4

Radiation Reaction

In the previous sections, almost all the results are obtained under the supposition that a character of the emitter motion has been prescribed. The physical factors providing this motion are deliberately ignored. Evidently, such statement of the problem is not quite correct. Really, the electromagnetic radiation always carries away certain amounts of the emitter energy and momentum. These losses either alter the corresponding characteristics of the emitter motion or have to be compensated by an external field providing the given motion.

Alterations in dynamic characteristics of a charged particle conditioned by the radiation field are negligible only within a limited time interval when these alterations are sufficiently small. For instance, the absence of noticeable violations of the conditions for the wave–particle synchronism, which considerably influence the radiation parameters, may serve as a criterion of the neglect admissibility. However, being justified in many cases, this approximation evidently contradicts the problem of transferring an essential amount of the particle energy to the wave emitted. Under experimental conditions, this problem can be partly solved by the corresponding alterations in the medium parameters. In particular, an adiabatic diminution of the wave phase velocity in amplifiers of the TWT and free electron laser types serves this purpose as well as the phase velocity increase along the accelerating structure in ion liner accelerators.

The exact compensation for the alterations in the emitter parameters with the help of external fields is impossible. The matter is that the characteristics of the electromagnetic field “prescribing” the emitter motion (and there exist no other means of influencing a charged particle) essentialy differ from those of the microwave radiation field. Consequently, external field cannot provide fulfillment of the corresponding conservation laws. The simplest case of the emitter uniform and rectilinear motion in a material medium makes the only exception. Under such conditions, the emitter energy losses by Cherenkov radiation can rather precisely be compensated by an external electric field, spatially homogeneous, constant in time and directed along the emitter trajectory.

Below we describe the physical processes responsible for the radiation reaction force as well as the quantitative characteristics of the force influence on the emitter dynamics.

4.1 Conservation Laws

In general, the problem of compensating alterations in the emitter trajectory parameters by external fields has no solution. The basic physical reason of this consists in the principal difference between the mechanical conservation laws for a point charged particle and for the radiation field. The electrodynamic conservation laws are integral ones (i.e., they are not localized either in space or in time). In particular, the law of conservation of the field energy is reducible to the following statement: within a closed volume, temporal alterations in the total energy of the field and emitting particles are equal to the time average flux of Poynting vector through the surface bounding this volume. Were the radiation field strengths decreasing in a sufficiently quick manner with the growth of a distance from the emitter, it would be possible to neglect Poynting vector flux by choosing the bounding surface at a distance large enough. Then the supposition about preservation of the "field + emitter" system total energy within the given volume would be justified. However, it is clear that the radiation field does not satisfy this supposition because of the very definition of this field. Therefore, the time–average total flux of Poynting vector has to be equal to the rate of the emitter energy variation (naturally, time averaged as well).[1] The analogous considerations are also true for the particle linear and angular momenta.

The reasoning presented indicates that to preserve the equation of motion in the standard form, it must be complemented with a force providing fulfillment of the conservation laws – if only on an average. This force is called the radiation reaction force. The averaging should be included because variations in the particle energy and of the power flux of the electromagnetic radiation field do not coincide either in time or in space. Not containing the degrees of the radiation field freedom, this equation (with taking into account the radiation reaction force) is irreducible to the canonical (Hamiltonian) form. Therefore, sometimes they define the radiation reaction force as the radiation friction or the force of deceleration by radiation. The necessity of introducing the radiation reaction force becomes especially clear by using as an example the plane-wave scattering by a free charged particle (see Sect. 3.6). There we have considered not the emitter motion to be prescribed but the electromagnetic field stimulating this motion. Surely, in the problem examined, an

[1] In a condensed medium, a part of the emitter kinetic energy is spent on exciting the medium polarization oscillations (see Sect. 2.1) and also on ionization of the medium atoms [10].

intensity of the transmitted wave has to be less than that of the incident one due to emergence of a scattered wave. However, this simultaneously implies the incident wave momentum decrease. At the same time, the scattered wave does not carry away any momentum (at least, in the dipole approximation). Consequently, an extra momentum, taken away from the incident wave and not carried away by the scattered one, has to be absorbed by the particle. The momentum absorbtion may be interpreted as the incident wave pressure. As a rule, this pressure may be presented as Lorentz force, described by the vector product of the magnetic field strength of the wave under emission and the particle velocity driven by the wave electric field. However, this explanation lacks in an important detail. That is, as the particle velocity is $\pi/2$-shifted in phase with respect to the wave electric field (and with the corresponding magnetic field as well), the average value of Lorentz force is precisely equal to zero. The only possible way out of this contradiction could consist in supposing that the particle acceleration conditioned by the electric field of the wave be phase–shifted with respect to the latter. In its turn, this implies the necessity to introduce an additional friction force – the radiation reaction – into the equation of motion.

