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On low rank 4d $\mathcal{N} = 2$ SCFTs

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Abstract: There are two major ways of constructing 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs): the first one is putting a 6d $(2, 0)$ theory on a punctured Riemann surface (class-S theory), and the second one is putting type IIB string theory on a 3d canonical singularity. As there are interests on low rank theories, we search all the possibilities from above two constructions. Most of those theories are engineered by class-S theory with irregular singularities, and we find a universal formula for the rank of theory so that a complete search is possible. We then compute various physical quantities of those theories, such as the central charges, flavor symmetry, associated vertex operator algebra and Higgs branch, etc. One of interesting consequence of our results are the prediction of many new isomorphism of 2d vertex operator algebra.

Keywords: Supersymmetric Effective Theories, Supersymmetric Gauge Theory, Supersymmetry and Duality

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1 Introduction

Four dimensional $\mathcal{N} = 2$ superconformal field theories (SCFTs) [1–3] constantly provide invaluable insights into the understanding of quantum field theory in the past thirty years. The studies of them also gave intimate connections to various branches of mathematics such as invariants of four manifolds, integrable systems, algebraic geometry, Hodge theory, vertex operator algebras (VOA) and etc. One of major advancement in last ten years is that one can engineer a large class of new 4d SCFTs by geometric methods: one class (class-S) is the compactification of 6d $(2, 0)$ theories on a punctured Riemann surface [4–8], and the other class is the compactification of type IIB string theory on a 3d canonical singularity [9–12]. Resulting theories are usually strongly coupled, so conventional perturbative methods are of little usage, yet geometric tools can help us understand lots of fundamental properties of these theories. One can often obtain the Seiberg-Witten geometry, central charges, the corresponding 2d VOA (and the Higgs branch from it), 3d mirror [13–15], relation with 5d theories [16–18] and etc. It is such an amazing fact that one can easily compute those quantities given the strongly coupled nature of these theories.

On the other hand, there are also interest in the complete classification of 4d $\mathcal{N}=2$ theories at small rank. By rank we mean the dimension of the Coulomb branch (CB). Classification of rank one and two theories are discussed in [19–28]. These classifications are mostly based on studying constraints on CB. Their efforts lead to some new theories which have not been noticed before [22, 25]. It would be interesting to construct these theories by geometric methods, as one can use geometric tools to further study those theories, e.g. understanding their Schur sector and Higgs branch.

Given the large landscape of the geometric engineered theories, it is not easy to systematically scan the theory space. In fact, most of low rank theories are constructed using class $\mathcal{S}$ theory with irregular singularities, and no simple formula for the rank is known. What makes the exhaustive search possible is the CB dimension formulae (2.7) and (2.8). Using these dimension formulae, we list all rank one, two and three geometric engineered theories. Indeed, many new theories found in [22, 25] appear also in our search. Using general results of geometric construction, one can get new information of these theories, such as the corresponding VOA and the Higgs branch, summarized in section 3.

Let us point out several interesting consequences of our findings:

- There is a sequence of rank one AD theories with the following flavor symmetry (see also [29])
$$A_0 \subset A_1 \subset A_2 \subset G_2 \subset B_3 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8,$$
and the corresponding VOA is the affine Kac-Moody algebra $V_k(\mathfrak{g})$ at level $k=-h/6-1$. Here h is the Coxeter number of the flavor group. This sequence is different from the famous Deligne exceptional series $A_1 \subset A_2 \subset G_2 \subset D_4 \subset F_4 \subset E_6 \subset E_7 \subset E_8$ [30] at level $-h^\vee/6-1$, when the group is non-simply-laced. It would be interesting to further study properties of this sequence.

- There is a sequence of theories with flavor symmetry group $\mathrm{SO}(4N+8)$, $N \geq 2$ and possible enhancement, i.e. E_8, $\mathrm{SO}(20)$, $\mathrm{SO}(24)$, and etc. These theories are 4d counterpart of 5d theory engineered by the UV limit of $\mathrm{SU}(N)$ gauge group with $2N+3$ flavors [31], and can also be engineered by compactifying 6d A_{2N+1} $(2,0)$ theory on a sphere with three punctures labelled by the partition $[N^2,2],[(N+1)^2],[1^{2N+2}]$. They are very likely to have the maximal flavor symmetry at their own rank: E_8 for rank one theories, $\mathrm{SO}(20)$ for rank two theories, $\mathrm{SO}(24)$ for rank three theories and etc. These classes of theory are special in that they seem to be the theory with maximal flavor symmetry for a given rank, i.e E_8 for rank one, $\mathrm{SO}(20)$ for rank two, etc.

- It is also easy to find many other sequence of theories, such as theories with flavor symmetry G_2, F_4. Some of those theories are studied in [32], our construction would give the VOA and the associated Higgs branch for these theories.

- Many of the new theories found in [19] are constructed using discrete gauging of the known theories. In our findings, those new theories are typically constructed using outer automorphism twist. The relation between discrete gauging and outer automorphism twist deserves further study.

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- Since a 4d theory can be realized in many different geometric ways, this often implies the highly nontrivial isomorphism for W algebras (see section 4). It would be interesting to prove those isomorphisms rigorously.

This paper is organized as follows. Section 2 reviews the geometric construction and how to compute various physical quantities such as SW geometries, central charges, etc. Section 3 summarizes the list of rank one, two and three theories from geometry. Section 4 lists isomorphisms for W algebras derived from our constructions. Section 5 discusses some results on theories with higher rank and summarize some rank four theories. Finally a conclusion is given in section 6.

2 Geometric engineering of 4d $\mathcal{N} = 2$ SCFTs

There are two main classes of 4d $\mathcal{N} = 2$ SCFTs. The first class, which is called class-S, is constructed by putting 6d $(2, 0)$ theories on a Riemann surface with regular or irregular singularities [4–8]. The second class is engineered by putting the type IIB string theory on a 3d-canonical singularities [9]. There are several advantages of these constructions:

1. The Coulomb branch solution (the SW geometry) is given by the spectral curve of the underlying Hitchin system for class-S construction, and is identified with the mini-versal deformation for type IIB construction.

2. There are simple formula for the physical data like central charges a, c, k_F and others.

3. The Higgs branch or more generally the Schur sector can be derived for a large portion of class-S theories, as their corresponding 2d VOA is known.

One must keep in mind that for a generic 4d $\mathcal{N} = 2$ theory the above information is not easy to derive, hence the above two constructions are extremely valuable for the study of 4d $\mathcal{N} = 2$ SCFTs. In the following, we will review these two constructions and how to compute various physical quantities.

2.1 Class-S theories

2.1.1 Theories with regular singularities only

One can engineer 4d $\mathcal{N} = 2$ SCFTs by putting 6d $(2, 0)$ theory of type $\mathfrak{j}$ on a genus g Riemann surface with only regular singularities [4, 5, 33–35]. In the untwisted case, the regular singularities are labeled by the nilpotent orbit f of $\mathfrak{j}$,[1] so an untwisted theory is specified by the data

$$(\Sigma_g, \mathfrak{j}, f_1, f_2, \ldots, f_n). \tag{2.1}$$

For a twisted theory, one denotes the outer-automorphism twist of $\mathfrak{j}$ by o, and the invariant Lie algebra obtained after the outer-automorphism by $\mathfrak{g}^\vee$. Let $\mathfrak{g}$ be the Langlands dual of $\mathfrak{g}^\vee$. There are two kinds of regular singularities for a twisted theory: twisted puncture f^t

[1] We label the regular singularity by the Nahm label, its (Spaltanstein) dual is also called Hitchin label which appears in the corresponding Hitchin system.

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labelled by the nilpotent orbit of $\mathfrak{g}$ and untwisted puncture f labelled by the nilpotent orbit of $\mathfrak{j}$, therefore a twisted theory is specified by the data

$$(\Sigma_g, \mathfrak{j}, o, f_1^t, f_2^t, \ldots, f_n^t, f_1, \ldots, f_s). \tag{2.2}$$

Notice that the number of twisted punctures is constrained by o. For example, n has to be even if o is Z_2.

Physical data such as CB spectrum, flavor symmetry and central charges are determined by the local data of the nilpotent orbit f together with global data of Σ_g [35–43]. The main feature of this class of theories is that the scaling dimensions of CB operators are almost all **integrals**, while the number of CB operators, or the rank of the untwisted theory, is given by

$$\dim \mathcal{B} = \frac{1}{2}\left(\sum \dim(f_i^D) + (g-1)\dim(\mathfrak{j})\right). \tag{2.3}$$

Here f_i^D is the dual orbit of f_i. On the other hand, the rank of a twisted theory is given by

$$\dim \mathcal{B} = \frac{1}{2}\left(\sum [dim(f_i^{t,D}) + (\dim \mathfrak{j} - \dim \mathfrak{g})] + \sum \dim(f_i^D) + (g-1)\dim(\mathfrak{g}^\vee)\right). \tag{2.4}$$

For the twisted theory, the dual orbit for the twisted puncture (which is labeled by nilpotent orbit of $\mathfrak{g}$) is a nilpotent orbit of $\mathfrak{g}^\vee$.

***Example*:** consider the class-S theory $(\Sigma_0, A_9, [5^2], [4^2, 2], [1^{10}])$. This rank three theory has Coulomb branch operators with dimension 10, 8 and 6. The flavor symmetry group is SO(24) which is maximal in all rank three theories we found. The a and c central charges are 337/24 and 101/6.

2.1.2 Theories with one irregular singularity

To get theories with fractional scaling dimension CB operators, one need to add irregular singularities on the Riemann surface. Yet only spheres with one irregular and one regular singularity lead to non-trivial SCFTs in the IR. The untwisted theories are labelled by the following data

$$(\mathfrak{j}, b, k, f), \tag{2.5}$$

where $\mathfrak{j} = ADE$ specifies the $6d$ $(2,0)$ theory, b is a positive integer which are determined by $\mathfrak{j}$, $k > -b$ is also an integer. $\mathfrak{j}$, k and b together determine the form of the irregular singularity, while the regular singularity is labelled by a nilpotent orbit f of $\mathfrak{j}$. The allowed value for b, k are listed in table 1, where the correspondence with the three-fold singularities [7, 9] is used.

For the twisted theory, one denote an outer-automorphism twist of $\mathfrak{j}$ by o. Let $\mathfrak{g}^\vee$ be the invariant Lie algebra after the outer-automorphism twist, and let $\mathfrak{g}$ be the Langlands dual of $\mathfrak{g}^\vee$. The twisted theories are labelled by data

$$(\mathfrak{j}, o, b_t, k_t, f) \tag{2.6}$$

The allowed set for b_t, k_t is summarized in table 2. f now labels a nilpotent orbit of $\mathfrak{g}$. Details of such theories can be found in [8].

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j	b	Singularity	Spectral curve at SCFT point	$\Delta[z]$	μ
A_{N-1}	N	$x_1^2+x_2^2+x_3^N+z^k=0$	$x^N+z^k=0$	$\frac{N}{N+k}$	$(N-1)(k-1)$
	$N-1$	$x_1^2+x_2^2+x_3^N+x_3z^k=0$	$x^N+xz^k=0$	$\frac{N-1}{N+k-1}$	$N(k-1)+1$
D_N	$2N-2$	$x_1^2+x_2^{N-1}+x_2x_3^2+z^k=0$	$x^{2N}+x^2z^k=0$	$\frac{2N-2}{2N+k-2}$	$N(k-1)$
	N	$x_1^2+x_2^{N-1}+x_2x_3^2+z^kx_3=0$	$x^{2N}+z^{2k}=0$	$\frac{N}{N+k}$	$2k(N-1)-N$
E_6	12	$x_1^2+x_2^3+x_3^4+z^k=0$	$x^{12}+z^k=0$	$\frac{12}{12+k}$	$6k-6$
	9	$x_1^2+x_2^3+x_3^4+z^kx_3=0$	$x^{12}+x^3z^k=0$	$\frac{9}{9+k}$	$8k-6$
	8	$x_1^2+x_2^3+x_3^4+z^kx_2=0$	$x^{12}+x^4z^k=0$	$\frac{8}{8+k}$	$9k-6$
E_7	18	$x_1^2+x_2^3+x_2x_3^3+z^k=0$	$x^{18}+z^k=0$	$\frac{18}{18+k}$	$7k-7$
	14	$x_1^2+x_2^3+x_2x_3^3+z^kx_3=0$	$x^{18}+x^4z^k=0$	$\frac{14}{14+k}$	$9k-7$
E_8	30	$x_1^2+x_2^3+x_3^5+z^k=0$	$x^{30}+z^k=0$	$\frac{30}{30+k}$	$8k-8$
	24	$x_1^2+x_2^3+x_3^5+z^kx_3=0$	$x^{30}+x^6z^k=0$	$\frac{24}{24+k}$	$10k-8$
	20	$x_1^2+x_2^3+x_3^5+z^kx_2=0$	$x^{30}+x^{10}z^k=0$	$\frac{20}{20+k}$	$12k-8$

Table 1. Three-fold isolated quasi-homogenous singularities of cDV type corresponding to the $(\mathfrak{j}, b, k)$ irregular punctures of the regular-semisimple type in [7]. These 3d singularity is very useful in extracting the CB spectrum [9].

$\mathfrak{j}/o$	$\mathfrak{g}$	b_t	SW geometry at SCFT point	Spectral curve at SCFT point	$\Delta[z]$
$A_{2N}/\mathbb{Z}_2$	$C_N^{(1)}$	$4N+2$	$x_1^2+x_2^2+x^{2N+1}+z^{k'+\frac{1}{2}}=0$	$x^{2N+1}+z^{k'+\frac{1}{2}}=0$	$\frac{4N+2}{4N+2k'+3}$
		$2N$	$x_1^2+x_2^2+x^{2N+1}+xz^{k'}=0$	$x^{2N+1}+xz^{k'}=0$	$\frac{2N}{k'+2N}$
$A_{2N-1}/\mathbb{Z}_2$	B_N	$4N-2$	$x_1^2+x_2^2+x^{2N}+xz^{k'+\frac{1}{2}}=0$	$x^{2N}+xz^{k'+\frac{1}{2}}=0$	$\frac{4N-2}{4N+2k'-1}$
		$2N$	$x_1^2+x_2^2+x^{2N}+z^{k'}=0$	$x^{2N}+z^{k'}=0$	$\frac{2N}{2N+k'}$
$D_{N+1}/\mathbb{Z}_2$	$C_N^{(2)}$	$2N+2$	$x_1^2+x_2^N+x_2x_3^2+x_3z^{k'+\frac{1}{2}}=0$	$x^{2N+2}+z^{2k'+1}=0$	$\frac{2N+2}{2k'+2N+3}$
		$2N$	$x_1^2+x_2^N+x_2x_3^2+z^{k'}=0$	$x^{2N+2}+x^2z^{k'}=0$	$\frac{2N}{k'+2N}$
$D_4/\mathbb{Z}_3$	G_2	12	$x_1^2+x_2^3+x_2x_3^2+x_3z^{k'\pm\frac{1}{3}}=0$	$x^8+z^{2k'\pm\frac{2}{3}}=0$	$\frac{12}{12+3k'\pm1}$
		6	$x_1^2+x_2^3+x_2x_3^2+z^{k'}=0$	$x^8+x^2z^{k'}=0$	$\frac{6}{6+k'}$
$E_6/\mathbb{Z}_2$	F_4	18	$x_1^2+x_2^3+x_3^4+x_3z^{k'+\frac{1}{2}}=0$	$x^{12}+x^3z^{k'+\frac{1}{2}}=0$	$\frac{18}{18+2k'+1}$
		12	$x_1^2+x_2^3+x_3^4+z^{k'}=0$	$x^{12}+z^{k'}=0$	$\frac{12}{12+k'}$
		8	$x_1^2+x_2^3+x_3^4+x_2z^{k'}=0$	$x^{12}+x^4z^{k'}=0$	$\frac{8}{8+k'}$

Table 2. Seiberg-Witten geometry of twisted theories at the SCFT point. Relations between k_t and k' which appears in SW geometry are: $k_t=2k'+1$ if $z^{k'+\frac{1}{2}}$ appears in SW geometry, $k_t=3k'\pm1$ if $z^{k'\pm\frac{1}{3}}$ in SW geometry, and finally $k_t=k'$ if $z^{k'}$ in SW geometry.

SW curve and CB spectrum: the SW curve can be identified with the spectral curve of the underlying Hitchin system, and the detailed scaling dimensions can be found from the underlying Newton polygon [44]. For an untwisted theory specified by $(\mathfrak{j}, b, k, f)$, its rank can be described by the following formula (derivation of the formula is summarized in appendix A and more details will be presented in [45])

$$\boxed{\dim\mathcal{B}=\frac{1}{2}\left[(h_{\mathfrak{j}}\frac{k}{b}+h_{\mathfrak{j}}-1)\mathrm{rank}(\mathfrak{j})-f_0-\dim\mathcal{O}_{\mathrm{prin}}+\dim\mathcal{O}_f^{\mathrm{Hitchin}}\right]}, \tag{2.7}$$

where f_0 is the number of mass deformations from the irregular singularity computed in [8, 44], $h_{\mathfrak{j}}$ is the Coxeter number of $\mathfrak{j}$ and $\mathcal{O}_f^{\mathrm{Hitchin}}$ is the Spaltenstein dual of the nilpotent

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j/o	A_{2N}/Z_2	A_{2N-1}/Z_2	D_{N+1}/Z_2	E_6/Z_2	D_4/Z_3
h_θ	$4N+2$	$4N-2$	$2N+2$	18	12

Table 3. The h_θ for different twisted AD theories.

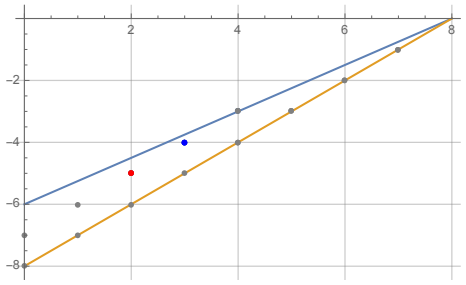

Figure 1. Newton Polygons for the theory $(A_7, 8, -6, [2^2, 1^4])$. Grey dots represents terms beyond the corresponding pole order. The red dot corresponds to the CB operator, while the blue dot is the mass parameter.

orbit f. Similarly for a twisted theory specified by $(\mathrm{j}, o, \mathfrak{g}, b_t, k_t, f)$, its rank is [45]

$$\boxed{\dim\mathcal{B} = \frac{1}{2}\left[\left(h_\theta \frac{k_t}{b_t} + h_\theta - 1\right)\mathrm{rank}(\mathfrak{g}) - f_0 - \dim\mathcal{O}_{\mathrm{prin}} + \dim\mathcal{O}_f^{\mathrm{Hitchin}}\right]}, \tag{2.8}$$

where h_θ take value in table 3 instead of the Coxeter number.[2]

Example 1: $(A_7, 8, -6, [2^2, 1^4])$ theory. In this example, $\mathfrak{g} = A_7$, $b = 8$, $k = -6$ and $f = [2^2, 1^4]$. Then $f_0 = 1$, $\dim\mathcal{O}_{\mathrm{prin}} = 56$ and $\mathcal{O}_f^{\mathrm{Hitchin}} = [4, 2^2]$ with dimension 52. According to (2.7), its rank is

$$\frac{1}{2}\left[\left(8 \times \frac{-6}{8} + 8 - 1\right) \times 7 - 1 - 56 + 52\right] = 1. \tag{2.9}$$

The SW curve of this theory looks like $x^8 + z^{-6} = \sum_{i=1}^{7} \phi_{d_i}(z) x^{8-d_i}$, where $\{d_i\} = \{2, 3, \cdots, 8\}$ are degrees of Casimir of $\mathfrak{su}(8)$. For $f = [2^2, 1^4]$ the pole order is $\{p_2, p_3, \cdots, p_8\} = \{1, 2, 3, 4, 5, 5, 6\}$. The SW curve can be visualized by a Newton polygon (figure 1) where each term $x^a z^b$ in the curve is represented by a point (a, b) inside the polygon [44]. In this example there are only two points in the polygon representing the term $ux^2z^{-5} + mx^3z^{-4}$, so the SW curve is

$$x^8 + z^{-6} = ux^2z^{-5} + mx^3z^{-4}, \tag{2.10}$$

where u has conformal dimension 2 and m has conformal dimension 1. In the end, the theory has one dimension 2 CB operator and one mass parameter.

Example 2: $(\mathfrak{e}_6, 12, -7, A_4)$ theory. In this example, $\mathfrak{g} = \mathfrak{e}_6$, $b = 12$, $k = -7$ and $f = A_4$. Then $f_0 = 0$, $\dim\mathcal{O}_{\mathrm{prin}} = 72$ and $\mathcal{O}_f^{\mathrm{Hitchin}} = A_3$ with dimension 52. Using (2.7), we get its rank being

$$\frac{1}{2}\left[\left(12 \times \frac{-7}{12} + 12 - 1\right) \times 6 - 0 - 72 + 52\right] = 2. \tag{2.11}$$

[2] h_θ is called the twisted Coxeter number in [46].

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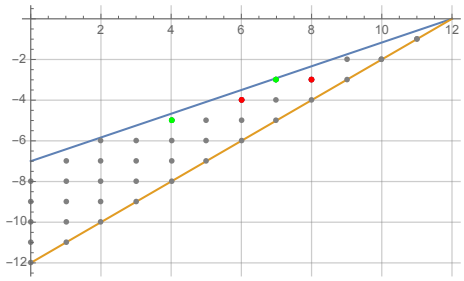

Figure 2. Newton Polygons for $(\mathfrak{e}_6, 12, -7, A_4)$ theory. Grey dots represents terms beyond the corresponding pole order. Red dots correspond to CB operators, while green dots are mass parameters.

For $f = A_4$ there is an extra degree 4 differential $\phi_4(z)$ in the SW curve, so the set of degrees $\{d_i\} = \{2, 4, 5, 6, 8, 9, 12\}$. The pole order structure after imposing constraints is $\{p_2, p_4, p_5, p_6, p_8, p_9, p_{12}\} = \{1, 3, 3, 4, 5, 5, 7\}$ [39]. Its Newton polygon is shown in figure 2, and the SW curve is

$$x^{12} + z^{-7} = u_1 x^6 z^{-4} + u_2 x^8 z^{-3} + c_1 x^7 z^{-3} + c_2 x^4 z^{-5}, \tag{2.12}$$

where two CB operators u_1, u_2 have conformal dimension $\frac{8}{5}, \frac{6}{5}$ respectively. The coupling c_1 and c_2 have conformal dimension $\frac{1}{5}$ and $\frac{4}{5}$ respectively.

***Example 3*:** $(\mathfrak{e}_6, \mathbb{Z}_2, \mathfrak{f}_4, 8, -7, 0)$ theory. In this example, $\mathfrak{j} = \mathfrak{e}_6$, $o = \mathbb{Z}_2$, $\mathfrak{g} = \mathfrak{f}_4$, $b_t = 8$, $k_t = -7$ and $f = 0$. Then $f_0 = 1$, $\dim \mathcal{O}_{\text{prin}} = 48$ and $\mathcal{O}_f^{\text{Hitchin}} = F_4$ with dimension 48. According to (2.8), its rank is

$$\frac{1}{2}\left[\left(18 \times \frac{-7}{8} + 18 - 1\right) \times 4 - 1 - 48 + 48\right] = 2. \tag{2.13}$$

For twisted punctures, p_5 are p_9 are half-integer. For $f = 0$ the pole order structure is $\{p_2, p_5, p_6, p_8, p_9, p_{12}\} = \{1, \frac{9}{2}, 5, 7, \frac{17}{2}, 11\}$. The Newton polygon is drawn in figure 3, and the SW curve is,

$$x^{12} + x^4 z^{-7} = u_1 z^{-11} + u_2 x^3 z^{-\frac{17}{2}} + m x^7 z^{-\frac{9}{2}} \tag{2.14}$$

All in all, the theory has two CB operators with conformal dimension 5 and 4 and one mass parameter. The (a, c) central charges of this theory are $(\frac{14}{3}, \frac{16}{3})$, and the flavor symmetry is $(\mathfrak{f}_4)_5 \times \mathfrak{u}(1)$.[3] Using this data we identify this theory as one of the theories in table 6 of [25], and we provide its SW curve as well.

Remark: our dimension formulae (2.7) and (2.8) actually computes the dimension of the Hitchin base, then identify it as the rank of the theory. Another way of computing the rank of the theory is to count the number of graded Coulomb branch operators by using the pole order of each individual differential in the SW curve. In rare occasions, the number of graded Coulomb branch operators is greater than the dimension of the Hitchin base. That is because the dimension of the Hitchin base is the sum of the virtual dimension of each individual differential. However, the virtual dimension of each individual differential might be negative, therefore the dimension of the Hitchin base is smaller than the number of the graded Coulomb branch operators in such cases.

[3] Our normalization of flavor anomaly is half of the one in [25].

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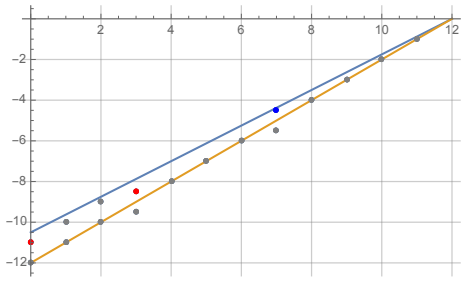

Figure 3. Newton Polygons for $(\mathfrak{e}_6, \mathbb{Z}_2, \mathfrak{f}_4, 8, -7, 0)$ theory. Grey dots represents terms beyond the corresponding pole order. Red dots correspond to CB operators, while the blue dot represents the mass parameter.

Central charges a_{4d} and c_{4d} : the a_{4d} and c_{4d} central charges are computed by using the following relations. The first one relates central charges with the CB spectrum [47, 48],

$$\boxed{2a_{4d} - c_{4d} = \frac{1}{4}\sum_i (2[u_i] - 1),} \tag{2.15}$$

where $[u_i]$ is the dimension of the CB operator u_i and i runs over all CB operators. The other two relations require data of the corresponding VOA [49, 50]

$$\boxed{\begin{aligned} c_{4d} &= -\frac{1}{12} c_{2d}, \\ a_{4d} - c_{4d} &= -\frac{1}{48}\mathcal{G}. \end{aligned}} \tag{2.16}$$

Here c_{2d} is the central charge of the corresponding VOA, $\mathcal{G}$ the growth function of its vacuum module. Any two of these relations can be used to work out a_{4d} and c_{4d} while the third one provides a consistency check. Here we will mainly discuss computing central charges using relations (2.16), that is, computing the central charge and growth of the corresponding VOA.

For an untwisted (twisted) theory labelled by $(\mathfrak{j}, b, k, f)$ ($(\mathfrak{j}, o, \mathfrak{g}, b_t, k_t, f)$ for twisted), if the irregular singularity has no mass deformation ($f_0 = 0$), the corresponding VOA is the W-algebra $W_{\tilde{k}}(\mathfrak{j}, f)$ with $\tilde{k} = -h^\vee_{\mathfrak{j}} + \frac{b}{k+b}$ ($W_{\tilde{k}}(\mathfrak{g}, f)$ with $\tilde{k} = -h^\vee_{\mathfrak{g}} + \frac{1}{m}\frac{b_t}{k_t+b_t}$ for twisted, where m is the lacety number of $\mathfrak{g}$). The central charge $c_{W_{\tilde{k}}(\mathfrak{g},f)}$ is given by [51]

$$c_{W_{\tilde{k}}(\mathfrak{g},f)} = \dim\mathfrak{g}^0 - \frac{1}{2}\dim\mathfrak{g}^{\frac{1}{2}} - \frac{12}{\tilde{k} + h^\vee_{\mathfrak{g}}}\left|\rho - (\tilde{k} + h^\vee_{\mathfrak{g}})\frac{h}{2}\right|^2. \tag{2.17}$$

Here the $\mathfrak{sl}_2$ triple (e, f, h) is fixed by the Dynkin grading. The Weyl vector ρ is sum of all fundamental weights. $\dim\mathfrak{g}^0$ and $\dim\mathfrak{g}^{\frac{1}{2}}$ are the number of positive roots with grading 0 and $\frac{1}{2}$. One way to compute them is by looking at the decomposition of the adjoint representation of $\mathfrak{g}$ under the subalgebra $\Lambda(\mathfrak{su}(2)) \oplus \mathfrak{f}_\Lambda$, where Λ is the embedding map $\Lambda : \mathfrak{su}(2) \to \mathfrak{g}$,

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_j \otimes \mathcal{R}_j \tag{2.18}$$

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where V_j is the spin-j representation and R_j in some representation of $\mathfrak{f}_\Lambda$, then

$$\dim\mathfrak{g}^0 = \sum_{j\in\mathbb{Z}_{\geq 0}} \dim\mathcal{R}_j \qquad \dim\mathfrak{g}^{\frac{1}{2}} = \sum_{j\in\frac{1}{2}+\mathbb{Z}_{\geq 0}} \dim\mathcal{R}_j. \tag{2.19}$$

If the irregular singularity has $f_0 \neq 0$ mass parameters, the corresponding VOA should be an extension of $W_{\tilde{k}}(\mathfrak{g}, f) \oplus M(1)^{\oplus f_0}$, where we denote the Heisenberg algebra by $M(1)$, therefore the $2d$ central charge equals to $c_{W_{\tilde{k}}(\mathfrak{g},f)} + f_0$. Combining these two situations, we see that the $4d$ central charge c_{4d} is

$$\boxed{c_{4d} = -\frac{1}{12}\left(\dim\mathfrak{g}^0 - \frac{1}{2}\dim\mathfrak{g}^{\frac{1}{2}} - \frac{12}{\tilde{k}+h^\vee_\mathfrak{g}}\left|\rho - (\tilde{k}+h^\vee_\mathfrak{g})\frac{h}{2}\right|^2\right) - \frac{f_0}{12},} \tag{2.20}$$

with $\tilde{k} = -h^\vee_\mathfrak{g} + \frac{b}{k+b}$ for untwisted theories and $\tilde{k} = -h^\vee_\mathfrak{g} + \frac{1}{m}\frac{b_t}{k_t+b_t}$ for twisted theories. The data relevant for h can be found from the weighted Dynkin diagram associated with the nilpotent orbit f.

On the other hand, computing the growth function of the vacuum module of a VOA requires more work. For a W-algebra $W_{\tilde{k}}(\mathfrak{g}, f)$ at boundary admissible level $\tilde{k} = -h^\vee + \frac{h^\vee}{u}$, the growth function of its vacuum module can be found in math literature (e.g. theorem 2.16 of [52]). If the VOA is non-admissible or an extension of $W_{\tilde{k}}(\mathfrak{g}, f) \oplus M(1)^{\oplus f_0}$, the growth function of its vacuum module is conjectured to have the following universal form [53]

$$\boxed{\mathcal{G} = \text{rank}\mathfrak{g} + f_0 - \frac{d(n)}{u} + \dim\mathcal{O}_{\text{prin}} - \dim\mathcal{O}_f.} \tag{2.21}$$

Here n and u is related to the level $\tilde{k}$ by

$$\frac{n}{u} = \tilde{k} + h^\vee, \quad \gcd(n, u) = 1. \tag{2.22}$$

The number $d(n)$ depends only on $\mathfrak{g}$ and n. To determine $d(n)$ one first consider the case $f = trivial$ so that the VOA is an extension of $V_{\tilde{k}}(\mathfrak{g}) \oplus u(1)^{\oplus f_0}$ with $2d$ central charge

$$\frac{\tilde{k}\dim\mathfrak{g}}{\tilde{k}+h^\vee} + f_0. \tag{2.23}$$

Together with the CB spectrum of the corresponding 4d theory, one can determine the growth function $\mathcal{G}$ of its vacuum module, hence determine $d(n)$. Results of $d(n)$ are listed in tables of appendix B. Here we also include the data for irregular singularities with mass deformation which were not discussed in [53].

Example: consider the $A_3/\mathbb{Z}_2$ twisted theory $\mathcal{T} \equiv (A_3, \mathbb{Z}_2, B_2, 4, -1, [3, 1^2])$. Its CB spectrum is $\{4/3, 4/3\}$, so

$$2a_{4d} - c_{4d} = \frac{1}{4} \times 2\left(2 \times \frac{4}{3} - 1\right) = \frac{5}{6}. \tag{2.24}$$

The irregular singularity has a mass deformation and the corresponding VOA is an extension of $W_{-3+\frac{2}{3}}(B_2, [3, 1^2]) \oplus M(1)$. Then we have

$$c_{2d} = -\frac{1}{12}, \tag{2.25}$$

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then central charges of this theory should be

$$a_{4d} = \frac{11}{12}, \qquad c_{4d} = 1. \tag{2.26}$$

For a consistency check, notice that the growth function of the vacuum module is

$$\mathcal{G} = 2 \times (2 \times 2 + 1) + 1 - \frac{d(2)}{3} - 6 = 4, \tag{2.27}$$

and indeed $a_{4d} - c_{4d} = -\mathcal{G}/48 = -1/12$. From central charges and CB spectrum, one conjecture that the VOA $V_{\mathcal{T}}$ is isomophsic to two copies of $V_{-\frac{4}{3}}(\mathfrak{su}(2))$,

$$W_{-3+\frac{2}{3}}(B_2, [3, 1^2]) \oplus M(1) \subset V_{\mathcal{T}} \cong V_{-\frac{4}{3}}(\mathfrak{su}(2)) \oplus V_{-\frac{4}{3}}(\mathfrak{su}(2)) \tag{2.28}$$

where $V_{-\frac{4}{3}}(\mathfrak{su}(2))$ means that the simple affine vertex algebra at level $k = -\frac{4}{3}$.

Flavor central charge: the flavor symmetry of our theories are determined by both the regular singularity f and the number of mass deformations f_0 of the irregular singularity. Given f and an embedding $\Lambda : \mathfrak{su}(2) \to \mathfrak{g}$, Let $\mathfrak{f}_\Lambda$ be the centralizer of $\Lambda(\mathfrak{su}(2))$ in $\mathfrak{g}$, then the naive flavor symmetry algebra of the theory is $\mathfrak{f}_\Lambda \oplus \mathfrak{u}(1)^{\oplus f_0}$. Notice that the full flavor symmetry may get enhanced to a larger algebra with $\mathfrak{f}_\Lambda \oplus \mathfrak{u}(1)^{\oplus f_0}$ as subalgebra.

Usually $\mathfrak{f}_\Lambda$ decomposes into several simple factors

$$\mathfrak{f}_\Lambda = \oplus_i \mathfrak{f}_i, \tag{2.29}$$

and we are interested in the 4d flavor anomaly $k_{4d,i}$ for each simple factor $\mathfrak{f}_i$. Using the 4d/VOA correspondence, this means that there are some simple factors of affine vertex algebra as the subalgebra of the W-algebra corresponding the theory in consideration,

$$\oplus_i V_{k_i}(\mathfrak{g}_i) \oplus M(1)^{\oplus f_0} \subset W_{\tilde{k}}(\mathfrak{g}, f) \oplus M(1)^{\oplus f_0} \subset V(\mathcal{T}). \tag{2.30}$$

One can compute the level of each affine vertex subalgebra $V_{k_i}(\mathfrak{g}_i)$ in the following way (see e.g. [35, 54]). Let T^a and T^b be generators of $\mathfrak{f}_i$ such that $\mathrm{tr}_{\mathfrak{f}_i} T^a T^b = h^\vee_{\mathfrak{f}_i} \delta^{ab}$. Denote by n the natural embedding $n : \mathfrak{f}_\Lambda \to \mathfrak{g}$. The level k_i of the affine vertex subalgebra $V_{k_i}(\mathfrak{g}_i)$ is then

$$k_i(\mathfrak{f}_i)\delta^{ab} = \frac{\tilde{k}_{2d}}{h^\vee_{\mathfrak{g}}} \mathrm{tr}_{\mathfrak{g}} n(T^a) n(T^b) + \sum_j 2j \mathrm{tr}_{\mathcal{R}_j} T^a T^b, \tag{2.31}$$

where R_j's are $\mathfrak{f}_\Lambda$ representations appear in the decomposition

$$\mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_{\geq 0}} V_j \otimes \mathcal{R}_j. \tag{2.32}$$

Once we have 2d level k_i, the 4d flavor anomaly is $k_{4d,i} = -k_i$ by the 4d/VOA correspondence.

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Example: consider the theory $(\mathfrak{e}_6, 9, -4, (A_2, \mathbb{Z}_2))$, one has $\mathfrak{f}_\Lambda = \mathfrak{su}(2)$ and $\tilde{k} = -12 + \frac{9}{5}$. The decomposition of the adjoint representation of $\mathfrak{e}_6$ is

$$\mathfrak{e}_6 \to (\mathbf{1};\mathbf{3}) + (\mathbf{3};\mathbf{1}) + (\mathbf{4};\mathbf{2}) + (\mathbf{5};\mathbf{1}) + (\mathbf{6};\mathbf{2}) + (\mathbf{7};\mathbf{1}) + (\mathbf{9};\mathbf{1}) + (\mathbf{10};\mathbf{2}) + (\mathbf{11};\mathbf{1}), \quad (2.33)$$

where the first entry in the bracket is the dimension of the $\mathfrak{su}(2)$ representation V_j, and the second entry the dimension of the representation of $\mathfrak{f}_\Lambda$. Normalizing the Dynkin index of the representation of $\mathfrak{su}(2)$ as the following

$$2\mathrm{tr}_{\mathrm{adj}} T^a T^b = 4, \qquad 2\mathrm{tr}_{\mathbf{2}} T^a T^b = 1, \quad (2.34)$$

the flavor anomaly of $\mathfrak{f}_\Lambda$ is then

$$k_{4d} = -k_{2d} = \frac{17}{10}. \quad (2.35)$$

We need to emphasize that equation (2.31) does not work for $\mathfrak{u}(1)$ factor in $\mathfrak{f}_\Lambda$. However, there is a way to compute the level of $\mathfrak{su}(1)$ in $\mathfrak{f}_\Lambda$ in terms of "pyramids" [55, 56]. We refer to [57] for the details.

2.2 Three dimensional canonical singularities

One can also construct 4d $\mathcal{N} = 2$ SCFTs by putting type IIB string theory on a 3d canonical singularity [9–12]. Two most interesting classes of canonical singularities are isolated quasi-homogeneous hypersurface singularity defined by a polynomial f and a complete intersection singularity defined by two polynomials (f_1, f_2).

Hypersurface singularity: given a hypersurface singularity $f(z_1, z_2, z_3, z_4)$, one can compute the CB spectrum as follows [9]: the quasi-homogeneity of f means that there is a C^* action

$$f(\lambda^{q_i} z_i) = \lambda f, \quad \sum q_i < 1, \quad q_i > 0. \quad (2.36)$$

One can also define a Jacobian algebra J_f

$$J_f = \frac{\mathbb{C}[z_1, z_2, z_3, z_4]}{\{\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_4}\}}. \quad (2.37)$$

Let $\{\phi_\alpha\}$ be a monomial basis of J_f, the SW geometry is then

$$F = f + \sum \lambda_\alpha \phi_\alpha. \quad (2.38)$$

The scaling dimension for λ_α can be worked out

$$\Delta(\lambda_\alpha) = \frac{(1 - Q(\phi_\alpha))}{\sum_i q_i}. \quad (2.39)$$

Here $Q(\phi_\alpha)$ is the weight for the monomial ϕ_α. For the SW curve, only λ_α with scaling dimension greater or equal to zero would be preserved as physical deformation parameters.

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The dimension of the charge lattice $\mu = 2r + f$ (Here r is the rank of the theory, and f the number of mass parameters) is given by the dimension of J_f (also called the Milnor number), which are related to weights of z_i's by the following formula

$$\mu = \left(1 - \frac{1}{q_1}\right)\left(1 - \frac{1}{q_2}\right)\left(1 - \frac{1}{q_3}\right)\left(1 - \frac{1}{q_4}\right). \tag{2.40}$$

Since one can find the number of mass parameters using the scaling formula, one can easily find the rank of the theory. The central charge can be computed as follows

$$a = \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5r}{24}, \quad c = \frac{R(B)}{3} + \frac{r}{6}, \tag{2.41}$$

where $R(A)$ and $R(B)$ are determined by the CB spectrum

$$R(A) = \sum_i ([u_i] - 1), \quad R(B) = \frac{1}{4}\mu[u]_{\max}. \tag{2.42}$$

Here $[u]_{\max}$ is the maximal scaling dimension, μ is the dimension of J_f (Milnor number) defined before.

