{m,\mathbb{Q}_p}(g_p) \tag{7.13}$$

where $g = (g_\infty, g_2, \dots)$ is now in $\mathrm{SL}_2(\mathbb{A})$ and we demand that $\chi_{m,\mathbb{Q}_p}(\mathbb{1}) = 1$ for finite primes p. As is evident by (7.9) and (7.10), each individual χ_m is in turn also given by a sum of terms so by the same reasoning, each term is replaced with an appropriate adelic function.

The Bruhat decomposition now gives (for $g = (g_\infty, \mathbb{1}, \dots))$

$$\begin{aligned}\widehat{f}_0(y) &= \sum_{w\in\mathcal{W}} C_w = \sum_{w\in\mathcal{W}} \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_{\mathbb{A}}(wng) \\ &= \sum_{m\in\mathbb{Z}}\sum_{w\in\mathcal{W}} C_{m,w} = \sum_{m\in\mathbb{Z}}\sum_{w\in\mathcal{W}} \int\limits_{w^{-1}B(\mathbb{Q})w\cap N(\mathbb{Q})\backslash N(\mathbb{A})} \chi_{m,\mathbb{A}}(wng).\end{aligned} \tag{7.14}$$

For each m, there are contributions from the trivial Weyl word $\mathbb{1}$ and the non-trivial Weyl word w_{long}. Starting with $m = 0$ and noting that $y > 0 \Rightarrow y = |y|_\infty$ we have the canonical adelization

$$\begin{aligned}\chi_{0,\mathbb{A}}(g) =& \chi_{0,\mathbb{A}}\left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v^{-1}\end{pmatrix} k\right) = \\ =& A|v^2|^3 + B|v^2| = A\prod_{p\leq\infty}|v_p^2|_p^3 + B\prod_{p\leq\infty}|v_p^2|_p\end{aligned} \tag{7.15}$$

where u and v are adeles and in particular $v_\infty = y^{1/2}$. This is a valid adelization, since if we set $g = (g_\infty, \mathbb{1}, \dots)$ we get $u_p = 0$ and $v_p = 1$ and we reproduce $\chi_0(g)$. This leads (after some algebra and p-adic calculus) to

$$C_{0,\mathbb{1}} = Ay^3 + By \quad \text{and} \quad C_{0,w_{\text{long}}} = A\frac{3\pi}{8}\frac{\zeta(5)}{\zeta(6)}y^{-2} + B\pi\frac{\zeta(1)}{\zeta(2)} \tag{7.16}$$

The two terms in $C_{0,\mathbb{1}}$ are in agreement with Eq. (7.6) but the two terms in $C_{w_{\text{long}},\mathbb{1}}$ have incorrect powers of y as well as the divergent factor $\zeta(1)$. We expect these terms to cancel with contributions from $m \neq 0$.

Continuing with $m \neq 0$, it is less clear what to choose as the adelization of $\chi_{m,\mathbb{Q}_p}$. Proceeding with the notion of additive characters from definition 4.13 for the adelization of the factor $e^{2\pi imx}$, one can argue that the contributions vanish $C_{m,\mathbb{1}}$ as one ends up integrating the character over a full period. For $w = w_{\text{long}}$, we still require the adelization of the Bessel function. There exists the notion of p-adic Bessel functions by generalization of Eq. (4.163) [18, 71], but postponing this discussion for now and focusing on the archimedean place, one is led to five copies of the integral (c.f. [40, Lemma B.12])

$$\int_{-\infty}^{\infty} \mathrm{d}n \left(\frac{2\pi|m|_\infty y}{(n+x)^2+y^2}\right)^{-a} K_b\left(\frac{2\pi|m|_\infty y}{(n+x)^2+y^2}\right) e^{2\pi im\frac{-(n+x)}{(n+x)^2+y^2}} \tag{7.17}$$

corresponding to the five terms in Eq. (7.10) with a and b taking values according to table 7.1. No success was had in solving these integrals and it is furthermore interesting to note that four of the five terms are divergent on their own, as can be checked using

a	b	$a+b+\frac{1}{2}$	Divergent?
-1	0	$-1/2$	No
1	0	$3/2$	Yes
0	1	$3/2$	Yes
2	1	$7/2$	Yes
$-1/2$	$7/2$	$7/2$	Yes

Table 7.1: Values specifying the precise form of the five integrals Eq. (7.17).

the small argument approximation for the Bessel functions

$$\begin{cases} K_0(z) \sim -\ln z \\ K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu}, \quad \operatorname{Re}\nu > 0 \end{cases} \tag{7.18}$$

leading to that the integral diverges if and only if

$$a + b + 1/2 \geq 0. \tag{7.19}$$

Remark 7.1.
For the integral with $a = -1/2$ and $b = 7/2$, a primitive function to the integrand was found using Wolfram Mathematica by virtue of the fact that K_ν has a finite series expansion for half integer ν. Since the Bessel identity Eq. (5.58) derived in Section 5.4 has some resemblance with the integrand and works by shifting the order of a Bessel function by one half, one possibility potentially worth exploring could be to look for Bessel identities (using the method of Section 5.4) which fit the integrand above thus allowing one to shift the order of the Bessel function to a half integer and find a primitive function. There is however no guarantee that this will work, as the Bessel identities derived in the spirit of Section 5.4 equate Fourier coefficients for automorphic forms, in particular Eisenstein series and the function f at hand which gives rise to Eq. (7.17) is not an automorphic form in the strict sense of definition 2.4. It is rather more likely that just as the Eisenstein series give rise to Bessel identities, the more general functions of which $f = \mathcal{E}^{(10)}_{(0,1)}$ is an example would give rise to wholly new Bessel identities involving integrals like Eq. (7.17) which are inaccessible by studying Eisenstein series alone.

If the adelic lift described above is correct then we expect an equality between Eq. (7.6) and Eq. (7.14). We have already seen that $C_{0,\mathbb{1}}$ reproduces the first two terms in Eq. (7.14). We must therefore have that the five contributions coming from Eq. (7.17) (each multiplied with their corresponding p-adic integrals) should cancel the incorrect terms coming from $C_{0,w_{\text{long}}}$, reproduce the two missing powers of y in Eq. (7.6) as well as the infinite sum of Bessel bilinears. This may seem like a stretch and it is wholly possible that this adelization procedure is too coarse for the function $f = \mathcal{E}^{(10)}_{(0,1)}$ especially since it is not understood how to think about $\mathcal{E}^{(10)}_{(0,1)}$ representation theoretically.

Remark 7.2.

Since we need the power behaved terms in $f = \mathcal{E}^{(10)}_{(0,1)}$ we could consider taking the weak coupling limit $y \to \infty$ in which the non-perturbative terms $\hat{f}^P_{m,-m}(y)$ vanish, which would simplify the integration in Eq. (7.17). There seems however to be a problem with exchanging the limit and the integration as is also remarked in [40, page 13].

More generally, it seems as if the hierarchy of functions $\mathcal{E}^{(D)}_{(p,q)}$ in the low energy effective action Eq. (3.4) with their partial BPS-protection belong to a class of functions that generalize the notion of automorphic forms, with Eisenstein series at the "bottom". Exactly what these functions are and how they can be understood from representation theoretically is a question which will likely require a significant research in collaboration with the mathematics community.

Lastly, an important check is to see whether or not the particular solution Eq. (3.78) (and equally well the full solution) for $\mathcal{E}^{(7)}_{(0,1)}$ found in Section 3.4 satisfies the higher rank tensorial differential equations derived in [36, 37, 38].

7.2 Generalizing the SL_n formalism

The physical value of the formalism presented in Chapter 6 is that it provides a closed formula for the calculation of Fourier coefficients in maximal parabolics (thus in particular for the three degeneration limits string perturbation, M-theory and decompactification of Section 3.3) for the Eisenstein series $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$ on $G(\mathbb{R}) = \mathrm{SL}_5(\mathbb{R})$ appearing in $D = 7$. These Fourier coefficients are calculated in [29] using Poisson resummation and thus relies on a lattice sum representation for $\mathcal{E}^{(7)}_{(0,0)}$ and $\mathcal{E}^{(7)}_{(1,0)}$. In general, finding an integer parametrization of a coset $P_\beta(\mathbb{Z})\backslash G(\mathbb{Z})$ and rewriting the maximal parabolic Eisenstein series Eq. (2.53) as a lattice sum is a difficult problem which is avoided in the SL_n-formalism of Chapter 6. It would be valuable to have access to a result analogous to theorem 6.2 for all Cremmer-Julia groups in table 3.1. This is currently being pursued by Gourevitch-Gustafsson-Kleinschmidt-Persson-Sahi for $G = \mathrm{E}_6, \mathrm{E}_7, \mathrm{E}_8$.