Surely, the radiation reaction force may (and must) be interpreted as a result of the particle proper field action. During the emitter uniform rectilinear motion in vacuum, such an interaction does not occur. This follows even from the condition of symmetry of a vacuum environment with respect to the emitter.[2] In general, the calculations of the radiation backward influence on the emitter under the conditions of its acceleration is complicated by some difficulties of the fundamental nature, the discussion of which is beyond the scope of this monograph. Such calculations are reliable solely in the case of the emitter acceleration low enough. Below we will restrict ourselves to the simplifying considerations following from the conservation laws.

As regards a reference frame where the particle velocity is low, one may consider the radiation to be of the dipole nature and the energy flux carried away with the radiation to be proportional to the average square of the emitter acceleration. Consequently, the radiation reaction force $\mathbf{F}_{\mathrm{rad}}$ must meet the condition

$$\left\langle \mathbf{F}_{\mathrm{rad}}\mathbf{v} + \frac{2q^2}{3c^3}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)^2 \right\rangle = 0\,, \tag{4.1}$$

where $\langle\ \rangle$ signifies the time average.

As the right–hand side of (4.1) is equal to zero, the expression in the angular brackets must represent the total derivative with respect to time of a certain function of velocity and its derivatives. Presenting the acceleration squared as

$$\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right) - \mathbf{v}\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} \tag{4.2}$$

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[2] In the meantime, we are not going to consider here the problem of the radiation reaction during the emitter motion in a material medium.

we now rewrite (4.1) in the following form:

$$\left\langle \left(\mathbf{F}_{\mathrm{rad}} - \frac{2q^2}{3c^3} \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2} \right) \mathbf{v} \right\rangle = 0 \; . \tag{4.3}$$

Judging by this formula, one may put in a rather general case

$$\mathbf{F}_{\mathrm{rad}} = -\mu \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \frac{2q^2}{3c^3} \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2} \; . \tag{4.4}$$

The first addendum, containing an undetermined factor μ, has the form of a nonrelativistic inertial force. It indicates that at least a part of the emitter inertial mass has to be conditioned by the electromagnetic field surrounding the particle (but again on an average). However, this is rather the problem of interpretation because no other mass except for the mass directly measured in the experiment and already included into the equation of motion can be ascribed to the emitter.[3]

Thus, we have demonstrated that in the system under consideration the equation of motion must be complemented with the radiation reaction force, proportional to the time derivative of the particle acceleration:

$$\mathbf{F}_{\mathrm{rad}} = \frac{2q^2}{3c^3} \frac{\mathrm{d}^2 \mathbf{v}}{\mathrm{d}t^2} \; . \tag{4.5}$$

The procedure of the time averaging used in the presented "derivation" of (4.5), based on the demand of fulfilling the energy conservation law. This can produce an impression that (4.5) is valid only in the cases of the emitter periodic (or quasi–periodic) motion. However, the same expression could be obtained by examining the acceleration enduring for a sufficiently short-time interval, in which the rate of the emitter kinetic energy diminution remains approximately constant. The necessity of averaging is conditioned by a nonlocalized character of the energy conservation law. That is, the radiation friction work on the charged particle per unit time is equal not only to the radiation power flux through a closed surface (the radiation power), but it also includes the field energy variation within the volume enclosed in this surface. By taking into account this variation, one can avoid many contradictions caused by applying (4.5). Anyway, the radiation friction concept is justified for sufficiently slow variations in dynamic parameters of the "radiation + emitter" system when, at least in one reference frame, the radiation reaction force is small in comparison with the forces of external fields acting upon the particle. The universal criterion of such smallness looks like

[3] Any model of the emitter, where the mechanical and electromagnetic components of the particle mass are singled out, contradicts its presentation as an elementary particle.

$$2\pi r_c \ll \lambda \; , \tag{4.6}$$

where $r_c = q^2/mc^2$ is the particle classical radius and λ is the characteristic wavelength of the field acting upon the particle or its path curvature radius. As for an electron $r_c = 2.8 \times 10^{-15}$ m, the condition (4.6) is surely fulfilled for all parameters of motion interesting applications.