Complete intersection singularity: given a complete intersection singularity $f = (f_1, f_2)$, and a C^* action

$$f_1(\lambda^{w_i} z_i) = \lambda f_1(z_i), \qquad f_2(\lambda^{w_i} z_i) = \lambda^d f_2(z_i), \tag{2.43}$$

there is a distinguished $(3, 0)$ form Ω satisfying

$$\Omega \wedge df_1 \wedge df_2 = dz_1 \wedge dz_2 \wedge \ldots \wedge dz_5, \tag{2.44}$$

which has weight $\sum w_i - 1 - d$ under the C^* action. To define a sensible 4d SCFT, we require this weight to be positive, which leads to a constraint on w_i's

$$\sum w_i > 1 + d. \tag{2.45}$$

The SW solution is described by the mini-versal deformation of the singularity [10, 11]

$$F(\lambda, z_i) = f(z_i) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha, \tag{2.46}$$

where ϕ_α is the monomial basis of the Jacobi module of f, and μ is the Milnor number of J_f. The coefficient λ_α is identified with the parameters on Coulomb branch. The scaling dimension of λ_α is determined by the requirement that Ω have dimension one, because the integration of Ω over the middle homology cycle of the Milnor fibration gives the mass of BPS particle. At the end the dimension of ϕ_α is computed by the following formulae

$$\begin{aligned} \phi_\alpha = [\psi_\alpha, 0]: \quad [\lambda_\alpha] &= \frac{1 - Q_\alpha}{\sum w_i - 1 - d}, \\ \phi_\alpha = [0, \psi_\alpha]: \quad [\lambda_\alpha] &= \frac{d - Q_\alpha}{\sum w_i - 1 - d}. \end{aligned} \tag{2.47}$$

Here Q_α is the C^* weight of the monomial ψ_α. The dimension of charge lattice μ can also be found from the weights [10, 11].

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3 Summary of theories with rank one, two and three

Now we can scan the theory space reviewed in the last section for SCFTs with small rank. Theories from spheres with one irregular singularity and one regular singularity with rank one, two and three are listed in table 4, 5, and 7 respectively. For each theory, we list its central charges, Coulomb branch spectrum, flavor symmetry and the corresponding VOA. The complete lists with constructions are summarized in appendix C.1, C.2 and C.3. The predictions of VOA isomorphisms are list in section 4. We also scan the theory engineered by compactifying A_n type theory on sphere with three or more regular singularities, and results are shown in table 9 and 10. The corresponding VOA of theories with regular singularities only is constructed rigorously by mathematicians [58].

We also search the rank one, two, three theories from the list of hypersurface and complete intersection singularities. They do not provide new possibilities, but we also include the construction in our tables. The class is labeled by two Dynkin diagrams (G, G') so that the 3-fold singularity is $f_G(x, y) + f_{G'}(w, z) = 0$, and $f_{G=ADE}$ is the corresponding ADE singularity.

All theories in table 4, 5, and 7 satisfy the following bound

$$\frac{12\left(\sum_i (u_i - 1)^2\right)}{u_{\max}(u_{\max} - 1)} \geq 2r + f, \tag{3.1}$$

where $\{u_i\}$ is the spectrum of CB operators, $u_{\max}$ is the maximal dimension of the CB operators, r is the rank of the theory and f is the rank of the flavor symmetry. Theories saturating this bound is marked with $*$ in the tables. Those theories seem special to us.

Notice that some theories could be the decoupled sum of two lower rank theories. They are summarized in table 6 and 8. Below we provide detailed analysis for one example of this situation.

Example: consider the theory with CB spectrum $\{\frac{8}{5}, \frac{6}{5}, \frac{6}{5}\}$ in table 8. It is expected to be comprised of two decoupled theories with CB spectrum $\{\frac{8}{5}, \frac{6}{5}\}$ and $\{\frac{6}{5}\}$ respectively. Using the corresponding VOA, one can denote these theories by $\mathcal{T}[W_{-7+\frac{7}{5}}(B_4, [5, 2^2])]$, $\mathcal{T}[V_{-2+\frac{2}{5}}(\mathfrak{su}(2))]$ and $\mathcal{T}[\mathrm{Vir}(2, 5)]$ repectively. Since all of these VOAs are admissible, one can compute its Schur index using the formula in [50] and check if it is the product of the indices of the two component theories.

For nilpotent orbit $[5, 2^2]$, the flavor symmetry is SU(2) and the adjoint representation of SO(9) decomposes as

$$\mathrm{adj}_{\mathrm{SO}(9)} = V_0 \otimes U_1 \oplus 2V_1 \oplus V_{\frac{3}{2}} \otimes U_{\frac{1}{2}} \oplus V_{\frac{5}{2}} \otimes U_{\frac{1}{2}} \oplus V_3, \tag{3.2}$$

where V_j is the spin j representation of $\mathfrak{su}(2)$ and U_j is the spin j representation of the flavor SU(2). The Schur index (the normalized vacuum character of $W_{-7+\frac{7}{5}}(B_4, [5, 2^2])$) is then

$$\mathcal{I}_{\mathcal{T}[W_{-7+\frac{7}{5}}(B_4,[5,2^2])]} = \mathrm{PE}\left[\frac{(q - q^5)\chi_1(z) + (q^2 - q^4)}{(1 - q)(1 - q^5)}\right], \tag{3.3}$$

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA	$\mathcal{M}_H$/Asso. Var.	Singularity
$*(\frac{43}{120},\frac{11}{30})$	$\frac{6}{5}$	–	$W_{-\frac{8}{5}}(\mathfrak{su}(2),[2])\cong \text{Vir}(2,5)$	–	(A_1,A_2)
$*(\frac{11}{24},\frac{1}{2})$	$\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}$	$V_{-\frac{4}{3}}(\mathfrak{su}(2))$	$[2]$	(A_1,A_3)
$*(\frac{7}{12},\frac{2}{3})$	$\frac{3}{2}$	$\mathfrak{su}(3)_{\frac{3}{2}}$	$V_{-\frac{3}{2}}(\mathfrak{su}(3))$	$[2,1]$	(A_1,D_4)
$*(\frac{23}{24},\frac{7}{6})$	2	$\mathfrak{so}(8)_2$	$V_{-2}(\mathfrak{so}(8))$	$[2^2,1^4]$	
$(\frac{23}{24},\frac{7}{6})$	2	$\mathfrak{so}(7)_2$	$V_{-2}(\mathfrak{so}(7))$	$[3,1^4]$	
$(\frac{23}{24},\frac{7}{6})$	2	$(\mathfrak{g}_2)_2$	$V_{-2}(\mathfrak{g}_2)$	$G_2(a_1)$	
$(\frac{3}{4},\frac{3}{4})$	2	$\mathfrak{su}(2)_{\frac{3}{2}}$	$W_{-6+2}(D_4,[3,2^2,1])$	$\overline{\mathcal{O}}_{[3^2,1^2]}\cap S_{[3,2^2,1]}$	
$*(\frac{41}{24},\frac{13}{6})$	3	$(\mathfrak{e}_6)_3$	$V_{-3}(\mathfrak{e}_6)$	A_1	
$(\frac{41}{24},\frac{13}{6})$	3	$(\mathfrak{f}_4)_3$	$V_{-3}(\mathfrak{f}_4)$	$\tilde{A}_1$	
$(\frac{17}{12},\frac{19}{12})$	3	$\mathfrak{sp}(2)_2\times\mathfrak{u}(1)_{f_0}$	$W_{-4+\frac{3}{2}}(C_3,[2,1^4])\oplus M(1)$		
$*(\frac{59}{24},\frac{19}{6})$	4	$(\mathfrak{e}_7)_4$	$V_{-4}(\mathfrak{e}_7)$	A_1	
$(\frac{25}{12},\frac{29}{12})$	4	$\mathfrak{sp}(3)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+7}(\mathfrak{e}_7,4A_1)\oplus M(1)$		
$*(\frac{95}{24},\frac{31}{6})$	6	$(\mathfrak{e}_8)_6$	$V_{-6}(\mathfrak{e}_8)$	A_1	

Table 4. List of rank one class-S theories on a sphere with one irregular and one regular punctures with central charges (a,c), CB spectrum Δ_{Coulomb}, flavor symmetry algebra $\mathfrak{f}$, one corresponding VOA and Higgs branch. Theories with $*$ saturate the bound (3.1). $\mathfrak{u}(1)$ with a subscript f_0 means the $\mathfrak{u}(1)$ algebra from the irregular singularity. $W\oplus M(1)^{\oplus f_0}$ means the actual VOA is possibly an extension of $W\oplus M(1)^{\oplus f_0}$. In the last coulomb, a single nilpotent orbit f means that the corresponding Higgs branch or associated variety is the closure of the nilpotent orbit $\overline{\mathcal{O}}_f$, while $S_{f'}$ means the Slodowy slice of f'.

where $\text{PE}[f(x,y,z,\cdots)]=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f(x^n,y^n,z^n,\cdots)\right)$ being the plethystic exponential and $\chi_1(z)$ is the character of the adjoint representation of SU(2). On the other hand, the indices of $\mathcal{T}[V_{-2+\frac{2}{5}}(\mathfrak{su}(2))]$ and $\mathcal{T}[\text{Vir}(2,5)]$ are

$$\mathcal{I}_{\mathcal{T}[V_{-2+\frac{2}{5}}(\mathfrak{su}(2))]}=\text{PE}\left[\frac{(q-q^5)\chi_1(z)}{(1-q)(1-q^5)}\right],\quad \mathcal{I}_{\mathcal{T}[\text{Vir}(2,5)]}=\text{PE}\left[\frac{q^2-q^4}{(1-q)(1-q^5)}\right]. \tag{3.4}$$

It is clear that

$$\mathcal{I}_{\mathcal{T}[W_{-7+\frac{7}{5}}(B_4,[5,2^2])]}=\mathcal{I}_{\mathcal{T}[V_{-2+\frac{2}{5}}(\mathfrak{su}(2))]}\mathcal{I}_{\mathcal{T}[\text{Vir}(2,5)]}. \tag{3.5}$$

This also leads to a possible isomorphism of VOAs

$$W_{-7+\frac{7}{5}}(B_4,[5,2^2])\cong V_{-2+\frac{2}{5}}(\mathfrak{su}(2))\oplus\text{Vir}(2,5). \tag{3.6}$$

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA	$\mathcal{M}_H$/Asso. Var.	Singularity
$*(\frac{67}{84},\frac{17}{21})$	$\frac{10}{7},\ \frac{8}{7}$	–	$W_{-2+\frac{2}{7}}(\mathfrak{su}(2),[2])$	–	(A_1,A_4)
$*(\frac{11}{12},\frac{23}{24})$	$\frac{3}{2},\ \frac{5}{4}$	$\mathfrak{u}(1)_{f_0}$	$W_{-5+\frac{5}{4}}(\mathfrak{su}(5),[3,2])$		(A_1,A_5)
$(\frac{7}{6},\frac{4}{3})$	$\frac{3}{2},\ \frac{3}{2}$	$\mathfrak{su}(3)_3$	$W_{-30+\frac{30}{4}}(\mathfrak{e}_8,D_4(a_1)+A_2)$	$\overline{\mathcal{O}}_{A_4+2A_1}\cap S_{D_4(a_1)+A_2}$	
$*(\frac{19}{20},1)$	$\frac{8}{5},\ \frac{6}{5}$	$\mathfrak{su}(2)_{\frac{8}{5}}$	$V_{-2+\frac{2}{5}}(\mathfrak{su}(2))$	$[2]$	(A_1,D_5)
$*(\frac{13}{12},\frac{7}{6})$	$\frac{5}{3},\ \frac{4}{3}$	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}$	$W_{-30+\frac{30}{9}}(\mathfrak{e}_8,E_7(a_3))$	$\overline{\mathcal{O}}_{E_8(a_6)}\cap S_{E_7(a_3)}$	(A_1,D_6)
$(\frac{7}{4},2)$	$2,\ 2$	$\mathfrak{u}(1)^5_{f_0}$	$W_{-8+1}(D_5,[7,3])\oplus M(1)^{\oplus 5}$		$(\sum_{i=1}^5 x_i^2,\sum_{i=1}^5 ix_i^2)$
$(\frac{7}{4},2)$	$2,\ 2$	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2\times\mathfrak{su}(2)_6$	$W_{-18+6}(\mathfrak{e}_7,A_2+2A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)}\cap S_{A_2+2A_1}$	
$(\frac{7}{4},2)$	$2,\ 2$	$\mathfrak{su}(2)_8\times\mathfrak{su}(2)_2$	$W_{-30+6}(\mathfrak{e}_8,D_5(a_1)+A_1)$	$\overline{\mathcal{O}}_{E_8(a_7)}\cap S_{D_5(a_1)+A_1}$	
$(\frac{7}{4},2)$	$2,\ 2$	$\mathfrak{su}(2)_{10}$	$W_{-30+6}(\mathfrak{e}_8,A_4+A_2+A_1)$	$\overline{\mathcal{O}}_{E_8(a_7)}\cap S_{A_4+A_2+A_1}$	
$(\frac{19}{12},\frac{5}{3})$	$2,\ 2$	$\mathfrak{sp}(2)_2$	$V_{-3+1}(\mathfrak{so}(5))$	$[3,1^2]$	
$(\frac{163}{120},\frac{17}{12})$	$\frac{12}{5},\ \frac{6}{5}$	$\mathfrak{su}(2)_{\frac{17}{10}}$	$W_{-12+\frac{9}{5}}(\mathfrak{e}_6,A_5)$	$\overline{\mathcal{O}}_{E_6(a_3)}\cap S_{A_5}$	
$*(\frac{7}{4},2)$	$\frac{5}{2},\ \frac{3}{2}$	$\mathfrak{su}(5)_{\frac{5}{2}}$	$V_{-5+\frac{5}{2}}(\mathfrak{su}(5))$	$[2^2,1]$	$D_2(\mathrm{SU}(5))$
$(\frac{19}{12},\frac{5}{3})$	$\frac{5}{2},\ \frac{3}{2}$	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$W_{-4+\frac{3}{4}}(C_3,[4,1^2])\oplus M(1)$		
$(\frac{13}{8},\frac{7}{4})$	$\frac{8}{3},\ \frac{4}{3}$	$\mathfrak{so}(3)_{\frac{4}{3}}\times\mathfrak{su}(2)_{\frac{11}{6}}$	$W_{-5+\frac{5}{3}}(C_4,[2^3,1^2])$	$\overline{\mathcal{O}}_{[3^2,2]}\cap S_{[2^3,1^2]}$	
$(\frac{47}{24},\frac{13}{6})$	$3,\ \frac{3}{2}$	$\mathfrak{su}(3)_3\times\mathfrak{su}(2)_2$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,3A_1)$	$\overline{\mathcal{O}}_{A_2+2A_1}\cap S_{3A_1}$	
$*(\frac{29}{12},\frac{17}{6})$	$3,\ 2$	$\mathfrak{su}(6)_3\times\mathfrak{u}(1)_{f_0}$	$V_{-6+3}(\mathfrak{su}(6))\oplus M(1)$		
$(2,2)$	$3,\ 2$	$\mathfrak{su}(2)_4$	$W_{-4+1}(\mathfrak{g}_2,A_1)$	$\overline{\mathcal{O}}_{G_2(a_1)}\cap S_{A_1}$	
$(\frac{61}{24},\frac{17}{6})$	$3,\ \frac{5}{2}$	$\mathfrak{sp}(3)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$V_{-4+\frac{3}{2}}(C_3)\oplus M(1)$		
$(2,\frac{13}{6})$	$\frac{10}{3},\ \frac{4}{3}$	$\mathfrak{sp}(2)_{\frac{13}{6}}$	$W_{-10+\frac{10}{3}}(D_6,[3,2^4,1])$	$\overline{\mathcal{O}}_{[3^4]}\cap S_{[3,2^4,1]}$	
$*(\frac{37}{12},\frac{11}{3})$	$4,\ 2$	$\mathfrak{spin}(12)_4$	$V_{-10+6}(D_6)$	$[2^4,1^4]$	
$(\frac{71}{24},\frac{41}{12})$	$4,\ 2$	$\mathfrak{su}(3)_4\times\mathfrak{su}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$W_{-12+4}(\mathfrak{e}_6,3A_1)\oplus M(1)^{\oplus 2}$		
$(\frac{71}{24},\frac{41}{12})$	$4,\ 2$	$(\mathfrak{g}_2)_4\times\mathfrak{su}(2)_{\frac{5}{2}}$	$W_{-30+10}(\mathfrak{e}_8,A_2+3A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)+A_1}\cap S_{A_2+3A_1}$	
$(\frac{17}{6},\frac{19}{6})$	$4,\ 2$	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$W_{-10+3}(D_6,[3,2^4,1])\oplus M(1)^{\oplus 2}$		
$(\frac{83}{24},\frac{47}{12})$	$4,\ 3$	$\mathfrak{sp}(3)_3\times\mathfrak{su}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+7}(\mathfrak{e}_7,(3A_1)')\oplus M(1)$		
$*(\frac{53}{12},\frac{16}{3})$	$5,\ 3$	$\mathfrak{spin}(12)_5\times\mathfrak{u}(1)^2_{f_0}$	$V_{-10+5}(D_6)\oplus M(1)^{\oplus 2}$		
$(\frac{95}{24},\frac{53}{12})$	$5,\ 3$	$\mathfrak{sp}(4)_3\times\mathfrak{u}(1)_{f_0}$	$W_{-6+\frac{5}{2}}(C_5,[2,1^8])\oplus M(1)$		
$(\frac{14}{3},\frac{16}{3})$	$5,\ 4$	$(\mathfrak{f}_4)_5\times\mathfrak{u}(1)_{f_0}$	$V_{-9+4}(\mathfrak{f}_4)\oplus M(1)$		
$*(\frac{101}{12},\frac{31}{3})$	$10,\ 4$	$(\mathfrak{e}_8)_{10}$	$V_{-30+20}(\mathfrak{e}_8)$	$2A_1$	

Table 5. List of rank two class-S theories on a sphere with one irregular and one regular punctures with central charges (a,c), CB spectrum Δ_{Coulomb}, flavor symmetry algebra $\mathfrak{f}$, one corresponding VOA and Higgs branch. Theories with $*$ saturate the bound (3.1). $\mathfrak{u}(1)$ with a subscript f_0 means the $\mathfrak{u}(1)$ algebra from the irregular singularity. $W\oplus M(1)^{\oplus f_0}$ means the actual VOA is possibly an extension of $W\oplus M(1)^{\oplus f_0}$. In the last coulomb, a single nilpotent orbit f means that the corresponding Higgs branch or associated variety is the closure of the nilpotent orbit $\overline{\mathcal{O}}_f$, while $S_{f'}$ means the Slodowy slice of f'.

(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA
$*(\frac{43}{60},\frac{11}{15})=2(\frac{43}{120},\frac{11}{30})$	$\frac{6}{5},\ \frac{6}{5}$	–	$W_{-12+\frac{6}{5}}(\mathfrak{e}_6,E_6(a_1))$
$*(\frac{11}{12},1)=2(\frac{11}{24},\frac{1}{2})$	$\frac{4}{3},\ \frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{su}(2)_{\frac{4}{3}}$	$W_{-9+\frac{5}{3}}(B_5,[7,1^4])$
$*(\frac{7}{6},\frac{4}{3})=2(\frac{7}{12},\frac{2}{3})$	$\frac{3}{2},\ \frac{3}{2}$	$\mathfrak{su}(3)_{\frac{3}{2}}\times\mathfrak{su}(3)_{\frac{3}{2}}$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,A_2)$
$(\frac{23}{12},\frac{7}{3})=2(\frac{23}{24},\frac{7}{6})$	$2,\ 2$	$(\mathfrak{g}_2)_2\times(\mathfrak{g}_2)_2$	$W_{-30+10}(\mathfrak{e}_8,2A_2)$

Table 6. Rank 2 theories which are likely to be two decouple rank 1 theories.

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA	$\mathcal{M}_H$/Asso. Var.	Singularity
$(\frac{3}{2},\frac{13}{8})$	$\frac{5}{4},\frac{3}{2},\frac{3}{2}$	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+\frac{9}{4}}(\mathfrak{e}_7, D_5+A_1)+M(1)$		
$(\frac{7}{4},2)$	$\frac{3}{2},\frac{3}{2},\frac{3}{2}$	$\mathfrak{su}(2)^3_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+\frac{9}{2}}(\mathfrak{e}_7, D_4(a_1))+M(1)$		
$*(\frac{91}{72},\frac{23}{18})$	$\frac{14}{9},\frac{4}{3},\frac{10}{9}$	–	$W_{-2+\frac{2}{9}}(\mathfrak{su}(2),[2])$	–	(A_1,A_6)
$*(\frac{167}{120},\frac{43}{30})$	$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$\mathfrak{u}(1)_{f_0}$	$W_{-4+\frac{4}{5}}(\mathfrak{su}(4),[3,1])$		(A_1,A_7)
$(\frac{75}{56},\frac{19}{14})$	$\frac{12}{7},\frac{9}{7},\frac{8}{7}$	–	$W_{-3+\frac{3}{7}}(\mathfrak{su}(3),[3])$	–	(A_1,E_6)
$*(\frac{81}{56},\frac{3}{2})$	$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$\mathfrak{su}(2)_{\frac{12}{7}}$	$V_{-2+\frac{2}{7}}(\mathfrak{su}(2))$	$[2]$	(A_1,D_7)
$*(\frac{19}{12},\frac{5}{3})$	$\frac{7}{4},\frac{3}{2},\frac{5}{4}$	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$W_{-5+\frac{5}{4}}(\mathfrak{su}(5),[3,1^2])$		(A_1,D_8)
$*(\frac{3}{2},\frac{31}{20})$	$\frac{9}{5},\frac{7}{5},\frac{6}{5}$	$\mathfrak{u}(1)_{\frac{9}{5}}$	$W_{-3+\frac{3}{5}}(\mathfrak{su}(3),[2,1])$	$\overline{\mathcal{O}}_{[3]}\cap S_{[2,1]}$	(A_1,E_7)
$*(\frac{15}{8},2)$	$2,\frac{3}{2},\frac{3}{2}$	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)^2_{f_0}$	$W_{-3+\frac{1}{2}}(\mathfrak{su}(3),[2,1])+M(1)^{\oplus 2}$		
$(\frac{61}{24},\frac{17}{6})$	$2,2,2$	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)^4_{f_0}$	$W_{-5+1}(\mathfrak{su}(5),[3,1^2])+M(1)^{\oplus 4}$		
$(\frac{61}{24},\frac{17}{6})$	$2,2,2$	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_1$	$W_{-30+6}(\mathfrak{e}_8, A_4+A_2)$	$\overline{\mathcal{O}}_{E_8(a_7)}\cap S_{A_4+A_2}$	
$(\frac{19}{8},\frac{5}{2})$	$2,2,2$	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2$	$W_{-6+2}(D_4,[2^2,1^4])$	$\overline{\mathcal{O}}_{[3^2,1^2]}\cap S_{[2^2,1^4]}$	
$(\frac{53}{24},\frac{13}{6})$	$2,2,2$	$\mathfrak{u}(1)$	$W_{-10+2}(D_6,[5,3^2,1])$	$\overline{\mathcal{O}}_{[5^2,1^2]}\cap S_{[5,3^2,1]}$	
$*(\frac{15}{8},2)$	$\frac{9}{4},\frac{3}{2},\frac{5}{4}$	$\mathfrak{su}(3)_{\frac{9}{4}}$	$V_{-3+\frac{3}{4}}(\mathfrak{su}(3))$	$[3]$	$D_4(\mathrm{SU}(3))$
$*(\frac{25}{12},\frac{9}{4})$	$\frac{7}{3},\frac{5}{3},\frac{4}{3}$	$\mathfrak{su}(3)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{5}{3}}$	$W_{-5+\frac{5}{3}}(\mathfrak{su}(5),[2,1^3])$	$\overline{\mathcal{O}}_{[3,2]}\cap S_{[2,1^3]}$	
$(\frac{103}{60},\frac{107}{60})$	$\frac{12}{5},\frac{6}{5},\frac{6}{5}$	$\mathfrak{su}(2)_{\frac{17}{10}}$	$W_{-18+\frac{14}{5}}(\mathfrak{e}_7, D_6(a_2))$	$\overline{\mathcal{O}}_{E_7(a_5)}\cap S_{D_6(a_2)}$	
$(\frac{85}{42},\frac{25}{12})$	$\frac{18}{7},\frac{12}{7},\frac{8}{7}$	$\mathfrak{su}(2)_{\frac{25}{14}}$	$W_{-12+\frac{12}{7}}(\mathfrak{e}_6, A_5)$	$\overline{\mathcal{O}}_{E_6(a_3)}\cap S_{A_5}$	
$(\frac{55}{24},\frac{5}{2})$	$\frac{8}{3},\frac{5}{3},\frac{4}{3}$	$\mathfrak{su}(4)_{\frac{8}{3}}$	$V_{-4+\frac{4}{3}}(\mathfrak{su}(4))$	$[3,1]$	$D_3(\mathrm{SU}(4))$
$(\frac{21}{8},\frac{17}{6})$	$\frac{8}{3},\frac{7}{3},\frac{4}{3}$	$\mathfrak{sp}(2)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{4}{3}}$	$W_{-8+\frac{8}{3}}(D_5,[2^4,1^2])$	$\overline{\mathcal{O}}_{[3^3,1]}\cap S_{[2^4,1^2]}$	
$(\frac{32}{15},\frac{133}{60})$	$\frac{14}{5},\frac{8}{5},\frac{6}{5}$	$\mathfrak{su}(2)_{\frac{19}{10}}$	$W_{-8+\frac{8}{5}}(D_5,[5,2^2,1])$	$\overline{\mathcal{O}}_{[5^2]}\cap S_{[5,2^2,1]}$	
$(\frac{8}{3},\frac{17}{6})$	$3,2,\frac{3}{2}$	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)^2_{f_0}$	$W_{-10+\frac{3}{2}}(D_6,[7,2^2,1])+M(1)^{\oplus 2}$		
$*(\frac{25}{8},\frac{7}{2})$	$3,2,2$	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)^3_{f_0}$	$V_{-4+1}(\mathfrak{su}(4))+M(1)^{\oplus 3}$		
$(\frac{73}{24},\frac{10}{3})$	$3,\frac{5}{2},\frac{3}{2}$	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-8+\frac{5}{2}}(D_5,[2^4,1^2])+M(1)$		
$(\frac{27}{8},\frac{7}{2})$	$3,3,2$	$(\mathfrak{g}_2)_3$	$V_{-4+1}(\mathfrak{g}_2)$	$G_2(a_1)$	
$(\frac{29}{8},4)$	$3,\frac{7}{2},\frac{3}{2}$	$\mathfrak{so}(7)_{\frac{7}{2}}\times\mathfrak{u}(1)_{f_0}$	$V_{-5+\frac{3}{2}}(B_3)+M(1)$		
$(\frac{49}{12},\frac{53}{12})$	$3,\frac{9}{2},\frac{3}{2}$	$\mathfrak{su}(2)_{\frac{9}{2}}\times\mathfrak{su}(2)_{\frac{11}{2}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+\frac{9}{2}}(\mathfrak{e}_7, 2A_2+A_1)+M(1)$		
$(\frac{77}{24},\frac{7}{2})$	$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$\mathfrak{spin}(7)_{\frac{10}{3}}$	$V_{-5+\frac{5}{3}}(\mathfrak{so}(7))$	$[3^2,1]$	$D_3(\mathrm{SO}(7))$
$(\frac{77}{24},\frac{7}{2})$	$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$\mathfrak{sp}(3)_{\frac{8}{3}}$	$V_{-4+\frac{4}{3}}(C_3)$	$[3^2]$	$D_3(\mathrm{Sp}(3))$
$(\frac{19}{6},\frac{41}{12})$	$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$\mathfrak{sp}(2)_6\times\mathfrak{su}(2)_{\frac{13}{6}}$	$W_{-10+\frac{10}{3}}(D_6,[3,2^2,1^5])$	$\overline{\mathcal{O}}_{[3^4]}\cap S_{[3,2^2,1^5]}$	

Table 7. List of rank three class-S theories on a sphere with one irregular and one regular punctures with central charges (a,c), CB spectrum Δ_{Coulomb}, flavor symmetry algebra $\mathfrak{f}$, one corresponding VOA and Higgs branch. Theories with $*$ saturate the bound (3.1). $\mathfrak{u}(1)$ with a subscript f_0 means the $\mathfrak{u}(1)$ algebra from the irregular singularity. $W\oplus M(1)^{\oplus f_0}$ means the actual VOA is possibly an extension of $W\oplus M(1)^{\oplus f_0}$. In the last coulomb, a single nilpotent orbit f means that the corresponding Higgs branch or associated variety is the closure of the nilpotent orbit $\overline{\mathcal{O}}_f$, while $S_{f'}$ means the Slodowy slice of f'. (continues).

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA	$\mathcal{M}_H$/Asso. Var.	Singularity
$*(\frac{7}{2},4)$	$\frac{7}{2},\frac{5}{2},\frac{3}{2}$	$\mathfrak{su}(7)_{\frac{7}{2}}$	$V_{-7+\frac{7}{2}}(\mathfrak{su}(7))$	$[2^3,1]$	$D_2(\mathrm{SU}(7))$
$(\frac{71}{24},\frac{46}{15})$	$\frac{18}{5},\frac{12}{5},\frac{6}{5}$	$\mathfrak{su}(2)_{\frac{23}{5}}$	$W_{-18+\frac{14}{5}}(\mathfrak{e}_7, A_5+A_1)$	$\overline{\mathcal{O}}_{E_7(a_5)}\cap S_{A_5+A_1}$	
$(\frac{83}{24},\frac{11}{3})$	$4,2,2$	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{su}(2)_4$	$W_{-30+8}(\mathfrak{e}_8, A_3+2A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)+A_2}\cap S_{A_3+2A_1}$	
$(\frac{83}{24},\frac{11}{3})$	$4,\frac{8}{3},\frac{4}{3}$	$\mathfrak{su}(2)_5\times\mathfrak{su}(2)_4$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7, 2A_2+A_1)$	$\overline{\mathcal{O}}_{A_3+A_2+A_1}\cap S_{2A_2+A_1}$	
$*(\frac{107}{24},\frac{31}{6})$	$4,3,2$	$\mathfrak{su}(8)_4\times\mathfrak{u}(1)_{f_0}$	$V_{-8+4}(\mathfrak{su}(8))+M(1)$		
$(\frac{101}{24},\frac{14}{3})$	$4,3,2$	$\mathfrak{sp}(3)_3\times\mathfrak{u}(1)^2_{f_0}$	$W_{-10+3}(D_6,[2^6])+M(1)^{\oplus 2}$		
$(\frac{25}{6},\frac{55}{12})$	$4,3,2$	$\mathfrak{sp}(2)_3\times\mathfrak{su}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$W_{-10+3}(D_6,[3,2^2,1^5])+M(1)^{\oplus 2}$		
$(\frac{15}{4},\frac{15}{4})$	$4,3,2$	$\mathfrak{su}(2)_{\frac{15}{2}}$	$W_{-9+2}(\mathfrak{f}_4, A_2+\tilde{A}_1)$	$\overline{\mathcal{O}}_{F_4(a_3)}\cap S_{A_2+\tilde{A}_1}$	
$(\frac{39}{8},\frac{11}{2})$	$4,4,2$	$\mathfrak{so}(7)_4\times\mathfrak{u}(1)_6\times\mathfrak{u}(1)^2_{f_0}$	$W_{-12+4}(\mathfrak{e}_6, 2A_1)+M(1)^{\oplus 2}$		
$(\frac{39}{8},\frac{11}{2})$	$4,4,2$	$\mathfrak{so}(7)_4\times\mathfrak{su}(2)_{12}$	$W_{-30+10}(\mathfrak{e}_8, A_2+2A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)+A_1}\cap S_{A_2+2A_1}$	
$(\frac{47}{12},\frac{17}{4})$	$\frac{14}{3},\frac{8}{3},\frac{4}{3}$	$\mathfrak{sp}(3)_{\frac{17}{6}}$	$W_{-5+\frac{5}{3}}(C_4,[2,1^6])$	$\overline{CO}_{[3^2,2]}\cap S_{[2,1^6]}$	
$(\frac{17}{3},\frac{19}{3})$	$5,\frac{7}{2},3$	$\mathfrak{sp}(5)_{\frac{7}{2}}\times\mathfrak{u}(1)_{f_0}$	$V_{-6+\frac{5}{2}}(C_5)+M(1)$		
$(\frac{143}{24},\frac{20}{3})$	$5,4,3$	$\mathfrak{so}(9)_5\times\mathfrak{su}(2)_3\times\mathfrak{u}(1)_{f_0}$	$W_{-18+7}(\mathfrak{e}_7, 2A_1)+M(1)$		
$(\frac{19}{4},\frac{21}{4})$	$6,2,2$	$\mathfrak{sp}(3)_{\frac{7}{2}}$	$W_{-18+6}(\mathfrak{e}_7, 4A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)}\cap S_{4A_1}$	
$*(\frac{45}{8},\frac{13}{2})$	$6,3,2$	$(\mathfrak{e}_6)_6$	$V_{-12+6}(\mathfrak{e}_6)$	A_2	
$(\frac{133}{24},\frac{35}{6})$	$6,4,2$	$\mathfrak{sp}(2)_7$	$W_{-30+8}(\mathfrak{e}_8, 2A_2+2A_1)$	$\overline{\mathcal{O}}_{D_4(a_1)+A_2}\cap S_{2A_2+2A_1}$	
$(\frac{43}{6},\frac{97}{12})$	$6,6,2$	$(\mathfrak{f}_4)_6\times\mathfrak{su}(2)_{\frac{7}{2}}$	$W_{-30+12}(\mathfrak{e}_8, 3A_1)$	$\overline{\mathcal{O}}_{2A_2}\cap S_{3A_1}$	
$*(\frac{75}{8},11)$	$9,5,3$	$(\mathfrak{e}_7)_9\times\mathfrak{u}(1)_{f_0}$	$V_{-18+9}(\mathfrak{e}_7)+M(1)$		
$(\frac{75}{8},11)$	$9,5,3$	$(\mathfrak{e}_7)_9$	$W_{-30+15}(\mathfrak{e}_8, A_1)$	$\overline{\mathcal{O}}_{A_2+A_1}\cap S_{A_1}$	

Table 7. List of rank three class-S theories on a sphere with one irregular and one regular punctures with central charges (a,c), CB spectrum Δ_{Coulomb}, flavor symmetry algebra $\mathfrak{f}$, one corresponding VOA and Higgs branch. Theories with $*$ saturate the bound (3.1). $\mathfrak{u}(1)$ with a subscript f_0 means the $\mathfrak{u}(1)$ algebra from the irregular singularity. $W\oplus M(1)^{\oplus f_0}$ means the actual VOA is possibly an extension of $W\oplus M(1)^{\oplus f_0}$. In the last coulomb, a single nilpotent orbit f means that the corresponding Higgs branch or associated variety is the closure of the nilpotent orbit $\overline{\mathcal{O}}_f$, while $S_{f'}$ means the Slodowy slice of f'.

4 Implications on isomorphism of W-algebras

In our construction, it is common that one can engineer the same 4d $\mathcal{N}=2$ SCFTs by quite different data. Since two 4d theories are very likely to be the same if they share the same CB spectrum, central charges and the rank of flavor symmetry (there might be enhancement of the flavor symmetry), using observables such as CB spectrum, central charges and flavor symmetry, one can identify theories from different construction. Because each data $(\mathfrak{j},b,k,f)$ or $(\mathfrak{j},o,\mathfrak{g},b_t,k_t,f)$ leads to a corresponding VOA, the duality of 4d theories implies possible isomorphisms of W-algebras.

In table 11, 12, 13 and 14 we list different W-algebras which correspond to the same 4d theory. In each table we list the CB spectrum, central charges and flavor symmetry of the 4d

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA
$*(\frac{43}{40},\frac{11}{10})=3(\frac{43}{120},\frac{11}{30})$	$\frac{6}{5},\frac{6}{5},\frac{6}{5}$	–	$W_{-18+\frac{6}{5}}(\mathfrak{e}_7,E_7(a_1))$
$(\frac{157}{120},\frac{41}{30})=(\frac{19}{20},1)+(\frac{43}{120},\frac{11}{30})$	$\frac{8}{5},\frac{6}{5},\frac{6}{5}$	$\mathfrak{su}(2)_{\frac{8}{5}}$	$W_{-7+\frac{7}{5}}(B_4,[5,2^2])$
$*(\frac{11}{8},\frac{3}{2})=3(\frac{11}{24},\frac{1}{2})$	$\frac{4}{3},\frac{4}{3},\frac{4}{3}$	$\mathfrak{su}(2)^3_{\frac{4}{3}}$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,D_4(a_1))$
$(\frac{37}{24},\frac{5}{3})=(\frac{13}{12},\frac{7}{6})+(\frac{11}{24},\frac{1}{2})$	$\frac{5}{3},\frac{4}{3},\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}\times\mathfrak{su}(2)_{\frac{4}{3}}$	$W_{-18+\frac{14}{6}}(\mathfrak{e}_7,D_5)+M(1)$
$(\frac{25}{12},\frac{9}{4})=(\frac{13}{8},\frac{7}{4})+(\frac{11}{24},\frac{1}{2})$	$\frac{8}{3},\frac{4}{3},\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{11}{6}}\times\mathfrak{su}(2)_{\frac{8}{3}}\times\mathfrak{su}(2)_{\frac{4}{3}}$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,(A_3+A_1)')$
$(\frac{61}{24},\frac{17}{6})=(\frac{47}{24},\frac{13}{6})+(\frac{7}{12},\frac{2}{3})$	$3,\frac{3}{2},\frac{3}{2}$	$\mathfrak{su}(2)_3\times\mathfrak{su}(2)_2\times\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}_{f_0}$	$W_{-18+\frac{9}{2}}(\mathfrak{e}_7,(A_3+A_1)')+M(1)$

Table 8. Rank 3 theories that are likely to be several decoupled lower rank theories. Data with the same color come from the same theory.

(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	Fixtures
$(\frac{7}{4},2)$	$2,2$	$\mathfrak{su}(2)^5$	$[1^2],[1^2],[1^2],[1^2],[1^2]$
$*(\frac{37}{12},\frac{11}{3})$	$4,2$	$\mathfrak{so}(12)$	$[3,1],[2^2],[2^2],[1^4]$
$(3,\frac{7}{2})$	$4,2$	$\mathfrak{so}(8)\times\mathfrak{su}(2)$	$[2^2],[2^2],[2^2],[2,1^2]$
$(\frac{15}{4},\frac{9}{2})$	$4,3$	$\mathfrak{su}(8)\times\mathfrak{su}(2)$	$[2,1^2],[1^4],[1^4]$
$*(\frac{53}{12},\frac{16}{3})$	$5,3$	$\mathfrak{so}(14)\times\mathfrak{u}(1)$	$[2^2,1],[2^2,1],[1^5]$
$(\frac{61}{12},\frac{37}{6})$	$5,4$	$\mathfrak{su}(10)$	$[3,2],[1^5],[1^5]$
$(5,6)$	$6,3$	$\mathfrak{e}_6\times\mathfrak{su}(2)$	$[2^3],[2^3],[2^2,1^2]$
$(\frac{23}{4},7)$	$6,4$	$\mathfrak{so}(16)\times\mathfrak{su}(2)$	$[3^2],[2^2,1^2],[1^6]$
$(7,\frac{17}{2})$	$8,4$	$\mathfrak{e}_7\times\mathfrak{su}(2)$	$[4^2],[2^4],[2^3,1^2]$
$(\frac{101}{12},\frac{31}{3})$	$8,6$	$\mathfrak{so}(20)$	$[4^2],[3^2,2],[1^8]$
$*(\frac{101}{12},\frac{31}{3})$	$10,4$	$\mathfrak{e}_8$	$[5^2],[3^3,1],[2^5]$
$(11,\frac{27}{2})$	$12,6$	$\mathfrak{e}_8\times\mathfrak{su}(2)$	$[6^2],[4^3],[2^5,1^2]$

Table 9. Rank two theories constructed from sphere with three or more regular singularities of A type. We list central charges, Coulomb branch spectrum, flavor symmetry and type of regular singularities. Theories with $*$ saturate the bound (3.1).

theory (aka the affine vertex subalgebra in VOA) in the first three columns. Isomorphic VOAs are listed in the subsequent columns. Note that in the table when we write $W\oplus M(1)^{\oplus n}$, we mean that the actual W-algebra might be an extension of $W\oplus M(1)^{\oplus f_0}$.[4] W-algebras marked with $*$ means the naive flavor symmetry may enhance to the flavor symmetry listed in the third column. Finally, theories with only one corresponding VOA found are not listed here.

