# Gauged Linear Sigma Models and Mirror Symmetry

Wei Gu

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Physics

Eric Sharpe, Chair
Lara Anderson
James Gray
Leo Piilonen

May 14, 2019
Blacksburg, Virginia

Keywords: Supersymmetry, Topological Field Theory

Gauged Linear Sigma Models and Mirror Symmetry

Wei Gu

(ABSTRACT)

This thesis is devoted to the study of gauged linear sigma models (GLSMs) and mirror symmetry. The first chapter of this thesis aims to introduce some basics of GLSMs and mirror symmetry. The second chapter contains the author's contributions to new exact results for GLSMs obtained by applying supersymmetric localization. The first part of that chapter concerns supermanifolds. We use supersymmetric localization to show that A-twisted GLSM correlation functions for certain supermanifolds are equivalent to corresponding A-twisted GLSM correlation functions for hypersurfaces. The second part of that chapter defines associated Cartan theories for non-abelian GLSMs by studying partition functions as well as elliptic genera. The third part of that chapter focuses on $\mathcal{N}$=(0,2) GLSMs. For those deformed from $\mathcal{N}$=(2,2) GLSMs, we consider A/2-twisted theories and formulate the genus-zero correlation functions in terms of Jeffrey-Kirwan-Grothendieck residues on Coulomb branches, which generalize the Jeffrey-Kirwan residue prescription relevant for the $\mathcal{N}$=(2,2) locus. We reproduce known results for abelian GLSMs, and can systematically calculate more examples with new formulas that render the quantum sheaf cohomology relations and other properties manifest. We also include unpublished results for counting deformation parameters. The third chapter is about mirror symmetry. In the first part of the third chapter, we propose an extension of the Hori-Vafa mirrror construction [25] from abelian (2,2) GLSMs they considered to non-abelian (2,2) GLSMs with connected gauge groups, a potential solution to an old problem. We formally show that topological correlation functions of B-twisted mirror LGs match those of A-twisted gauge theories. In this thesis, we study two examples, Grassmannians and two-step flag manifolds, verifying in each case that the mirror correctly reproduces details ranging from the number of vacua and correlations functions to quantum cohomology relations. In the last part of the third chapter, we propose an extension of the Hori-Vafa construction [25] of (2,2) GLSM mirrors to (0,2) theories obtained from (2,2) theories by special tangent bundle deformations. Our ansatz can systematically produce the (0,2) mirrors of toric varieties and the results are consistent with existing examples which were produced by laborious guesswork. The last chapter briefly discusses some directions that the author would like to pursue in the future.

# Gauged Linear Sigma Models and Mirror Symmetry

Wei Gu

## (GENERAL AUDIENCE ABSTRACT)

In this thesis, I summarize my work on gauged linear sigma models (GLSMs) and mirror symmetry. We begin by using supersymmetric localization to show that A-twisted GLSM correlation functions for certain supermanifolds are equivalent to corresponding A-twisted GLSM correlation functions for hypersurfaces. We also define associated Cartan theories for non-abelian GLSMs. We then consider $\mathcal{N}$=(0,2) GLSMs. For those deformed from $\mathcal{N}$=(2,2) GLSMs, we consider A/2-twisted theories and formulate the genus-zero correlation functions on Coulomb branches. We reproduce known results for abelian GLSMs, and can systematically compute more examples with new formulas that render the quantum sheaf cohomology relations and other properties are manifest. We also include unpublished results for counting deformation parameters. We then turn to mirror symmetry, a duality between seemingly-different two-dimensional quantum field theories. We propose an extension of the Hori-Vafa mirrror construction [25] from abelian (2,2) GLSMs to non-abelian (2,2) GLSMs with connected gauge groups, a potential solution to an old problem. In this thesis, we study two examples, Grassmannians and two-step flag manifolds, verifying in each case that the mirror correctly reproduces details ranging from the number of vacua and correlations functions to quantum cohomology relations. We then propose an extension of the Hori-Vafa construction [25] of (2,2) GLSM mirrors to (0,2) theories obtained from (2,2) theories by special tangent bundle deformations. Our ansatz can systematically produce the (0,2) mirrors of toric varieties and the results are consistent with existing examples. We conclude with a discussion of directions that we would like to pursue in the future.

# Acknowledgments

I am deeply grateful to my advisor Eric Sharpe for his constant support throughout my graduate career, as well as collaboration on many papers. We had a lot of discussions and debates, and many new ideas and insights came from those discussions and debates. He also taught me many professional skills regarding how to give a talk, how to write a paper and other countless things that I could not include here.

I am also grateful to my collaborators in my graduate career, Cyril Closset, Jirui Guo and Bei Jia. I would like to thank Hadi Parsian and Hao Zou for their discussions and questions to motivate me to think about the projects we have worked on.

I would also like to thank some additional faculty: Lara Anderson, Paul Aspinwall, James Gray, Djordje Minic, Ilarion Melnikov, Kentaro Hori, Ronen Plesser, Mauricio Romo, Yongbin Ruan, Cumrun Vafa, Guangbo Xu as well as many other physicists and mathematicians for very beneficial discussions at different times.

I would also like to thank some postdocs, Yu Deng, Xin Gao, Paul Oehlmann, Chen Sun for their correspondence and fun discussions. I hope they can find faculty positions soon.

I am grateful to the members of ICTS especially Jianxin Lu for his funding support and for giving me an office position which helped me through some very difficult times before starting graduate study at Virginia Tech. I also learnt string theory from his class in the University of Science and Technology of China. I also thank Yang-Hui He for working with me. Without his support, I could not continue my academic career and transfer to serious research, and I have to thank my collaborator Da Zhou for the project we worked on with Yang-Hui He, as the improved programming is due to Da. Before moving to Virginia Tech, Oleg Evnin also supported me through difficult times, I thank him for his help.

I would also like to thank the other people in my office. I learned a lot of math from Mohsen Karkheiran. I also received a lot of help from Wei Cui, He Feng, and Juntao Wang. I would also like to thank my current roommates Wenzheng Dong, Ziang Feng and Zhongshun Zhang for the food they cooked for me.

Last I could not possibly be writing this thesis if there had not been my mother Zhihua, as well as my grandfather Guijun, who always cook delicious food when I am back home and always give me support. This thesis is dedicated to them.

# Contents

# Chapter 1

# Introduction

String theory is the leading attempt to unify general relativity and quantum field theory into one coherent framework. String theory has had made many successes, and also makes some novel predictions. Perhaps the most novel of those predictions is that spacetime should have ten dimensions, instead of four. It was proposed in [1] that the four dimensions we observe should arise after the ten dimensions are rolled up or "compactified" on a six-dimensional manifold. Demanding four-dimensional spacetime geometry (at high energies) constrains the properties of that six-dimensional internal space, typically to be a 'Calabi-Yau' manifold.

In a compactification, properties of low-energy quantum field theories are determined by the geometry of the internal six-dimensional space, as shown in [1]. Over the three decades since that paper was written, the subject of string compactifications has undergone significant developments. New insights and techniques have led to many developments in this subject. Methods to study string compactifications can be classified in two broad categories: study geometries (which are also called target spaces) directly by using mathematical tools with some constraints from physics; study two-dimensional worldsheet propagating in some target space. These two methods yield mutually consistent results; however each method has its own strengths. This thesis is largely devoted to the author's contributions to this subject by using the second method, however. it is worth mentioning some new developments from the first method briefly. I apologize that I can not include all of the developments in this thesis.

Physicists usually start by solving for vacuum configurations of effective theories of string theory or M theory which preserve some supersymmetry. In simple cases, this requires that the compacification spaces be Calabi-Yau manifolds [1]. Along with these "physical tools," traditional mathematical tools such as algebraic geometry are also necessary for studying these manifolds. A good review of this can be found at [2]. Surprisingly, string dualities [3] give new insights into Calabi-Yau manifolds that are not visible from classical mathematics. For example, mirror symmetry is a duality which relates two different Calabi-Yau manifolds. A proposal for a geometrical construction of mirrors can be found in [4]. With these mathematical tools physicists also extend string phenomenology from the perturbative regime to nonperturbative physics and more aspects of particle physics can be "derived" from string theory [5, 6]. It turns out the number of Calabi-Yau manifolds is huge and statistical methods would be useful in classifying these Calabi-Yau manifolds, such as [7–9]. Recently, some physicists have applied machine learning to Calabi-Yau manifolds, see [10] for more about this direction and references therein.

Another approach to the study of Calabi-Yau manifolds is from the worldsheet perspective [11]. More specifically, one can define two-dimensional worldsheet theories known as nonlinear sigma models (NLSMs) which describe strings propagating on a space. It is a theory of maps from the worldsheet into the target space. Its topological twists were studied by Witten [13]. Witten gave a detailed correspondence between operators in the (topologically-twisted) worldsheet theory and the cohomology of the target space. More generally, such two-dimensional theories encode the geometry and topology of their target spaces. A different class of two-dimensional theories which also are related to target-space geometries are known as Landau-Ginzburg (LG) models. Their massless spectra are often similar to those of nonlinear sigma models [14], which motivated the conjecture [15] that there exists an equivalence between NLSMs and LG models.

The NLSM/LG correspondence was explained in [16,17]. In [16], Witten constructed gauged linear sigma models (GLSMs) and noted that NLSMs and LG models are often two different phases of the same GLSM. The paper [17] used mirror symmetry to examine the same problem. Since Witten's work, there has been a great deal of work on GLSMs. However, for many years, nonperturabtive effects in GLSMs could be only understood via mirror symmetry. This problem was solved within last decade due to the development of supersymmetric localization. Inspired by Pestun's paper on localization in four-dimensional supersymmetric theories [18], people extended localization to other cases including two-dimensional gauge theories with various backgrounds [19]. Many new exact results have been obtained by applying supersymmetric localization to GLSMs, which are reviewed in chapter 2 of this thesis, along with some new unpublished results in section 2.2 and section 2.3.6 and some detailed derivations of one-loop determinants in appendices.

As emphasised by Witten in [16], mirror symmetry can be interpreted in terms of exchanging chiral multiplets annd twisted chiral multiplets of (2,2) supersymmetry. Following this picture, Morrison and Plesser [20] explained mirror symmetry as a duality between pairs of GLSMs that generalizes the mathematical picture of Batyrev-Borisov [21–23]. (Some new progress in this direction [24].) In 2000, Hori-Vafa [25] applied T-duality to abelian gauged linear sigma models to obtain mirror Landau-Ginzburg theories. This formulation of mirror symmetry not only includes Calabi-Yau manifolds but also more general toric varieties which have a GLSM description. However, they did not provide a general ansatz for mirrors to non-abelian GLSMs at that time. This open problem was solved in [26] last year. We also extended the Hori-Vafa (2,2) mirror construction to (0,2) cases in [27]. Chapter 3 describes these new developments, largely followng my work with Eric Sharpe. In addition, several unpublished detailed computations are also included.

Despite a vast literature and substantial progress however, many open questions remain and some other new methods should be involved to understand more fully the dynamics of gauged linear sigma models and mirror symmetry. The last chapter briefly summarizes some of these open questions. In the reminder of this chapter we will review some basics of GLSMs and mirror symmetry which will be used in later chapters.

## 1.1 Review of Gauged Linear Sigma Models

We will review some basics of (2,2) and (0,2) GLSMs. The material here can also be found in [16, 28].

### 1.1.1 Review of (2,2) GLSMs

In this section, we briefly review some basics of physical $\mathcal{N} = (2,2)$ GLSMs. In this section, we assume the worldsheet is flat. Our notation and conventions in this section largely follow [16, 28].

#### $(2,2)$ Supersymmetry Algebra

The generators of the $(2,2)$ supersymmetry algebra are four supercharges $Q_\pm$, $\overline{Q}_\pm$, spacetime translations $P$, $H$, and Lorentz rotation $M$, as well as two $R$-symmetries $U(1)_V$ with generators $F_V$ and $U(1)_A$ with generator $F_A$. These satisfy the following algebraic relations:

$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0, \tag{1.1.1}$$

$$\left\{Q_\pm, \overline{Q}_\pm\right\} = 2\left(H \mp P\right), \tag{1.1.2}$$

$$\left\{\overline{Q}_+, \overline{Q}_-\right\} = 2Z, \qquad \left\{Q_+, Q_-\right\} = 2Z^*, \tag{1.1.3}$$

$$\left\{Q_+, \overline{Q}_-\right\} = 2\widetilde{Z}, \qquad \left\{Q_+, \overline{Q}_-\right\} = 2\widetilde{Z}^*, \tag{1.1.4}$$

$$[M, Q_\pm] = \mp Q_\pm, \qquad \left[M, \overline{Q}_\pm\right] = \mp \overline{Q}_\pm, \tag{1.1.5}$$

$$[F_V, Q_\pm] = -Q_\pm, \qquad \left[F_V, \overline{Q}_\pm\right] = \overline{Q}_\pm. \tag{1.1.6}$$

$$[F_A, Q_\pm] = \mp Q_\pm, \qquad \left[F_A, \overline{Q}_\pm\right] = \pm \overline{Q}_\pm, \tag{1.1.7}$$

$$[M, F_V] = 0, \qquad [M, F_A] = 0, \tag{1.1.8}$$

where $Z$ and $\widetilde{Z}$ are central charges, and the hermiticity of the generators is dictated by $Q_\pm^\dagger = \overline{Q}_\pm$.

To have two $U(1)$ $R$-symmetries in the theory, we set both $Z$ and $\widetilde{Z}$ to zero. This corresponds to $superconformal$ field theory, and a superconformal nonlinear sigma model typically has target spaces that are Calabi-Yau manifolds. In this paper, we consider theories possessing at least one $R$-symmetry.

**Twisted $(2,2)$ Supersymmetry Algebra** The (2,2) SUSY algebra has a rich structure and can be topological twisted. We require $Z$ or $\widetilde{Z}$ to be vanishing in order to have at least one $R$-symmetry. From Eq. (1.1.6), (1.1.7), and (1.1.8), one can observe that the generators $M$, $F_V$ and $F_A$ share similar commutators with other generators while they have vanishing commutators among themselves. From quantum mechanics, one can linearly combine them to obtain some other new well-defined operators, and two nontrivial and interesting combinations are the following

$$M'_A = M + F_V, \qquad M'_B = M + F_A. \tag{1.1.9}$$

We define

$$Q_A = \overline{Q}_+ + Q_-, \qquad Q_B = \overline{Q}_+ + \overline{Q}_-. \tag{1.1.10}$$

If we treat $M'_A$ or $M'_B$ as a new generator of Lorentz rotations, along with $Q_A$ or $Q_B$, one can obtain the $A$ or $B$-model respectively, with BRST operators $Q_A$, $Q_B$

$$Q_A^2 \;=\; Q_B^2 = 0, \tag{1.1.11}$$

$$[M'_A, Q_A] = 0, \qquad [M'_B, Q_B] = 0, \tag{1.1.12}$$

$$[M'_A, Q_+] = -2Q_+, \qquad [M'_B, Q_+] = -2Q_+, \tag{1.1.13}$$

$$\left[M'_A, \overline{Q}_-\right] = 2\overline{Q}_-, \qquad [M'_B, Q_-] = 2Q_-. \tag{1.1.14}$$

From the above, one can easily see that some supercharges become worldsheet scalars while the others become worldsheet vectors. Therefore, one can define twisted GLSMs on arbitary curved worldsheets. These are topological quantum field theories which were studied initially by Witten [13].

**Observables of $(2,2)$ GLSMs**

The language of superspace is convenient to use when studying *supersymmetric* quantum field theories. These superspaces can be represented in terms of superspace coordinates $x^0$, $x^1$, $\theta^\pm$, $\overline{\theta}^\pm$. One can introduces a representation of the supercharges in terms of superspace coordinates as

$$Q_\pm = \frac{\partial}{\partial\theta^\pm} + i\overline{\theta}^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right), \qquad \overline{Q}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} - i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{1.1.15}$$

Then one can introduce covariant derivative in superspace which commute with the supercharges

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - i\overline{\theta}^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right), \qquad \overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + i\theta^\pm\left(\frac{\partial}{\partial x^0} \pm \frac{\partial}{\partial x^1}\right). \tag{1.1.16}$$

The $R$-symmetry generators are

$$F_V = \theta^+ \frac{\partial}{\partial\theta^+} + \theta^- \frac{\partial}{\partial\theta^-} - \overline{\theta}^+ \frac{\partial}{\partial\overline{\theta}^+} - \overline{\theta}^- \frac{\partial}{\partial\overline{\theta}^-}, \quad F_A = \theta^+ \frac{\partial}{\partial\theta^+} - \theta^- \frac{\partial}{\partial\theta^-} - \overline{\theta}^+ \frac{\partial}{\partial\overline{\theta}^+} + \overline{\theta}^- \frac{\partial}{\partial\overline{\theta}^-}. \quad (1.1.17)$$

We describe supersymmetric multiplets of fields in terms of a superfield, a function on superspace. Short representations, known as chiral superfields and twisted chiral superfields. A chiral superfield is defined by

$$\overline{D}_\pm \Phi = 0, \quad (1.1.18)$$

and can be expanded as

$$\Phi = \phi + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- + 2\theta^+\theta^- F + \ldots, \quad (1.1.19)$$

where $F$ is a complex auxiliary field and $+\ldots$ stands for terms only involving the derivatives of $\phi$ and $\psi$. The conjugate of a superfield $\Phi$, $\overline{\Phi}$ can be defined in a similar fashion as $D_\pm\overline{\Phi} = 0$. We call this an anti-chiral superfield. Following the notation of [25], we can introduce twisted chiral superfields which obey the following constraint

$$\overline{D}_+ Y = D_- Y = 0. \quad (1.1.20)$$

A twisted superfield $Y$ can be expressed

$$Y = y + \sqrt{2}\theta^+\overline{\chi}_+ + \sqrt{2}\overline{\theta}^-\chi_- + 2\theta^+\overline{\theta}^- G + \ldots, \quad (1.1.21)$$

where $G$ is a complex auxiliary field and $+\ldots$ has no extra fields and only involves the derivatives of the component fields. We can also define the twisted anti-chiral superfield $\overline{Y}$ which satisfies $D_+\overline{Y} = \overline{D}_-\overline{Y} = 0$. $\overline{Y}$ is the hermitian conjugate of a twisted chiral superfield $Y$.

Gauged linear sigma models have gauge dynamics, therefore we need to introduce vector superfields. A vector superfield $\mathcal{V}$, in the Wess-Zumino gauge, can be written in terms of component fields

$$\begin{aligned} \mathcal{V} &= \theta^-\overline{\theta}^- (v_0 - v_1) + \theta^+\overline{\theta}^+ (v_0 + v_1) - \theta^-\overline{\theta}^+\sigma + \theta^+\overline{\theta}^-\overline{\sigma}, \\ &\quad + \sqrt{2}i\overline{\theta}^+\overline{\theta}^- \left(\theta^-\lambda_- + \theta^+\lambda_+\right) + 2\theta^-\theta^+\overline{\theta}^+\overline{\theta}^- D \end{aligned} \quad (1.1.22)$$

where $v_\mu$ is a vector field, the Dirac fermions $\lambda_\pm$ and its conjugate $\overline{\lambda}_\pm$ are superpartners, and $\sigma$ and $\overline{\sigma}$ are scalars. In a nonabelian gauge theory, all the fields are in the adjoint representation of the gauge group. The gauge covariant derivatives are defined as

$$\mathcal{D}_\pm = e^{-V} D_\pm e^V, \qquad \overline{\mathcal{D}}_\pm = e^{-V}\overline{D}_\pm e^V. \quad (1.1.23)$$

One can then define the superfield strength

$$\begin{aligned} \Sigma &= \frac{1}{2}\left\{\overline{\mathcal{D}}_+, \mathcal{D}_-\right\} \\ &= \sigma + i\sqrt{2}\theta^+\overline{\lambda}_- - i\sqrt{2}\overline{\theta}^-\lambda_- + 2\theta^+\overline{\theta}^-(D - iF_{01}) + \ldots, \end{aligned} \quad (1.1.24)$$

where $F_{01}$ is the field strength of the $v_\mu$ field. One can easily check that the superfield $\Sigma$ is a twisted chiral superfield obeying $\overline{\mathcal{D}}_+\Sigma = \mathcal{D}_-\Sigma = 0$.

**Observables of the $A$-model** The BRST transformations of the vector superfield components in the $A$-model are

$$[Q_A, v_+] = \sqrt{2}i\overline{\lambda}_+, \qquad [Q_A, v_-] = \sqrt{2}i\lambda_-, \tag{1.1.25}$$

$$[Q_A, \sigma] = 0, \qquad [Q_A, \overline{\sigma}] = -\sqrt{2}i\overline{\lambda}_- - \sqrt{2}i\lambda_+, \tag{1.1.26}$$

$$\{Q_A, \lambda_+\} = \sqrt{2}i\left[D + iF_{01} + \frac{i}{2}[\sigma, \overline{\sigma}]\right], \qquad \left\{Q_A, \overline{\lambda}_+\right\} = \sqrt{2}D_+\sigma, \tag{1.1.27}$$

$$\left\{Q_A, \overline{\lambda}_-\right\} = -\sqrt{2}i\left[D + iF_{01} - \frac{i}{2}[\sigma, \overline{\sigma}]\right], \qquad \{Q_A, \lambda_-\} = \sqrt{2}D_-\sigma, \tag{1.1.28}$$

where $v_\pm = v_0 \pm v_1$, $D_\pm = D_0 \pm D_1$. One can easily show that the observables of $A$-model $GLSM$ are functions of the $\sigma$ fields. In a nonabelian gauge groups, one should use gauge invariant combinations of sigmas such as trace of powers of sigma.

The BRST transformations of the components of a charged chiral superfield in the $A$-model are

$$[Q_A, \phi] = \sqrt{2}\psi_-, \tag{1.1.29}$$

$$\{Q_A, \psi_+\} = \sqrt{2}iD_+\phi + \sqrt{2}F, \qquad \{Q_A, \psi_-\} = \sqrt{2}\sigma\phi. \tag{1.1.30}$$

One can assign $R$-charges for these superfields at the classical level,

$$R_V(\Sigma) = 0, \qquad R_V(\Phi_i) = r_i, \qquad R_A(\Sigma) = 2. \tag{1.1.31}$$

Note that $R_A$ is generally anomalous in quantum field theory for twisted $A$-twisted theories unless the theory flows to a nontrivial SCFT, such as a nonlinear sigma model on a Calabi-Yau.

**Observables of the $B$-model** The $B$ models we mainly consider in this thesis are $B$-twisted Landau-Ginzburg theories. These theories are functions of the lowest components field of the chiral superfields Y, which in mirror constructions will typically obey a periodicity condition $Y = Y + 2\pi in$. (Note that the chiral superfields in the mirror frame correspond to twisted chiral superfields in the original frame). The BRST transformations of the components of a chiral superfield

$$[Q_B, y] = 0, \tag{1.1.32}$$

$$\{Q_B, \bar{\chi}_+\} = \sqrt{2}iD_+y, \qquad \{Q_B, \chi_-\} = -\sqrt{2}iD_-y. \tag{1.1.33}$$

From Eq. (1.1.32) and (1.1.33), one can find that the observables of the $B$-model are functions of the lowest component field of the chiral superfields Y.

The R-charge assigned to the $Y$ field is

$$R_A(\exp(-Y)) = 2 \qquad R_V(Y) = 0, \tag{1.1.34}$$

Given the Y fields' periodicities, note that $e^{-y}$ is a well-defined observable in $B$-model.

**Lagrangian of Landau-Ginzburg Theories** The Landau-Ginzburg theories, we consider have $k$ auxiliary fields $\Sigma_a$ and $N$ chiral superfields $Y_i$. The general Lagrangian of this system is given by

$$L = \int d^4\theta \left( K\left(Y_i, \overline{Y}_i\right) - \sum_a \frac{1}{2e_a^2} \overline{\Sigma}_a \Sigma_a \right) + \left( \int d^2\theta f\left(Y_i, \Sigma_a, t_b\right) + c.c \right), \tag{1.1.35}$$

where $t_a = r_a - i\theta_a$ are FI parameters, if Landau Ginzburg theories are mirror to GLSMs. The Kahler potential can be written as

$$K\left(Y_i, \overline{Y}_i\right) = -\frac{1}{2} \sum_i \left(Y_i + \overline{Y}_i\right) \log\left(Y_i + \overline{Y}_i\right). \tag{1.1.36}$$

The superpotential is

$$f\left(Y_i, \Sigma_a, t_b\right) = \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{N} Q_i^a Y_i - t^a \right) + \sum_i \exp(-Y_i). \tag{1.1.37}$$

In [25], the superpotential is split into a "pertubative" piece

$$\sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{N} Q_i^a Y_i - t^a \right), \tag{1.1.38}$$

and a "nonpertubative" piece

$$\sum_i \exp(-Y_i). \tag{1.1.39}$$

Their nomenclature reflects how their superpotential is derived from a gauged linear sigma model. We will review the Hori-Vafa mirror construction in section 1.2.

**Lagrangians for $(2,2)$ GLSMs**

**Abelian Gauged Linear Sigma Models** One can consider abelian gauged linear sigma models for compact simplicial toric varieties [28]. The theory has gauge group $U(1)^k$ with $N > k$ charged chiral superfields $\Phi_i$, the charge matrix is $Q_i^a$, and vector superfields denoted by $\mathcal{V}_a$. The bare Lagrangian is given by

$$L = \int d^4\theta \left( \sum_{i=1}^{N} \overline{\Phi}_i e^{2\sum_{a=1}^{k} Q_i^a \mathcal{V}_a} \Phi_i - \sum_a \frac{1}{2e_a^2} \overline{\Sigma}^a \Sigma^a \right) + \frac{1}{2} \left( -\int d^2\widetilde{\theta} \sum_a t_0^a \Sigma_a + c.c \right), \tag{1.1.40}$$

where the $t_0^a$ are

$$t_0^a = r_0^a - i\theta^a. \tag{1.1.41}$$

The bare FI parameters are renormalized to cancel one-loop divergences. The dependence of the bare parameter $r_0$ on the cutoff $\Lambda_{UV}$ is

$$r_0^a = \frac{1}{2\pi}\sum_{i=1}^{N} Q_i^a \log\left(\frac{\Lambda_{UV}}{\Lambda}\right). \tag{1.1.42}$$

$\Lambda$ is a dynamical scale that replaces $r$ as a physical parameter to describe theories with $\sum_{i=1}^{N} Q_i^a \neq 0$.

The theory above at least has global symmetry $U(1)^{N-k}$, thus we can turn on twisted masses in the theory. It means we can add some extra mass terms in the classical lagrangian as follows

$$L_m = \int d^4\theta \sum_{i=1}^{N} \widetilde{\mathbf{m}}_i \overline{\Phi}_i \Phi_i, \tag{1.1.43}$$

where the twisted masses $\widetilde{\mathbf{m}}_i = -2\widetilde{m}_i\theta^-\overline{\theta}^+$ have vanishing vector $R$-charge and axial $R$-charge two. Only $N-k$ of them are independent parameters.

To describe a hypersurface in the toric variety, we add a superpotential to the GLSM

$$L_W = \int d^2\theta \sum_m P_m G_m(\Phi_i) + c.c., \tag{1.1.44}$$

where $G_m(\Phi_i)$ are polynomials in $\Phi_i$, the $P_m$ fields have vector $R$-charge two and the superpotential is gauge-invariant.

When the matter fields are heavy, one can integrate them out, leaving a pure Coulomb branch theory

$$\widetilde{W}_{eff}(\Sigma_1,\ldots,\Sigma_k) = -\sum_{a=1}^{k}(\Sigma_a + \widetilde{m}_a)\left(\sum_{i=1}^{N} Q_i^a\left(\log\left(\frac{\sum_{b=1}^{k} Q_i^b\Sigma_b + \widetilde{m}_i}{\mu}\right) - 1\right) + t^a\right) \tag{1.1.45}$$

where we have included twisted masses $\widetilde{m}$, and $\sum_{a=1}^{k}\widetilde{m}_a Q_i^a = \widetilde{m}_i$. One can show that the twisted superpotential above breaks the axial $R$-symmetry in general except in the special case that $\sum_i Q_i^a = 0$. Define $t_i$ by $t_a = \sum_{i=1}^{N} Q_i^a t_i$.

From equation (1.1.45), we can obtain the vacua by solving

$$\exp\frac{\partial\widetilde{W}_{eff}(\Sigma_1,\ldots,\Sigma_k)}{\partial\Sigma_a} = 1, \tag{1.1.46}$$

thus, we have

$$\prod_{i=1}^{N}\left(\sum_{a=1}^{k} Q_i^a\sigma_a + \widetilde{m}_i\right)^{Q_i^a} = q^a, \qquad \text{where} \quad q^a = e^{-t^a} \tag{1.1.47}$$

Eq. (1.1.47) gives the chiral ring relations. We will set $\mu = 1$ in the above equation and also in the rest of this paper.

**Phases of Gauged Linear Sigma Models** In the early literature [15], there were some heuristic ideas regarding the correspondence between certain Calabi-Yau nonlinear sigma models and Landau-Ginzburg models. Witten's paper [16] clarified the relationship by demonstrating that those nonlinear sigma models and Landau-Ginzburg models are different phases of the same GLSMs. Other evidence for this picture can be found in [17], which used mirror symmetry in the analysis.

In special cases, some (hybrid) Landau-Ginzburg models can be interpreted geometrically. In this section we will review an example that appeared in [29].

This gauged linear sigma model has a total of six chiral superfields, four ($\phi_i, i \in \{1, \cdots, 4\}$) of charge 1 corresponding to homogenous coordinates on $\mathbb{P}^3$, and two $(p_1, p_2)$ of charge -2 corresponding to two hypersurfaces in $\mathbb{P}^3$.

The D-term of the Lagrangian for this GLSM reads

$$\sum_{i=1}^{4} \mid \phi_i \mid^2 -2 \mid p_1 \mid^2 -2 \mid p_2 \mid^2 = r. \tag{1.1.48}$$

• When $r \gg 0$, then we see that not all the $\phi_i$ can vanish, corresponding to their interpretation as homogeneous coordinates on $\mathbb{P}^3$. By considering the restriction of the F-terms, we recover the geometric interpretation of this gauged linear sigma model as a complete intersection.