As a real physical meaning is appropriate to $\mathbf{F}_{\text{rad}}$ (at least, within the framework of the perturbation theory), this force has to be interpreted as the spatial component of a 4-vector f_i, orthogonal to the 4-velocity u_i and reducing to

$$\left(\frac{2q^2}{3c}\right)\left(\frac{\mathrm{d}^2 u_i}{\mathrm{d}s^2}\right) \; , \qquad \text{if} \quad u_{1,2,3} = 0 \; .$$

As $u_i^2 \equiv -1$ and $u_i\,(\mathrm{d}u_i/\mathrm{d}s) \equiv 0$, this vector has the form [1]:

$$f_i = \frac{2q^2}{3c}\left[\frac{\mathrm{d}^2 u_i}{\mathrm{d}s^2} - u_i\left(\frac{\mathrm{d}u_k}{\mathrm{d}s}\right)^2\right] \; . \tag{4.7}$$

The expression obtained permits us to write the radiation reaction force in the three–dimensional form in an arbitrary inertial reference frame where $\mathbf{v} \neq 0$:

$$\mathbf{F}_{\text{rad}} = -P_0\mathbf{v} + \frac{2q^2}{3c^3\gamma}\frac{\mathrm{d}}{\mathrm{d}t}\gamma^3\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \; . \tag{4.8}$$

Here P_0 is the radiation power (or, to be more precise, it is the work performed by the particle against the radiation reaction force per unit time):

$$P_0 = \frac{2q^2}{3c}\left[\gamma^4\left(\frac{\mathrm{d}\mathbf{v}}{c\mathrm{d}t}\right)^2 - \gamma\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\right] \; . \tag{4.9}$$

As the factor γ is raised to the high power on the right-hand side of (4.8), we can limit ourselves in the relativistic case to the first addendum. This, in the literal sense, attaches to $\mathbf{F}_{\text{rad}}$ the meaning of the friction directed against the emitter velocity. This force is conditioned by the recoil momentum of the radiation that is directed mainly along $\mathbf{v}$. Consequently, the expression for the radiation losses is simplified also because the relativistic particle acceleration consists in variations in the velocity direction but not in its magnitude. Therefore, the emitter acceleration and the radiation power may be presented as

$$\frac{\mathrm{d}\mathbf{v}}{c\mathrm{d}t} \approx \frac{\mathbf{F}_\perp}{m\gamma c} \quad \text{and} \quad P_0 \approx \frac{2q^2F_\perp^2}{3m^2c^3}\gamma^2 \; . \tag{4.10}$$

Here $\mathbf{F}_\perp$ is the force transverse to the instantaneous velocity $\mathbf{v}$.

As this force is almost independent of the emitter energy in the relativistic case, the relativistic particle power losses are proportional to γ^2 for fixed curvature of the trajectory.

4.2 Radiation Reaction and Emitter Proper Field

The considerations above clearly indicate that the radiation reaction force is not a force in the literal sense of the word because it is conditioned not only by variations in the mechanical characteristics of the emitter motion but also by structural changes in the electromagnetic field surrounding the charged particle. As (4.8) and (4.10) indicate, a simple physical interpretation of this force as a recoil stimulated by the radiation is justifiable only in the ultra-relativistic case. In this connection, it would be appropriate to note that introducing the radiation friction on the basis of the conservation laws, we have not specified any concrete model of the emitter. At the same time, we have ignored the possibilities of variations in the energy of the particle internal degrees of freedom (for instance, the rotational motion). Both these simplifying suppositions are justifiable in the case of an elementary particle because, within the framework of the relativistic principles, it must be point one. At the same time, the force acting upon the emitter is the Lorentz force, coinciding with the electric field in the rest frame. The emitter proper electric field only can make the physical reason for the radiation friction (to be more precise, the proper field deviation from the Coulomb field, caused by the particle acceleration and finiteness of the velocity of light).

Calculation of the emitter proper field, which again yields (4.4), is usually based on applying either the retarded potentials ([1]) or the retarded field.