A Conventions and helpful formulae

A.1 Conventions

I employ the following notation

- $\prod_p \equiv \prod_{p<\infty} \equiv \prod_{p \text{ prime}}$ denotes the product over all primes.
- gcd denotes the greatest common divisor.
- For a set of numbers S equipped with the notion of multiplication, we define $S^\times$ as the largest subset of S such that all elements of $S^\times$ have multiplicative inverses. For a number field F we then have $F^\times = F \smallsetminus \{0\}$ where 0 is the additive inverse in F.
- For integers, $a, \ldots, b$ denotes $a, a+1, a+2, \ldots, b-1, b$.
- For indices, i_a—i_b is shorthand for $i_a i_{a+1} i_{a+2} \ldots i_{b-1} i_b$.
- $\mathbb{N}$ denotes the set $\{1, 2, \ldots\}$ and $\mathbb{N}_0$ denotes the set $\{0, 1, 2, \ldots\}$.
- $\mathbb{R}_+$ denotes the set $\{x \in \mathbb{R} : x > 0\}$.
- For $x \in \mathbb{A}$ we define $\mathbf{e}(x) \equiv \mathrm{e}^{2\pi \mathrm{i} x} \equiv \mathrm{e}^{2\pi \mathrm{i} x_\infty} \prod_p \mathrm{e}^{-2\pi \mathrm{i} x_p}$. For $x \in \mathbb{R}$ we simply have $\mathbf{e}(x) \equiv \mathrm{e}^{2\pi \mathrm{i} x}$.
- A prime x' on a variable generally denotes $x \neq 0$.
- We write $\sum_x \equiv \sum_{x \in \mathbb{Q}}$.
- A prime on a summation variable denotes $\sum_{x'} f(x) = \sum_{x \in \mathbb{Q} \smallsetminus \{0\}} f(x)$ and $\sum_{x' \in \mathbb{Z}} f(x) = \sum_{x \in \mathbb{Z} \smallsetminus \{0\}} f(x)$. Note that the prime is only used to indicate whether or not the zero element is included in the sum but the prime is omitted in the summand. We will also employ the same notation for when the summation variable is a vector, for example $\sum_{(m,n)' \in \mathbb{Z}^2} f(m,n) = \sum_{(m,n) \in \mathbb{Z}^2 \smallsetminus \{(0,0)\}} f(m,n)$.

- Characters $\psi : G \to S^1$ in this thesis are always given in terms of products of the complex exponentials $\mathbf{e}$. The bar in $\overline{\psi}$ stands for complex conjugation. This is sometimes also denoted ψ^{-1}.
- For a simple Lie group G, we write will sometimes write $\psi_{\alpha_{i_1},\ldots,\alpha_{i_k}}$ to denote a character which is sensitive to the simple roots $\alpha_{i_1}, \ldots, \alpha_{i_k}$ and carry unit charge (see definition 2.19). For example, given $n = \begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}$ we have $\psi_{\alpha_1,\alpha_3}(n) = \mathbf{e}(x_1 + x_3)$.
- For Whittaker functions W_ψ (see definition 2.21) we will similarly write $W_{\alpha_{i_1},\ldots,\alpha_{i_k}}$ for when $\psi = \psi_{\alpha_{i_1},\ldots,\alpha_{i_k}}$. We may also write $W_{m_1,\ldots,m_r}$ for when the character ψ carries charges $m_1, \ldots, m_r$ with respect to the simple roots of $\mathfrak{g}$ where r is the rank of $\mathfrak{g}$. For example, given $n = \begin{pmatrix} 1 & x_1 & x_4 & x_6 \\ & 1 & x_2 & x_5 \\ & & 1 & x_3 \\ & & & 1 \end{pmatrix}$ we have $W_{m_1,0,m_3}(ng) = \mathbf{e}(m_1 x_1 + m_3 x_3) W_{m_1,0,m_3}(g)$ (see remark 2.23).

A.2 Formulae

There exists an adelic analog of the real Dirac delta "comb"

$$\delta(x) = \sum_{m\in\mathbb{Z}} \mathbf{e}(mx) \quad \text{for} \quad x \in \mathbb{R} \tag{A.1}$$

which reads

$$\delta_{\mathbb{A}}(x) = \sum_{q\in\mathbb{Q}} \mathbf{e}(mx) \quad \text{for} \quad x \in \mathbb{A}. \tag{A.2}$$

The sum goes over rational numbers m diagonally embedded into the adeles $\mathbb{A}$. In the main text, the subscript $\mathbb{A}$ is dropped.

B Two lemmas for matrix minors

Lemma B.1
An $m \times n$ matrix M obeys

$$M\left(\begin{smallmatrix} r_1 & \dots & r_k \\ c_1 & \dots & c_k \end{smallmatrix}\right) M\left(\begin{smallmatrix} r_2 & \dots & r_k \\ d_2 & \dots & d_k \end{smallmatrix}\right) = \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} r_1 & r_2 & \dots & r_k \\ c_a & d_2 & \dots & d_k \end{smallmatrix}\right) M\left(\begin{smallmatrix} r_2 & \dots & r_a & r_{a+1} & \dots & r_k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) \tag{B.1}$$

where $r_i \in \{1, \dots, m\}$ and $c_i \in \{1, \dots, n\}$ and $d_i \in \{1, \dots, n\}$ and $k \in \mathbb{N}$.

Proof. For readability, we will drop r and write r_i just as i. To prove the identity, we expand both sides using Laplace expansion. Start by expanding the first factor of the left hand side as

$$\begin{aligned} \text{LHS} &= M\left(\begin{smallmatrix} 1 & \dots & k \\ c_1 & \dots & c_k \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & k \\ d_2 & \dots & d_k \end{smallmatrix}\right) = \\ &= \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} 1 \\ c_a \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & a & a+1 & \dots & k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & k \\ d_2 & \dots & d_k \end{smallmatrix}\right). \end{aligned} \tag{B.2}$$

Expand the first factor in the sum of the right hand side as

$$\begin{aligned} \text{RHS} &= \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} 1 & 2 & \dots & k \\ c_a & d_2 & \dots & d_k \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & a & a+1 & \dots & k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) = \\ &= \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} 2 & \dots & a & a+1 & \dots & k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) \\ &\quad \Bigg(\underbrace{M\left(\begin{smallmatrix} 1 \\ c_a \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & k \\ d_2 & \dots & d_k \end{smallmatrix}\right)}_{\text{I}} + \underbrace{\sum_{b=2}^{k} (-1)^{b+1} M\left(\begin{smallmatrix} b \\ c_a \end{smallmatrix}\right) M\left(\begin{smallmatrix} 1 & \dots & b-1 & b+1 & \dots & k \\ d_2 & \dots & d_b & d_{b+1} & \dots & d_k \end{smallmatrix}\right)}_{\text{II}} \Bigg). \end{aligned} \tag{B.3}$$

The term labelled I corresponds to the left hand side. The term labelled II vanishes according to

$$0 = M\left(\begin{smallmatrix} b & 2 & \dots & k \\ c_1 & c_2 & \dots & c_k \end{smallmatrix}\right) = \sum_{a=1}^{k} (-1)^{a+1} M\left(\begin{smallmatrix} b \\ c_a \end{smallmatrix}\right) M\left(\begin{smallmatrix} 2 & \dots & a & a+1 & \dots & k \\ c_1 & \dots & c_{a-1} & c_{a+1} & \dots & c_k \end{smallmatrix}\right) \tag{B.4}$$

for $b \in \{2, \dots, k\}$, due to antisymmetry of minors. □

Remark B.2.

The lemma holds true as stated but the assertion is trivial unless $k \le \min\{m,n\}$ and all r's, all c's as well as all d's are different.

Remark B.3.
A special case of the lemma is the identity

$$M\begin{pmatrix} r_1 & \cdots & r_k & r_{k+1} \\ c_1 & \cdots & c_k & r_{k+1} \end{pmatrix} M\begin{pmatrix} r_2 & \cdots & r_{k+1} \\ r_2 & \cdots & r_{k+1} \end{pmatrix} = \sum_{a=1}^{k} (-1)^{a+1} M\begin{pmatrix} r_1 & r_2 & \cdots & r_{k+1} \\ c_a & r_2 & \cdots & r_{k+1} \end{pmatrix} M\begin{pmatrix} r_2 & \cdots & r_a & r_{a+1} & \cdots & r_k & r_{k+1} \\ c_1 & \cdots & c_{a-1} & c_{a+1} & \cdots & c_k & r_{k+1} \end{pmatrix} \tag{B.5}$$

which will be used in the proof of lemma B.4.