In some cases, one finds that the W-algebra is isomorphic to an affine vertex algebra. This phenomenon is called collapsing level [59, 60]. One may also read off possible collapsing levels of W-algebras from our tables. For example,

$$\begin{aligned} &W_{-7+\frac{7}{3}}(\mathfrak{su}(7),[3,1^4])\cong V_{-\frac{8}{3}}(\mathfrak{su}(4)), \qquad &&W_{-7+\frac{7}{4}}(\mathfrak{su}(7),[4,1^3])\cong V_{-\frac{9}{4}}(\mathfrak{su}(3)),\\ &W_{-9+\frac{9}{2}}(\mathfrak{su}(9),[2,1^7])\cong V_{-\frac{7}{2}}(\mathfrak{su}(7)), \qquad &&W_{-9+\frac{9}{7}}(\mathfrak{su}(9),[7,1^2])\cong V_{-\frac{12}{7}}(\mathfrak{su}(2)). \end{aligned} \tag{4.1}$$

[4]This happens when there are f_0 mass deformations from the irregular singularity.

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(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	Fixtures
$*(\frac{25}{8},\frac{7}{2})$	$3,2,2$	$\mathfrak{su}(4)\times\mathfrak{u}(1)^3$	$[2,1],[2,1],[2,1],[2,1],[1^3]$
$*(\frac{89}{24},\frac{25}{6})$	$4,2,2$	$\mathfrak{so}(8)\times\mathfrak{u}(1)$	$[3,1],[3,1],[2^2],[2^2],[2^2]$
$*(\frac{107}{24},\frac{31}{6})$	$4,3,2$	$\mathfrak{su}(8)\times\mathfrak{u}(1)$	$[3,1],[3,1],[1^4],[1^4]$
$(\frac{35}{8},5)$	$4,3,2$	$\mathfrak{su}(6)\times\mathfrak{su}(2)\times\mathfrak{u}(1)$	$[3,1],[2^2],[2,1^2],[1^4]$
$(\frac{103}{24},\frac{29}{6})$	$4,3,2$	$\mathfrak{su}(4)\times\mathfrak{su}(2)^2\times\mathfrak{u}(1)$	$[2^2],[2^2],[2,1^2],[2,1^2]$
$(\frac{39}{8},\frac{11}{2})$	$4,4,2$	$\mathfrak{su}(4)\times\mathfrak{su}(2)^3$	$[2^2],[2^2],[2^2],[1^4]$
$(\frac{45}{8},\frac{13}{2})$	$4,4,3$	$\mathfrak{su}(4)^3$	$[1^4],[1^4],[1^4]$
$(\frac{51}{8},\frac{15}{2})$	$5,4,3$	$\mathfrak{su}(10)\times\mathfrak{su}(2)$	$[3,1^2],[1^5],[1^5]$
$(\frac{25}{4},\frac{29}{4})$	$5,4,3$	$\mathfrak{su}(7)\times\mathfrak{su}(3)\times\mathfrak{u}(1)$	$[2^2,1],[2,1^3],[1^5]$
$*(\frac{45}{8},\frac{13}{2})$	$6,3,2$	$\mathfrak{e}_6$	$[5,1],[3^2],[2^3],[2^3]$
$*(\frac{51}{8},\frac{15}{2})$	$6,4,2$	$\mathfrak{so}(16)$	$[5,1],[3^2],[3^2],[1^6]$
$(\frac{145}{24},\frac{41}{6})$	$6,4,2$	$\mathfrak{so}(10)$	$[4,2],[3^2],[3^2],[2^3]$
$(6,\frac{27}{4})$	$6,4,2$	$\mathfrak{so}(8)\times\mathfrak{u}(1)$	$[3^2],[3^2],[3^2],[3,2,1]$
$(\frac{163}{24},\frac{47}{6})$	$6,4,3$	$\mathfrak{so}(10)\times\mathfrak{su}(2)^2\times\mathfrak{u}(1)$	$[2^3],[2^2,1^2],[2^2,1^2]$
$(\frac{59}{8},\frac{17}{2})$	$6,5,3$	$\mathfrak{su}(6)\times\mathfrak{su}(4)$	$[2^3],[2^3],[2,1^4]$
$(\frac{15}{2},\frac{35}{4})$	$6,5,3$	$\mathfrak{su}(9)\times\mathfrak{u}(1)$	$[3,2,1],[2^3],[1^6]$
$(\frac{199}{24},\frac{59}{6})$	$6,5,4$	$\mathfrak{su}(12)$	$[4,2],[1^6],[1^6]$
$(\frac{65}{8},\frac{19}{2})$	$6,5,4$	$\mathfrak{su}(8)\times\mathfrak{su}(4)$	$[3^2],[2,1^4],[1^6]$
$*(\frac{179}{24},\frac{26}{3})$	$7,4,3$	$\mathfrak{e}_6\times\mathfrak{u}(1)^2$	$[3^2,1],[2^3,1],[2^3,1]$
$*(\frac{199}{24},\frac{59}{6})$	$7,5,3$	$\mathfrak{so}(18)\times\mathfrak{u}(1)$	$[3^2,1],[3^2,1],[1^7]$
$(\frac{75}{8},11)$	$7,6,4$	$\mathfrak{su}(10)\times\mathfrak{u}(1)$	$[4,3],[2^3,1],[1^7]$
$(\frac{223}{24},\frac{65}{6})$	$8,5,4$	$\mathfrak{e}_6\times\mathfrak{su}(2)^2\times\mathfrak{u}(1)$	$[4^2],[2^3,1^2],[2^3,1^2]$
$(\frac{73}{8},\frac{21}{2})$	$8,6,3$	$\mathfrak{so}(12)\times\mathfrak{u}(1)$	$[3^2,2],[3^2,2],[2^4]$
$(\frac{254}{24},\frac{73}{6})$	$8,6,4$	$\mathfrak{so}(20)\times\mathfrak{su}(2)$	$[4^2],[3^2,1^2],[1^8]$
$(\frac{79}{8},\frac{23}{2})$	$8,6,4$	$\mathfrak{so}(12)\times\mathfrak{su}(4)$	$[4^2],[2^4],[2^2,1^4]$
$(\frac{39}{4},\frac{45}{4})$	$9,6,3$	$\mathfrak{e}_6\times\mathfrak{u}(1)$	$[3^3],[3^3],[3^2,2,1]$
$*(\frac{75}{8},11)$	$9,5,3$	$\mathfrak{e}_7\times\mathfrak{u}(1)$	$[4^2,1],[3^3],[2^4,1]$
$(\frac{105}{8},\frac{31}{2})$	$9,8,6$	$\mathfrak{su}(12)$	$[5,4],[3^3],[1^9]$
$(\frac{269}{24},\frac{79}{6})$	$10,6,4$	$\mathfrak{e}_7\times\mathfrak{su}(2)\times\mathfrak{u}(1)$	$[5^2],[3^3,1],[2^4,1^2]$
$(\frac{293}{24},\frac{85}{6})$	$10,8,4$	$\mathfrak{so}(14)\times\mathfrak{su}(2)$	$[5^2],[3^2,2^2],[2^5]$
$(\frac{337}{24},\frac{101}{6})$	$10,8,6$	$\mathfrak{so}(24)$	$[5^2],[4^2,2],[1^{10}]$
$(\frac{27}{2},\frac{63}{4})$	$12,8,4$	$\mathfrak{e}_7\times\mathfrak{u}(1)$	$[6^2],[3^4],[3^3,2,1]$
$(\frac{119}{8},\frac{35}{2})$	$12,8,6$	$\mathfrak{e}_7\times\mathfrak{su}(4)$	$[6^2],[4^3],[2^4,1^4]$
$(\frac{147}{8},\frac{43}{2})$	$14,12,6$	$\mathfrak{so}(18)$	$[7^2],[5^2,4],[2^7]$
$(21,\frac{99}{4})$	$18,12,6$	$\mathfrak{e}_8\times\mathfrak{u}(1)$	$[9^2],[6^3],[3^5,2,1]$

Table 10. Rank three theories constructed from sphere with three or more regular singularities of A type. We list central charges, Coulomb branch spectrum, flavor symmetry and type of regular singularities. Theories with $*$ saturate the bound (3.1).

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These results are examples of theorem 8.7 of [57]. One also finds other possible collapsing levels of W-algebras whose nilpotent orbits are not labelled by the partition of form $[s, 1^q]$ as above. For example,

$$W_{-8+\frac{8}{3}}(\mathfrak{su}(8), [2^4]) \cong V_{-\frac{8}{3}}(\mathfrak{su}(4)), \qquad W_{-9+\frac{9}{4}}(\mathfrak{su}(9), [3^3]) \cong V_{-\frac{9}{4}}(\mathfrak{su}(3)). \tag{4.2}$$

These are examples of collapsing levels proved recently in [61]. There are more examples of collapsing levels from our tables, some of which even involve non-admissible W-algebras.

In [62], the authors construct a new rank 2 Argyres-Douglas theory, the Schur index of their proposed theory is,

$$\mathcal{I}_{\mathrm{AD}(C_2)} = \mathrm{PE}\left[\frac{1}{1-q}\sum_{i\in 3\mathbb{N}}(\chi^{C_2}_{\mathrm{adj}}q^{i+1} + \chi^{C_2}_{\mathbf{4}}q^{i+\frac{3}{2}} - \chi^{C_2}_{\mathbf{4}}q^{i+\frac{5}{2}} - \chi^{C_2}_{\mathrm{adj}}q^{i+3})\right] \tag{4.3}$$

This theory has the CB spectrum $(\frac{4}{3}, \frac{10}{3})$, central charge $(a, c) = (2, \frac{13}{6})$ and flavor symmetry group $\mathrm{Sp}(4)_{\frac{13}{3}}$. Using our results, one corresponding VOA of this theory is $W_{-10+\frac{10}{3}}(\mathfrak{so}(12), [3, 2^4, 1])$. Since this W-algebra is admissible, its normalised character can be written down easily,

$$\mathcal{I}_{W_{-10+\frac{10}{3}}(\mathfrak{so}(12),[3,2^4,1])}(q, z) = \mathrm{PE}\left[\frac{q\chi^{C_2}_{\mathrm{adj}}(z) + q^{\frac{3}{2}}\chi^{C_2}_{\mathbf{4}}(z) - q^{\frac{5}{2}}\chi^{C_2}_{\mathbf{4}}(z) - q^3\chi^{C_2}_{\mathrm{adj}}(z)}{(1-q)(1-q^3)}\right], \tag{4.4}$$

which matches $\mathcal{I}_{AD(C_2)}$ exactly. From this vacuum module of W-algebra, we can recognize this character not equal to the vacuum character of the affine vertex algebra $V_{-\frac{13}{6}}(\mathfrak{sp}(4))$ because of the extra terms $q^{\frac{3}{2}}\chi^{C_2}_{\mathbf{4}}(z) - q^{\frac{5}{2}}\chi^{C_2}_{\mathbf{4}}(z)$. The Higgs branch (associated variety of $W_{-10+\frac{10}{3}}(\mathfrak{so}(12), [3, 2^4, 1])$) is $\overline{\mathcal{O}}_{[3^4]} \cap S_{[3,2^4,1]}$.

This example might suggest that the VOA for Z_2 twisted A_{2n} theory with the regular puncture being full type might not be the simple Kac-Moody algebra as predicted in [50], and there should be a finite extension. There are some strange features of A_{2n}/Z_2 twisted theories, and this fact might also be one of it.

5 Some results on rank four theories

It is possible to scan the rank four theories, although it takes significant more time. In this section, a different approach is taken, i.e. we look at those theories with special flavor symmetry, namely only the special type of regular singularities would be considered.

Since there are universal formulae of dimensions of trivial, minimal, subregular and principal nilpotent orbit of $\mathfrak{g}$ [63], one can have a universal treatment for AD theories whose regular singularity is of such type. Below we list values of k(or k_t) of rank $\dim \mathcal{B}$ theories with $f_0 = 0$ when the regular puncture is chosen to be full, minimal, subregular and principal. Since k should be an integer and $f_0 = 0$, there are only finite number of choices for a given rank $\dim \mathcal{B}$.

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Δ_{Coulomb}	(a,c)	AKM subalgebra	$W_{\tilde{k}}(\mathfrak{g},f)$	$W_{\tilde{k}}(\mathfrak{g},f)$	$W_{\tilde{k}}(\mathfrak{g},f)$
$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	–	$W_{-3+\frac{3}{5}}(\mathfrak{su}(3),[3])$ $W_{-6+\frac{8}{5}}(D_4,[7,1])$ $W_{-12+\frac{12}{5}}(\mathfrak{e}_6,A_4+A_1)$ $W_{-18+\frac{14}{15}}(\mathfrak{e}_7,E_7)$	$W_{-7+\frac{7}{5}}(\mathfrak{su}(7),[5,2])$ $W_{-10+\frac{9}{5}}(D_6,[9,3])$ $W_{-12+\frac{9}{10}}(\mathfrak{e}_6,E_6)$ $W_{-30+\frac{24}{25}}(\mathfrak{e}_8,E_8)$	$W_{-8+\frac{8}{5}}(\mathfrak{su}(8),[5,3])$ $W_{-6+\frac{6}{5}}(D_4,[5,3])$ $W_{-18+\frac{18}{5}}(\mathfrak{e}_7,A_4+A_2)$
$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathfrak{su}(2)_{-\frac{4}{3}}$	$W_{-4+\frac{4}{3}}(\mathfrak{su}(4),[2^2])$ $W_{-7+\frac{7}{3}}(\mathfrak{su}(7),[3,2^2])$ $W_{-2+\frac{1}{3}}(\mathfrak{su}(2),[2])+M(1)(*)$ $W_{-7+\frac{7}{3}}(B_4,[3^2,1^3])$ $W_{-5+\frac{5}{3}}(C_4,[3^2,1^2])$ $W_{-10+\frac{10}{3}}(D_6,[3^3,1^3])$ $W_{-8+\frac{7}{3}}(D_5,[9,1])+M(1)(*)$ $W_{-18+\frac{14}{3}}(\mathfrak{e}_7,A_3+A_2)$ $W_{-30+\frac{30}{9}}(\mathfrak{e}_8,D_7(a_1))(*)$	$W_{-4+\frac{4}{3}}(\mathfrak{su}(4),[4])+M(1)(*)$ $W_{-8+\frac{8}{3}}(\mathfrak{su}(8),[3^2,1^2])$ $W_{-5+\frac{5}{3}}(B_3,[3,2^2])$ $W_{-9+\frac{5}{3}}(B_5,[7,2^2])$ $W_{-6+\frac{4}{3}}(D_4,[4^2])$ $W_{-6+\frac{4}{3}}(D_4,[5,1^3])$ $W_{-8+\frac{8}{3}}(D_5,[3^2,2^2])$ $W_{-18+\frac{14}{3}}(\mathfrak{e}_7,E_6)$ $W_{-30+\frac{20}{3}}(\mathfrak{e}_8,A_4+A_2)$	$W_{-5+\frac{5}{3}}(\mathfrak{su}(5),[3,1^2])$ $W_{-8+\frac{8}{6}}(\mathfrak{su}(8),[6,2])+M(1)(*)$ $W_{-9+\frac{5}{3}}(B_5,[5^2,1])(*)$ $W_{-4+\frac{4}{3}}(C_3,[2^3])$ $W_{-4+\frac{4}{3}}(D_3,[3,1^3])$ $W_{-4+\frac{4}{6}}(D_3,(5,1))+M(1)(*)$ $W_{-12+\frac{8}{6}}(\mathfrak{e}_6,E_6)(*)$ $W_{-18+\frac{14}{6}}(\mathfrak{e}_7,E_7(a_4))+M(1)(*)$ $W_{-9+\frac{4}{3}}(\mathfrak{f}_4,F_4(a_2))+M(1)(*)$
$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathfrak{su}(2)_{-\frac{3}{2}}\times\mathfrak{u}(1)^2$	$W_{-4+\frac{3}{2}}(\mathfrak{su}(4),[1^4])+M(1)$ $W_{-8+\frac{8}{2}}(D_5,[3^2,2^2])+M(1)$	$W_{-7+\frac{6}{3}}(\mathfrak{su}(7),[4,3])+M(1)^{\oplus 2}(*)$ $W_{-10+\frac{8}{3}}(D_6,[6^2])+M(1)^{\oplus 2}(*)$	$W_{-4+\frac{3}{2}}(D_3,[2^2,1^2])+M(1)$
$\frac{3}{2}$	$(\frac{7}{12},\frac{2}{3})$	$\mathfrak{su}(3)_{-\frac{3}{2}}$	$W_{-3+\frac{3}{2}}(\mathfrak{su}(3),[1^3])$ $W_{-6+\frac{6}{2}}(\mathfrak{su}(6),[4,1^2])+M(1)(*)$ $W_{-9+\frac{9}{2}}(\mathfrak{su}(9),[2^3,1^3])$ $W_{-8+\frac{8}{3}}[D_5,(7,1^3])+M(1)(*)$ $W_{-12+\frac{9}{4}}(\mathfrak{e}_6,D_4)$ $W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_4+A_1)$	$W_{-3+\frac{3}{6}}(\mathfrak{su}(3),[3])+M(1)^{\oplus 2}(*)$ $W_{-6+\frac{6}{4}}(\mathfrak{su}(6),[3^2])+M(1)(*)$ $W_{-9+\frac{8}{6}}(\mathfrak{su}(9),[6,3])+M(1)^{\oplus 2}(*)$ $W_{-4+\frac{3}{2}}(C_3,[2^2,1^2]+M(1)(*)$ $W_{-6+\frac{6}{8}}D_4,[7,1])+M(1)^{\oplus 2}(*)$ $W_{-18+\frac{14}{4}}(\mathfrak{e}_7,D_5(a_1))+M(1)(*)$	$W_{-5+\frac{5}{2}}(\mathfrak{su}(5),[2,1^3])$ $W_{-7+\frac{7}{2}}(\mathfrak{su}(7),[2^2,1^4])$ $W_{-2+\frac{1}{2}}(\mathfrak{su}(2),[1^2])+M(1)(*)$ $W_{-6+\frac{5}{2}}(C_5,[2^4,1^2]+M(1)(*)$ $W_{-12+\frac{9}{8}}(\mathfrak{e}_6,A_2+A_1)$ $W_{-18+\frac{14}{4}}(\mathfrak{e}_7,A_4+A_2)+M(1)(*)$
2	$(\frac{23}{24},\frac{7}{6})$	$\mathfrak{so}(8)_{-2}$	$W_{-3+\frac{3}{3}}(\mathfrak{su}(3),[1^3])+M(1)^{\oplus 2}(*)$ $W_{-6+\frac{6}{2}}(\mathfrak{su}(6),[2,1^4])+M(1)(*)$ $W_{-8+\frac{8}{2}}(\mathfrak{su}(8),[2^2,1^4])+M(1)(*)$ $W_{-9+\frac{8}{3}}(\mathfrak{su}(9),[3,2^3])+M(1)^{\oplus 2}(*)$ $W_{-4+\frac{3}{3}}(D_3,[3,1^3])+M(1)^{\oplus 3}(*)$ $W_{-10+\frac{6}{7}}(D_6,[2^2,1^8])$ $W_{-30+\frac{30}{3}}(\mathfrak{e}_8,D_4(a_1))$	$W_{-4+\frac{4}{2}}(\mathfrak{su}(4),[1^4])+M(1)(*)$ $W_{-6+\frac{6}{3}}(\mathfrak{su}(6),[3,1^3])+M(1)^{\oplus 2}(*)$ $W_{-8+\frac{8}{4}}(\mathfrak{su}(8),[4,2^2])+M(1)^{\oplus 3}(*)$ $W_{-7+2}(B_4,[3^2,1^3])+M(1)^{\oplus 2}(*)$ $W_{-6+\frac{4}{2}}(D_4,[1^8])$ $W^{-10+\frac{6}{2}}(D_6,[3^3,1^3])+M(1)^{\oplus 2}(*)$ $W_{-18+\frac{14}{2}}(\mathfrak{e}_7,A_2+A_1)+M(1)(*)$	$W_{-4+\frac{4}{4}}(\mathfrak{su}(4),[2^2])+M(1)^{\oplus 3}(*)$ $W_{-6+\frac{6}{3}}(\mathfrak{su}(6),[2^3])+M(1)^{\oplus 2}(*)$ $W_{-9+\frac{9}{3}}(\mathfrak{su}(9),[3^2,1^3])+M(1)^{\oplus 2}(*)$ $W_{-9+5}(B_5,(3,1^8))$ $W_{-6+\frac{4}{4}}(D_4,[5,3])+M(1)^{\oplus 4}(*)$ $W_{-4+\frac{4}{2}}(D_3,[1^6])+M(1)(*)$ $W_{-9+4}(\mathfrak{f}_4,\tilde{A}_1)+M(1)(*)$
2	$(\frac{23}{24},\frac{7}{6})$	$(\mathfrak{g}_2)_{-2}$	$W_{-4+2}(\mathfrak{g}_2,0)$ $W_{-18+6}(\mathfrak{e}_7,2A_2)$	$W_{-9+3}(\mathfrak{f}_4,\tilde{A}_2)$ $W_{-30+6}(\mathfrak{e}_8,E_6(a_3))$	$W_{-18+6}(\mathfrak{e}_7,A_2+3A_1)$
2	$(\frac{23}{24},\frac{7}{6})$	$\mathfrak{su}(3)_{-2}$	$W_{-30+6}(\mathfrak{e}_8,D_4+A_2)$	$W_{-9+3}(\mathfrak{f}_4,A_2)$	
2	$(\frac{23}{24},\frac{7}{6})$	$\mathfrak{so}(7)_{-2}$	$W_{-5+3}(B_3,[1^7])$ $W_{-18+6}(\mathfrak{e}_7,A_3)$	$W_{-30+12}(\mathfrak{e}_8,A_2+2A_1)$	$W_{-12+6}(\mathfrak{e}_6,2A_1)$
2	$(\frac{3}{4},\frac{3}{4})$	$\mathfrak{su}(2)_{-\frac{3}{2}}$	$W_{-6+2}(D_4,[3,2^2,1])$ $W_{-6+2}(C_5,[3^2,2,1^2])$	$W_{-30+8}(\mathfrak{e}_8,A_3+A_2+A_1)$	$W_{-3+1}(B_2,[2^2,1])$
2	$(1,\frac{5}{4})$	$\mathfrak{su}(3)_{-2}\times\mathfrak{u}(1)^3$	$W_{-5+\frac{4}{2}}(\mathfrak{su}(5),[2,1^3])+M(1)^{\oplus 2}$	$W_{-7+\frac{6}{3}}(\mathfrak{su}(7),[3,2^2])+M(1)^{\oplus 3}(*)$	
2	$(1,\frac{5}{4})$	$\mathfrak{so}(7)_{-2}\times\mathfrak{su}(2)_{-\frac{1}{2}}$	$W_{-30+10}(\mathfrak{e}_8,A_3+A_1)$	$W_{-9+5}(B_5,[2^2,1^7])$	
3	$(\frac{41}{24},\frac{13}{6})$	$(\mathfrak{e}_6)_{-3}$	$W_{-30+\frac{30}{2}}(\mathfrak{e}_8,A_2)$	$W_{-8+\frac{8}{1}}(D_5,[1^{10}])+M(1)(*)$	$W_{-6+\frac{6}{2}}(D_4,[1^8])+M(1)^{\oplus 2}(*)$
3	$(\frac{17}{12},\frac{19}{12})$	$\mathfrak{sp}(2)_{-2}\times\mathfrak{u}(1)$	$W_{-4+\frac{3}{2}}(C_3,[2,1^4])+M(1)$	$W_{-6+\frac{5}{2}}(C_5,[2^3,1^4])+M(1)$	
4	$(\frac{59}{24},\frac{19}{6})$	$(\mathfrak{e}_7)_{-4}$	$W_{-12+8}(\mathfrak{e}_6,0)+M(1)(*)$	$W_{-30+20}(\mathfrak{e}_8,A_1)$	
4	$(\frac{25}{12},\frac{29}{12})$	$\mathfrak{sp}(3)_{-\frac{5}{2}}\times\mathfrak{u}(1)$	$W_{-18+7}(\mathfrak{e}_7,4A_1)+M(1)$	$W_{-9+4}(\mathfrak{f}_4,A_1)+M(1)$	

Table 11. The VOA isomorphism from rank one theories.

Example: for $(A_{n-1},n,k,f=triv)$ theories, the dimension formula requires that $k=-n+1+\frac{2\text{dim}\mathcal{B}}{n-1}$. Since k should be an integer and $f_0=0$, $n-1$ must be a divisor of $2\dim\mathcal{B}$ such that n and k are coprime. Let the integer $q=\frac{2\dim\mathcal{B}}{n-1}+1$ satisfy $\gcd(n,q)=1$, then $k=-n+q$. The CB spectrum is the following set

$$\{\Delta_{\text{Coulomb}}\}=\left\{\Delta_{l,j}=l-j\frac{n}{q}\mid(l,j)\in\mathbb{Z}^2,2\leq l\leq n,1\leq j\leq\lfloor\frac{(l-1)q}{n}\rfloor\right\}. \tag{5.1}$$

The flavor symmetry should be A_n, and the corresponding VOA is $V_{-n+\frac{n}{q}}(\mathfrak{su}(n))$. The 4d central charge c_{4d} is

$$c_{4d}=\frac{1}{12}(q-1)(n^2-1). \tag{5.2}$$

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Δ_{Coulomb}	(a,c)	AKM subalgebra	$W^{\hat{k}}(\mathfrak{g},f)$	$W^{\hat{k}}(\mathfrak{g},f)$	$W^{\hat{k}}(\mathfrak{g},f)$
$\frac{6}{5},\frac{6}{5}$	$(\frac{43}{60},\frac{11}{15})$	–	$W_{-12+\frac{12}{10}}(\mathfrak{e}_6,E_6(a_1))$ $W_{-4+\frac{4}{5}}(C_3,[4,2])$	$W_{-12+\frac{9}{5}}(\mathfrak{e}_6,E_6(a_3))$ $W_{-5+\frac{3}{5}(B_3,[7])}$	$W_{-7+\frac{7}{5}}(B_4,[5,3,1])$
$\frac{4}{3},\frac{4}{3}$	$(\frac{111}{12},1$	$\mathfrak{su}(2)_{-\frac{4}{3}}\times\mathfrak{su}(2)_{-\frac{4}{3}}$	$W_{-8+\frac{5}{3}}(D_5,[5,3,1^2])+M(1)(*)$ $W_{-18+\frac{14}{3}}(\mathfrak{e}_7,D_4(a_1)+A_1)$ $W_{-3+\frac{2}{3}}(B_2,[3,1^2])+M(1)(*)$ $W_{-5+\frac{5}{3}}(C_4,[2^4])$	$W_{-8+\frac{8}{3}}(D_5,[3^2,1^4])$ $W_{-18+\frac{14}{6}}(\mathfrak{e}_7,D_5+A_1)+M(1)(*)$ $W_{-7+\frac{2}{3}}(B_4,[9])+M(1)^{\oplus 2}(*)$ $W_{-9+\frac{4}{3}}(\mathfrak{f}_4,B_3)+M(1)(*)$	$W_{-8+\frac{8}{6}}(D_5,[5^2])+M(1)(*)$ $W_{-5+\frac{5}{3}}(B_3,[3,1^4])$ $W_{-9+\frac{5}{3}}(B_5,[7,1^4])$
$\frac{10}{7},\frac{8}{7}$	$(\frac{67}{84},\frac{17}{21})$	–	$W_{-2+\frac{2}{7}}(\mathfrak{su}(2),[2])$ $W_{-10+\frac{6}{7}}(D_6,[11,1])$ $W^{-30+\frac{30}{7}}(\mathfrak{e}_8,A_6+A_1)$ $W_{-9+\frac{9}{7}}(\mathfrak{f}_4,F_4(a_2))$	$W_{-5+\frac{5}{7}}(\mathfrak{su}(5),[5])$ $W^{-8+\frac{8}{7}}(D_5,[7,3])$ $W_{-30+\frac{20}{21}}(\mathfrak{e}_8,E_8)$	$W_{-9+\frac{9}{7}}(\mathfrak{su}(9),[7,2])$ $W_{-10+\frac{10}{7}}(D_6,[7,5])$ $W_{-3+\frac{3}{7}}(B_2,[5])$
$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathfrak{u}(1)$	$W_{-2+\frac{2}{8}}(\mathfrak{su}(2),[2])+M(1)$ $W_{-7+\frac{7}{4}}(\mathfrak{su}(7),[4,2,1])$ $W_{-18+\frac{14}{8}}(\mathfrak{e}_7,E_7(a_3))+M(1)$ $W_{-6+\frac{5}{4}}(C_5,[4^2,2]+M(1)$	$W_{-5+\frac{5}{4}}(\mathfrak{su}(5),[3,2])$ $W_{-9+\frac{9}{4}}(\mathfrak{su}(9),[4,3,2])$ $W_{-30+\frac{30}{8}}(\mathfrak{e}_8,E_6(a_1)+A_1)$ $W_{-3+\frac{3}{4}}(\mathfrak{su}(3),[2,1])$	$W_{-6+\frac{6}{8}}(\mathfrak{su}(6),[6])+M(1)$ $W_{-12+\frac{9}{4}}(\mathfrak{e}_6,A_4+A_1)$ $W_{-4+\frac{3}{4}}(C_3,[4,2]+M(1)$
$\frac{3}{2},\frac{3}{2}$	$(\frac{17}{6},\frac{4}{3})$	$\mathfrak{su}(3)_{-\frac{3}{2}}\times\mathfrak{su}(3)_{-\frac{3}{2}}$	$W_{-8+\frac{5}{2}}(D_5,[3^2,1^4])+M(1)(*)$	$W_{-6+\frac{6}{4}}(D_4,[3^2,1^2])+M(1)^{\oplus 2}(*)$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,A_2)$
$\frac{8}{5},\frac{6}{5}$	$(\frac{19}{20},1)$	$\mathfrak{su}(2)_{-\frac{8}{3}}$	$W_{-2+\frac{2}{5}}(\mathfrak{su}(2),[1^2])$ $W_{-4+\frac{3}{5}}(\mathfrak{su}(4),[4])+M(1)(*)$ $W_{-6+\frac{6}{5}}(D_4,[4^2])$ $W_{-12+\frac{12}{5}}(\mathfrak{e}_6,A_4)$	$W_{-7+\frac{7}{5}}(\mathfrak{su}(7),[5,1^2])$ $W_{-4+\frac{3}{5}}(D_3,[5,1])+M(1)(*)$ $W_{-6+\frac{6}{5}}(D_4,[5,1^3])$ $W_{-3+\frac{2}{5}}(B_2,[5])+M(1)(*)$	$W_{-8+\frac{8}{5}}(\mathfrak{su}(8),[4^2])$ $W_{-10+\frac{6}{5}}(D_6,[9,1^3])$ $W_{-8+\frac{8}{10}}(D_5,[9,1])+M(1)(*)$
$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathfrak{su}(2)_{-\frac{5}{3}}\times\mathfrak{u}(1)$	$W_{-2+\frac{2}{6}}(\mathfrak{su}(2),[1^2])+M(1)$ $\mathcal{T}[W_{-7+\frac{7}{3}}(\mathfrak{su}(7),[3,2,1^2])]$ $W_{-3+\frac{2}{6}}(\mathfrak{su}(3),[2,1])+M(1)(*)$ $W_{-10+\frac{10}{3}}(D_6,[3^2,2^2,1^2])$ $W_{-18+\frac{14}{6}}(\mathfrak{e}_7,A_6)+M(1)$ $W_{-7+\frac{7}{3}}(B_4,[3,2^2,1^2])$	$W_{-4+\frac{4}{3}}(\mathfrak{su}(4),[2,1^2])$ $W_{-8+\frac{8}{3}}(\mathfrak{su}(8),[3,2^2,1])$ $W_{-5+\frac{4}{6}}(\mathfrak{su}(5),[5])+M(1)^{\oplus 2}(*)$ $W_{-10+\frac{10}{12}}(D_6,[11,1])+M(1)^{\oplus 2}(*)$ $W_{-30+\frac{30}{9}}(\mathfrak{e}_8,E_7(a_3))$ $W_{-4+\frac{4}{3}}(C_3,[2^2,1^2])$	$W_{-5+\frac{5}{3}}(\mathfrak{su}(5),[2^2,1])$ $W_{-8+\frac{8}{6}}(\mathfrak{su}(8),[6,1^2])+M(1)$ $W_{-4+\frac{4}{3}}(D_3,[2^2,1^2])$ $W_{-18+\frac{14}{6}}(\mathfrak{e}_7,D_6(a_1))+M(1)$ $W_{-30+\frac{20}{9}}(\mathfrak{e}_8,A_4+2A_1)$ $W_{-9+\frac{4}{3}}(\mathfrak{f}_4,C_3)+M(1)$
$2,2$	$(\frac{7}{4},2)$	$\mathfrak{su}(2)_{-2}\times\mathfrak{su}(2)_{-2}\times\mathfrak{u}(1)^3$	$W_{-4+\frac{4}{4}}(\mathfrak{su}(4),[2,1^2])+M(1)^{\oplus 3}(*)$ $W_{-6+\frac{6}{3}}(\mathfrak{su}(6),[2^2,1^2])+M(1)^{\oplus 2}$ $W_{-8+\frac{8}{4}}(\mathfrak{su}(8),[3^2,2])+M(1)^{\oplus 3}(*)$ $W_{-6+\frac{4}{4}}(D_4,[4^2])+M(1)^{\oplus 4}(*)$	$W_{-5+\frac{5}{8}}(\mathfrak{su}(5),[3,2])+M(1)^{\oplus 4}(*)$ $W_{-8+\frac{8}{4}}(\mathfrak{su}(8),[4,2,1^2])+M(1)^{\oplus 3}(*)$ $W_{-9+\frac{9}{3}}(\mathfrak{su}(9),[3,2^2,1^2])+M(1)^{\oplus 2}$ $W_{-6+\frac{4}{4}}(D_4,[5,1^3])+M(1)^{\oplus 4}(*)$	$W_{-10+3}(D_6,[3^2,2^2,1^2])+M(1)^{\oplus 2}(*)$ $W_{-4+1}(D_3,[2^2,1^2])+M(1)^{\oplus 3}(*)$ $W_{-8+1}(D_5,[7,3])]+M(1)^{\oplus 5}(*)$
$2,2$	$(\frac{19}{12},\frac{5}{3})$	$\mathfrak{sp}(2)_{-2}$	$W_{-6+\frac{4}{2}}(D_4,(2^4))$ $W_{-6+\frac{4}{2}}(D_4,[3,1^5])$ $W_{-9+3}(B_5,[3^2,1^5])$	$W_{-30+\frac{24}{5}}(\mathfrak{e}_8,A_3+A_2)$ $W_{-3+1}(B_2,[1^5])$ $W_{-6+2}(C_5,[3^2,1^4])$	$W_{-9+3}(B_5,[3,2^4])$ $W_{-6+2}(C_5,[2^5])$
$\frac{12}{5},\frac{6}{5}$	$(\frac{163}{120},\frac{17}{12})$	$\mathfrak{su}(2)_{-\frac{17}{10}}$	$W_{-12+\frac{9}{5}}(\mathfrak{e}_6,A_5)$	$W_{-7+\frac{7}{5}}(B_4,[4^2,1])$	$W_{-4+\frac{4}{5}}(C_3,[4,1^2])$
$\frac{5}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathfrak{su}(5)_{-\frac{5}{2}}$	$W_{-5+\frac{5}{5}}(\mathfrak{su}(5),[1^5])$ $W_{-4+\frac{3}{2}}(\mathfrak{su}(4),[1^4])+M(1)(*)$ $W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_4)$	$W_{-7+\frac{7}{2}}(\mathfrak{su}(7),[2,1^5])$ $W_{-4+\frac{3}{2}}(D_3,[1^6])+M(1)(*)$	$W_{-9+\frac{9}{2}}(\mathfrak{su}(9),[2^2,1^5])$ $W_{-9+\frac{5}{2}}(B_5,[5,1^6])+M(1)(*)$
$\frac{5}{2},\frac{3}{2}$	$(\frac{19}{12},\frac{5}{3})$	$\mathfrak{su}(2)_{-\frac{7}{4}}\times\mathfrak{u}(1)$	$W_{-4+\frac{3}{4}}(C_3,[4,1^2])+M(1)$	$W_{-6+\frac{5}{4}}(C_5,[4,3^2])+M(1)$	
$\frac{8}{3},\frac{4}{3}$	$(\frac{13}{8},\frac{7}{4})$	$\mathfrak{su}(2)_{-\frac{11}{6}}\times\mathfrak{su}(2)_{-\frac{8}{3}}$	$W_{-8+\frac{5}{3}}(D_5,[5,2^2,1])+M(1)(*)$ $W_{-5+\frac{5}{3}}(B_3,[2^2,1^3])$ $W_{-5+\frac{5}{3}}(C_4,[2^3,1^2])$	$W_{-8+\frac{8}{3}}(D_5,[3,2^2,1^3])$ $W_{-3+\frac{2}{3}}(B_2,[2^2,1])+M(1)(*)$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,A_3+2A_1)$ $W_{-7+\frac{4}{3}}(B_4,[4^2,1])+M(1)(*)$
$3,\frac{3}{2}$	$(\frac{47}{24},\frac{13}{6})$	$\mathfrak{su}(2)_{-2}\times\mathfrak{su}(3)_{-3}\times\mathfrak{u}(1)$	$W_{-8+\frac{5}{2}}(D_5,[3,2^2,1^3])+M(1)(*)$ $W_{-5+\frac{3}{2}}(B_3,[2^2,1^3])+M(1)(*)$	$W_{-6+\frac{6}{4}}(D_4,[3,2^2,1])+M(1)^{\oplus 2}(*)$ $W_{-9+\frac{5}{2}}(B_5,[3^2,2^2,1])+M(1)(*)$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,3A_1)$
$3,2$	$(\frac{29}{12},\frac{17}{6})$	$\mathfrak{su}(6)_{-3}\times\mathfrak{u}(1)$	$W_{-6+\frac{6}{2}}(\mathfrak{su}(6),[1^6])+M(1)$ $W_{-5+\frac{4}{2}}(\mathfrak{su}(5),[1^5])+M(1)^{\oplus 2}(*)$	$W_{-8+\frac{8}{2}}(\mathfrak{su}(8),[2,1^6])+M(1)$	$W_{-18+\frac{14}{2}}(\mathfrak{e}_7,A_2)+M(1)$
$3,2$	$(2,2)$	$\mathfrak{su}(2)_{-4}$	$W_{-12+\frac{12}{4}}(\mathfrak{e}_6,2A_2+A_1)$	$W_{-4+1}(\mathfrak{g}_2,A_1)$	
$3,\frac{5}{2}$	$(\frac{61}{24},\frac{17}{6})$	$\mathfrak{sp}(3)_{-\frac{5}{2}}\times\mathfrak{u}(1)$	$W_{-18+\frac{14}{4}}(\mathfrak{e}_7,D_4)+M(1)$	$W_{-4+\frac{3}{2}}(C_3,[1^6])+M(1)$	$W_{-6+\frac{5}{2}}(C_5,[2^2,1^6])+M(1)$
$\frac{10}{3},\frac{4}{3}$	$(2,\frac{13}{6})$	$\mathfrak{sp}(2)_{-\frac{13}{6}}$	$W_{-10+\frac{10}{3}}(D_6,[3,2^4,1])$	$W_{-7+\frac{7}{3}}(B_4,[2^4,1])$	$W_{-4+\frac{4}{3}}(C_3,[2,1^4])$
$4,2$	$(\frac{37}{12},\frac{11}{3})$	$\mathfrak{so}(12)_{-4}$	$W_{-10+6}(D_6,[1^{12}])$ $W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_3)$	$W_{-8+\frac{8}{2}}(D_5,[1^{10}])+M(1)(*)$	$W_{-9+5}(B_5,[1^{11}])$
$4,2$	$(\frac{17}{6},\frac{19}{6})$	$\mathfrak{sp}(2)_{-\frac{5}{2}}\times\mathfrak{u}(1)$	$W_{-10+3}(D_6,[3,2^4,1])+M(1)^{\oplus 2}$	$W_{-7+2}(B_4,[2^4,1])+M(1)^{\oplus}(*)$	$W_{-9+5}(B_5,[1^{11}])$
$5,3$	$(\frac{53}{12},\frac{16}{3})$	$(\mathfrak{so}_{13})_{-\frac{5}{2}}$	$W_{-18+\frac{18}{2}}(\mathfrak{e}_7,A_1)+M(1)(*)$	$W_{-30+\frac{30}{2}}(\mathfrak{e}_8,2A_1)$	
$5,4$	$(\frac{14}{3},\frac{16}{3})$	$(\mathfrak{f}_4)_{-\frac{5}{2}}\times\mathfrak{u}(1)$	$W_{-18+\frac{14}{2}}(\mathfrak{e}_7,(3A_1)'')+M(1)$	$W_{-9+4}(\mathfrak{f}_4,0)+M(1)$	

Table 12. The VOA isomorphism from rank two theories.