• The $r \ll 0$ phase is a hybrid Landau-Ginzburg model. The D-term constraint implies that not all the $p_a$ can vanish and they act as homogeneous coordinates on a $\mathbb{P}^1$, except that these homogeneous coordinates have charge 2 rather than charge 1.

Because of these nonminimal charges, the hybrid Landau-Ginzburg model is ultimately going to describe a (branched) double cover. The superpotential

$$W = p_1 G_1 + p_2 G_2$$

(where the $G_1$ and $G_2$ are quadric polynomials) can be rewritten in the form

$$W = \sum_{ij} \phi_i A^{ij}(p) \phi_j, \tag{1.1.49}$$

where $A^{ij}$ is a rank 4 symmetric matrix with entries linear in the $p$'s. Away from the locus where $A$ drops rank, *i.e*, away from the hypersurface $\det A = 0$, the $\phi$ fields are all massive, leaving only the $p$ massless, which all have charge -2. A GLSM with nonminimal charges describes a gerbe [30–33]. Furthermore, in [29], they claimed that this is actually a branched double cover of $\mathbb{P}^1$ which is torus. One can also refer to [34–36] for more examples.

### 1.1.2 $\mathcal{N}=(0,2)$ GLSMs

The contents of this section were adapted, with minor modifications, with permission from JHEP, from our publication [42]. In this section, we review some basics of $\mathcal{N}=(0,2)$ GLSMs. Our notation and conventions are slightly different from section 1.1.1 and will instead largely follow [42]. The physical results we will compute do not rely on the notation.

#### $\mathcal{N}=(0,2)$ Curved-Space Supersymmetry

We wish to consider $\mathcal{N}=(0,2)$ supersymmetric gauge theories with an $R$-symmetry, denoted $U(1)_R$. In this section, we explain how to preserve supersymmetry on any closed orientable Riemann surface $\Sigma_{\mathbf{g}}$. We then discuss $\mathcal{N}=(0,2)$ supersymmetric multiplets, Lagrangians and observables on curved space.

#### Background Supergravity and the Half-Twist

Consider any $\mathcal{N}=(0,2)$ supersymmetric field theory with an $R$-symmetry. The theory possesses a conserved $\mathcal{R}$-symmetry current $j_\mu^{(R)}$ which sits in the $\mathcal{N}=(0,2)$ $\mathcal{R}$-multiplet [48] together with the right-moving supercurrent $S_+^\mu$, $\widetilde{S}_+^\mu$ and the energy-momentum tensor $T_{\mu\nu}$. Such a theory can be coupled to a $(0,2)$ background supergravity multiplet containing a metric $g_{\mu\nu}$, two gravitini $\psi_{-\mu}$, $\widetilde{\psi}_{-\mu}$ and a $U(1)_R$ gauge field $A_\mu^{(R)}$. At first order around flat space, $g_{\mu\nu}=\delta_{\mu\nu}+\Delta g_{\mu\nu}$, the supergravity multiplet couples to the $\mathcal{R}$-multiplet according to:

$$\mathscr{L}_{SUGRA}=-\frac{1}{2}\Delta g_{\mu\nu}T^{\mu\nu}+A_\mu^{(R)}j^\mu_{(R)}-\frac{1}{2}\left(S_+^\mu\psi_{-\mu}-\widetilde{S}_+^\mu\widetilde{\psi}_{-\mu}\right)\ . \tag{1.1.50}$$

Curved-space rigid supersymmetry is best understood in terms of a supersymmetric background for the metric and its superpartners [45, 50]. A background $(\Sigma_{\mathbf{g}}, g_{\mu\nu}, A_\mu^{(R)})$ is supersymmetric if and only if the supersymmetry variations of the gravitini vanish for some non-trivial supersymmetry parameters. In the present case, we must have:

$$(\nabla_\mu - iA_\mu^{(R)})\zeta_-=0\ ,\qquad\qquad (\nabla_\mu + iA_\mu^{(R)})\widetilde{\zeta}_-=0\ . \tag{1.1.51}$$

Note that the spinors $\zeta_-$, $\widetilde{\zeta}_-$ have $R$-charge $\pm1$, respectively. One can derive these equations by studying linearized supergravity along the lines of [46, 47]. (See also [51].) The only way to solve (1.1.51) on $\Sigma_{\mathbf{g}}$ is by setting the gauge field $A_\mu=\pm\frac{1}{2}\omega_\mu$, with $\omega_\mu$ the spin connection. This preserves either $\zeta_-$ or $\widetilde{\zeta}_-$. (The only obvious exception is when $\Sigma_{\mathbf{g}=1}$ is a flat torus.) We choose to preserve $\widetilde{\zeta}_-$:

$$A_\mu^{(R)}=\frac{1}{2}\omega_\mu\ ,\qquad \zeta_-=0\ ,\qquad \partial_\mu\widetilde{\zeta}_-=0\ . \tag{1.1.52}$$

Since $\widetilde{\zeta}_-$ is a constant, it is obviously well-defined globally on $\Sigma_{\mathbf{g}}$. This supersymmetric background is known as the half-twist [16] and it preserves one supercharge $\widetilde{Q}_+$ on any $\Sigma_{\mathbf{g}}$. It follows from (1.1.52) that

$$\frac{1}{2\pi}\int_\Sigma dA^{(R)} = -\frac{1}{8\pi}\int_\Sigma d^2x\sqrt{g}\,\mathrm{R} = g-1\ , \tag{1.1.53}$$

where R is the Ricci scalar of $g_{\mu\nu}$, and therefore the $R$-charge is quantized in units of $\frac{1}{g-1}$. In particular, the $R$-charge is integer-quantized on the Riemann sphere.

### Supersymmetry Multiplets

Since the supersymmetry parameter $\widetilde{\zeta}_-$ is covariantly conserved, the supersymmetry variations in curved space can be obtained from the flat space expressions by replacing derivatives by covariant derivatives. Let us denote by $\delta$ the supercharge $\widetilde{Q}_+$ acting on fields. Importantly, $\delta$ is nilpotent:

$$\delta^2 = 0\ . \tag{1.1.54}$$

The half-twist effectively assigns to every field a shifted spin

$$S = S_0 + \frac{1}{2}R\ , \tag{1.1.55}$$

where $S_0$ and $R$ are the flat-space spin and the flat-space $R$-charge, respectively. It is convenient to use notation adapted to the twist. We also use the covariant derivatives

$$D_\mu\varphi_{(s)} = (\partial_\mu - is\omega_\mu)\,\varphi_{(s)}\ , \tag{1.1.56}$$

acting on a field of twisted spin $s$. the curved-space conventions can be found at [41].

**General multiplet** Let $\mathcal{S}_s$ be a general long multiplet of $\mathcal{N}=(0,2)$ supersymmetry with $2+2$ complex components, with $s$ the spin of the lowest component:

$$\mathcal{S}_s = (\mathbf{C}\ ,\ \chi_{\bar{1}}\ ,\ \widetilde{\chi}\ ,\ \mathbf{v}_{\bar{1}})\ . \tag{1.1.57}$$

The four components of (1.1.57) have spin $(s, s-1, s, s-1)$, respectively. The curved-space supersymmetry transformations are:

$$\begin{aligned} &\delta\mathbf{C} = -i\widetilde{\chi}\ , &&\delta\chi_{\bar{1}} = 2i\mathbf{v}_{\bar{1}} + 2D_{\bar{1}}\mathbf{C}\ ,\\ &\delta\widetilde{\chi} = 0\ , &&\delta\mathbf{v}_{\bar{1}} = D_{\bar{1}}\widetilde{\chi}\ . \end{aligned} \tag{1.1.58}$$

Note that $\delta$ is a scalar—it commutes with the spin operator. All the supersymmetry multiplets of interest to us are made out of one or two general multiplets subject to some conditions.

**Chiral multiplets** The simplest $\mathcal{N}=(0,2)$ multiplets are the chiral multiplet $\Phi_i$ and the antichiral multiplet $\widetilde{\Phi}_i$. In flat space, they contains a complex scalar and a spin $-\frac{1}{2}$ fermion. After twisting, one has:

$$\Phi_i = (\phi_i \ , \ \mathcal{C}_i) \ , \qquad\qquad \widetilde{\Phi}_i = \left(\widetilde{\phi}_i \ , \ \widetilde{\mathcal{B}}_i\right) \ . \tag{1.1.59}$$

If $\Phi_i$, $\widetilde{\Phi}_i$ are assigned integer $R$-charges $r_i$ and $-r_i$, the components (1.1.59) have twisted spins

$$\left(\frac{r_i}{2}, \frac{r_i}{2} - 1\right) \tag{1.1.60}$$

and

$$\left(-\frac{r_i}{2}, -\frac{r_i}{2}\right) , \tag{1.1.61}$$

respectively. The supersymmetry transformations rules are:

$$\begin{aligned} &\delta\phi_i = 0 \ , && \delta\widetilde{\phi}_i = \widetilde{\mathcal{B}}_i \ , \\ &\delta\mathcal{C}_i = 2iD_{\bar{1}}\phi^i \ , && \delta\widetilde{\mathcal{B}}_i = 0 \ . \end{aligned} \tag{1.1.62}$$

Note that $\Phi$ and $\widetilde{\Phi}$ can be understood as a general multiplets (1.1.57) satisfying the constraints $\widetilde{\chi} = 0$ and $\chi_{\bar{1}} = 0$, respectively.

Given any holomorphic function $\mathcal{F}(\Phi_i)$ of the chiral multiplets $\Phi_i$, one can construct a new chiral multiplet as long as $\mathcal{F}$ itself has definite $R$-charge, and similarly with the anti-chiral multiplets:

$$(\mathcal{F} \ , \ \mathcal{C}^{\mathcal{F}}) = \left(\mathcal{F}(\phi) \ , \ \frac{\partial\mathcal{F}}{\partial\phi_i}\mathcal{C}_i\right) \ , \qquad (\widetilde{\mathcal{F}}, \widetilde{\mathcal{B}}^{\mathcal{F}}) = \left(\widetilde{\mathcal{F}}(\widetilde{\phi}) \ , \ \frac{\partial\widetilde{\mathcal{F}}}{\partial\widetilde{\phi}_i}\widetilde{\mathcal{B}}_i\right) \ . \tag{1.1.63}$$

**Fermi multiplets** Another important multiplet is the Fermi multiplet, whose lowest flat-space component is a spin $+\frac{1}{2}$ fermion. For each Fermi multiplet $\Lambda_I$, we have a function $\mathcal{E}_I(\phi)$ holomorphic in the chiral fields of the theory. Similarly, an anti-Fermi multiplet $\widetilde{\Lambda}_I$ comes with an anti-holomorphic function $\widetilde{\mathcal{E}}(\widetilde{\phi})$. For the elementary Fermi multiplets, these functions must be specified as part of the data defining the $\mathcal{N}=(0,2)$ theory. In order to preserve the $R$-symmetry, they must have $R$-charges $R[\mathcal{E}_I] = r_I + 1$, with $r_I$ is the $R$-charge of $\Lambda$. Similarly, the charge-conjugate multiplet $\widetilde{\Lambda}$ has $R$-charge $-r_I$ and $R[\widetilde{\mathcal{E}}_I] = -r_I - 1$.

A Fermi multiplet $\Lambda_I$ of $R$-charge $r_I$ has components:

$$\Lambda_I = (\Lambda_I \ , \ \mathcal{G}_I) \ , \qquad E_I = \left(\mathcal{E}_I, \mathcal{C}_I^E\right) \ , \tag{1.1.64}$$

where $E_I$ is the chiral multiplet of lowest component $\mathcal{E}_I$. The spins of (1.1.64) are

$$\left(\frac{r_I}{2} + \frac{1}{2}, \frac{r_I}{2} - \frac{1}{2}\right) \tag{1.1.65}$$

and

$$\left(\frac{r_I}{2}+\frac{1}{2},\frac{r_I}{2}-\frac{1}{2}\right), \tag{1.1.66}$$

respectively, and the supersymmetry transformations are given by:

$$\begin{aligned} &\delta\Lambda_I = 2\mathcal{E}_I \ , && \delta\mathcal{E}_I = 0 \ , \\ &\delta\mathcal{G}_I = 2\mathcal{C}_I^E - 2iD_{\bar{1}}\Lambda_I \ , && \delta\mathcal{C}_I^E = 2iD_{\bar{1}}\mathcal{E}_I \ . \end{aligned} \tag{1.1.67}$$

Similarly, for an anti-Fermi multiplet $\widetilde{\Lambda}_I$ of $R$-charge $-r_I$, we have the components:

$$\widetilde{\Lambda}_I = \left(\widetilde{\Lambda}_I \ , \ \widetilde{\mathcal{G}}_I\right) \ , \qquad \widetilde{E}_I = \left(\widetilde{\mathcal{E}}_I \ , \ \widetilde{\mathcal{B}}_I^E\right) \ , \tag{1.1.68}$$

of spins

$$\left(-\frac{r_I}{2}+\frac{1}{2},-\frac{r_I}{2}+\frac{1}{2}\right) \tag{1.1.69}$$

and

$$\left(-\frac{r_I}{2}-\frac{1}{2},-\frac{r_I}{2}-\frac{1}{2}\right), \tag{1.1.70}$$

respectively, and

$$\begin{aligned} &\delta\widetilde{\Lambda}_I = \widetilde{\mathcal{G}}_I \ , && \delta\widetilde{\mathcal{E}}_I = \widetilde{\mathcal{B}}_I^E \ , \\ &\delta\widetilde{\mathcal{G}}_I = 0 \ , && \delta\widetilde{\mathcal{B}}_I^E = 0 \ . \end{aligned} \tag{1.1.71}$$

The product of a chiral multiplet $\Phi_i$ of $R$-charge $r_i$ with a Fermi multiplet $\Lambda_I$ of $R$-charge $r_I$ gives another Fermi multiplet of $R$-charge $r_i + r_I$, with components:

$$\Lambda^{(\Phi\Lambda)} = \left(\phi_i\Lambda_I \ , \ \phi_i\mathcal{G}_I - \mathcal{C}_i\Lambda_I\right) \ , \qquad E_I^{(\Phi\Lambda)} = \left(\phi_i\mathcal{E}_I \ , \phi_i\mathcal{C}_I^E + \mathcal{C}_i\mathcal{E}_I\right) \ . \tag{1.1.72}$$

Similarly, for the charge-conjugate multiplet:

$$\widetilde{\Lambda}^{(\Phi\Lambda)} = \left(\widetilde{\phi}_i\widetilde{\Lambda}_I \ , \ \widetilde{\phi}_i\widetilde{\mathcal{G}}_I + \widetilde{\mathcal{B}}_i\widetilde{\Lambda}_I\right) \ , \qquad \widetilde{E}_I^{(\Phi\Lambda)} = \left(\widetilde{\phi}_i\widetilde{\mathcal{E}}_I \ , \ \widetilde{\phi}_i\widetilde{\mathcal{B}}_I^E + \widetilde{\mathcal{B}}_i\widetilde{\mathcal{E}}_I^E\right) \ . \tag{1.1.73}$$

**Vector multiplet** Consider a compact Lie group $\mathbf{G}$ and its Lie algebra $\mathfrak{g}$. The associated vector multiplet is built out of two $\mathfrak{g}$-valued general multiplets $(\mathcal{V}, \mathcal{V}_1)$ of spins 0 and 1, subject to the gauge transformations:

$$\delta_\Omega\mathcal{V} = \frac{i}{2}(\Omega - \widetilde{\Omega}) + \frac{i}{2}[\Omega + \widetilde{\Omega}, \mathcal{V}] \ , \qquad \delta_\Omega\mathcal{V}_1 = \frac{1}{2}\partial_1(\Omega + \widetilde{\Omega}) + \frac{i}{2}[\Omega + \widetilde{\Omega}, \mathcal{V}_1] \ , \tag{1.1.74}$$

where $\Omega$ and $\widetilde{\Omega}$ are $\mathfrak{g}$-valued chiral and antichiral multiplets of vanishing $R$-charge. [1] One can use (1.1.74) to fix a Wess-Zumino (WZ) gauge, wherein the vector multiplet has components:

$$\mathcal{V} = (0 \ , \ 0 \ , \ 0 \ , \ a_{\bar{1}}) \ , \qquad \mathcal{V}_1 = \left(a_1 \ , \ \widetilde{\lambda} \ , \ \lambda_1, \ D\right) \ , \tag{1.1.75}$$

[1] The addition rules implicit in (1.1.74) are obtained by embedding $\Omega$, $\widetilde{\Omega}$ into general multiplets.

The non-zero components have spin $-1$ and $(1,0,1,0)$, respectively. Under the residual gauge transformations $\Omega = \widetilde{\Omega} = (\omega, 0)$, we have

$$\delta_\omega a_\mu = \partial_\mu \omega + i[\omega, a_\mu] \ , \quad \delta_\omega \lambda_1 = i[\omega, \lambda_1] \ , \quad \delta_\omega \widetilde{\lambda} = i[\omega, \widetilde{\lambda}] \ , \quad \delta_\omega D = i[\omega, D] \ . \tag{1.1.76}$$

The supersymmetry transformations are:

$$\begin{aligned} &\delta a_{\bar{1}} = 0 \ , && \delta a_1 = -i\lambda_1 \ , && \\ &\delta\widetilde{\lambda} = -i(D - 2if_{1\bar{1}}) \ , && \delta\lambda_1 = 0 \ , && \delta D = -2D_{\bar{1}}\lambda_1 \ , \end{aligned} \tag{1.1.77}$$

where we defined the field strength

$$f_{1\bar{1}} = \partial_1 a_{\bar{1}} - \partial_{\bar{1}} a_1 - i[a_1, a_{\bar{1}}] \ , \tag{1.1.78}$$

and the covariant derivative $D_\mu$ is also gauge-covariant. Here and henceforth, $\delta$ denotes the supersymmetry variation in WZ gauge, which includes a compensating gauge transformation.

**Field strength multiplet** From the vector multiplet (1.1.75), one can build a Fermi and an anti-Fermi multiplet:

$$\mathcal{Y} = \Big(2\lambda_1 \ , \ 2i(2if_{1\bar{1}})\Big) \ , \qquad \widetilde{\mathcal{Y}} = \Big(\widetilde{\lambda} \ , \ -i(D - 2if_{1\bar{1}})\Big) \ , \tag{1.1.79}$$

of $R$-charge 1 (that is, the multiplets $\mathcal{Y}$ and $\widetilde{\mathcal{Y}}$ have twisted spin 1 and 0, respectively), with $\mathcal{E}_{\mathcal{Y}} = 0$. These field strength multiplets are $\mathfrak{g}$-valued. [2]

**Charged chiral and Fermi multiplets** Consider the chiral multiplets $\Phi_i$ in the representations $\mathfrak{R}_i$ of the gauge algebra $\mathfrak{g}$, the Fermi multiplets $\Lambda_I$ in the representations $\mathfrak{R}_I$ of $\mathfrak{g}$, and similarly for the charge conjugate multiplets $\widetilde{\Phi}_i$ and $\widetilde{\Lambda}_I$. Under a gauge transformation (1.1.74), we have

$$\delta_\Omega \Phi = i\Omega\Phi \ , \qquad \delta_\Omega \widetilde{\Phi} = -i\widetilde{\Phi}\widetilde{\Omega} \ , \qquad \delta_\Omega \Lambda = i\Omega\Lambda \ , \qquad \delta_\Omega \widetilde{\Lambda} = -i\widetilde{\Lambda}\widetilde{\Omega} \ , \tag{1.1.80}$$

with $\Omega$, $\widetilde{\Omega}$ valued in the corresponding representations. The supersymmetry transformations in WZ gauge are given by (1.1.62), (1.1.67) and (1.1.71) with the understanding that the covariant derivative $D_\mu$ is also gauge-covariant

[2] Our definition of $\mathcal{Y}$ in (1.1.79) a slightly idiosyncratic. There is a unique definition for $\mathcal{Y}$ in flat space, namely $(2\lambda_1 \ , \ i(D + 2if_{1\bar{1}}))$, but in curved space with one supercharge the present choice is also consistent. The present choice is the same as in [41].

**Supersymmetric Lagrangians**

There are four types of supersymmetric Lagrangians we can consider on curved space:

1. $\mathbf{v}$-*term.* Given a general multiplet $\mathcal{S}_1$ of twisted spin $s = 1$ with components (1.1.57), we can built the supersymmetric Lagrangian
$$\mathscr{L}_{\mathbf{v}} = \mathbf{v}_{1\bar{1}} \tag{1.1.81}$$
from the top component. It is clear from (1.1.58) that the corresponding action is both $\delta$-closed and $\delta$-exact.

2. $\mathcal{G}$-*term.* From a fermi multiplet $\Lambda$ with $s = 1$ (that is, $R$-charge $\frac{1}{2}$) and $\mathcal{E} = 0$, we have the supersymmetric Lagrangian
$$\mathscr{L}_{\mathcal{G}} = \mathcal{G}\ . \tag{1.1.82}$$
This term is *not* $\delta$-exact.

3. $\widetilde{\mathcal{G}}$-*term.* From an antifermi multiplet $\widetilde{\Lambda}$ with $s = 0$ (that is, $R$-charge $\frac{1}{2}$), we can similarly build
$$\mathscr{L}_{\mathcal{G}} = \widetilde{\mathcal{G}}\ . \tag{1.1.83}$$
We see from (1.1.71) that this term is both $\delta$-closed and $\delta$-exact.

4. *Improvement Lagrangian.* This term is special to curved space. Given a conserved current multiplet [46], the Lagrangian
$$\mathscr{L}_{\mathcal{J}} = A_\mu^{(R)} j^\mu + \frac{1}{4}\mathrm{R} J\ , \tag{1.1.84}$$
is supersymmetric upon using (1.1.52).

In the remainder of this section, we spell out the various Lagrangians that we shall need in this paper.

**Kinetic terms** All the standard kinetic terms are $\mathbf{v}$-terms and are therefore $\delta$-exact. Consider a $\mathfrak{g}$-valued vector multiplet. The standard supersymmetric Yang-Mills Lagrangian reads:
$$\mathscr{L}_{YM} = \frac{1}{e_0^2}\left(\frac{1}{2}(2if_{1\bar{1}})^2 - \frac{1}{2}D^2 - 2i\widetilde{\lambda} D_{\bar{1}}\lambda_1\right)\ . \tag{1.1.85}$$
Here and below, the appropriate trace over $\mathfrak{g}$ is implicit. The Lagrangian (1.1.85) is $\delta$-exact:
$$\mathscr{L}_{YM} = \frac{1}{e_0^2}\delta\left(\frac{1}{2i}\widetilde{\lambda}(D + 2if_{1\bar{1}})\right)\ . \tag{1.1.86}$$

Consider charged chiral multiplets $\Phi_i$ of $R$-charges $r_i$, transforming in representation $\mathfrak{R}_i$ of $\mathfrak{g}$. Their kinetic term reads

$$\mathscr{L}_{\widetilde{\Phi}\Phi} = D_\mu \widetilde{\phi}^i D^\mu \phi_i + \frac{r_i}{4} \mathrm{R} \widetilde{\phi}^i \phi_i + \widetilde{\phi}^i D \phi_i + 2i \widetilde{\mathcal{B}}^i D_1 \mathcal{C}_i - 2i \widetilde{\phi}^i \lambda_1 \mathcal{C}_i + i \widetilde{\mathcal{B}}^i \widetilde{\lambda} \phi_i \ , \tag{1.1.87}$$

where the vector multiplet fields $(a_\mu, \widetilde{\lambda}, \lambda_1, D)$ are suitably $\mathfrak{R}_i$-valued. The Lagrangian (1.1.87) is more conveniently written as:

$$\begin{aligned} \mathscr{L}_{\widetilde{\Phi}\Phi} &= \delta\left(2i\widetilde{\phi}^i D_1 \mathcal{C}_i + i\widetilde{\phi}^i \widetilde{\lambda} \phi_i\right) , \\ &= \widetilde{\phi}^i \left(-4 D_1 D_{\bar{1}} + D - 2i f_{1\bar{1}}\right) \phi_i + 2i \widetilde{\mathcal{B}}^i D_1 \mathcal{C}_i - 2i \widetilde{\phi}^i \lambda_1 \mathcal{C}_i + i \widetilde{\mathcal{B}}^i \widetilde{\lambda} \phi_i \ . \end{aligned} \tag{1.1.88}$$

Similarly, for charged Fermi multiplets $\Lambda_I$ of $R$-charges $r_I$ in representations $\mathfrak{R}_I$ of $\mathfrak{g}$, we have

$$\begin{aligned} \mathscr{L}_{\widetilde{\Lambda}\Lambda} &= \delta\left(-\widetilde{\Lambda}^I \mathcal{G}_I + \frac{1}{2} \widetilde{\mathcal{E}}^I \Lambda_I\right) , \\ &= -2i \widetilde{\Lambda}^I D_{\bar{1}} \Lambda_I - \widetilde{\mathcal{G}}^I \mathcal{G}_I + \widetilde{\mathcal{E}}^I \mathcal{E}_I + 2 \widetilde{\Lambda}^I \frac{\partial \mathcal{E}_I}{\partial \phi^i} \mathcal{C}^i + \frac{1}{2} \widetilde{\mathcal{B}}^i \frac{\partial \widetilde{\mathcal{E}}^I}{\partial \widetilde{\phi}^i} \Lambda_I \ , \end{aligned} \tag{1.1.89}$$

including the $\mathfrak{R}_I$-valued gauge field in the covariant derivatives $D_{\bar{1}}$. The holomorphic functions $\mathcal{E}_I(\phi)$ transform in the same representations $\mathfrak{R}_I$ as $\Lambda_I$.

**Superpotential terms** To each Fermi multiplet $\Lambda_I$, one can associate a holomorphic function of the chiral multiplets $J_I = J_I(\Phi)$, transforming in the representation $\bar{\mathfrak{R}}_I$ conjugate to $\mathfrak{R}_I$ and with $R$-charge $1 - r_I$. From these $\mathcal{N} = (0, 2)$ superpotential terms, one can build the $\mathcal{G}$-term Lagrangian (1.1.82) according to:

$$\mathscr{L}_{J_I} = i \sum_I \mathcal{G}^{(J_I)} = i \mathcal{G}^I J_I + i \Lambda^I \frac{\partial J_I}{\partial \phi^i} \mathcal{C}^i \ . \tag{1.1.90}$$

Note that this Lagrangian is not $\delta$-exact. Supersymmetry implies that

$$\mathcal{E}^I J_I = 0 \ . \tag{1.1.91}$$

Similarly, from the charge conjugate anti-holomorphic functions $\widetilde{J}_I = \widetilde{J}_I(\widetilde{\Phi})$ one builds the $\widetilde{\mathcal{G}}$-term:

$$\mathscr{L}_{\widetilde{J}_I} = -i \sum_I \widetilde{\mathcal{G}}^{(\widetilde{J}_I)} = -i \widetilde{\mathcal{G}}^I \widetilde{J}_I + i \widetilde{\Lambda}^I \frac{\partial \widetilde{J}_I}{\partial \widetilde{\phi}^i} \widetilde{\mathcal{B}}^i = \delta\left(-i \widetilde{\Lambda}^I \widetilde{J}_I\right) \ , \tag{1.1.92}$$

which is $\delta$-closed and $\delta$-exact.

**Fayet-Iliopoulos terms** Consider a gauge theory with Abelian factors $U(1)_A \subset \mathbf{G}$. From (1.1.79), we construct the gauge invariant Fermi multiplets

$$\mathcal{Y}_A = \mathrm{tr}_A(\mathcal{Y}) \ , \qquad \widetilde{\mathcal{Y}}_A = \mathrm{tr}_A(\widetilde{\mathcal{Y}}) \ , \tag{1.1.93}$$

where $\mathrm{tr}_A$ is the projection onto the $U(1)_A$ factor. These Fermi multiplet have vanishing $\mathcal{E}$-potential but they admit $J$-potentials. In the present work, we restrict ourselves to the case of a constant $J_{\mathcal{Y}_A} = J_A$ in the classical Lagrangian:

$$J_A = \tau_A \equiv \frac{\theta_A}{2\pi} + i\xi_A \ , \qquad\qquad \widetilde{J}_A = \widetilde{\tau}_A \equiv -2i\xi_A \ . \tag{1.1.94}$$

Here $\xi_A$ and $\theta_A$ are the Fayet-Iliopoulos (FI) and $\theta$-angles, respectively. (The unusual definition of $\widetilde{\tau}$ is on par with (1.1.79).) The corresponding supersymmetric Lagrangian reads

$$\mathcal{L}_{FI} = \frac{1}{2}\left(\tau\mathcal{G}^{\mathcal{Y}} + \widetilde{\tau}\widetilde{\mathcal{G}}^{\widetilde{\tau}}\right) = i\frac{\theta^A}{2\pi}\,\mathrm{tr}_A(2if_{1\bar{1}}) - \xi^A\,\mathrm{tr}_A(D) \ . \tag{1.1.95}$$

Note that the coupling $\widetilde{\tau}$ is $\delta$-exact while the coupling $\tau$ is not.

### GLSM Field Content and Anomalies

Consider a general $\mathcal{N} = (0,2)$ GLSM with a gauge group $\mathbf{G}$, and let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$. The gauge sector consists of a $\mathfrak{g}$-valued vector multiplet $(\mathcal{V}, \mathcal{V}_1)$. If $\mathbf{G}$ contains $U(1)$ factors,

$$\prod_{A=1}^{n} U(1)_A \subset \mathbf{G} \ , \tag{1.1.96}$$

we turn the FI parameters (1.1.94). Let us also define the quantity:

$$q_A = \exp(2\pi i\tau_A) \ . \tag{1.1.97}$$

The matter sector consists of chiral multiplets $\Phi_i$ of $R$-charges $r_i$ in representations $\mathfrak{R}_i$ of $\mathfrak{g}$, and of Fermi multiplets $\Lambda_I$ of $R$-charges $r_I$ in representations $\mathfrak{R}_I$ of $\mathfrak{g}$. To each $\Lambda_I$, we associate two chiral multiplets $\mathcal{E}_I = \mathcal{E}_I(\Phi)$ and $J_I = J_I(\Phi)$ constructed out of the chiral multiplets $\Phi_i$, satisfying $\mathcal{E}^I J_I = 0$, with $R$-charges

$$R[\mathcal{E}_I] = r_I + 1 \ , \qquad R[J_I] = 1 - r_I \ , \tag{1.1.98}$$

and such that $\mathrm{Tr}(\widetilde{\Lambda}^I\mathcal{E}_I)$ and $\mathrm{Tr}(\Lambda^I J_I)$ are gauge invariant.