Lemma B.4
An $m \times n$ matrix M obeys the following identity involving the determinant of a $k \times k$-matrix of minors of M

$$\begin{vmatrix} M\begin{pmatrix} r_1 & r_2 & \cdots & r_{k+1} \\ c_1 & r_2 & \cdots & r_{k+1} \end{pmatrix} & M\begin{pmatrix} r_1 & r_2 & \cdots & r_{k+1} \\ c_2 & r_2 & \cdots & r_{k+1} \end{pmatrix} & \cdots & M\begin{pmatrix} r_1 & r_2 & \cdots & r_{k+1} \\ c_k & r_2 & \cdots & r_{k+1} \end{pmatrix} \\ M\begin{pmatrix} r_2 & r_3 & \cdots & r_{k+1} \\ c_1 & r_3 & \cdots & r_{k+1} \end{pmatrix} & M\begin{pmatrix} r_2 & r_3 & \cdots & r_{k+1} \\ c_2 & r_3 & \cdots & r_{k+1} \end{pmatrix} & \cdots & M\begin{pmatrix} r_2 & r_3 & \cdots & r_{k+1} \\ c_k & r_3 & \cdots & r_{k+1} \end{pmatrix} \\ \vdots & \vdots & \ddots & \vdots \\ M\begin{pmatrix} r_k & r_{k+1} \\ c_1 & r_{k+1} \end{pmatrix} & M\begin{pmatrix} r_k & r_{k+1} \\ c_2 & r_{k+1} \end{pmatrix} & \cdots & M\begin{pmatrix} r_k & r_{k+1} \\ c_k & r_{k+1} \end{pmatrix} \end{vmatrix} = \\ = M\begin{pmatrix} r_1 & r_2 & \cdots & r_k & r_{k+1} \\ c_1 & c_2 & \cdots & c_k & r_{k+1} \end{pmatrix} M\begin{pmatrix} r_2 & \cdots & r_{k+1} \\ r_2 & \cdots & r_{k+1} \end{pmatrix} \ldots M\begin{pmatrix} r_k & r_{k+1} \\ r_k & r_{k+1} \end{pmatrix} \tag{B.6}$$

where $r_i \in \{1, \ldots, m\}$ and $c_i \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$.

Proof. The proof works by induction. The base case $k = 1$ trivial. Assume that the formula holds for $k \le q-1$ for some $q-1 \in \mathbb{N}$. Expanding the determinant for $k = q$ along the first row and using the induction hypothesis on the remaining $(q-1) \times (q-1)$-determinants gives

$$\begin{aligned} \text{LHS} = \sum_{a=1}^{q} (-1)^{a+1} M\begin{pmatrix} r_1 & r_2 & \cdots & r_{q+1} \\ c_a & r_2 & \cdots & r_{q+1} \end{pmatrix} M\begin{pmatrix} r_2 & \cdots & r_a & r_{a+1} & \cdots & r_q & r_{q+1} \\ c_1 & \cdots & c_{a-1} & c_{a+1} & \cdots & c_q & r_{q+1} \end{pmatrix} \\ M\begin{pmatrix} r_3 & \cdots & r_{q+1} \\ r_3 & \cdots & r_{q+1} \end{pmatrix} \ldots M\begin{pmatrix} r_q & r_{q+1} \\ r_q & r_{q+1} \end{pmatrix} = \\ = M\begin{pmatrix} r_1 & \cdots & r_q & r_{q+1} \\ c_1 & \cdots & c_q & r_{q+1} \end{pmatrix} M\begin{pmatrix} r_2 & \cdots & r_{q+1} \\ r_2 & \cdots & r_{q+1} \end{pmatrix} M\begin{pmatrix} r_3 & \cdots & r_{q+1} \\ r_3 & \cdots & r_{q+1} \end{pmatrix} \ldots M\begin{pmatrix} r_q & r_{q+1} \\ r_q & r_{q+1} \end{pmatrix} \end{aligned} \tag{B.7}$$

where we have used Eq. (B.5). Peano's axiom of induction now establishes the lemma. □

C Euler products and twisted characters

This appendix contains details and explanations for Section 6.1, which is why we restrict to the field $F = \mathbb{Q}$ with the corresponding ring of adeles $\mathbb{A} = \mathbb{A}_\mathbb{Q}$.

An Euler product is a product over the primes. The p-adic norm is denoted $|\cdot|_p$ and is defined for the p-adic numbers $\mathbb{Q}_p$. The absolute value norm or "infinity norm" is denoted $|\cdot|_\infty$ and is defined for real numbers $\mathbb{R} = \mathbb{Q}_\infty$. The p-adic numbers as well as the real numbers (being completions of the rational numbers) all contain the rational numbers: $\mathbb{Q} \subset \mathbb{Q}_p$ for all p prime. The norm of an adele $x = (x_\infty, x_2, x_3, x_5, \dots) \in \mathbb{A}$ is denoted $|\cdot|$ (without ornaments) and is the product of norms at the local places

$$|x| = \prod_{p \le \infty} |x_p|_p \,. \tag{C.1}$$

The rational numbers $\mathbb{Q}$ are diagonally embedded into the adeles $\mathbb{A}$

$$\mathbb{Q} \subset \mathbb{A} \quad \text{in the sense that} \quad (q, q, q, q, \dots) \in \mathbb{A} \quad \text{for} \quad q \in \mathbb{Q} \,. \tag{C.2}$$

Product of norms For a rational number $x \in \mathbb{Q}$ with a decomposition into primes as

$$x = \pm \prod_p p^{m^{(p)}} \,, \tag{C.3}$$

we get a particularly simple result for the adelic norm of x, namely

$$\prod_{p \le \infty} |x|_p = |x|_\infty \prod_p p^{-m^{(p)}} = |x|_\infty |x|_\infty^{-1} = 1 \,. \tag{C.4}$$

This is most often used as

$$x \in \mathbb{Q} \quad \Rightarrow \quad \prod_p |x|_p = |x|_\infty^{-1} \,. \tag{C.5}$$

Greatest common divisor For a set of natural numbers $\{x_i\}$ where each x_i has a decomposition into primes as

$$x_i = \prod_p p^{m_i^{(p)}} , \tag{C.6}$$

one can express the greatest common divisor k as

$$k \equiv \gcd(\{x_i\}) = \prod_p p^{\min_i \left\{ m_i^{(p)} \right\}}. \tag{C.7}$$

Together with

$$|x_i|_p = p^{-m_i^{(p)}} , \tag{C.8}$$

we are led to the expression

$$k = \prod_p p^{\min_i \left\{ m_i^{(p)} \right\}} = \prod_p \min_i \left\{ p^{m_i^{(p)}} \right\} = \prod_p \min_i \left\{ |x_i|_p^{-1} \right\} = \prod_p \left(\max_i \{ |x_i|_p \} \right)^{-1} . \tag{C.9}$$

We also have the formula

$$|k|_p = \max_i \left\{ |x_i|_p \right\} . \tag{C.10}$$

Note that

$$\gcd(x_1, \ldots, x_n, 0) = \gcd(x_1, \ldots, x_n) , \tag{C.11}$$

since every (nonzero) integer divides 0. Additionally, we define

$$\gcd(x) = x \quad \forall x \in \mathbb{Z} , \tag{C.12}$$

including $x = 0$.

Divisor sum We have the identity

$$\prod_p \frac{1 - p^{-s} \, |m|_p^s}{1 - p^{-s}} = \sum_{d|m} d^{-s} \equiv \sigma_{-s}(m) , \tag{C.13}$$

for $s \in \mathbb{C}$ and $m \in \mathbb{Z}$ and $d|m$ denotes all divisors d of $|m|_\infty$.

The completed Riemann zeta function The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad \mathrm{Re}(s) > 1 \tag{C.14}$$

can be written as an Euler product as

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \quad \mathrm{Re}(s) > 1 \tag{C.15}$$

and can be analytically continued to the whole complex plane except at $s = 0$ and $s = 1$ where it has simple poles. This is done by defining the completed Riemann zeta function

$$\xi(s) \equiv \Gamma\left(\frac{s}{2}\right)\pi^{-s/2}\zeta(s) \tag{C.16}$$

which obeys the functional relation

$$\xi(s) = \xi(1-s) \tag{C.17}$$

as shown by Riemann.

p-adic gaussian The p-adic gaussian $\gamma_p : \mathbb{Q}_p \to \{0, 1\}$ is defined as

$$\gamma_p(x) = \begin{cases} 1, & |x|_p \leq 1 \\ 0, & |x|_p > 1 \end{cases} = \begin{cases} 1, & x \in \mathbb{Z}_p \\ 0, & x \notin \mathbb{Z}_p \,. \end{cases} \tag{C.18}$$

For a rational number x we then get

$$\prod_p \gamma_p(x) = \begin{cases} 1, & x \in \mathbb{Z}_p \forall p \\ 0, & \text{else} \end{cases} = \begin{cases} 1, & x \in \mathbb{Z} \\ 0, & \text{else}\,. \end{cases} \tag{C.19}$$

Notice also that for rational numbers $x_1, \ldots, x_n \in \mathbb{Q}$ and picking an $x \in \mathbb{Q}$ such that for all primes p

$$|x|_p = \max\{|x_1|_p, \ldots, |x_n|_p\}\,, \tag{C.20}$$

we have

$$\gamma_p(x) = \prod_{i=1}^{n} \gamma_p(x_i). \tag{C.21}$$

A consequence of this is that for an eulerian function depending only on the p-adic norms of its argument

$$f(x) = \prod_p f_p(|x|_p)\,, \tag{C.22}$$

then with x as in Eq. (C.20), we have

$$f(x)\prod_p \gamma_p(x) = \prod_p f_p(|x|_p)\gamma_p(x) = \prod_p f_p(|k|_p)\gamma_p(x) = f(k)\prod_p \gamma_p(x)\,, \tag{C.23}$$

where

$$k = \gcd(|x_1|_\infty, \ldots, |x_n|_\infty)\,. \tag{C.24}$$

This equation makes sense as $\prod_p \gamma_p(x)$ ensures that the left- and right hand sides are nonzero only when each x_i is integer for which k is well defined. We now see how a sum over rationals with x as in Eq. (C.20) can collapse to a sum over integers due to the p-adic gaussian

$$\sum_{x_1,\ldots,x_n} f(x)\prod_p \gamma_p(x) = \sum_{x_1,\ldots,x_n} f(k)\prod_p \gamma_p(x) = \sum_{x_1,\ldots,x_n\in\mathbb{Z}} f(k). \tag{C.25}$$