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Δ_{Coulomb}	(a,c)	AKM subalgebra	$W_{\bar{k}}(\mathfrak{g},f)$	$W_{\bar{k}}(\mathfrak{g},f)$	$W_{\bar{k}}(\mathfrak{g},f)$
$\frac{6}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{43}{30},\frac{11}{10})$	–	$W_{-4+\frac{4}{5}}(\mathfrak{g}_2,G_2(a_1))$ $W_{-18+\frac{14}{5}}(\mathfrak{e}_7,E_7(a_5))$	$W_{-4+\frac{2}{5}}(\mathfrak{g}_2,G_2)$	$W_{-18+\frac{18}{15}}(\mathfrak{e}_7,E_7(a_1))$
$\frac{4}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{11}{8},\frac{3}{2})$	$\mathfrak{su}(2)^3_{-\frac{4}{3}}$	$W_{-12+\frac{12}{9}}(\mathfrak{e}_6,D_5)+M(1)^{\oplus 2}$	$W_{-12+\frac{8}{3}}(\mathfrak{e}_6,D_4(a_1))+M(1)$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,D_4(a_1))$
$\frac{14}{9},\frac{4}{3},\frac{10}{9}$	$(\frac{91}{72},\frac{23}{18})$	–	$W^{-2+\frac{2}{7}}(\mathfrak{su}(2),[2])$ $W_{-30+\frac{20}{9}}(\mathfrak{e}_8,E_8(b_4))$	$W_{-7+\frac{7}{9}}(\mathfrak{su}(7),[7])$ $W_{-5+\frac{5}{9}}(B_3,[7])$	$W_{-10+\frac{10}{9}}(D_6,[9,3])$ $W_{-4+\frac{4}{9}}(C_3,[6])$
$\frac{8}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{157}{120},\frac{41}{30})$	$\mathfrak{su}(2)_{-\frac{8}{5}}$	$W_{-7+\frac{7}{5}}(B_4,[5,2^2])$	$W_{-4+\frac{4}{5}}(C_3,[3^2])$	
$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathfrak{u}(1)$	$W_{-2+\frac{2}{10}}(\mathfrak{su}(2),[2])+M(1)$ $W_{-9+\frac{9}{5}}(\mathfrak{su}(9),[5,3,1])$ $W_{-3+\frac{3}{5}}(B_2,[3,1^2])$ $W_{-6+\frac{6}{5}}(\mathfrak{su}(6),[4,2])$	$W_{-4+\frac{4}{5}}(\mathfrak{su}(4),[3,1])$ $W_{-4+\frac{4}{5}}(D_3,[3^2])$ $W_{-9+\frac{9}{5}}(B_5,[5,3^2])$	$W_{-8+\frac{8}{10}}(\mathfrak{su}(8),[8])+M(1)$ $W_{-8+\frac{8}{5}}(D_5,[5,3,1^2])$ $W_{-6+\frac{6}{5}}(C_5,[4^2,2])$
$\frac{5}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{37}{24},\frac{5}{3})$	$\mathfrak{su}(2)_{-\frac{5}{3}}\times\mathfrak{su}(2)_{-\frac{4}{3}}\times\mathfrak{u}(1)_{f_0}$	$W_{-18+\frac{14}{6}}(\mathfrak{e}_7,D_5)+M(1)$	$W_{-8+\frac{5}{3}}(D_5,[4^2,1^2])+M(1)(*)$	
$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathfrak{su}(2)_{-\frac{12}{7}}$	$W_{-2+\frac{2}{7}}(\mathfrak{su}(2),[1^2])$ $W--8+\frac{8}{7}(D_5,[7,1^3])$ $W_{-30+\frac{30}{7}}(\mathfrak{e}_8,A_6)$	$W_{-9+\frac{9}{7}}(\mathfrak{su}(9),[7,1^2])$ $W_{-10+\frac{10}{7}}(D_6,[6^2])$ $W_{-9+\frac{9}{7}}(\mathfrak{f}_4,B_3)$	$W_{-6+\frac{5}{7}}(\mathfrak{su}(6),[6])+M(1)(*)$ $W_{-12+\frac{8}{7}}(\mathfrak{e}_6,E_6(a_1))+M(1)(*)$ $W_{-9+\frac{9}{7}}(\mathfrak{f}_4,C_3)$
$\frac{12}{7},\frac{9}{7},\frac{8}{7}$	$(\frac{75}{56},\frac{19}{14})$	–	$W_{-3+\frac{3}{7}}(\mathfrak{su}(3),[3])$ $W_{-12+\frac{12}{7}}(\mathfrak{e}_6,E_6(a_3))$ $W_{-5+\frac{5}{7}}(C_4,[6,2])$	$W_{-4+\frac{4}{7}}(\mathfrak{su}(4),[4])$ $W_{-12+\frac{12}{14}}(\mathfrak{e}_6,E_6)$ $W_{-30+\frac{24}{7}}(\mathfrak{e}_8,E_8(a_3))$	$W_{-4+\frac{4}{7}}(D_3,[5,1])$ $W_{-30+\frac{24}{7}}(\mathfrak{e}_8,E_8(b_6))$
$\frac{7}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathfrak{su}(2)_{-\frac{7}{4}}\times\mathfrak{u}(1)$	$W_{-2+\frac{2}{8}}(\mathfrak{su}(2),[1^2])+M(1)$ $W_{-9+\frac{9}{5}}(\mathfrak{su}(9),[4,3,1^2])$ $W_{-4+\frac{3}{4}}(D_3,[3^2])+M(1)(*)$ $W_{-4+\frac{3}{4}}(C_3,[3^2])+M(1)$	$W_{-5+\frac{5}{4}}(\mathfrak{su}(5),[3,1^2])$ $W_{-4+\frac{3}{4}}(\mathfrak{su}(4),[3,1])+M(1)(*)$ $W_{-8+\frac{5}{4}}(D_5,[5^2])+M(1)(*)$ $W_{-6+\frac{5}{4}}(C_5,[4^2,1^2])+M(1)$	$W_{-7+\frac{7}{4}}(\mathfrak{su}(7),[3^2,1])$ $W_{-7+\frac{6}{8}}(\mathfrak{su}(7),[7])+M(1)^{\oplus 2}(*)$ $W_{-12+\frac{9}{4}}(\mathfrak{e}_6,A_4)$
$\frac{7}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{13}{8},\frac{7}{4})$	$\mathfrak{su}(2)_{-\frac{7}{4}}\times\mathfrak{u}(1)^2$	$W_{-6+\frac{5}{4}}(\mathfrak{su}(6),[4,1^2])+M(1)$	$W_{-8+\frac{7}{4}}(\mathfrak{su}(8),[4,3,1])+M(1)(*)$	
$\frac{9}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathfrak{u}(1)$	$W_{-3+\frac{3}{5}}(\mathfrak{su}(3),[2,1])$ $W_{-18+\frac{18}{5}}(\mathfrak{e}_7,A_4+A_1)$	$W_{-7+\frac{7}{5}}(\mathfrak{su}(7),[4,3])$ $W_{-18+\frac{18}{30}}(\mathfrak{e}_7,E_7)+M(1)$	$W_{-3+\frac{2}{5}}(\mathfrak{su}(3),[3])+M(1)$ $W_{-8+\frac{8}{5}}(\mathfrak{su}(8),[5,2,1])$
$2,\frac{3}{2},\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathfrak{u}(1)^3$	$W_{-3+\frac{3}{6}}(\mathfrak{su}(3),[2,1])+M(1)^{\oplus 2}$ $W_{-6+\frac{6}{4}}(\mathfrak{su}(6),[3,2,1])+M(1)$ $W_{-3+\frac{1}{2}}(B_2,[3,1^2])+M(1)^{\oplus 2}$ $W_{-5+\frac{1}{2}}(B_3,[7])+M(1)^{\oplus 3}$	$W_{-4+\frac{4}{8}}(\mathfrak{su}(4),[4])+M(1)^{\oplus 3}$ $W_{-9+\frac{9}{6}}(\mathfrak{su}(9),[6,2,1])+M(1)^{\oplus 2}$ $W_{-10+\frac{6}{4}}(D_6,[7,3,1^2])+M(1)^{\oplus 2}$	$W_{-9+\frac{9}{6}}(\mathfrak{su}(9),[5,4])+M(1)^{\oplus 2}$ $W_{-4+\frac{4}{8}}(D_3,[5,1])+M(1)^{\oplus 3}$ $W_{-18+\frac{14}{4}}(\mathfrak{e}_7,A_4+A_1)+M(1)$
$2,2,2$	$(\frac{61}{24},\frac{17}{6})$	$\mathfrak{su}(2)_{-2}\times\mathfrak{su}(2)_{-2}\times\mathfrak{u}(1)_{-2}\times\mathfrak{u}(1)_{f_0}$	$W_{-5+\frac{5}{5}}(\mathfrak{su}(5),[3,1^2])+M(1)^{\oplus 4}(*)$ $W_{-8+1}(D_5,[7,1^3])+M(1)^{\oplus 5}(*)$	$W_{-6+\frac{6}{6}}(\mathfrak{su}(6),[4,2])+M(1)^{\oplus 5}(*)$ $\mathcal{T}[W_{-10+1}(D_6,[9,3])]+M(1)^{\oplus 6}(*)$	$W_{-8+\frac{8}{4}}(\mathfrak{su}(8),[3^2,1^2])+M(1)^{\oplus 3}$
$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathfrak{su}(2)_{-2}\times\mathfrak{su}(2)_{-2}\times\mathfrak{su}(2)_{-2}$	$W_{-30+\frac{24}{6}}(\mathfrak{e}_8,D_4(a_1)+A_1)$ $W_{-12+\frac{12}{6}}(\mathfrak{e}_6,(A_4+A_1)+M(1)^{\oplus 2}(*)$	$W_{-9+3}(B_5,[3,2^2,1^4])$ $\mathcal{T}[W_{-8+\frac{8}{4}}(D_5,[3^2,2^2])]+M(1)^{\oplus}(*)$	$W_{-6+2}(C_5,[2^4,1^2])$ $W_{-6+\frac{4}{2}}(D_4,[2^2,1^4])$
$2,2,2$	$(\frac{53}{24},\frac{13}{6})$	$\mathfrak{u}(1)$	$W_{-10+\frac{6}{3}}(D_6,[5,3^2,1])$	$W_{-30+\frac{24}{6}}(\mathfrak{e}_8,D_5+A_2)$	$W_{-5+1}(B_3,[3^2,1])$
$\frac{9}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{15}{8},2)$	$\mathfrak{su}(3)_{-\frac{9}{4}}$	$W^{-3+\frac{3}{4}}(\mathfrak{su}(3),[1^3])$ $W_{-18+\frac{18}{8}}(\mathfrak{e}_7,A_6)+M(1)(*)$	$W_{-7+\frac{7}{4}}(\mathfrak{su}(7),[4,1^3])$ $W_{-30+\frac{30}{8}}(\mathfrak{e}_8,E_6(a_1))$	$W_{-9+\frac{9}{4}}(\mathfrak{su}(9),[3^3])$ $W_{-6+\frac{5}{4}}(\mathfrak{su}(6),[3^2])+M(1)(*)$
$\frac{7}{3},\frac{5}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathfrak{su}(3)_{-\frac{7}{3}}\times\mathfrak{u}(1)$	$W_{-5+\frac{5}{3}}(\mathfrak{su}(5),[2,1^3])$ $W_{-3+\frac{2}{3}}(\mathfrak{su}(3),[1^3])+M(1)$	$W_{-7+\frac{7}{3}}(\mathfrak{su}(7),[2^3,1])$ $W_{-5+\frac{4}{3}}(\mathfrak{su}(5),[2^2,1])+M(1)(*)$	$W_{-8+\frac{8}{3}}(\mathfrak{su}(8),[3,2,1^3])$ $W_{-30+\frac{20}{3}}(\mathfrak{e}_8,A_4+A_1)$
$\frac{7}{3},\frac{5}{3},\frac{4}{3}$	$(\frac{17}{8},\frac{7}{3})$	$\mathfrak{su}(3)_{-\frac{7}{3}}\times\mathfrak{u}(1)^2$	$W_{-6+\frac{5}{3}}(\mathfrak{su}(6),[3,1^3])+M(1)$	$W_{-8+\frac{7}{3}}(\mathfrak{su}(8),[3,2^2,1])+M(1)(*)$	
$\frac{12}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{103}{60},\frac{107}{60})$	$\mathfrak{su}(2)_{-\frac{17}{10}}$	$W_{-18+\frac{14}{5}}(\mathfrak{e}_7,D_6(a_2))$	$W_{-4+\frac{4}{5}}(\mathfrak{g}_2,\tilde{A}_1)$	
$\frac{18}{7},\frac{12}{7},\frac{8}{7}$	$(\frac{85}{42},\frac{25}{12})$	$\mathfrak{su}(2)_{-\frac{25}{14}}$	$W_{-12+\frac{12}{7}}(\mathfrak{e}_6,A_5)$	$W_{-30+\frac{24}{7}}(\mathfrak{e}_8,A_7)$	$W_{-5+\frac{5}{7}}(C_4,(6,1^2)$
$\frac{8}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathfrak{su}(2)_{-\frac{11}{6}}\times\mathfrak{su}(2)_{-\frac{8}{3}}\times\mathfrak{su}(2)_{-\frac{4}{3}}$	$W_{-12+\frac{8}{3}}(\mathfrak{e}_6,A_3+A_1)+M(1)$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,(A_3+A_1)')$	
$\frac{8}{3},\frac{5}{3},\frac{4}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathfrak{su}(4)_{-\frac{8}{3}}$	$W_{-4+\frac{4}{3}}(\mathfrak{su}(4),[1^4])$ $W_{-6+\frac{5}{3}}(\mathfrak{su}(6),[2^3])+M(1)$ $W_{-7+\frac{7}{4}}(B_4,[3,1^6])$	$W_{-7+\frac{7}{3}}(\mathfrak{su}(7),[3,1^4])$ $W_{-4+\frac{4}{3}}(D_3,[1^6])$	$W_{-8+\frac{8}{3}}(\mathfrak{su}(8),[2^4])$ $W_{-10+\frac{10}{3}}(D_6,[3^2,1^6])$
$\frac{8}{3},\frac{7}{3},\frac{4}{3}$	$(\frac{21}{8},\frac{17}{6})$	$\mathfrak{sp}(2)_{-\frac{7}{3}}\times\mathfrak{u}(1)_{f_0}$	$W_{-8+\frac{8}{3}}(D_5,[2^4,1^2])$ $W_{-5+\frac{5}{3}}(C_4,[2^2,1^4])$	$W_{-8+\frac{5}{3}}(D_5,[5,1^5])+M(1)$	$W_{-3+\frac{2}{3}}(B_2,[1^5])+M(1)$
$\frac{14}{5},\frac{8}{5},\frac{6}{5}$	$(\frac{32}{15},\frac{133}{60})$	$\mathfrak{su}(2)_{-\frac{19}{10}}$	$W_{-8+\frac{8}{5}}(D_5,[5,2^2,1])$ $W_{-6+\frac{6}{5}}(C_5,[4,3^2])$	$W_{-3+\frac{8}{3}}(B_2,[2^2,1])$	$W_{-9+\frac{9}{5}}(B_5,[4^2,3])$

Table 13. The VOA isomorphism from rank three theories.

In table 17 we summarized all rank four theories with special nilpotent orbit discussed in this section. One immediate implication is the following VOA isomorphisms

$$W_{-9+\frac{3}{5}}(\mathfrak{f}_4,F_4)\cong \mathrm{Vir}(2,5)^{\oplus 4}, \tag{5.3}$$

and

$$W_{-9+\frac{5}{7}}(B_5,[11])\cong W_{-2+\frac{2}{7}}(\mathfrak{su}(2),[2])^{\oplus 2}. \tag{5.4}$$

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Δ_{Coulomb}	(a,c)	AKM subalgebra	$W_{\bar{k}}(\mathfrak{g},f)$	$W_{\bar{k}}(\mathfrak{g},f)$	$W_{\bar{k}}(\mathfrak{g},f)$
$3,\frac{5}{2},\frac{3}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathfrak{sp}(2)_{-\frac{5}{2}}\times\mathfrak{u}(1)^2$	$W_{-8+\frac{5}{2}}(D_5,[2^4,1^2])+M(1)$ $W_{-6+\frac{6}{4}}(D_4,[2^4])+M(1)^{\oplus 2}$	$W_{-6+\frac{6}{4}}(D_4,[3,1^5])+M(1)^{\oplus 2}$ $W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_3+A_2)$	$W_{-10+\frac{10}{4}}(D_6,[3^3,1^3])+M(1)^{\oplus 2}$ $W_{-9+\frac{5}{2}}(B_5,[3^2,1^5])+M(1)$
$3,2,\frac{3}{2}$	$(\frac{3}{8},\frac{17}{6})$	$\mathfrak{su}(2)_{-2}\times\mathfrak{u}(1)^2$	$W_{-10+\frac{6}{4}}[D_6,(7,2^2,1])+M(1)^{\oplus 2}$	$W_{-3+\frac{1}{2}}(B_2,[2^2,1])+M(1)^{\oplus 2}$	
$3,2,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathfrak{su}(4)_{-3}\times\mathfrak{u}(1)^3$	$\mathcal{T}[W_{-4+\frac{4}{4}}(\mathfrak{su}(4),[1^4])]+M(1)^{\oplus 3}$ $W_{-9+\frac{9}{3}}(\mathfrak{su}(9),[3,2,1^4])+M(1)^{\oplus 2}$ $W_{-9+\frac{8}{4}}(\mathfrak{su}(9),[3^3])+M(1)^{\oplus 4}(*)$	$W_{-6+\frac{6}{3}}(\mathfrak{su}(6),[2,1^4])+M(1)^{\oplus 2}$ $W_{-9+\frac{9}{3}}(\mathfrak{su}(9),[2^4,1])+M(1)^{\oplus 2}$ $W_{-4+\frac{3}{3}}(D_3,[1^6])+M(1)^{\oplus 3}$	$W_{-8+\frac{8}{4}}(\mathfrak{su}(8),[4,1^4])+M(1)^{\oplus 3}$ $W_{-7+\frac{6}{3}}(\mathfrak{su}(7),[2^3,1])+M(1)^{\oplus 3}(*)$ $W_{-10+\frac{6}{2}}(D_6,[3^2,1^6])+M(1)^{\oplus 2}$
$3,3,2$	$(\frac{27}{8},\frac{7}{2})$	$(\mathfrak{g}_2)_3$	$W_{-12+3}(\mathfrak{e}_6,2A_2)$	$W_{-4+1}(\mathfrak{g}_2,0)$	
$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$(\frac{19}{6},\frac{41}{12})$	$\mathfrak{sp}(2)_{-\frac{13}{6}}\times\mathfrak{su}(2)_{-\frac{8}{3}}$	$W_{-10+\frac{10}{3}}(D_6,[3,2^2,1^5])$	$W_{-7+\frac{7}{3}}(B_4,[2^2,1^5])$	
$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathfrak{so}(7)_{-\frac{10}{3}}$	$W_{-8+\frac{8}{3}}(D_5,[3,1^7])$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,(A_3+A_1)'')$	$W_{-5+\frac{5}{3}}(B_3,[1^7])$
$\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathfrak{sp}(3)_{-\frac{8}{3}}$	$W_{-10+\frac{10}{3}}(D_6,[2^6])$	$W_{-4+\frac{4}{3}}(C_3,[1^6])$	
$\frac{7}{2},\frac{5}{2},\frac{3}{2}$	$(\frac{7}{2},4)$	$\mathfrak{su}(7)_{-\frac{7}{2}}$	$W_{-7+\frac{7}{2}}(\mathfrak{su}(7),[1^7])$	$W_{-9+\frac{9}{2}}(\mathfrak{su}(9),[2,1^7])$	$W_{-6+\frac{5}{2}}(\mathfrak{su}(6),[1^6])+M(1)(*)$
$\frac{7}{2},3,\frac{3}{2}$	$(\frac{29}{8},4)$	$\mathfrak{so}(7)_{-\frac{7}{2}}\times\mathfrak{u}(1)$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,2A_1)$	$W_{-5+\frac{3}{2}}(B_3,[1^7])+M(1)$	$W_{-8+\frac{5}{2}}(D_5,[3,1^7])+M(1)$
$\frac{18}{5},\frac{12}{5},\frac{6}{5}$	$(\frac{71}{24},\frac{46}{15})$	$\mathfrak{su}(2)_{-\frac{23}{5}}$	$W_{-18+\frac{14}{5}}(\mathfrak{e}_7,A_5+A_1)$	$W_{-4+\frac{4}{5}}(\mathfrak{g}_2,A_1)$	
$4,\frac{8}{3},\frac{4}{3}$	$(\frac{83}{24},\frac{11}{3})$	$\mathfrak{su}(2)_{-5}\times\mathfrak{su}(2)_{-4}$	$W_{-12+\frac{8}{3}}(\mathfrak{e}_6,2A_2+A_1)+M(1)(*)$	$W_{-18+\frac{14}{3}}(\mathfrak{e}_7,2A_2+A_1)$	
$4,2,2$	$(\frac{83}{24},\frac{11}{3})$	$\mathfrak{sp}(2)_{-\frac{5}{2}}\times\mathfrak{su}(2)_{-4}$	$W_{-30+\frac{24}{3}}(\mathfrak{e}_8,A_3+2A_1)$	$W_{-9+3}(B_5,[2^4,1^3])$	$W_{-6+2}(C_5,[2^3,1^4])$
$4,3,2$	$(\frac{107}{24},\frac{31}{6})$	$\mathfrak{su}(8)_{-4}\times\mathfrak{u}(1)$	$W_{-8+\frac{8}{2}}(\mathfrak{su}(8),[1^8])+M(1)$	$W_{-7+\frac{6}{2}}(\mathfrak{su}(7),[1^7])+M(1)^{\oplus 2}(*)$	
$4,3,2$	$(\frac{25}{6},\frac{55}{12})$	$\mathfrak{sp}(3)_{-3}\times\mathfrak{su}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2$	$W_{-7+2}(B_4,[2^2,1^5])+M(1)^{\oplus 2}$	$W_{-10+\frac{6}{2}}(D_6,[3,2^2,1^5])+M(1)^{\oplus 2}$	
$4,4,2$	$(\frac{39}{8},\frac{11}{2})$	$\mathfrak{so}(7)_{-4}\times\mathfrak{su}(2)_{-12}$	$W_{-12+\frac{8}{2}}(\mathfrak{e}_6,2A_1)+M(1)^{\oplus 2}$	$W_{-30+\frac{30}{3}}(\mathfrak{e}_8,A_2+2A_1)$	
$6,2,2$	$(\frac{19}{4},\frac{21}{4})$	$\mathfrak{sp}(3)_{-\frac{7}{2}}$	$W_{-30+\frac{24}{4}}(\mathfrak{e}_8,D_4+A_1)$	$W_{-9+3}(\mathfrak{f}_4,A_1)$	$W_{-18+\frac{18}{3}(\mathfrak{e}_7,4A_1)}$
$6,3,2$	$(\frac{45}{8},\frac{13}{2})$	$(\mathfrak{e}(6))_{-6}$	$W_{-12+\frac{12}{2}}(\mathfrak{e}_6,0)$	$W_{-30+\frac{24}{2}}(\mathfrak{e}_8,A_2)$	

Table 14. The VOA isomorphism from rank three AD theories-continued.

$\mathfrak{g}$	b	k $(f=\mathcal{O}_{\text{triv}})$	k $(f=\mathcal{O}_{\text{min}})$	k $(f=\mathcal{O}_{\text{subreg}})$	k $(f=\mathcal{O}_{\text{prin}})$
A_{n-1}	n	$-n+1+\frac{2\dim\mathcal{B}}{n-1}$	$-n+1+\frac{2\dim\mathcal{B}+2}{n-1}$	$-1+\frac{2\dim\mathcal{B}}{n-1}$	$1+\frac{2\dim\mathcal{B}}{n-1}$
D_n	n	$-n+\frac{1}{2}+\frac{2\dim\mathcal{B}+1}{2(n-1)}$	$-n+\frac{1}{2}+\frac{2\dim\mathcal{B}+3}{2(n-1)}$	$-\frac{3}{2}+\frac{2\dim\mathcal{B}+3}{2(n-1)}$	$\frac{1}{2}+\frac{2\dim\mathcal{B}+1}{2(n-1)}$
	$2n-2$	$-2n+3+\frac{2\dim\mathcal{B}}{n}$	$-2n+3+\frac{2\dim\mathcal{B}+2}{n}$	$-3+\frac{2(\dim\mathcal{B}+3)}{n}$	$1+\frac{2\dim\mathcal{B}}{n}$
E_6	12	$-11+\frac{\dim\mathcal{B}}{3}$	$-11+\frac{\dim\mathcal{B}+1}{3}$	$-3+\frac{\dim\mathcal{B}+1}{3}$	$1+\frac{\dim\mathcal{B}}{3}$
	9	$-8+\frac{\dim\mathcal{B}-1}{4}$	$-8+\frac{\dim\mathcal{B}}{4}$	$-2+\frac{\dim\mathcal{B}}{4}$	$1+\frac{\dim\mathcal{B}-1}{4}$
	8	$-7+\frac{2\dim\mathcal{B}-3}{9}$	$-7+\frac{2\dim\mathcal{B}-1}{9}$	$-2+\frac{2\dim\mathcal{B}+2}{9}$	$1+\frac{2\dim\mathcal{B}-3}{9}$
E_7	18	$-17+\frac{2\dim\mathcal{B}}{7}$	$-17+\frac{2\dim\mathcal{B}+2}{7}$	$-4+\frac{2\dim\mathcal{B}+1}{7}$	$1+\frac{2\dim\mathcal{B}}{7}$
	14	$-13+\frac{2\dim\mathcal{B}-2}{9}$	$-13+\frac{2\dim\mathcal{B}}{9}$	$-3+\frac{2\dim\mathcal{B}}{9}$	$1+\frac{2\dim\mathcal{B}-2}{9}$
E_8	30	$-29+\frac{\dim\mathcal{B}}{4}$	$-29+\frac{\dim\mathcal{B}+1}{4}$	$-7+\frac{\dim\mathcal{B}+3}{4}$	$1+\frac{\dim\mathcal{B}}{4}$
	24	$-23+\frac{\dim\mathcal{B}-1}{5}$	$-23+\frac{\dim\mathcal{B}}{5}$	$-5+\frac{\dim\mathcal{B}}{5}$	$1+\frac{\dim\mathcal{B}-1}{5}$
	20	$-19+\frac{\dim\mathcal{B}-2}{6}$	$-19+\frac{\dim\mathcal{B}-1}{6}$	$-4+\frac{\dim\mathcal{B}-1}{6}$	$1+\frac{\dim\mathcal{B}-2}{6}$

Table 15. Values of k for a rank $\dim\mathcal{B}$ untwisted theories with $f_0=0$. The requirements of $k\in\mathbb{Z}$ and $\gcd(b,k)=1$ determine possible theories for a given $\dim\mathcal{B}$.

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j/o	$\mathfrak{g}$	b_t	k_t $(f=\mathcal{O}^{\text{Hitchin}}_{\text{prin}})$	k_t $(f=\mathcal{O}^{\text{Hitchin}}_{\text{subreg}})$	k_t $(f=\mathcal{O}^{\text{Hitchin}}_{\text{min}})$	k_t $(f=\mathcal{O}^{\text{Hitchin}}_{\text{triv}})$
$A_{2n-1}/\mathbb{Z}_2$	B_n	$4n-2$	$-4n+3+\frac{2\dim\mathcal{B}}{n}$	$-4n+3+\frac{2\dim\mathcal{B}+2}{n}$	$-2n-1+\frac{2(\dim\mathcal{B}+1)}{n}$	$-2n+3+\frac{2\dim\mathcal{B}}{n}$
		$2n$	$-2n+\frac{1}{2}+\frac{4\dim\mathcal{B}+1}{2(2n-1)}$	$-2n+\frac{1}{2}+\frac{4\dim\mathcal{B}+5}{2(2n-1)}$	$-n-1+\frac{4\dim\mathcal{B}+2}{2(2n-1)}$	$-n+1+\frac{4\dim\mathcal{B}+2}{2(2n-1)}$
$D_{n+1}/\mathbb{Z}_2$	$C_n^{(2)}$	$2n+2$	$-2n-1+\frac{2\dim\mathcal{B}}{n}$	$-2n-1+\frac{2\dim\mathcal{B}+2}{n}$	$-5+\frac{2(\dim\mathcal{B}+1)}{n}$	$-1+\frac{2\dim\mathcal{B}}{n}$
		$2n$	$-2n+1+\frac{2\dim\mathcal{B}-1}{n+1}$	$-2n+1+\frac{2\dim\mathcal{B}+1}{n+1}$	$-5+\frac{2\dim\mathcal{B}+7}{n+1}$	$-1+\frac{2\dim\mathcal{B}+1}{n+1}$
$D_4/\mathbb{Z}_3$	G_2	12	$-11+\dim\mathcal{B}$	$-10+\dim\mathcal{B}$	$-10+\dim\mathcal{B}$	$-5+\dim\mathcal{B}$
		6	$-5+\frac{\dim\mathcal{B}-1}{2}$	$-5+\frac{\dim\mathcal{B}}{2}$	$-5+\frac{\dim\mathcal{B}}{2}$	$\frac{-5+\dim\mathcal{B}}{2}$
$E_6/\mathbb{Z}_2$	F_4	18	$-17+\frac{\dim\mathcal{B}}{2}$	$-17+\frac{\dim\mathcal{B}+1}{2}$	$-11+\frac{\dim\mathcal{B}+1}{2}$	$-5+\frac{\dim\mathcal{B}}{2}$
		12	$-11+\frac{\dim\mathcal{B}-1}{3}$	$-11+\frac{\dim\mathcal{B}}{3}$	$-7+\frac{\dim\mathcal{B}}{3}$	$-3+\frac{\dim\mathcal{B}-1}{3}$
		8	$-7+\frac{2\dim\mathcal{B}-5}{9}$	$-7+\frac{2\dim\mathcal{B}-3}{9}$	$-5+\frac{2\dim\mathcal{B}+3}{9}$	$-2+\frac{2\dim\mathcal{B}-2}{9}$

Table 16. Values of k_t for a rank $\dim\mathcal{B}$ twisted theories with $f_0=0$. The requirements of $k_t\in\mathbb{Z}$ and $\gcd(b_t,k_t)=1$ determine possible theories for a given $\dim\mathcal{B}$.

(a,c)	Δ_{Coulomb}	$\mathfrak{f}$	VOA	$\mathcal{M}_H$/Asso. Var.	Singularity
$*(\frac{43}{30},\frac{22}{15})$	$\frac{6}{5},\frac{6}{5},\frac{6}{5},\frac{6}{5}$	–	$W_{-9+\frac{3}{5}}(\mathfrak{f}_4,F_4)$	–	
$*(\frac{67}{42},\frac{34}{21})$	$\frac{10}{7},\frac{10}{7},\frac{8}{7},\frac{8}{7}$	–	$W_{-9+\frac{5}{7}}(B_5,[11])$	–	
$*(\frac{115}{66},\frac{58}{33})$	$\frac{18}{11},\frac{16}{11},\frac{14}{11},\frac{12}{11}$	–	$W_{-2+\frac{2}{11}}(\mathfrak{su}(2),[2])$	–	(A_1,A_8)
$*(\frac{15}{8},\frac{23}{12})$	$\frac{5}{3},\frac{3}{2},\frac{4}{3},\frac{7}{6}$	$\mathfrak{u}(1)$	$W_{-5+\frac{5}{6}}(\mathfrak{su}(5),[4,1])$	$\overline{\mathcal{O}}_{[5]}\cap S_{[4,1]}$	
$*(\frac{35}{18},2)$	$\frac{16}{9},\frac{14}{9},\frac{4}{3},\frac{10}{9}$	$\mathfrak{su}(2)_{\frac{16}{9}}$	$V_{-\frac{16}{9}}(\mathfrak{su}(2))$	$[2]$	(A_1,D_9)
$*(\frac{91}{48},\frac{23}{12})$	$\frac{15}{8},\frac{3}{2},\frac{5}{4},\frac{9}{8}$	–	$W_{-3+\frac{3}{8}}(\mathfrak{su}(3),[3])$	–	
$*(2,2)$	$2,\frac{4}{3},\frac{4}{3},\frac{4}{3}$	–	$W_{-6+\frac{2}{3}}(D_4,[7,1])$	–	
$*(\frac{38}{15},\frac{8}{3})$	$\frac{12}{5},\frac{9}{5},\frac{7}{5},\frac{6}{5}$	$\mathfrak{su}(3)_{\frac{12}{5}}$	$V_{-\frac{12}{5}}(\mathfrak{su}(3))$	$[3]$	$D_5(\mathrm{SU}(3))$
$(\frac{19}{6},\frac{10}{3})$	$\frac{14}{5},\frac{12}{5},\frac{8}{5},\frac{6}{5}$	$\mathfrak{so}(5)_{\frac{12}{5}}$	$V_{-\frac{12}{5}}(B_2)$	$[5]$	$D_5(\mathrm{SO}(5))$
$*(\frac{11}{3},4)$	$\frac{10}{3},\frac{7}{3},\frac{5}{3},\frac{4}{3}$	$\mathfrak{su}(5)_{\frac{10}{3}}$	$V_{-\frac{10}{3}}(\mathfrak{su}(5))$	$[3,2]$	$D_3(\mathrm{SU}(5))$
$(\frac{25}{6},\frac{9}{2})$	$\frac{10}{3},\frac{8}{3},\frac{7}{3},\frac{4}{3}$	$\mathfrak{so}(6)_{\frac{10}{3}}\times\mathfrak{su}(2)_{\frac{7}{3}}$	$W_{-8+\frac{8}{3}}(D_5,[2^2,1^6])$	$\overline{\mathcal{O}}_{[3^3,1]}\cap S_{[2^2,1^6]}$	
$(\frac{133}{30},\frac{14}{3})$	$\frac{18}{5},\frac{16}{5},\frac{12}{5},\frac{6}{5}$	$(\mathfrak{g}_2)_{\frac{16}{5}}$	$V_{-\frac{16}{5}}(\mathfrak{g}_2)$	$G_2(a_1)$	
$*(\frac{13}{3},\frac{14}{3})$	$4,2,2,2$	$\mathfrak{so}(8)_4$	$V_{-4}(\mathfrak{so}(8))$	$[3^2,1^2]$	
$(\frac{29}{6},\frac{31}{6})$	$4,3,2,2$	$\mathfrak{sp}(3)_3\times\mathfrak{u}(1)_2$	$W_{-6+2}(C_5,[2^2,1^6])$	$\overline{\mathcal{O}}_{[3^2,2^2]}\cap S_{[2^2,1^6]}$	
$(\frac{43}{8},\frac{23}{4})$	$4,4,2,2$	$\mathfrak{so}(7)_4\times\mathfrak{su}(2)_{\frac{5}{2}}$	$W_{-9+3}(B_5,[2^2,1^7])$	$\overline{\mathcal{O}}_{[3^3,1^2]}\cap S_{[2^2,1^7]}$	
$*(\frac{35}{6},\frac{20}{3})$	$\frac{9}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2}$	$\mathfrak{su}(9)_{\frac{9}{2}}$	$V_{-\frac{9}{2}}(\mathfrak{su}(9))$	$[2^4,1]$	$D_2(\mathrm{SU}(9))$
$(\frac{35}{6},\frac{77}{12})$	$\frac{9}{2},\frac{7}{2},3,\frac{3}{2}$	$\mathfrak{su}(6)_{\frac{9}{2}}$	$W_{-12+\frac{9}{2}}(\mathfrak{e}_6,A_1)$	$\overline{\mathcal{O}}_{A_2+2A_1}\cap S_{A_1}$	
$(\frac{11}{2},6)$	$\frac{14}{3},\frac{10}{3},\frac{8}{3},\frac{4}{3}$	$\mathfrak{so}(9)_{\frac{14}{3}}$	$V_{-\frac{14}{3}}(B_4)$	$[3^3]$	$D_3\mathrm{SO}(9)$
$(\frac{13}{2},7)$	$6,4,2,2$	$\mathfrak{sp}(4)_{\frac{7}{2}}$	$W_{-6+2}(C_5,[2,1^8])$	$\overline{\mathcal{O}}_{[3^2,2^2]}\cap S_{[2,1^8]}$	
$(\frac{47}{6},\frac{26}{3})$	$6,6,2,2$	$(\mathfrak{f}_4)_6$	$V_{-6}(\mathfrak{f}_4)$	$F_4(a_3)$	
$*(\frac{107}{6},\frac{62}{3})$	$15,9,5,3$	$(\mathfrak{e}_8)_{15}$	$V_{-15}(\mathfrak{e}_8)$	A_2+A_1	

Table 17. Rank four theories with regular singularity being full, minimal, subregular or regular. Central charges (a,c), CB spectrum, flavor symmetry algebra $\mathfrak{f}$, one corresponding VOA and the Higgs branch are listed. Theories with $*$ saturate the bound (3.1).

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6 Conclusion

In this paper, we perform a systematical search for small rank theories in the space of geometrically engineered 4d $\mathcal{N} = 2$ SCFTs. Some of the rank one and rank two theories constructed in [22, 25] are found here, and new properties of these theories such as the corresponding VOA and the Higgs branch can be obtained. The main new input is the dimension formula (2.7) and (2.8) for the CB. It is now possible to study these theories with arbitrary rank.

Let us now list some interesting questions for further study

1. The main feature of the new theories found in [22, 25] is that undeformable singularities except the I_1 type appear in generic deformations. These theories also show up in our twisted constructions. While these special undeformable singularities are related to discrete gauging [64], they are related to the outer automophsim twist in our construction. It would be interesting to further clarify the relation between the discrete gauging and the outer automorphism twist.

2. For generic rank it is difficult to classify all possible theories, therefore one should try to look for bounds on the rank of flavor symmetry, the maximal scaling dimension, and etc at a given rank.

3. If the mass parameter of the irregular singularity is not zero, the corresponding VOA is not completely determined (might be an extension between the W algebra and Heisenberg algebra). it is interesting to find the full corresponding VOA. The isomorphism of VOAs found in this paper might be a clue for the general correspondence.