Anomaly cancelation imposes further constraints on the matter content and on the $R$-charge assignment. Let us decompose the gauge algebra $\mathfrak{g}$ into semi-simple factors $\mathfrak{g}_\alpha$ and Abelian factors $u(1)_A$, $\mathfrak{g} \cong (\oplus_\alpha\mathfrak{g}_\alpha) \oplus (\oplus_A u(1)_A)$. The vanishing of the non-Abelian gauge anomalies requires

$$\sum_i T_{\mathfrak{R}_i^{(\alpha)}} - \sum_I T_{\mathfrak{R}_I^{(\alpha)}} - T_{\mathfrak{g}_\alpha} = 0 \ , \qquad \forall\alpha \ , \tag{1.1.99}$$

where $\mathfrak{R}^{(\alpha)}$ denotes the representation of $\mathfrak{g}_\alpha$ obtained by projecting the representation $\mathfrak{R}$ of $\mathfrak{g}$ onto $\mathfrak{g}_\alpha$, while $T_{\mathfrak{R}^{(\alpha)}}$ denotes the Dynkin index of $\mathfrak{R}^{(\alpha)}$ and $T_{\mathfrak{g}_\alpha}$ stands for the index of the adjoint representation of $\mathfrak{g}_\alpha$. For instance, one has $T_{\text{fund}} = T_{\overline{\text{fund}}} = \frac{1}{2}$ and $T_{su(N)} = N$ for the fundamental, antifundamental and adjoint representations of $su(N)$. In order to cancel the $U(1)^2$ gauge anomalies, we also need

$$\sum_i \dim \mathfrak{R}_i \, Q_i^A \, Q_i^B - \sum_I \dim \mathfrak{R}_I \, Q_I^A \, Q_I^B = 0 \; , \qquad \forall A, B \; , \tag{1.1.100}$$

where $Q_i^A$ and $Q_I^A$ are the $U(1)_A$ charges of the chiral and Fermi multiplets, respectively. In addition, the $U(1)_R$-gauge anomalies should vanish:

$$\sum_i \dim \mathfrak{R}_i \, (r_i - 1) \, Q_i^A - \sum_I \dim \mathfrak{R}_I \, r_I \, Q_I^A = 0 \; , \qquad \forall A \; . \tag{1.1.101}$$

Let us also note that the FI parameters $\xi_A$ often run at one-loop with $\beta$-functions:

$$\beta^A \equiv \mu \frac{d\tau^A}{d\mu} = -\frac{b_0^A}{2\pi i} \; , \qquad\qquad b_0^A = \sum_i \mathrm{tr}_{\mathfrak{R}_i}(t_A) \; , \tag{1.1.102}$$

due to contributions from the charged chiral multiplets.

### Pseudo-Topological Observables

Consider an $\mathcal{N} = (0, 2)$ theory in curved space, with a certain twist by the $R$-symmetry. The flat-space theory has an $\mathcal{R}$-multiplet [48] that includes the stress-energy tensor $T_{\mu\nu}$ and the $R$-symmetry current $j_\mu$, and we can define a "twisted" stress-energy tensor:

$$\mathcal{T}_{zz} = T_{zz} - \frac{i}{2}\partial_z j_z \; , \qquad \mathcal{T}_{z\bar{z}} = T_{z\bar{z}} - \frac{i}{2}\partial_z j_{\bar{z}} \; , \qquad \mathcal{T}_{\bar{z}\bar{z}} = T_{\bar{z}\bar{z}} + \frac{i}{2}\partial_{\bar{z}} j_{\bar{z}} \; , \tag{1.1.103}$$

which is conserved because $T_{\mu\nu}$ and $j_\mu$ are conserved. The operator $\mathcal{T}_{zz}$ is $\widetilde{Q}_+$-closed, while $\mathcal{T}_{z\bar{z}}$ and $\mathcal{T}_{\bar{z}\bar{z}}$ are also $\widetilde{Q}_+$-exact. By a standard arguments, it follows that correlation functions of $\widetilde{Q}_+$-closed operators are independent of the Hermitian structure on the two-dimensional manifold $\Sigma$, while they may depend holomorphicaly on its complex structure moduli [16].

The supersymmetric observables are also (locally) holomorphic functions of the various couplings. It is clear that they are holomorphic in the superpotential couplings appearing in $J_I$, and in the FI parameters $J_A = \tau_A$, since the anti-holomorphic couplings $\widetilde{J}_I$ and $\widetilde{J}_A$ are $\delta$-exact. To understand the dependence on the $\mathcal{E}_I$-potential couplings, note that any deformation of $\widetilde{\mathcal{E}}_I$ by $\Delta\widetilde{\mathcal{E}}_I(\widetilde{\phi})$ deforms the classical Lagrangian (1.1.89) by a $\delta$-exact operator:

$$\Delta\mathscr{L} = \Delta\widetilde{\mathcal{E}}^I \mathcal{E}_I + \frac{1}{2}\mathcal{B}^i \partial_i(\Delta\widetilde{\mathcal{E}}_I) \, \Lambda^I = \frac{1}{2}\delta\left(\Delta\widetilde{\mathcal{E}}^I \Lambda_I\right) \; . \tag{1.1.104}$$

More generally, it follows from (1.1.71) that $\widetilde{\mathcal{E}}$-deformations commute with the supersymmetry. On the other hands, deformations of the holomorphic potentials $\mathcal{E}_I$ commute with the supercharge up to terms holomorphic in $\Delta\mathcal{E}_I$. Since $\mathcal{E}_I$ only enters the Lagrangian through $\delta$-exact terms, this implies that supersymmetric observables depend holomorphically on the $\mathcal{E}_I$-couplings.

We are interested in the special class of $\delta$-closed operators with non-singular OPEs that we discussed in the introduction, and we would like to consider their correlations functions of the Riemann sphere:

$$\langle \mathcal{O}_a \mathcal{O}_b \cdots \rangle_{\mathbb{P}^1} \ . \tag{1.1.105}$$

In the next section, we will further restrict ourselves to the case of the A/2-twisted pseudo-chiral ring of $\mathcal{N}=(0,2)$ theories with an $\mathcal{N}=(2,2)$ locus. We leave more general studies of arbitrary $\mathcal{N}=(0,2)$ pseudo-chiral rings for future work.

#### Supersymmetric Locus and Zero modes on the Sphere

A configuration of bosonic fields from the vector, chiral and Fermi multiplets preserves the single supercharge on curved space if and only if the fields satisfy the supersymmetry equations:

$$D = 2if_{1\bar{1}} \ , \qquad D_{\bar{z}}\phi_i = 0 \ , \qquad \mathcal{E}_I(\phi) = \widetilde{\mathcal{E}}_I(\widetilde{\phi}) = 0 \ . \tag{1.1.106}$$

In particular, the chiral field $\phi_i$ is an holomorphic section of an holomorphic vector bundle determined by its $R$- and gauge-charges. Such configurations will dominate the path integral. In the special case of an A/2-twisted GLSM with an $\mathcal{N}=(2,2)$ locus—to be discussed in the next section—we will argue that the path integral for pseudo-topological supersymmetric observables can be further localized into Coulomb branch configurations, in which case the charged chiral multiplets are massive and localize to $\phi_i = 0$. We still have to sum over all the topological sectors, with fluxes:

$$\frac{1}{2\pi}\int d^2x\sqrt{g}\,(-2if_{1\bar{1}}) \equiv k \in i\mathfrak{h} \ . \tag{1.1.107}$$

Note that we generally have fermionic zero modes, in addition to the bosonic zero modes that solve the second equation in (1.1.106). For future reference, let us summarize the counting of zero modes on the sphere. (The generalization to any genus is straightforward.) Consider a charged chiral multiplet $\Phi_i$ of $R$-charge $r$ and gauge charges $\rho_i$ (the weights of the representation $\mathfrak{R}_i$), in a particular flux sector (1.1.107), together with its charge conjugate multiplet $\widetilde{\Phi}_i$. Let us define:

$$\mathbf{r}_{\rho_i} = r_i - \rho_i(k) \ . \tag{1.1.108}$$

The scalar field component $\phi^{(\rho_i)}$ is a section of a line bundle $\mathcal{O}(-\mathbf{r}_{\rho_i})$ over $\mathbb{P}^1$, with first Chern class $-\mathbf{r}_{\rho_i}$. Its zero-modes are holomorphic sections of $\mathcal{O}(-\mathbf{r}_{\rho_i})$, which exist if and only if $\mathbf{r}_{\rho_i} \leq 0$. The analysis for the other chiral multiplet fields $\mathcal{C}_{\bar{1}}, \widetilde{\phi}$ and $\widetilde{\mathcal{B}}$ is similar. For each weight $\rho_i$ of the representation $\mathfrak{R}_i$, one has the following zero-modes:

$$\Phi_{\rho_i} \to \begin{cases} -\mathbf{r}_{\rho_i} + 1 & \text{zero-modes of } (\phi \,, \widetilde{\phi} \,, \widetilde{\mathcal{B}})^{(\rho_i)} & \text{if } \ \mathbf{r}_{\rho_i} \leq 0 \,, \\ \mathbf{r}_{\rho_i} - 1 & \text{zero-modes of } \mathcal{C}_{\bar{1}}^{(\rho_i)} & \text{if } \ \mathbf{r}_{\rho_i} \geq 1 \,. \end{cases} \tag{1.1.109}$$

Similarly, for a Fermi multiplet $\Lambda_I$ and its charge conjugate $\widetilde{\Lambda}_I$, with $R$-charge $r_I$ and gauge representation $\mathfrak{R}_I$, one finds:

$$\Lambda_{\rho_I} \to \begin{cases} \mathbf{r}_{\rho_I} & \text{zero-modes of } \widetilde{\Lambda}_I & \text{if } \ \mathbf{r}_{\rho_I} \geq 1 \,, \\ -\mathbf{r}_{\rho_I} & \text{zero-modes of } \Lambda_I & \text{if } \ \mathbf{r}_{\rho_I} \leq 0 \,, \end{cases} \tag{1.1.110}$$

where we defined $\mathbf{r}_{\rho_I} = r_I - \rho_I(k)$. The zero-modes (1.1.109)-(1.1.110) are present if we turn off all interactions, while most of then are generally lifted by the gauge and $\mathcal{E}_I$ couplings. In addition, we also have $\mathrm{rk}(\mathbf{G})$ gaugino zero modes $\widetilde{\lambda}_a$ from the vector multiplet (1.1.75).

## 1.2 Review of the Mirror Construction for Abelian Gauged Linear Sigma Models

Let us quickly review the mirror ansatz for abelian (2,2) GLSMs for Fano toric varieties in [25]. The contents of this section were adapted, with minor modifications, with permission from JHEP, from our publication [27].

### 1.2.1 General Aspects

First, we consider a GLSM with gauge group $U(1)^k$ and $N$ chiral superfields, with charges encoded in charge matrix $(Q_i^a)$.

Following [25], the mirror is a theory with $k$ superfields $\Sigma_a$, as many as $U(1)$s in the original GLSM, and $N$ twisted chiral fields $Y_i$, as many as chiral multiplets in the original GLSM, of periodicity $2\pi i$, with superpotential

$$W \;=\; \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{N} Q_i^a Y_i - t_a \right) \;+\; \mu \sum_{i=1}^{N} \exp(-Y_i), \tag{1.2.1}$$

where $\mu$ is a scale factor.

In the expression above, the $\Sigma_a$ act effectively as Lagrange multipliers, generating constraints

$$\sum_{i=1}^{N} Q_i^a Y_i = t_a \tag{1.2.2}$$

originating with the D terms of the original theory. We can solve these constraints formally[3] by writing

$$Y_i \;=\; \sum_{A=1}^{N-k} V_i^A \theta_A \;+\; \tilde{t}_i \tag{1.2.3}$$

where $\theta_A$ are the surviving physical degrees of freedom, $\tilde{t}_i$ are solutions of

$$\sum_{i=1}^{N} Q_i^a \tilde{t}_i \;=\; t_a, \tag{1.2.4}$$

and $V_i^A$ is a rank-$(N-k)$ matrix solving

$$\sum_{i=1}^{N} Q_i^a V_i^A \;=\; 0. \tag{1.2.5}$$

(The rank requirement goes hand-in-hand with the statement that there are $N-k$ independent $\theta_A$'s.) The periodicity of the $Y_i$'s will lead to interpretations of the space of $\theta_A$'s in terms of LG orbifolds and character-valued fields, as we shall review later. Note that for $t_i$, $V_i^A$ satisfying the equation above,

$$\sum_{i=1}^{N} Q_i^a Y_i \;=\; \sum_i Q_i^a \left( \sum_A V_i^A \theta_A + \tilde{t}_i \right) \;=\; t_a,$$

and so the $V_i^A$ encode a solution of the D-term constraints.

After integrating out the Lagrange multipliers, the superpotential can be rewritten as

$$W \;=\; \mu \sum_{i=1}^{N} \left( e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A) \right). \tag{1.2.6}$$

In this language, the (2,2) mirror map between A- and B-model operators is (partially) defined by

$$\sum_{a=1}^{k} Q_i^a \sigma_a \;\leftrightarrow\; \mu \exp(-Y_i) \;=\; \mu e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A), \tag{1.2.7}$$

which can be derived by differentiating (1.2.1) with respect to $Y_i$. (See for example [25][section 3.2], where this is derived as the equations of motion of the mirror theory. In the next section, we will also see that this map is consistent with axial R symmetries.) In fact, this overdetermines the map – only a subset of the $Y_i$'s will be independent variables solving the constraints (1.2.2). As we will see explicitly later, the redundant equations are equivalent to

[3] The expressions given here are entirely formal, and there can be subtleties. For example, if the entries in $V_i^A$ are fractional, then as is well-known, the mirror may have orbifolds.

chiral ring relations (as must follow since they all arise as the same equations of motion in the mirror), and are also specified by the equations of motion derived from the superpotential $W$ above.

In appendix B we will briefly outline a variation on the usual GLSM-based mirror derivation. Regardless of how the B-model mirror superpotential is obtained, it can be checked by comparing closed-string A model correlation functions between the mirror and the original A-twisted GLSM using supersymmetric localization. For (2,2) theories, this can be done at arbitrary genus using the methods of [52, 53], whereas for (0,2) theories, we can only apply analogous tests at genus zero. We will perform such correlation function checks later in this paper.

### R Charges

Let us take a moment to consider R charges. In the A-twisted theory, the axial R-charge is in general broken by nonperturbative effects, so that under an axial symmetry transformation, anomalies induce a shift in the theta angle[4] by

$$\theta^a \ \mapsto \ \theta^a + 2\alpha \sum_i Q_i^a, \quad t_a \ \mapsto \ t_a + 2i\alpha \sum_i Q_i^a,$$

for $\alpha$ parametrizing axial R symmetry rotations. The shift above can formally be described as

$$\tilde{t}_i \ \mapsto \ \tilde{t}_i + 2i\alpha,$$

(using the relation between $\tilde{t}_i$ and $t^a$ in (1.2.4)). In the same vein, under the same axial R symmetry, the mirror field $Y_i$ transforms as

$$Y_i \ \mapsto \ Y_i + 2i\alpha,$$

so that $\exp(-Y_i)$ has axial R-charge 2. If we take $\Sigma_a$ to also have axial R-charge 2, then it is easy to verify that the entire mirror superpotential (1.2.1) has axial R-charge 2, as desired, taking the $t$'s to have nonzero R-charge as described. In addition, the operator mirror map (1.2.7) is also consistent with axial R-charges in that case.

### Twisted Masses

One can also consider adding twisted masses. Recall that a twisted mass can be thought of as the vev of a vector multiplet, gauging some flavor symmetry. Taking the vev removes the gauge field, gauginos, and auxiliary field, and replaces them with a single mass parameter $\tilde{m}$, corresponding to the vev of the $\sigma$ field. In the notation of [16][equ'n (2.19)], this means,

[4] This should not be confused with the fundamental field $\theta_A$ defined earlier.

for a single $U(1)$ flavor symmetry that acts on a field $\phi_i$ with charge $Q_{F,i}$, we add terms to the action of the form

$$-2|\tilde{m}|^2 \sum_i Q_{F,i}^2 |\phi_i|^2 \; - \; \sqrt{2} \sum_i Q_{F,i} \left( \overline{\tilde{m}} \overline{\psi}_{+,i} \psi_{-,i} + \tilde{m} \overline{\psi}_{-,i} \psi_{+,i} \right).$$

In the present case, for a toric variety with no superpotential, there are at least as many flavor symmetries as chiral superfields modulo gauged $U(1)$s, *i.e.* at least $N-k$ $U(1)$ flavor symmetries. (There can also be nonabelian components.) For simplicity, we will simply allow for a twisted mass $\tilde{m}_i$ associated to each chiral superfield, and will not try to distinguish between those related by gauge $U(1)$s.

Including twisted masses $\tilde{m}_i$, the full mirror superpotential (before integrating out $\Sigma$'s) takes the form

$$W \;=\; \sum_{i=1}^{N} \left( \sum_{a=1}^{k} \Sigma_a Q_i^a + \tilde{m}_i \right) \left( Y_i - \tilde{t}_i \right) \; + \; \mu \sum_{i=1}^{N} \exp(-Y_i). \tag{1.2.8}$$

This expression manifestly has consistent axial R-charge 2 (using the 'modified' R-charge that acts on $\tilde{t}_i$). It differs from the more traditional expression [25][equ'n (3.86)]

$$W \;=\; \sum_{a=1}^{k} \Sigma_a \left( \sum_i Q_i^a Y_i - t_a \right) + \sum_{i=1}^{N} \tilde{m}_i Y_i \; + \; \mu \sum_{i=1}^{N} \exp(-Y_i), \tag{1.2.9}$$

by a constant term (proportional to $\sum_{i=1}^{N} \tilde{m}_i \tilde{t}_i$), and so defines the same physics.

After including twisted masses, the operator mirror map becomes

$$\sum_{a=1}^{k} Q_i^a \sigma_a \; + \; \tilde{m}_i \; \leftrightarrow \; \mu \exp(-Y_i).$$

Note that both sides of this expression are consistent with the (modified) R-charge assignments described above.

Generically in this paper we will absorb $\mu$ into a redefinition of the $Y_i$'s, and so not write it explicitly, but we mention it here for completeness.

Finally, we should remind the reader that in addition to the superpotential above, one may also need to take an orbifold to define the theory, as is well-known. This will happen if, for example, some of the entries in $(V_i^A)$ are fractions, in order to reflect ambiguities in taking the roots implicit in resulting expressions such as $\exp(-V_i^A \theta_A)$.

### 1.2.2 Example with Twisted Masses

To give another perspective, in this section we will review the (2,2) mirror to the GLSM for $\text{Tot}(\mathcal{O}(-n) \to \mathbb{P}^2)$, for $n \leq 3$ (and no superpotential), and to make this interesting, we will include twisted masses $\tilde{m}_i$, correspnding to phase rotations of each field.

The charge matrix for this GLSM is

$$Q \;=\; (1,1,1,-n),$$

and following the usual procedure, the D terms constrain the dual (twisted) chiral superfields as

$$Y_1 + Y_2 + Y_3 - nY_p \;=\; t.$$

The standard procedure at this point is to eliminate $Y_p$, and write the dual potential in terms of $Y_{1-3}$, taking a $\mathbb{Z}_n$ orbifold to account for the fractional coefficients of the $Y_i$ and its periodicity. In other words,

$$Y_p \;=\; \frac{1}{n}\left(Y_1 + Y_2 + Y_3 - t\right),$$

hence the (2,2) superpotential is given by

$$\begin{aligned} W &= \sum_i \tilde{m}_i Y_i + \exp(-Y_1) + \exp(-Y_2) + \exp(-Y_3) + \exp(-Y_p), \\ &= \sum_i \tilde{m}_i Y_i + (\exp(-Y_1/n))^n + (\exp(-Y_2/n))^n + (\exp(-Y_3/n))^n \\ &\qquad + \exp(-t/n)\exp(-Y_1/n)\exp(-Y_2/n)\exp(-Y_3/n). \end{aligned}$$

Phrased more simply, if we define $Z_i = \exp(-Y_i/n)$, then the (2,2) mirror theory is, as expected, a $\mathbb{Z}_n$ orbifold with superpotential

$$W \;=\; -\sum_i \tilde{m}_i n \ln Z_i + Z_1^n + Z_2^n + Z_3^n + \exp(-t/n) Z_1 Z_2 Z_3,$$

with the understanding that the fundamental fields are $Y_i$s not $Z_i$s. (For hypersurfaces, the fundamental fields will change.)

Later, we will use the matrices $(V_i^A)$ extensively, so in that language, the change of variables above is encoded in

$$(V_i^A) \;=\; \left[\begin{array}{cccc} 1 & 0 & 0 & 1/n \\ 0 & 1 & 0 & 1/n \\ 0 & 0 & 1 & 1/n \end{array}\right].$$

Then, we write $Y_i = V_i^A \theta_A$, and so

$$Y_1 = \theta_1, \quad Y_2 = \theta_2, \quad Y_3 = \theta_3, \quad Y_p = (1/n)(\theta_1 + \theta_2 + \theta_3 - t).$$

Let us next discuss the operator mirror map. This is given by

$$\begin{aligned} \exp(-Y_1) = Z_1^n &\;\leftrightarrow\; \sigma, \\ \exp(-Y_2) = Z_2^n &\;\leftrightarrow\; \sigma, \\ \exp(-Y_3) = Z_3^n &\;\leftrightarrow\; \sigma, \\ \exp(-Y_p) = Z_1 Z_2 Z_3 \exp(-t/n) &\;\leftrightarrow\; -n\sigma. \end{aligned}$$

### 1.2.3 (2,2) in (0,2) Language

Now, let us describe (2,2) mirrors in (0,2) language, as preparation for describing more general (0,2) mirrors. Let $(\Sigma_a, \Upsilon_a)$ be the (0,2) chiral and Fermi components of $\Sigma_a$, and $(Y_i, F_i)$ the (0,2) chiral and Fermi components of $Y_i$. Then, the (2,2) superpotential (1.2.8) is given in (0,2) superspace by

$$W \;=\; \sum_{a=1}^{k} \left[ \Upsilon_a \left( \sum_{i=1}^{N} Q_i^a Y_i - t_a \right) \;+\; \sum_{i=1}^{N} \Sigma_a Q_i^a F_i \right] \;-\; \mu \sum_{i=1}^{N} F_i \exp(-Y_i) \;+\; \sum_{i=1}^{N} \tilde{m}_i F_i. \tag{1.2.10}$$

We integrate out $\Sigma_a$, $\Upsilon_a$ to get the constraints

$$\sum_{i=1}^{N} Q_i^a Y_i \;=\; t_a, \quad \sum_{i=1}^{N} Q_i^a F_i \;=\; 0,$$

which we solve with the $V_i^A$ by writing

$$Y_i \;=\; \sum_{A=1}^{N-k} V_i^A \theta_A \;+\; \tilde{t}_i, \quad F_i \;=\; \sum_{A=1}^{N-k} V_i^A G_A,$$

where $(\theta_A, G_A)$ are the chiral and Fermi components of the (2,2) chiral superfields $\theta_A$. After integrating out the constraints, the (0,2) superpotential becomes

$$W \;=\; \sum_{i=1}^{N} \sum_{A=1}^{N-k} G_A V_i^A \left( \tilde{m}_i - \mu \exp(-Y_i) \right) \;=\; \sum_{i=1}^{N} \sum_{A=1}^{N-k} G_A V_i^A \left( \tilde{m}_i - \mu e^{\tilde{t}_i} \prod_{B=1}^{N-k} \exp(-V_i^B \theta_B) \right). \tag{1.2.11}$$

As is standard, we remind that reader that depending upon the entries in $(V_i^A)$, the mirror may be a LG orbifold, which are required to leave $W$ invariant.

In this language, the (2,2) mirror map between A- and B-model operators is (partially) defined by

$$\sum_{a=1}^{k} Q_i^a \sigma_a + \tilde{m}_i \;\leftrightarrow\; \mu \exp(-Y_i) \;=\; \mu e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A), \tag{1.2.12}$$

which can be derived by differentiating (1.2.10) with respect to $F_i$.

In most of the rest of this paper, we will absorb $\mu$ into a field redefinition of the $Y_i$s for simplicity, but we include it here for completeness.

# Chapter 2

# Some New Developments in Gauged Linear Sigma Models

This chapter contains some of our results on properties of gauged linear sigma models. The contents of this chapter were adapted, with minor modifications, with permission from JHEP and arXiv, from our publication [42] in JHEP and our paper [65] on the arXiv. Section 2.1 is from [65] while section 2.3 is in [42]. We also include unpublished results in section 2.2.

## 2.1 Exact Results in (2,2) GLSMs and Applications

In this section, we apply supersymmetric localization to study gauged linear sigma models (GLSMs) describing supermanifold target spaces. We use the localization method to show that A-twisted GLSM correlation functions for certain supermanifolds are equivalent to A-twisted GLSM correlation functions for hypersurfaces in ordinary spaces under certain conditions. We also argue that physical two-sphere partition functions are the same for these two types of target spaces. Therefore, we reproduce the claim of [49, 55] that A-twisted NLSM correlation functions for certain supermanifolds are equivalent to A-twisted NLSM correlation functions for hypersurfaces in ordinary spaces under certain conditions.

### 2.1.1 Review of GLSMs for Toric Varieties

We will briefly review some aspects of GLSMs for toric varieties and how to compute correlation functions via supersymmetric localization on the Coulomb branch in some concrete examples.

Consider a GLSM with gauge group $U(1)^k$ and $N$ chiral superfield $\Phi_i$ of gauge charges $Q_i^a$

and vector R charges[1] $R_i$, where $a = 1, \ldots k$ and $i = 1, \ldots, N$. The lowest component of $\Phi_i$ is a bosonic scalar $\phi_i$, and we call this $\Phi_i$ an *even* chiral superfield. The Lagrangian and general discussions of this model can be found in the literature, see *e.g.* [16, 28].

In this GLSM, the global symmetry is of the form

$$\times_\alpha U(N_\alpha), \tag{2.1.1}$$

where all $N_\alpha \geq 0$ and $\sum_\alpha N_\alpha = N$, and we do not include the R-symmetries. However, if we have a superpotential, the global symmetry will be smaller [54]. For example, in the GLSM for the quintic, the global symmetry is $U(1)$. Another similar example can be found in appendix A in Hori's paper [39] [2].

The correlation function for a general operator $\mathcal{O}(\sigma)$ can be calculated via localization on the Coulomb branch as [41]

$$\langle \mathcal{O}(\sigma) \rangle = (-1)^{N_*} \sum_{\boldsymbol{m}} \oint_{\mathrm{JK-Res}} \prod_{a=1}^{k} \left( \frac{\mathrm{d}\sigma_a}{2\pi i} \right) \mathcal{O}(\sigma) Z_{\boldsymbol{m}}^{1-\mathrm{loop}} q^{\boldsymbol{m}}, \tag{2.1.2}$$

where $q^{\boldsymbol{m}} = e^{-t^a m_a}$, in which:

$$t^a = r^a + i\theta^a,$$
$$r^a = r_0^a + \sum_i Q_i^a \ln \frac{\mu}{\Lambda},$$

and $Z_{\boldsymbol{m}}^{1-\mathrm{loop}}$ is the one loop determinant. For abelian gauge theories, it is known that

$$Z_{\boldsymbol{m}}^{1-\mathrm{loop}} = \prod_i (Q_i^a \sigma_a + \tilde{m}_i)^{R_i - 1 - Q_i(\boldsymbol{m})},$$

in which

$$Q_i(\boldsymbol{m}) = Q_i^a m_a,$$

and $\tilde{m}_i$ are the twisted masses associated to the global symmetry. The overall factor $(-1)^{N_*}$, where $N_*$ is the number of $p$ fields, comes from the assignment for the fields with R-charge 2 [28, 41]. We will later see this overall factor would automatically show up from the redefinition of $q$'s in the supermanifold case in following sections. The special case $N_* = 0$, corresponds to a toric space without a superpotential.

Next, we will apply the above formula to calculate several concrete examples.

<u>GLSM for $\mathbb{CP}^4$</u>

[1] In order to make this GLSM to be A-twistable on a two-sphere, the vector R-charges, denoted as $R_V$, should be integers [89, 123].

[2] To clarify, here when we speak of 'global symmetry,' we include gauge symmetries as a special case, which is a somewhat nonstandard choice.

In this model, we have five chiral superfields with $U(1)$ charges and $R_V$-charges given by

| $Q$ | 1 | 1 | 1 | 1 | 1 |
|---|---|---|---|---|---|
| $R_V$ | 0 | 0 | 0 | 0 | 0 |

and it has a $SU(5)$ flavor symmetry. For simplicity, we set twisted masses to zero.