SL_2 **Whittaker function** The ramified (meaning m not necessarily unity) $\mathrm{SL}_2(\mathbb{A})$ Whittaker function evaluated at

$$g = (g_\infty, \mathbb{1}, \mathbb{1}, \ldots) = \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix} k, \mathbb{1}, \mathbb{1}, \ldots\right) \tag{C.26}$$

written as an Euler product reads

$$\begin{aligned} W_\alpha\left(2s\Lambda-\rho, m; g\right) &= W_\alpha\left(2s\Lambda-\rho, m; \left(\left(\begin{smallmatrix} 1 & x \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y & \\ & \frac{1}{y} \end{smallmatrix}\right) k, \mathbb{1}, \mathbb{1}, \ldots\right)\right) = \\ &= \mathbf{e}(mx) B_s(m,y) \prod_p \gamma_p(m)\left(1-p^{-2s}\right)\frac{1-p^{-2s+1}|m|_p^{2s-1}}{1-p^{-2s+1}}\,, \end{aligned} \tag{C.27}$$

where α is the simple root and Λ is the fundamental weight. Here

$$B_s(m,y) \equiv \frac{2\pi^s}{\Gamma(s)} y^{1/2}|m|_\infty^{s-1/2} K_{s-1/2}\left(2\pi|m|_\infty y\right) \tag{C.28}$$

should be seen as the archimedean SL_2-Whittaker function and each factor in the Euler product as the non-archimedean Whittaker functions. The product

$$\prod_p \gamma_p(m) \tag{C.29}$$

restricts to $m \in \mathbb{Z}$ as explained above. The expression can then be written as

$$W_\alpha\left(2s\Lambda-\rho, m; g\right) = \mathbf{e}(mx)\frac{2}{\xi(2s)} y^{1/2}|m|_\infty^{s-1/2}\sigma_{1-2s}(m) K_{s-1/2}\left(2\pi|m|_\infty y\right) \tag{C.30}$$

where the divisor sum makes sense since $m \in \mathbb{Z}$. Since the argument of the γ_p's and the productand giving rise to the divisor sum are always equal, the p-adic gaussians will restrict it to an integer and ensure that the divisor sum always makes sense, even if the argument was a complicated expression to begin with.

Notice how the factors of the Eulerian expression for the Riemann zeta function in the non-archimedean part combines with $\pi^s/\Gamma(s)$ in the archimedean part to form a completed Riemann zeta function.

Twisted character Let $m \in \mathbb{Q}$ and $\psi_{p,m}$ be an additive character on $\mathbb{Q}_p$ defined as

$$\psi_{\infty,m}(x) = \mathrm{e}^{2\pi \mathrm{i} mx}\,; \qquad m, x \in \mathbb{R} \quad \text{for real numbers} \tag{C.31}$$

$$\psi_{p,m}(x) = \mathrm{e}^{-2\pi \mathrm{i}[mx]_p}\,; \quad m, x \in \mathbb{Q}_p \quad \text{for } p\text{-adic numbers}\,. \tag{C.32}$$

A unitary multiplicative character on the unipotent radical $N(\mathbb{A})$ of the Borel subgroup of $\mathrm{SL}_n(\mathbb{A})$ can then be parametrized by $m_1, \ldots, m_{n-1} \in \mathbb{Q}$ as

$$\psi(n) = \psi(\mathrm{e}^{\sum_{\alpha\in\Delta_+} u_\alpha E_\alpha}) = \psi(\mathrm{e}^{\sum_{\alpha\in\Pi} u_\alpha E_\alpha}) = \prod_{p\leq\infty} \prod_{i=1}^{n} \psi_{p,m_i}\left((u_{\alpha_i})_p\right)\,, \tag{C.33}$$

where Δ_+ is the set of positive roots and $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\} \subset \Delta_+$ is the set of simple roots. The second equality is due to the fact that the additive character is only sensitive to the abelianization of $N(\mathbb{A})$. In the final equality, $(x_{\alpha_i})_p$ denotes the p-adic (or real) component of the adelic coordinate x_{α_i}.

For an element $a \in A(\mathbb{A})$, we would like to evaluate the twisted character

$$\psi^a(n) \equiv \psi(ana^{-1}). \tag{C.34}$$

Let $x_\alpha(t) = \exp(tE_\alpha)$ where $t \in \mathbb{A}$ and E_α is the positive Chevalley generator for the root α. For $t \in \mathbb{A}^\times$, define $w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t)$ and $h_\alpha(t) = w_\alpha(t)w_\alpha(1)^{-1}$. An element $a \in A(\mathbb{A})$ is then parametrized by [49]

$$a = \prod_{i=1}^{n-1} h_{\alpha_i}(y_i) \qquad y_i \in \mathbb{A}^\times\,, \tag{C.35}$$

where the different generators h_α commute for all simple roots $\alpha \in \Pi$ and are multiplicative in t_α. For a simple root α and a root β, we have that [49]

$$h_\alpha(y)x_\beta(u)h_\alpha(y)^{-1} = x_\beta(y^{\beta(H_\alpha)}u)\,. \tag{C.36}$$

Since the character ψ on N is sensitive only to the x_β with β a simple root, it is enough to consider

$$h_{\alpha_i}(y)x_{\alpha_j}(u)h_{\alpha_i}(y)^{-1} = x_{\alpha_j}(y^{A_{ij}}u) \tag{C.37}$$

for the simple roots α_i and α_j, where A_{ij} is the Cartan matrix.

We then have that

$$\begin{aligned}\psi(ana^{-1}) = \psi\left(\exp\left(\sum_{j=1}^{n-1}\left(\prod_{i=1}^{n-1} y_i^{A_{ij}}\right) x_{\alpha_j} E_{\alpha_j}\right)\right) = \\ = \psi\left(\exp\left(\frac{y_1^2}{y_2} x_{\alpha_1} E_{\alpha_1} + \sum_{j=2}^{n-2} \frac{y_j^2}{y_{j-1}y_{j+1}} x_{\alpha_j} E_{\alpha_j} + \frac{y_{n-1}^2}{y_{n-2}} x_{\alpha_{n-1}} E_{\alpha_{n-1}} + \dots\right)\right).\end{aligned} \tag{C.38}$$

We can interpret the transformation $\psi \to \psi^a$ as that the parameters m_i transform according to

$$m_i \to m_i' = \left(\frac{y_i^2}{y_{i-1}y_{i+1}}\right) m_i, \quad i = 1, \dots, n-1\,, \tag{C.39}$$

where we have defined $y_0 = y_n = 1$. Note that starting with rational parameters m_i, the transformed parameters m_i' are no longer necessarily rational.

D Parametrizing Γ_i and Λ_j

For this appendix, we will work over a general number field F rather than over the rationals specifically. Recall the definitions

$$\Gamma_i(\psi_0) := \begin{cases} (\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) & 1 \leq i \leq n-2 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_0} & i = n-1\,, \end{cases} \tag{D.1}$$

where

$$(\mathrm{SL}_{n-i}(F))_{\hat{Y}} = \left\{ \left(\begin{smallmatrix} 1 & \xi^{\intercal} \\ 0 & h \end{smallmatrix} \right) : h \in \mathrm{SL}_{n-i-1}(F), \xi \in F^{n-i-1} \right\} ; \tag{D.2}$$

and

$$\Lambda_j(\psi_0) := \begin{cases} (\mathrm{SL}_j(F))_{\hat{X}} \backslash \mathrm{SL}_j(F) & 2 \leq j \leq n-1 \\ (T_{\psi_0} \cap T_{\psi_{\alpha_1}}) \backslash T_{\psi_0} & j = 1\,, \end{cases} \tag{D.3}$$

where

$$(\mathrm{SL}_j(F))_{\hat{X}} = \left\{ \left(\begin{smallmatrix} h & \xi \\ 0 & 1 \end{smallmatrix} \right) : h \in \mathrm{SL}_{j-1}(F), \xi \in F^{j-1} \right\} . \tag{D.4}$$

T denotes the diagonal matrices in $\mathrm{SL}_n(F)$ and T_ψ denotes the stabilizer of a character ψ in T. In this appendix, we will find convenient representatives for these coset spaces. We begin with a lemma.