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A Derivation of dimension formula

Here we provide the derivation of the dimension formula (2.7) and (2.8). Starting with a non-twisted theory labelled by $(\mathfrak{j}, b, k, f)$, if there is irregular singularity only (i.e. f is chosen to be principal), the same theory can also be engineered by putting type IIB theory on a three-fold singularity which are listed in the third column of table 1. One can compute the (hyper-Kahler) dimension of its Coulomb branch by using the following formula [9–12]

$$r_0 = \frac{\mu - f_0}{2},$$

where μ is the Milnor number which can be computed using the weight data of singularity [9], and f_0 is the number of mass parameter computed in [8, 44]. The tables of μ in each cases

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can also be found in the last column of table I [65] and we also list them in the last column of table 1 for reader's convenience. Adding a regular singularity with Nahm label f, will change the Coulomb branch dimension into

$$r = r_0 + \frac{1}{2}\dim \mathcal{O}_f^{\text{Hitchin}},$$

where $\mathcal{O}_f^{\text{Hitchin}}$ is the dual nilpotent orbit of $\mathcal{O}_f$. Finally one can check case by case that the Milnor number μ for non-twisted cases can also be written uniformly as

$$\mu = \left(h_{\mathfrak{j}}\frac{k}{b} + h_{\mathfrak{j}} - 1\right)\text{rank}(\mathfrak{j}) - \dim \mathcal{O}_{\text{prin}} \tag{A.1}$$

Example: when $\mathfrak{j} = A_{N-1}$, b can be either N or $N-1$ as table 1. If there is no regular puncture, the corresponding 3-fold singularity is

$$\begin{aligned} x^2 + y^2 + z^N + w^k = 0, \quad b = N \\ x^2 + y^2 + z^N + zw^k = 0, \quad b = N-1 \end{aligned}$$

For $b = n$, the Milnor number $\mu = (N-1)(k-1)$. On the other hand, since $h = N$, $\text{rank}(\mathfrak{j}) = N-1$ and $\dim \mathcal{O}_{\text{prin}} = N^2 - N$, we have

$$(h_{\mathfrak{j}}\frac{k}{b} + h_{\mathfrak{j}} - 1)\text{rank}(\mathfrak{j}) - \dim \mathcal{O}_{\text{prin}} = (k-1)(N-1) = \mu \tag{A.2}$$

For $b = n-1$, the Milnor number is $\mu = N(k-1)+1$, which also agrees with equation (A.1)

$$(h_{\mathfrak{j}}\frac{k}{b} + h_{\mathfrak{j}} - 1)\text{rank}(\mathfrak{j}) - \dim \mathcal{O}_{\text{prin}} = Nk + (N-1)^2 - N(N-1) = N(k-1)+1 = \mu. \tag{A.3}$$

The dimension formula for twisted cases is a direct generalization of the untwisted one and can be verified by the dimension of the Hitchin fiber mentioned in the following paragraph.

Finally let us mention that there is a different way of counting the Coulomb branch dimension. The Hitchin moduli space has a Hitchin fibration $f : M_H \to B$, and the dimension of the fiber is the same as the dimension of the base B, which is also the dimension of the Coulomb branch. The dimension formula of the Hitchin fiber can also be found in math literature [46] for both untwisted and twisted cases, which is exactly the formula we found using physics arguments. This provids a cross check of the dimension formula.

B List of $d(n)$

In this section we list $d(n)$ used to compute the growth function for theories engineered using one irregular and one regular singularities.

C List of AD theories

One can choose k, b and regular puncture such that the CB dimension formular satisfied. In the following, we list all rank1, rank2, rank3 non-twisted AD theories with type ADE from the CB dimension formular. We list the folowing data:

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$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
A_2	$d(2)=3,\ d(1)=0,$
A_3	$d(3)=8,\ d(2)=3,\ d(1)=0$
A_4	$d(4)=15,\ d(2)=6,\ d(3)=8,\ d(1)=0$
A_5	$d(5)=24,\ d(4)=15,\ d(2)=6,\ d(1)=0$
A_6	$d(6)=35,\ d(5)=24,\ d(3)=16,\ d(2)=9,\ d(1)=0$
A_7	$d(7)=48,\ d(6)=35,\ d(3)=16,\ d(2)=9$
A_8	$d(8)=63,\ d(7)=48,\ d(4)=30,\ d(2)=12$
A_9	$d(9)=80,\ d(8)=63,\ d(4)=30,\ d(3)=24$

$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
B_1	$d(1)=3\,\text{or}\,0$
B_2	$d(3)=10,\ d(2)=3,\ d(1)=6\,\text{or}\,0$
B_3	$d(5)=21,\ d(3)=11\,\text{or}\,8,\ d(1)=9\,\text{or}\,0$
B_4	$d(7)=36,\ d(4)=15,\ d(2)=6,\ d(1)=12$
B_5	$d(9)=55,\ d(5)=27\,\text{or}\,24,\ d(3)=21$

$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
C_1	$d(2)=3,\ d(1)=0$
C_2	$d(3)=10,\ d(1)=0$
C_3	$d(4)=21,\ d(3)=16$
C_4	$d(5)=36$
C_5	$d(6)=55,\ d(5)=48, d(2)=21$

$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
D_3	$d(4)=15,\ d(3)=8,\ d(2)=6,\ d(1)=0$
D_4	$d(6)=28,\ d(4)=18,\ d(3)=8,\ d(2)=12,\ d(1)=0$
D_5	$d(8)=45,\ d(5)=24.\ d(4)=18,\ d(2)=12,\ d(1)=0$
D_6	$d(10)=66,\ d(6)=38,\ d(5)=24,\ d(3)=16,\ d(2)=18,\ d(1)=0$

$\mathfrak{g}$	$d_{\mathfrak{g}}(n)$
$\mathfrak{e}_6$	$d(12)=78,\ d(9)=56,\ d(8)=45,\ d(6)=36,\ d(4)=18,\ d(3)=24,\ d(2)=12$
$\mathfrak{e}_7$	$d(18)=133,\ d(14)=99,\ d(9)=56,\ d(7)=48,\ d(6)=39,\ d(3)=24,\ d(2)=21$
$\mathfrak{e}_8$	$d(30)=248,\ d(24)=190,\ d(20)=156,\ d(15)=112,\ d(12)=92,\ d(10)=72$ $d(8)=66,\ d(6)=40,\ d(5)=48,\ d(4)=36,\ d(3)=32,\ d(2)=24$
$\mathfrak{g}_2$	$d(4)=14,\ d(2)=12(u=3k),\ d(2)=4(u\neq 3k),\ d(1)=8$
$\mathfrak{f}_4$	$d(9)=52,\ d(6)=30,\ d(3)=24(u=3k-1),\ d(3)=12(u=3k+1),\ d(4)=21$ $d(2)=18,\ d(1)=12$

Table 18. Data used to compute the growth function of Kac-Moody algebra $V_{-h^\vee+\frac{n}{u}}(\mathfrak{g})$, see formula. (2.21).

- The parameters which describe the Higgs field of the Hitchin system near the irregular singularity b and k.
- The Hitchin label (HL) and Nahm label (NL) of nilpotent orbits, for A_{n-1} type, we only list the Nahm label denoted by $\mathcal{O}_f$.
- The number of mass deformation f_0 and the number of exact marginal deformation f_1 [44]. For E_8 type, $f_0=0$ for all cases, we only list the f_1.
- The dimension of nilpotent orbits correspond to Hichin label and Nahm label denoted by HD and ND respectively
- The CB operator scaling dimension Δ_{Coulomb}.
- The flavor symmetry $\mathfrak{f}$ correspond to regular puncture and flavor central charge k_{2d} for each simple factor.
- Conformal central charge (a_{4d}, c_{4d}).
- The notation "Theory" label AD theories.
- $\mathfrak{sp}(r)$ means C_r.

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C.1 Rank 1

n	k	$\mathcal{O}_f$	f_0	f_1	ND	Δ_{Coulomb}	$\mathfrak{f}$	(a,c)	Theory
2	1	(1^2)	0	0	0	$\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-\frac{4}{3}}(\mathfrak{su}(2),(1^2))]$
	2	(1^2)	1	1	0	$\frac{3}{2}$	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-\frac{3}{2}}(\mathfrak{su}(2)),(1^2)]+\mathfrak{u}(1)$
	3	(2)	0	0	2	$\frac{6}{5}$	–	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-\frac{8}{5}}(\mathfrak{su}(2),(2))]$
	4	(2)	1	1	2	$\frac{4}{3}$	$\mathfrak{u}(1)_{f_0}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-\frac{5}{3}}(\mathfrak{su}(2),(2))]+\mathfrak{u}(1)$
3	-1	(1^3)	0	0	0	$\frac{3}{2}$	$\mathfrak{su}(3)_{\frac{3}{2}}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-3+\frac{3}{2}}(\mathfrak{su}(3),(1^3))]$
	0	(1^3)	2	2	0	2	$\mathfrak{su}(3)_2\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-3+\frac{3}{3}}(\mathfrak{su}(3),(1^3))]+2\mathfrak{u}(1)$
	2	(3)	0	0	6	$\frac{6}{5}$	–	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-3+\frac{3}{5}}(\mathfrak{su}(3),(3))]$
	3	(3)	2	2	6	$\frac{3}{2}$	$\mathfrak{u}(1)^2_{f_0}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-3+\frac{3}{6}}(\mathfrak{su}(3),(3))]+2\mathfrak{u}(1)$
4	-2	(1^4)	1	1	0	2	$\mathfrak{su}(4)_2\times\mathfrak{u}(1)_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{2}}(\mathfrak{su}(4),(1^4))]+\mathfrak{u}(1)$
	-1	(2^2)	0	0	8	$\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(\mathfrak{su}(4),(2^2))]$
	0	(2^2)	3	3	8	2	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)^3_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{4}}(\mathfrak{su}(4),(2^2))]+3\mathfrak{u}(1)$
	2	(4)	1	1	12	$\frac{4}{3}$	$\mathfrak{u}(1)_{f_0}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-4+\frac{4}{6}}(\mathfrak{su}(4),(4))]+\mathfrak{u}(1)$
5	-3	$(2,1^3)$	0	0	8	$\frac{3}{2}$	$\mathfrak{su}(3)_{\frac{3}{2}}\times\mathfrak{u}(1)_0$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-5+\frac{5}{2}}(\mathfrak{su}(5),(2,1^3))]$
	-2	$(3,1^2)$	0	0	14	$\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{u}(1)_0$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(\mathfrak{su}(5),(3,1^2))]$
6	-4	$(2,1^4)$	1	1	10	2	$\mathfrak{su}(4)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-6+\frac{6}{2}}(\mathfrak{su}(6),(2,1^4))]+\mathfrak{u}(1)$
	-3	$(3,1^3)$	2	2	18	2	$\mathfrak{su}(3)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-6+\frac{6}{3}}(\mathfrak{su}(6),(3,1^3))]+2\mathfrak{u}(1)$
	-3	(2^3)	2	2	18	2	$\mathfrak{su}(3)_2\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-6+\frac{6}{3}}(\mathfrak{su}(6),(2^3))]+2\mathfrak{u}(1)$
	-2	$(4,1^2)$	1	1	24	$\frac{3}{2}$	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(\mathfrak{su}(6),(4,1^2))]+\mathfrak{u}(1)$
	-2	(3^2)	1	1	24	$\frac{3}{2}$	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(\mathfrak{su}(6),(3^2))]+\mathfrak{u}(1)$
7	-5	$(2^2,1^3)$	0	0	20	$\frac{3}{2}$	$\mathfrak{su}(2)_0\times\mathfrak{su}(3)_{\frac{3}{2}}\times\mathfrak{u}(1)_0$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-7+\frac{7}{2}}(\mathfrak{su}(7),(2^2,1^4))]$
	-4	$(3,2^2)$	0	0	30	$\frac{4}{3}$	$\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{u}(1)_0$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(\mathfrak{su}(7),(3,2^2))]$
	-2	$(5,2)$	0	0	38	$\frac{6}{5}$	$\mathfrak{u}(1)_0$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-7+\frac{7}{5}}(\mathfrak{su}(7),(5,2))]$
8	-6	$(2^2,1^4)$	1	1	24	2	$\mathfrak{su}(2)_0\times\mathfrak{su}(4)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-8+\frac{8}{2}}(\mathfrak{su}(8),(2^2,1^4))]+\mathfrak{u}(1)$
	-5	$(3^2,1^2)$	0	0	40	$\frac{4}{3}$	$\mathfrak{su}(2)_0\times\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{u}(1)_0$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(\mathfrak{su}(8),(3^2,1^2))]$
	-4	$(4,2^2)$	3	3	44	2	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^3_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(\mathfrak{su}(8),(4,2^2))]+3\mathfrak{u}(1)$
	-3	$(5,3)$	0	0	50	$\frac{6}{5}$	$\mathfrak{u}(1)_0$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-8+\frac{8}{5}}(\mathfrak{su}(8),(5,3))]$
	-2	$(6,2)$	1	1	52	$\frac{4}{3}$	$\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-8+\frac{8}{6}}(\mathfrak{su}(8),(6,2))]+\mathfrak{u}(1)$
9	-7	$(2^3,1^3)$	0	0	36	$\frac{3}{2}$	$\mathfrak{su}(3)_0\times\mathfrak{su}(3)_{\frac{3}{2}}\times\mathfrak{u}(1)_0$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-9+\frac{9}{2}}(\mathfrak{su}(9),(2^3,1^3))]$
	-6	$(3^2,1^3)$	2	2	48	2	$\mathfrak{su}(3)_2\times\mathfrak{su}(2)_0\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-9+\frac{9}{3}}(\mathfrak{su}(9),(3^2,1^3))]+2\mathfrak{u}(1)$
	-6	$(3,2^3)$	2	2	48	2	$\mathfrak{su}(3)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-9+\frac{9}{3}}(\mathfrak{su}(9),(3,2^3))]+2\mathfrak{u}(1)$
	-3	$(6,3)$	2	2	66	$\frac{3}{2}$	$\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-9+\frac{9}{6}}(\mathfrak{su}(9),(6,3))]+2\mathfrak{u}(1)$

Table 19. A_{n-1}, $b=n$.

n	k	$\mathcal{O}_f$	f_0	f_1	ND	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
4	-1	$(2,1^2)$	1	0	6	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathcal{T}[W_{-4+\frac{3}{2}}(\mathfrak{su}(4),(1^4))]+\mathfrak{u}(1)$
5	-2	$(2,1^3)$	2	1	8	$\mathfrak{su}(3)_2\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)^2_{f_0}$	2	$(1,\frac{5}{4})$	$\mathcal{T}[W_{-5+\frac{4}{2}}(\mathfrak{su}(5),(2,1^3))]+2\mathfrak{u}(1)$
	-1	$(3,2)$	1	0	16	$\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{1}{2},\frac{7}{12})$	$\mathcal{T}[W_{-5+\frac{4}{3}}(\mathfrak{su}(5),(3,2))]+\mathfrak{u}(1)$
6	-3	$(2^2,1^2)$	1	0	16	$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{2}{3},\frac{5}{6})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(\mathfrak{su}(6),(2^2,1^2))]+\mathfrak{u}(1)$
7	-4	$(2^2,1^3)$	2	1	20	$\mathfrak{su}(3)_2\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)^2_{f_0}$	2	$(\frac{25}{24},\frac{4}{3})$	$\mathcal{T}[W_{-7+\frac{6}{2}}(\mathfrak{su}(7),(2^2,1^3))]+2\mathfrak{u}(1)$
	-3	$(3,2^2)$	3	2	30	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)^3_{f_0}$	2	$(1,\frac{5}{4})$	$\mathcal{T}[W_{-7+\frac{6}{3}}(\mathfrak{su}(7),(3,2^2))]+3\mathfrak{u}(1)$
	-2	$(4,3)$	2	1	36	$\mathfrak{u}(1)_1\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathcal{T}[W_{-7+\frac{6}{4}}(\mathfrak{su}(7),(4,3))]+2\mathfrak{u}(1)$
8	-5	$(2^3,1^2)$	1	0	30	$\mathfrak{su}(3)_{\frac{3}{2}}\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{17}{24},\frac{11}{12})$	$\mathcal{T}[W_{-8+\frac{7}{2}}(\mathfrak{su}(8),(2^3,1^2))]+\mathfrak{u}(1)$
	-4	$(3^2,2)$	1	0	42	$\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{13}{24},\frac{2}{3})$	$\mathcal{T}[W_{-8+\frac{7}{3}}(\mathfrak{su}(8),(3^2,2))]+\mathfrak{u}(1)$
9	-6	$(2^3,1^3)$	2	1	36	$\mathfrak{su}(3)_2\times\mathfrak{su}(3)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)^2_{f_0}$	2	$(\frac{13}{12},\frac{17}{12})$	$\mathcal{T}[W_{-9+\frac{8}{2}}(\mathfrak{su}(9),(2^3,1^3))]+2\mathfrak{u}(1)$

Table 20. A_{n-1}, $b=n-1$.

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n	k	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	-1	$(3,3)$	$(2^2,1^2)$	10	6	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{\frac{1}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathcal{T}[W_{-4+\frac{3}{2}}(D_3,(2^2,1^2))] + \mathfrak{u}(1)$
	0	$(3,1^3)$	$(3,1^3)$	8	8	3	2	$\mathfrak{su}(2)_1 \times \mathfrak{u}(1)^3_{f_0}$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{3}{3}}(D_3,(3,1^3))] + 3\mathfrak{u}(1)$
4	-3	$(7,1)$	(1^8)	24	0	0	1	$\mathfrak{spin}(8)_2$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-6+\frac{4}{1}}(D_4,(1^8))]$
	-2	$(3^2,1^2)$	$(3^2,1^2)$	18	18	0	3	$\mathfrak{u}(1)_0 \times \mathfrak{u}(1)_0$	1	$(\frac{5}{24},\frac{1}{6})$	$\mathcal{T}[W_{-6+\frac{4}{2}}(D_4,(3^2,1^2))]$
	-2	$(3^2,1^2)$	$(3,2^2,1)$	18	16	0	3	$\mathfrak{su}(2)_{\frac{3}{2}}$	2	$(\frac{3}{4},\frac{3}{4})$	$\mathcal{T}[W_{-6+\frac{4}{2}}(D_4,(3,2^2,1))]$
	-1	(2^4)	(4^2)	12	20	0	1	$\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-6+\frac{4}{3}}(D_4,(4^2))]$
	-1	$(3,1^5)$	$(5,1^3)$	12	20	0	1	$\mathfrak{so}(3)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-6+\frac{4}{3}}(D_4,(5,1^3))]$
	0	$(2^2,1^4)$	$(5,3)$	10	22	4	3	$\mathfrak{u}(1)^4_{f_0}$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-6+\frac{4}{4}}(D_4,(5,3))] + 4\mathfrak{u}(1)$
	1	(1^8)	$(7,1)$	0	24	0	1	–	$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-6+\frac{4}{5}}(D_4,(7,1))]$
5	-4	$(9,1)$	(1^{10})	40	0	1	0	$\mathfrak{spin}(10)_3 \times \mathfrak{u}(1)_{f_0}$	3	$(\frac{41}{24},\frac{13}{6})$	$\mathcal{T}[W_{-8+\frac{5}{1}}(D_5,(1^{10}))] + \mathfrak{u}(1)$
	-3	$(4^2,1^2)$	$(3^2,2^2)$	32	28	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathcal{T}[W_{-8+\frac{5}{2}}(D_5,(3^2,2^2))] + \mathfrak{u}(1)$
	-1	$(3,1^7)$	$(7,1^3)$	16	36	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-8+\frac{5}{4}}(D_5,(7,1^3))] + \mathfrak{u}(1)$
	1	(1^{10})	$(9,1)$	0	40	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-8+\frac{5}{6}}(D_5,(9,1))] + \mathfrak{u}(1)$
6	-5	$(9,3)$	$(2^2,1^8)$	58	18	0	1	$\mathfrak{spin}(8)_2 \times \mathfrak{su}(2)_0$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-10+\frac{6}{1}}(D_6,(2^2,1^8))]$
	-4	$(5,3^2,1)$	$(3^3,1^3)$	50	42	2	1	$\mathfrak{su}(2)_1 \times \mathfrak{su}(2)_1 \times \mathfrak{u}(1)^2_{f_0}$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-10+\frac{6}{2}}(D_6,(3^3,1^3))] + 2\mathfrak{u}(1)$
	-2	(2^6)	(6^2)	30	54	2	1	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8},\frac{3}{4})$	$\mathcal{T}[W_{-10+\frac{6}{4}}(D_6,(6^2))] + 2\mathfrak{u}(1)$
	-1	$(2^2,1^8)$	$(9,3)$	18	58	0	1	–	$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-10+\frac{6}{5}}(D_6,(9,3))]$

Table 21. D_n, $b = n$.

n	k	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	-2	$(5,1)$	(1^6)	12	0	1	1	$\mathfrak{su}(4)_2 \times \mathfrak{u}(1)_{f_0}$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{2}}(D_3,(1^6))] + \mathfrak{u}(1)$
	-1	$(3,1^3)$	$(3,1^3)$	8	8	0	0	$\mathfrak{so}(3)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(D_3,(3,1^3))]$
	2	(1^6)	$(5,1)$	0	12	1	1	$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-4+\frac{4}{6}}(D_3,(5,1))] + \mathfrak{u}(1)$
4	-4	$(7,1)$	(1^8)	24	0	2	0	$\mathfrak{spin}(8)_3 \times \mathfrak{u}(1)^2_{f_0}$	3	$(\frac{41}{24},\frac{13}{6})$	$\mathcal{T}[W_{-6+\frac{6}{2}}(D_4,(1^8))] + 2\mathfrak{u}(1)$
	-1	$(2^2,1^4)$	$(5,3)$	10	22	0	0	–	$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-6+\frac{6}{5}}(D_4,(5,3))]$
	2	(1^8)	$(7,1)$	0	24	2	0	$\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-6+\frac{6}{8}}(D_4,(7,1))] + 2\mathfrak{u}(1)$
5	-6	$(7,3)$	$(2^2,1^6)$	38	14	1	1	$\mathfrak{su}(4)_2 \times \mathfrak{su}(2)_1 \times \mathfrak{u}(1)_{f_0}$	2	$(\frac{25}{24},\frac{4}{3})$	$\mathcal{T}[W_{-8+\frac{8}{2}}(D_5,(2^2,1^6))] + \mathfrak{u}(1)$
	-5	$(4^2,1^2)$	$(3^2,2^2)$	32	28	0	0	$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{u}(1)_0$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(D_5,(3^2,2^2))]$
	-4	$(3^2,2^2)$	$(4^2,1^2)$	28	32	1	3	$\mathfrak{su}(2)_1 \times \mathfrak{u}(1)_0 \times \mathfrak{u}(1)_{f_0}$	1	$(\frac{7}{24},\frac{1}{3})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(D_5,(4^2,1^2))] + \mathfrak{u}(1)$
6	-8	$(9,3)$	$(2^2,1^8)$	58	18	2	0	$\mathfrak{spin}(8)_3 \times \mathfrak{su}(2)_2 \times \mathfrak{u}(1)^2_{f_0}$	3	$(\frac{43}{24},\frac{7}{3})$	$\mathcal{T}[W_{-10+\frac{10}{2}}(D_6,(2^2,1^8))] + 2\mathfrak{u}(1)$
	-7	$(5,3^2,1)$	$(3^3,1^3)$	50	42	0	0	$\mathfrak{so}(3)_{\frac{2}{3}} \times \mathfrak{so}(3)_0$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6,(3^3,1^3))]$

Table 22. D_n, $b = 2n-2$.

b	k	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-7	A_2+2A_1	A_4+A_1	50	62	0	0	$\mathfrak{u}(1)_0$	$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-12+\frac{12}{5}}(\mathfrak{e}_6,A_4+A_1)]$
12	-10	D_5	$2A_1$	68	32	0	1	$\mathfrak{so}(7)_2 \times \mathfrak{u}(1)_0$	2	$(\frac{23}{24},\frac{7}{6})$	$\mathcal{T}[W_{-12+\frac{12}{2}}(\mathfrak{e}_6,2A_1)]$
12	-9	$D_5(a_1)$	A_2+A_1	64	46	2	1	$\mathfrak{su}(3)_2 \times \mathfrak{u}(1)_3 \times \mathfrak{u}(1)^2_{f_0}$	2	$(\frac{13}{12},\frac{17}{12})$	$\mathcal{T}[W_{-12+\frac{12}{3}}(\mathfrak{e}_6,A_2+A_1)] + 2\mathfrak{u}(1)$
12	-4	$2A_1$	D_5	32	68	0	2	$\mathfrak{u}(1)_0$	1	$(\frac{5}{24},\frac{1}{6})$	$\mathcal{T}[W_{-12+\frac{12}{8}}(\mathfrak{e}_6,D_5)]$
12	-6	A_2+A_1	$D_5(a_1)$	46	64	2	3	$\mathfrak{u}(1)_3 \times \mathfrak{u}(1)^2_{f_0}$	1	$(\frac{1}{3},\frac{5}{12})$	$\mathcal{T}[W_{-12+\frac{12}{6}}(\mathfrak{e}_6,D_5(a_1))] + 2\mathfrak{u}(1)$
9	-8	E_6	0	72	0	0	0	$(\mathfrak{e}_6)_3$	3	$(\frac{41}{24},\frac{13}{6})$	$\mathcal{T}[W_{-3}(\mathfrak{e}_6,0)]$
	-7	$D_5(a_1)$	A_2+A_1	64	46			$\mathfrak{su}(3)_{\frac{3}{2}} \times \mathfrak{u}(1)_0$	$\frac{3}{2}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-12+\frac{9}{2}}(\mathfrak{e}_6,A_2+A_1)]$
	-5	$2A_2$	D_4	48	60			$\mathfrak{su}(3)_{\frac{3}{2}}$	$\frac{3}{2}$	$(\frac{7}{12},\frac{2}{3})$	$\mathcal{T}[W_{-12+\frac{9}{4}}(\mathfrak{e}_6,D_4)]$
	1	0	E_6	0	72			–	$\frac{6}{5}$	$(\frac{43}{120},\frac{11}{30})$	$\mathcal{T}[W_{-12+\frac{9}{10}}(\mathfrak{e}_6,E_6)]$
8	-7	E_6	0	72	0	1	0	$(\mathfrak{e}_6)_4 \times \mathfrak{u}(1)_{f_0}$	4	$(\frac{59}{24},\frac{19}{6})$	$\mathcal{T}[W_{-4}(\mathfrak{e}_6,0)] + \mathfrak{u}(1)$
	1	0	E_6	0	72			$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24},\frac{1}{2})$	$\mathcal{T}[W_{-12+\frac{8}{9}}(\mathfrak{e}_6,E_6)] + \mathfrak{u}(1)$

Table 23. E_6.

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b	k	HL	NL	HD	ND	(f_0, f_1)	$\mathfrak{f}$	Δ_{Coulomb}	(a, c)	Theory
18	-13	$A_3+A_2+A_1$	A_4+A_2	100	106	$(0,0)$	$\mathfrak{su}(2)_0$	$\frac{6}{5}$	$(\frac{43}{120}, \frac{11}{30})$	$\mathcal{T}[W_{-18+\frac{18}{5}}(\mathfrak{e}_7, A_4+A_2)]$
18	-16	$E_7(a_2)$	$2A_1$	122	52	$(1,0)$	$\mathfrak{so}(9)_3 \times \mathfrak{su}(2)_1 \times \mathfrak{u}(1)_{f_0}$	3	$(\frac{43}{24}, \frac{7}{3})$	$\mathcal{T}[W_{-18+\frac{18}{2}}(\mathfrak{e}_7, 2A_1)] + \mathfrak{u}(1)$
	-14	$D_5(a_1)+A_1$	A_3+A_2	108	98		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{\frac{1}{6}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{2}{3}, \frac{5}{6})$	$\mathcal{T}[W_{-18+\frac{18}{4}}(\mathfrak{e}_7, A_3+A_2)] + \mathfrak{u}(1)$
18	-15	A_6	A_2+3A_1	114	84	$(0,2)$	$(\mathfrak{g}_2)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7, A_2+3A_1)]$
	-15	D_5+A_1	$2A_2$	114	84		$(\mathfrak{g}(2))_2 \times \mathfrak{su}(2)_0$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7, 2A_2)]$
	-15	$D_6(a_1)$	A_3	114	84		$\mathfrak{so}(7)_2 \times \mathfrak{su}(2)_0$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7, A_3)]$
18	-12	$D_4(a_1)$	$E_7(a_5)$	94	112	$(1,2)$	$\mathfrak{u}(1)_{f_0}$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-18+\frac{18}{6}}(\mathfrak{e}_7, E_7(a_5))] + \mathfrak{u}(1)$
	-12	$(D_4(a_1), \mathbb{Z}_2)$	$D_6(a_2)$	94	110		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	2	$(\frac{3}{4}, \frac{3}{4})$	$\mathcal{T}[W_{-18+\frac{18}{6}}(\mathfrak{e}_7, D_6(a_2))] + \mathfrak{u}(1)$
	-6	$2A_1$	$E_7(a_2)$	52	122		$\mathfrak{u}(1)_{f_0}$	1	$(\frac{1}{4}, \frac{1}{4})$	$\mathcal{T}[W_{-18+\frac{18}{12}}(\mathfrak{e}_7, E_7(a_2))] + \mathfrak{u}(1)$
14	-13	E_7	0	126	0	$(0,0)$	$(\mathfrak{e}_7)_4$	4	$(\frac{59}{24}, \frac{19}{6})$	$\mathcal{T}[W_{-4}(\mathfrak{e}_7, 0)]$
	-11	$D_5(a_1)+A_1$	A_3+A_2	108	98		$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{u}(1)_0$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7, A_3+A_2)]$
	-5	$(3A_1)''$	E_6	54	120		$\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-18+\frac{14}{9}}(\mathfrak{e}_7, E_6)]$
	1	0	E_7	0	126		–	$\frac{6}{5}$	$(\frac{43}{120}, \frac{11}{30})$	$\mathcal{T}[W_{-18+\frac{14}{15}}(\mathfrak{e}_7, E_7)]$
14	-12	$E_6(a_1)$	A_2+A_1	118	76	$(1,0)$	$\mathfrak{su}(4)_2 \times \mathfrak{u}(1)_0 \times \mathfrak{u}(1)_{f_0}$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7, A_2+A_1)] + \mathfrak{u}(1)$
	-12	$(E_6(a_1), \mathbb{Z}_2)$	$4A_1$	118	70		$\mathfrak{sp}(3)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	4	$(\frac{25}{12}, \frac{29}{12})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7, 4A_1)] + \mathfrak{u}(1)$
	-10	A_4	$D_5(a_1)$	100	106		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_0 \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$\mathcal{T}[W_{-18+\frac{14}{4}}(\mathfrak{e}_7, D_5(a_1))] + \mathfrak{u}(1)$
	-10	$A_3+A_2+A_1$	A_4+A_2	100	106		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{5}{6})$	$\mathcal{T}[W_{-18+\frac{14}{4}}(\mathfrak{e}_7, A_4+A_2)] + \mathfrak{u}(1)$
	-8	A_2+2A_1	$E_7(a_4)$	82	116		$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-18+\frac{14}{6}}(\mathfrak{e}_7, E_7(a_4))] + \mathfrak{u}(1)$

Table 24. E_7.

b	k	Hitchin label	Nahm label	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a, c)	Theory
30	-28	$E_8(a_3)$	A_2	234	0	0	$(\mathfrak{e}_6)_3$	3	$(\frac{41}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-30+\frac{30}{2}}(\mathfrak{e}_8, A_2)]$
	-28	$(E_8(a_3), \mathbb{Z}_2)$	$3A_1$	234			$(\mathfrak{f}_4)_3 \times \mathfrak{su}(2)_{\frac{1}{2}}$	3	$(\frac{7}{4}, \frac{9}{4})$	$\mathcal{T}[W_{-30+\frac{30}{2}}(\mathfrak{e}_8, 3A_1)]$
	-26	$E_6(a_1)+A_1$	A_4+A_1	218			$\mathfrak{su}(3)_{\frac{3}{2}} \times \mathfrak{u}(1)_0$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$\mathcal{T}[W_{-30+\frac{30}{4}}(\mathfrak{e}_8, A_4+A_1)]$
30	-27	$E_8(b_5)$	$D_4(a_1)$	226	0	1	$\mathfrak{so}(8)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8, D_4(a_1))]$
	-27	$(E_8(b_5, S_3))$	$2A_2+A_1$	226			$(\mathfrak{g}_2)_2 \times \mathfrak{su}(2)_1$	2	$(\frac{25}{24}, \frac{4}{3})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8, 2A_2+A_1)]$
	-27	$(E_8(b_5, \mathbb{Z}_2))$	A_3+A_1	226			$\mathfrak{so}(7)_2 \times \mathfrak{su}(2)_{\frac{1}{2}}$	2	$(1, \frac{5}{4})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8, A_3+A_1)]$
	-21	A_3+A_2	$D_7(a_1)$	178			$\mathfrak{u}(1)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-30+\frac{30}{9}}(\mathfrak{e}_8, D_7(a_1))]$
30	-25	A_6	D_4+A_2	210	0	3	$\mathfrak{su}(3)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8, D_4+A_2)]$
	-25	$D_6(a_1)$	$E_6(a_3)$	210			$(\mathfrak{g}_2)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8, E_6(a_3))]$
	-25	$(D_6(a_1), \mathbb{Z}_2)$	A_5	210			$(\mathfrak{g}_2)_2 \times \mathfrak{su}(2)_{\frac{1}{2}}$	2	$(1, \frac{5}{4})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8, A_5)]$
30	-6	A_1	$E_8(a_1)$	58	0	1	–	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-30+\frac{30}{24}}(\mathfrak{e}_8, E_8(a_1))]$
24	-23	E_8	0	240	0	0	$(\mathfrak{e}_8)_6$	6	$(\frac{95}{24}, \frac{31}{6})$	$\mathcal{T}[W_{-30+\frac{24}{1}}(\mathfrak{e}_8, 0)]$
	1	0	E_8	0			–	$\frac{6}{5}$	$(\frac{43}{120}, \frac{11}{30})$	$\mathcal{T}[W_{-30+\frac{24}{25}}(\mathfrak{e}_8, E_8)]$
24	-22	$E_8(b_4)$	A_2+2A_1	230	0	1	$\mathfrak{so}(7)_2 \times \mathfrak{su}(2)_0$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{24}{2}}(\mathfrak{e}_8, A_2+2A_1)]$
24	-21	$E_7(a_3)$	A_4	220	0	1	$\mathfrak{su}(5)_2$	2	$(\frac{43}{24}, \frac{11}{6})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8, A_4)]$
	-21	$E_8(b_6)$	$D_4(a_1)+A_2$	220			$\mathfrak{su}(3)_0$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8, D_4(a_1)+A_2)]$
	-21	$(E_8(b_6), \mathbb{Z}_2)$	$A_3+A_2+A_1$	220			$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{su}(2)_0$	2	$(\frac{3}{4}, \frac{3}{4})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8, A_3+A_2+A_1)]$
20	-19	$E_8(a_1)$	A_1	238	0	0	$(\mathfrak{e}_7)_4$	4	$(\frac{59}{24}, \frac{19}{6})$	$\mathcal{T}[W_{-30+20}(\mathfrak{e}_8, A_1)]$
	-17	D_5+A_2	A_4+A_2	214			$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{su}(2)_0$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-30+\frac{20}{3}}(\mathfrak{e}_8, A_4+A_2)]$

Table 25. E_8.

n	k_t	C-partition	B-partition	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a, c)	Theory
1	1	(2)	(1^3)	2	0	0	0	$\mathfrak{so}(3)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$W_{-1+\frac{1}{3}}(B_1, (1^3))$
	3	(1^2)	(3)	0	2	0	0	–	$\frac{6}{5}$	$(\frac{43}{120}, \frac{11}{30})$	$W_{-1+\frac{1}{5}}(B_1, (3))$
2	-3	(2^2)	$(3, 1^2)$	6	6	0	2	$\mathfrak{u}(1)_0$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-3+1}(B_2, (3, 1^2))]$
	-3	$((2^2), \mathbb{Z}_2)$	$(2^2, 1)$	6	4	0	2	$\mathfrak{su}(2)_{\frac{3}{2}}$	2	$(\frac{3}{4}, \frac{3}{4})$	$\mathcal{T}[W_{-3+1}(B_2, (2^2, 1))]$
3	-7	(3^2)	$(3, 2^2)$	14	12	0	0	$\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(B_3, (3, 2^2))]$
4	-11	$(4, 2^2)$	$(3^2, 1^3)$	26	22	0	0	$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{u}(1)_0$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(B_4, (3^2, 1^3))]$
5	-15	$(4^2, 2)$	$(3^3, 1^2)$	42	36	0	2	$\mathfrak{su}(2)_0 \times \mathfrak{u}(1)_0$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-9+3}(B_5, (3^3, 1^2))]$
	-15	$((4^2, 2), \mathbb{Z}_2)$	$(3^2, 2^2, 1)$	42	34	0	2	$\mathfrak{su}(2)_1 \times \mathfrak{u}(1)_0$	2	$(\frac{3}{4}, \frac{3}{4})$	$\mathcal{T}[W_{-9+3}(B_5, (3^2, 2^2, 1))]$

Table 26. $A_{2n-1}/\mathbb{Z}_2$, $b_t = 4n-2$.