Then from the formula (2.1.2), we obtain:

$$\langle \mathcal{O}(\sigma) \rangle = \sum_{k \geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)}{\sigma^{5+5k}} q^k.$$

If take $\mathcal{O}(\sigma) = \sigma^{5k+4}$, we could immediately obtain

$$\langle \sigma^{5k+4} \rangle = q^k,$$

and this equation encodes the chiral ring relation as

$$\sigma^5 = q.$$

<u>GSLM for Tot $(\mathcal{O}(-\mathrm{d}) \to \mathbb{CP}^4)$</u>

Tot $(\mathcal{O}(-\mathrm{d}) \to \mathbb{CP}^4)$ is the total space of the bundle $\mathcal{O}(-d) \to \mathbb{CP}^4$. For the special case when $d = 5$, it is also called $V^+$ model as in [28]. In this example, we have six chiral superfields with $U(1)$ charges and $R_V$-charges given by

| $Q$ | 1 | 1 | 1 | 1 | 1 | $-d$ |
|---|---|---|---|---|---|---|
| $R_V$ | 0 | 0 | 0 | 0 | 0 | 0 |

This model has flavor symmetry $SU(5) \times U(1)$. We require $\sum_i Q_i \geq 0$ so this system has a geometric phase corresponding to a weak coupling limit. Then we have

$$\begin{aligned}\langle \mathcal{O}(\sigma) \rangle &= \sum_{k} \oint_{JK-Res} \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)}{\sigma^{5+5k}(-d\sigma)^{1-dk}} q^k, \\ &= \sum_{k \geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)}{\sigma^{6+(5-d)k}(-d)^{1-dk}} q^k.\end{aligned}$$

For the special case $d = 5$, we can further obtain the following chiral ring relation:

$$\langle \sigma^5 \rangle = -\frac{1}{5} \frac{1}{1 + 5^5 q}.$$

<u>GLSM for Hypersurface in $\mathbb{CP}^4$</u>

This model is defined by six chiral superfields with $U(1)$ charges and $R_V$-charges given by:

| $Q$ | 1 | 1 | 1 | 1 | 1 | $-d$ |
|---|---|---|---|---|---|---|
| $R_V$ | 0 | 0 | 0 | 0 | 0 | 2 |

which we also require $\sum_i Q_i \geq 0$. It has no flavor symmetry. We have

$$\begin{aligned} \langle \mathcal{O}(\sigma) \rangle &= (-1)^1 \sum_k \oint_{JK-Res} \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(-d\sigma)^2}{\sigma^{5+5k}(-d\sigma)^{1-dk}} q^k, \\ &= -\sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(-d)^{1+dk}}{\sigma^{4+(5-d)k}} q^k. \end{aligned}$$

In particular, if $d = 5$, then it satisfies the Calabi-Yau condition. Then,

$$\langle \mathcal{O}(\sigma) \rangle = -\sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(-5)^{1+5k}}{\sigma^4} q^k.$$

Take $\mathcal{O}(\sigma) = \sigma^3$, then we can obtain

$$\left\langle \sigma^3 \right\rangle = \frac{5}{1 + 5^5 q}.$$

This correlation function is in agreement with $\langle \sigma^3 \left(-(-5\sigma)^2\right) \rangle$ in the previous $V^+$-model [28].

### 2.1.2 GLSMs for Complex Kähler Supermanifolds

A supermanifold $X$ of dimension $N|M$ is locally described by $N$ even coordinates and $M$ odd coordinates together with compatible transition functions. If it is further a split supermanifold, then it can be viewed as the total space of an odd vector bundle $V$ of rank $M$ over a $N$-dimensional manifold, which is along the even directions and denoted $X_{\text{red}}$:

$$X \simeq \mathrm{Tot}(V \to X_{\text{red}}).$$

For more rigorous definitions of supermanifolds and split supermanifolds, we recommend [56]. According to the fundamental structure theorem [56], every smooth supermanifold can be split, so even the split case is still considerable.

To build up a $(2,2)$ GLSM as a UV-complete theory of a NLSM for a complex Kähler supermanifold $\mathcal{M}$, we only consider those toric supermanifolds [55] obeying certain constraints, which we will give later as Eq. (2.1.4). We obtain this from the GLSM perspective, but it can be derived from NLSMs [49]. By toric supermanifold, we mean that $\mathcal{M}$ has a complexified symmetry group $(\mathbb{C}^*)^k$ and can be obtained as a symplectic reduction of a super vector space by an abelian gauge group, which is realized in a GLSM by gauging a group action on a super vector space (corresponding to matter fields). It was pointed out in [55] that this kind of supermanifold is also split. Therefore, we can still take advantage of the bundle structure of split supermanifolds in our construction. One example of these toric supermanifolds is $\mathbb{CP}^{4|1}$, which is defined by

$$\big\{[x_1, x_2, x_3, x_4, x_5, \theta] \mid (x_1, x_2, x_3, x_4, x_5, \theta) \sim (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5, \lambda^d \theta)\big\}. \tag{2.1.3}$$

This is a different geometry than $\mathbb{CP}^4$. For example, on $\mathbb{CP}^{4|1}$ we can choose a patch where $\{x_1, \ldots, x_5\}$ all vanish, while the odd coordinate is nonzero.

### The Model

In order to construct the GLSM for a toric supermanifold described by a $U(1)^k$ gauge theory, we can follow the construction of $V_+$-model [28] but change the statistical properties along the bundle directions. In other words, we view fields along bundle direction as ghosts. In [57], there is a formal discussion about building GLSM for supermanifolds. Here we only focus on toric supermanifolds. More specifically, we have two sets of chiral superfields:

- $N + 1$ *(Grassmann) even chiral superfields* $\Phi_i$ with $U(1)^k$ gauge charges $Q_i^a$ and R-charges $R_i$, whose lowest components are bosonic scalars;
- $M$ *(Grassmann) odd chiral superfields* $\tilde{\Phi}_\mu$ with gauge $U(1)^k$ charges $\tilde{Q}_\mu^a$ and R-charges $\tilde{R}_\mu$, whose lowest components are fermionic scalars. [3]

In the above, we impose an analogue of a Fano requirement for the supermanifold, requiring that for each index $a$

$$\sum_i Q_i^a - \sum_\mu \tilde{Q}_\mu^a \geq 0, \tag{2.1.4}$$

and in later sections we impose this condition implicitly. (We will derive this condition from the worldsheet beta function later in this section.)

Associated to the gauge group $U(1)^k$, there are $k$ vector superfields: $V_a,\ a = 1, \ldots, k$. The total Lagrangian consists of five parts[4]:

$$\mathcal{L} = \mathcal{L}_{\text{kin}}^{\text{even}} + \mathcal{L}_{\text{kin}}^{\text{odd}} + \mathcal{L}_{\text{gauge}} + \mathcal{L}_W + \mathcal{L}_{\tilde{W}}.$$

As advertised in the introduction, we will consider vanishing superpotential in this paper, i.e. $W = 0$. Take the classical twisted superpotential to be a linear function[5]

$$\tilde{W} = -\sum_a t^a \Sigma_a. \tag{2.1.5}$$

[3]For general discussions, we use tilde '$\sim$' to indicate the odd chiral superfields and their charges.
[4] For a comprehensive expression for the Lagrangian, please refer to [57][section 2].
[5] We use notations of [12].

In the above Lagrangian, the even kinetic part, the gauge part and the twisted superpotential part share the same form as in a GLSM for an ordinary target space. The odd kinetic part is defined in the same fashion as the even part [57]:

$$\mathcal{L}_{\text{kin}}^{\text{odd}} = \int d^4\theta \sum_{\mu} \bar{\tilde{\Phi}}_\mu e^{2Q_\mu^a V_a} \tilde{\Phi}_\mu. \tag{2.1.6}$$

The equations of motion for the auxiliary fields $D^a$ inside vector superfields are

$$D^a = -e^2 \Big( \sum_i Q_i^a |\phi_i|^2 + \sum_\mu \tilde{Q}_\mu^a \bar{\tilde{\phi}}_\mu \tilde{\phi}_\mu - r^a \Big), \tag{2.1.7}$$

where $r^a$ are the FI parameters. Since $W = 0$, the equations of motion for the auxiliary fields $F_{i/\mu}$ inside even/odd chiral superfields are

$$F_i = 0, \quad F_\mu = 0.$$

The potential energy is

$$U = \frac{1}{2e^2} D^2 + 2|\sigma|^2 \Big( \sum_i Q_i^2 |\phi_i|^2 + \sum_\mu \tilde{Q}_\mu^2 \bar{\tilde{\phi}}_\mu \tilde{\phi}_\mu \Big).$$

Semiclassically, we can discuss low energy physics by requiring $U = 0$, *i.e.* $\sigma = 0$ and $D = 0$, which is

$$\sum_i Q_i^a |\phi_i|^2 + \sum_\mu \tilde{Q}_\mu^a \bar{\tilde{\phi}}_\mu \tilde{\phi}_\mu - r^a = 0.$$

In the case with one $U(1)$, we often require a geometric phase where $r \gg 0$ defined by $(\mathbb{C}^{N+1|M} - Z)/\mathbb{C}^*$. [6] Returning to the general case, in the phase $r^a \gg 0$ for all $a \in \{1, \ldots, k\}$, the above condition requires that not all $\phi_i$ or $\phi_\mu$ can vanish, then the target space is a super-version of toric variety, $X$, which we will call a super toric variety:

$$X \simeq \frac{\mathbb{C}^{N+1|M} - Z}{(\mathbb{C}^*)^k}, \tag{2.1.8}$$

where the torus action $(\mathbb{C}^*)^k$ is defined as, for each $a$,

$$\Big( \ldots, \phi_i, \ldots, \tilde{\phi}_\mu, \ldots \Big) \mapsto \Big( \ldots, \lambda^{Q_i^a} \phi_i, \ldots, \lambda^{\tilde{Q}_\mu^a} \tilde{\phi}_\mu, \ldots \Big), \quad \lambda \in \mathbb{C}^*.$$

As in the case for ordinary toric varieties, we have global symmetry for GLSM for super toric variety. For the general case, (2.1.8), the maximum torus of the global symmetry would be:

$$U(1)^{N+1} \times U(1)^M.$$

[6] To be thorough, we also need to define theory at other phases. For example, there exists another phase called nongeometric phase corresponding to $r \leq 0$ [101]. However, supersymmetric localization are calculated at UV, which corresponds to a geometric phase in this paper under the condition Eq. (2.1.4).

Since we are not considering superpotentials in our models, this symmetry will not break. The one-loop correction to the $D$-terms can be calculated as in [57]:

$$\left\langle -\frac{D^a}{e^2} \right\rangle_{1-\text{loop}} = \frac{1}{2}\sum_i Q_i^a \ln\left(\frac{\Lambda^2}{Q_i^b Q_i^c \bar{\sigma}_b \sigma_c}\right) - \frac{1}{2}\sum_\mu \tilde{Q}_\mu^a \ln\left(\frac{\Lambda^2}{\tilde{Q}_\mu^b \tilde{Q}_\mu^c \bar{\sigma}_b \sigma_c}\right). \tag{2.1.9}$$

Therefore, the effective FI-parameters are given as

$$\begin{aligned} r_{\text{eff}}^a &= r^a - \frac{1}{2}\sum_i Q_i^a \ln\left(\frac{\Lambda^2}{Q_i^b Q_i^c \bar{\sigma}_b \sigma_c}\right) - \frac{1}{2}\sum_\mu \tilde{Q}_\mu^a \ln\left(\frac{\Lambda^2}{\tilde{Q}_\mu^b \tilde{Q}_\mu^c \bar{\sigma}_b \sigma_c}\right), \\ &= r^a + \frac{1}{2}\left[\sum_i Q_i^a \ln\left(Q_i^b Q_i^c \bar{\sigma}_b \sigma_c\right) - \sum_\mu \tilde{Q}_\mu^a \ln\left(\tilde{Q}_\mu^b \tilde{Q}_\mu^c \bar{\sigma}_b \sigma_c\right)\right] \\ &\qquad - \left(\sum_i Q_i^a - \sum_\mu \tilde{Q}_\mu^a\right) \ln \Lambda, \end{aligned}$$

where $a = 1, \ldots, k$. Introduce the physical scale $\mu$ and from dimensional analysis,

$$\tilde{Q}_\mu^b \tilde{Q}_\mu^c \bar{\sigma}_b \sigma_c = C\mu^2, \quad \tilde{Q}_\mu^b \tilde{Q}_\mu^c \bar{\sigma}_b \sigma_c = \tilde{C}\mu^2,$$

where $C$ and $\tilde{C}$ are nonzero constants. Then from the definition of the beta function, we have

$$\beta^a = \mu \frac{\partial r_{\text{eff}}^a}{\partial \mu} = \sum_i Q_i^a - \sum_\mu \tilde{Q}_\mu^a.$$

This is where we get the constraints Eq. (2.1.4). In particular, if the charges satisfy

$$\sum_i Q_i^a - \sum_\mu \tilde{Q}_\mu^a = 0, \tag{2.1.10}$$

$\beta = 0$ and the correction will be $\Lambda$ independent, and it will give us a conformal field theory. When we compare GLSMs for supermanifolds to related GLSMs for hypersurfaces (or complete intersections) in the next section, we will see that this condition corresponds to the Calabi-Yau condition for the hypersurfaces (or complete intersections):

$$\sum_i Q_i^a = \sum_\mu \tilde{Q}_\mu^a. \tag{2.1.11}$$

For convenience, we will refer to both conditions, (2.1.10) and (2.1.11), as the Calabi-Yau condition. This is also a hint that indicates there exists a close relationship between those two models [49, 55].

**Chiral Ring Relation**

From the effective value of $r$, we could also write down the effective twisted superpotential:

$$\tilde{W}_{\text{eff}}(\Sigma_a) = -t^a\Sigma_a - \Sigma_a\left[\sum_i Q_i^a \ln\left(\frac{Q_i^b\Sigma_b}{\Lambda}\right) - \sum_\mu \tilde{Q}_\mu^a \ln\left(\frac{\tilde{Q}_\mu^b\Sigma_b}{\Lambda}\right)\right]. \tag{2.1.12}$$

The above one-loop corrected effective twisted potential (2.1.12) can be rewritten in terms of physical scale [12], $\mu$, as

$$\tilde{W}_{\text{eff}}(\Sigma_a) = -t^a\Sigma_a - \Sigma_a\left[\sum_i Q_i^a \left(\ln\frac{Q_i^b\Sigma_b}{\mu} - 1\right) - \sum_\nu \tilde{Q}_\nu^a \left(\ln\frac{\tilde{Q}_\nu^b\Sigma_b}{\mu} - 1\right)\right].$$

The Coulomb branch vacua are found by solving

$$\exp\left(\frac{\partial\tilde{W}_{\text{eff}}}{\partial\sigma_a}\right) = 1,$$

and we can read off the chiral ring relations as

$$q_a \equiv e^{-t_a} = \prod_i \left(\frac{Q_i^b\sigma_b}{\mu}\right)^{Q_i^a} \prod_\nu \left(\frac{\tilde{Q}_\nu^b\sigma_b}{\mu}\right)^{-\tilde{Q}_\nu^a}.$$

This is an exact relation where all the $\sigma$'s satisfy. Usually, we set the physical scale $\mu = 1$, then the above relation can be simply written as

$$q_a = \prod_i \left(Q_i^b\sigma_b\right)^{Q_i^a} \prod_\nu \left(\tilde{Q}_\nu^b\sigma_b\right)^{-\tilde{Q}_\nu^a}. \tag{2.1.13}$$

We will see in the next section that the GLSM for the hypersurface corresponding to this supermanifold has the chiral ring relation:

$$\tilde{q}_a = \prod_i \left(Q_i^b\sigma_b\right)^{Q_i^a} \prod_\nu \left(-\tilde{Q}_\nu^b\sigma_b\right)^{-\tilde{Q}_\nu^a}. \tag{2.1.14}$$

It is easy to see that the two chiral ring relations are related by

$$q_a = (-1)^{\sum_\nu \tilde{Q}_\nu^a}\tilde{q}_a.$$

Actually, the factor $(-1)^{\sum_\nu \tilde{Q}_\nu^a}$ will show up repeatedly in next section, and we will call this the map connecting the GLSM for a supermainfold to the corresponding GLSM for a hypersurface (or complete intersection).

**Supersymmetric Localization for Supermanifolds**

In this section, we want to focus on calculations of correlation functions for supermanifolds. Here we only list results of GLSM for supermanifolds on $S^2$ and it can be generalized to higher genus cases (at fixed complex structure) as in [41, 52, 58]. Similar to the calculations given in section 2.1.1, we could also use supersymmetric localization on the Coulomb branch for supermanifolds. However, here we have several Grassmann odd chiral superfields, and it will also contribute to the one-loop determinants. As we are considering the abelian case in this paper, the one-loop determinants for the gauge fields is trivial by the same argument as in [41, 53]. The one-loop determinant for chiral superfields can be written as the product of even and odd parts:

$$Z_{\mathbf{k}}^{1-\text{loop}} = Z_{\mathbf{k},\text{even}}^{1-\text{loop}} \cdot Z_{\mathbf{k},\text{odd}}^{1-\text{loop}},$$

where

$$Z_{\mathbf{k},\text{even}}^{1-\text{loop}} = \prod_i \left(Q_i^a \sigma_a + \tilde{m}_i\right)^{R_i - 1 - Q_i(\mathbf{k})}, \tag{2.1.15a}$$

$$Z_{\mathbf{k},\text{odd}}^{1-\text{loop}} = \prod_\mu \left(\tilde{Q}_\mu^a \sigma_a + \tilde{m}_\mu\right)^{-\tilde{R}_\mu + 1 + \tilde{Q}_\mu(\mathbf{k})}. \tag{2.1.15b}$$

In the above, $R_i$ and $\tilde{R}_\mu$ are the $R_V$ charges for even chiral superfields and odd chiral superfields, respectively, and they are all integers. In appendix A.7, we have discussed the assignments of $R_V$-charges. Roughly speaking, except for the P-fields, $R_V$-charges for odd chiral superfields should be proportional to those for even chiral superfields. Since we are considering twisted models without superpotential in this paper, specifically without the $P$-fields arising in descriptions of hypersurfaces, $R_V$-charges for both even and odd chiral superfields should all be assigned to be zero in twisted models. This $R_V$-charge assignment is also consistent with the large volume limit requirement [59].

Before we get to the one-loop determinant for odd chiral superfields, (2.1.15b), let us briefly review the derivation of (2.1.15a) following [41, 53]. For Grassmann even superfields $\Phi^i = (\phi^i, \psi^i, \dots)$, the one loop determinant from supersymmetric localization is given by

$$Z_{\text{even}}^{1-\text{loop}} = \prod_i \frac{\det \Delta_{\psi^{\text{i}}}}{\det \Delta_{\phi^{\text{i}}}},$$

where $\det \Delta_\phi$ in the denominator comes from a Gaussian integral while $\det \Delta_\psi$ in the numerator comes from a Grassmann integral. Because of supersymmetry, the only thing that will survive from the ratio above is the zero modes of $\psi$, which determine (2.1.15a). It is straightforward to generalize above story for Grassmann odd chiral superfields. For odd chiral superfields $\Phi^\mu = (\phi^\mu, \psi^\mu, \dots)$, the statistical properties of the components $\phi^\mu$ and $\psi^\mu$ are changed, $\phi^\mu$ becomes Grassmann odd while $\phi^\mu$ becomes Grassmann even. At the same time, the operators, $\Delta_\psi$ and $\Delta_\phi$, have the same form as those for even chiral superfields [57].

Therefore, we can use [41,53] to compute the one-loop determinant for odd chiral superfields:

$$Z_{\text{odd}}^{1-\text{loop}} = \prod_{\mu} \frac{\det \Delta_{\phi^\mu}}{\det \Delta_{\psi^\mu}},$$

which leads to (2.1.15b).

Once we have the one-loop determinant for both even and odd chiral superfields, (2.1.15a) and (2.1.15b), the correlation function for a general operator $\mathcal{O}(\sigma)$ can also be obtained by

$$\langle \mathcal{O}(\sigma) \rangle = \sum_{\mathbf{k}} \oint_{\text{JK}-\text{Res}} \prod_{a=1}^{k} \left( \frac{\mathrm{d}\sigma_a}{2\pi i} \right) \mathcal{O}(\sigma) Z_{\mathbf{k},\text{even}}^{1-\text{loop}} Z_{\mathbf{k},\text{odd}}^{1-\text{loop}} q^{\mathbf{k}}. \tag{2.1.16}$$

Here, the JK-residue calculation is also done at the geometric phase.

**Elliptic Genera**

The elliptic genus is a powerful tool to extract physical quantities of target spaces, such as Witten indices. It is the partition function on a torus with twisted boundary conditions, which reduces to the Witten index in a certain parameter limit [60, 61, 63]. There are many discussions of elliptic genera in the literature. In this section we will follow the localization computations in [60, 62] and generalize their discussions to supermanifolds[7]. In the next section, we will use our generalizations for supermanifolds to compare with the hypersurface case, which should provide a consistency check that those two models are indeed equivalent to each other under certain conditions.

In [60, 62], the elliptic genus was computed from supersymmetric localization to be

$$Z_{T^2}(\tau, z) = -\sum_{u_j \in \mathfrak{M}^+_{\text{sing}}} \oint_{u=u_j} \mathrm{d}u \frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \prod_{\Phi_i} \frac{\theta_1(q, y^{R_i/2-1} x^{Q_i})}{\theta_1(q, y^{R_i/2} x^{Q_i})}.$$

Here, we turn off the holonomy of the global symmetry on torus. In the above,

$$y = e^{2\pi i z} \quad \text{and} \quad x_a = e^{2\pi i u_a} \tag{2.1.17}$$

come from the R symmetry and gauge symmetry.

The idea is to use supersymmetric localization to transform the path integral of a torus partition function into a residue integral over zero-modes of vector chiral superfields. In the integrand, the elliptic genus consists of three parts: one-loop determinant for (even) chiral superfields, non-zero modes of vector superfields and twisted chiral superfields. For supermanifolds, we need to include the one-loop determinant for odd chiral superfields with

[7] We expect that one can also follow a different approach as in [63] to get a similar result for supermanifolds.

the same twisted boundary conditions on the torus. From supersymmetric localization, the one-loop determinant for odd chiral superfields are almost the same as that for even chiral superfields, except it should have an overall $-1$ exponent.

Now we argue that we would have a very similar formula for elliptic genera for supermanifolds, and the only difference is to include the one-loop determinant for odd chiral superfields. The result is

$$Z_{T^2}(\tau, z) = -\sum_{u_j \in \mathfrak{M}^+_{\text{sing}}} \oint_{u=u_j} \mathrm{d}u \frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \prod_{\Phi_i} \frac{\theta_1(q, y^{R_i/2-1} x^{Q_i})}{\theta_1(q, y^{R_i/2} x^{Q_i})} \prod_{\tilde{\Phi}_\mu} \frac{\theta_1(q, y^{R_\mu/2} x^{\tilde{Q}_\mu})}{\theta_1(q, y^{R_\mu/2-1} x^{\tilde{Q}_\mu})}. \tag{2.1.18}$$

Our argument mainly follows [60], and we follow the notation of that reference. First, we shall note that with twisted boundary condition on torus, the one-loop determinant for odd chiral superfields can be calculated from localization:

$$Z_{\tilde{\Phi}_\mu, \tilde{Q}_\mu} = \prod_{m,n} \frac{\left| m + n\tau + \frac{R_\mu}{2} z + \tilde{Q}_\mu u \right|^2 + i\tilde{Q}_\mu D}{\left( m + n\tau + (1 - \frac{R_\mu}{2}) z - \tilde{Q}_\mu u \right) \left( m + n\bar{\tau} + \frac{R_\mu}{2} \bar{z} + \tilde{Q}_\mu \bar{u} \right)},$$

and when $D = 0$, it can be written in terms of theta functions as inside the integral above.

The starting point is

$$Z_{T^2} = \int_{\mathbb{R}} \mathrm{d}D \int_{\mathfrak{M}} \mathrm{d}^2 u f_{e,g}(u, \bar{u}, D) \exp\left[ -\frac{1}{2e^2} D^2 - i\zeta D \right],$$

but with a different $D$-term here, which is given in Eq. (2.1.7). Following the procedure in [60], we want to integrate over $D$ and simplify the integral over $u$. After introducing odd chiral superfields, we can still take certain parameter limits to reduce the integral above to $\mathfrak{M} \setminus \Delta_\epsilon$ and then obtain the residue integral formula. Integrating out $D$, we have

$$Z_{T^2} = \int_{\mathfrak{M}} \mathrm{d}^2 u F_{e,g}(u, \bar{u}),$$

with

$$\begin{aligned} F_{e,0} = & C_{u,e} \int_{\mathbb{C}^{M_*|N_*}} \mathrm{d}^{2M_*} \phi_i \mathrm{d}^{2N_*} \tilde{\phi}_\mu \exp\left[ -\frac{1}{g} \sum_i |Q_i(u - u_*)|^2 |\phi_i|^2 - \frac{1}{g} \sum_\mu |\tilde{Q}_\mu(u - u_*)|^2 |\tilde{\phi}_\mu|^2 \right] \\ & \times \exp\left[ -\frac{e^2}{2} (\sum_i Q_i |\phi_i|^2 + \sum_\mu \tilde{Q}_\mu |\tilde{\phi}_\mu|^2 - \zeta)^2 \right]. \end{aligned}$$

Here we use $N_*$ to denote the number of odd chiral superfields which has a zero-mode $\tilde{\phi}_\mu$ at $u_*$. It is easy to see that the odd chiral superfields do not affect arguments in [60] as we can expand those odd chiral superfields in the exponent up to linear terms, and the integrals over them are just finite constants before taking the limit $e \to 0$. Therefore, we shall take $\epsilon \to 0$ and then $e \to 0$, also denoted as $\lim_{e,\epsilon\to 0}$, and then the integral will reduce to

$$Z_{T^2} = \lim_{e,\epsilon\to 0} \int_{\mathfrak{M}\setminus\Delta_\epsilon} \mathrm{d}^2 u F_{e,0}(u,\bar{u}).$$

Once we have the relation above, then other derivations are the same as in [60] and we obtain the formula, Eq. (2.1.18), for elliptic genera of supermanifolds.

In principle, we can also turn on the holonomy of global symmetry for supermanifolds on torus. We will return to this point later. Before going to the next section, we shall mention that the elliptic genus we calculate here has a natural generalization by including odd chiral superfields. The authors are not aware of a corresponding mathematical notion for supermanifolds, and leave that for future work.

### 2.1.3 Comparison with GLSMs for Hypersurfaces

The main goal of this section is to reproduce the claim of [49, 55], namely that an A-twisted NLSM on a supermanifold is equivalent to an A-twisted NLSM on a hypersurface (or a complete intersection). Instead of discussing these two NLSMs, we consider the corresponding GLSMs, namely GLSMs for supermanifolds and GLSMs for hypersurfaces (or complete intersections). However, here is a subtlety: the GLSM FI parameter $t$ is different from the NLSM parameter $\tau$, reflecting the difference between algebraic and flat coordinates. They are related by the mirror map [16, 28]. Therefore, we need to show the mirror map for supermanifolds is the same as the mirror map for the corresponding hypersurfaces. This is indicated by matching the physical two-sphere partition functions [64].

Before working through concrete calculations, let us argue that our calculations are plausible. As mentioned in section 2.1.1 and 2.1.2, the GLSM for supermanifolds we considered in this paper has no superpotential and so the global symmetry for target space is all kept, while the GLSM for a hypersurface will have fewer global symmetries. Therefore, there are more global parameters for the supermanifold case. Further, the statement we want to reproduce was proposed for NLSMs, which correspond to the Higgs branch of a GLSM. However, in this section our calculations are all done on Coulomb branches, for example the correlation functions, (2.1.2) and (2.1.16). To probe properties of correlation functions on a Higgs branch, those real twisted masses, $\tilde{m}$, shall be set to zero. This can be achieved as correlation functions are holomorphic function in $\tilde{m}$ [41]. Following above logic, our results can be used to derive the statement in [49, 55].

In the last section, when we calculated the one-loop correction, the antisymmetric property for odd chiral superfields leads to a minus sign in front of the correction even though we assign positive charges to both even and odd chiral superfields at first. This minus sign is essential to demonstrate equivalent relations between a GLSM for a supermanifold and for a corresponding hypersurface (or complete intersection).

In the following, we will study some concrete examples. In those examples, it is not necessary to impose the Calabi-Yau condition (2.1.10). In this sense, we also generalize the statement in [49, 55] to non-Calabi-Yau cases. What we will use to compare are mainly chiral ring relations, correlation functions and elliptic genera.

### Hypersurface in $\mathbb{CP}^N$ vs. $\mathbb{CP}^{N|1}$

First, let us recall the chiral ring relations for the GLSM for the hypersurface case. In this model, we introduce the superpotential:

$$W = PG(\Phi),$$

where $G(\Phi)$ is a degree $d$ polynomial of $\Phi$'s, and $P$ is a chiral superfield with $U(1)$ charge $-d$ and R-charge 2. Then the twisted superpotential with one-loop correction is:

$$\tilde{W} = -t\Sigma - \Sigma\left[(N+1)\left(\ln\frac{\Sigma}{\mu} - 1\right) - d\left(\ln\frac{-d\Sigma}{\mu} - 1\right)\right].$$

From

$$\exp\left(\frac{\partial\tilde{W}}{\partial\sigma}\right) = 1,$$

we obtain

$$q \equiv e^{-t} = \left(-\frac{d\sigma}{\mu}\right)^{-d}\left(\frac{\sigma}{\mu}\right)^{N+1} = (-1)^d\left(\frac{d\sigma}{\mu}\right)^{-d}\left(\frac{\sigma}{\mu}\right)^{N+1}.$$

Setting $\mu = 1$, we would get

$$q = (-1)^d\,(d\sigma)^{-d}\,\sigma^{N+1}.$$

Let us compare to the analogous result for the supermanifold $\mathbb{CP}^{N|1}$. We can read the chiral ring relation from Eq. (2.1.13) with one $U(1)$ and only one odd chiral superfield with $U(1)$ charge $d$,

$$q = \sigma^{N+1}\,(d\sigma)^{-d}.$$

Comparing above two chiral ring relations, they are the same up to a factor $(-1)^d$.

Without loss of generality, we can take $N = 4$. We will look at the relation between the correlation functions for GLSMs for hypersurfaces of degree $d$ in $\mathbb{CP}^4$ and those on $\mathbb{CP}^{4|1}$, which is defined as in Eq. (2.1.3). In the supermanifold case, we shall have fields with $U(1)$ charges: $(1, 1, 1, 1, 1, d)$. Using Eq. (2.1.16), we obtain

$$\langle\mathcal{O}(\sigma)\rangle = \sum_{k\geq 0}\oint\frac{\mathrm{d}\sigma}{2\pi i}\frac{\mathcal{O}(\sigma)(d\sigma)^{1+dk}}{\sigma^{5+5k}}q^k. \tag{2.1.19}$$

Comparing with the hypersurface case, if we redefine $q$ as

$$\tilde{q} = (-1)^d q,$$

then the correlation functions for the supermanifold are exactly the same as those for the hypersurface.

