Generalized GRMHD equations and their implications

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Abstract

We derived a set of 3+1 formalism of generalized general relativistic magnetohydrodynamic (GRMHD) equations to study phenomena of plasmas around rotating black holes. One of the equations with respect to the generalized Ohm's law shows electromotive forces due to gravitation, centrifugal force, and frame-dragging effect in a plasma around a black hole. In this paper, we summarize the equations briefly, and also mention the gravitational magnetic reconnection which can be caused by the gravitational electromotive force and charge separation even in a case of zero resistivity.

1 Introduction

Numerical simulations of general relativistic magnetohydrodynamics (GRMHD) of plasmas around black holes have confirmed a mechanism of a magnetically driven relativistic jet from a disk around a black hole [1, 2]. All of these GRMHD simulations of jet formation showed artificial appearance of magnetic islands, which are caused through magnetic reconnections due to numerical resistivity. In spite of the numerical inconsistency, these numerical results clearly suggested spontaneous formation of anti-parallel magnetic configuration, which means magnetic reconnection is caused easily in the black hole magnetospheres. The magnetic reconnection would change the global magnetic configuration drastically and influence the global dynamics of plasmas around the black holes. Thus, calculations including resistivity, the cause of magnetic reconnection, are highly expected. It is noted that causality is broken and artificial wave instability is caused when we use the standard Ohm's law, where an inertia of charge and moment of current are neglected [3]. To guarantee causality, we have to use generalized GRMHD including generalized relativistic Ohm's law [3, 4]. The generalized GRMHD equations were introduced on the basis of two-fluid approximation of plasma in Kerr metric by Khanna [5]. A more generalized equations from the general relativistic Vlasov-Boltzmann equation in time-varying space-time were formulated by Meier [6]. It was proofed that causality is satisfied for plasmas whose plasma parameter is much greater than unity [3, 7]. In this paper, we summarize the generalized GRMHD equations derived by Koide [7]. We show the electromotive forces due to gravitation, centrifugal force, and frame-dragging effect around the black hole, which are indicated by the generalized Ohm's law of plasmas around rotating black holes. The gravitational electromotive force can cause the magnetic reconnection even in a case of zero resistivity.

2 Generalized GRMHD equations

We summarize the generalized GRMHD equations of plasmas in the space-time, $x^\mu = (t, x^1, x^2, x^3)$ around a black hole where metric ds^2 is given by $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ (Equations (18), (24), and (59) with Equations (25) and (58) of Koide [7]). Throughout this paper, we use the unit system, where light speed is unity. The generalized GRMHD equations are as follows:

$$\nabla_\nu (\rho U^\nu) = 0, \tag{1}$$

$$\nabla_\nu T^{\mu\nu} = 0, \tag{2}$$

$$\frac{1}{ne}\nabla_\nu K^{\mu\nu} = \frac{1}{2ne}\nabla^\mu(\Delta\mu p - \Delta p) + \left(U^\nu - \frac{\Delta\mu}{ne}J^\nu\right) F^\mu{}_\nu - \eta\left[J^\mu - \rho_{\rm e}'(1+\Theta)U^\mu\right], \tag{3}$$

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and Maxwell equations

$$\nabla_\nu \, {}^*F^{\mu\nu} = 0, \tag{4}$$

$$\nabla_\nu F^{\mu\nu} = J^\mu, \tag{5}$$

where

$$T^{\mu\nu} \equiv p g^{\mu\nu} + h U^\mu U^\nu + \frac{\mu h^\ddagger}{(ne)^2} J^\mu J^\nu + \frac{2\mu \Delta h}{ne}(U^\mu J^\nu + J^\mu U^\nu) + F^\mu{}_\sigma F^{\mu\sigma} - \frac{1}{4} g^{\mu\nu} F^{\kappa\lambda} F_{\kappa\lambda},$$

$$K^{\mu\nu} \equiv \frac{\mu h^\ddagger}{ne}(U^\mu J^\nu + J^\mu U^\nu) + \frac{\Delta h}{2} U^\mu U^\nu - \frac{\mu \Delta h^\sharp}{(ne)^2} J^\mu J^\nu.$$