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n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	2	(2)	(1^3)	2	0	1	0	$\mathfrak{so}(3)_{\frac{3}{4}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$W_{-1+\frac{1}{4}}(B_1, (1^3))$
	4	(1^2)	(3)	0	2	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$W_{-1+\frac{1}{6}}(B_1, (3))$
3	-5	(6)	(1^7)	18	0	0	1	$\mathfrak{so}(7)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-5+3}(B_3, (1^7))]$
	-4	(3^2)	$(3, 2^2)$	14	12	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8}, \frac{3}{4})$	$\mathcal{T}[W_{-5+\frac{3}{2}}(B_3, (3, 2^2))] + \mathfrak{u}(1)$
4	-7	(8)	(1^9)	32	0	1	0	$\mathfrak{so}(9)_3 \times \mathfrak{u}(1)_{f_0}$	3	$(\frac{41}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-7+4}(B_4, (1^9))] + \mathfrak{u}(1)$
	-6	$(4, 2^2)$	$(3^2, 1^3)$	26	22	2	1	$\mathfrak{so}(3)_1 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^2_{f_0}$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-7+2}(B_4, (3^2, 1^3))] + 2\mathfrak{u}(1)$
5	-9	$(8, 2)$	$(3, 1^8)$	48	18	0	1	$\mathfrak{so}(8)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-9+5}(B_5, (3, 1^8))]$
	-9	$((8, 2), \mathbb{Z}_2)$	$(2^2, 1^7)$	48	16	0	1	$\mathfrak{so}(7)_2 \times \mathfrak{su}(2)_{\frac{1}{2}}$	2	$(1, \frac{5}{4})$	$\mathcal{T}[W_{-9+5}(B_5, (2^2, 1^7))]$
	-8	$(4^2, 1^2)$	$(5, 2^2, 1^2)$	40	38	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{5}{8}, \frac{3}{4})$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5, (5, 2^2, 1^2))] + \mathfrak{u}(1)$
	-7	$(3^2, 1^4)$	$(7, 2^2)$	30	44	0	1	$\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-9+\frac{5}{3}}(B_5, (7, 2^2))]$
	-7	(2^5)	$(5^2, 1)$	30	44	0	1	$\mathfrak{u}(1)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-9+\frac{5}{3}}(B_5, (5^2, 1))]$

Table 27. $A_{2n-1}/\mathbb{Z}_2$, $b_t = 2n$.

n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	-1	(3)	(1^2)	2	0	0	0	$\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$W_{-2+\frac{2}{3}}(C_1, (1^2))$
	1	(1^3)	(2)	0	2	0	0	$-$	$\frac{6}{5}$	$(\frac{43}{120}, \frac{11}{30})$	$W_{-2+\frac{2}{5}}(C_1, (2))$
3	-5	$(3^2, 1)$	(2^3)	14	12	0	0	$\mathfrak{so}(3)_{\frac{2}{3}}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(C_3, (2^3)]$
4	-7	$(5, 2^2)$	$(3^2, 1^2)$	26	22	0	0	$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{su}(2)_0$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(C_4, (3^2, 1^2)]$
5	-9	$(5, 3^2)$	$(3^2, 2^2)$	42	36	0	2	$\mathfrak{su}(2)_0 \times \mathfrak{u}(1)$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-6+2}(C_5, (3^2, 2^2)]$
	-9	$((5, 3^2), \mathbb{Z}_2)$	$(3^2, 2, 1^2)$	42	34	0	2	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{su}(2)_0$	2	$(\frac{3}{4}, \frac{3}{4})$	$\mathcal{T}[W_{-6+2}(C_5, (3^2, 2, 1^2)]$

Table 28. $D_{n+1}/\mathbb{Z}_2$, $b_t = 2n + 2$.

n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	0	(3)	(1^2)	2	0	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$W_{-2+\frac{1}{2}}(C_1, (1^2)) + M(1)$
3	-4	$(5, 1^2)$	$(2^2, 1^2)$	16	10	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_0 \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$\mathcal{T}[W_{-4+\frac{3}{2}}(C_3, (2^2, 1^2)] + \mathfrak{u}(1)$
	-4	$((5, 1^2), \mathbb{Z}_2)$	$(2, 1^4)$	16	6	1	0	$\mathfrak{sp}(2)_2 \times \mathfrak{u}(1)_{f_0}$	3	$(\frac{17}{12}, \frac{19}{12})$	$\mathcal{T}[W_{-4+\frac{3}{2}}(C_3, (2, 1^4)] + \mathfrak{u}(1)$
5	-8	$(7, 3, 1)$	$(2^4, 1^2)$	46	28	1	0	$\mathfrak{so}(4)_0 \times \mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}$	$(\frac{7}{12}, \frac{2}{3})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(C_5, (2^4, 1^2)] + \mathfrak{u}(1)$
	-8	$((7, 3, 1), \mathbb{Z}_2)$	$(2^3, 1^4)$	46	24	1	0	$\mathfrak{sp}(2)_2 \times \mathfrak{so}(3)_0 \times \mathfrak{u}(1)_{f_0}$	3	$(\frac{17}{12}, \frac{19}{12})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(C_5, (2^3, 1^4)] + \mathfrak{u}(1)$

Table 29. $D_{n+1}/\mathbb{Z}_2$, $b_t = 2n$.

b_t	k_t	HL	NL	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-10	G_2	0	12	0	1	$(\mathfrak{g}_2)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-4+2}(\mathfrak{g}_2, 0)]$
	-4	0	G_2	0	0	1	$-$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-4+\frac{1}{2}}(\mathfrak{g}_2, G_2)]$

Table 30. $D_4/\mathbb{Z}_3$, $b_t = 12$, $b_t = 6$.

b_t	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-11	F_4	0	48	0	0	0	$(\mathfrak{f}_4)_3$	3	$(\frac{41}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-9+6}(\mathfrak{f}_4, 0)]$
	-10	B_3	A_2	42	30	0	2	$\mathfrak{su}(3)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-9+3}(\mathfrak{f}_4, A_2)]$
	-10	C_3	$\tilde{A}_2$	42	30	0	2	$(\mathfrak{g}_2)_2$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-9+3}(\mathfrak{f}_4, \tilde{A}_2)]$
	-8	A_2	B_3	30	42	0	1	$\mathfrak{su}(2)_0$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-9+\frac{3}{2}}(\mathfrak{f}_4, B_3)]$
	-8	$\tilde{A}_2$	C_3	30	42	0	1	$\mathfrak{su}(2)_{\frac{3}{2}}$	$\frac{3}{2}$	$(\frac{23}{24}, \frac{11}{12})$	$\mathcal{T}[W_{-9+\frac{3}{2}}(\mathfrak{f}_4, C_3)]$
	-3	0	F_4	0	48	0	1	$-$	1	$(\frac{5}{24}, \frac{1}{6})$	$\mathcal{T}[W_{-9+\frac{2}{3}}(\mathfrak{f}_4, F_4)]$
8	-7	$F_4(a_1)$	$\tilde{A}_1$	46	22	1	0	$\mathfrak{su}(4)_2 \times \mathfrak{u}(1)_{f_0}$	2	$(\frac{23}{24}, \frac{7}{6})$	$\mathcal{T}[W_{-9+4}(\mathfrak{f}_4, \tilde{A}_1)] + \mathfrak{u}(1)$
	-7	$(F_4(a_1), \mathbb{Z}_2)$	A_1	46	16	1	0	$\mathfrak{sp}(3)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	4	$(\frac{25}{12}, \frac{29}{12})$	$\mathcal{T}[W_{-9+4}(\mathfrak{f}_4, A_1)] + \mathfrak{u}(1)$
	-5	$A_1 + \tilde{A}_1$	$F_4(a_2)$	28	44	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}$	$(\frac{11}{24}, \frac{1}{2})$	$\mathcal{T}[W_{-9+\frac{4}{3}}(\mathfrak{f}_4, F_4(a_2))] + \mathfrak{u}(1)$

Table 31. $E_6/\mathbb{Z}_2$.

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n	k	$\mathcal{O}_f$	f_0	f_1	ND	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
2	3	(1^2)	0	0	0	$\mathfrak{su}(2)_{\frac{8}{5}}$	$\frac{8}{5},\frac{6}{5}$	$(\frac{19}{20},1)$	$\mathcal{T}[W_{-2+\frac{2}{5}}(\mathfrak{su}(2),(1^2))]$
	4	(1^2)	1	1	0	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-2+\frac{2}{6}}(\mathfrak{su}(2),(1^2))]+\mathfrak{u}(1)$
	5	(2)	0	0	2	–	$\frac{10}{7},\frac{8}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-2+\frac{2}{7}}(\mathfrak{su}(2),(2))]$
	6	(2)	1	1	2	$\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-2+\frac{2}{8}}(\mathfrak{su}(2),(2))]+\mathfrak{u}(1)$
3	1	$(2,1)$	0	0	4	$\mathfrak{u}(1)_{\frac{3}{2}}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-3+\frac{3}{4}}(\mathfrak{su}(3),(2,1))]$
4	-1	$(2,1^2)$	0	0	6	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{\frac{4}{3}}$	$\frac{4}{3},\frac{5}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(\mathfrak{su}(4),(2,1^2))]$
	0	$(2,1^2)$	3	3	6	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}^3$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-4+\frac{4}{4}}(\mathfrak{su}(4),(2,1^2))]+3\mathfrak{u}(1)$
5	-3	(1^5)	0	0	0	$\mathfrak{su}(5)_{\frac{5}{2}}$	$\frac{5}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-5+\frac{5}{2}}(\mathfrak{su}(5),(1^5))]$
	-2	$(2^2,1)$	0	0	12	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{\frac{5}{3}}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(\mathfrak{su}(5),(2^2,1))]$
	-1	$(3,2)$	0	0	16	$\mathfrak{u}(1)_{\frac{5}{4}}$	$\frac{5}{4},\frac{3}{2}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-5+\frac{5}{4}}(\mathfrak{su}(5),(3,2))]$
	0	$(3,2)$	4	4	16	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}^4$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-5+\frac{5}{5}}(\mathfrak{su}(5),(3,2))]+4\mathfrak{u}(1)$
	2	(5)	0	0	20	–	$\frac{10}{7},\frac{8}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-5+\frac{5}{7}}(\mathfrak{su}(5),(5))]$
6	-4	(1^6)	1	1	0	$\mathfrak{su}(6)_3\times\mathfrak{u}(1)_{f_0}$	$3,2$	$(\frac{29}{12},\frac{17}{6})$	$\mathcal{T}[W_{-6+\frac{6}{2}}(\mathfrak{su}(6),(1^6))]+\mathfrak{u}(1)$
	-3	$(2^2,1^2)$	2	2	16	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}^2$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-6+\frac{6}{3}}(\mathfrak{su}(6),(2^2,1^2))]+2\mathfrak{u}(1)$
	2	(6)	1	1	30	$\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-6+\frac{6}{8}}(\mathfrak{su}(6),(6))]+\mathfrak{u}(1)$
7	-5	$(2,1^5)$	0	0	12	$\mathfrak{su}(5)_{\frac{5}{2}}\times\mathfrak{u}(1)_0$	$\frac{5}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-7+\frac{7}{2}}(\mathfrak{su}(7),(2,1^5))]$
	-4	$(3,2,1^2)$	0	0	28	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{\frac{4}{3}}\times\mathfrak{u}(1)_0$	$\frac{4}{3},\frac{5}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(\mathfrak{su}(7),(3,2,1^2))]$
	-3	$(4,2,1)$	0	0	34	$\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)_0$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-7+\frac{7}{4}}(\mathfrak{su}(7),(4,2,1))]$
	-2	$(5,1^2)$	0	0	36	$\mathfrak{su}(2)_{\frac{8}{5}}\times\mathfrak{u}(1)_0$	$\frac{6}{5},\frac{8}{5}$	$(\frac{19}{20},1)$	$\mathcal{T}[W_{-7+\frac{7}{5}}(\mathfrak{su}(7),(5,1^2))]$
8	-6	$(2,1^6)$	1	1	14	$\mathfrak{su}(6)_3\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$3,2$	$(\frac{29}{12},\frac{17}{6})$	$\mathcal{T}[W_{-8+\frac{8}{2}}(\mathfrak{su}(8),(2,1^6))]+\mathfrak{u}(1)$
	-5	$(3,2^2,1)$	0	0	38	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{\frac{10}{3}}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(\mathfrak{su}(8),(3,2^2,1))]$
	-4	$(4,2,1^2)$	3	3	42	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}^3$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-8+\frac{8}{4}}(\mathfrak{su}(8),(4,2,1^2))]+3\mathfrak{u}(1)$
	-4	$(3^2,2)$	3	3	42	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}^3$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-8+\frac{8}{4}}(\mathfrak{su}(8),(3^2,2))]+3\mathfrak{u}(1)$
	-3	(4^2)	0	0	48	$\mathfrak{su}(2)_{\frac{8}{5}}$	$\frac{8}{5},\frac{6}{5}$	$(\frac{19}{20},1)$	$\mathcal{T}[W_{-8+\frac{8}{5}}(\mathfrak{su}(8),(4^2))]$
	-2	$(6,1^2)$	1	1	50	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{5}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-8+\frac{8}{6}}(\mathfrak{su}(8),(6,1^2))]+\mathfrak{u}(1)$
9	-7	$(2^2,1^5)$	0	0	28	$\mathfrak{su}(5)_{\frac{5}{2}}\times\mathfrak{su}(2)_0\times\mathfrak{u}(1)_0$	$\frac{5}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-9+\frac{9}{2}}(\mathfrak{su}(9),(2^2,1^5))]$
	-6	$(3,2^2,1^2)$	2	2	46	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)\times\mathfrak{u}(1)_{f_0}^2$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-9+\frac{9}{3}}(\mathfrak{su}(9),(3,2^2,1^2))]+2\mathfrak{u}(1)$
	-5	$(4,3,2)$	0	0	58	$\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)_0$	$\frac{5}{4},\frac{3}{2}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-9+\frac{9}{4}}(\mathfrak{su}(9),(4,3,2))]$
	-2	$(7,2)$	0	0	68	$\mathfrak{u}(1)_0$	$\frac{8}{7},\frac{10}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-9+\frac{9}{7}}(\mathfrak{su}(9),(7,2))]$

Table 32. A_{n-1}, $b=n$.

n	k	$\mathcal{O}_f$	f_0	f_1	ND	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	1	$(2,1)$	1	0	4	$\mathfrak{u}(1)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-3+\frac{2}{3}}(\mathfrak{su}(3),(2,1))]+\mathfrak{u}(1)$
4	-1	(1^4)	1	0	0	$\mathfrak{su}(4)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-4+\frac{3}{2}}(\mathfrak{su}(4),(1^4))]+\mathfrak{u}(1)$
	2	(4)	1	0	12	$\mathfrak{u}(1)_{f_0}$	$\frac{8}{5},\frac{6}{5}$	$(\frac{19}{20},1)$	$\mathcal{T}[W_{-4+\frac{3}{5}}(\mathfrak{su}(4),(4))]+\mathfrak{u}(1)$
5	-2	(1^5)	2	1	0	$\mathfrak{su}(5)_3\times\mathfrak{u}(1)_{f_0}^2$	$3,2$	$(\frac{29}{12},\frac{17}{6})$	$\mathcal{T}[W_{-5+\frac{4}{2}}(\mathfrak{su}(5),(1^5))]+2\mathfrak{u}(1)$
	-1	$(3,1^2)$	1	0	14	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{5}{3}$	$(\frac{9}{8},\frac{5}{4})$	$\mathcal{T}[W_{-5+\frac{4}{3}}(\mathfrak{su}(5),(3,1^2))]+\mathfrak{u}(1)$
	2	(5)	2	1	20	$\mathfrak{u}(1)_{f_0}^2$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-5+\frac{4}{6}}(\mathfrak{su}(5),(5))]+2\mathfrak{u}(1)$
6	-3	$(2,1^4)$	1	0	10	$\mathfrak{su}(4)_{\frac{5}{2}}\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},\frac{3}{2}$	$(\frac{43}{24},\frac{25}{12})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(\mathfrak{su}(6),(2,1^4))]+\mathfrak{u}(1)$
	-2	$(3,2,1)$	1	0	22	$\mathfrak{u}(1)_{\frac{5}{3}}\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{9}{8},\frac{5}{4})$	$\mathcal{T}[W_{-6+\frac{5}{3}}(\mathfrak{su}(6),(3,2,1))]+\mathfrak{u}(1)$
	-1	$(4,2)$	1	0	26	$\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{4},\frac{3}{2}$	$(\frac{23}{24},\frac{25}{24})$	$\mathcal{T}[W_{-6+\frac{5}{4}}(\mathfrak{su}(6),(4,2))]+\mathfrak{u}(1)$
7	-4	$(2,1^5)$	2	1	12	$\mathfrak{su}(5)_3\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}^2$	$3,2$	$(\frac{59}{24},\frac{35}{12})$	$\mathcal{T}[W_{-7+\frac{6}{2}}(\mathfrak{su}(7),(2,1^5))]+2\mathfrak{u}(1)$
	-3	$(3,2,1^2)$	3	2	28	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}^3$	$2,2$	$(\frac{43}{24},\frac{25}{12})$	$\mathcal{T}[W_{-7+\frac{6}{3}}(\mathfrak{su}(7),(3,2,1^2))]+3\mathfrak{u}(1)$
8	-5	$(2^2,1^4)$	1	0	24	$\mathfrak{su}(4)_{\frac{5}{2}}\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},\frac{3}{2}$	$(\frac{11}{6},\frac{13}{6})$	$\mathcal{T}[W_{-8+\frac{7}{2}}(\mathfrak{su}(8),(2^2,1^4))]+\mathfrak{u}(1)$
	-4	$(3^2,1^2)$	1	0	40	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{5}{3}$	$(\frac{7}{6},\frac{4}{3})$	$\mathcal{T}[W_{-8+\frac{7}{3}}(\mathfrak{su}(8),(3^2,1^2))]+\mathfrak{u}(1)$
9	-6	$(2^2,1^5)$	2	1	28	$\mathfrak{su}(5)_3\times\mathfrak{su}(2)_1\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}^2$	$3,2$	$(\frac{5}{2},3)$	$\mathcal{T}[W_{-9+\frac{8}{2}}(\mathfrak{su}(9),(2^2,1^5))]+2\mathfrak{u}(1)$
	-5	$(3^2,2,1)$	1	0	52	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{7}{6},\frac{4}{3})$	$\mathcal{T}[W_{-9+\frac{8}{3}}(\mathfrak{su}(9),(3^2,2,1))]+\mathfrak{u}(1)$
	-4	$(4,3,2)$	4	3	58	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}^4$	$2,2$	$(\frac{43}{24},\frac{25}{12})$	$\mathcal{T}[W_{-9+\frac{8}{4}}(\mathfrak{su}(9),(4,3,2))]+4\mathfrak{u}(1)$
	-3	$(5,4)$	1	0	64	$\mathfrak{u}(1)_1\times\mathfrak{u}(1)_{f_0}$	$\frac{8}{5},\frac{6}{5}$	$(\frac{119}{120},\frac{13}{12})$	$\mathcal{T}[W_{-9+\frac{8}{5}}(\mathfrak{su}(9),(5,4))]+\mathfrak{u}(1)$

Table 33. A_{n-1}, $b=n-1$.

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n	k	HL	NL	HD	ND	f_0	f_1	f	Δ_{Coulomb}	(a,c)	Theory
3	-1	$(5,1)$	(1^6)	12	0	1	0	$\mathfrak{su}(4)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{2}, \frac{3}{2}$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-4+\frac{3}{2}}(D_3, (1^6))] + \mathfrak{u}(1)$
	0	$(3,3)$	$(2^2,1^2)$	10	6	3	2	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^3_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-4+\frac{3}{3}}(D_3, (2^2,1^2))] + 3\mathfrak{u}(1)$
	2	(1^6)	$(5,1)$	0	12	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{8}{5}, \frac{6}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-4+\frac{3}{5}}(D_3, (5,1))] + \mathfrak{u}(1)$
4	-2	$(4,4)$	(2^4)	20	12	0	3	$\mathfrak{sp}(2)_2$	$2, 2$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-6+\frac{4}{2}}(D_4, (2^4))]$
	-2	$(5,1^3)$	$(3,1^5)$	20	12	0	3	$\mathfrak{sp}(2)_2$	$2, 2$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-6+\frac{4}{2}}(D_4, (3,1^5))]$
	0	(2^4)	(4^2)	12	20	4	3	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)^4_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-6+\frac{4}{4}}(D_4, (4^2))] + 4\mathfrak{u}(1)$
	0	$(3,1^5)$	$(5,1^3)$	12	20	4	3	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)^4_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-6+\frac{4}{4}}(D_4, (5,1^3))] + 4\mathfrak{u}(1)$
5	-3	$(5,3,1^2)$	$(3^2,1^4)$	34	26	1	0	$\mathfrak{su}(2)^2_{\frac{3}{2},\frac{3}{2}} \times \mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{7}{6}, \frac{4}{3})$	$\mathcal{T}[W_{-8+\frac{5}{2}}(D_5, (3^2,1^4))] + \mathfrak{u}(1)$
	-3	$(5,3,1^2)$	$(3,2^2,1^3)$	34	24	1	0	$\mathfrak{su}(2)_2 \times \mathfrak{so}(3)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, 3$	$(\frac{47}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-8+\frac{5}{2}}(D_5, (3,2^2,1^3))] + \mathfrak{u}(1)$
	-2	$(3^2,1^4)$	$(5,3,1^2)$	26	34	1	0	$\mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-8+\frac{5}{3}}(D_5, (5,3,1^2))] + \mathfrak{u}(1)$
	-2	$(3^2,1^4)$	$(5,2^2,1)$	26	32	1	0	$\mathfrak{su}(2)_{\frac{11}{6}} \times \mathfrak{u}(1)_{f_0}$	$\frac{8}{3}, \frac{4}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-8+\frac{5}{3}}(D_5, (5,2^2,1))] + \mathfrak{u}(1)$
	0	$(2^2,1^6)$	$(7,3)$	14	38	5	4	$\mathfrak{u}(1)^5_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-8+1}(D_5, (7,3))] + 5\mathfrak{u}(1)$
6	-5	$(11,1)$	(1^{12})	60	0	0	1	$\mathfrak{spin}(12)_4$	$2, 4$	$(\frac{37}{12}, \frac{11}{3})$	$\mathcal{T}[W_{-10+6}(D_6, (1^{12}))]$
	-4	$(5^2,1^2)$	$(3^2,2^2,1^2)$	52	40	2	1	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)^2 \times \mathfrak{u}(1)^2_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-10+3}(D_6, (3^2,2^2,1^2))] + 2\mathfrak{u}(1)$
	-4	$(5^2,1^2)$	$(3,2^4,1)$	52	36	2	1	$\mathfrak{sp}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$2, 4$	$(\frac{17}{6}, \frac{19}{6})$	$\mathcal{T}[W_{-10+3}(D_6, (3,2^4,1))] + 2\mathfrak{u}(1)$
	-3	$(3^2,2^2,1^2)$	$(5^2,1^2)$	40	52	0	5	$\mathfrak{u}(1)_0 \times \mathfrak{u}(1)_0$	$1, 1$	$(\frac{5}{12}, \frac{1}{3})$	$\mathcal{T}[W_{-10+2}(D_6, (5^2,1^2))]$
	-1	$(3,1^9)$	$(9,1^3)$	20	56	0	1	$\mathfrak{so}(3)_{\frac{4}{5}}$	$\frac{8}{5}, \frac{6}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-10+\frac{6}{5}}(D_6, (9,1^3))]$
	1	(1^{12})	$(11,1)$	0	60	0	1	–	$\frac{8}{7}, \frac{10}{7}$	$(\frac{67}{84}, \frac{17}{21})$	$\mathcal{T}[W_{-10+\frac{6}{7}}(D_6, (11,1))]$

Table 34. D_n, $b = n$.

n	k	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	-1	$(3,3)$	$(2^2,1^2)$	10	6	0	0	$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)_{\frac{2}{3}}$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(D_3, (2^2,1^2))]$
4	-2	$(3^2,1^2)$	$(3^2,1^2)$	18	18	2	0	$\mathfrak{u}(1) \times \mathfrak{u}(1) \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{7}{6}, \frac{4}{3})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(D_4, (3^2,1^2))] + 2\mathfrak{u}(1)$
	-2	$(3^2,1^2)$	$(3,2^2,1)$	18	16	2	0	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)^2_{f_0}$	$3, \frac{3}{2}$	$(\frac{47}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(D_4, (3,2^2,1))] + 2\mathfrak{u}(1)$
	-1	(2^4)	(4^2)	12	20	0	0	$\mathfrak{su}(2)_{\frac{8}{5}}$	$\frac{6}{5}, \frac{8}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-6+\frac{6}{5}}(D_4, (4^2))]$
	-1	$(3,1^5)$	$(5,1^3)$	12	20	0	0	$\mathfrak{so}(3)_{\frac{4}{5}}$	$\frac{8}{5}, \frac{6}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-6+\frac{6}{5}}(D_4, (5,1^3))]$
5	-6	$(9,1)$	(1^{10})	40	0	1	1	$\mathfrak{spin}(10)_4 \times \mathfrak{u}(1)_{f_0}$	$2, 4$	$(\frac{37}{12}, \frac{11}{3})$	$\mathcal{T}[W_{-8+\frac{8}{2}}(D_5, (1^{10}))] + \mathfrak{u}(1)$
	-5	$(5,3,1^2)$	$(3^2,1^4)$	34	26	0	0	$\mathfrak{su}(2)^2_{\frac{4}{3},\frac{4}{3}} \times \mathfrak{u}(1)_0$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-8+\frac{8}{3}}(D_5, (3^2,1^4))]$
	-5	$(5,3,1^2)$	$(3,2^2,1^3)$	34	24	0	0	$\mathfrak{su}(2)_{\frac{11}{6}} \times \mathfrak{so}(3)_{\frac{4}{3}}$	$\frac{4}{3}, \frac{8}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(D_5, (3,2^2,1^3))]$
	-4	$(3^3,1)$	$(3^3,1)$	30	30	1	3	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)_{f_0}$	$2, 2$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(D_5, (3^3,1))] + \mathfrak{u}(1)$
	-2	$(2^4,1^2)$	(5^2)	20	36	1	1	$\mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-8+\frac{8}{6}}(D_5, (5^2))] + \mathfrak{u}(1)$
	-1	$(2^2,1^6)$	$(7,3)$	14	38	0	0	–	$\frac{10}{7}, \frac{8}{7}$	$(\frac{67}{84}, \frac{17}{21})$	$\mathcal{T}[W_{-8+\frac{8}{7}}(D_5, (7,3))]$
	2	(1^{10})	$(9,1)$	0	40	1	1	$\mathfrak{u}(1)_{f_0}$	$\frac{6}{5}, \frac{8}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-8+\frac{8}{10}}(D_5, (9,1))] + \mathfrak{u}(1)$
6	-8	$(11,1)$	(1^{12})	60	0	2	0	$\mathfrak{spin}(12)_5 \times \mathfrak{u}(1)^2_{f_0}$	$3, 5$	$(\frac{53}{12}, \frac{16}{3})$	$\mathcal{T}[W_{-10+\frac{10}{2}}(D_6, (1^{12}))] + 2\mathfrak{u}(1)$
	-7	$(5^2,1^2)$	$(3^2,2^2,1^2)$	52	40	0	0	$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)^2$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6, (3^2,2^2,1^2))]$
	-7	$(5^2,1^2)$	$(3,2^4,1)$	52	36	0	0	$\mathfrak{sp}(2)_{\frac{13}{6}}$	$\frac{4}{3}, \frac{10}{3}$	$(2, \frac{13}{6})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6, (3,2^4,1))]$
	-6	$(4^2,3,1)$	(3^4)	48	44	2	0	$\mathfrak{so}(4)_{\frac{5}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}, \frac{5}{2}$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-10+\frac{10}{4}}(D_6, (3^4))] + 2\mathfrak{u}(1)$
	-6	$(5,3,2^2)$	$(4^2,1^4)$	48	44	2	0	$\mathfrak{su}(2)_1 \times \mathfrak{su}(2)^2_{\frac{3}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{5}{4}, \frac{3}{2})$	$\mathcal{T}[W_{-10+\frac{10}{4}}(D_6, (4^2,1^4)] + 2\mathfrak{u}(1)$
	-3	$(2^4,1^4)$	$(7,5)$	28	56	0	0	–	$\frac{8}{7}, \frac{10}{7}$	$(\frac{67}{84}, \frac{17}{21})$	$\mathcal{T}[W_{-10+\frac{10}{7}}(D_6, (7,5))]$
	2	(1^{12})	$(11,1)$	0	60	2	0	$\mathfrak{u}(1)^2_{f_0}$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-10+\frac{10}{12}}(D_6, (11,1))] + 2\mathfrak{u}(1)$

Table 35. D_n, $b = 2n - 2$.

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b	k	HL	NL	HD	ND	(f_0, f_1)	$\mathfrak{f}$	Δ_{Coulomb}	(a, c)	Theory
12	-7	A_3	A_4	52	60	$(0,0)$	$\mathfrak{su}(2)_{\frac{8}{5}} \times \mathfrak{u}(1)_0$	$\frac{6}{5}, \frac{8}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-12+\frac{12}{5}}(\mathfrak{e}_6, A_4)]$
12	-10	$E_6(a_1)$	A_1	70	22	$(0,1)$	$\mathfrak{su}(6)_3$	$3, 2$	$(\frac{29}{12}, \frac{17}{6})$	$\mathcal{T}[W_{-12+\frac{12}{2}}(\mathfrak{e}_6, A_1)]$
	-2	A_1	$E_6(a_1)$	22	70		–	$\frac{6}{5}, \frac{6}{5}$	$(\frac{43}{60}, \frac{11}{15})$	$\mathcal{T}[W_{-12+\frac{12}{10}}(\mathfrak{e}_6, E_6(a_1))]$
12	-9	$E_6(a_3)$	A_2	66	42	$(2,1)$	$\mathfrak{su}(3)_2 \times \mathfrak{su}(3)_2 \times \mathfrak{u}(1)^2_{f_0}$	$2, 2$	$(\frac{23}{12}, \frac{7}{3})$	$\mathcal{T}[W_{-12+\frac{12}{3}}(\mathfrak{e}_6, A_2)] + 2\mathfrak{u}(1)$
	-9	$(E_6(a_3), \mathbb{Z}_2)$	$3A_1$	66	40		$\mathfrak{su}(3)_4 \times \mathfrak{su}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$4, 2$	$(\frac{71}{24}, \frac{41}{12})$	$\mathcal{T}[W_{-12+\frac{12}{3}}(\mathfrak{e}_6, 3A_1)] + 2\mathfrak{u}(1)$
12	-8	$D_4(a_1)$	$D_4(a_1)$	58	58	$(0,2)$	$\mathfrak{u}(1)_0 \times \mathfrak{u}(1)_0$	$1, 1$	$(\frac{5}{12}, \frac{1}{3})$	$\mathcal{T}[W_{-12+\frac{12}{4}}(\mathfrak{e}_6, D_4(a_1))]$
	-8	$(D_4(a_1), S_3)$	$2A_2 + A_1$	58	54		$\mathfrak{su}(2)_4$	$3, 2$	$(2, 2)$	$\mathcal{T}[W_{-12+\frac{12}{4}}(\mathfrak{e}_6, 2A_2 + A_1)]$
9	-7	$E_6(a_3)$	A_2	66	42	$(0,0)$	$\mathfrak{su}(3)_{\frac{3}{2}} \times \mathfrak{su}(3)_{\frac{3}{2}}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{7}{6}, \frac{4}{3})$	$\mathcal{T}[W_{-12+\frac{9}{2}}(\mathfrak{e}_6, A_2)]$
	-7	$(E_6(a_3), \mathbb{Z}_2)$	$3A_1$	66	40		$\mathfrak{su}(3)_3 \times \mathfrak{su}(2)_2$	$3, \frac{3}{2}$	$(\frac{47}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-12+\frac{9}{2}}(\mathfrak{e}_6, 3A_1)]$
	-5	$A_2 + 2A_1$	$A_4 + A_1$	50	62		$\mathfrak{u}(1)_{\frac{9}{4}}$	$\frac{5}{4}, \frac{3}{2}$	$(\frac{11}{12}, \frac{23}{24})$	$\mathcal{T}[W_{-12+\frac{9}{4}}(\mathfrak{e}_6, A_4 + A_1)]$
	-4	A_2	$E_6(a_3)$	42	66		–	$\frac{6}{5}, \frac{6}{5}$	$(\frac{43}{60}, \frac{11}{15})$	$\mathcal{T}[W_{-12+\frac{9}{5}}(\mathfrak{e}_6, E_6(a_3))]$
	-4	$(A_2, \mathbb{Z}_2)$	A_5	42	64		$\mathfrak{su}(2)_{\frac{17}{10}}$	$\frac{6}{5}, \frac{12}{5}$	$(\frac{163}{120}, \frac{17}{12})$	$\mathcal{T}[W_{-12+\frac{9}{5}}(\mathfrak{e}_6, A_5)]$

Table 36. E_6.

b	k	HL	NL	HD	ND	f_0, f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a, c)	Theory
18	-16	$E_7(a_1)$	A_1	124	34	$(1,0)$	$\mathfrak{so}(12)_5 \times \mathfrak{u}(1)_{f_0}$	$5, 3$	$(\frac{53}{12}, \frac{16}{3})$	$\mathcal{T}[W_{-18+\frac{18}{2}}(\mathfrak{e}_7, A_1)] + \mathfrak{u}(1)$
	-14	$E_6(a_3)$	$D_4(a_1) + A_1$	110	96		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{29}{24}, \frac{17}{12})$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7, D_4(a_1) + A_1)] + \mathfrak{u}(1)$
	-14	$(E_6(a_3), \mathbb{Z}_2)$	$A_3 + 2A_1$	110	94		$\mathfrak{su}(2)_2 \times \mathfrak{su}(2)_3 \times \mathfrak{u}(1)_{f_0}$	$3, \frac{3}{2}$	$(2, \frac{9}{4})$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7, A_3 + 2A_1)] + \mathfrak{u}(1)$
	-10	$A_2 + 2A_1$	$E_7(a_4)$	82	116		$\mathfrak{u}(1)_{f_0}$	$\frac{5}{4}, \frac{3}{2}$	$(\frac{23}{24}, \frac{25}{24})$	$\mathcal{T}[W_{-18+\frac{18}{8}}(\mathfrak{e}_7, E_7(a_4))] + \mathfrak{u}(1)$
18	-15	$E_7(a_4)$	$A_2 + 2A_1$	116	82	$(0,2)$	$\mathfrak{su}(2)_6 \times \mathfrak{su}(2)^2_2$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7, A_2 + 2A_1)]$
18	-12	$D_4(a_1) + A_1$	$E_6(a_3)$	96	110	$(1,2)$	$\mathfrak{su}(2)_1 \times \mathfrak{u}(1)_{f_0}$	$1, 1$	$(\frac{1}{2}, \frac{1}{2})$	$\mathcal{T}[W_{-18+\frac{18}{6}}(\mathfrak{e}_7, E_6(a_3))] + \mathfrak{u}(1)$
	-6	$(3A_1)''$	E_6	54	120		$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, 1$	$(\frac{19}{24}, \frac{5}{6})$	$\mathcal{T}[W_{-18+\frac{18}{12}}(\mathfrak{e}_7, E_6)] + \mathfrak{u}(1)$
14	-11	$E_6(a_3)$	$D_4(a_1) + A_1$	110	96	$(0,0)$	$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7, D_4(a_1) + A_1)]$
	-11	$(E_6(a_3), \mathbb{Z}_2)$	$A_3 + 2A_1$	110	94		$\mathfrak{su}(2)_{\frac{8}{3}} \times \mathfrak{su}(2)_{\frac{11}{6}}$	$\frac{8}{3}, \frac{4}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7, A_3 + 2A_1)]$
14	-12	E_6	$(3A_1)''$	120	54	$(1,0)$	$(\mathfrak{f}_4)_5 \times \mathfrak{u}(1)_{f_0}$	$5, 4$	$(\frac{14}{3}, \frac{16}{3})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7, (3A_1)'')] + \mathfrak{u}(1)$
	-12	$E_7(a_3)$	A_2	120	66		$\mathfrak{su}(6)_3 \times \mathfrak{u}(1)_{f_0}$	$3, 2$	$(\frac{29}{12}, \frac{17}{6})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7, A_2)] + \mathfrak{u}(1)$
	-12	$(E_7(a_3), \mathbb{Z}_2)$	$(3A_1)'$	120	64		$\mathfrak{sp}(3)_3 \times \mathfrak{su}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	$3, 4$	$(\frac{83}{24}, \frac{47}{12})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7, (3A_1)')] + \mathfrak{u}(1)$
	-10	$(A_5)''$	D_4	102	96		$\mathfrak{sp}(3)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	$3, \frac{5}{2}$	$(\frac{61}{24}, \frac{17}{6})$	$\mathcal{T}[W_{-18+\frac{14}{4}}(\mathfrak{e}_7, D_4)] + \mathfrak{u}(1)$
	-8	A_3	$D_6(a_1)$	84	114		$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{14}{6}}(\mathfrak{e}_7, D_6(a_1))] + \mathfrak{u}(1)$
	-8	$2A_2$	$D_5 + A_1$	84	114		$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-18+\frac{14}{6}}(\mathfrak{e}_7, D_5 + A_1)] + \mathfrak{u}(1)$
	-8	$A_2 + 3A_1$	A_6	84	114		$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-18+\frac{14}{6}}(\mathfrak{e}_7, A_6)] + \mathfrak{u}(1)$
	-6	A_2	$E_7(a_3)$	66	120		$\mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{5}{4}$	$(\frac{11}{12}, \frac{23}{24})$	$\mathcal{T}[W_{-18+\frac{14}{8}}(\mathfrak{e}_7, E_7(a_3))] + \mathfrak{u}(1)$

Table 37. E_7.

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b	k	Hitchin label	Nahm label	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
30	-23	$A_4+A_2+A_1$	A_6+A_1	196	0	0	$\mathfrak{su}(2)_0$	$\frac{8}{7},\frac{10}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-30+\frac{30}{7}}(\mathfrak{e}_8,A_6+A_1)]$
30	-28	$E_8(a_2)$	$2A_1$	236	0	0	$\mathfrak{so}(13)_5$	$5,3$	$(\frac{53}{12},\frac{16}{3})$	$\mathcal{T}[W_{-30+\frac{30}{2}}(\mathfrak{e}_8,2A_1)]$
	-26	$E_7(a_3)$	A_4	220			$\mathfrak{su}(5)_{\frac{5}{2}}$	$\frac{3}{2},\frac{5}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_4)]$
	-26	$E_8(b_6)$	$D_4(a_1)+A_2$	220			$\mathfrak{su}(3)_3$	$\frac{3}{2},\frac{3}{2}$	$(\frac{7}{6},\frac{4}{3})$	$\mathcal{T}[W_{-30+\frac{30}{4}}(\mathfrak{e}_8,D_4(a_1)+A_2)]$
	-26	$(E_8(b_6),\mathbb{Z}_2)$	$A_3+A_2+A_1$	220			$\mathfrak{su}(2)^2_{12,2}$	$\frac{3}{2},3$	$(\frac{47}{24},\frac{13}{6})$	$\mathcal{T}[W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_3+A_2+A_1)]$
	-22	A_4+A_1	$E_6(a_1)+A_1$	188			$\mathfrak{u}(1)_{\frac{3}{2}}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-30+\frac{30}{8}}(\mathfrak{e}_8,E_6(a_1)+A_1)]$
30	-27	$E_7(a_1)$	A_3	228	0	1	$\mathfrak{so}(11)_4$	$2,4$	$(\frac{37}{12},\frac{11}{3})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8,A_3)]$
	-27	$E_8(a_5)$	$2A_2$	228			$(\mathfrak{g}_2)^2_{2,2}$	$2,2$	$(\frac{23}{12},\frac{7}{3})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8,2A_2)]$
	-27	$(E_8(a_5),\mathbb{Z}_2)$	A_2+3A_1	228			$(\mathfrak{g}_2)_4\times\mathfrak{su}(2)_{\frac{5}{2}}$	$4,2$	$(\frac{71}{24},\frac{41}{12})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8,A_2+3A_1)]$
	-21	A_4	$E_7(a_3)$	180			$\mathfrak{su}(2)_{\frac{5}{3}}$	$\frac{4}{3},\frac{5}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{30}{9}}(\mathfrak{e}_8,E_7(a_3))]$
30	-25	A_6+A_1	$A_4+A_2+A_1$	212	0	3	$\mathfrak{su}(2)_{10}$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8,A_4+A_2+A_1)]$
	-25	$E_7(a_4)$	$D_5(a_1)+A_1$	212			$\mathfrak{su}(2)^2_{8,2}$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8,D_5(a_1)+A_1)]$
30	-18	$2A_2$	$E_8(a_5)$	156	0	1	–	$1,1$	$(\frac{5}{12},\frac{1}{3})$	$\mathcal{T}[W_{-30+\frac{30}{12}}(\mathfrak{e}_8,E_8(a_5))]$
30	-10	$2A_1$	$E_8(a_2)$	92	0	3	–	$1,1$	$(\frac{5}{12},\frac{1}{3})$	$\mathcal{T}[W_{-30+\frac{30}{20}}(\mathfrak{e}_8,E_8(a_2))]$
24	-22	$E_8(a_4)$	A_2+A_1	232	0	1	$\mathfrak{su}(6)_3$	$2,3$	$(\frac{29}{12},\frac{17}{6})$	$\mathcal{T}[W_{-30+\frac{24}{2}}(\mathfrak{e}_8,A_2+A_1)]$
24	-21	$D_7(a_1)$	A_3+A_2	222	0	1	$\mathfrak{sp}(2)_2\times\mathfrak{u}(1)_0$	$2,2$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8,A_3+A_2)]$
24	-18	A_4+2A_1	$D_7(a_2)$	192	0	3	$\mathfrak{u}(1)_0$	$1,1$	$(\frac{5}{12},\frac{1}{3})$	$\mathcal{T}[W_{-30+\frac{24}{6}}(\mathfrak{e}_8,D_7(a_2))]$
20	-19	E_8	0	240	0	0	$(\mathfrak{e}_8)_{10}$	$10,4$	$(\frac{101}{12},\frac{31}{3})$	$\mathcal{T}[W_{-30+\frac{20}{1}}(\mathfrak{e}_8,0)]$
	-17	$D_7(a_2)$	A_4+2A_1	216			$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{\frac{6}{5}}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-30+\frac{20}{3}}(\mathfrak{e}_8,A_4+2A_1)]$
	1	0	E_8	0			–	$\frac{8}{7},\frac{10}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-30+\frac{20}{21}}(\mathfrak{e}_8,E_8)]$

Table 38. E_8.

n	k_t	C-partition	B-partition	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	3	(2)	(1^3)	2	0	0	0	$\mathfrak{so}(3)_{\frac{4}{5}}$	$\frac{8}{5},\frac{6}{5}$	$(\frac{19}{20},1)$	$W_{-1+\frac{1}{5}}(B_1,(1^3))$
	5	(1^2)	(3)	0	2	0	0	–	$\frac{10}{7},\frac{8}{7}$	$(\frac{67}{84},\frac{17}{21})$	$W_{-1+\frac{1}{7}}(B_1,(3))$
2	-3	(4)	(1^5)	8	0	0	2	$\mathfrak{sp}(2)_2$	$2,2$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-3+1}(B_2,(1^5))]$
	1	(1^4)	(5)	0	8	0	0	–	$\frac{8}{7},\frac{10}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-3+\frac{3}{7}}(B_2,(5))]$
3	-7	$(4,2)$	$(3,1^4)$	16	10	0	0	$\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3},\frac{4}{3}$	$(\frac{11}{12},1)$	$\mathcal{T}[W_{-5+\frac{5}{3}}(B_3,(3,1^4))]$
	-7	$((4,2),\mathbb{Z}_2)$	$(2^2,1^3)$	16	8	0	0	$\mathfrak{su}(2)_{\frac{11}{6}}\times\mathfrak{so}(3)_{\frac{4}{3}}$	$\frac{4}{3},\frac{8}{3}$	$(\frac{13}{8},\frac{7}{4})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(B_3,(2^2,1^3))]$
	-5	$(2^2,1^2)$	$(5,1^2)$	10	16	0	4	$\mathfrak{u}(1)_0$	$1,1$	$(\frac{5}{12},\frac{1}{3})$	$\mathcal{T}[W_{-5+1}(B_3,(5,1^2))]$
4	-11	(4^2)	$(3,2^2,1^2)$	28	20	0	0	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)$	$\frac{4}{3},\frac{5}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(B_4,(3,2^2,1^2))]$
	-11	$((4^2),\mathbb{Z}_2)$	$(2^4,1)$	28	16	0	0	$\mathfrak{sp}(2)_{\frac{13}{6}}$	$\frac{4}{3},\frac{10}{3}$	$(2,\frac{13}{6})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(B_4,(2^4,1))]$
	-9	(2^4)	$(5,3,1)$	20	28	0	0	–	$\frac{6}{5},\frac{6}{5}$	$(\frac{43}{60},\frac{11}{15})$	$\mathcal{T}[W_{-7+\frac{7}{5}}(B_4,(5,3,1))]$
	-9	$((2^4),\mathbb{Z}_2)$	$(4^2,1)$	20	26	0	0	$\mathfrak{su}(2)_{\frac{17}{10}}$	$\frac{6}{5},\frac{12}{5}$	$(\frac{163}{120},\frac{17}{12})$	$\mathcal{T}[W_{-7+\frac{7}{5}}(B_4,(4^2,1))]$
5	-15	$(6,2^2)$	$(3^2,1^5)$	44	30	0	2	$\mathfrak{sp}(2)_2\times\mathfrak{u}(1)_0$	$2,2$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-9+3}(B_5,(3^2,1^5))]$
	-15	(5^2)	$(3,2^4)$	44	30	0	2	$\mathfrak{sp}(2)_2$	$2,2$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-9+3}(B_5,(3,2^4))]$

Table 39. $A_{2n-1}/\mathbb{Z}_2$, $b_t=4n-2$.