In particular, if we take $d = 5$, the hypersurface will be the quintic. Correlation functions are

$$\langle \mathcal{O}(\sigma) \rangle = -\sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(-5\sigma)^2}{\sigma^{5+5k}(-5\sigma)^{1-5k}} \tilde{q}^k = -\sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(-5)^{1+5k}}{\sigma^4} \tilde{q}^k.$$

Then, correspondingly, correlation functions for the supermanifold are:

$$\langle \mathcal{O}(\sigma) \rangle = \sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)(5\sigma)^{1+5k}}{\sigma^{5+5k}} q^k = \sum_{k\geq 0} \oint \frac{\mathrm{d}\sigma}{2\pi i} \frac{\mathcal{O}(\sigma)5^{1+5k}}{\sigma^4} q^k.$$

We shall see that the $\tilde{q}$ and $q$ are related by

$$\tilde{q} = (-1)^5 q,$$

then it is easy to observe that those correlation functions on both models are exactly the same. It is in this sense that we claim we have reproduced the statement in [49, 55].

Further, we can compare their elliptic genera. The quintic example is already calculated in [60], which is

$$Z_{T^2}(\tau, z) = -\frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \oint_{u=0} \mathrm{d}u \frac{\theta_1(q, x^{-5})}{\theta_1(q, yx^{-5})} \left( \frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)} \right)^5,$$

and we can generalize it to a more general hypersurface of degree $d$:

$$Z_{T^2}(\tau, z) = -\frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \oint_{u=0} \mathrm{d}u \frac{\theta_1(q, x^{-d})}{\theta_1(q, yx^{-d})} \left( \frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)} \right)^5.$$

For the supermanifold $\mathbb{CP}^{4|1}$, from the formula (2.1.18), the elliptic genus is

$$Z_{T^2}(\tau, z) = -\frac{i\eta(q)^3}{\theta_1(q, y^{-1})} \oint_{u=0} \mathrm{d}u \frac{\theta_1(q, x^{d})}{\theta_1(q, y^{-1}x^{d})} \left( \frac{\theta_1(q, y^{-1}x)}{\theta_1(q, x)} \right)^5.$$

Using the following property of theta functions:

$$\theta_1(\tau, x) = -\theta_1(\tau, x^{-1}), \tag{2.1.20}$$

we conclude that the elliptic genera for both models are exactly the same without turning on the holonomy of global symmetry on torus.

As the first example, we have shown the equivalent relations between GLSM for hypersurface in $\mathbb{CP}^N$ and on supermanifold $\mathbb{CP}^{N|1}$. For elliptic genera, the R charge assignment can be more general which is discussed in appendix A.7. We also show in appendix A.3 that the equivalent relation for their elliptic genera is still valid. One can extend to more general cases and details can be found in [65].

## 2.2 Non-abelian GLSMs and Their Associated Cartan Theories

Supersymmetric localization can also be applied to non-abelian gauged linear sigma models, and the exact results one obtains give us more insight into these theories. In [66], the two-sphere partition function was utilized to argue that the physics of the nonabelian theory corresponds to an abelian theory which they called the associated Cartan theory. This observation was first made by mathematicians in [67] from a different point of view, and resonated with an early proposal by Hori and Vafa [25] for mirror symmetry. The paper [66] also demonstrated that the phase boundaries of non-abelian GLSMs can be obtained from the secondary fan of the associated Cartan theory.

### 2.2.1 Matching Two-Sphere Partition Functions

The two-sphere partition function in the Coulomb branch representation is

$$Z_{S^2}(q_a, \bar{q}_a) = \frac{1}{\mid W \mid} \sum_{\Updownarrow \in \mathbb{Z}^{rank(G)}} \int \left( \prod_{a=1}^{rankG} d\tau_a \right) Z_{class} Z_{gauge} \prod_i Z_{\Phi_i}, \tag{2.2.1}$$

where the classical factor $Z_{class}$ and the one-loop determinants $Z_\Phi$ and $Z_{gauge}$ are given by

$$Z_{class} = \exp(-\mathbf{r} \cdot \tau - i\theta), \tag{2.2.2}$$

$$Z_{gauge} = \prod_{\alpha \in roots(G)} \left( \frac{(\alpha_\mu m_\mu)^2}{4} - (\alpha_\mu \tau_\mu)^2 \right), \tag{2.2.3}$$

$$Z_{\Phi_i} = \prod_{\rho^i \in R_i} \frac{\Gamma\left(\frac{q_i}{2} - \rho^i_a(\tau_a + \frac{m_a}{2})\right)}{\Gamma\left(\frac{1-q_i}{2} + \rho^i_a(\tau_a - \frac{m_a}{2})\right)}. \tag{2.2.4}$$

The $\rho^i_a$ denote the weights of the representation $R$ of G. Note that $\alpha_\mu$ and $\rho^i_a$ are vectors in the weight lattice $\cong \mathbb{Z}^{rank(G)}$. $W$ denotes the Weyl group of G and $\mid W \mid$, its cardinality. The $q_i$ denote the R charges of the matter fields.

One can easily find that the partition function of a non-abelian GLSM is closely related to the partition function of an *abelian*-like GLSM from the structure of 2.2.1. The maximal torus of G is the abelian gauge group $U(1)^{rank(G)}$ which is the abelian group for the associated Cartan theory. The chiral matter fields of the GLSM correspond to $\dim(R_i)$ chiral superfields with charges specified by the weights of the representation which are the vectors $\rho^i_a$. Notice that we can rewrite $Z_{gauge}$ as

$$Z_{gauge} = \prod_{\alpha \in roots} \left( \frac{(\alpha_\mu m_\mu)^2}{4} - (\alpha_\mu \tau_\mu)^2 \right) = \left( \prod_{\alpha \in roots} e^{i\pi \sum_\mu m_\mu} \right) \left( \prod_{\alpha \in roots} \frac{\Gamma\left(1 - \alpha_\mu\left(\tau_\mu + \frac{m_\mu}{2}\right)\right)}{\Gamma\left(\alpha_\mu\left(\tau_\mu - \frac{m_\mu}{2}\right)\right)} \right). \tag{2.2.5}$$

This suggests that one needs to include additional $\dim(G) - \mathrm{rank(G)}$ chiral multiplets with gauge charge $(\alpha_\mu)$ and R-charge 2 in the associated Cartan theory. The paper [66] also suggests that the central charges of the IR limits of the non-abelian theory and its associated Cartan match. The central charges are encoded in the modular transformations of the elliptic genera, and in the next section we will show that the elliptic genus of the non-abelian theory is the same as the associated Cartan theory, which is a stronger assertion than equality of central charges.

### 2.2.2 Matching Elliptic Genera

In this section, which contains unpublished new results, we show that non-abelian GLSMs have the same elliptic genera as their associated Cartan theories. We define both theories in the UV and do the calculation by localization. The main difference between a non-abelian GLSM and its associated abelian GLSM is that the one-loop determinant in the non-abelian GLSM has off-diagonal terms in the vector multiplet corresponding to the W-bosons. The associated abelian Cartan theory has $\dim(G) - \mathrm{rank(G)}$ chiral matter fields with gauge charge $\alpha$ and vector R-charge 2. Here we prove their elliptic genera are the same.

Following [60, 62] and [63], we find that in the non-abelian theory, the off-diagonal terms contribute

$$Z_{\mathrm{off}}\left(\tau, z, u\right) = \prod_{\alpha:\mathrm{roots}} \frac{\theta_1\left(q, x^{\alpha}\right)}{\theta_1\left(q, y^{-1}x^{\alpha}\right)}. \tag{2.2.6}$$

By contrast, in associated abelian theory, the one-loop determinant of the corresponding $\dim(G) - \mathrm{rank(G)}$ chiral matter fields is

$$Z_{\Phi,\alpha}\left(\tau, z, u\right) = \prod_{\alpha:\mathrm{roots}} \frac{\theta_1\left(q, x^{\alpha}\right)}{\theta_1\left(q, yx^{\alpha}\right)}, \tag{2.2.7}$$

where we have used the fact that $\theta_1\left(q|z\right) = -\theta_1\left(q|-z\right)$ and the structure of the roots $\alpha$. Thus, we will have

$$Z_{\Phi,\alpha}\left(\tau, z, u\right) = \frac{\prod_{\alpha:\mathrm{roots}} \theta_1\left(q, x^{\alpha}\right)}{\prod_{-\alpha:\mathrm{roots}} \theta_1\left(q, yx^{-\alpha}\right)} = (-1)^{\sum \alpha} \prod_{\alpha:\mathrm{roots}} \frac{\theta_1\left(q, x^{\alpha}\right)}{\theta_1\left(q, y^{-1}x^{\alpha}\right)}. \tag{2.2.8}$$

This is exactly the off-diagonal one-loop determinant term. We have not considered twisted sectors of the Weyl obrifolds in the elliptic genus computations above. This is because the locus on which the supersymmetric theory localizes does not intersect the Weyl orbifold fixed points.

## 2.3 Exact Results: $A/2$-Twisted GLSM with a $\mathcal{N} = (2,2)$ Locus

In this section, we consider a $\mathcal{N} = (0,2)$ GLSM with a $\mathcal{N} = (2,2)$ locus. We compute $A/2$-twisted correlation functions exactly using supersymmetric localization, and then compare to previous results in [111, 117, 118, 120]. In terms of $(0,2)$ multiplets, the theory contains a $\mathfrak{g}$-valued vector multiplet, a chiral multiplet $\Sigma$ in the adjoint representation of $\mathfrak{g}$, and pairs of chiral and Fermi multiplets $(\Phi_i, \Lambda_I)$, with $i = I$, transforming in representations $\mathfrak{R}_i$ of $\mathfrak{g}$. On the $\mathcal{N} = (2,2)$ locus, the $\mathcal{E}_I$ and $J_I$ potentials read

$$\mathcal{E}_I = \Sigma\Phi_i \; , \qquad J_I = \partial_{\Phi_i} W(\Phi) \; , \qquad (I = i) \; , \tag{2.3.1}$$

where $\Sigma$ acts on $\Phi_i$ in the representation $\mathfrak{R}_i$, and $W$ is the $\mathcal{N} = (2,2)$ superpotential. More generally, any properly gauge-covariant holomorphic functions $\mathcal{E}_I$, $J_I$ are allowed as long as (1.1.91) is satisfied. (On the $\mathcal{N} = (2,2)$ locus, $\mathcal{E}^I J_I = 0$ follows from the gauge invariance of $W$.)

We choose to assign the following $R$-charges to the matter fields: [8]

$$R[\Sigma] = 0 \; , \qquad R[\Phi_i] = r_i \; , \qquad R[\Lambda_i] = r_i - 1 \; , \qquad r_i \in \mathbb{Z} \; . \tag{2.3.2}$$

This assignment automatically satisfies (1.1.101). The corresponding curved-space theory realizes the so-called A/2-twist, generalizing the $A$-twist off the $\mathcal{N} = (2,2)$ locus. The potential functions $\mathcal{E}_I$ and $J_I$ must have $R$-charges $r_i$ and $2 - r_i$, respectively. On the $\mathcal{N} = (2,2)$ locus, there also exists an axial-like $R$-symmetry $U(1)_A$ at the classical level. In $\mathcal{N} = (0,2)$ language, it corresponds to an alternative $R$-charge assignment

$$R_{\rm ax}[\Sigma] = 2 \; , \qquad R_{\rm ax}[\Phi_i] = 0 \; , \qquad R_{\rm ax}[\Lambda_i] = 1 \; . \tag{2.3.3}$$

We restrict ourselves to theories that preserve that $R_{\rm ax}$ off the $(2,2)$ locus as well. This means that $\mathcal{E}_I$ remains linear in $\Sigma$ while $J_I$ cannot depend on $\Sigma$ at all. Note that $R_{\rm ax}$ is generally anomalous at one-loop.

We would like to compute the correlation functions

$$\langle \mathcal{O}(\sigma) \rangle_{\mathbb{P}^1}^{(A/2)} \tag{2.3.4}$$

in the A/2-twisted theory on the sphere, where $\mathcal{O}(\sigma)$ is any gauge invariant polynomial in the scalar $\sigma$ in the $(0,2)$ chiral multiplet $\Sigma$. These are the simplest operators in the A/2-type pseudo-chiral ring. The presence of the $R_{\rm ax}$ symmetry leads to simple selection rules for (2.3.4). The gauge anomaly of $R_{\rm ax}$ assigns $R_{\rm ax}$ charge

$$R_{\rm ax}[q_A] = 2b_0^A \; , \tag{2.3.5}$$

[8] In the examples we will consider, the $R$-charges $r_i$ will all be either 0 or 2.

to the abelian gauge couplings (1.1.97), where $b_0^A$ equals the $\beta$-function coefficient (1.1.102). Moreover, $R_{\rm ax}$ suffers from a "gravitational" anomaly upon twisting. Due to the presence of zero-modes on the sphere, the path integral measure picks up a non-zero $R_A$ charge:

$$R_{\rm ax}[Z^{(A/2)}_{\mathbb{P}^1}] = -2d_{\rm grav} \ , \qquad d_{\rm grav} = -\dim(\mathfrak{g}) - \sum_i (r_i - 1)\dim(\mathfrak{R}_i) \ . \tag{2.3.6}$$

Therefore, the standard ghost number selection rules of the $A$-model remain valid away from the $(2,2)$ locus. Note that the coefficient $b_0^A$ in (2.3.5) also controls the FI parameter $\beta$-function (1.1.102).

The A/2-twisted correlation functions (2.3.4) can be computed by supersymmetric localization on the "Coulomb branch" spanned by the scalar field $\sigma$ in the chiral multiplet $\Sigma$. As we will show, the recent results of [41] can be extended to the $\mathcal{N} = (0,2)$ theory, provided some genericity condition is satisfied.

### 2.3.1 The $\mathcal{N} = (0,2)$ Coulomb Branch

Consider the Coulomb branch consisting of diagonal VEVs $\sigma$:

$$\sigma = {\rm diag}(\sigma_a) \ , \qquad a = 1, \cdots, {\rm rk}(\mathbf{G}) \ , \tag{2.3.7}$$

and similarly for the charge-conjugate field $\widetilde{\sigma}$. The Coulomb branch has the form $\mathfrak{M} \cong \mathfrak{h}_{\mathbb{C}}/W$, with $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ and $W$ the Weyl group of $\mathbf{G}$. Let us also denote by $\widetilde{\mathfrak{M}} \cong \mathfrak{h}_{\mathbb{C}} \cong \mathbb{C}^{{\rm rk}(\mathbf{G})}$ the covering space of $\mathfrak{M}$. At a generic point on $\mathfrak{M}$, the gauge group is Higgsed to its Cartan subgroup $\mathbf{H}$,

$$\mathbf{G} \to \mathbf{H} = \prod_{a=1}^{{\rm rank}(G)} U(1)_a \ , \tag{2.3.8}$$

with Lie algebra $\mathfrak{h}$. Consider the holomorphic potentials $\mathcal{E}_I = \mathcal{E}_I(\sigma, \phi)$, linear in $\sigma$, of $R$-charge $r_I = r_i - 1$ with $I = i$, which transform in the same representations $\mathfrak{R}_I$ of $\mathfrak{g}$ as $\Lambda_I$. Here and in the rest of this section, we identify the indices $i = I, j = J$, etc. On the Coulomb branch, we have

$$\mathcal{E}_I = \sigma_a \, E_I^a(\phi) \ , \tag{2.3.9}$$

and the matter multiplets $\Phi_I, \Lambda_I$ acquire masses

$$M_{IJ} = \partial_J \mathcal{E}_I \big|_{\phi=0} = \sigma_a \, \partial_J E_I^a \big|_{\phi=0} \ . \tag{2.3.10}$$

Note that (2.3.10) transforms in the representation $\mathfrak{R}_I \otimes \bar{\mathfrak{R}}_J$ of $\mathfrak{g}$. Gauge- and $U(1)_R$-invariance implies that the mass matrix (2.3.10) is block-diagonal (up to a relabeling of the indices), with each block consisting of fields transforming in the same gauge representation and having the same $R$-charge. Let us denote by $\gamma = \{I_\gamma\} \subset \{I\}$ the subset of indices corresponding to each of these blocks, so that we can partition the indices as $\{I\} = \cup_\gamma \{I_\gamma\}$, and let $\mathfrak{R}_\gamma = \mathfrak{R}_{I_\gamma}$ be the corresponding gauge representations. We also denote by $r_\gamma$ the corresponding $R$-charge. (That is, the chiral and Fermi multiplets $\Phi_{I_\gamma}$ and $\Lambda_{I_\gamma}$ have $R$-charges $r_\gamma$ and $r_\gamma - 1$, respectively.) Each block is diagonal in representation space, and we introduce the notation:

$$M_{I_\gamma J_\gamma}{}^{\rho_\gamma \rho'_\gamma} = \delta^{\rho_\gamma \rho'_\gamma} \left(M_{(\gamma,\, \rho_\gamma)}\right)_{I_\gamma J_\gamma} \ , \tag{2.3.11}$$

for each block. In (2.3.11), $\rho_\gamma, \rho'_\gamma$ are indices running over the weights of the representation $\mathfrak{R}_\gamma$. We also write

$$\det M_{(\gamma,\, \rho_\gamma)} = \det_{I_\gamma J_\gamma} \left(M_{(\gamma,\, \rho_\gamma)}\right)_{I_\gamma J_\gamma} \ . \tag{2.3.12}$$

In the following, we shall assume that

$$\det M_{(\gamma,\, \rho_\gamma)} \neq 0 \ , \qquad \forall (\gamma,\, \rho_\gamma) \ , \tag{2.3.13}$$

at any *generic* point on the Coulomb branch. This ensures that all the matter fields are massive on $\widetilde{\mathfrak{M}}$ except at special loci of positive codimensions. In particular, the condition (2.3.13) rules out theories with $\mathcal{E}_I = 0$.

At a generic point on the Coulomb branch, we can integrate out the matter fields to obtain an effective $J_a$-potential:

$$J_a^{\text{eff}} = \tau^a - \frac{1}{2\pi i} \sum_\gamma \sum_{\rho_\gamma \in \mathfrak{R}_\gamma} \rho_\gamma^a \log \left(\det M_{(\gamma,\, \rho_\gamma)}\right) - \frac{1}{2} \sum_{\alpha > 0} \alpha^a \ , \tag{2.3.14}$$

where $\rho_\gamma$ are the weights of $\mathfrak{R}_\gamma$ and $\alpha$ are the positive simples roots of $\mathfrak{g}$. The classical couplings $(\tau^a) \in \mathfrak{h}^*_{\mathbb{C}}$ are the complexified parameters of the effective theory, which are obtained from the parameters $\tau^A$ by embedding the central sub-algebra $\mathfrak{c}^*_{\mathbb{C}} \subset \mathfrak{h}^*_{\mathbb{C}} \subset \mathfrak{g}^*_{\mathbb{C}}$ of the dual of $\mathfrak{g}$ into $\mathfrak{h}^*_{\mathbb{C}}$. The second set of terms in (2.3.14) arises from integrating out the chiral and Fermi multiplets [115], and the last term is the contribution from the $W$-bosons multiplets. From (2.3.14), we read off the effective FI parameter on the Coulomb branch. In particular, we are interested in the effective FI parameter at infinity on the Coulomb branch. Denoting by $R$ the overall radius of $\widetilde{\mathfrak{M}} \cong \mathbb{C}^r$, we define:

$$\xi_{\text{eff}}^{\text{UV}} = \xi + \frac{1}{2\pi} b_0 \lim_{R \to \infty} \log R \ , \qquad b_0 = \sum_i \sum_{\rho_i \in \mathfrak{R}_i} \rho_i \ , \tag{2.3.15}$$

where $b_0 \in i\mathfrak{h}^*$ is equivalent to $(b_0^A) \in i\mathfrak{c}^*$ defined in (1.1.102).

### 2.3.2 $A/2$-Twisted Correlation Functions

The correlation functions (2.3.4) can be computed explicitly as a sum over flux sectors on the sphere, with each summand given by a generalized Jeffrey-Kirwan (JK) residue on

$\widetilde{\mathfrak{M}} \cong \mathbb{C}^{\mathrm{rk}(\mathbf{G})}$. We find:

$$\langle \mathcal{O}(\sigma) \rangle_{\mathbb{P}^1}^{(A/2)} = \frac{(-1)^{N_*}}{|W|} \sum_{k \in \Gamma_{\mathbf{G}^\vee}} q^k \,\text{JKG-Res}\big[\xi_{\text{eff}}^{\text{UV}}\big]\, \mathcal{Z}_k^{\text{1-loop}}(\sigma)\, \mathcal{O}(\sigma)\, d\sigma_1 \wedge \cdots \wedge d\sigma_{\mathrm{rk}(\mathbf{G})} \ , \quad (2.3.16)$$

with

$$\mathcal{Z}_k^{\text{1-loop}}(\sigma) = (-1)^{\sum_{\alpha>0}(\alpha(k)+1)} \prod_{\alpha>0} \alpha(\sigma)^2 \prod_\gamma \prod_{\rho_\gamma \in \mathfrak{R}_\gamma} \left(\det M_{(\gamma,\rho_\gamma)}\right)^{r_\gamma - 1 - \rho_\gamma(k)} \ . \quad (2.3.17)$$

Here and in the next subsection, we explain the notation used in (2.3.16)-(2.3.17). The derivation of the formula is discussed in subsection 2.3.4.

The overall factor in (2.3.16) is Weyl symmetry factor, with $|W|$ the order of the Weyl group of $\mathbf{G}$. The sign factor $(-1)^{N_*}$ is a sign ambiguity. In the examples we shall consider with chiral multiplets of $R$-charges 0 and 2 only, we should take $N_*$ to be the number of chiral multiplets of $R$-charge 2 [41].

The sum in (2.3.16) is over the GNO-quantized magnetic fluxes $k \in \Gamma_{\mathbf{G}^\vee} \subset i\mathfrak{h}$. The integral lattice $\Gamma_{\mathbf{G}^\vee} \cong \mathbb{Z}^{\mathrm{rk}(\mathbf{G})}$ can be obtained from $\Gamma_{\mathbf{G}}$, the weight lattice of electric charges of $\mathbf{G}$ within the vector space $i\mathfrak{h}^*$, by

$$\Gamma_{\mathbf{G}^\vee} = \{\, k \,:\, \rho(k) \in \mathbb{Z} \ \ \forall \rho \in \Gamma_{\mathbf{G}} \,\} \ , \quad (2.3.18)$$

where $\rho(k) = \sum_a \rho^a k_a$ is given by the canonical pairing of the dual vector spaces. Let us also introduce the notation $\vec{k} \in \mathbb{Z}^n$ to denote the fluxes in the free part (1.1.96) of the center of $\mathbf{G}$. We define

$$q^k \equiv \exp(2\pi i \sum_{A=1}^{n} (\vec{\tau})_A (\vec{k})_A) = \exp(2\pi i \tau(k)) \ . \quad (2.3.19)$$

Here $\vec{\tau} \in \mathbb{C}^n$ denotes the complexified FI parameter, while $\tau$ is the same FI parameter viewed as an element of $\mathfrak{h}^*_{\mathbb{C}}$.

Each summand in (2.3.16) is given by a (conjectured) generalization of the JK residue, the JKG residue, upon which we elaborate shortly. That residue depends on the argument $\xi_{\text{eff}}^{\text{UV}}$ in (2.3.16), the effective FI parameter in the UV defined in (2.3.15).

The integrand is a rk($\mathbf{G}$)-form on $\widetilde{\mathfrak{M}} \cong \mathbb{C}^{\mathrm{rk}(\mathbf{G})}$ written in the coordinates $\sigma_a$. The expression (2.3.17) is the contribution from the massive fields on the Coulomb branch. The first product in (2.3.17) runs over all the positive simple roots $\alpha > 0$ of $\mathfrak{g}$ and corresponds to the $W$-bosons. The second product in (2.3.17) is the contribution from the matter multiplets $\Phi_I, \Lambda_I$, with the partition of indices $\{I\} = \cup_\gamma \{I_\gamma\}$ as explained above (2.3.11), and another product over all the weights $\rho_\gamma$ of the representation $\mathfrak{R}_\gamma$ of $\mathfrak{g}$, for each $\gamma$. The polynomials $\det M_{(\gamma,\rho_\gamma)}$ were defined in (2.3.11)-(2.3.12).

### 2.3.3 The Jeffrey-Kirwan-Grothendiek Residue

Let us introduce the collective label $\mathcal{I}_\gamma = (\gamma, \rho_\gamma)$ for the field components in each block $\gamma$. In any given flux sector, the integrand in (2.3.16) is a meromorphic $(0, r)$-form on [9] $\widetilde{\mathfrak{M}} \cong \mathfrak{h}_\mathbb{C} \cong \mathbb{C}^r$ with potential singularities at:

$$\cup_\gamma \mathcal{H}_{\mathcal{I}_\gamma} \subset \mathbb{C}^r \; , \qquad \mathcal{H}_{\mathcal{I}_\gamma} \cong \{\sigma \in \mathbb{C}^r \,|\, \det M_{\mathcal{I}_\gamma} = 0\} \; . \tag{2.3.20}$$

Each $\mathcal{H}_{\mathcal{I}_\gamma}$ is a divisor (codimension-one subvariety [10]) of $\mathbb{C}^r$ and all these divisors intersect at $\sigma = 0$. Let us denote by

$$P_{\mathcal{I}_\gamma}(\sigma) = \det M_{\mathcal{I}_\gamma}(\sigma) \in \mathbb{C}[\sigma_1, \cdots, \sigma_r] \tag{2.3.21}$$

the homogeneous polynomials of degree $d_\gamma$ associated to (2.3.20). (For each $\gamma$, every $P_{\mathcal{I}_\gamma}$ has the same degree.) To each $P_{\mathcal{I}_\gamma}$, we associate the charge vector $Q_{\mathcal{I}_\gamma} \in i\mathfrak{h}^*$, which is the $U(1)^r$ gauge charge of the field component $\mathcal{I}_\gamma$ under the Cartan subalgebra $\mathfrak{h}$—that is:

$$Q^a_{\mathcal{I}_\gamma} = \rho^a_\gamma \; , \tag{2.3.22}$$

if $\mathcal{I}_\gamma = (\gamma, \rho_\gamma)$. In any flux sector with flux $k$, the actual singularities consist of the subset of the potentials singularities (2.3.20) at $P_{\mathcal{I}_\gamma}$ such that

$$\rho_\gamma(k) - r_\gamma \geq 0 \; . \tag{2.3.23}$$

We shall *assume* that, in any given flux sector, the set of charge vectors $\mathbf{Q} \subset \{Q_{\mathcal{I}_\gamma}\}$ associated to the actual singularities is *projective*—that is, the vectors $\mathbf{Q}$ are contained within a half-space of $i\mathfrak{h}^*$. Note that a non-projective $\mathbf{Q}$ signals the presence of dangerous gauge invariant operators which may take an arbitrarily large VEV [41].

We define a "Jeffrey-Kirwan-Grothendieck" (JKG) residue as a simple generalization of the Jeffrey-Kirwan residue. Let us first recall the definition of the Grothendieck residue [104] specialized to our case. Given $r$ homogeneous polynomials $P_b$, $b = 1, \cdots, r$, in $\mathbb{C}[\sigma_1, \cdots, \sigma_r]$, of degrees $d_b$, such that $P_1 = \cdots = P_r = 0$ if and only if $\sigma_1 = \cdots = \sigma_r = 0$, let us define a $(r, 0)$-form on $\mathbb{C}^r$:

$$\omega^{(P)} = \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_r}{P_1(\sigma) \cdots P_r(\sigma)} \; . \tag{2.3.24}$$

Let $D_b$ be the divisor in $\mathbb{C}^r$ corresponding to $P_b = 0$, and let $D_P = \cup_b D_b$. The form (2.3.24) is holomorphic on $\mathbb{C}^r \backslash D_P$. The Grothendieck residue of $f\,\omega^{(P)}$ at $\sigma = 0$, with $f = f(\sigma)$ any holomorphic function, is given by:

$$\mathrm{Res}_{(0)}\, f\,\omega^{(P)} = \frac{1}{(2\pi i)^r} \oint_{\Gamma_\epsilon} f\,\omega^{(P)} \; , \tag{2.3.25}$$

[9]Here and in the rest of this section, we write $r = \mathrm{rk}(\mathbf{G})$ to avoid clutter.

[10]We use the terms "divisor" and "codimension-one variety" interchangeably. That is, all our divisors are effective.

with a real $r$-dimensional contour:

$$\Gamma_\epsilon = \left\{\sigma \in \mathbb{C}^r \,\middle|\, |P_{l_b}| = \epsilon_b \ , \ b = 1, \cdots, r\right\} \ , \tag{2.3.26}$$

oriented by $d(\arg(P_{l_1})) \wedge \cdots \wedge d(\arg(P_{l_r})) \geq 0$, with $\epsilon_b > 0$, $\forall b$. The residue (2.3.25) only depends on the homology class of $\Gamma_\varepsilon$ in $H_n(\mathbb{C}^r \backslash D_P)$. Note that, if $f$ is an homogenous polynomial of degree $d_0$, the residue (2.3.25) vanishes unless $d_0 = \sum_{b=1}^r (d_b - 1)$.