Here, we used the two-fluid model, where we assumed the plasma consists of positively charged particles with charge e and mass m_+ and negatively charged particles with charge $-e$ and mass m_-. We used the typical mass of plasma particle $m = m_+ + m_-$, normalized reduced mass $\mu = m_+ m_- / m^2$, and normalized mass difference $\Delta\mu = (m_+ - m_-)/m$. The variables ρ, h, p, $n \equiv \rho/m$, Δp, and Δh are mass density, enthalpy density, pressure, number density, pressure difference of two fluids, and difference of two fluid enthalpy densities. Furthermore, ∇_μ, U^μ, and J^μ are covariant derivative, 4–velocity, and 4–current density, respectively, and $F_{\mu\nu}$ is the electromagnetic strength tensor and ${}^*F_{\mu\nu}$ is dual tensor of $F_{\mu\nu}$. We also use the variables related to enthalpy density,

$$h^\ddagger = h - \Delta\mu h \qquad \text{and} \qquad \Delta h^\sharp = \Delta\mu h - \frac{1 - 3\mu}{2\mu} \Delta h.$$

The variable η indicates resistivity and Θ is the rate of equipartition with respect to the thermalized energy due to friction (for detail, see Appendix A of Koide [4]). It is noted that Equation (5) yields the equation of continuity with respect to the current,

$$\nabla_\nu J^\nu = 0. \tag{6}$$

In addition, we assume the plasma consists of two perfect fluids with the equal specific heat ratio, Γ. The equations of states are

$$h = n^2 \left[\frac{m_+}{n_+} + \frac{m_-}{n_-} + \frac{\Gamma}{2(\Gamma - 1)} \left\{ \left(\frac{1}{n_+^2} + \frac{1}{n_-^2} \right) p + \left(\frac{1}{n_+^2} - \frac{1}{n_-^2} \right) \Delta p \right\} \right], \tag{7}$$

$$\Delta h = 2\mu m n^2 \left[\frac{1}{n_+} - \frac{1}{n_-} + \frac{\Gamma}{2(\Gamma - 1)} \left\{ \left(\frac{1}{m_+ n_+^2} - \frac{1}{m_- n_-^2} \right) p + \left(\frac{1}{m_+ n_+^2} + \frac{1}{m_- n_-^2} \right) \Delta p \right\} \right], \tag{8}$$

where

$$n_\pm \equiv \left[n^2 \mp \frac{2 m_\mp n}{em} U^\nu J_\nu - \left(\frac{m_\mp}{em} \right)^2 J^\nu J_\nu \right]^{1/2} \tag{9}$$

corresponds to the particle number density of each charged fluid (see Equations (74) – (78) of Koide [7]).

In the generalized GRMHD equations and the Maxwell equations, there are three types of terms including covariant derivatives: (i) contravariant vector like ρU^μ and J^μ, (ii) anti-symmetric 2nd rank tensor like $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$, and (iii) symmetric 2nd rank tensor like $T^{\mu\nu}$ and $K^{\mu\nu}$. With respect to any contravariant vector A^μ and any anti-symmetric 2nd rank tensor $A^{\mu\nu}$, we have

$$\nabla_\nu A^\nu = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} A^\nu \right), \tag{10}$$

$$\nabla_\nu A^{\mu\nu} = -\frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} A^{\mu\nu} \right), \tag{11}$$

where g is the determinant of the metric $(g_{\mu\nu})$. As for an arbitrary symmetric 2nd rank tensor $S^{\mu\nu}$, the derivative is written by the Christoffel symbol, $\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}(-\partial_\sigma g_{\mu\nu} + \partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu})$ as

$$\nabla_\nu S^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g} S^{\mu\nu} \right) + \Gamma^\mu_{\sigma\nu} S^{\sigma\nu}. \tag{12}$$

We assume that off-diagonal spatial elements of the metric $g_{\mu\nu}$ vanish, $g_{ij} = 0$ $(i \neq j)$. Writing non-zero components by

$$g_{00} = -h_0^2, \qquad g_{ii} = h_i^2, \qquad g_{i0} = g_{0i} = -h_i^2\omega_i, \tag{13}$$

we have

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -h_0^2 dt^2 + \sum_{i=1}^{3}\left[h_i^2(dx^i)^2 - 2h_i^2\omega_i dt dx^i\right]. \tag{14}$$