It is also supposed that the particle motion varies slowly enough. In other words, the change of the particle acceleration during the time required for light to pass through the charge distribution is small in comparison with the acceleration itself. Further, at the limit of the zero particle "radius," one gets grounds for calling (4.4) "an exact expression in a certain sense" [1]. All the same, it remains not quite satisfactory. Really, taking into account $\mathbf{F}_{\mathrm{rad}}$ in the equation of motion for a free particle results in

$$m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{2q^2}{3c^3}\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} \; . \tag{4.11}$$

The latter, in addition to the trivial solution $\mathbf{v} = \mathrm{const}$, possesses another one, physically senseless: both the acceleration and velocity of the particle are exponentially increasing with the characteristic time $\tau = 2q^2/3mc^2$ [1].

As we avoid using the retarded potentials because of the above–given reasons, our starting point in calculations of the emitter proper field is the system of Maxwell equations for the electric field Fourier–amplitudes

$$\mathbf{E}(\mathbf{k}, t) = \frac{1}{8\pi^3}\int \exp(-\mathrm{i}\mathbf{k}\mathbf{r})\mathbf{E}(\mathbf{r}, t)\,\mathrm{d}\mathbf{r} \; .$$

The current and charge densities for a point charged particle, moving along the trajectory $\mathbf{r}_0\,(t)$ with the velocity $\mathbf{v}\,(t) \equiv \mathrm{d}\mathbf{r}_0/\mathrm{d}t$, can be written as

$$\rho(\mathbf{r}, t) = q\delta(\mathbf{r} - \mathbf{r}_0) \qquad \rho(\mathbf{k}, t) = \frac{1}{8\pi^3}\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}_0\right) \tag{4.12a}$$

$$\mathbf{j}(\mathbf{r},t) = q\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_0) \qquad \mathbf{j}(\mathbf{k},t) = \frac{\mathbf{v}}{8\pi^3}\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}_0\right) \ . \tag{4.12b}$$

Making use of the latter relations, one gets the following equation for $\mathbf{E}\,(\mathbf{k},t)$:

$$\begin{aligned}&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k^2c^2\right)\mathbf{E}(\mathbf{k},t) \\ &\qquad = -\frac{q}{2\pi^2}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} - \mathrm{i}\left(\mathbf{k}\mathbf{v}\right)\mathbf{v} + \mathrm{i}\mathbf{k}c^2\right)\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}_0(t)\right) \ .\end{aligned} \tag{4.13}$$

It should be noted that (4.13) is written down for the total electric field without singling out its longitudinal and transverse components.

According to the method applied in [2], let us present the total field as a sum of the static Coulomb field of the charged particle located at the point $\mathbf{r}_0\,(t)$ and an additional field $\mathbf{E}'$:

$$\mathbf{E}(\mathbf{k},t) = -\frac{\mathrm{i}q\mathbf{k}}{2\pi^2k^2}\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r}_0\right) + \mathbf{E}'(\mathbf{k},t) \ . \tag{4.14}$$

For $\mathbf{E}'\,(\mathbf{k},t)$ one gets the equation:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}+k^2c^2\right)\mathbf{E}'(\mathbf{k},t) = -\frac{q}{2\pi^2k^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{k}\times\left[\mathbf{U}\times\mathbf{k}\right]\right] \ , \tag{4.15}$$

where $\mathbf{U} = \mathbf{v}\exp\left(-\mathrm{i}\mathbf{k}\mathbf{r_0}\,(t)\right)$.

By the direct check it is easy to see that (4.15) admits the solution satisfying zero initial conditions:

$$\begin{aligned}\mathbf{E}'(\mathbf{k},t) = \frac{q^2}{2\pi^2c^2k^4}&\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{k}\times\left[\mathbf{k}\times\mathbf{U}\right]\right]\right. \\ &-\cos(kct)\left(\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{k}\times\left[\mathbf{k}\times\mathbf{U}\right]\right]\right)_{t=0} \\ &\left.-\int_0^t\cos\left(kc(t-t')\right)\frac{\mathrm{d}^2}{\mathrm{d}t'^2}\left[\mathbf{k}\times\left[\mathbf{k}\times\mathbf{U}\right]\right]\mathrm{d}t'\right\} \ .\end{aligned} \tag{4.16}$$

To restore the time dependence of the emitter proper field at the point of the particle location, the right–hand side of (4.16) must be integrated over space of $\mathbf{k}$ values with the multiplier $\exp\left(\mathrm{i}\mathbf{k}\mathbf{r}_0\,(t)\right)$. The integral is to be taken over the spherical volume $|\mathbf{k}| < k_{\mathrm{m}}$, where the condition $\mathbf{k}\left(\mathbf{r}_0\,(t) - \mathbf{r}_0\,(t')\right) \ll 1$ holds, so that the exponent may be considered equal to unit. Integration all over the directions of $\mathbf{k}$ yields:

$$\int\mathbf{k}\,\mathrm{d}\Omega = 0\,; \qquad \int\left[\mathbf{k}\times\left[\mathbf{k}\times\mathbf{U}\right]\right]\,\mathrm{d}\Omega = -\frac{8\pi}{3}\mathbf{U} \ .$$