Consider an arrangement of $s \geq r$ distinct irreducible divisors $\mathcal{H}_{\mathcal{I}_\gamma} \cong \{\sigma \,|\, P_{\mathcal{I}_\gamma} = 0\}$ of $\mathfrak{h}_\mathbb{C} \cong \mathbb{C}^r$, intersecting at $\sigma = 0$, and denote by $D_\mathbf{P}$ their union. To each $\mathcal{I}_\gamma$ is associated the charge $Q_{\mathcal{I}_\gamma} \in i\mathfrak{h}^*$. We denote this data by:

$$\mathbf{P} = \{P_{\mathcal{I}_\gamma}\} \ , \qquad \qquad \mathbf{Q} = \{Q_{\mathcal{I}_\gamma}\} \ , \tag{2.3.27}$$

with $\mathbf{Q}$ projective. Let $R_\mathbf{P}$ be the space of rational holomorphic $(r, 0)$-forms with poles on $D_\mathbf{P}$, and let $S_\mathbf{P} \subset R_\mathbf{P}$ be the linear span of

$$\omega_{S,P_0} = d\sigma_1 \wedge \cdots \wedge d\sigma_r \prod_{P_b \in P_S} \frac{P_0}{P_b} \ , \tag{2.3.28}$$

where $P_S = \{P_1, \cdots, P_r\} \subset \mathbf{P}$ denotes any subset of $r$ distinct polynomials in $\mathbf{P}$ associated to $r$ distinct charges $Q_S = \{Q_1, \cdots, Q_r\} \subset \mathbf{Q}$, while $P_0$ is any homogeneous polynomial of degree $d_0 = \sum_{b=1}^r (d_b - 1)$, with $d_b$ the degree of $P_b$. The JKG-residue on $S_\mathbf{P}$ is defined by

$$\text{JKG-Res}[\eta] \ \omega_S = \begin{cases} \text{sign}\left(\det(Q_S)\right) \ \text{Res}_{(0)} \, \omega_S & \text{if} \quad \eta \in \text{Cone}(Q_S) \ , \\ 0 & \text{if} \quad \eta \notin \text{Cone}(Q_S) \ , \end{cases} \tag{2.3.29}$$

in terms of a vector $\eta \in \mathfrak{h}^*$. Here, $\text{Cone}(Q_S)$ denotes the positive span of the $r$ linearly independent vectors $Q_S$ in $\mathfrak{h}^*$. In (2.3.16), we have this same JKG-residue with $\eta = \xi_{\text{eff}}^{\text{UV}}$.

On the $\mathcal{N} = (2, 2)$ locus, the divisors $\mathcal{H}_{\mathcal{I}_\gamma}$ are hyperplanes perpendicular to $Q_{\mathcal{I}_\gamma}$, with

$$P_{\mathcal{I}_\gamma} = \left(Q_{\mathcal{I}_\gamma}(\sigma)\right)^{d_{\mathcal{I}_\gamma}} \ , \tag{2.3.30}$$

and the JKG-residue reduces to an ordinary Jeffrey-Kirwan residue, reproducing previous results for the $A$-twisted GLSM.

### 2.3.4 Derivation of the JKG Residue Formula

In this subsection, we sketch a derivation of the residue formula (2.3.16), closely following previous works [41, 62], to which we refer for more details. We shall leave one important technical step—the proper cell decomposition of the Coulomb branch—as a conjecture. More generally, we would like to stress that the JKG residue has not yet been defined satisfactorily at the mathematical level. We hope that the present work will lead to further investigation of this new conjectured residue.

### Generalities

We use the kinetic terms of section 1.1.2 in the localizing action:

$$\mathscr{L}_{\text{loc}} = \frac{1}{e^2}\left(\mathscr{L}_{YM} + \mathscr{L}_{\widetilde{\Sigma}\Sigma}\right) + \frac{1}{g^2}\left(\mathscr{L}_{\widetilde{\Phi}\Phi} + \mathscr{L}_{\widetilde{\Lambda}\Lambda}\right) \ , \tag{2.3.31}$$

with $e$ and $g$ some dimensionless parameters that we can take arbitrarily small. With the standard reality condition $\widetilde{\sigma} = \bar{\sigma}$, the kinetic term for the chiral multiplet $\Sigma$ localizes to

$$\partial_\mu \sigma = 0 \ , \qquad\qquad [\sigma, \widetilde{\sigma}] = 0 \ . \tag{2.3.32}$$

We therefore localize onto the Coulomb branch discussed in subsection 2.3.1. We also have a sum over gauge fluxes,

$$k = \frac{1}{2\pi}\int_{\mathbb{P}^1} da \ , \tag{2.3.33}$$

with $k$ in the flux lattice (2.3.18). In each topological sector, let us define

$$\hat{D} = -i\left(D - 2if_{1\bar{1}}\right) \ , \tag{2.3.34}$$

with $\hat{D}$ a real field corresponding to fluctuations around the supersymmetric value $\hat{D} = 0$. At a generic points on the Coulomb branch, all the other matter field are massive, while for special values of $\sigma$ corresponding to

$$P_{\mathcal{I}_\gamma}(\sigma) = 0 \ , \tag{2.3.35}$$

with $P_{\mathcal{I}_\gamma}$ defined in (2.3.21), we have additional bosonic zero modes and the localized path integral would be singular. To regularize these singularities, it is useful to keep the constant mode of $\hat{D}$ in intermediate computations [62].

We also have the fermionic zero modes $\widetilde{\lambda}$ from the Coulomb branch vector multiplets, and the fermionic zero modes $\widetilde{\mathcal{B}}^\Sigma$ from $\Sigma$—corresponding to (1.1.109) with $\mathbf{r} = 0$. The path integral localizes to:

$$Z_{\text{GLSM}} = \frac{1}{|W|}\sum_k q^k \int \prod_a^{\text{rk}(\mathbf{G})} \left[d^2\sigma_a \, d\hat{D}_a \, d\widetilde{\lambda}_a \, d\widetilde{\mathcal{B}}^\Sigma_a\right] \, \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \widetilde{\lambda}, \widetilde{\mathcal{B}}^\Sigma, \hat{D}) \ , \tag{2.3.36}$$

where $\mathcal{Z}_k(\sigma, \widetilde{\sigma}, \widetilde{\lambda}, \widetilde{\mathcal{B}}^\Sigma, \hat{D})$ is the result of integrating out the matter fields and W-bosons in the supersymmetric background:

$$\mathcal{V}_0 = (\widetilde{\lambda}_a \ , \hat{D}_a) \ , \qquad \Sigma_0 = (\sigma_a \ , \widetilde{\sigma}_a \ , \widetilde{\mathcal{B}}^\Sigma_a) \ . \tag{2.3.37}$$

Supersymmetry implies the relation:

$$\delta \mathcal{Z}_k = \left(\hat{D}_a \frac{\partial}{\partial \widetilde{\lambda}_a} + \widetilde{\mathcal{B}}^\Sigma_a \frac{\partial}{\partial \widetilde{\sigma}_a}\right) \mathcal{Z}_k = 0 \ . \tag{2.3.38}$$

In the limit $e, g \to 0$, we have

$$\mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D}) \equiv \mathcal{Z}_k(\sigma, \widetilde{\sigma}, 0, 0, \hat{D}) = \lim_{e\to 0} e^{-S_0}\, \mathcal{Z}_k^{\text{massive}}(\sigma, \widetilde{\sigma}, \hat{D})\, \mathcal{Z}_k^{\text{1-loop}}(\sigma, \widetilde{\sigma}, \hat{D})\ . \tag{2.3.39}$$

Here, $e^{-S_0}$ is the classical contribution, with

$$S_0 = \text{vol}(S^2) \left( \frac{1}{2e^2} \hat{D}^2 - \frac{1}{2} \widetilde{\tau}(\hat{D}) \right)\ , \tag{2.3.40}$$

(setting $e_0 = 1$ in (1.1.85)), while $\mathcal{Z}_k^{\text{massive}}$ is the contribution from non-zero modes, which is trivial when $\hat{D} = 0$, and $\mathcal{Z}_k^{\text{1-loop}}$ is the zero-mode contribution, which reduces to (2.3.17) when $\hat{D} = 0$. These one-loop contributions are derived and discussed in Appendix A.4.

The insertion of any pseudo-chiral operator $\mathcal{O}(\sigma)$ does not modify the derivation. It simply corresponds to inserting the same factor $\mathcal{O}(\sigma)$ with constant $\sigma$ in the integrand (2.3.45).

**The Rank-One Case**

Consider first the case of a rank-one gauge group. We choose $\mathbf{G} = U(1)$ for simplicity, but the generalization is straightforward. We have matter fields $\Phi_i, \Lambda_i$ with gauge charges $Q_i$ and $R$-charges $r_i$ and $r_i - 1$, organized in blocks $\Phi_\gamma$. We have the one-loop contributions

$$\mathcal{Z}_k^{\text{massive}}(\sigma, \widetilde{\sigma}, \hat{D}) = \prod_\gamma \prod_{\lambda_{(\gamma,k)}} \frac{\det(\lambda_{(\gamma,k)} + |M_\gamma|^2)}{\det(\lambda_{(\gamma,k)} + |M_\gamma|^2 + iQ_\gamma \hat{D})} \tag{2.3.41}$$

with $\lambda_{(\gamma,k)} > 0$, and

$$\mathcal{Z}_k^{\text{1-loop}}(\sigma, \widetilde{\sigma}, \hat{D}) = \prod_\gamma \mathcal{Z}_{k,\gamma}^{\text{1-loop}} \tag{2.3.42}$$

with

$$\mathcal{Z}_{k,\gamma}^{\text{1-loop}} = \begin{cases} (\det M_\gamma)^{r_\gamma - 1 - Q_\gamma k} & \text{if } \ r_\gamma - Q_\gamma k \geq 1\ , \\ \left( \frac{\det \bar{M}_\gamma}{\det\left(|M_\gamma|^2 + iQ_\gamma \hat{D}\right)} \right)^{1 - r_\gamma + Q_\gamma k} & \text{if } \ r_\gamma - Q_\gamma k < 1\ , \end{cases} \tag{2.3.43}$$

from the zero modes. The singular locus on the Coulomb branch corresponds to $\det M_\gamma = 0$, for each $\gamma$. This is simply $\sigma = 0$ in the present case, but it is useful to suppose that $\det M_\gamma$ have more general roots.

In each flux sector, we remove a small neighborhood $\Delta_{\epsilon,k}$ of the singular locus, of size $\epsilon > 0$, and we decompose this neighborhood as

$$\Delta_{\epsilon,k} = \Delta_{\epsilon,k}^{(+)} \cup \Delta_{\epsilon,k}^{(-)} \cup \Delta_{\epsilon,k}^{(\infty)}\ , \tag{2.3.44}$$

where $\Delta_{\epsilon,k}^{(\pm)}$ corresponds to the neighborhood of the singularities the positively and negatively charged matter fields ($Q_\gamma > 0$ and $Q_\gamma < 0$, respectively), as well as the neighborhood of

$\sigma = \infty$. We assume that our theory is such that we can always separate the singularities from positively and negatively charged fields, for any given $k$. (Such singularities "projective singularities" in the sense defined below (2.3.23).)

Using (2.3.38), one can perform the integration over the fermionic zero modes in (2.3.45), to obtain:

$$Z_{\text{GLSM}} = \sum_k q^k \int_\Gamma \frac{d\hat{D}}{\hat{D}} \oint_{\partial\Delta_{\epsilon,k}} d\sigma\, \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D}) \ . \tag{2.3.45}$$

For each $\gamma$ block, the Hermitian matrix $|M_\gamma|^2$ can be diagonalized with eigenvalues $m_\gamma^2 > 0$. The absence of chiral multiplet tachyonic modes requires that

$$\text{Im}(Q_\gamma \hat{D}) < m_\gamma^2 \ , \qquad \forall\gamma, \forall m_\gamma^2 \ . \tag{2.3.46}$$

This determines the $\hat{D}$ contour of integration $\Gamma$ exactly like in [41]. There is an important contribution from infinity, which is controlled by the effective FI parameter (2.3.15). We have a twofold freedom in choosing $\Gamma$ (corresponding to the sign of $\eta$ in (2.3.29)) and we can choose $\eta = \xi_{\text{eff}}^{\text{UV}}$ so that the contribution from $\partial\Delta_{\epsilon,k}^{(\infty)}$ vanishes [41]. In that case, performing the $\hat{D}$ integral picks the contributions from $\partial\Delta_{\epsilon,k}^{(+)}$ or $\partial\Delta_{\epsilon,k}^{(-)}$ according to the sign of $\xi_{\text{eff}}^{\text{UV}}$:

$$Z_{\text{GLSM}}^{(+)} = \sum_k q^k \oint_{\partial\Delta_{\epsilon,k}^{(+)}} d\sigma\, \mathcal{Z}_k^{\text{1-loop}}(\sigma) \ , \quad Z_{\text{GLSM}}^{(-)} = -\sum_k q^k \oint_{\partial\Delta_{\epsilon,k}^{(-)}} d\sigma\, \mathcal{Z}_k^{\text{1-loop}}(\sigma) \ . \tag{2.3.47}$$

The first equality corresponds to $\eta = \xi_{\text{eff}}^{\text{UV}} > 0$ and the second equality corresponds to $\eta = \xi_{\text{eff}}^{\text{UV}} < 0$. When $b_0 = 0$, $\xi_{\text{eff}}^{\text{UV}}$ can be tuned to be of either sign and the two formulas (2.3.47) are equal as formal series [41]. The result (2.3.47) can be written as the JKG residue (2.3.29).

### The General Case

In the general case, one can perform the fermionic integral in (2.3.45) explicitly to obtain:

$$Z_{\text{GLSM}} = \frac{1}{|W|} \sum_k q^k \int \prod_a^{\text{rk}(\mathbf{G})} \left[ d\sigma_a\, d\widetilde{\sigma}_a\, d\hat{D}_a \right] \det_{ab}(h_{ab})\, \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D}) \ , \tag{2.3.48}$$

with $h_{ab}$ a two-tensor on $\widetilde{\mathfrak{M}}$ that satisfies

$$\partial_{\widetilde{\sigma}_a} h_{bc} - \partial_{\widetilde{\sigma}_c} h_{ba} = 0 \ , \qquad \partial_{\widetilde{\sigma}_a} \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D}) = \hat{D}^b h_{ba}\, \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D}) \ , \tag{2.3.49}$$

with $\mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D})$ given in (2.3.39). The only difference with the discussion in [41] is that $h_{ab}$ need not be symmetric. One way to motivate this result is to note that the low-energy effective action on the Coulomb branch should take the form

$$S_{\text{eff}} \propto -\hat{D}^a \widetilde{J}_a^{\text{eff}} + \widetilde{\lambda}^a \frac{\partial \widetilde{J}_a^{\text{eff}}}{\partial \sigma_b} \widetilde{\mathcal{B}}_b^\Sigma \ , \tag{2.3.50}$$

with $\widetilde{J}_a^{\text{eff}}$ the anti-holomorphic effective superpotential. Therefore, we have $h_{ab} = \frac{\partial \widetilde{J}_a}{\partial \sigma_b}$ and the properties (2.3.49) follow. More generally, the $h_{ab}$ in (2.3.48) may depend on $\hat{D}_a$ but the above properties are preserved and follow from supersymmetry. We may define a form

$$\nu(V) = V^a h_{ab} d\widetilde{\sigma}^b \tag{2.3.51}$$

for any $V$ valued in $\mathfrak{h}_{\mathbb{C}}$, in terms of which (2.3.49) reads

$$\bar{\partial}\nu = 0 \ , \qquad \bar{\partial}\mathcal{Z}_k = \nu(D)\mathcal{Z}_k \ , \tag{2.3.52}$$

with $\bar{\partial}$ the Dolbeault operator on $\widetilde{\mathfrak{M}}$. In any flux sector, we define $\Delta_{\epsilon,k}$ to be the union of the small neighborhoods of size $\epsilon$ around the divisors $\mathcal{H}_{\mathcal{I}_\gamma}$ in (2.3.20) such that (2.3.23) holds, and of the neighborhood of $\sigma = \infty$. We have

$$Z_{\text{GLSM}} = \frac{1}{|W|} \lim_{e,\epsilon \to 0} \sum_k q^k \int_{\Gamma \ltimes \widetilde{\mathfrak{M}} \setminus \Delta_{\epsilon,k}} \mu_{(k)} \ , \tag{2.3.53}$$

where $r = \text{rk}(\mathbf{G})$ and $\mu_{(k)}$ is a top-form:

$$\mu_{(k)} = \frac{1}{r!} \mathcal{Z}_k(\sigma, \widetilde{\sigma}, \hat{D})\, d^r\sigma \wedge \nu(d\hat{D})^{\wedge r} \ . \tag{2.3.54}$$

From here onward, one may follow [41] almost verbatim. The main difficulty lies in dealing with the boundaries of $\Delta_{\epsilon,k}$, the tubular neighborhood of the singular locus that should be excised from $\widetilde{\mathfrak{M}}$. We conjecture that a sufficiently good cell decomposition exists, such that the manipulations of [41, 62] can be repeated while replacing the singular hyperplanes by singular divisors. This would establish the JKG residue prescription in the regular case, that is when the number $s$ of singular divisors equals $r$. (The prescription for the non-regular case, $s > r$, is a further conjecture, motivated by examples.)

### 2.3.5 Abelian Examples and Quantum Sheaf Cohomology

For abelian (0,2) supersymmetric gauged linear sigma models which are deformations of (2,2) theories, there are already extensive results in the literature (see *e.g.* [68–70]). For example, for models describing toric varieties with a deformation of the tangent bundle, both the quantum sheaf cohomology rings and expressions for correlation functions are known.

However, the methods of supersymmetric localization give new and much simplified derivations of those expressions. In this section, we will outline several examples in this new language.

#### Projective Spaces $\mathbb{P}^{N-1}$

In this section, we will discuss $\mathbb{P}^{N-1}$. Now, the tangent bundle of $\mathbb{P}^{N-1}$ is rigid, it admits no deformations. We can formally try to deform it, which will act as a warm-up example for more general toric varieties, but in this special case, field redefinitions can remove our deformations.

The tangent bundle of $\mathbb{P}^{N-1}$ is defined by the sequence

$$0 \longrightarrow \mathcal{O} \stackrel{*}{\longrightarrow} \mathcal{O}(1)^N \longrightarrow T\mathbb{P}^{N-1} \longrightarrow 0$$

where $*$ is given by multiplication by homogeneous coordinates, and is encoded in

$$\overline{D}_+ \Lambda^i \;=\; \Sigma \Phi^i$$

where the $\Lambda$ are the Fermi superfields corresponding to the $\mathcal{O}(1)$ line bundle elements, and $\Phi^i$ the chiral superfields corresponding to homogeneous coordinates. Formally, we could try to deform the bundle by taking

$$\overline{D}_+ \Lambda^i \;=\; A^i_j \Sigma \Phi^j,$$

where $A$ is an invertible $N \times N$ matrix. Physically, we can use field redefinitions of the $\Phi$'s to remove the $A$ dependence, so physically these $A$'s have no meaning, agreeing with the mathematical fact that the tangent bundle of $\mathbb{P}^{N-1}$ has no deformations. However, we can formally use this as a test case to develop the technology.

Using earlier results, the contribution from the deformed chiral and Fermi superfields above has the form

$$\left( \frac{1}{\det E} \right)^{\mathfrak{m}+1}$$

where

$$E^i_j \;=\; A^i_j \sigma,$$

so we get the following expression for correlation functions:

$$\langle \sigma_1 \cdots \sigma_n \rangle \;=\; \sum_{\mathfrak{m} \in \mathbb{Z}} \int \frac{d\sigma}{2\pi i} \frac{1}{(\det E)^{\mathfrak{m}+1}} q^{\mathfrak{m}} \sigma^n, \tag{2.3.55}$$

$$= \begin{cases} (\det A)^{-1} q^{(n+1-N)/N} & \text{if } n+1 = N(k+1) \text{ for some } k \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.56}$$

Note in particular that the OPE's $\det E = q$ are obeyed, as expected in this example [70]. Also note that if we make the (2,2) locus completely explicit by taking $A$ to be the identity matrix, then the result above reproduces that in *e.g.* [53][section 5.1], as

$$\det E \;=\; (\det A)\, \sigma^N.$$

$\mathbb{P}^1 \times \mathbb{P}^1$

The toric variety $\mathbb{P}^1 \times \mathbb{P}^1$ is the simplest example of a toric variety with nontrivial tangent bundle deformations, and so is often used as a prototype for many discussions of quantum sheaf cohomology.

Mathematically, we can describe a deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$ as the cokernel $\mathcal{E}$ below:

$$0 \longrightarrow \mathcal{O}^2 \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0$$

where

$$* = \left[ \begin{array}{cc} Ax & Bx \\ Cy & Dy \end{array} \right],$$

for $x$ and $y$ vectors of homogeneous coordinates on the two projective space factors.

Now, following the methods we have described so far, correlation functions in the A/2 twist of this theory are of the form

$$\langle f(\sigma, \widetilde{\sigma}) \rangle = \sum_{\mathfrak{m}_1 \in \mathbb{Z}} \sum_{\mathfrak{m}_2 \in \mathbb{Z}} \mathrm{JKG} - \mathrm{Res}_{\sigma=\tilde{\sigma}=0} \left( \frac{1}{\det E} \right)^{\mathfrak{m}_1+1} \left( \frac{1}{\det \widetilde{E}} \right)^{\mathfrak{m}_2+1} f(\sigma, \widetilde{\sigma}) q^{\mathfrak{m}_1} \widetilde{q}^{\mathfrak{m}_2}, \quad (2.3.57)$$

for

$$E = A\sigma + B\widetilde{\sigma}, \quad \widetilde{E} = C\sigma + D\widetilde{\sigma}.$$

The quantum sheaf cohomology ring relations (OPE's in the A/2 twist) of this model were derived in *e.g.* [68–70] and take the form

$$\det E = q, \quad \det \widetilde{E} = \widetilde{q}.$$

These relations can be more or less immediately read off from the result (2.3.57) for the correlation functions in terms of residues. Specifically, note that inserting a factor of *e.g.* $\det E$ in the correlator is equivalent to shifting $\mathfrak{m}_1$ by 1, which in turn is equivalent to shifting the exponent of $q$ by 1. Thus, formally in the correlation function, inserting $\det E$ is equivalent to inserting $q$, and similarly inserting $\det \widetilde{E}$ is equivalent to inserting $\widetilde{q}$.

Results for correlation functions are also straightforward to derive. We can evaluate the residue above as iterated residues, first computed at the zeroes of $\det E$, namely,

$$\begin{aligned} \sigma = \; & (2 \det A)^{-1} \Big( \det A + \det B - \det(A+B) \\ & \pm \big[ (\det A)^2 - 2(\det A)(\det(A+B)) + (\det(A+B))^2 - 2(\det A)(\det B) \\ & -2(\det(A+B))(\det B) + (\det B)^2 \big]^{1/2} \Big) \tilde{\sigma} \end{aligned}$$

and then taking residues at $\tilde{\sigma} = 0$. For this model, two- and four-point functions were independently computed using Cech cohomology techniques, and the results of the residue computation match the results of Cech cohomology perfectly.

For completeness, let us summarize the form of the two-point functions. These are given by

$$\langle \sigma\sigma \rangle = -\alpha^{-1}\Gamma_1, \quad \langle \sigma\widetilde{\sigma} \rangle = \alpha^{-1}\Delta, \quad \langle \widetilde{\sigma}\widetilde{\sigma} \rangle = -\alpha^{-1}\Gamma_2,$$

where

$$\gamma_{AB} \;=\; \det(A+B) - \det A - \det B, \quad \gamma_{CD} \;=\; \det(C+D) - \det C - \det D,$$

$$\begin{aligned} \Gamma_1 &= \gamma_{AB} \det D \;-\; \gamma_{CD} \det B, \\ \Gamma_2 &= \gamma_{CD} \det A \;-\; \gamma_{AB} \det C, \end{aligned}$$

$$\Delta \;=\; (\det A)(\det D) \;-\; (\det B)(\det C),$$

$$\alpha \;=\; \Delta^2 \;-\; \Gamma_1 \Gamma_2.$$

It was argued in [91] that the singular locus of these correlation functions, *i.e.* the locus $\{\alpha = 0\}$, coincides with the locus on which the bundle degenerates. Indeed, in general, the standard lore is that singularities in (0,2) theories are determined by singularities in the bundle, not in the base, so this result matches expectations.

### 2.3.6 Comments on Deformations

In this section, we study the relevant deformations of the effective theory, and suggest the number of deformation parameters of the A/2 theory may be smaller than the number of physical moduli. Some previous work on this matter can be found in [116, 119]. We will not give a complete analysis, rather, we only give some heuristic arguments.

$\mathbb{P}^1 \times \mathbb{P}^1$

The formula for correlation functions of (0,2) theories suggests that there are fewer moduli in the topological theory than in the physical theory We will use the quantum sheaf cohomology relations to see the basic idea. The quantum sheaf cohomology relations for a (0,2) deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ are

$$\det\left(A\sigma + B\widetilde{\sigma}\right) = q^1, \qquad \det\left(C\sigma + D\widetilde{\sigma}\right) = q^2. \tag{2.3.58}$$

To count the deformations, we factor the determinants in the form

$$\det\left(A\sigma + B\widetilde{\sigma}\right) \sim \left(a_1\sigma + b_1\widetilde{\sigma}\right)\left(a_2\sigma + b_2\widetilde{\sigma}\right), \qquad \det\left(C\sigma + D\widetilde{\sigma}\right) \sim \left(c_1\sigma + d_1\widetilde{\sigma}\right)\left(c_2\sigma + d_2\widetilde{\sigma}\right), \tag{2.3.59}$$

We can perform the following field redefinitions

$$\sigma \to \sigma + x\widetilde{\sigma}, \qquad \widetilde{\sigma} \to \widetilde{\sigma} + y\sigma. \tag{2.3.60}$$

When we choose $x = -\frac{b_1}{a_1}$ and $y = -\frac{c_1}{d_1}$, the quantum sheaf cohomology relations above become

$$\left(a_1 - \frac{b_1 c_1}{d_1}\right)\sigma\left(\left(a_2 - \frac{b_2 c_1}{d_1}\right)\sigma + \left(b_2 - \frac{b_1 a_2}{a_1}\right)\widetilde{\sigma}\right) = q^1, \tag{2.3.61}$$

$$\left(d_1 - \frac{b_1 c_1}{a_1}\right)\widetilde{\sigma}\left(\left(d_2 - \frac{b_1 c_2}{a_1}\right)\widetilde{\sigma} + \left(c_2 - \frac{c_1 d_2}{d_1}\right)\sigma\right) = q^2. \tag{2.3.62}$$

If we redefine $q^1$ as $q^1(a_1 - \frac{b_1 c_1}{d_1})(a_2 - \frac{b_2 c_1}{d_1})$ and similarly for $q^2$, we obtain

$$\sigma\left(\sigma + \lambda\widetilde{\sigma}\right) = q^1, \qquad \widetilde{\sigma}\left(\widetilde{\sigma} + \omega\sigma\right) = q^2. \tag{2.3.63}$$

Thus we see that quantum sheaf cohomology ring relations only depend upon two parameters $(\lambda, \omega)$ instead of six determinants [106]. Therefore, we suggest that the number of moduli of the A/2 theory is two rather than six.

#### Hirzebruch surface $\mathbb{F}_n$

The charge matrix for target fields is different with the $\mathbb{P}^1 \times \mathbb{P}^1$ except for the n=0 case, but they share a common submatrix which is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.3.64}$$

For Hirzebruch surface there are three matrices, one has rank two while the others have rank one. They are the following

$$M_X = \sigma A + \widetilde{\sigma} B, \qquad M_W = \sigma\gamma_1 + \widetilde{\sigma}\beta_1, \qquad M_S = \sigma\gamma_2 + \widetilde{\sigma}\beta_2. \tag{2.3.65}$$

On the (2,2) locus, $A = I$, $B = 0$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = n$ and $\gamma_2 = 0$. We have omitted the nonlinear deformations as they do not contribute to the correlation functions. As for $\mathbb{P}^1 \times \mathbb{P}^1$, we can factorize the determinants as follows:

$$\det M_X = \left(a_1\sigma + b_1\widetilde{\sigma}\right)\left(a_2\sigma + b_2\widetilde{\sigma}\right), \qquad M_W = c_1\left(n\sigma + \widetilde{\sigma}\right) + d_1\widetilde{\sigma}, \qquad M_S = c_2\widetilde{\sigma} + d_2\sigma. \tag{2.3.66}$$

The coefficients above can be expressed in terms of the original parameters, for example we have $c_2 = \gamma_2$ and so on. The detailed expression of the coefficient are not important. Again, we perform a linear field redefinition

$$\sigma \to \sigma + x\widetilde{\sigma}, \qquad \widetilde{\sigma} \to \widetilde{\sigma} + y\sigma. \tag{2.3.67}$$

When $x = -\frac{b_1}{a_1}$ and $y = -\frac{d_2}{c_2}$, the above matrices become the following

$$\det M_X = \left(a_1 - \frac{b_1 d_2}{c_2}\right)\sigma\left(\left(a_2 - \frac{b_2 d_2}{c_2}\right)\sigma + \left(b_2 - \frac{a_2 b_1}{a_1}\right)\widetilde{\sigma}\right),$$

$$M_W = \frac{\left(c_1 n - \frac{d_2}{c_2} - \frac{d_1 d_2}{c_2}\right)}{n}\left(n\sigma + \widetilde{\sigma}\right) + \left[c_1\left(1 - n\frac{b_1}{a_1} + d_1 - \frac{\left(c_1 n - \frac{d_2}{c_2} - \frac{d_1 d_2}{c_2}\right)}{n}\right)\right]\widetilde{\sigma},$$

$$M_S = \left(c_2 - \frac{d_2 b_1}{a_1}\right)\widetilde{\sigma}.$$

Again we can redefine $q$ to write the determinants as follows

$$\det M_X = \sigma\left(\sigma + \omega_1\widetilde{\sigma}\right), \quad M_W = (n\sigma + \widetilde{\sigma}) + \omega_2\sigma, \qquad M_S = \widetilde{\sigma}. \tag{2.3.68}$$

Thus, for Hirzebruch surfaces, the number of moduli of the A/2 theory is two.

# Chapter 3

# Some New Progress in Mirror Symmetry

This chapter contains some of our results on mirror symmetry. The contents of this chapter were adapted, with minor modifications, with permission from JHEP and arXiv, from our publication [27] and article [26] on the arXiv.

Many aspects of ordinary mirror symmetry are well understood. For example, the Batyrev-Borisov construction [23] describes mirrors to complete intersections in projective spaces, and the Hori-Vafa construction [25] describes Landau-Ginzburg mirrors to abelian gage theories. One open problem has been to understand non-abelian versions of these constructions: there is no known analogue of Batyrev-Borisov for complete intersections in Grassmannians, and the Hori-Vafa construction was only known for abelian theories. In section 3.1 we will describe a proposal for a non-abelian extension of the Hori-Vafa construction (adapted here from our work [26]).