When we define the lapse function α and shift vector β^i by

$$\alpha = \left[h_0^2 + \sum_{i=1}^{3}(h_i\omega_i)^2\right]^{1/2}, \quad \beta^i = \frac{h_i\omega_i}{\alpha}, \tag{15}$$

the line element ds is written by

$$ds^2 = -\alpha^2 dt^2 + \sum_{i=1}^{3}(h_i dx^i - \alpha\beta^i dt)^2. \tag{16}$$

Using "zero-angular-momentum observer (ZAMO) frame" $\hat{x}^\mu$, where the line element is $ds^2 = -d\hat{t}^2 + \sum_i(\hat{x}^i)^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, we have the 3+1 formalism of the generalized GRMHD and the Maxwell equations. As for equations including only derivatives with respect to contravariant vector or anti-symmetric 2nd rank tensor, we obtain their 3+1 formalism easily using Equations (10) and (11). With respect to an equation including terms of derivative of symmetric 2nd rank tensor like Equation (12),

$$\nabla_\nu S^{\mu\nu} = H^\mu, \tag{17}$$

the 3+1 formalism is given by

$$\frac{\partial}{\partial t}\hat{S}^{00} + \frac{1}{h_1h_2h_3}\sum_j \frac{\partial}{\partial x^j}\left[\frac{\alpha h_1h_2h_3}{h_j}\left(\hat{S}^{0j} + \beta^j\hat{S}^{00}\right)\right] + \sum_j \frac{1}{h_j}\frac{\partial\alpha}{\partial x^j}\hat{S}^{j0}$$
$$+\sum_{j,k}\alpha\beta^k(G_{kj}\hat{S}^{kj} - G_{jk}\hat{S}^{jj}) + \sum_{j,k}\sigma_{jk}\hat{S}^{jk} = \alpha\hat{H}^0, \tag{18}$$
$$\frac{\partial}{\partial t}\hat{S}^{i0} + \frac{1}{h_1h_2h_3}\sum_j \frac{\partial}{\partial x^j}\left[\frac{\alpha h_1h_2h_3}{h_j}\left(\hat{S}^{ij} + \beta^j\hat{S}^{i0}\right)\right] + \frac{1}{h_i}\frac{\partial\alpha}{\partial x^i}\hat{S}^{00}$$
$$-\sum_j \alpha\left[G_{ij}\hat{S}^{ij} - G_{ji}\hat{S}^{jj} + \beta^j(G_{ij}\hat{S}^{0i} - G_{ji}\hat{S}^{0j})\right] + \sum_j \sigma_{ji}\hat{S}^{0j} = \alpha\hat{H}^i, \tag{19}$$

where the alphabetic index (i, j, and k) runs from 1 to 3, $G_{ij} \equiv -\frac{1}{h_ih_j}\frac{\partial h_i}{\partial x^j}$, and $\sigma_{ij} \equiv \frac{1}{h_j}\frac{\partial}{\partial x^j}(\alpha\beta^i)$ (see Appendix A of Koide [7]).

3 Discussion

Using Equations (3) and (19), we obtain the intuitive 3+1 form of spatial part of the generalized general relativistic Ohm's law,