Lemma D.1
Let $S_k(F)$ denote the set of all $k \times k$ matrices m over the field F satisfying $\dim\ker m = 1$. *The coset space $\mathrm{GL}_k(F)\backslash S_k(F)$ can then be parametrized as*

$$\mathrm{GL}_k(F)\backslash S_k = \bigcup_{a=0}^{k-1} \left\{ \left(\begin{smallmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & v & I_{k-a-1} \end{smallmatrix} \right) : v \in F^{k-a-1} \right\} . \tag{D.5}$$

Proof. We will use induction to prove the lemma. Assume that the result holds up to and including matrices of size $k \times k$ and consider the coset space $\mathrm{GL}_{k+1}(F)\backslash S_{k+1}(F)$. Start with a matrix $m_{k+1} \in S_{k+1}(F)$. Left action of the group $\mathrm{GL}_{k+1}(F) \ni h_{k+1}$ taking $m_{k+1} \to h_{k+1}m_{k+1}$ is equivalent to performing Gauss elimination among the rows of m_{k+1}. Since we have $\dim\ker m_{k+1} = 1$ we can bring m_{k+1} to the form

$$m_{k+1} \to \left(\begin{smallmatrix} 0 & 0 \\ v & m \end{smallmatrix} \right), \tag{D.6}$$

where $v \in F^k$ and m satisfies $\dim\ker m_k \leq 1$. We cannot have $\dim\ker m \geq 2$ as we could then perform additional row manipulations to produce two zero rows in m and hence another zero row in m_{k+1} which violates $m_{k+1} \in S_{k+1}(F)$.

- **Case 1:** $\dim\ker m = 0$:
  Here m is invertible and we can bring m_{k+1} to the form

$$\left(\begin{smallmatrix} 0 & 0 \\ v & m_k \end{smallmatrix}\right) \to \left(\begin{smallmatrix} 0 & 0 \\ v & I_k \end{smallmatrix}\right) \tag{D.7}$$

  having relabelled v. We get the contribution

$$\left.\left\{ \begin{pmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & v & I_{k+1-a-1} \end{pmatrix} : v \in F^{k+1-a-1} \right\}\right|_{a=0} . \tag{D.8}$$

- **Case 2:** $\dim\ker m = 1$:
  We now have $m = m_k$ for some $m_k \in S_k$ and we can apply the induction assumption which leads us to consider matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ v^{(1)} & 0 & 0 & 0 \\ v^{(2)} & I_a & 0 & 0 \\ v^{(3)} & 0 & u & I_{k-a-1} \end{pmatrix}, \quad \text{where} \quad a \in [0, k-2] \cap \mathbb{Z} . \tag{D.9}$$

  We see that we must have $v^{(1)} \neq 0$ and with further row manipulations we can thus bring this to the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ v^{(1)} & 0 & 0 & 0 \\ v^{(2)} & I_a & 0 & 0 \\ v^{(3)} & 0 & u & I_{k-a-1} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & u & I_{k-a-1} \end{pmatrix} . \tag{D.10}$$

  We get the contributions

$$\bigcup_{a=0}^{k-2} \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-1} \end{pmatrix} : v \in F^{k-a-1} \right\} = \bigcup_{a=1}^{k-1} \left\{ \begin{pmatrix} 0 & 0 & 0 \\ I_a & 0 & 0 \\ 0 & v & I_{k+1-a-1} \end{pmatrix} : v \in F^{k+1-a-1} \right\} . \tag{D.11}$$

  This combines with the contribution from case 1 to give the form stated in the lemma.

That the base case $k = 1$ has the correct form is trivial. Peano's axiom of induction now establishes the lemma. □

Lemma D.2
The coset space

$$(\mathrm{SL}_{n-i}(F))_{\hat{Y}} \backslash \mathrm{SL}_{n-i}(F) \quad 1 \leq i \leq n-2 \tag{D.12}$$

can be parametrized as

$$
\begin{aligned}
&(\mathrm{SL}_{n-i}(F))_{\hat{Y}}\backslash \mathrm{SL}_{n-i}(F) = \\
&= \left\{ \begin{pmatrix} x'^{-1} & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{n-i-2} \end{pmatrix} : x' \in F^\times, y \in F, v \in F^{n-i-2} \right\} \\
&\cup \bigcup_{a=0}^{n-i-2} \left\{ \begin{pmatrix} 0 & 0 & (-1)^{a+1}x'^{-1} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{n-i-a-2} \end{pmatrix} : x' \in F^\times, v \in F^{n-i-a-2} \right\} .
\end{aligned}
\tag{D.13}
$$

Proof. Denote $k \equiv n - i$. Consider a matrix

$$
G = \begin{pmatrix} s & T^\mathsf{T} \\ B & m \end{pmatrix} \in \mathrm{SL}_k(F) , \tag{D.14}
$$

where s is a scalar, m is a $(k-1) \times (k-1)$-matrix and T and B (for "top" and "bottom") are $(k-1)$-column vectors. The action of an element

$$
M = \begin{pmatrix} 1 & \xi^\mathsf{T} \\ 0 & h \end{pmatrix} \in (\mathrm{SL}_k(F))_{\hat{Y}} \tag{D.15}
$$

on G is

$$
G \to MG = \begin{pmatrix} s+\xi^\mathsf{T} B & T^\mathsf{T}+\xi^\mathsf{T} m \\ hB & hm \end{pmatrix} . \tag{D.16}
$$

Parametrizing the coset space $(\mathrm{SL}_k(F))_{\hat{Y}}\backslash \mathrm{SL}_k(F)$ amounts to choosing $\xi \in F^{k-1}$ and $h \in \mathrm{SL}_{k-1}(F)$ such that the product MG takes a particularly nice form, manifestly with at most k degrees of freedom which is the dimension of this coset space.

Even though $h \in \mathrm{SL}_{k-1}(F)$ we will proceed with $h \in \mathrm{GL}_{k-1}(F)$ and restore the unit determinant of h at the end by left multiplication of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix}$ where $0 \neq x' = (\det h)^{-1}$. By having $h \in \mathrm{GL}_{k-1}(F)$ we are free to perform Gauss elimination among the bottom $k-1$ rows in G.

We consider the two cases $\dim\ker m = 0$ and $\dim\ker m = 1$. Note that the cases $\dim\ker m \geq 2$ do not arise as with row elimination it would then be possible to produce two zero-rows in m and hence a zero-row in G which violates $G \in \mathrm{SL}_k(F)$.

- **Case 1:** $\dim\ker m = 0$:
  We choose $h = m^{-1}$ and $\xi^\mathsf{T} = -T^\mathsf{T} m^{-1}$. Since h has full rank, we can redefine $hB \to B$ without loss of generality and redefine $s + \xi^\mathsf{T} B \to s$. This leads to the representative

  $$
  G \to \begin{pmatrix} s & 0 \\ B & I_{k-1} \end{pmatrix} . \tag{D.17}
  $$

  We now restore the unit determinant to h

  $$
  G \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix} \begin{pmatrix} s & 0 \\ B & I_{k-1} \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{k-2} \end{pmatrix} , \tag{D.18}
  $$

  where we have split the $(k-1)$-vector into a scalar y and a $(k-2)$-vector v. The condition $\det G = 1$ now sets $s = x'^{-1}$ leading to

$$G = \begin{pmatrix} x'^{-1} & 0 & 0 \\ y & x' & 0 \\ v & 0 & I_{k-2} \end{pmatrix}. \tag{D.19}$$

This is a nice form of the representative G which manifestly has k degrees of freedom.

- **Case 2:** $\dim \ker m = 1$:
We can no longer choose $h = m^{-1}$. Having promoted h to be an element of $\mathrm{GL}_{k-1}(F)$, we can make use of lemma D.1 which leads us to consider representatives of the form

$$G \to \begin{pmatrix} s & T^{(1)\mathrm{T}} & T^{(2)} & T^{(3)\mathrm{T}} \\ B^{(1)} & 0 & 0 & 0 \\ B^{(2)} & I_a & 0 & 0 \\ B^{(3)} & 0 & v & I_{k-a-2} \end{pmatrix} \quad \text{for} \quad a \in [0, k-2] \cap \mathbb{N} \quad \text{and} \quad v \in F^{k-a-2}. \tag{D.20}$$

We see that $B^{(1)} \neq 0$ in order for G to remain non-singular. With further row elimination we can therefore bring this to the form

$$\begin{pmatrix} s & T^{(1)\mathrm{T}} & T^{(2)} & T^{(3)\mathrm{T}} \\ B^{(1)} & 0 & 0 & 0 \\ B^{(2)} & I_a & 0 & 0 \\ B^{(3)} & 0 & v & I_{k-a-2} \end{pmatrix} \to \begin{pmatrix} s & T^{(1)\mathrm{T}} & T^{(2)} & T^{(3)\mathrm{T}} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}. \tag{D.21}$$