These formulae indicate that Coulomb field component vanishes. Hence, the emitter proper field may be written as

$$\mathbf{E}\left(\mathbf{r}_0(t),t\right) = -\frac{4qk_\mathrm{m}}{3\pi c^2}\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} + \frac{4q}{3\pi c^2}\left(\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t}\right)_0 \frac{\sin k_\mathrm{m} ct}{ct}$$
$$+\frac{4q}{3\pi c^2}\int_0^t \mathrm{d}t' \frac{\mathrm{d}^2\mathbf{U}}{\mathrm{d}t'^2}\int_0^{k_\mathrm{m}} \cos\left(kc(t-t')\right)\,\mathrm{d}k\,. \qquad (4.17)$$

In the particle rest system, $\mathrm{d}\mathbf{U}/\mathrm{d}t$ coincides with $\mathrm{d}\mathbf{v}/\mathrm{d}t$. Therefore, after multiplying by q, the first addendum represents the inertial force, corresponding to the emitter electromagnetic mass. In accordance with the above–given considerations, it has to be included into the total inertial force. However, if $k_\mathrm{m} \to \infty$, this term is formally divergent. [4]

In calculations of the third addendum, the equality

$$\lim_{k_\mathrm{m}\to\infty}\int_0^{k_\mathrm{m}} \cos\left(kc(t-t')\right)\,\mathrm{d}k = \frac{\pi}{c}\delta(t-t')$$

is to be used. This results in the following expression for the electric field acting upon the emitter:

$$q\mathbf{E}\left(\mathbf{r}_0,t\right) = \frac{2q^2}{3c^3}\frac{\mathrm{d}^2\mathbf{v}}{\mathrm{d}t^2} + \frac{4q^2}{3\pi c^2}\left(\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}\right)_0 \frac{\sin k_\mathrm{m} ct}{ct}$$
$$+\ \text{the terms going to zero when} \quad k_\mathrm{m} \to \infty\,. \qquad (4.18)$$

If the time is long enough, the second term in (4.18) vanishes so that one again obtains (4.4) describing the radiation friction force.

In its essence, the above–given derivation does not differ from the traditional one based on the smallness of the electromagnetic signal retardation [6]. Nevertheless, in our opinion, the limitations on applicability of the notion of radiation friction more distinctly indicate themselves during derivation presented. First, as a result of a not quite correct (but necessary) procedure of the emitter mass renormalization, there arises an additional addendum in the form of $\sin k_\mathrm{m} ct/ct$, which has no definite limit when $k_\mathrm{m} \to \infty$, but vanishes for large values of t. This makes it meaningless to solve the equation of motion of the (4.11) type under any possible initial conditions. Thus, the above–mentioned paradox concerning the charged particle "self–acceleration" during the time τ can be avoided. Second, the limiting transition $k_\mathrm{m} \to \infty$ is correct only under a sufficiently smooth behavior of the function $\mathrm{d}^2\mathbf{v}/\mathrm{d}t^2$. The latter supposition deprives the vector $\mathbf{F}_\mathrm{rad}$ of the sense of an instantaneous force acting upon the emitter. In the long run, the inequality of the (4.6) type serves as a criterion of applicability of the "radiation friction" notion.

The methodological difficulties presented are conditioned by the enforced introduction of the concept of a "point" charged particle into electrodynamics, where the intrinsic fundamental length (the particle classical radius r_c exists). For solving these problems, one should address to quantum electrodynamics,

[4] The necessity of the particle mass renormalization arises from infiniteness of the point particle proper field energy.

where the notion of the Compton wavelength $\hbar/mc$ arises (the latter is by two orders of magnitude larger r_c). Discussion of these problems is beyond the scope of this book. We just would like to emphasize that the similar difficulties still remain within the framework of the quantum theory as well.