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n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	4	(2)	(1^3)	2	0	1	0	$\mathfrak{so}(3)_{\frac{5}{6}} \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{3}, \frac{4}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$W_{-1+\frac{1}{6}}(B_1, (1^3))$
2	-1	(2^2)	$(3, 1^2)$	6	6	1	0	$\mathfrak{u}(1)_{\frac{1}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-3+\frac{2}{3}}(B_2, (3, 1^2))] + \mathfrak{u}(1)$
	-1	$((2^2), \mathbb{Z}_2)$	$(2^2, 1)$	6	4	1	0	$\mathfrak{su}(2)_{\frac{11}{6}} \times \mathfrak{u}(1)_{f_0}$	$\frac{8}{3}, \frac{4}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-3+\frac{2}{3}}(B_2, (2^2, 1))] + \mathfrak{u}(1)$
	1	(1^4)	(5)	0	8	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{6}{5}, \frac{8}{5}$	$(\frac{19}{20}, 1)$	$\mathcal{T}[W_{-3+\frac{2}{5}}(B_2, (5))] + \mathfrak{u}(1)$
3	-4	$(4, 2)$	$(3, 1^4)$	16	10	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{7}{6}, \frac{4}{3})$	$\mathcal{T}[W_{-5+\frac{3}{2}}(B_3, (3, 1^4))] + \mathfrak{u}(1)$
	-4	$((4, 2), \mathbb{Z}_2)$	$(2^2, 1^3)$	16	8	1	0	$\mathfrak{su}(2)_2 \times \mathfrak{so}(3)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, 3$	$(\frac{47}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-5+\frac{3}{2}}(B_3, (2^2, 1^3))] + \mathfrak{u}(1)$
	-1	(1^6)	(7)	0	18	0	1	$-$	$\frac{6}{5}, \frac{6}{5}$	$(\frac{43}{60}, \frac{11}{15})$	$\mathcal{T}[W_{-5+\frac{3}{5}}(B_3, (7))]$
4	-6	(4^2)	$(3, 2^2, 1^2)$	28	20	2	1	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1) \times \mathfrak{u}(1)^2_{f_0}$	$2, 2$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-7+2}(B_4, (3, 2^2, 1^2))] + 2\mathfrak{u}(1)$
	-6	$((4^2), \mathbb{Z}_2)$	$(2^4, 1)$	28	16	2	1	$\mathfrak{sp}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$2, 4$	$(\frac{17}{6}, \frac{19}{6})$	$\mathcal{T}[W_{-7+2}(B_4, (2^4, 1))] + 2\mathfrak{u}(1)$
	-5	(2^4)	$(5, 3, 1)$	20	28	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-7+\frac{4}{3}}(B_4, (5, 3, 1))] + \mathfrak{u}(1)$
	-5	$((2^4), \mathbb{Z}_2)$	$(4^2, 1)$	20	26	1	0	$\mathfrak{su}(2)_{\frac{11}{6}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{8}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-7+\frac{4}{3}}(B_4, (4^2, 1))] + \mathfrak{u}(1)$
	-2	(1^8)	(9)	0	32	2	1	$\mathfrak{u}(1)^2_{f_0}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-7+\frac{2}{3}}(B_4, (9))] + 2\mathfrak{u}(1)$
5	-9	(10)	(1^{11})	50	0	0	1	$\mathfrak{so}(11)_4$	$2, 4$	$(\frac{37}{12}, \frac{11}{3})$	$\mathcal{T}[W_{-9+5}(B_5, (1^{11}))]$
	-8	$(6, 2, 1^2)$	$(5, 1^6)$	42	32	1	0	$\mathfrak{su}(4)_{\frac{5}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{2}, \frac{3}{2}$	$(\frac{7}{4}, 2)$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5, (5, 1^6))] + \mathfrak{u}(1)$
	-8	$(4^2, 2)$	$(3^3, 1^2)$	42	36	1	0	$\mathfrak{so}(3)_{\frac{3}{2}} \times \mathfrak{u}(1)_{\frac{1}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{3}{2}$	$(\frac{7}{6}, \frac{4}{3})$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5, (3^3, 1^2))] + \mathfrak{u}(1)$
	-8	$((4^2, 2), \mathbb{Z}_2)$	$(3^2, 2^2, 1)$	42	34	1	0	$\mathfrak{su}(2)_{\frac{3}{2}} \times \mathfrak{u}(1)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$3, \frac{3}{2}$	$(\frac{47}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5, (3^2, 2^2, 1))] + \mathfrak{u}(1)$
	-7	$(4, 2, 1^4)$	$(7, 1^4)$	32	42	0	1	$\mathfrak{su}(2)_{\frac{4}{3}} \times \mathfrak{su}(2)_{\frac{4}{3}}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-9+\frac{5}{3}}(B_5, (7, 1^4))]$

Table 40. $A_{2n-1}/\mathbb{Z}_2$, $b_t = 2n$.

n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	1	(3)	(1^2)	2	0	0	0	$\mathfrak{su}(2)_{\frac{5}{8}}$	$\frac{8}{5}, \frac{6}{5}$	$(\frac{19}{20}, 1)$	$W_{-2+\frac{2}{5}}(C_1, (1^2))$
	3	(1^3)	(2)	0	2	0	0	$-$	$\frac{10}{7}, \frac{8}{7}$	$(\frac{67}{84}, \frac{17}{21})$	$W_{-2+\frac{2}{7}}(C_1, (2))$
2	1	(1^5)	(4)	0	8	0	0	$-$	$\frac{10}{7}, \frac{8}{7}$	$(\frac{67}{84}, \frac{17}{21})$	$W_{-3+\frac{3}{7}}(C_2, (4))$
3	-5	$(5, 1^2)$	$(2^2, 1^2)$	16	10	0	0	$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)_{\frac{2}{3}}$	$\frac{4}{3}, \frac{5}{3}$	$(\frac{13}{12}, \frac{7}{6})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(C_3, (2^2, 1^2)]$
	-5	$((5, 1^2), \mathbb{Z}_2)$	$(2, 1^4)$	16	6	0	0	$\mathfrak{sp}(2)_{\frac{13}{6}}$	$\frac{4}{3}, \frac{10}{3}$	$(2, \frac{13}{6})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(C_3, (2, 1^4)]$
	-3	$(3, 1^4)$	$(4, 2)$	10	16	0	0	$-$	$\frac{6}{5}, \frac{6}{5}$	$(\frac{43}{60}, \frac{11}{15})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(C_3, (4, 2)]$
	-3	$((3, 1^4), \mathbb{Z}_2)$	$(4, 1^2)$	10	14	0	0	$\mathfrak{su}(2)_{\frac{17}{10}}$	$\frac{12}{5}, \frac{6}{5}$	$(\frac{163}{120}, \frac{17}{12})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(C_3, (4, 1^2)]$
4	-7	$(5, 3, 1)$	(2^4)	28	20	0	0	$\mathfrak{so}(4)_{\frac{4}{3}}$	$\frac{4}{3}, \frac{4}{3}$	$(\frac{11}{12}, 1)$	$\mathcal{T}[W_{-5+\frac{5}{3}}(C_4, (2^4)]$
	-7	$((5, 3, 1), \mathbb{Z}_2)$	$(2^3, 1^2)$	28	18	0	0	$\mathfrak{so}(3)_{\frac{4}{3}} \times \mathfrak{su}(2)_{\frac{11}{6}}$	$\frac{8}{3}, \frac{4}{3}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W-5+\frac{5}{3}(C_4, (2^3, 1^2)]$
5	-9	$(7, 2^2)$	$(3^2, 1^4)$	44	30	0	2	$\mathfrak{sp}(2)_2 \times \mathfrak{su}(2)_0$	$2, 2$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-6+2}(C_5, (3^2, 1^4)]$
	-9	$(5^2, 1)$	(2^5)	44	30	0	2	$\mathfrak{so}(5)_2$	$2, 2$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-6+2}(C_5, (2^5)]$

Table 41. $D_{n+1}/\mathbb{Z}_2$, $b_t = 2n + 2$.

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n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	2	(1^3)	(2)	0	2	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$W_{-2+\frac{1}{4}}(C_1,(2))+M(1)$
3	-4	(7)	(1^6)	18	0	1	0	$\mathfrak{sp}(3)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},3$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-4+\frac{3}{2}}(C_3,(1^6)]+\mathfrak{u}(1)$
	-2	$(3,1^4)$	$(4,2)$	10	16	1	0	$\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-4+\frac{3}{4}}(C_3,(4,2)]+\mathfrak{u}(1)$
	-2	$((3,1^4),\mathbb{Z}_2)$	$(4,1^2)$	10	14	1	0	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},\frac{3}{2}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-4+\frac{3}{4}}(C_3,(4,1^2)]+\mathfrak{u}(1)$
5	-8	$(9,1^2)$	$(2^2,1^6)$	48	18	1	0	$\mathfrak{sp}(3)_{\frac{5}{2}}\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},3$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(C_5,(2^2,1^6)]+\mathfrak{u}(1)$
	-8	$((9,1^2),\mathbb{Z}_2)$	$(2,1^8)$	48	10	1	0	$\mathfrak{sp}(4)_3\times\mathfrak{u}(1)_{f_0}$	$3,5$	$(\frac{95}{24},\frac{53}{12})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(C_5,(2,1^8)]+\mathfrak{u}(1)$
	-6	$(3^3,1^2)$	$(4^2,2)$	36	42	1	0	$\mathfrak{u}(1)_0\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{4}$	$(\frac{11}{12},\frac{23}{24})$	$\mathcal{T}[W_{-6+\frac{5}{4}}(C_5,(4^2,2)]+\mathfrak{u}(1)$
	-6	$((3^3,1^2),\mathbb{Z}_2)$	$(4,3^2)$	36	40	1	0	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{5}{2}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-6+\frac{5}{4}}(C_5,(4,3^2)]+\mathfrak{u}(1)$

Table 42. $D_{n+1}/\mathbb{Z}_2$, $b_t=2n$.

b_t	k_t	HL	NL	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-8	$G_2(a_1)$	$G_2(a_1)$	10	0	1	$-$	$1,1$	$(\frac{5}{12},\frac{1}{3})$	$\mathcal{T}[W_{-4+1}(\mathfrak{g}_2,G_2(a_1))]$
	-8	$(G_2(a_1),S_3)$	A_1	10	0	1	$\mathfrak{su}(2)_4$	$3,2$	$(2,2)$	$\mathcal{T}[W_{-4+1}(\mathfrak{g}_2,A_1)]$

Table 43. $D_4/\mathbb{Z}_3$, $b_t=12$, $b_t=6$.

b_t	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(c,a)	Theory
18	-15	$F_4(a_2)$	$A_1+\tilde{A}_1$	44	28	0	2	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_8$	$2,2$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-9+3}(\mathfrak{f}_4,A_1+\tilde{A}_1)]$
	-11	$\tilde{A}_1+A_1$	$F_4(a_2)$	28	44	0	0	$-$	$\frac{8}{7},\frac{10}{7}$	$(\frac{67}{84},\frac{17}{21})$	$\mathcal{T}[W_{-9+\frac{9}{7}}(\mathfrak{f}_4,F_4(a_2))]$
8	-7	F_4	0	48	0	1	0	$(\mathfrak{f}_4)_5\times\mathfrak{u}(1)_{f_0}$	$5,4$	$(\frac{14}{3},\frac{16}{3})$	$\mathcal{T}[W_{-9+4}(\mathfrak{f}_4,0)]+\mathfrak{u}(1)$
	-5	A_2	B_3	30	42	1	0	$\mathfrak{su}(2)_{\frac{4}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{4}{3}$	$(\frac{11}{12},1)$	$\mathcal{T}[W_{-9+\frac{4}{3}}(\mathfrak{f}_4,B_3)]+\mathfrak{u}(1)$
	-5	$\tilde{A}_2$	C_3	30	42	1	0	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3}$	$(\frac{13}{12},\frac{7}{6})$	$\mathcal{T}[W_{-9+\frac{4}{3}}(\mathfrak{f}_4,C_3)]+\mathfrak{u}(1)$

Table 44. $E_6/\mathbb{Z}_2$.

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n	k	$\mathcal{O}_f$	f_0	f_1	ND	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
2	5	(1^2)	0	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}$	$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-2+\frac{2}{7}}(\mathfrak{su}(2),(1^2))]$
	6	(1^2)	1	1	0	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$\frac{7}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-2+\frac{2}{8}}(\mathfrak{su}(2),(1^2))]+\mathfrak{u}(1)$
	7	(2)	0	0	2	–	$\frac{14}{9},\frac{4}{3},\frac{10}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-2+\frac{2}{9}}(\mathfrak{su}(2),(2))]$
	8	(2)	1	1	2	$\mathfrak{u}(1)_{f_0}$	$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-2+\frac{2}{10}}(\mathfrak{su}(2),(2))]+\mathfrak{u}(1)$
3	1	(1^3)	0	0	0	$\mathfrak{su}(3)_{\frac{9}{4}}$	$\frac{9}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-3+\frac{3}{4}}(\mathfrak{su}(3),(1^3))]$
	2	$(2,1)$	0	0	4	$\mathfrak{u}(1)_{\frac{9}{5}}$	$\frac{9}{5},\frac{6}{5},\frac{7}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathcal{T}[W_{-3+\frac{3}{5}}(\mathfrak{su}(3),(2,1))]$
	3	$(2,1)$	2	2	4	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)^2_{f_0}$	$2,\frac{3}{2},\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-3+\frac{3}{6}}(\mathfrak{su}(3),(2,1))]+2\mathfrak{u}(1)$
	4	(3)	0	0	6	–	$\frac{12}{7},\frac{9}{7},\frac{8}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-3+\frac{3}{7}}(\mathfrak{su}(3),(3))]$
4	-1	(1^4)	0	0	0	$\mathfrak{su}(4)_{\frac{8}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{5}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(\mathfrak{su}(4),(1^4))]$
	0	(1^4)	3	3	0	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)^3_{f_0}$	$3,2,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-4+\frac{4}{4}}(\mathfrak{su}(4),(1^4))]+3\mathfrak{u}(1)$
	1	$(3,1)$	0	0	10	$\mathfrak{u}(1)_{\frac{8}{5}}$	$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(\mathfrak{su}(4),(3,1))]$
	3	(4)	0	0	12	–	$\frac{12}{7},\frac{8}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-4+\frac{4}{7}}(\mathfrak{su}(4),(4))]$
	4	(4)	3	3	12	$\mathfrak{u}(1)^3_{f_0}$	$2,\frac{3}{2},\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-4+\frac{4}{8}}(\mathfrak{su}(4),(4))]+3\mathfrak{u}(1)$
5	-2	$(2,1^3)$	0	0	8	$\mathfrak{su}(3)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{5}{3}}$	$\frac{5}{3},\frac{7}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(\mathfrak{su}(5),(2,1^3))]$
	-1	$(3,1^2)$	0	0	14	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{\frac{4}{5}}$	$\frac{5}{4},\frac{3}{2},\frac{7}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-5+\frac{5}{4}}(\mathfrak{su}(5),(3,1^2))]$
	0	$(3,1^2)$	4	4	14	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)^4_{f_0}$	$2,2,2$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-5+\frac{5}{5}}(\mathfrak{su}(5),(3,1^2))]+4\mathfrak{u}(1)$
6	-3	$(2,1^4)$	2	2	10	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)^2_{f_0}$	$2,3,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-6+\frac{6}{3}}(\mathfrak{su}(6),(2,1^4))]+2\mathfrak{u}(1)$
	-2	$(3,2,1)$	1	1	22	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},2,\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-6+\frac{6}{4}}(\mathfrak{su}(6),(3,2,1))]+\mathfrak{u}(1)$
	-1	$(4,2)$	0	0	26	$\mathfrak{u}(1)_{\frac{6}{5}}$	$\frac{6}{5},\frac{7}{5},\frac{8}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-6+\frac{6}{5}}(\mathfrak{su}(6),(4,2))]$
	0	$(4,2)$	5	5	26	$\mathfrak{u}(1)_2\times\mathfrak{u}(1)^5_{f_0}$	$2,2,2$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-6+\frac{6}{6}}(\mathfrak{su}(6),(4,2))]+5\mathfrak{u}(1)$
7	-5	(1^7)	0	0	0	$\mathfrak{su}(7)_{\frac{7}{2}}$	$\frac{7}{2},\frac{5}{2},\frac{3}{2}$	$(\frac{7}{2},4)$	$\mathcal{T}[W_{-7+\frac{7}{2}}(\mathfrak{su}(7),(1^7))]$
	-4	$(3,1^4)$	0	0	22	$\mathfrak{su}(4)_{\frac{8}{3}}\times\mathfrak{u}(1)_0$	$\frac{4}{3},\frac{8}{3},\frac{5}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(\mathfrak{su}(7),(3,1^4))]$
	-4	$(2^3,1)$	0	0	24	$\mathfrak{su}(3)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{7}{3}}$	$\frac{7}{3},\frac{4}{3},\frac{5}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(\mathfrak{su}(7),(2^3,1))]$
	-3	$(4,1^3)$	0	0	30	$\mathfrak{su}(3)_{\frac{9}{4}}\times\mathfrak{u}(1)_0$	$\frac{3}{2},\frac{9}{4},\frac{5}{4}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-7+\frac{7}{4}}(\mathfrak{su}(7),(4,1^3))]$
	-3	$(3^2,1)$	0	0	32	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{\frac{7}{4}}$	$\frac{7}{4},\frac{3}{2},\frac{5}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-7+\frac{7}{4}}(\mathfrak{su}(7),(3^2,1))]$
	-2	$(4,3)$	0	0	36	$\mathfrak{u}(1)_{\frac{7}{5}}$	$\frac{7}{5},\frac{9}{5},\frac{6}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathcal{T}[W_{-7+\frac{7}{5}}(\mathfrak{su}(7),(4,3))]$
	2	(7)	0	0	42	–	$\frac{14}{9},\frac{4}{3},\frac{10}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-7+\frac{7}{9}}(\mathfrak{su}(7),(7))]$
8	-6	(1^8)	1	1	0	$\mathfrak{su}(8)_4\times\mathfrak{u}(1)_{f_0}$	$4,3,2$	$(\frac{107}{24},\frac{31}{6})$	$\mathcal{T}[W_{-8+\frac{8}{2}}(\mathfrak{su}(8),(1^8))]+\mathfrak{u}(1)$
	-5	$(3,2,1^3)$	0	0	34	$\mathfrak{su}(3)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{5}{3}}\times\mathfrak{u}(1)_0$	$\frac{5}{3},\frac{7}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(\mathfrak{su}(8),(3,2,1^3))]$
	-5	(2^4)	0	0	32	$\mathfrak{su}(4)_{\frac{8}{3}}$	$\frac{8}{3},\frac{5}{3},\frac{4}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(\mathfrak{su}(8),(2^4))]$
	-4	$(4,1^4)$	3	3	36	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^3_{f_0}$	$2,3,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(\mathfrak{su}(8),(4,1^4))]+3\mathfrak{u}(1)$
	-4	$(3^2,1^2)$	3	3	40	$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)^3_{f_0}$	$2,2,2$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(\mathfrak{su}(8),(3^2,1^2))]+3\mathfrak{u}(1)$
	-3	$(5,2,1)$	0	0	48	$\mathfrak{u}(1)_{\frac{9}{5}}\times\mathfrak{u}(1)_0$	$\frac{6}{5},\frac{9}{5},\frac{7}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathcal{T}[W_{-8+\frac{8}{5}}(\mathfrak{su}(8),(5,2,1))]$
	2	(8)	1	1	56	$\mathfrak{u}(1)_{f_0}$	$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-8+\frac{8}{10}}(\mathfrak{su}(8),(8))]+\mathfrak{u}(1)$
9	-7	$(2,1^7)$	0	0	16	$\mathfrak{su}(7)_{\frac{7}{2}}\times\mathfrak{u}(1)_0$	$\frac{7}{2},\frac{5}{2},\frac{3}{2}$	$(\frac{7}{2},4)$	$\mathcal{T}[W_{-9+\frac{9}{2}}(\mathfrak{su}(9),(2,1^7))]$
	-6	$(3,2,1^4)$	2	2	40	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$2,3,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-9+\frac{9}{3}}(\mathfrak{su}(9),(3,2,1^4))]+2\mathfrak{u}(1)$
	-6	$(2^4,1)$	2	2	40	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)_3\times\mathfrak{u}(1)^2_{f_0}$	$3,2,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-9+\frac{9}{3}}(\mathfrak{su}(9),(2^4,1))]+2\mathfrak{u}(1)$
	-5	$(4,3,1^2)$	0	0	56	$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{\frac{5}{4}}\times\mathfrak{u}(1)_0$	$\frac{5}{4},\frac{3}{2},\frac{7}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-9+\frac{9}{4}}(\mathfrak{su}(9),(4,3,1^2))]$
	-5	(3^3)	0	0	54	$\mathfrak{su}(3)_{\frac{9}{4}}$	$\frac{9}{4},\frac{5}{4},\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-9+\frac{9}{4}}(\mathfrak{su}(9),(3^3))]$
	-4	$(5,3,1)$	0	0	62	$\mathfrak{u}(1)\times\mathfrak{u}(1)_0$	$\frac{8}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-9+\frac{9}{5}}(\mathfrak{su}(9),(5,3,1))]$
	-3	$(6,2,1)$	2	2	64	$\mathfrak{u}(1)\times\mathfrak{u}(1)_0\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2},2,\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-9+\frac{9}{6}}(\mathfrak{su}(9),(6,2,1))]+2\mathfrak{u}(1)$
	-3	$(5,4)$	2	2	64	$\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2},2,\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-9+\frac{9}{6}}(\mathfrak{su}(9),(5,4))]+2\mathfrak{u}(1)$
	-2	$(7,1^2)$	0	0	66	$\mathfrak{su}(2)_{\frac{12}{7}}\times\mathfrak{u}(1)_0$	$\frac{8}{7},\frac{10}{7},\frac{12}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-9+\frac{9}{7}}(\mathfrak{su}(9),(7,1^2))]$

Table 45. A_{n-1}, $b=n$.

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n	k	$\mathcal{O}_f$	f_0	f_1	ND	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	1	(1^3)	1	0	0	$\mathfrak{su}(3)_{\frac{7}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{3}, \frac{5}{3}, \frac{4}{3}$	$(\frac{25}{12}, \frac{9}{4})$	$\mathcal{T}[W_{-3+\frac{2}{3}}(\mathfrak{su}(3), (1^3))] + \mathfrak{u}(1)$
	3	(3)	1	0	6	$\mathfrak{u}(1)_{f_0}$	$\frac{9}{5}, \frac{7}{5}, \frac{6}{5}$	$(\frac{3}{2}, \frac{31}{20})$	$\mathcal{T}[W_{-3+\frac{2}{5}}(\mathfrak{su}(3), (3))] + \mathfrak{u}(1)$
4	1	$(3,1)$	1	0	10	$\mathfrak{u}(1)_{\frac{7}{4}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{4}, \frac{3}{2}, \frac{5}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-4+\frac{3}{4}}(\mathfrak{su}(4), (3,1))] + \mathfrak{u}(1)$
5	-1	$(2^2,1)$	1	0	12	$\mathfrak{su}(2)_{\frac{7}{3}} \times \mathfrak{u}(1)_{\frac{7}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{3}, \frac{4}{3}, \frac{5}{3}$	$(\frac{25}{12}, \frac{9}{4})$	$\mathcal{T}[W_{-5+\frac{4}{3}}(\mathfrak{su}(5), (2^2,1))] + \mathfrak{u}(1)$
6	-3	(1^6)	1	0	0	$\mathfrak{su}(6)_{\frac{7}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{2}, \frac{5}{2}, \frac{3}{2}$	$(\frac{7}{2}, 4)$	$\mathcal{T}[W_{-6+\frac{5}{2}}(\mathfrak{su}(6), (1^6))] + \mathfrak{u}(1)$
	-2	$(3,1^3)$	1	0	18	$\mathfrak{su}(3)_{\frac{7}{3}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{3}, \frac{7}{3}, \frac{4}{3}$	$(\frac{17}{8}, \frac{7}{3})$	$\mathcal{T}[W_{-6+\frac{5}{3}}(\mathfrak{su}(6), (3,1^3))] + \mathfrak{u}(1)$
	-2	(2^3)	1	0	18	$\mathfrak{su}(3)_{\frac{8}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{8}{3}, \frac{5}{3}, \frac{4}{3}$	$(\frac{55}{24}, \frac{5}{2})$	$\mathcal{T}[W_{-6+\frac{5}{3}}(\mathfrak{su}(6), (2^3))] + \mathfrak{u}(1)$
	-1	$(4,1^2)$	1	0	24	$\mathfrak{su}(2)_{\frac{7}{4}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{4}, \frac{3}{2}, \frac{7}{4}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-6+\frac{5}{4}}(\mathfrak{su}(6), (4,1^2))] + \mathfrak{u}(1)$
	-1	(3^2)	1	0	24	$\mathfrak{su}(2)_{\frac{9}{4}} \times \mathfrak{u}(1)_{f_0}$	$\frac{9}{4}, \frac{5}{4}, \frac{3}{2}$	$(\frac{15}{8}, 2)$	$\mathcal{T}[W_{-6+\frac{5}{4}}(\mathfrak{su}(6), (3^2))] + \mathfrak{u}(1)$
	2	(6)	1	0	30	$\mathfrak{u}(1)_{f_0}$	$\frac{12}{7}, \frac{10}{7}, \frac{8}{7}$	$(\frac{81}{56}, \frac{3}{2})$	$\mathcal{T}[W_{-6+\frac{5}{7}}(\mathfrak{su}(6), (6))] + \mathfrak{u}(1)$
7	-4	(1^7)	2	1	0	$\mathfrak{su}(7)_4 \times \mathfrak{u}(1)^2_{f_0}$	$4, 3, 2$	$(\frac{107}{24}, \frac{31}{6})$	$\mathcal{T}[W_{-7+\frac{6}{2}}(\mathfrak{su}(7), (1^7))] + 2\mathfrak{u}(1)$
	-3	$(3,1^4)$	3	2	22	$\mathfrak{su}(4)_3 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^3_{f_0}$	$2, 3, 2$	$(\frac{19}{6}, \frac{43}{12})$	$\mathcal{T}[W_{-7+\frac{6}{3}}(\mathfrak{su}(7), (3,1^4))] + 3\mathfrak{u}(1)$
	-3	$(2^3,1)$	3	2	24	$\mathfrak{su}(3)_3 \times \mathfrak{u}(1)_3 \times \mathfrak{u}(1)^3_{f_0}$	$3, 2, 2$	$(\frac{25}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-7+\frac{6}{3}}(\mathfrak{su}(7), (2^3,1))] + 3\mathfrak{u}(1)$
	-2	$(4,2,1)$	2	1	34	$\mathfrak{u}(1)_2 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}, 2, \frac{3}{2}$	$(\frac{23}{12}, \frac{25}{12})$	$\mathcal{T}[W_{-7+\frac{6}{4}}(\mathfrak{su}(7), (4,2,1))] + 2\mathfrak{u}(1)$
	-1	$(5,2)$	1	0	38	$\mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{6}{5}, \frac{7}{5}, \frac{8}{5}$	$(\frac{43}{30}, \frac{91}{60})$	$\mathcal{T}[W_{-7+\frac{6}{5}}(\mathfrak{su}(7), (5,2))] + \mathfrak{u}(1)$
	2	(7)	2	1	42	$\mathfrak{u}(1)^2_{f_0}$	$\frac{7}{4}, \frac{3}{2}, \frac{5}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-7+\frac{6}{8}}(\mathfrak{su}(7), (7))] + 2\mathfrak{u}(1)$
8	-5	$(2,1^6)$	1	0	14	$\mathfrak{su}(6)_{\frac{7}{2}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{2}, \frac{5}{2}, \frac{3}{2}$	$(\frac{85}{24}, \frac{49}{12})$	$\mathcal{T}[W_{-8+\frac{7}{2}}(\mathfrak{su}(8), (2,1^6))] + \mathfrak{u}(1)$
	-4	$(3,2^2,1)$	1	0	38	$\mathfrak{su}(2)_{\frac{7}{3}} \times \mathfrak{u}(1) \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{3}, \frac{4}{3}, \frac{5}{3}$	$(\frac{17}{8}, \frac{7}{3})$	$\mathcal{T}[W_{-8+\frac{7}{3}}(\mathfrak{su}(8), (3,2^2,1))] + \mathfrak{u}(1)$
	-3	$(4,3,1)$	1	0	46	$\mathfrak{u}(1)_{\frac{7}{4}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{4}, \frac{3}{2}, \frac{5}{4}$	$(\frac{13}{8}, \frac{7}{4})$	$\mathcal{T}[W_{-8+\frac{7}{4}}(\mathfrak{su}(8), (4,3,1))] + \mathfrak{u}(1)$
	-2	$(5,3)$	1	0	50	$\mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{5}, \frac{9}{5}, \frac{6}{5}$	$(\frac{37}{24}, \frac{49}{30})$	$\mathcal{T}[W_{-8+\frac{7}{5}}(\mathfrak{su}(8), (5,3))] + \mathfrak{u}(1)$
9	-6	$(2,1^7)$	2	1	16	$\mathfrak{su}(7)_4 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^2_{f_0}$	$4, 3, 2$	$(\frac{9}{2}, \frac{21}{4})$	$\mathcal{T}[W_{-9+\frac{8}{2}}(\mathfrak{su}(9), (2,1^7))] + 2\mathfrak{u}(1)$
	-5	$(3^2,1^3)$	1	0	48	$\mathfrak{su}(3)_{\frac{7}{3}} \times \mathfrak{su}(2)_1 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{3}, \frac{7}{3}, \frac{4}{3}$	$(\frac{13}{6}, \frac{29}{12})$	$\mathcal{T}[W_{-9+\frac{8}{3}}(\mathfrak{su}(9), (3^2,1^3))] + \mathfrak{u}(1)$
	-5	$(3,2^3)$	1	0	48	$\mathfrak{su}(3)_{\frac{8}{3}} \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)_{f_0}$	$\frac{8}{3}, \frac{5}{3}, \frac{4}{3}$	$(\frac{7}{3}, \frac{31}{12})$	$\mathcal{T}[W_{-9+\frac{8}{3}}(\mathfrak{su}(9), (3,2^3))] + \mathfrak{u}(1)$
	-4	$(4,3,1^2)$	4	3	56	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)_2 \times \mathfrak{u}(1)_1 \times \mathfrak{u}(1)^4_{f_0}$	$2, 2, 2$	$(\frac{31}{12}, \frac{35}{12})$	$\mathcal{T}[W_{-9+\frac{8}{4}}(\mathfrak{su}(9), (4,3,1^2))] + 4\mathfrak{u}(1)$
	-4	(3^3)	4	3	54	$\mathfrak{su}(3)_3 \times \mathfrak{u}(1)^4_{f_0}$	$3, 2, 2$	$(\frac{25}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-9+\frac{8}{4}}(\mathfrak{su}(9), (3^3))] + 4\mathfrak{u}(1)$

Table 46. A_{n-1}, $b = n - 1$.

n	k	Hitchin label	Nahm label	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	0	$(5,1)$	(1^6)	12	0	3	2	$\mathfrak{su}(4)_3 \times \mathfrak{u}(1)^3_{f_0}$	$3, 2, 2$	$(\frac{25}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-4+\frac{3}{3}}(D_3, (1^6))] + 3\mathfrak{u}(1)$
	1	$(2^2,1^2)$	(3^2)	6	10	1	0	$\mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{5}{4}, \frac{7}{4}, \frac{3}{2}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-4+\frac{3}{4}}(D_3, (3^2))] + \mathfrak{u}(1)$
	3	(1^6)	$(5,1)$	0	12	3	2	$\mathfrak{u}(1)^3_{f_0}$	$2, \frac{3}{2}, \frac{3}{2}$	$(\frac{15}{8}, 2)$	$\mathcal{T}[W_{-4+\frac{3}{6}}(D_3, (5,1))] + 3\mathfrak{u}(1)$
4	-2	$(5,3)$	$(2^2,1^4)$	22	10	0	3	$\mathfrak{su}(2)_2 \times \mathfrak{su}(2)_2 \times \mathfrak{su}(2)_2$	$2, 2, 2$	$(\frac{19}{8}, \frac{5}{2})$	$\mathcal{T}[W_{-6+\frac{4}{2}}(D_4, (2^2,1^4))]$
5	-3	$(5,5)$	$(2^4,1^2)$	36	20	1	0	$\mathfrak{sp}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)_{\frac{3}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, 3, \frac{5}{2}$	$(\frac{73}{24}, \frac{10}{3})$	$\mathcal{T}[W_{-8+\frac{5}{2}}(D_5, (2^4,1^2))] + \mathfrak{u}(1)$
	-3	$(7,1^3)$	$(3,1^7)$	36	16	1	0	$\mathfrak{spin}(7)_{\frac{7}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{7}{2}, 3$	$(\frac{29}{8}, 4)$	$\mathcal{T}[W_{-8+\frac{5}{2}}(D_5, (3,1^7))] + \mathfrak{u}(1)$
	-2	$(3^2,2^2)$	$(4^2,1^2)$	28	32	1	0	$\mathfrak{su}(2)_{\frac{5}{3}} \times \mathfrak{u}(1)_{\frac{1}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{4}{3}, \frac{5}{3}, \frac{4}{3}$	$(\frac{37}{24}, \frac{5}{3})$	$\mathcal{T}[W_{-8+\frac{5}{3}}(D_5, (4^2,1^2))] + \mathfrak{u}(1)$
	-2	$(5,1^5)$	$(5,1^5)$	28	28	1	0	$\mathfrak{sp}(2)_{\frac{7}{3}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{3}, \frac{8}{3}, \frac{4}{3}$	$(\frac{21}{8}, \frac{17}{6})$	$\mathcal{T}[W_{-8+\frac{5}{3}}(D_5, (5,1^5))] + \mathfrak{u}(1)$
	-1	$(2^4,1^2)$	(5^2)	20	36	1	0	$\mathfrak{u}(1) \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{7}{4}, \frac{5}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-8+\frac{5}{4}}(D_5, (5^2))] + \mathfrak{u}(1)$
	0	$(3,1^7)$	$(7,1^3)$	16	36	5	4	$\mathfrak{so}(3)_1 \times \mathfrak{u}(1)^5_{f_0}$	$2, 2, 2$	$(\frac{61}{24}, \frac{17}{6})$	$\mathcal{T}[W_{-8+1}(D_5, (7,1^3))] + 5\mathfrak{u}(1)$
6	-4	(6^2)	(2^6)	54	30	2	1	$\mathfrak{sp}(3)_3 \times \mathfrak{u}(1)^2_{f_0}$	$2, 3, 4$	$(\frac{101}{24}, \frac{14}{3})$	$\mathcal{T}[W_{-10+\frac{6}{2}}(D_6, (2^6))] + 2\mathfrak{u}(1)$
	-4	$(7,3,1^2)$	$(3^2,1^6)$	54	34	2	1	$\mathfrak{su}(4)_3 \times \mathfrak{u}(1) \times \mathfrak{u}(1)^2_{f_0}$	$3, 2, 2$	$(\frac{25}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-10+\frac{6}{2}}(D_6, (3^2,1^6))] + 2\mathfrak{u}(1)$
	-4	$(7,3,1^2)$	$(3,2^2,1^5)$	54	32	2	1	$\mathfrak{sp}(2)_3 \times \mathfrak{su}(2)_{\frac{5}{2}} \times \mathfrak{u}(1)^2_{f_0}$	$3, 2, 4$	$(\frac{25}{6}, \frac{55}{12})$	$\mathcal{T}[W_{-10+\frac{6}{2}}(D_6, (3,2^2,1^5))] + 2\mathfrak{u}(1)$
	-3	$(3^3,1^3)$	$(5,3^2,1)$	42	50	0	5	$\mathfrak{u}(1)$	$2, 2, 2$	$(\frac{53}{24}, \frac{13}{6})$	$\mathcal{T}[W_{-10+\frac{6}{3}}(D_6, (5,3^2,1))]$
	-2	$(3^2,1^6)$	$(7,3,1^2)$	34	54	2	1	$\mathfrak{u}(1) \times \mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2}, 2, \frac{3}{2}$	$(\frac{15}{8}, 2)$	$\mathcal{T}[W_{-10+\frac{6}{4}}(D_6, (7,3,1^2))] + 2\mathfrak{u}(1)$
	-2	$(3^2,1^6)$	$(7,2^2,1)$	34	52	2	1	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)^2_{f_0}$	$3, \frac{3}{2}, 2$	$(\frac{8}{3}, \frac{17}{6})$	$\mathcal{T}[W_{-10+\frac{6}{4}}(D_6, (7,2^2,1))] + 2\mathfrak{u}(1)$
	0	$(2^2,1^8)$	$(9,3)$	18	58	6	5	$\mathfrak{u}(1)^6_{f_0}$	$2, 2, 2$	$(\frac{61}{24}, \frac{17}{6})$	$\mathcal{T}[W_{-10+1}(D_6, (9,3))] + 6\mathfrak{u}(1)$

Table 47. D_n, $b = n$.