Next, using the ξ-freedom we can bring this to the form

$$\begin{aligned} &\begin{pmatrix} s & T^{(1)\mathrm{T}} & T^{(2)} & T^{(3)\mathrm{T}} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \to \begin{pmatrix} 1 & \xi^{(1)} & \xi^{(2)\mathrm{T}} & \xi^{(3)\mathrm{T}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_a & 0 \\ 0 & 0 & 0 & I_{k-a-2} \end{pmatrix} \begin{pmatrix} s & T^{(1)\mathrm{T}} & T^{(2)} & T^{(3)\mathrm{T}} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \\ &= \begin{pmatrix} s+\xi^{(1)} & T^{(1)\mathrm{T}}+\xi^{(2)\mathrm{T}} & T^{(2)}+\xi^{(3)\mathrm{T}}v & T^{(3)\mathrm{T}}+\xi^{(3)\mathrm{T}} \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \to \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}, \end{aligned} \tag{D.22}$$

with a suitable choice of ξ and having redefined $T^{(2)}$. We now restore the unit determinant to h

$$\begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & I_{k-2} \end{pmatrix} \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix} \to= \begin{pmatrix} 0 & 0 & T^{(2)} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}. \tag{D.23}$$

The condition $\det G = 1$ now sets $T^{(2)} = (-1)^{a+1}x'^{-1}$, leading to the representative

$$\begin{pmatrix} 0 & 0 & (-1)^{a+1}x'^{-1} & 0 \\ x' & 0 & 0 & 0 \\ 0 & I_a & 0 & 0 \\ 0 & 0 & v & I_{k-a-2} \end{pmatrix}. \tag{D.24}$$

□

Lemma D.3

The coset space

$$(\mathrm{SL}_j(F))_{\hat{X}}\backslash \mathrm{SL}_j(F) \quad 2 \leq j \leq n-1 \tag{D.25}$$

can be parametrized as

$$\begin{aligned} &(\mathrm{SL}_j(F))_{\hat{X}}\backslash \mathrm{SL}_j(F) = \\ &= \left\{ \begin{pmatrix} I_{j-2} & 0 & 0 \\ 0 & x' & 0 \\ v^\mathsf{T} & y & x'^{-1} \end{pmatrix} : x' \in F^\times, y \in F, v \in F^{j-2} \right\} \\ &\cup \bigcup_{a=0}^{j-2} \left\{ \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{j-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & (-1)^{j+a+1}x'^{-1} & v^\mathsf{T} & 0 \end{pmatrix} : x' \in F^\times, v \in F^{j-a-2} \right\}. \end{aligned} \tag{D.26}$$

Proof. Denote $k \equiv j$. Consider a matrix

$$G = \begin{pmatrix} m & T \\ B^\mathsf{T} & s \end{pmatrix} \in \mathrm{SL}_k(F), \tag{D.27}$$

where s is a scalar, m is a $(k-1) \times (k-1)$-matrix and T and B (for "top" and "bottom") are $(k-1)$-column vectors. The action of an element

$$M = \begin{pmatrix} h & h\xi \\ 0 & 1 \end{pmatrix} \in (\mathrm{SL}_k(F))_{\hat{X}} \tag{D.28}$$

on G is

$$G \to MG = \begin{pmatrix} h(m+\xi B^\mathsf{T}) & h(T+s\xi) \\ B^\mathsf{T} & s \end{pmatrix}. \tag{D.29}$$

Parametrizing the coset space $(\mathrm{SL}_k(F))_{\hat{X}}\backslash \mathrm{SL}_k(F)$ amounts to choosing $\xi \in F^{k-1}$ and $h \in \mathrm{SL}_{k-1}(F)$ such that the product MG takes a particularly nice form, manifestly with at most k degrees of freedom which is the dimension of this coset space.

Even though $h \in \mathrm{SL}_{k-1}(F)$ we will proceed with $h \in \mathrm{GL}_{k-1}(F)$ and restore the unit determinant of h at the end by left multiplication of the matrix $\begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $0 \neq x' = (\det h)^{-1}$. By having $h \in \mathrm{GL}_{k-1}(F)$ we are free to perform Gauss elimination among the top $k-1$ rows in G.

We consider the two cases $s \neq 0$ and $s = 0$.

- **Case 1:** $s = s' \neq 0$:
  We choose $\xi = \frac{-1}{s'}T$. This leads to the representative

$$G \to \begin{pmatrix} m - \frac{1}{s'}TB^\mathsf{T} & 0 \\ B^\mathsf{T} & s' \end{pmatrix}. \tag{D.30}$$

  From the condition

$$1 = \det G = \det\left(m - \frac{1}{s'}TB^\mathsf{T}\right)s' \tag{D.31}$$

  we get that

$$\det\left(m - \frac{1}{s'}TB^\mathsf{T}\right) \neq 0,$$

and hence the matrix $m - \frac{1}{s'}TB^\mathsf{T}$ can be inverted using our h-freedom which leads to the representative

$$\begin{pmatrix} m-\frac{1}{s'}TB^\mathsf{T} & 0 \\ B^\mathsf{T} & s' \end{pmatrix} \to \begin{pmatrix} I_{k-1} & 0 \\ B^\mathsf{T} & s' \end{pmatrix}. \tag{D.32}$$

We now restore the unit determinant to h

$$G \to \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_{k-1} & 0 \\ B^\mathsf{T} & s' \end{pmatrix} = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ v^\mathsf{T} & y & s' \end{pmatrix}, \tag{D.33}$$

where we have split B into a scalar y and a $(k-2)$-vector v. The condition $\det G = 1$ now sets $s' = x'^{-1}$ leading to

$$G = \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ v^\mathsf{T} & y & x'^{-1} \end{pmatrix}. \tag{D.34}$$

This is a nice form of the representative G which manifestly has k degrees of freedom.

- **Case 2:** $s = 0$:
We can no longer eliminate T with our ξ-freedom. The group element G takes the form

$$G = \begin{pmatrix} m & T' \\ B'^\mathsf{T} & 0 \end{pmatrix}, \tag{D.35}$$

where the vectors T' and B' must be non-zero (as indicated by the primes) in order for G to be non-singular. A ξ-transformation takes the form

$$\begin{pmatrix} m & T' \\ B'^\mathsf{T} & 0 \end{pmatrix} \to \begin{pmatrix} m+\xi B'^\mathsf{T} & T' \\ B'^\mathsf{T} & 0 \end{pmatrix}. \tag{D.36}$$

We now consider the $k-1$ distinct cases labelled by $a \in [0, k-2] \cap \mathbb{N}$ defined by that B'^T takes the form $B'^\mathsf{T} = \begin{pmatrix} 0_{1\times a} & b' & v \end{pmatrix}$, where v is a $k-a-2$-vector and $0 \neq b' \in F$. The ξ-transformation then lets us eliminate the $(a+1)^{\text{th}}$ column of m. This works since the $(a+1)^{\text{th}}$ column of the matrix $\xi B'^\mathsf{T}$ is $b'\xi$ where $b' \neq 0$ by assumption. We are led to the representative

$$\begin{pmatrix} m & T' \\ B'^\mathsf{T} & 0 \end{pmatrix} \to \begin{pmatrix} m_1 & 0 & m_2 & T' \\ 0 & b' & v^\mathsf{T} & 0 \end{pmatrix}, \tag{D.37}$$

where m_1 is a $(k-1) \times a$-matrix and m_2 is a $(k-1) \times (k-a-2)$-matrix.

The $(k-1) \times (k-1)$ matrix $\begin{pmatrix} m_1 & 0 & m_2 \end{pmatrix}$ clearly has column-rank at most $k-2$. Since column-rank and row-rank for matrices are equal, we know that the row-rank is also at most $k-2$ and with row manipulations we can thus produce a zero row

$$\begin{pmatrix} m_1 & 0 & m_2 & T' \\ 0 & b' & v^\mathsf{T} & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & t' \\ m_{21} & 0 & m_{22} & T^- \\ 0 & b' & v^\mathsf{T} & 0 \end{pmatrix}, \tag{D.38}$$

where $0 \neq t' \in F$ in order for G to be non-singular. With further row manipulations we can then eliminate the vector T^- and bring this to the form

$$\begin{pmatrix} 0 & 0 & 0 & t' \\ m_{21} & 0 & m_{22} & T^- \\ 0 & b' & v^\intercal & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ m_{21} & 0 & m_{22} & 0 \\ 0 & b' & v^\intercal & 0 \end{pmatrix}. \tag{D.39}$$

The $(k-2) \times (k-2)$-matrix $\begin{pmatrix} m_{21} & m_{22} \end{pmatrix}$ must have full rank in order for G to be non-singular and can thus be inverted, leading to the representative

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ m_{21} & 0 & m_{22} & 0 \\ 0 & b' & v^\intercal & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & b' & v^\intercal & 0 \end{pmatrix}. \tag{D.40}$$

We permute the first $k-1$ rows to get

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & b' & v^\intercal & 0 \end{pmatrix} \to \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^\intercal & 0 \end{pmatrix}. \tag{D.41}$$

Lastly, we restore the unit determinant to h

$$\begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^\intercal & 0 \end{pmatrix} \to \begin{pmatrix} I_{k-2} & 0 & 0 \\ 0 & x' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b' & v^\intercal & 0 \end{pmatrix} \to = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & b' & v^\intercal & 0 \end{pmatrix}. \tag{D.42}$$

The condition $\det G = 1$ now sets $b' = (-1)^{k+a+1} x'^{-1}$, leading to the representative

$$\begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & 0 & I_{k-a-2} & 0 \\ 0 & 0 & 0 & x' \\ 0 & (-1)^{k+a+1} x'^{-1} & v^\intercal & 0 \end{pmatrix}. \tag{D.43}$$

□

Remark D.4.
Another way of parametrizing the coset $(\mathrm{SL}_k(F))_{\hat{X}} \backslash \mathrm{SL}_k(F)$ is to parametrize the coset $\mathrm{SL}_k(F)/(\mathrm{SL}_k(F))_{\hat{X}}$ which works analogously to how the coset $(\mathrm{SL}_k(F))_{\hat{Y}} \backslash \mathrm{SL}_k(F)$ was parametrized in lemma D.2 and then invert the resulting matrices.