Another open problem has been (0,2) analogues of these constructions. In section 3.2, we will discuss a proposal for an extension of Hori-Vafa [25] to (0,2) theories, for certain families of deformations of (2,2) theories. This work was previously published in [27].

## 3.1 A Proposal for Nonabelian Mirror Symmetry

### 3.1.1 The Proposal Itself

In [26], we made a proposal for a non-abelian of Hori-Vafa's work [25], which can be described as follows. Consider an A-twisted (2,2) GLSM with gauge group $G$ (of dimension $n$ and rank $r$) and matter in some representation $\mathcal{R}$ (of dimension $N$). For simplicity, in this paper we will assume $G$ is connected. For the moment, we will assume that the A-twisted gauge theory has no superpotential, and we will consider generalizations later in this section. We propose that the mirror is an orbifold of a Landau-Ginzburg model. We will describe the Landau-Ginzburg model first, then the orbifold. The Landau-Ginzburg model has the following matter fields:

- $r$ (twisted) chiral superfields $\sigma_a$, corresponding to a choice of Cartan subalgebra of the Lie algebra of $G$,
- $N$ (twisted) chiral superfields $Y_i$, each of imaginary periodicity $2\pi i$ as in [25][section 3.1], which we will discuss further in a few paragraphs,
- $n-r$ (twisted) chiral superfields $X_{\tilde{\mu}}$,

with superpotential[1]

$$\begin{aligned} W &= \sum_{a=1}^{r} \Sigma_a \left( \sum_{i=1}^{N} \rho_i^a Y_i \;-\; \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \ln X_{\tilde{\mu}} \;-\; t_a \right) \\ &\quad + \sum_{i=1}^{N} \exp\left(-Y_i\right) \;+\; \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}}, \end{aligned} \tag{3.1.1}$$

where the $\rho_i^a$ are the weight vectors for the representation $\mathcal{R}$, and the $\alpha_{\tilde{\mu}}^a$ are root vectors for the Lie algebra of $G$. (We will sometimes write[2] $X_{\tilde{\mu}}$ in terms of $Z_{\tilde{\mu}} = -\ln X_{\tilde{\mu}}$ for convenience, but $X_{\tilde{\mu}}$ is the fundamental[3] field.) (In passing, in later sections, we will slightly modify our index notation: $i$ will be broken into flavor and color components, and $\tilde{\mu}$ will more explicitly

[1] If we want to be careful about QFT scales, the second line should be written

$$\sum_{i=1}^{N} \mu \exp\left(-Y_i\right) \;+\; \sum_{\tilde{\mu}=1}^{n-r} \mu X_{\tilde{\mu}},$$

where $\mu$ is a pertinent mass scale.

[2] Taking into account QFT mass scales, $Z_{\tilde{\mu}} = -\ln\left(\mu X_{\tilde{\mu}}\right)$.

[3] We use the term 'fundamental field' to implicitly indicate the form of the path integral integration measure. In the present case, in the B model, the integration measure (over constant zero modes) has the form

$$\int \left( \prod_a d^2 \sigma_a \right) \left( \prod_i d^2 Y_i \right) \left( \prod_{\tilde{\mu}} d^2 X_{\tilde{\mu}} \right).$$

Similarly, the holomorphic top-form takes the form

$$\wedge_a d\sigma_a \wedge_i dY_i \wedge_{\tilde{\mu}} dX_{\tilde{\mu}}.$$

Later in this section when we discuss mirrors to fields with R-charges, we will say that different fields are 'fundamental', for example, for a mirror to a field of R-charge two, the fundamental field would be $\exp(-Y)$. In such a case, instead of $d^2 Y$, one would have $d^2 \exp(-Y)$.

reflect the adjoint representation, in the form of $\tilde{\mu} \mapsto \mu\nu$, for $\mu$, $\nu$ ranging over the same values as $a$.. However, for the moment, this convention is very efficient for outlining the proposal.) For reasons we will discuss in section 3.1.3, we believe that the loci $\{X_{\tilde{\mu}} = 0\}$ are dynamically excluded by *e.g.* diverging potentials. (This will lead to an algebraic derivation of the excluded locus in A model Coulomb branch computations.) Finally, for reasons of brevity, we will often omit the term 'twisted' when describing the $\Sigma$, $Y$, and $X$ superfields; the reader should add it as needed from context.

Strictly speaking, the $\sigma_a$ should be understood as curvatures of vector multiplets of the original A-twisted gauge theory: $\sigma_a \propto \overline{D}_+ D_- V_a$ for vector multiplets $V_a$, just as in [25]. This means the $\sigma$ terms in the superpotential above encode theta angle terms such as $\theta F_{z\overline{z}}$, which tie into periodicities of the $Y$ fields (to which we will return next). The reader should also note that our notation is slightly nonstandard: whereas other papers use $\Sigma$, we use $\sigma$ to denote both the twisted chiral superfield (the curvature of $V$) as well as the lowest component of the superfield. (As these $\sigma$'s often occur inside and next to summation symbols, we feel our slightly nonstandard notation will improve readability.) In the limit that the gauge coupling of the original theory becomes infinite, the $\sigma_a$ become Lagrange multipliers, from the form of the kinetic terms [25][equ'n (3.69)]. As a result, we will often speak of integrating them out. (On occasion, we will utilize the fact that we are in a TFT to integrate out other fields as well.)

We have not carefully specified to which two-dimensional (2,2) supersymmetric theories the ansatz above should apply. Certainly we feel it should apply to theories with isolated vacua and theories describing compact CFTs, and we have checked numerous examples of this form. In addition, later we will also see it reproduces results for non-regular theories (in the sense of [39]), as well as results for theories that flow to free twisted chiral multiplets. In any event, as we have not provided a proof of the ansatz above, we can not completely nail down a range of validity.

To clarify the $Y$ periodicities,

$$Y_i \sim Y_i + 2\pi i,$$

so that the $Y$'s take values in a torus of the form $\mathbb{C}^{\dim \mathcal{R}}/2\pi i\mathbb{Z}$, and the superpotential terms $\exp(-Y_i)$ are well-defined. Then, schematically, the contraction $\rho Y$ has periodicity $2\pi i M$ for $M$ the weight lattice. For abelian cases, this is merely the usual affine shift by $2\pi i$ that appeared in [25], independently for each $Y$, but may be a little more complicated in nonabelian cases. Furthermore, in our conventions, the weight lattice is normalized so that the theta angle periodicities of the original gauge theory are of the form $2\pi M$, since phases picked up by $\rho Y$ are absorbed into theta angles. Finally, note that in the first line of the superpotential above, the log branch cut ambiguity effective generates shifts of weight lattice periodicities by roots.

So far we have described the Landau-Ginzburg model. The proposed mirror is an orbifold of the theory above, by the Weyl group $W$, acting on the $\Sigma_a$, $Y_i$, and $X_\mu$. (The action can be essentially inferred from the quantum numbers, and we will describe it in explicit detail in examples.)

The superpotential above, written in terms of root and weight vectors, is invariant under this Weyl group action simply because the Weyl group permutes root vectors into other root vectors and weight vectors into other weight vectors for any finite-dimensional representation, see *e.g.* [71][chapter VIII.1], [72][chapter 14.1]. As a result, the Weyl group maps $Z_{\tilde{\mu}}$'s to other $Z_{\tilde{\mu}}$'s, and $Y_i$'s to other $Y_i$'s, consistently with changes in $\rho_i^a$ and $\alpha_{\tilde{\mu}}^a$. Furthermore, we take each Weyl group reflection to also act in the same way on the $\Sigma$'s as on the root and weight lattices, so that the combinations

$$\sum_a \Sigma_a \alpha_{\tilde{\mu}}^a, \quad \sum_a \Sigma_a \rho_i^a$$

are permuted at the same time and in the same way as the $X_{\tilde{\mu}}$ and $Y_i$. This guarantees that the superpotential remains invariant under the Weyl group, a fact we shall also check explicitly in examples.

In practice, in the examples in this paper, the Weyl group will act by permutations and sign flips, and so it will be straightforward to check that, so long as the $\Sigma$'s are also permuted and sign-flipped, the superpotential is invariant. In addition, in some of the examples we shall compute, we shall also see alternate representations of the superpotential above, in which the $\Sigma_a$ terms above involve nontrivial matrix multiplications, rather than just root and weight vectors. In such cases, we will check explicitly that the superpotential is again invariant under the Weyl group action.

We require the mirror Landau-Ginzburg model admit a B twist, which constrains the orbifold. After all, to define the B twist in a closed string theory, the orbifold must be such that the square of the holomorphic top-form is invariant [113]. (It is sometimes said that the B model is only defined for Calabi-Yau's, but as discussed in [113], the Calabi-Yau condition for existence of the closed string B model can be slightly weakened.) In the present case, each element of the Weyl group acts by exchanging some of the fields, possibly with signs. Under such (signed) interchanges, a holomorphic top-form will change by at most a sign; a square of the holomorphic top-form will be invariant. Therefore, this Weyl orbifold will always be compatible with the B twist.

In our proposal, we have deliberately not specified the Kähler potential. As we are working with topologically-twisted theories, and the space of $\sigma$s, $Y$s, and $X$s is topologically trivial, the Kähler potential is essentially irrelevant. One suspects that in a physical, untwisted, nonabelian mirror, the Kähler potential terms would reflect nonabelian T-duality, just as the kinetic terms in the Hori-Vafa proposal [25] reflected abelian duality. We do briefly outline an idea of how one might go about proving this proposal in section 3.1.2, we make a few tentative suggestions for a possible form of the Kähler potential. It would be interesting to pursue this in future work.

Our proposal only refers to the Lie algebra of the A model gauge theory, not the gauge group. Different gauge groups with the same Lie algebra can encode different nonperturbative physics, see *e.g.* [29–31, 33, 88]. Here, we conjecture that the different Lie groups with the same Lie algebra (and matter content) are described in the mirror by rescalings of the mirror roots $\alpha^a_{\tilde{\mu}}$ and matter weights $\rho^a_i$. Integrating out the $\Sigma$'s would then result in gerbe structures as discussed in analogous abelian cases in [31]. (See *e.g.* [74][section 4.5] for related observations in other theories.)

By computing the critical locus along $Y$'s and $Z$'s, we also find the operator mirror map, in the sense of [27]:

$$\exp(-Y_i) = \sum_{a=1}^{r} \Sigma_a \rho^a_i, \tag{3.1.2}$$

$$X_{\tilde{\mu}} = \sum_{a=1}^{r} \Sigma_a \alpha^a_{\tilde{\mu}}. \tag{3.1.3}$$

We interpret the right-hand side as defining A model Coulomb branch operators, which this map shows us how to relate to B model operators.

In principle, to make the ansatz above useful for general cases, one would like to be able to evaluate Landau-Ginzburg correlation functions on general orbifolds. Many Landau-Ginzburg computations are known, especially massless spectrum computations in conformal models [34, 75, 76], and more recently [35], but correlation function computations on orbifolds are not, to our knowledge, understood in complete generality. On the other hand, in many simple cases we can get by with less. In particular, in the examples in this paper, the critical points of the superpotential are not located at orbifold fixed points. (This is essentially because of the assumption that $X_{\tilde{\mu}} \neq 0$ mentioned earlier. One of the effects of this assumption is to make the superpotential well-defined – although it has poles where any $X_{\tilde{\mu}}$ vanishes, it becomes ill-defined when multiple $X_{\tilde{\mu}}$ vanish, an issue which we will return to in the discussion about "excluded loci," where we will discuss this as a regularization issue.) In any event, since the Weyl group will interchange the $\Sigma_a$, the orbifold fixed-point locus will lie where some $X_{\tilde{\mu}}$ vanish. Rescaling the worldsheet metric in the B-twisted theory, one quickly finds that the bosonic contribution to the path integral is of the form [123][section 2.2], [78]

$$\lim_{\lambda \to \infty} \int_X d\phi \exp\left( - \sum_i |\lambda \partial_i W|^2 \right),$$

and so vanishes unless the critical locus intersects the fixed-point locus. As a result, since in this paper we are computing *e.g.* correlation functions of untwisted operators on genus zero worldsheets, we are able to consistently omit contributions from twisted sectors in the computations presented here.

As a consistency check, let us specialize to the case that $G = U(1)^r$. In this case, there is no Weyl orbifold, there are no fields $X_{\tilde{\mu}}$, and the mirror is defined by the fields $\Sigma_a$ and $Y_i$ with superpotential

$$W \;=\; \sum_{a=1}^{r} \sigma_a \left( \sum_{i=1}^{N} Q_i^a Y_i \;-\; t_a \right) \;+\; \sum_{i=1}^{N} \exp\left(-Y_i\right),$$

since the weight vectors $\rho_i^a$ reduce to the charge matrix $Q_i^a$. This is precisely the mirror of an abelian GLSM discussed in [25], as expected.

Let us now return to the nonabelian theory. If the fields $\phi_i$ of the original A model have twisted masses $\tilde{m}_i$, then the mirror proposal is the same orbifold but with a different superpotential, given by

$$\begin{aligned} W \;&=\; \sum_{a=1}^{r} \Sigma_a \left( \sum_{i=1}^{N} \rho_i^a Y_i \;+\; \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a Z_{\tilde{\mu}} \;-\; t_a \right) \\ &\quad - \sum_{i=1}^{N} \tilde{m}_i \left( Y_i \;-\; \sum_a \rho_i^a t_a \right) \\ &\quad + \sum_{i=1}^{N} \exp\left(-Y_i\right) \;+\; \sum_{\tilde{\mu}=1}^{n-r} X_{\tilde{\mu}}. \end{aligned} \tag{3.1.4}$$

Computing the critical locus along the $Y$'s and $X$'s yields the operator mirror map including twisted masses and R-charges:

$$\exp(-Y_i) \;=\; -\tilde{m}_i + \sum_{a=1}^{r} \Sigma_a \rho_i^a, \tag{3.1.5}$$

$$X_{\tilde{\mu}} \;=\; \sum_{a=1}^{r} \Sigma_a \alpha_{\tilde{\mu}}^a. \tag{3.1.6}$$

We can also formally derive quantum cohomology relations in a similar fashion. The critical locus for $\sigma_a$ is

$$\sum_{i=1}^{N} \rho_i^a Y_i \;+\; \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a Z_{\tilde{\mu}} \;=\; t_a, \tag{3.1.7}$$

and exponentiating gives

$$\left[ \prod_i \left(\exp(-Y_i)\right)^{\rho_i^a} \right] \left[ \prod_{\tilde{\mu}} \left(X_{\tilde{\mu}}\right)^{\alpha_{\tilde{\mu}}^a} \right] \;=\; q_a. \tag{3.1.8}$$

Applying the operator mirror map equations above, this becomes

$$\left[ \prod_i \left( \sum_b \sigma_b \rho_i^b - \tilde{m}_i \right)^{\rho_i^a} \right] \left[ \prod_{\tilde{\mu}} \left( \sum_b \sigma_b \alpha_{\tilde{\mu}}^b \right)^{\alpha_{\tilde{\mu}}^a} \right] \;=\; q_a. \tag{3.1.9}$$

In practice, we will see later in section 3.1.2 that

$$\left[\prod_{\tilde{\mu}}\left(\sum_b \sigma_b \alpha^b_{\tilde{\mu}}\right)^{\alpha^a_{\tilde{\mu}}}\right]$$

is a $\Sigma$-independent constant matching that discussed in [89][setion 10], so we can write the quantum cohomology relations as either

$$\prod_i \left(\sum_b \sigma_b \rho_i^b - \tilde{m}_i\right)^{\rho_i^a} \;=\; \tilde{q}_a, \tag{3.1.10}$$

or equivalently in the mirror

$$\prod_i \exp\left(-\rho_i^a Y_i\right) \;=\; \tilde{q}_a, \tag{3.1.11}$$

where $\tilde{q}_a$ differs from $q_a$ by the constant discussed above.

As written above, our proposal is for the mirror to an A-twisted gauge theory with no superpotential. Let us now consider the case that the A-twisted theory has a superpotential. In this case, one must specify nonzero R charges for the fields, so that the superpotential has R charge two. Furthermore, in order for the A twist to exist, those R charges must be integral (see *e.g.* [123], [89][section 3.4], [42][section 2.1]). (Technically, on Riemann surfaces of nonzero genus, this requirement can be slightly relaxed, but in order to have results valid for all genera, we will assume the most restrictive form, namely the genus zero result that R charges are integral.)

Given an A-twisted gauge theory with superpotential and suitable R charges, we can now define the mirror. Both the A-twisted theory and its mirror will be independent of the details of the (A model) superpotential (which is BRST exact in the A model, see *e.g.* [123][section 3.1]), though not independent of the R charges of the fields. Our proposal is that the B-twisted mirror has exactly the same form as discussed above – same number of fields, same mirror superpotential – but with one minor quirk, that the choice of fundamental field changes. Specifically, if a field $\phi_i$ of the A model has nonzero R-charge $r_i$, then the fundamental field in the mirror is

$$X_i \;\equiv\; \exp(-(r_i/2)Y_i),$$

and in the expressions above, we take $Y_i$ to mean

$$Y_i \;=\; -\frac{2}{r_i}\ln X_i.$$

Furthermore, in this case, ultimately because of the periodicity of $Y_i$, the mirror theory with field $X_i$ has a cyclic orbifold of order $2/r_i$ (which we assume to be an integer), for the same reasons as discussed elsewhere in Hori-Vafa [25] mirrors. (Of course, if the original field has $r_i = 0$, then there is no change in fundamental field, and so no orbifold in the mirror.) The reader should also note that field redefinitions in the mirror may introduce additional orbifolds, which is essentially what happens in the Hori-Vafa mirror to the quintic, for example.

Note that for the A-twisted theory to exist, every $r_i$ must be an integer, and for the orbifold in the B model mirror to be well-defined, we must require $2/r_i$ (for nonzero $r_i$) to also be an integer. Also taking into account a positivity condition discussed in [89][section 3.4], this means we are effectively restricted to the choices $r_i \in \{0, 1, 2\}$ in our proposal. If a gauge theory has a superpotential that is incompatible with such choices of R charges, then either the A twist does not exist or our proposed mirror does not apply.

In principle, in the language of the dictionary above, mirrors to the W bosons act like mirrors to fields of R charge two, and are the fields $X_{\tilde{\mu}}$ rather than the $Z_{\tilde{\mu}}$. Also, since $2/2 = 1$, there is no orbifold (beyond the Weyl group orbifold) associated with the $X_{\tilde{\mu}}$ specifically.

Finally, we should mention that the axial R symmetry of the A model theory appears here following the same pattern as in [25][equ'n (3.30)]. Specifically, under $\mathrm{R}_{\mathrm{axial}}$,

$$Y_i \mapsto Y_i - 2i\alpha, \tag{3.1.12}$$

and

$$X_{\tilde{\mu}} \mapsto X_{\tilde{\mu}} \exp\left(+2i\alpha\right), \tag{3.1.13}$$

so that for example the superpotential terms

$$\sum_i \exp\left(-Y_i\right) + \sum_{\tilde{\mu}} X_{\tilde{\mu}}$$

have charge 2, as one would expect. In that vein, note that

$$\exp\left(-(r_i/2)Y_i\right) \mapsto \exp\left(-(r_i/2)Y_i\right) \exp\left(+ir_i\alpha\right), \tag{3.1.14}$$

as one would expect for a field of R charge $r_i$. Similarly, the effect of the R charge on the $\sigma$ terms is to generate a term

$$(-2i\alpha) \sum_a \sigma_a \left( \sum_i \rho_i^a + \sum_{\tilde{\mu}} \alpha_{\tilde{\mu}}^a \right) = (-2i\alpha) \sum_a \sigma_a \left( \sum_i \rho_i^a \right) \tag{3.1.15}$$

(since the sum over $\alpha_{\tilde{\mu}}^a$ will vanish), reflecting the fact that if the A model theory has an axial anomaly, then an $\mathrm{R}_{\mathrm{axial}}$ rotation will shift theta angles.

### 3.1.2 Justification for the Proposal

The bulk of this paper will be spent checking examples, which to our minds will be the best verification of the proposal, but before working through those examples, we wanted to briefly describe the origin of some of the details of the proposal above, as well as perform some consistency tests, such as a general comparison of correlation functions.

**General remarks**

At least for the authors, one of the motivations for this work was to find a UV realization of factors of the form

$$\prod_{a<b} (\sigma_a - \sigma_b)$$

appearing in integration measures, such as the Hori-Vafa conjecture for nonabelian mirrors in [25][appendix A], and later in expressions for supersymmetric partition functions of non-abelian gauge theories in [43, 44]. Later, [83] studied $S^2$ partition functions of Hori-Vafa mirrors, and in section 4 of that paper, applied the same methods to predict the form of partition functions of the mirror of a $U(k)$ gauge theory (with $k > 1$) corresponding to a Grassmannian, where again they found factors in the integration measure of the same form (albeit squared[4]), a result we will duplicate later.

We reproduce such factors via the fields $X_{\tilde{\mu}}$, the mirrors to the W bosons. The basic idea originates in an observation in [66][section 2], which relates the partition function of a nonabelian theory to that of an associated 'Cartan theory,' an abelian gauge theory in which the nonabelian gauge group is replaced by its Cartan torus, and in addition to the chiral multiplets of the nonabelian theory, one adds an additional set of chiral multiplets of R charge two corresponding to the nonzero roots of the Lie algebra. It is briefly argued that the $S^2$ partition function of the original nonabelian theory matches the $S^2$ partition function of the associated Cartan theory. In effect, we are taking this observation a step further, by dualizing the associated Cartan theory in the sense of [25] to construct this proposal for nonabelian mirrors.

We take the Weyl orbifold to get the right moduli space: the Coulomb branch moduli space is not quite just the moduli space of a $U(1)^r$ gauge theory, as one should also identify $\sigma$ fields related by the Weyl group. Note the Weyl group does not survive the adjoint Higgsing; instead, we taking the orbifold so as to reproduce the correct Coulomb branch. This is analogous[5] to the $c = 1$ boson at self-dual radius: as one moves away from the self-dual point, the $SU(2)$ is broken to $U(1)$ on both sides, and the Weyl orbifold allows one to forget about radii that are smaller, since they are all Weyl equivalent to larger radii. Another example is the construction of the $u$ plane in four-dimensional $N = 2$ Seiberg-Witten theory. In the present case, we will see for example in the case of Grassmannians that to get the correct number of vacua, one has to quotient by the Weyl group.

[4] In open string computations, one gets factors of $\prod_{a<b}(\sigma_a - \sigma_b)$, whereas in closed string computations, one typically gets factors of $\prod_{a<b}(\sigma_a - \sigma_b)^2$. As this paper is focused on closed string computations, we will see the latter.

[5] We would like to thank I. Melnikov for providing this analogy.

**Suggestions of a route towards a proof**

We do not claim to have a rigorous proof of the proposal of this paper, but there is a simple idea for a proof. Given a (2,2) supersymmetric GLSM, imagine moving to a generic point on the Coulomb branch, described by a Weyl-group orbifold of an abelian gauge theory, with gauge group equal to the Cartan of the original theory. Now, apply abelian duality to this abelian gauge theory[6]. One will T-dualize the original matter fields (which become the $Y_i$) as well as the W bosons (which become the $X_{\mu\nu}$).

For later purposes, it will be instructive to fill in a few steps. That said, we emphasize that we are not claiming we have a rigorous demonstration. Our goal here is merely to suggest a program, and to investigate the form of a possible Kähler potential to justify certain plausibility arguments elsewhere.

For ordinary matter fields $\Phi$, of charge $\rho^a$ under the $a$th $U(1)$, T-duality in this context [20, 25] says that the field should be described by an 'intermediate' Lagrangian density of the form [25][equ'n (3.9)]

$$L_\Phi \: = \: \int d^4\theta \left( \exp\left( 2 \sum_a \rho^a V_a \: + \: B \right) \: - \: \frac{1}{2} \left( Y + \overline{Y} \right) B \right). \tag{3.1.16}$$

Reviewing the analysis of [25][section 3.1], if one integrates over $Y$, one gets constraints

$$\overline{D}_+ D_- B \: = \: 0 \: = \: D_+ \overline{D}_- B, \tag{3.1.17}$$

which are solved by taking

$$B \: = \: \Psi \: + \: \overline{\Psi}. \tag{3.1.18}$$

Plugging back in, one finds

$$L_\Phi \: = \: \int d^4\theta \, \exp\left( 2 \sum_a \rho^a V_a \: + \: \Psi \: + \: \overline{\Psi} \right) \: = \: \int d^4\theta \, \overline{\Phi} \exp\left( 2 \sum_a \rho^a V_a \right) \Phi, \tag{3.1.19}$$

the original Lagrangian, for $\Phi = \exp(\Psi)$.

If one instead integrates over $B$ first, then one recovers the dual theory, as follows. Integrating out $B$ first yields

$$B \: = \: -2 \sum_a \rho^a V_a \: + \: \ln \left( \frac{Y + \overline{Y}}{2} \right), \tag{3.1.20}$$

[6] In other words, to any nonabelian gauge theory we can associate a toric variety or stack, defined by matter fields plus W bosons at a generic point on the Coulomb branch. Our proposal seems consistent with abelian duality for that toric variety.

and plugging this in we find

$$L_\Phi \; = \; \int d^4\theta \left( \frac{Y+\overline{Y}}{2} \; + \; \left(Y+\overline{Y}\right) \sum_a \rho^a V_a \; - \; \frac{1}{2} \left(Y+\overline{Y}\right) \ln \left( \frac{Y+\overline{Y}}{2} \right) \right), \tag{3.1.21}$$

$$= \; \int d^4\theta \left( \left(Y+\overline{Y}\right) \sum_a \rho^a V_a \; - \; \left( \frac{Y+\overline{Y}}{2} \right) \ln \left( \frac{Y+\overline{Y}}{2} \right) \right). \tag{3.1.22}$$

Since $Y$ is a twisted chiral superfield, the first term can be written

$$\int d^4\theta \left( Y \sum_a \rho^a V_a \right) \; = \; \int d^2\theta \sum_a \sigma_a \rho^a Y, \tag{3.1.23}$$

where $\sigma_a = \overline{D}_+ D_- V_a$, and so this term contributes to the superpotential.

Equating the two forms (3.1.18), (3.1.20) for $B$, one finds

$$Y + \overline{Y} \; = \; 2\overline{\Phi} \exp \left( 2 \sum_a \rho^a V_a \right) \Phi, \tag{3.1.24}$$

and from the Kähler potential term above, we see that the metric seen by the kinetic terms for $Y$ components is

$$ds^2 \; = \; \frac{|dy|^2}{2(y+\overline{y})}, \tag{3.1.25}$$

where $y$ is the scalar part of $Y$.

So far this analysis is entirely standard. Now, let us think about the analogous analysis for T-duals of the W bosons. Here, we take the W bosons to be described by[7] chiral superfields $W_{\tilde{\mu}}$ and the Lagrangian density

$$L_W \; = \; \int d^4\theta \, \overline{W}_{\tilde{\mu}} \exp \left( 2 \sum_a \alpha^a_{\tilde{\mu}} V_a \right) W_{\tilde{\mu}}. \tag{3.1.26}$$

[7] We emphasize that the W bosons are described by ordinary chiral superfields and not twisted chiral superfields, unlike the $\Sigma$ superfield. A recent technical discussion of this is in [41][appendix C.4], which observes that the W bosons are in ordinary chirals (with nonzero R charge), not twisted chirals. In addition, this phenomenon can be understood very simply as follows. Consider for example a two-dimensional $SU(2)$ gauge theory. Use the (adjoint-valued) $\sigma$'s to Higgs the $SU(2)$ to a $U(1)$. The W bosons must be charged under that $U(1)$. However, only an ordinary chiral multiplet can be charged under an (ordinary) vector multiplet. If the W bosons were twisted chirals, then under a gauge transformation, they would be multiplied by factors of the form $\exp \Lambda$ for $\Lambda$ an ordinary chiral multiplet, hence gauge transformations would mix twisted chirals (the W bosons) with ordinary chirals (the gauge transformation parameter). Since only ordinary chirals can be charged under the remaining $U(1)$, the W bosons are in fact in ordinary chiral multiplets, not twisted chiral multiplets.

Proceeding as before, we can consider the intermediate Lagrangian density

$$L_W \;=\; \int d^4\theta \left( \exp\left( 2 \sum_a \alpha^a_{\tilde{\mu}} V_a \;+\; B_{\tilde{\mu}} \right) \;-\; \frac{1}{2} \left( Z_{\tilde{\mu}} + \overline{Z}_{\tilde{\mu}} \right) B_{\tilde{\mu}} \right). \tag{3.1.27}$$

Our analysis will closely follow the pattern for $\Phi$, $Y$. Integrating over the $Z_{\tilde{\mu}}$ recovers the original Lagrangian density (3.1.26). Integrating out the $B_{\tilde{\mu}}$, one finds

$$L_W \;=\; \int d^4\theta \left( \left( Z_{\tilde{\mu}} + \overline{Z}_{\tilde{\mu}} \right) \sum_a \alpha^a_{\tilde{\mu}} V_a \;-\; \left( \frac{Z_{\tilde{\mu}} + \overline{Z}_{\tilde{\mu}}}{2} \right) \ln \left( \frac{Z_{\tilde{\mu}} + \overline{Z}_{\tilde{\mu}}}{2} \right) \right). \tag{3.1.28}$$

The first term can be rewritten as a superpotential contribution. The primary difference here is that we take the fundamental field to be $X_{\tilde{\mu}} = \exp(-Z_{\tilde{\mu}})$. In terms of $X_{\tilde{\mu}}$, the kinetic term takes the form

$$\int d^4\theta \left( \frac{\ln X_{\tilde{\mu}} + \ln \overline{X}_{\tilde{\mu}}}{2} \right) \ln \left( - \frac{\ln X_{\tilde{\mu}} + \ln \overline{X}_{\tilde{\mu}}}{2} \right), \tag{3.1.29}$$

and from this Kähler potential it is straightforward to compute that the metric for the kinetic terms has the form

$$ds^2 \;=\; \frac{|dx|^2}{2|x|^2 \ln |x|^2}. \tag{3.1.30}$$

To resolve subtleties in renormalization, in [25], it was noted that the kinetic terms were written in terms of a bare field $Y_0$ related to a renormalized field by [25][equ'n (3.23)]

$$Y_0 = \ln(\Lambda_{UV}/\mu) + Y, \tag{3.1.31}$$

and then in a suitable limit, the metric on the $Y$'s becomes flat. The analogue here is to write $X_0 = (\mu/\Lambda_{UV})X$, so that the metric for the kinetic term becomes

$$\frac{|dx|^2}{|x|^2 \left( -2 \ln(\Lambda_{UV}/\mu) + \ln |x|^2 \right)}. \tag{3.1.32}$$

Even in the analogous scaling limit however, this metric diverges as $x \to 0$, suggesting that the kinetic terms dynamically forbid $x = 0$.