4.3 Radiation Friction and Charged Particle Dynamics Radiation Cooling

The radiation friction forces are supposed to be much weaker than the external ones. Nevertheless, they can influence essentially the motion and dynamics of charged particles. This is clear even by intuition. Really, the radiation friction work upon the emitter monotonously, even if slowly, diminishes the particle energy. This causes violation of the conditions for synchronism, which determine the spectral–angular composition of the radiation emitted. For instance, as a result of such violation, infinitely narrow spectral lines, characteristic of the emitter periodic motion, in reality have a finite "natural" width. Under certain conditions, this effect can be of principal importance. At least, it limits the radiation source spectral brightness (or the scattering resonant cross section). However, as it will be demonstrated below, other effects usually determine the spectral line width in the processes of the radiation emission by intense beams. Such factors are either finiteness of the radiation emission time and/or the coherency of individual emitters, which are not quite independent.

Nevertheless, in many cases the radiation reaction forces are of a principal importance. They, for example, should be taken into account to explain the nature of a force acting upon the particle and directed along the incident wave Poynting vector (the light pressure). Really, for a small enough amplitude of the incident plane–polarized wave, one may suppose that the charged particle is moving strictly along the electric field $E = E_0 \sin \omega t$, obeying the equation of motion:

$$mc\frac{\mathrm{d}\beta}{\mathrm{d}t} = qE_0 \sin \omega t + \frac{2q^2}{3c^2}\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} \, . \tag{4.19}$$

It is easy to prove that the solution driven by the field E has the form:

$$\beta(t) \approx -\frac{qE_0}{mc\omega}\left(\cos \omega t - \frac{4\pi r_c}{3\lambda}\sin \omega t\right) \quad \text{for} \quad r_c \ll \lambda \, . \tag{4.20}$$

The second term, proportional to a small ratio of the emitter classical radius r_c to the wavelength $\lambda = 2\pi c/\omega$, is due to the radiation friction and provides a small phase shift between the electric field and particle acceleration. So, the time-average force proportional to the particle velocity and directed along the incident wave vector is nonzero only because of the radiation friction:

$$\langle F \rangle \approx \frac{E_0^2 r_c^2}{3} = \frac{P_0 \sigma_0}{c} \, . \tag{4.21}$$

Here P_0 is the density of the incident wave energy flow; σ_0 is the Thomson scattering cross section [1].

At the same time, this result is obtainable directly from the law of conservation of momentum. It is altogether predictable because the radiation deceleration force has been introduced for ensuring the fulfillment of this law.

The radiation friction influence on relativistic particles is especially noticeable. Increasing as the square of the particle energy, the radiation friction can become the major force acting upon the electron.[5] In this case, making use of (4.10), one may present the energy variation of the emitter along its trajectory as a result of the work performed by the radiation friction force only [1]:

$$\frac{\mathrm{d}\gamma}{\mathrm{d}s} = -\gamma^2 \frac{2q^2 \mathbf{F}_\perp^2}{3m^2c^3} , \tag{4.22}$$

where $\mathbf{F}_\perp$ is an external force transverse to the particle trajectory. Choosing the initial condition $\gamma = \gamma_0$ when $s \to -\infty$ one gets

$$\gamma^{-1} = \gamma_0^{-1} + \frac{2q^2}{3m^2c^3} \int_{-\infty}^{s} F_\perp^2 \, \mathrm{d}s .$$

The limit of $\gamma_0 \to \infty$ indicates that the emitter energy at the point s remains finite (Pomeranchuk's theorem). In other words, when a charged particle has passed through an external field, its energy cannot exceed the value:

$$\gamma_{\max} = \frac{3m^2c^3}{2q^2} \left(\int_{-\infty}^{+\infty} F_\perp^2 \, \mathrm{d}s \right)^{-1} . \tag{4.23}$$

However, to avoid misunderstandings, one should keep in mind the following. The electric field work upon the particle all over its passage through the external field is supposed to be much less than the total radiation losses. As it is easy to see, this phenomenon is conditioned by the strong dependence of the radiation friction force versus the emitter energy.

The fundamental changes in the emitter motion characteristics are somewhat less evident if the particle radiation losses are compensated by an external electric field. In this case, the particle energy remains constant or even, on an average, increases according to a prescribed law. Such conditions are typical of installations for accelerating or storing relativistic electrons or positrons. One of their most important characteristics is the beam brightness defined as the particle density in the phase space. In particular, it is the brightness that determines the possibilities of the beam focusing. In their turn, these characteristics influence the effectiveness of the beam application in the high–energy experiments. Great significance of the beam brightness consideration follows from the fact, purely of the theoretical interest at first sight: the behavior of individual noninteracting particles (or the ones interacting via the self–consistent field) can be described by Hamilton canonical equations:

[5] This does not contradict the requirement that the radiation friction force is small, which has to be satisfied only in the reference frame where the particle is in rest.