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n	k	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
3	-1	$(5,1)$	(1^6)	12	0	0	0	$\mathfrak{su}(4)_{\frac{8}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{5}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(D_3,(1^6))]$
	1	$(2^2,1^2)$	(3^2)	6	10	0	0	$\mathfrak{u}(1)$	$\frac{6}{5},\frac{8}{5},\frac{7}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(D_3,(3^2))]$
	3	(1^6)	$(5,1)$	0	12	0	0	–	$\frac{12}{7},\frac{8}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-4+\frac{4}{7}}(D_3,(5,1))]$
4	-2	(4^2)	(2^4)	20	12	2	0	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$3,\frac{3}{2},\frac{5}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(D_4,2^4))]+2\mathfrak{u}(1)$
	-2	$(5,1^3)$	$(3,1^5)$	20	12	2	0	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$\frac{5}{2},3,\frac{3}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-6+\frac{6}{4}}(D_4,(3,1^5))]+2\mathfrak{u}(1)$
5	-5	(5^2)	$(2^4,1^2)$	36	20	0	0	$\mathfrak{sp}(2)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{4}{3}}$	$\frac{4}{3},\frac{8}{3},\frac{7}{3}$	$(\frac{21}{8},\frac{17}{6})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(D_5,(2^4,1^2))]$
	-5	$(7,1^3)$	$(3,1^7)$	36	16	0	0	$\mathfrak{spin}(7)_{\frac{10}{3}}$	$\frac{4}{3},\frac{10}{3},\frac{8}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathcal{T}[W_{-8+\frac{8}{3}}(D_5,(3,1^7))]$
	-4	$(4^2,1^2)$	$(3^2,2^2)$	32	28	1	3	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)_2\times\mathfrak{u}(1)_{f_0}$	$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathcal{T}[W_{-8+\frac{8}{4}}(D_5,(3^2,2^2))]+\mathfrak{u}(1)$
	-3	$(3^2,1^4)$	$(5,3,1^2)$	26	34	0	0	$\mathfrak{u}(1)$	$\frac{6}{5},\frac{8}{5},\frac{7}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-8+\frac{8}{5}}(D_5,(5,3,1^2))]$
	-3	$(3^2,1^4)$	$(5,2^2,1)$	26	32	0	0	$\mathfrak{su}(2)_{\frac{19}{10}}$	$\frac{14}{5},\frac{6}{5},\frac{8}{5}$	$(\frac{32}{15},\frac{133}{60})$	$\mathcal{T}[W_{-8+\frac{8}{5}}(D_5,(5,2^2,1))]$
	-1	$(3,1^7)$	$(7,1^3)$	16	36	0	0	$\mathfrak{so}(3)_{\frac{6}{7}}$	$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-8+\frac{8}{7}}(D_5,(7,1^3))]$
6	-7	$(6,6)$	(2^6)	54	30	0	0	$\mathfrak{sp}(3)_{\frac{8}{3}}$	$\frac{4}{3},\frac{10}{3},\frac{8}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6,(2^6))]$
	-7	$(7,3,1^2)$	$(3^2,1^6)$	54	34	0	0	$\mathfrak{su}(4)_{\frac{8}{3}}\times\mathfrak{u}(1)_0$	$\frac{8}{3},\frac{4}{3},\frac{5}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6,(3^2,1^6))]$
	-7	$(7,3,1^2)$	$(3,2^2,1^5)$	54	32	0	0	$\mathfrak{sp}(2)_{\frac{13}{6}}\times\mathfrak{su}(2)_{\frac{8}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{10}{3}$	$(\frac{19}{6},\frac{41}{12})$	$\mathcal{T}[W_{-10+\frac{10}{3}}(D_6,(3,2^2,1^5))]$
	-6	$(5,3^2,1)$	$(3^3,1^3)$	50	42	2	0	$\mathfrak{so}(3)_{\frac{5}{2}}\times\mathfrak{so}(3)_{\frac{3}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2},3,\frac{5}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-10+\frac{10}{4}}(D_6,(3^3,1^3))]+2\mathfrak{u}(1)$
	-3	(2^6)	(6^2)	30	54	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}$	$\frac{8}{7},\frac{10}{7},\frac{12}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-10+\frac{10}{7}}(D_6,(6^2))]$
	-1	$(2^2,1^8)$	$(9,3)$	18	58	0	0	–	$\frac{14}{9},\frac{4}{3},\frac{10}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-10+\frac{10}{9}}(D_6,(9,3))]$

Table 48. D_n, $b=2n-2$.

b	k	HL	NL	HD	ND	(f_0,f_1)	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-5	A_2	$E_6(a_3)$	42	66	$(0,0)$	–	$\frac{12}{7},\frac{8}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-12+\frac{12}{7}}(\mathfrak{e}_6,E_6(a_3))]$
	-5	$(A_2,\mathbb{Z}_2)$	A_5	42	64		$\mathfrak{su}(2)_{\frac{25}{14}}$	$\frac{12}{7},\frac{8}{7},\frac{18}{7}$	$(\frac{85}{42},\frac{25}{12})$	$\mathcal{T}[W_{-12+\frac{12}{7}}(\mathfrak{e}_6,A_5)]$
12	-10	E_6	0	72	0	$(0,1)$	$(\mathfrak{e}_6)_6$	$6,3,2$	$(\frac{45}{8},\frac{13}{2})$	$\mathcal{T}[W_{-12+\frac{12}{2}}(\mathfrak{e}_6,0)]$
	2	0	E_6	0	72		–	$\frac{8}{7},\frac{9}{7},\frac{12}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-12+\frac{12}{14}}(\mathfrak{e}_6,E_6)]$
12	-9	D_5	$2A_1$	68	32	$(2,1)$	$\mathfrak{so}(7)_4\times\mathfrak{u}(1)_6\times\mathfrak{u}(1)^2_{f_0}$	$4,4,2$	$(\frac{39}{8},\frac{11}{2})$	$\mathcal{T}[W_{-12+\frac{12}{3}}(\mathfrak{e}_6,2A_1)]+2\mathfrak{u}(1)$
	-3	$2A_1$	D_5	32	68		$\mathfrak{u}(1)_{\frac{1}{6}}\times\mathfrak{u}(1)^2_{f_0}$	$\frac{4}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{11}{8},\frac{3}{2})$	$\mathcal{T}[W_{-12+\frac{12}{9}}(\mathfrak{e}_6,D_5)]+2\mathfrak{u}(1)$
12	-8	D_4	$2A_2$	60	48	$(0,2)$	$(\mathfrak{g}_2)_3$	$3,2,3$	$(\frac{27}{8},\frac{7}{2})$	$\mathcal{T}[W_{-12+\frac{12}{4}}(\mathfrak{e}_6,2A_2)]$
12	-6	A_2+2A_1	A_4+A_1	50	62	$(2,3)$	$\mathfrak{u}(1)_6\times\mathfrak{u}(1)^2_{f_0}$	$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathcal{T}[W_{-12+\frac{12}{6}}(\mathfrak{e}_6,A_4+A_1)]+2\mathfrak{u}(1)$
9	-7	D_5	$2A_1$	68	32	$(0,0)$	$\mathfrak{so}(7)_{\frac{7}{2}}\times\mathfrak{u}(1)_{\frac{9}{2}}$	$\frac{7}{2},\frac{3}{2},3$	$(\frac{29}{8},4)$	$\mathcal{T}[W_{-12+\frac{9}{2}}(\mathfrak{e}_6,2A_1)]$
	-5	A_3	A_4	52	60		$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{\frac{9}{8}}$	$\frac{5}{4},\frac{3}{2},\frac{7}{4}$	$(\frac{19}{12},\frac{5}{3})$	$\mathcal{T}[W_{-12+\frac{9}{4}}(\mathfrak{e}_6,A_4)]$
8	-5	$D_4(a_1)$	$D_4(a_1)$	58	58	$(1,0)$	$\mathfrak{u}(1)_{\frac{1}{3}}\times\mathfrak{u}(1)_{\frac{1}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{11}{8},\frac{3}{2})$	$\mathcal{T}[W_{-12+\frac{8}{3}}(\mathfrak{e}_6,D_4(a_1))]+\mathfrak{u}(1)$
	-5	$(D_4(a_1),S_3)$	$2A_2+A_1$	58	54		$\mathfrak{su}(2)_5\times\mathfrak{u}(1)_{f_0}$	$\frac{8}{3},4,\frac{4}{3}$	$(\frac{83}{24},\frac{11}{3})$	$\mathcal{T}[W_{-12+\frac{8}{3}}(\mathfrak{e}_6,2A_2+A_1)]+\mathfrak{u}(1)$
	-5	$(D_4(a_1),\mathbb{Z}_2)$	A_3+A_1	58	56		$\mathfrak{su}(2)_{\frac{11}{6}}\times\mathfrak{u}(1)_{\frac{1}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{8}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-12+\frac{8}{3}}(\mathfrak{e}_6,A_3+A_1)]+\mathfrak{u}(1)$
	-1	A_1	$E_6(a_1)$	22	70		$\mathfrak{u}(1)_{f_0}$	$\frac{8}{7},\frac{10}{7},\frac{12}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-12+\frac{8}{7}}(\mathfrak{e}_6,E_6(a_1))]+\mathfrak{u}(1)$

Table 49. E_6.

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b	k	HL	NL	HD	ND	f_0, f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
18	-13	A_4+A_1	A_4+A_1	104	104	$(0,0)$	$\mathfrak{u}(1)^2$	$\frac{6}{5},\frac{7}{5},\frac{9}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathcal{T}[W_{-18+\frac{18}{5}}(\mathfrak{e}_7, A_4+A_1)]$
18	-16	E_7	0	126	0	$(0,0)$	$(\mathfrak{e}_7)_9\times\mathfrak{u}(1)_{f_0}$	$9,5,3$	$(\frac{75}{8},11)$	$\mathcal{T}[W_{-18+\frac{18}{2}}(\mathfrak{e}_7,0)]+\mathfrak{u}(1)$
	-14	D_5	$(A_3+A_1)''$	112	86		$\mathfrak{so}(7)_{\frac{7}{2}}\times\mathfrak{u}(1)_{f_0}$	$3,\frac{7}{2},\frac{3}{2}$	$(\frac{11}{3},\frac{49}{12})$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7,(A_3+A_1)'')]+\mathfrak{u}(1)$
	-14	$E_7(a_5)$	$D_4(a_1)$	112	94		$\mathfrak{su}(2)^3_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{3}{2},\frac{3}{2}$	$(\frac{7}{4},2)$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7,D_4(a_1))]+\mathfrak{u}(1)$
	-14	$(E_7(a_5),S_3)$	$2A_2+A_1$	112	90		$\mathfrak{su}(2)_{\frac{9}{2}}\times\mathfrak{su}(2)_{\frac{11}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{9}{2},3,\frac{3}{2}$	$(\frac{49}{12},\frac{53}{12})$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7,2A_2+A_1)]+\mathfrak{u}(1)$
	-14	$(E_7(a_5),\mathbb{Z}_2)$	$(A_3+A_1)'$	112	92	$(1,0)$	$\mathfrak{su}(2)^3_{\frac{3}{2},3}\times\mathfrak{u}(1)_{f_0}$	$3,\frac{3}{2},\frac{3}{2}$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-18+\frac{9}{2}}(\mathfrak{e}_7,(A_3+A_1)')]+\mathfrak{u}(1)$
	-10	A_3	$D_6(a_1)$	84	114		$\mathfrak{su}(2)_{\frac{7}{4}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{4},\frac{3}{2},\frac{7}{4}$	$(\frac{13}{8},\frac{7}{4})$	$\mathcal{T}[W_{-18+\frac{18}{8}}(\mathfrak{e}_7,D_6(a_1))]+\mathfrak{u}(1)$
	-10	$2A_2$	D_5+A_1	84	114		$\mathfrak{su}(2)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{4},\frac{3}{2},\frac{3}{2}$	$(\frac{3}{2},\frac{13}{8})$	$\mathcal{T}[W_{-18+\frac{18}{8}}(\mathfrak{e}_7,D_5+A_1)]+\mathfrak{u}(1)$
	-10	A_2+3A_1	A_6	84	114		$\mathfrak{su}(2)_{\frac{9}{4}}\times\mathfrak{u}(1)_{f_0}$	$\frac{9}{4},\frac{3}{2},\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-18+\frac{18}{8}}(\mathfrak{e}_7,A_6)]+\mathfrak{u}(1)$
	2	0	E_7	0	126		$\mathfrak{u}(1)_{f_0}$	$\frac{9}{5},\frac{7}{5},\frac{6}{5}$	$(\frac{3}{2},\frac{31}{20})$	$\mathcal{T}[W_{-18+\frac{18}{20}}(\mathfrak{e}_7,E_7)]+\mathfrak{u}(1)$
18	-15	$E_6(a_1)$	A_2+A_1	118	76	$(0,2)$	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)_1$	$2,2,3$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7,A_2+A_1)]$
	-15	$(E_6(a_1),\mathbb{Z}_2)$	$4A_1$	118	70		$\mathfrak{sp}(3)_{\frac{7}{2}}$	$6,2,2$	$(\frac{19}{4},\frac{21}{4})$	$\mathcal{T}[W_{-18+\frac{18}{3}}(\mathfrak{e}_7,4A_1)]$
	-3	A_1	$E_7(a_1)$	34	124		$-$	$\frac{6}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{43}{40},\frac{11}{10})$	$\mathcal{T}[W_{-18+\frac{18}{15}}(\mathfrak{e}_7,E_7(a_1))]$
18	-12	A_3+A_2	$D_5(a_1)+A_1$	98	108	$(1,2)$	$\mathfrak{su}(2)_4\times\mathfrak{u}(1)_{f_0}$	$2,2,1$	$(\frac{43}{24},\frac{11}{6})$	$\mathcal{T}[W_{-18+\frac{18}{6}}(\mathfrak{e}_7,D_5(a_1)+A_1)]+\mathfrak{u}(1)$
18	-9	A_2+A_1	$E_6(a_1)$	76	118	$(0,6)$	$\mathfrak{u}(1)_0$	$1,1,1$	$(\frac{5}{8},\frac{1}{2})$	$\mathcal{T}[W_{-18+\frac{18}{9}}(\mathfrak{e}_7,E_6(a_1))]$
14	-11	D_5	$(A_3+A_1)''$	112	86	$(0,0)$	$\mathfrak{so}(7)_{\frac{10}{3}}$	$\frac{8}{3},\frac{10}{3},\frac{4}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7,(A_3+A_1)'')]$
	-11	$E_7(a_5)$	$D_4(a_1)$	112	94		$\mathfrak{su}(2)^3_{\frac{4}{3}}$	$\frac{4}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{11}{8},\frac{3}{2})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7,D_4(a_1))]$
	-11	$(E_7(a_5),S_3)$	$2A_2+A_1$	112	90		$\mathfrak{su}(2)^2_{4,5}$	$\frac{8}{3},\frac{4}{3},4$	$(\frac{83}{24},\frac{11}{3})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7,2A_2+A_1)]$
	-11	$(E_7(a_5),\mathbb{Z}_2)$	$(A_3+A_1)'$	112	92		$\mathfrak{su}(2)^3_{\frac{8}{3},\frac{4}{3},\frac{11}{6}}$	$\frac{8}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-18+\frac{14}{3}}(\mathfrak{e}_7,(A_3+A_1)')]$
	-9	$D_4(a_1)$	$E_7(a_5)$	94	112		$-$	$\frac{6}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{43}{40},\frac{11}{10})$	$\mathcal{T}[W_{-18+\frac{14}{5}}(\mathfrak{e}_7,E_7(a_5))]$
	-9	$(D_4(a_1),S_3)$	A_5+A_1	94	108		$\mathfrak{su}(2)_{\frac{23}{5}}$	$\frac{18}{5},\frac{12}{5},\frac{6}{5}$	$(\frac{71}{24},\frac{46}{15})$	$\mathcal{T}[W_{-18+\frac{14}{5}}(\mathfrak{e}_7,A_5+A_1)]$
	-9	$(D_4(a_1),\mathbb{Z}_2)$	$D_6(a_2)$	94	110		$\mathfrak{su}(2)_{\frac{17}{10}}$	$\frac{12}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{103}{60},\frac{107}{60})$	$\mathcal{T}[W_{-18+\frac{14}{5}}(\mathfrak{e}_7,D_6(a_2))]$
14	-12	$E_7(a_2)$	$2A_1$	122	52	$(1,0)$	$\mathfrak{so}(9)_5\times\mathfrak{su}(2)_3\times\mathfrak{u}(1)_{f_0}$	$5,3,4$	$(\frac{143}{24},\frac{20}{3})$	$\mathcal{T}[W_{-18+\frac{14}{2}}(\mathfrak{e}_7,2A_1)]+\mathfrak{u}(1)$
	-10	A_4+A_1	A_4+A_1	104	104		$\mathfrak{u}(1)^2\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{3}{2},2$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-18+\frac{14}{4}}(\mathfrak{e}_7,A_4+A_1)]+\mathfrak{u}(1)$
	-8	$(A_3+A_1)''$	D_5	86	112		$\mathfrak{su}(2)^2_{\frac{4}{3},\frac{4}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{4}{3},\frac{5}{3},\frac{4}{3}$	$(\frac{37}{24},\frac{5}{3})$	$\mathcal{T}[W_{-18+\frac{14}{6}}(\mathfrak{e}_7,D_5)]+\mathfrak{u}(1)$

Table 50. E_7.

b	k	HL	NL	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(c,a)	Theory
30	$k=-23$	D_4+A_2	A_6	198	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}\times\mathfrak{su}(2)_0$	$\frac{8}{7},\frac{10}{7},\frac{12}{7}$	$(\frac{81}{56},\frac{3}{2})$	$\mathcal{T}[W_{-30+\frac{30}{7}}(\mathfrak{e}_8,A_6)]$
30	$k=-28$	$E_8(a_1)$	A_1	238	0	0	$(\mathfrak{e}_7)_9$	$9,5,3$	$(\frac{75}{8},11)$	$\mathcal{T}[W_{-30+\frac{30}{2}}(\mathfrak{e}_8,A_1)]$
	$k=-26$	$D_7(a_1)$	A_3+A_2	222			$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)_1$	$\frac{3}{2},3,\frac{5}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-30+\frac{30}{4}}(\mathfrak{e}_8,A_3+A_2)]$
	$k=-22$	$D_5(a_1)$	$E_6(a_1)$	190			$\mathfrak{su}(3)_{\frac{9}{4}}$	$\frac{3}{2},\frac{5}{4},\frac{9}{4}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-30+\frac{30}{8}}(\mathfrak{e}_8,E_6(a_1))]$
30	$k=-27$	$E_8(b_4)$	A_2+2A_1	230	0	1	$\mathfrak{so}(7)_4\times\mathfrak{su}(2)_{12}$	$4,4,2$	$(\frac{39}{8},\frac{11}{2})$	$\mathcal{T}[W_{-30+\frac{30}{3}}(\mathfrak{e}_8,A_2+2A_1)]$
30	$k=-25$	$E_6(a_1)$	$D_5(a_1)$	214	0	3	$\mathfrak{su}(4)_3$	$2,2,3$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8,D_5(a_1))]$
	$k=-25$	$(E_6(a_1),\mathbb{Z}_2)$	D_4+A_1	214			$\mathfrak{sp}(3)_{\frac{7}{2}}$	$6,2,2$	$(\frac{19}{4},\frac{21}{4})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8,D_4+A_1)]$
	$k=-25$	D_5+A_2	A_4+A_2	214			$\mathfrak{su}(2)_2\times\mathfrak{su}(2)_1$	$2,2,2$	$(\frac{61}{24},\frac{17}{6})$	$\mathcal{T}[W_{-30+\frac{30}{5}}(\mathfrak{e}_8,A_4+A_2)]$
24	$k=-17$	$D_4(a_1)+A_2$	$E_8(b_6)$	184	0	0	$-$	$\frac{12}{7},\frac{8}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-30+\frac{24}{7}}(\mathfrak{e}_8,E_8(b_6))]$
	$k=-17$	$(D_4(a_1)+A_2,\mathbb{Z}_2)$	A_7	184			$\mathfrak{su}(2)_{\frac{25}{14}}$	$\frac{18}{7},\frac{12}{7},\frac{8}{7}$	$(\frac{85}{42},\frac{25}{12})$	$\mathcal{T}[W_{-30+\frac{24}{7}}(\mathfrak{e}_8,A_7)]$
24	$k=-22$	$E_8(a_3)$	A_2	234	0	1	$(\mathfrak{e}_6)_6$	$6,2,3$	$(\frac{45}{8},\frac{13}{2})$	$\mathcal{T}[W_{-30+\frac{24}{2}}(\mathfrak{e}_8,A_2)]$
	$k=-22$	$(E_8(a_3),\mathbb{Z}_2)$	$3A_1$	234			$(\mathfrak{f}_4)_6\times\mathfrak{su}(2)_{\frac{7}{2}}$	$6,6,2$	$(\frac{43}{6},\frac{97}{12})$	$\mathcal{T}[W_{-30+\frac{24}{2}}(\mathfrak{e}_8,3A_1)]$
	$k=-10$	A_2	$E_8(a_3)$	114			$-$	$\frac{12}{7},\frac{8}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-30+\frac{24}{14}}(\mathfrak{e}_8,E_8(a_3))]$
24	$k=-21$	$E_8(a_6)$	$D_4(a_1)+A_1$	224	0	1	$\mathfrak{su}(2)^3_{2,2,2}$	$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8,D_4(a_1)+A_1)]$
	$k=-21$	$(E_8(a_6),S_3)$	A_3+2A_1	224			$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{su}(2)_4$	$4,2,2$	$(\frac{83}{24},\frac{11}{3})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8,A_3+2A_1)]$
	$k=-21$	$(E_8(a_6),\mathbb{Z}_2)$	$2A_2+2A_1$	224			$\mathfrak{sp}(2)_7$	$6,4,2$	$(\frac{133}{24},\frac{35}{6})$	$\mathcal{T}[W_{-30+\frac{24}{3}}(\mathfrak{e}_8,2A_2+2A_1)]$
24	$k=-18$	A_4+A_2	D_5+A_2	194	0	3	$\mathfrak{u}(1)_2$	$2,2,2$	$(\frac{53}{24},\frac{13}{6})$	$\mathcal{T}[W_{-30+\frac{24}{6}}(\mathfrak{e}_8,D_5+A_2)]$
20	$k=-17$	$E_6(a_1)+A_1$	A_4+A_1	218	0	0	$\mathfrak{su}(3)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{5}{6}}$	$\frac{4}{3},\frac{7}{3},\frac{5}{3}$	$(\frac{25}{12},\frac{9}{4})$	$\mathcal{T}[W_{-30+\frac{20}{3}}(\mathfrak{e}_8,A_4+A_1)]$
	$k=-11$	A_2+2A_1	$E_8(b_4)$	146			$-$	$\frac{4}{3},\frac{14}{9},\frac{10}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-30+\frac{20}{9}}(\mathfrak{e}_8,E_8(b_4))]$

Table 51. E_8.

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n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	5	(2)	(1^3)	2	0	0	0	$\mathfrak{so}(3)_{\frac{6}{7}}$	$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$(\frac{81}{56},\frac{3}{2})$	$W_{-1+\frac{1}{7}}(B_1,(1^3))$
2	-1	(2^2)	$(3,1^2)$	6	6	0	0	$\mathfrak{u}(1)_{\frac{2}{5}}$	$\frac{6}{5},\frac{8}{5},\frac{7}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-3+\frac{3}{5}}(B_2,(3,1^2))]$
	-1	$((2^2),\mathbb{Z}_2)$	$(2^2,1)$	6	4	0	0	$\mathfrak{su}(2)_{\frac{19}{10}}$	$\frac{6}{5},\frac{14}{5},\frac{8}{5}$	$(\frac{32}{15},\frac{133}{60})$	$\mathcal{T}[W_{-3+\frac{3}{5}}(B_2,(2^2,1))]$
3	-7	(6)	(1^7)	18	0	0	0	$\mathfrak{so}(7)_{\frac{10}{3}}$	$\frac{4}{3},\frac{10}{3},\frac{8}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(B_3,(1^7))]$
	-5	(2^3)	$(3^2,1)$	12	14	0	4	$\mathfrak{u}(1)_2$	$2,2,2$	$(\frac{53}{24},\frac{13}{6})$	$\mathcal{T}[W_{-5+1}(B_2,(3^2,1))]$
	-1	(1^6)	(7)	0	18	0	0	–	$\frac{4}{3},\frac{10}{9},\frac{14}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-5+\frac{5}{9}}(B_3,(7))]$
4	-11	$(6,2)$	$(3,1^6)$	30	14	0	0	$\mathfrak{su}(4)_{\frac{8}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{5}{3}$	$(\frac{55}{24},\frac{5}{2})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(B_4,(3,1^6))]$
	-11	$((6,2),\mathbb{Z}_2)$	$(2^2,1^5)$	30	12	0	0	$\mathfrak{sp}(2)_{\frac{13}{6}}\times\mathfrak{su}(2)_{\frac{8}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{10}{3}$	$(\frac{19}{6},\frac{41}{12})$	$\mathcal{T}[W_{-7+\frac{7}{3}}(B_4,(2^2,1^5))]$
	-9	$(3^2,1^2)$	$(5,2^2)$	22	26	0	0	$\mathfrak{su}(2)_{\frac{8}{5}}$	$\frac{8}{5},\frac{6}{5},\frac{6}{5}$	$(\frac{157}{120},\frac{41}{30})$	$\mathcal{T}[W_{-7+\frac{7}{5}}(B_4,(5,2^2))]$
	-7	$(2^2,1^4)$	$(7,1^2)$	14	30	0	6	$\mathfrak{u}(1)_0$	$1,1,1$	$(\frac{5}{8},\frac{1}{2})$	$\mathcal{T}[W_{-7+1}(B_4,(7,1^2))]$
5	-15	$(6,4)$	$(3,2^2,1^4)$	46	28	0	2	$\mathfrak{su}(2)^2_{2,2}\times\mathfrak{su}(2)_2$	$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathcal{T}[W_{-9+3}(B_5,(3,2^2,1^4))]$
	-15	$((6,4),\mathbb{Z}_2)$	$(2^4,1^3)$	46	24	0	2	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{so}(3)_2$	$2,2,4$	$(\frac{83}{24},\frac{11}{3})$	$\mathcal{T}[W_{-9+3}(B_5,(2^4,1^3))]$
	-13	$(3^2,2^2)$	$(5,3^2)$	36	42	0	0	$\mathfrak{u}(1)$	$\frac{6}{5},\frac{8}{5},\frac{7}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-9+\frac{9}{5}}(B_5,(5,3^2))]$
	-13	$((3^2,2^2),\mathbb{Z}_2)$	$(4^2,3)$	36	40	0	0	$\mathfrak{su}(2)_{\frac{19}{10}}$	$\frac{6}{5},\frac{8}{5},\frac{14}{5}$	$(\frac{32}{15},\frac{133}{60})$	$\mathcal{T}[W_{-9+\frac{9}{5}}(B_5,(4^2,3))]$

Table 52. $A_{2n-1}/\mathbb{Z}_2$, $b_t = 4n-2$.

n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
2	-1	(4)	(1^5)	8	0	1	0	$\mathfrak{sp}(2)_{\frac{7}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{7}{3},\frac{8}{3},\frac{4}{3}$	$(\frac{21}{8},\frac{17}{6})$	$\mathcal{T}[W_{-3+\frac{2}{3}}(B_2,(1^5))]+\mathfrak{u}(1)$
	0	(2^2)	$(3,1^2)$	6	6	2	1	$\mathfrak{u}(1)_{\frac{1}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2},2,\frac{3}{2}$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-3+\frac{1}{2}}(B_2,(3,1^2))]+2\mathfrak{u}(1)$
	0	$((2^2),\mathbb{Z}_2)$	$(2^2,1)$	6	4	2	1	$\mathfrak{su}(2)_2\times\mathfrak{u}(1)^2_{f_0}$	$\frac{3}{2},3,2$	$(\frac{8}{3},\frac{17}{6})$	$\mathcal{T}[W_{-3+\frac{1}{2}}(B_2,(2^2,1))]+2\mathfrak{u}(1)$
3	-4	(6)	(1^7)	18	0	1	0	$\mathfrak{so}(7)_{\frac{7}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{3}{2},\frac{7}{2},3$	$(\frac{29}{8},4)$	$\mathcal{T}[W_{-5+\frac{3}{2}}(B_3,(1^7))]+\mathfrak{u}(1)$
	0	(1^6)	(7)	0	18	3	2	$\mathfrak{u}(1)^3_{f_0}$	$\frac{3}{2},\frac{3}{2},2$	$(\frac{15}{8},2)$	$\mathcal{T}[W_{-5+\frac{1}{2}}(B_3,(7))]+3\mathfrak{u}(1)$
4	-6	$(6,2)$	$(3,1^6)$	30	14	2	1	$\mathfrak{su}(4)_3\times\mathfrak{u}(1)^2_{f_0}$	$3,2,2$	$(\frac{25}{8},\frac{7}{2})$	$\mathcal{T}[W_{-7+2}(B_4,(3,1^6))]+2\mathfrak{u}(1)$
	-6	$((6,2),\mathbb{Z}_2)$	$(2^2,1^5)$	30	12	2	1	$\mathfrak{sp}(2)_3\times\mathfrak{su}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)^2_{f_0}$	$3,2,4$	$(\frac{25}{6},\frac{55}{12})$	$\mathcal{T}[W_{-7+2}(B_4,(2^2,1^5))]+2\mathfrak{u}(1)$
	-5	$(3^2,1^2)$	$(5,2^2)$	22	26	1	0	$\mathfrak{su}(2)_{\frac{5}{3}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{3},\frac{4}{3},\frac{4}{3}$	$(\frac{37}{24},\frac{5}{3})$	$\mathcal{T}[W_{-7+\frac{4}{3}}(B_4,(5,2^2))]+\mathfrak{u}(1)$
5	-8	$(6,2^2)$	$(3^2,1^5)$	44	30	1	0	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)_{\frac{3}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},3,\frac{3}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5,(3^2,1^5))]+\mathfrak{u}(1)$
	-8	(5^2)	$(3,2^4)$	44	30	1	0	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{u}(1)_{f_0}$	$\frac{5}{2},3,\frac{3}{2}$	$(\frac{73}{24},\frac{10}{3})$	$\mathcal{T}[W_{-9+\frac{5}{2}}(B_5,(3,2^4))]+\mathfrak{u}(1)$

Table 53. $A_{2n-1}/\mathbb{Z}_2$, $b_t = 2n$.

n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	3	(3)	(1^2)	2	0	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}$	$\frac{12}{7},\frac{10}{7},\frac{8}{7}$	$(\frac{81}{56},\frac{3}{2})$	$W_{-2+\frac{2}{7}}(C_1,(1^2))$
2	-1	$(3,1^2)$	$(2,1^2)$	6	4	0	0	$\mathfrak{su}(2)_{\frac{19}{10}}$	$\frac{14}{5},\frac{8}{5},\frac{6}{5}$	$(\frac{32}{15},\frac{133}{60})$	$W_{-3+\frac{3}{5}}(C_2,(2,1^2))$
3	-5	(7)	(1^6)	18	0	0	0	$\mathfrak{sp}(3)_{\frac{8}{3}}$	$\frac{4}{3},\frac{10}{3},\frac{8}{3}$	$(\frac{77}{24},\frac{7}{2})$	$\mathcal{T}[W_{-4+\frac{4}{3}}(C_3,(1^6)]$
	-3	$(3,2^2)$	(3^2)	12	14	0	0	$\mathfrak{su}(2)_{\frac{8}{5}}$	$\frac{6}{5},\frac{8}{5},\frac{6}{5}$	$(\frac{157}{120},\frac{41}{30})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(C_3,(3^2)]$
	1	(1^7)	(6)	0	18	0	0	–	$\frac{10}{9},\frac{4}{3},\frac{14}{9}$	$(\frac{91}{72},\frac{23}{18})$	$\mathcal{T}[W_{-4+\frac{4}{9}}(C_3,(6)]$
4	-7	$(7,1^2)$	$(2^2,1^4)$	30	14	0	0	$\mathfrak{sp}(2)_{\frac{7}{3}}\times\mathfrak{u}(1)_{\frac{4}{3}}$	$\frac{8}{3},\frac{4}{3},\frac{7}{3}$	$(\frac{21}{8},\frac{17}{6})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(C_4,(2^2,1^4)]$
	-7	$((7,1^2),\mathbb{Z}_2)$	$(2,1^6)$	30	8	0	0	$\mathfrak{sp}(3)_{\frac{17}{6}}$	$\frac{8}{3},\frac{14}{3},\frac{4}{3}$	$(\frac{47}{12},\frac{17}{4})$	$\mathcal{T}[W_{-5+\frac{5}{3}}(C_4,(2,1^6)]$
	-3	$(3,1^6)$	$(6,2)$	14	30	0	0	–	$\frac{8}{7},\frac{12}{7},\frac{9}{7}$	$(\frac{75}{56},\frac{19}{14})$	$\mathcal{T}[W_{-5+\frac{5}{7}}(C_4,(6,2)]$
	-3	$((3,1^6),\mathbb{Z}_2)$	$(6,1^2)$	14	28	0	0	$\mathfrak{su}(2)_{\frac{25}{14}}$	$\frac{18}{7},\frac{8}{7},\frac{12}{7}$	$(\frac{85}{42},\frac{25}{12})$	$\mathcal{T}[W_{-5+\frac{5}{7}}(C_4,(6,1^2)]$
5	-9	$(7,3,1)$	$(2^4,1^2)$	46	28	0	2	$\mathfrak{so}(4)_2\times\mathfrak{su}(2)_2$	$2,2,2$	$(\frac{19}{8},\frac{5}{2})$	$\mathcal{T}[W_{-6+2}(C_5,(2^4,1^2)]$
	-9	$((7,3,1),\mathbb{Z}_2)$	$(2^3,1^4)$	46	24	0	2	$\mathfrak{sp}(2)_{\frac{5}{2}}\times\mathfrak{so}(3)_2$	$2,4,2$	$(\frac{83}{24},\frac{11}{3})$	$\mathcal{T}[W_{-6+2}(C_5,(2^3,1^4)]$
	-7	$(3^3,1^2)$	$(4^2,2)$	36	42	0	0	$\mathfrak{u}(1)_{\frac{2}{5}}$	$\frac{8}{5},\frac{6}{5},\frac{7}{5}$	$(\frac{167}{120},\frac{43}{30})$	$\mathcal{T}[W_{-6+\frac{6}{5}}(C_5,(4^2,2)]$
	-7	$((3^3,1^2),\mathbb{Z}_2)$	$(4,3^2)$	36	40	0	0	$\mathfrak{su}(2)_{\frac{19}{10}}$	$\frac{8}{5},\frac{6}{5},\frac{14}{5}$	$(\frac{32}{15},\frac{133}{60})$	$\mathcal{T}[W_{-6+\frac{6}{5}}(C_5,(4,3^2)]$

Table 54. $D_{n+1}/\mathbb{Z}_2$, $b_t = 2n+2$.

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n	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
1	2	(3)	(1^2)	2	0	1	0	$\mathfrak{su}(2)_{\frac{7}{4}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{4}, \frac{3}{2}, \frac{5}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$W_{-2+\frac{1}{4}}(C_1, (1^2)) + \mathfrak{u}(1)$
2	0	$(3, 1^2)$	$(2, 1^2)$	6	4	2	1	$\mathfrak{su}(2)_2 \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, 2, 3$	$(\frac{8}{3}, \frac{17}{6})$	$W_{-3+\frac{2}{4}}(C_2, (2, 1^2)) + 2\mathfrak{u}(1)$
3	-2	$(3, 2^2)$	(3^2)	12	14	1	0	$\mathfrak{su}(2)_{\frac{7}{4}} \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{5}{4}, \frac{7}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-4+\frac{3}{4}}(C_3, (3^2)] + \mathfrak{u}(1)$
5	-8	(11)	(1^{10})	50	0	1	0	$\mathfrak{sp}(5)_{\frac{7}{2}} \times \mathfrak{u}(1)_{f_0}$	$\frac{7}{2}, 3, 5$	$(\frac{17}{3}, \frac{19}{3})$	$\mathcal{T}[W_{-6+\frac{5}{2}}(C_5, (1^{10})] + \mathfrak{u}(1)$
	-6	$(5, 2^2, 1^2)$	$(4^2, 1^2)$	38	40	1	0	$\mathfrak{su}(2)_{\frac{7}{4}} \times \mathfrak{u}(1)_0 \times \mathfrak{u}(1)_{f_0}$	$\frac{3}{2}, \frac{5}{4}, \frac{7}{4}$	$(\frac{19}{12}, \frac{5}{3})$	$\mathcal{T}[W_{-6+\frac{5}{4}}(C_5, (4^2, 1^2)] + \mathfrak{u}(1)$

Table 55. $D_{n+1}/\mathbb{Z}_2$, $b_t = 2n$.

b_t	k_t	HL	NL	HD	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(a,c)	Theory
12	-8	G_2	0	12	0	1	$(\mathfrak{g}_2)_3$	$2, 3, 3$	$(\frac{27}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-4+1}(\mathfrak{g}_2, 0)]$
	-7	$G_2(a_1)$	$G_2(a_1)$	10	0	0	$-$	$\frac{6}{5}, \frac{6}{5}, \frac{6}{5}$	$(\frac{43}{40}, \frac{11}{10})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(\mathfrak{g}_2, G_2(a_1))]$
	-7	$(G_2(a_1), \mathbb{Z}_2)$	$\tilde{A}_1$	10	0	0	$\mathfrak{su}(2)_{\frac{17}{10}}$	$\frac{12}{5}, \frac{6}{5}, \frac{6}{5}$	$(\frac{103}{160}, \frac{107}{60})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(\mathfrak{g}_2, \tilde{A}_1)]$
	-7	$(G_2(a_1), S_3)$	A_1	10	0	0	$\mathfrak{su}(2)_{\frac{23}{5}}$	$\frac{18}{5}, \frac{12}{5}, \frac{6}{5}$	$(\frac{71}{24}, \frac{46}{15})$	$\mathcal{T}[W_{-4+\frac{4}{5}}(\mathfrak{g}_2, A_1)]$
	-2	0	G_2	0	0	1	$-$	$\frac{6}{5}, \frac{6}{5}, \frac{6}{5}$	$(\frac{43}{40}, \frac{11}{10})$	$\mathcal{T}[W_{-4+\frac{2}{5}}(\mathfrak{g}_2, G_2)]$

Table 56. $D_4/\mathbb{Z}_3$, $b_t = 12$, $b_t = 6$.

b_t	k_t	HL	NL	HD	ND	f_0	f_1	$\mathfrak{f}$	Δ_{Coulomb}	(c,a)	Theory
18	-15	$F_4(a_1)$	$\tilde{A}_1$	46	22	0	2	$\mathfrak{su}(4)_3$	$2, 2, 3$	$(\frac{25}{8}, \frac{7}{2})$	$\mathcal{T}[W_{-9+3}(\mathfrak{f}_4, \tilde{A}_1)]$
	-15	$(F_4(a_1), \mathbb{Z}_2)$	A_1	46	16	0	2	$\mathfrak{sp}(3)_{\frac{7}{2}}$	$2, 2, 6$	$(\frac{19}{4}, \frac{21}{4})$	$\mathcal{T}[W_{-9+3}(\mathfrak{f}_4, A_1)]$
	-11	A_2	B_3	30	42	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}$	$\frac{8}{7}, \frac{12}{7}, \frac{10}{7}$	$(\frac{81}{56}, \frac{3}{2})$	$\mathcal{T}[W_{-9+\frac{9}{7}}(\mathfrak{f}_4, B_3)]$
	-11	$\tilde{A}_2$	C_3	30	42	0	0	$\mathfrak{su}(2)_{\frac{12}{7}}$	$\frac{8}{7}, \frac{12}{7}, \frac{10}{7}$	$(\frac{81}{56}, \frac{3}{2})$	$\mathcal{T}[W_{-9+\frac{9}{7}}(\mathfrak{f}_4, C_3)]$
	-9	$\tilde{A}_1$	$F_4(a_1)$	22	46	0	5	$-$	$1, 1, 1$	$(\frac{5}{8}, \frac{1}{2})$	$\mathcal{T}[W_{-9+1}(\mathfrak{f}_4, F_4(a_1))]$
12	-9	$F_4(a_3)$	$F_4(a_3)$	40	40	0	1	$-$	$1, 1, 1$	$(\frac{5}{8}, \frac{1}{2})$	$\mathcal{T}[W_{-9+2}(\mathfrak{f}_4, F_4(a_3))]$
	-9	$(F_4(a_3), S_4)$	$A_2 + \tilde{A}_1$	40	34	0	1	$\mathfrak{su}(2)_{\frac{15}{2}}$	$4, 3, 2$	$(\frac{15}{4}, \frac{15}{4})$	$\mathcal{T}[W_{-9+2}(\mathfrak{f}_4, A_2 + \tilde{A}_1)]$
	-7	$A_1 + \tilde{A}_1$	$F_4(a_2)$	28	44	0	0	$-$	$\frac{6}{5}, \frac{7}{5}, \frac{8}{5}$	$(\frac{167}{120}, \frac{43}{30})$	$\mathcal{T}[W_{-9+\frac{6}{5}}(\mathfrak{f}_4, F_4(a_2))]$

Table 57. $E_6/\mathbb{Z}_2$.

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