We lastly provide parametrizations of $\Gamma_{n-1}(\psi_{\alpha_i})$ for the cases $1 \le i \le n-3$ and of $\Lambda_1(\psi_{\alpha_j})$ for the cases $3 \le j \le n-1$.

Lemma D.5
The coset space $\Gamma_{n-1}(\psi_{\alpha_i}) = (T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_i}}$ *can be parametrized as*

$$(T_{\psi_{\alpha_1}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_1}} = \left\{ \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & z' & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 & \\ & & & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\} \tag{D.44}$$

for $i = 1$ while for $2 \le i \le n-3$ it can be parametrized as

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}) \backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' &&&& \\ & 1 &&& \\ && \ddots && \\ &&& 1 & \\ &&&& z'^{-1} \end{pmatrix} : z' \in F^\times \right\}. \tag{D.45}$$

Further possible parametrizations are discussed in the proof below.

Proof. A stabilizer $T_{\psi_{\alpha_k}}$ is given by

$$T_{\psi_{\alpha_k}} = \left\{ \begin{pmatrix} t_1 &&& \\ & y_k && \\ && y_k & \\ &&& t_2 \end{pmatrix} \leftarrow \text{row } k : y_k \in F^\times, \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T'(y_k^{-2}) \right\}, \quad 1 \le k \le n-1 \tag{D.46}$$

where $T'(y_k^{-2})$ are the diagonal matrices in $\mathrm{GL}_{n-2}(F)$ with determinant y_k^{-2}. This is seen by the group action of a diagonal element $h \in T_{\alpha_k}$ on an upper triangular element $n \in N$,

$$hnh^{-1} = \begin{pmatrix} * &&& \\ & y_k && \\ && y_{k+1} & \\ &&& * \end{pmatrix} \begin{pmatrix} * & * & * \\ & 1 \; x_{i,i+1} & * \\ & 1 & * \\ && * \end{pmatrix} \begin{pmatrix} * &&& \\ & y_k^{-1} && \\ && y_{k+1}^{-1} & \\ &&& * \end{pmatrix} = \begin{pmatrix} * & * & * \\ & 1 \; \frac{y_k}{y_{k+1}} x_{i,i+1} & * \\ & 1 & * \\ && * \end{pmatrix}. \tag{D.47}$$

Since the expression $\psi_{\alpha_k}(n)$ only depends on $x_{i,i+1}$ we require $y_k = y_{k+1}$ in order to have $\psi(n) = \psi(hnh^{-1})$. We get that the intersection $T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}}$ can be written

$$T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}} = \left\{ \begin{pmatrix} t_1 &&&&& \\ & y_i &&&& \\ && y_i &&& \\ &&& t_2 && \\ &&&& y_{n-1} & \\ &&&&& y_{n-1} \end{pmatrix} : y_i, y_{n-1} \in F^\times ; \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \in T''(y_i^{-2} y_{n-1}^{-2}) \right\} \tag{D.48}$$

where $T''(y_i^{-2} y_{n-1}^{-2})$ are the diagonal matrices in $\mathrm{GL}_{n-4}(F)$ with determinant $y_i^{-2} y_{n-1}^{-2}$. Looking at the left action of an element of this intersection on an element of the stabilizer T_{α_i} gives

$$\begin{aligned} &\begin{pmatrix} t_1 &&&&& \\ & y_i &&&& \\ && y_i &&& \\ &&& t_2 && \\ &&&& y_{n-1} & \\ &&&&& y_{n-1} \end{pmatrix} \begin{pmatrix} u_1 &&&&& \\ & \alpha_i &&&& \\ && \alpha_i &&& \\ &&& u_2 && \\ &&&& \alpha_{n-1} & \\ &&&&& \alpha_n \end{pmatrix} = \\ &= \begin{pmatrix} t_1 u_1 &&&&& \\ & y_i \alpha_i &&&& \\ && y_i \alpha_i &&& \\ &&& t_2 u_2 && \\ &&&& y_{n-1} \alpha_{n-1} & \\ &&&&& y_{n-1} \alpha_n \end{pmatrix}. \end{aligned} \tag{D.49}$$

The job is now to choose t_1, y_i, t_2 and y_{n-1} to eliminate degrees of freedom. We pick $y_i = \alpha_i^{-1}$ and with y_{n-1} we can eliminate either α_{n-1} or α_n but not both. After fixing t_1 and t_2 as well we can arrive at a particular class of convenient final forms

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} I_a && \\ & z' & \\ && I_{n-3-a} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} z'^{-1} & \\ & 1 \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 1 & \\ & z'^{-1} \end{pmatrix} \tag{D.50}$$

where the allowed choices for a are $a = 0, \dots, i-2, i+1, \dots, n-3$.

For $i = 1$ we can choose

$$(T_{\psi_{\alpha_1}} \cap T_{\psi_{\alpha_{n-1}}})\backslash T_{\psi_{\alpha_1}} = \left\{ \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & z' & & & & \\ & & & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\} \tag{D.51}$$

while for $2 \leq i \leq n-3$ we can choose

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_{n-1}}})\backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}, \quad 2 \leq i \leq n-3. \tag{D.52}$$

∎

Lemma D.6
The coset space $\Lambda_1(\psi_{\alpha_i}) = (T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_1}})\backslash T_{\psi_{\alpha_i}}$ can be parametrized as

$$(T_{\psi_{\alpha_{n-1}}} \cap T_{\psi_{\alpha_1}})\backslash T_{\psi_{\alpha_{n-1}}} = \left\{ \begin{pmatrix} z' & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & z'^{-1} & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} : z' \in F^\times \right\} \tag{D.53}$$

for $i = n-1$ while for $3 \leq i \leq n-2$ it can be parametrized as

$$(T_{\psi_{\alpha_i}} \cap T_{\psi_{\alpha_1}})\backslash T_{\psi_{\alpha_i}} = \left\{ \begin{pmatrix} z' & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & z'^{-1} \end{pmatrix} : z' \in F^\times \right\}. \tag{D.54}$$

Further possible parametrizations can be deduced from the proof of lemma D.5.

Proof. Analogous to the proof of lemma D.5. ∎

Remark D.7.
The lemmas can be extended to include the cosets $\Gamma_{n-1}(\psi_{\alpha_{n-2}})$ and $\Gamma_{n-1}(\psi_{\alpha_{n-1}})$ which is trivial (as well as the corresponding Λ-cosets). They are however never needed in theorem 6.2.

E Proving Eq. (2.43)

After putting in explicit expressions for the axions and dilatons, proving Eq. (2.43) is equivalent to proving

$$\begin{aligned} X \equiv \epsilon\left(V_\mu, V_r, \ldots, V_n; V_\nu, V_r, \ldots, V_n\right) \epsilon\left(V_{r+1}, \ldots, V_n; V_{r+1}, \ldots, V_n\right) = \\ = \epsilon\left(V_\mu, V_{r+1}, \ldots, V_n; V_\nu, V_{r+1}, \ldots, V_n\right) \epsilon\left(V_r, \ldots, V_n; V_r, \ldots, V_n\right) \\ -\epsilon\left(V_\mu, V_{r+1}, \ldots, V_n; V_r, \ldots, V_n\right) \epsilon\left(V_\nu, V_{r+1}, \ldots, V_n; V_r, \ldots, V_n\right) \equiv Y - Z \end{aligned} \tag{E.1}$$

To this end, one may use Laplace expansion for the generalized Kronecker delta

$$\begin{aligned} \delta^{I\, i_1 \dots i_m}_{A\, a_1 \dots a_m} &= \delta^I_A \delta^{i_1 \dots i_m}_{a_1 \dots a_m} - \sum_{k=1}^{m} \delta^{i_k}_A \delta^{i_1 \dots i_{k-1}\, I\, i_{k+1} \dots i_m}_{a_1 \dots a_{k-1}\, a_k\, a_{k+1} \dots a_m} = \\ &= \delta^I_A \delta^{i_1 \dots i_m}_{a_1 \dots a_m} - \sum_{k=1}^{m} \delta^{I}_{a_k} \delta^{i_1 \dots i_{k-1}\, i_k\, i_{k+1} \dots i_m}_{a_1 \dots a_{k-1}\, A\, a_{k+1} \dots a_m} . \end{aligned} \tag{E.2}$$