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \nabla_{\mathbf{P}}\mathcal{H}\,; \qquad \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} = -\nabla\mathcal{H}\,. \tag{4.24}$$

The Hamiltonian $\mathcal{H}\,(t, \mathbf{r}, \mathbf{P})$ depends on time, coordinates, and components of the generalized momentum $\mathbf{P}$, conjugated with the coordinates.

As it will be demonstrated below, this is enough for the brightness to become an exact integral of motion under acting upon the beam by an arbitrary external field. This statement being a fundamental one is well–known as the Liouville theorem [31]. There is its another formulation: the shape of the six–dimensional phase space region where the particle image points are located can be changed almost in an arbitrary manner, but its total volume (the beam emittance) all the same remains constant. The point to be made here is that the formulations used by us relate to the continuum presentation of the beam. [6] This implies that an average distance between the image points is much smaller than the intervals d$\mathbf{r}$ and d$\mathbf{P}$, which are already physically small. A somewhat imperative character of the statement concerning the impossibility of changing the brightness is conditioned by the following fact: as external fields are of macroscopic nature, with their help, one cannot introduce a new particle into an already–occupied phase space cell. To do this, one has to move apart the particles, located there (to be more precise, their image points). Consequently, the beam particle stacking in magnetic systems must be accompanied by extending the phase space occupied by the particles (i.e., the beam emittance). It is also easily predictable that pair collisions between the beam particles, being of microscopic nature, violate the Liouville theorem. Such collisions cause the emittance increase and enlarge the system total entropy. [7]

The above–mentioned noncanonical nature of the radiation friction force implies violation of the Liouville theorem as well. In fact, this force is "personalized" i.e., the radiation reaction is considered to be acting directly and solely upon the emitter. However, it is also implied that the total power of radiation emitted by a system of particles is equal, at least on an average, to the sum of powers emitted by each particle (i.e., radiation is regarded as incoherent). This condition is to be examined below. At present, we will limit ourselves to studying the beams rarefied enough to guarantee the absence of the emitter coherence. Generally speaking, compatibility of this limitation with the notion of the continuous distribution of particles implies the radiation wavelength smallness in comparison with the characteristic distance between the emitters.

[6] This limitation is not necessary for the general Liouville theorem, formulated for the $6N$-dimensional phase space, N being a number of particles.

[7] However, there remains the possibility of diminishing the phase space of one macroscopic component of the beam system by the corresponding equivalent heightening of the emittance of the rest of the system. A refined method of the heavy–particle beam "electron cooling," bearing no relation to the radiative effects, works on this principle [32].

To quantitatively describe variations in the beam brightness, let us introduce the distribution function of the beam particles in the phase space $\Psi(\mathbf{r}, \mathbf{P}, t)$. This function describes the number of particles within the phase space cell $\mathrm{d}\mathbf{r}\mathrm{d}\mathbf{P}$ and satisfies the continuity equation of the form

$$\frac{\partial \Psi}{\partial t} + \nabla \left(\Psi \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right) + \nabla_{\mathbf{P}} \left(\Psi \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} \right) = 0 \, . \tag{4.25}$$

The image point velocities $\mathrm{d}\mathbf{r}/\mathrm{d}t$ and $\mathrm{d}\mathbf{P}/\mathrm{d}t$ in the phase space must be presented as functions of $\mathbf{r}$, $\mathbf{P}$, and t.

Introduction of the radiation friction force, non–Hamiltonian by nature, means that the generalized force

$$-\frac{P_0}{c^2}\mathbf{v} = -\frac{P_0}{\gamma m c^2}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right) \tag{4.26}$$

must be included into the second equation in this system (4.24) (as before, $P_0(\gamma, t)$ is the power of the radiation losses). Thus, (4.25) may be presented in the form:

$$\frac{\partial \Psi}{\partial t} + \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\nabla\Psi + \frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t}\nabla_{\mathbf{P}}\Psi = \Psi\nabla_{\mathbf{P}}\frac{P_0}{\gamma m c^2}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right) \, , \tag{4.27}$$

because the terms containing cross derivatives of $\mathcal{H}$ are mutually reducible.