$$\frac{1}{ne}\frac{\partial}{\partial t}\left(\frac{\mu h^\ddagger}{ne}\hat{J}^{\dagger j}\right) = -\frac{1}{ne}\left[\frac{1}{h_1h_2h_3}\sum_j\frac{\partial}{\partial x^j}\left[\frac{\alpha h_1h_2h_3}{h_j}(\hat{K}^{ij} + \beta^j\frac{\mu h^\ddagger}{ne}\hat{J}^{\dagger i})\right]\right.$$
$$\left.+\frac{2\mu h^\ddagger}{ne}\frac{1}{h_i}\frac{\partial\alpha}{\partial x^i}\rho_{\rm e}^\dagger - \sum_j\alpha\left\{G_{ij}\hat{K}^{ij} - G_{ji}\hat{K}^{jj} + \beta^j\frac{\mu h^\ddagger}{ne}\left(G_{ij}\hat{J}^{\dagger i} - G_{ji}\hat{J}^{\dagger j}\right)\right\} + \sum_j\frac{\mu h^\ddagger}{ne}\sigma_{ji}\hat{J}^{\dagger j}\right]$$
$$+\alpha\left[\frac{1}{2ne}\frac{1}{h_i}\frac{\partial}{\partial x^i}(\Delta\mu p - \Delta p) + \left(\hat{U}^\nu - \frac{\Delta\mu}{ne}\hat{J}^\nu\right)\hat{F}_{i\nu} - \eta[\hat{J}^i - \rho_{\rm e}'(1+\Theta)\hat{U}^i]\right], \tag{20}$$

where we defined the modified current density and modified charge density as,

$$\hat{J}^{\dagger i} \equiv \frac{ne}{\mu h^{\ddagger}}\hat{K}^{i0} = \gamma \hat{J}^i + \hat{\rho}_{\rm e}\hat{U}^i - \frac{\Delta h^{\sharp}}{neh^{\ddagger}}\hat{\rho}_{\rm e}\hat{J}^i + \frac{ne\Delta h}{2\mu h^{\ddagger}}\gamma U^i \approx \gamma \tilde{J}^i + (\tilde{\rho}_{\rm e} - \gamma\tilde{\rho}'_{\rm e})U^i, \tag{21}$$

$$\hat{\rho}_{\rm e}^{\dagger} \equiv \frac{ne}{2\mu h^{\ddagger}}\hat{K}^{00} = \hat{\rho}_{\rm e}\left(\gamma - \frac{\Delta h^{\sharp}}{2neh^{\ddagger}}\hat{\rho}_{\rm e}\right) + \frac{ne\Delta h}{4\mu h^{\ddagger}}\hat{\gamma}^2 \approx \gamma(\tilde{\rho}_{\rm e} - \gamma\tilde{\rho}'_{\rm e}/2). \tag{22}$$

The last three terms in the last bracket of the right-hand side of Equation (20) correspond the electromotive forces due to gravity, centrifugal force, and frame-dragging effect, respectively. The gravitational electromotive 3-force,

$$\boldsymbol{E}_{\rm grv} = \frac{2\mu h}{(ne)^2}\nabla(\ln\alpha)\rho_{\rm e}^{\dagger}\frac{1}{\gamma} \tag{23}$$

may cause the magnetic reconnection, while frame dragging and centrifugal electromotive forces never change the topology of the magnetic field configuration. The magnetic reconnection will be induced by the gravitational electromotive force in the following situation as an example. Let us consider a current sheet in an accretion disk around a black hole, which is thin and localized near the equatorial plane and whose current is directed radially. When the net electric charge is distributed at the equatorial current sheet locally, the local radial electric field is induced by the gravitational electromotive force $\boldsymbol{E} = \boldsymbol{E}_{\rm grv}$. When the direction of the gravitational electromotive force $\boldsymbol{E}_{\rm grv}$ is the same as that of the current density $\boldsymbol{J}$ of the current sheet, we can define the positive effective resistivity $\eta_{\rm grv}$, which satisfies $\boldsymbol{E}_{\rm grv} = \eta_{\rm grv}\boldsymbol{J}$. We recognize that this effective resistivity induces the magnetic reconnection. The sign of $\eta_{\rm grv}$ depends on the charge separation $\rho_{\rm e}^{\dagger}$ and the directions of current and gravity. This process shows that the charge causes the magnetic reconnection in the gravity. It is natural to consider that the charge separation is not kept stationary because of plasma oscillation and the gravitational magnetic reconnection is transient. With respect to the charge separation of plasmas around black holes, we found an new instability of charge separation using linear analysis of the generalized GRMHD equations [8]. This instability causes the charge separation with small-scale length in a disk around a black hole. Thus, the gravitational electromotive force due to the small-scale charge separation is expected to induce the magnetic reconnection in average.

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