In what follows, we will not write out the V's. The expressions are assumed to be fully contracted with vectors corresponding to the indices. For example, an index a_{r+1} is assumed to be contracted with $(V_{r+1})^{a_{r+1}}$ and in particular the index A is contracted with $(V_\mu)^A$ and I is contracted with $(V_\nu)_I$. We get

$$
\begin{aligned}
X &= \delta^{I\, i_r \cdots i_n}_{A a_r \cdots a_n} \delta^{j_{r+1} \cdots j_n}_{b_{r+1} \cdots b_n} = \\
&= \left(\delta^I_A \delta^{i_r \cdots i_n}_{a_r \cdots a_n} - \sum_{k=r}^{n} \delta^{i_k}_A \delta^{i_r \cdots i_{k-1}\, I \;\, i_{k+1} \cdots i_n}_{a_r \cdots a_{k-1} a_k a_{k+1} \cdots a_n} \right) \delta^{j_{r+1} \cdots j_n}_{b_{r+1} \cdots b_n} = \\
&= \delta^{i_r \cdots i_n}_{a_r \cdots a_n} \left(\delta^{I\, j_{r+1} \cdots j_n}_{A b_{r+1} \cdots b_n} + \sum_{k=r+1}^{n} \delta^{j_k}_A \delta^{j_{r+1} \cdots j_{k-1}\, I \;\, j_{k+1} \cdots j_n}_{b_{r+1} \cdots b_{k-1} b_k b_{k+1} \cdots b_n} \right) \\
&- \sum_{k=r}^{n} \delta^{i_r \cdots i_{k-1}\, I \;\, i_{k+1} \cdots i_n}_{a_r \cdots a_{k-1} a_k a_{k+1} \cdots a_n} \left(\delta^{i_k\, j_{r+1} \cdots j_n}_{A\, b_{r+1} \cdots b_n} + \sum_{l=r+1}^{n} \delta^{j_l}_A \delta^{j_{r+1} \cdots j_{l-1}\, i_k\, j_{l+1} \cdots j_n}_{b_{r+1} \cdots b_{l-1} b_l b_{l+1} \cdots b_n} \right) = \\
&= Y - Z + \delta^{i_r \cdots i_n}_{a_r \cdots a_n} \sum_{k=r+1}^{n} \delta^{j_k}_A \delta^{j_{r+1} \cdots j_{k-1}\, I \;\, j_{k+1} \cdots j_n}_{b_{r+1} \cdots b_{k-1} b_k b_{k+1} \cdots b_n} \\
&- \delta^{I \;\, i_{r+1} \cdots i_n}_{a_r a_{r+1} \cdots a_n} \sum_{l=r+1}^{n} \delta^{j_l}_A \delta^{j_{r+1} \cdots j_{l-1}\, i_r\, j_{l+1} \cdots j_n}_{b_{r+1} \cdots b_{l-1} b_l b_{l+1} \cdots b_n} \\
&- \sum_{k=r+1}^{n} \delta^{i_r \cdots i_{k-1}\, I \;\, i_{k+1} \cdots i_n}_{a_r \cdots a_{k-1} a_k a_{k+1} \cdots a_n} \delta^{j_k}_A \delta^{j_{r+1} \cdots j_{k-1}\, i_k\, j_{k+1} \cdots j_n}_{b_{r+1} \cdots b_{k-1} b_k b_{k+1} \cdots b_n} = \\
&= Y - Z + \sum_{k=r+1}^{n} \delta^{j_k}_A \Big(\delta^{i_r \cdots i_n}_{a_r \cdots a_n} \qquad \delta^{I \;\, j_{r+1} \cdots j_{k-1} j_{k+1} \cdots j_n}_{b_k b_{r+1} \cdots b_{k-1} b_{k+1} \cdots b_n} \\
&\qquad - \delta^{j_{r+1} \cdots j_{k-1}\, i_r\, j_{k+1} \cdots j_n}_{b_{r+1} \cdots b_{k-1} b_k b_{k+1} \cdots b_n} \qquad \delta^{I \;\, i_{r+1} \cdots i_n}_{a_r a_{r+1} \cdots a_n} \\
&\qquad - \delta^{i_{r+1} \cdots i_n}_{b_{r+1} \cdots b_n} \qquad \delta^{I \;\, j_r \cdots j_{k-1} j_{k+1} \cdots j_n}_{a_k a_r \cdots a_{k-1} a_{k+1} \cdots a_n} \Big) \equiv \\
&\equiv Y - Z + \sum_{k=r+1}^{n} \delta^{j_k}_A \left(R_1 - R_2 - R_3 \right).
\end{aligned}
\tag{E.3}
$$

The job is now to show that the remainder vanishes. We expand R_1

$$\begin{aligned}
R_1 &= \delta^{i_r \text{—} i_n}_{a_r \text{—} a_n}\, \delta^{I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{b_k\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n} = \\
&= \left(\delta^{i_r}_{a_r}\, \delta^{i_{r+1}\text{—}i_n}_{a_{r+1}\text{—}a_n} - \sum_{l=r+1}^{n} \delta^{i_l}_{a_r}\, \delta^{i_{r+1}\text{—}i_{l-1}\, i_r\, i_{l+1}\text{—}i_n}_{a_{r+1}\text{—}a_{l-1}\, a_l\, a_{l+1}\text{—}a_n} \right) \delta^{I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{b_k\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n} = \\
&= \delta^{i_{r+1}\text{—}i_n}_{a_{r+1}\text{—}a_n} \Bigg(\underbrace{\delta^{i_r\, I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{a_r\, b_k\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n}}_{R_3} + \underbrace{\delta^{i_r}_{b_k}\, \delta^{I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{a_r\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n}}_{\text{I}} + \\
&+ \underbrace{\sum_{m=r+1}^{k-1} \delta^{i_r}_{b_m}\, \delta^{I\; j_{r+1}\text{—}j_{m-1}\, j_m\, j_{m+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{a_r\, b_{r+1}\text{—}b_{m-1}\, a_r\; b_{m+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n}}_{\text{II}} \\
&+ \underbrace{\sum_{m=r+1}^{k-1} \delta^{i_r}_{b_m}\, \delta^{I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_{m-1}\, j_m\, j_{j+1}\text{—}j_n}_{a_r\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_{m-1}\, a_r\; b_{m+1}\text{—}b_n}}_{\text{III}} \Bigg) \\
&- \sum_{l=r+1}^{n} \delta^{i_{r+1}\text{—}i_{l-1}\, i_r\, i_{l+1}\text{—}i_n}_{a_{r+1}\text{—}a_{l-1}\, a_l\, a_{l+1}\text{—}a_n} \Bigg(\underbrace{\delta^{i_l\, I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{a_r\, b_k\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n}}_{R_2} \\
&+ \underbrace{\delta^{i_l}_{b_k}\, \delta^{I\; j_{r+1}\text{—}\; j_{k-1}\, j_{k+1}\text{—}j_n}_{a_r\, b_{r+1}\text{—}\; b_{k-1}\, b_{k+1}\text{—}\; b_n}}_{\text{I}} + \\
&+ \underbrace{\sum_{m=r+1}^{k-1} \delta^{i_l}_{b_m}\, \delta^{I\; j_{r+1}\text{—}j_{m-1}\, j_m\, j_{m+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_n}_{b_k\, b_{r+1}\text{—}b_{m-1}\, a_r\; b_{m+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_n}}_{\text{II}} \\
&+ \underbrace{\sum_{m=k+1}^{n} \delta^{i_l}_{b_m}\, \delta^{I\; j_{r+1}\text{—}j_{k-1}\, j_{k+1}\text{—}j_{m-1}\, j_m\, j_{m+1}\text{—}j_n}_{b_k\, b_{r+1}\text{—}b_{k-1}\, b_{k+1}\text{—}b_{m-1}\, a_r\; b_{m+1}\text{—}b_n}}_{\text{III}} \Bigg).
\end{aligned} \tag{E.4}$$

The terms marked I, II and III cancel out by writing

$$\begin{aligned}
\delta^{i_r}_{b_k}\, \delta^{i_{r+1}\text{—}i_n}_{a_{r+1}\text{—}a_n} &= \delta^{i_r\, i_{r+1}\text{—}i_n}_{b_k\, a_{r+1}\text{—}a_n} + \sum_{l=r+1}^{n} \delta^{i_l}_{b_k}\, \delta^{i_{r+1}\text{—}i_{l-1}\, i_r\, i_{l+1}\text{—}i_n}_{a_{r+1}\text{—}a_{l-1}\, a_l\, a_{l+1}\text{—}a_n} = \\
&= = \sum_{l=r+1}^{n} \delta^{i_l}_{b_k}\, \delta^{i_{r+1}\text{—}i_{l-1}\, i_r\, i_{l+1}\text{—}i_n}_{a_{r+1}\text{—}a_{l-1}\, a_l\, a_{l+1}\text{—}a_n}
\end{aligned} \tag{E.5}$$

and similarly for terms II and III with $\delta^{i_r}_{b_m}\, \delta^{i_{r+1}\text{—}i_n}_{a_{r+1}\text{—}a_n}$.

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