The left–hand side of (4.27) represents by itself the total derivative $\mathrm{d}\Psi/\mathrm{d}t$ along the particle phase trajectory determined by (4.24). In the absence of the radiation friction, this derivative is equal to zero, which depicts the essence of the Liouville theorem. As regards the right–hand side of this formula, making use of the equality

$$\gamma^2 = 1 + \frac{1}{m^2c^4}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right)^2$$

one may equate

$$\nabla_{\mathbf{P}}\frac{P_0}{\gamma}\left(\mathbf{P} - \frac{q}{c}\mathbf{A}\right) \approx 3\frac{P_0}{\gamma} + \gamma\frac{\partial}{\partial\gamma}\frac{P_0}{\gamma} = \frac{1}{\gamma^2}\frac{\partial P_0\gamma^2}{\partial\gamma} \, . \tag{4.28}$$

According to (4.10), $P_0 \propto \gamma^2$. Finally, the equation for Ψ takes the form:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{4P_0}{\gamma m c^2}\Psi \tag{4.29}$$

and

$$\Psi \propto \exp\int^t \frac{4P_0}{\mathcal{E}}\,\mathrm{d}t \, , \tag{4.30}$$

where $\mathcal{E}$ is the particle energy.

Thus, it has been demonstrated that the beam brightness exponentially increases in time. In the presence of the radiation friction force and in the absence of counteraction (e.g., the above–mentioned pair collisions), the six–dimensional phase space occupied by the beam particles would tend to zero whatever variations in the particle energy be [33]. The characteristic time of this contraction (or the "radiation cooling") $\tau_c \approx \mathcal{E}/P_0$ by the order of magnitude coincides with the time during which the charged particle would spend its total energy on the radiation emission, provided any compensation were absent.

Importance of the phenomenon of the radiation cooling to physics of accelerators is conditioned by the two factors. First, the radiation cooling substantially limits the influence of various disturbances that in an uncontrollable manner give irreversible rise to the beam emittance. Second, which is more important, the radiation cooling is prospective for accumulating light particles in the circulating beam without any increase of its emittance (surely, the accumulation time has to exceed τ_c).

However, the difficulties in realizing the radiation cooling should be mentioned as well. Surely, for the beam long-time existence in a storage ring or in an accelerator, the total six–dimensional emittance of the beam must not increase. At the same time, there also must not take place any increase of the three partial two–dimensional emittances, corresponding to the particle proper oscillations, i.e., small deviations of the particle from the equilibrium coordinates, independent of one another. In the absence of the radiation emission, the betatron oscillations, transverse to the equilibrium orbit and not related to the particle energy variations, play the role of such proper oscillations. Another independent type of oscillations consists in variations in the particle energy (synchrotron oscillations). Providing correlation between the betatron and synchrotron oscillations, the process of the radiation emission also causes redistribution of the partial emittances between these types of oscillations (the total damping decrement still remains equal to $4P_0/\mathcal{E}$). For instance, in the simplest magnetic system, consisting of a sequence of focusing and defocusing sectors, the partial emittances behave in the following way:

$$\varepsilon_z \propto \exp\left(-\int \frac{P_0}{\mathcal{E}}\,\mathrm{d}t\right) \qquad \text{vertical oscillations} \tag{4.31a}$$

$$\varepsilon_x \propto \exp\left((1-\alpha)\int \frac{P_0}{\mathcal{E}}\,\mathrm{d}t\right) \qquad \text{radial oscillations} \tag{4.31b}$$

$$\varepsilon_s \propto \exp\left((\alpha-4)\int \frac{P_0}{\mathcal{E}}\,\mathrm{d}t\right) \qquad \text{synchrotron oscillations .} \tag{4.31c}$$

Here α is the momentum compaction factor that characterizes the dependence of the equilibrium orbit perimeter versus the particle energy. In particular, in strong–focusing systems ($\alpha \ll 1$), the radial betatron oscillations are excited rather quickly at the expense of damping of the synchrotron oscillations. More detailed information about this effect and the methods of its suppression can be found in [33].

To conclude this section, it should be noted that the considerations above are relevant for a single particle isolated from the others. On the one hand, that seems justified for rare beams where the mean distance between particles exceeds essentially the radiation wavelength. On the other hand, the radiation field decreases with distance rather slowly especially in waveguide systems. So, each particle motion could be influenced by a great many others. This influence being systematic, the proper radiation reaction can be shadowed by the total (collective) radiation field. Then qualitatively new many-particle phenomena may develops, which are the subject of the following sections.