The take-away observation from the computation above is that the proposed kinetic terms for the W-boson mirrors have singularities at $X = 0$. Now, granted, a more rigorous analysis of duality might well work along the lines of nonabelian T-duality rather than abelian T-duality in a Cartan, and so yield different kinetic terms still, see *e.g.* [90] for a pertinent discussion of nonabelian T-duality. Furthermore, these kinetic terms will receive quantum corrections, that could even smooth out singularities of the form above, see *e.g.* [25, 87].

In passing, let us point out a few other consistency checks. As observed in [41][appendic C.4], in the A-twisted gauge theory, supersymmetric W bosons contribute to supersymmetric localization as chiral multiplets of R-charge two, so that the mirror should be a twisted chiral multiplet (same as the $X$ fields), and the R charge dictates that the mirror fields should appear linearly in the superpotential (as the fundamental field is $\exp(-(r/2)Y)$). The mass of the $X$ fields themselves is a bit off:

$$\frac{\partial^2 W}{\partial X_{\tilde{\mu}} \partial X_{\tilde{\nu}}} = \delta_{\tilde{\mu}\tilde{\nu}} \frac{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}}{X^2_{\tilde{\mu}}} = \delta_{\tilde{\mu}\tilde{\nu}} \frac{1}{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}} \tag{3.1.33}$$

after applying the mirror map, whereas the mass of a W boson is instead $\sum_a \sigma_a \alpha^a_{\tilde{\mu}}$. On the other hand,

$$\frac{\partial^2 W}{\partial \ln X_{\tilde{\mu}} \partial \ln X_{\tilde{\nu}}} = \delta_{\tilde{\mu}\tilde{\nu}} X_{\tilde{\mu}} = \delta_{\tilde{\mu}\tilde{\nu}} \sum_a \sigma_a \alpha^a_{\tilde{\mu}}, \tag{3.1.34}$$

after applying the mirror map, exactly right to match the mass of the W bosons, suggesting that the W boson mirrors are $\ln X_{\tilde{\mu}}$.

**Comparison of correlation functions**

In this section, we will give a formal outline of how (some) A model correlation functions match B model correlation functions in the proposed nonabelian mirror, by in the mirror formally integrating out the mirrors to the W bosons and the matter fields, yielding a theory of $\sigma$'s only. We will give several versions of this comparison, of varying levels of rigour. We will focus exclusively on correlation functions of Weyl-group-invariant untwisted-sector operators, which together with the fact that the Weyl group orbifold fixed points do not intersect superpotential critical points in the examples in this paper, will enable us to largely gloss over the Weyl group orbifold.

**First argument – iterated integrations out**

Begin with the basic mirror proposal, the Landau-Ginzburg orbifold described in section 3.1, with superpotential $W$ given in (3.1.4). If none of the critical points intersect fixed points of the Weyl group orbifold, then we can integrate out the $X_{\tilde{\mu}}$, as we shall outline next.

First, it is straightforward to compute from the superpotential (3.1.4) that

$$\frac{\partial W}{\partial X_{\tilde{\mu}}} = 1 - \frac{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}}{X_{\tilde{\mu}}}, \tag{3.1.35}$$

$$\frac{\partial^2 W}{\partial X_{\tilde{\mu}} \partial X_{\tilde{\nu}}} = \delta_{\tilde{\mu}\tilde{\nu}} \frac{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}}{X^2_{\tilde{\mu}}}, \tag{3.1.36}$$

a diagonal matrix of second derivatives. Evaluating on the critical locus (and identifying the field $\sigma_a$ with the mirror field $\sigma_a$, reflecting their common origin),

$$\frac{\partial^2 W}{\partial X^2_{\tilde{\mu}}} = \frac{1}{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}}. \tag{3.1.37}$$

So long as the determinant of the matrix of second derivatives is nonvanishing, the $X_{\tilde{\mu}}$ are massive, so it is consistent to integrate them out. (If the matrix of second derivatives were to have a zero eigenvalue somewhere, integrating out the $X_{\tilde{\mu}}$ would, of course, not be consistent.)

To integrate them out, we follow the same logic as [123][section 2.2], [78]. Briefly, the pertinent terms in the Lagrangian are of the form

$$\sum_{\tilde{\mu}} |\partial_{\tilde{\mu}} W|^2 \; + \; \psi_+^{\tilde{\mu}} \psi_-^{\tilde{\nu}} \partial_{\tilde{\mu}} \partial_{\tilde{\nu}} W \; + \; \text{c.c.} \tag{3.1.38}$$

Expanding the purely bosonic term about the critical locus $X^o_{\tilde{\mu}}$, given by

$$X^o_{\tilde{\mu}} \; = \; \sum_a \sigma_a \alpha^a_{\tilde{\mu}},$$

we write

$$\sum_{\tilde{\mu}} |\partial_{\tilde{\mu}} W|^2 \; = \; 0 \; + \; \left. |\partial_{\tilde{\nu}} \partial_{\tilde{\mu}} W|^2 \right|_{X^o} |\delta X_{\tilde{\nu}}|^2, \tag{3.1.39}$$

suppressing higher-order terms as in [123][section 2.2]. Performing the Gaussian integral over $\delta X_{\tilde{\nu}}$ yields a factor

$$\frac{1}{H_X \overline{H}_X},$$

for $H_X$ the determinant of the matrix of second derivatives with respect to the $X_{\tilde{\mu}}$, meaning

$$H_X \; = \; \prod_{\tilde{\mu}} \frac{1}{\sum_a \sigma_a \alpha^a_{\tilde{\mu}}}. \tag{3.1.40}$$

Repeating the same for the Yukawa interactions

$$\psi_+^{\tilde{\mu}} \psi_-^{\tilde{\nu}} \partial_{\tilde{\mu}} \partial_{\tilde{\nu}} W \; + \; \text{c.c.}$$

as in [123][section 2.2], [78] yields another factor of $\overline{H}_X H_X^g$ at genus $g$. Putting these factors together results in a net factor of $1/H_X^{1-g}$ in correlation functions on a genus $g$ worldsheet.

Another effect of integrating out the $X_{\tilde{\mu}}$ should be to modify the superpotential (3.1.4), evaluating the $X_{\tilde{\mu}}$ on the critical loci:

$$\begin{aligned} W_0 \; &= \; \sum_{a=1}^r \sigma_a \left( \sum_{i=1}^N \rho_i^a Y_i \; - \; \sum_{\tilde{\mu}=1}^{n-r} \alpha^a_{\tilde{\mu}} \ln \left( \sum_b \sigma_b \alpha^b_{\tilde{\mu}} \right) \; - \; t_a \right) \\ &\quad + \sum_{i=1}^N \exp\left(-Y_i\right) \; - \; \sum_{i=1}^N \tilde{m}_i Y_i \; + \; \sum_{\tilde{\mu}} \sum_a \sigma_a \alpha^a_{\tilde{\mu}}, \end{aligned} \tag{3.1.41}$$

$$\begin{aligned} &= \; \sum_{a=1}^r \sigma_a \left[ \sum_{i=1}^N \rho_i^a Y_i \; - \; \sum_{\tilde{\mu}=1}^{n-r} \alpha^a_{\tilde{\mu}} \left( \ln \left( \sum_b \sigma_b \alpha^b_{\tilde{\mu}} \right) \; - \; 1 \right) \; - \; t_a \right] \\ &\quad + \sum_{i=1}^N \exp\left(-Y_i\right) \; - \; \sum_{i=1}^N \tilde{m}_i Y_i \end{aligned} \tag{3.1.42}$$

(up to constant terms we have omitted). The $\sigma(\ln(\sigma)-1)$ term in the $\sigma_a$ constraint, originating from integrating out the $X_{\tilde{\mu}}$ fields, reflects the shift of the FI parameter described in [89][section 10]. We can simplify this constant by rewriting it as a sum over positive roots:

$$\begin{aligned}
& \sum_{\tilde{\mu}=1}^{n-r} \alpha_{\tilde{\mu}}^a \left( \ln \left( \sum_b \sigma_b \alpha_{\tilde{\mu}}^b \right) - 1 \right) \\
&= \sum_{\text{pos}'} \alpha_{\tilde{\mu}}^a \ln \left( \sum_b \sigma_b \alpha_{\tilde{\mu}}^b \right) - \sum_{\text{pos}'} \alpha_{\tilde{\mu}}^a \left( \ln \left( \sum_b \sigma_b \alpha_{\tilde{\mu}}^b \right) - \pi i \right), \qquad (3.1.43) \\
&= \sum_{\text{pos}'} i\pi \alpha_{\tilde{\mu}}^a, \qquad (3.1.44)
\end{aligned}$$

giving a shift of the theta angle matching that given in [89][equ'n (10.9)].

Altogether, the effect of integrating out the $X_{\tilde{\mu}}$ is to add a factor of $\overline{H}_X H_X^g / (H_X \overline{H}_X) = 1/H_X^{1-g}$ to correlation functions (at genus $g$):

$$\langle \mathcal{O} \rangle \;=\; \int [DY_i][D\sigma_a]\, \mathcal{O} \left( \prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha_{\tilde{\mu}}^a \right) \right)^{1-g} \exp(-S_0), \qquad (3.1.45)$$

or more simply, for the case of isolated vacua (and no contributions from orbifold twisted sectors),

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{|W|} \sum_{\text{vacua}} \frac{\mathcal{O}}{(\det \partial^2 W_0)^{1-g}} \left( \prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha_{\tilde{\mu}}^a \right) \right)^{1-g}. \qquad (3.1.46)$$

Note that we can rewrite the new factor above solely in terms of the positive roots:

$$\prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha_{\tilde{\mu}}^a \right) \;\propto\; \prod_{\text{pos}'\text{roots}} \left( \sum_a \sigma_a \alpha_{\tilde{\mu}}^a \right)^2. \qquad (3.1.47)$$

So far, we have glossed over the fact that there is a Weyl-group orbifold present. For genus zero computations, since $\det \partial^2 W_0$ and

$$\prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha_{\tilde{\mu}}^a \right)$$

are both invariant under the Weyl group, so long as $\mathcal{O}$ itself is also Weyl group invariant, the effect of the Weyl group orbifold is solely to contribute the overall factor of $1/|W|$, where $|W|$ is the order of the Weyl group. For genus $g > 0$, one should be more careful, as partition functions now contain sums over twisted sectors. However, the B model localizes on constant maps, so therefore so long as no critical points of the superpotential intersect the fixed point locus of the orbifold, we do not expect any twisted sector contributions to correlation functions of Weyl-group-invariant operators, even at genus $g > 0$.

Our analysis so far has been rather formal, but in fact, we will see in later that the results are consistent with concrete computations in the case of Grassmannians.

To review, so far we have argued that correlation functions (on a genus $g$ worldsheet of fixed complex structure) take the form

$$\langle \mathcal{O} \rangle \;=\; \frac{1}{|W|} \sum_{\text{vacua}} \frac{\mathcal{O}}{(\det \partial^2 W_0)^{1-g}} \left( \prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \right) \right)^{1-g} \tag{3.1.48}$$

(for isolated vacua), where $W_0$ is the superpotential (3.1.42).

Next, we integrate out the $Y_i$ fields, in the same fashion. It is straightforward to compute

$$\begin{aligned} \frac{\partial W_0}{\partial Y_i} &= \sum_a \sigma_a \rho^a_i \;-\; \exp\left(-Y_i\right) \;-\; \tilde{m}_i, \\ \frac{\partial^2 W_0}{\partial Y_i \partial Y_j} &= +\delta_{ij} \exp\left(-Y_i\right). \end{aligned}$$

The critical points $Y^o_i$ for $Y_i$ follow from the derivative above as

$$\exp\left(-Y^o_i\right) \;=\; \sum_a \sigma_a \rho^a_i \;-\; \tilde{m}_i. \tag{3.1.49}$$

Integrating out the superfield $\delta Y_i = Y_i - Y^o_i$ results in correlation functions with an extra factor of $1/H_Y^{1-g}$ for $H_Y$ the determinant of the matrix of second derivatives with respect to $Y$'s, namely

$$H_Y \;=\; \prod_{i=1}^{N} \exp\left(-Y_i\right),$$

and superpotential $W_{00}$ given by evaluating $W_0$ at $Y^o_i$, meaning

$$\begin{aligned} W_{00} \;=\;& -\sum_a \sum_i \sigma_a \rho^a_i \ln\left( \sum_b \sigma_b \rho^b_i \;-\; \tilde{m}_i \right) \;+\; \sum_a \sum_i \sigma_a \rho^a_i \;-\; \sum_a \sigma_a t_a \\ & - \sum_a \sum_{\tilde{\mu}} \sigma_a \alpha^a_{\tilde{\mu}} \ln\left( \sum_b \sigma_b \alpha^b_{\tilde{\mu}} \right) \;+\; \sum_{\tilde{\mu}} \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \\ & + \sum_i \tilde{m}_i \ln\left( \sum_b \sigma_b \rho^b_i \;-\; \tilde{m}_i \right), \end{aligned} \tag{3.1.50}$$

$$\begin{aligned} \;=\;& -\sum_a \sum_i \sigma_a \rho^a_i \ln\left( \sum_b \sigma_b \rho^b_i \;-\; \tilde{m}_i \right) \;+\; \sum_a \sum_i \sigma_a \rho^a_i \;-\; \sum_a \sigma_a t_a \\ & - \sum_{\text{pos}'} i\pi \alpha^a_{\tilde{\mu}} \sigma_a \;+\; \sum_i \tilde{m}_i \ln\left( \sum_b \sigma_b \rho^b_i \;-\; \tilde{m}_i \right), \end{aligned} \tag{3.1.51}$$

where we have used the simplification (3.1.44).

Concretely, this means correlation functions (for isolated vacua, away from fixed points of the orbifold) on a worldsheet of genus $g$ (and fixed complex structure) are given by

$$\langle \mathcal{O} \rangle \; = \; \frac{1}{|W|} \sum_{\rm vacua} \frac{\mathcal{O}}{(\det \partial^2 W_{00})^{1-g}} \left[ \prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \right) \right]^{1-g} \left[ \prod_{i=1}^N \left( \sum_a \sigma_a \rho_i^a \; - \; \tilde{m}_i \right) \right]^{g-1}, \tag{3.1.52}$$

where the matrix of second derivatives $\partial^2 W_{00}$ now consists solely of derivatives with respect to $\sigma$'s. We deal with the Weyl-group-orbifold in the same fashion as in the previous section: since the B model localizes on constant maps, and we assume that the critical points of the superpotential do not intersect the fixed points of the orbifold, there are no twisted sector contributions at any worldsheet genus.

Up to overall factors, the expression (3.1.52) B model correlation function for the mirror to the A-twisted gauge theory, matches [52][section 4], with $\Delta^2(\sigma)$ reproducing

$$\prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \right)$$

and $\exp(-2\mathcal{U}_0)$ reproducing

$$\prod_{i=1}^N \left( \sum_a \sigma_a \rho_i^a \; - \; \tilde{m}_i \right),$$

and $W_{00}$ matches the "$W_0$" given in [52][equ'ns (2.17), (2.19)]. Also up to factors, for genus zero worldsheets, the expression (3.1.52) also matches [41][equ'n (4.77)] for an A-twisted (2,2) supersymmetric gauge theory in two dimensions, where $Z_0^{1-{\rm loop}}$ encodes [41]

$$\left[ \prod_{\tilde{\mu}} \left( \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \right) \right] \left[ \prod_{i=1}^N \left( \sum_a \sigma_a \rho_i^a \; - \; \tilde{m}_i \right) \right]^{-1}.$$

In passing, a (0,2) supersymmetric version of the same A model result is given in [42][equ'n (3.63)].

There is another formal argument to demonstrate that correlation functions should match. If we integrate out the mirrors to the W bosons, but not other fields, then as shown in [27][section 4.1], $\det \partial^2 W_0$ matches the product of $Z_{1-{\rm loop}}$ and the Hessian that arise in A model computations, which together with the factor of

$$\prod_{\rm pos'roots} \left( \sum_a \sigma_a \alpha^a_{\tilde{\mu}} \right)^2$$

in correlation functions arising from integrating out the $X$ fields, implies that B model correlation functions match their A model counterparts, as expected.

Our computation of B model correlation functions glossed over cross-terms in the superpotential such as $\partial^2 W / \partial X_{\tilde{\mu}} \partial \sigma_a$. In the next subsection, we shall revisit this computation from another perspective, taking into account those cross-terms, and derive the same result for correlation functions that we have derived in this subsection.

**Second argument**

In this section, we will give a different formal derivation of the correlation functions, that will give the same result – the correlation functions in our B-twisted proposed mirror (of untwisted sector states) will match conventional computations on Coulomb branches of A-twisted gauge theories. Instead of sequentially integrating out the $X_{\tilde{\mu}}$, then the $Y_i$, let us formally consider instead a direct computation of correlation functions, assuming that critical loci are isolated (and distinct from fixed points of the orbifold). (If critical loci are not isolated, one could suitably deform the superpotential to make them isolated.)

Correlation functions are then of the form

$$\langle \mathcal{O} \rangle \; = \; \sum_{\rm vacua} \frac{\mathcal{O}}{H^{1-g}},$$

where $H$ is the determinant of the matrix of second derivatives. Write

$$H \; = \; \det \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right], \tag{3.1.53}$$

where $A$ is the submatrix of derivatives with respect to $X_{\tilde{\mu}}$ and $Y_i$,

$$\begin{aligned} \frac{\partial^2 W}{\partial X_{\tilde{\mu}} \partial X_{\tilde{\nu}}} & = \delta_{\tilde{\mu}\tilde{\nu}} \left( \sum_c \sigma_c \alpha_{\tilde{\mu}}^c \right)^{-1}, \\ \frac{\partial^2 W}{\partial Y_i \partial Y_j} & = \delta_{ij} \left( \sum_c \sigma_c \rho_i^c \; - \; \tilde{m}_i \right), \\ \frac{\partial^2 W}{\partial X_{\tilde{\mu}} \partial Y_i} & = 0, \end{aligned}$$

(on the critical locus,) $B = C^T$ are the matrices of derivatives of the form

$$\begin{aligned} \frac{\partial^2 W}{\partial X_{\tilde{\mu}} \partial \sigma_a} & = -\alpha_{\tilde{\mu}}^a \left( \sum_c \sigma_c \alpha_{\tilde{\mu}}^c \right)^{-1}, \\ \frac{\partial^2 W}{\partial Y_i \partial \sigma_a} & = \rho_i^a, \end{aligned}$$

and $D$ is the matrix of second derivatives with respect to $\sigma$'s. Since $\sigma$ only appears linearly in the superpotential, $D = 0$.

Putting this together, from [92], we can write

$$H = (\det A) \det \left( D - CA^{-1}B \right). \tag{3.1.54}$$

The factor $\det A$ we have seen previously: since $A$ is diagonal, it is straightforward to see that

$$\det A = \left[ \prod_{\tilde{\mu}} \left( \sum_c \sigma_c \alpha^c_{\tilde{\mu}} \right) \right]^{-1} \left[ \prod_i \left( \sum_c \sigma_c \rho^c_i - \tilde{m}_i \right) \right]. \tag{3.1.55}$$

Note first that

$$\left( \frac{1}{\det A} \right)^{1-g} \tag{3.1.56}$$

is the same factor that appears multiplying operators in correlation functions in our previous expression (3.1.52); we have duplicated it without any extra factors, despite the fact that our previous analysis omitted cross-terms such as $\partial^2 W / \partial X_{\tilde{\mu}} \partial \sigma_a$. The remaining factor,

$$\det \left( D - CA^{-1}B \right),$$

can be interpreted as the usual Hessian from some superpotential we shall label $W_{\rm eff}$, which we shall see next will coincide with the $W_{00}$ of the previous subsection.

Proceeding carefully, since $C = B^T$, $D - 0$, and $A$ is symmetric, the quantity $CA^{-1}B$ is a symmetric matrix, so we can define a function $W_{\rm eff}$ as follows:

$$\left( -CA^{-1}B \right)_{ab} = \frac{\partial^2 W_{\rm eff}}{\partial \sigma_a \partial \sigma_b}. \tag{3.1.57}$$

Computing the matrix multiplication, we find

$$\left( -CA^{-1}B \right)_{ab} = -\sum_{\tilde{\mu}} \frac{\alpha^a_{\tilde{\mu}} \alpha^b_{\tilde{\mu}}}{\sum_c \sigma_c \alpha^c_{\tilde{\mu}}} - \sum_i \frac{\rho^a_i \rho^b_i}{\sum_c \sigma_c \rho^c_i - \tilde{m}_i}. \tag{3.1.58}$$

Curiously, it can be shown that for the superpotential $W_{00}$ computed in the previous subsection,

$$\frac{\partial^2 W_{00}}{\partial \sigma_a \partial \sigma_b} = -\sum_{\tilde{\mu}} \frac{\alpha^a_{\tilde{\mu}} \alpha^b_{\tilde{\mu}}}{\sum_c \sigma_c \alpha^c_{\tilde{\mu}}} - \sum_i \frac{\rho^a_i \rho^b_i}{\sum_c \sigma_c \rho^c_i - \tilde{m}_i}, \tag{3.1.59}$$

the same as the result above, hence we can identify

$$W_{\rm eff} = W_{00} \tag{3.1.60}$$

(up to irrelevant terms annihilated by the second derivative).

Phrased more simply, by more carefully taking into account all fields and cross-terms, we reproduce the same result for correlation functions derived in the previous subsection, which itself matches results in the literature for A model correlation functions.

In fact, there is a more general statement of this form that can be made, that for B model correlation functions, sequential 'integrations-out' are equivalent to correlation function computations. Consider for simplicity a superpotential $W = W(x,y)$, a function of two variables. Assuming isolated critical points, correlation functions are weighted by a factor of

$$\det \begin{bmatrix} \frac{\partial^2 W}{\partial x^2} & \frac{\partial^2 W}{\partial x \partial y} \\ \frac{\partial^2 W}{\partial y \partial x} & \frac{\partial^2 W}{\partial y^2} \end{bmatrix} = \left( \frac{\partial^2 W}{\partial x^2} \right) \left( \frac{\partial^2 W}{\partial y^2} \right) - \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2,$$

$$= \left( \frac{\partial^2 W}{\partial x^2} \right) \left[ \frac{\partial^2 W}{\partial y^2} - \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 \left( \frac{\partial^2 W}{\partial x^2} \right)^{-1} \right],$$

(mimicking the form of the result in [92]). We claim, as an elementary result, that

$$\frac{\partial^2 W}{\partial y^2} - \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 \left( \frac{\partial^2 W}{\partial x^2} \right)^{-1} = \frac{\partial^2 W_0}{\partial y^2}, \tag{3.1.61}$$

where $W_0 = W(x_0(y), y)$, for $x_0$ the critical loci of $W$ defined by

$$\left. \frac{\partial W}{\partial x} \right|_{x = x_0(y)} = 0. \tag{3.1.62}$$

The trivial generalization to multiple variables establishes the equivalence of the two arguments described in this section.

To demonstrate this, we compute:

$$\frac{\partial W_0}{\partial y} = \frac{\partial W(x_0, y)}{\partial x_0} \frac{\partial x_0}{\partial y} + \frac{\partial W(x_0, y)}{\partial y},$$

$$\frac{\partial^2 W_0}{\partial y^2} = \frac{\partial^2 W(x_0, y)}{\partial y^2} + 2 \frac{\partial^2 W}{\partial x_0 \partial y} \frac{\partial x_0}{\partial y} + \frac{\partial^2 W}{\partial x_0^2} \left( \frac{\partial x_0}{\partial y} \right)^2.$$

From the fact that $\partial W(x_0, y) / \partial x_0 = 0$, we have that

$$\frac{\partial}{\partial y} \frac{\partial W(x_0, y)}{\partial x_0} = \frac{\partial^2 W}{\partial x_0^2} \frac{\partial x_0}{\partial y} + \frac{\partial^2 W}{\partial x_0 \partial y} = 0, \tag{3.1.63}$$

and plugging into the equation above we find

$$\frac{\partial^2 W_0}{\partial y^2} = \frac{\partial^2 W}{\partial y^2} + 2 \frac{\partial^2 W}{\partial x_0 \partial y} \left( - \frac{\partial^2 W}{\partial x_0 \partial y} \right) \left( \frac{\partial^2 W}{\partial x_0^2} \right)^{-1}$$

$$+ \frac{\partial^2 W}{\partial x_0^2} \left( - \frac{\partial^2 W}{\partial x_0 \partial y} \right)^2 \left( \frac{\partial^2 W}{\partial x_0^2} \right)^{-2}, \tag{3.1.64}$$

$$= \frac{\partial^2 W}{\partial y^2} - \left( \frac{\partial^2 W(x_0, y)}{\partial x_0 \partial y} \right)^2 \left( \frac{\partial^2 W(x_0, y)}{\partial x_0^2} \right)^{-1}, \tag{3.1.65}$$

as claimed, establishing the desired equivalence.

In passing, note in the argument above that since $\partial W_0 / \partial y = 0$, since $\partial W(x_0, y) / \partial x_0 = 0$, we also have that $\partial W(x_0, y) / \partial y = 0$.

### 3.1.3 Example: Grassmannian $G(k,n)$

For our first example, we will compute the prediction for the mirror to a Grassmannian. Other proposals for this case also exist in the literature, see *e.g.* [25, 79–83].

#### Predicted mirror

Here, the A-twisted gauge theory is a $U(k)$ gauge theory with $n$ chiral superfields in the fundamental representation. The resulting GLSM describes the Grassmannian $G(k,n)$ [37].

The mirror is predicted to be an $S_k$-orbifold of a Landau-Ginzburg model with matter fields $Y_{ia}$ ($i \in \{1, \cdots n\}$, $a \in \{1, \cdots k\}$), $X_{\mu\nu} = \exp(-Z_{\mu\nu})$, $\mu, \nu \in \{1, \cdots, k\}$, and superpotential

$$W = \sum_a \sigma_a \left( \sum_{ib} \rho^a_{ib} Y_{ib} + \sum_{\mu\nu} \alpha^a_{\mu\nu} Z_{\mu\nu} - t \right) + \sum_{ia} \exp\left(-Y_{ia}\right) + \sum_{\mu \neq \nu} X_{\mu\nu}, \qquad (3.1.66)$$

where[8]

$$\rho^a_{ib} = \delta^a_b, \quad \alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu,$$

$X_{\mu\nu} = \exp(-Z_{\mu\nu})$ is a fundamental field, and the $X_{\mu\nu}$, $Z_{\mu\nu}$ need not be (anti)symmetric but are only defined for $\mu \neq \nu$, as the diagonal entries would correspond to the elements of the Cartan subalgebra that we use to define constraints via $\sigma$'s. Between $X_{\mu\nu}$ and $Z_{\mu\nu}$, $X_{\mu\nu}$ is the fundamental field, but it will sometimes be convenient to work with its logarithm, so we retain $Z_{\mu\nu} \equiv -\ln X_{\mu\nu}$.

We orbifold the space of fields $\sigma_a$, $Y_{ia}$, and $Z_{\mu\nu}$ by the Weyl group of the gauge group. Now, the Weyl group of $U(k)$ is the symmetric group on $k$ entries. It acts on a Cartan torus by

[8] Let us illustrate the $\alpha$'s explicitly for the case of $U(3)$. Begin by describing the Cartan subalgebra of $U(3)$ as

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

Describe the W bosons as

$$\begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{bmatrix}.$$

Under a gauge transformation in the Cartan,

$$\begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix} \begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & 0 & A_{23} \\ A_{31} & A_{32} & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} 0 & a^{-1}A_{12}b & a^{-1}A_{13}c \\ b^{-1}A_{21}a & 0 & b^{-1}A_{23}c \\ c^{-1}A_{31}a & c^{-1}A_{32}b & 0 \end{bmatrix}.$$

Thus, we see that

$$\alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu.$$

permuting $U(1)$ elements. In the present case, that means the orbifold acts by permuting the $\sigma_a$, by making corresponding permutations of the $Y_{ia}$ (acting on the $a$ index, leaving the $i$ fixed), and correspondingly on the $X_{\mu\nu}$ (associated with root vectors). We will see concrete examples in the next subsections.

In passing, the Weyl group $S_k$ acts by interchanging fields, which will leave a holomorphic top-form on the space of fields $\sigma_a$, $Y_{ia}$, and $X_{\mu\nu}$ invariant up to a sign. (For example, for (two-dimensional) Calabi-Yau surfaces $M$, namely $T^4$ and $K3$, $S_k$ leaves invariant the holomorphic top-form on $M^k$ [93].) As discussed earlier and in [113], this is sufficient for the B twist to exist. A more general orbifold might not be compatible with the B twist, but as previously discussed, the Weyl orbifold is always compatible with the B twist.

We begin working in the untwisted sector of the orbifold. (Later we will observe that only the untwisted sector is relevant.) Integrating out the $\sigma_a$, we get constraints

$$\sum_i Y_{ia} \; - \; \sum_{\nu \neq a} (Z_{a\nu} - Z_{\nu a}) \; - \; t \; = \; 0,$$

which we use to eliminate $Y_{na}$:

$$Y_{na} \; = \; - \sum_{i=1}^{n-1} Y_{ia} \; + \; \sum_{\nu \neq a} (Z_{a\nu} - Z_{\nu a}) \; + \; t.$$

Define

$$\begin{aligned} \Pi_a \; &= \; \exp(-Y_{na}), & (3.1.67) \\ &= \; q \left( \prod_{i=1}^{n-1} \exp(+Y_{ia}) \right) \left( \prod_{\mu \neq a} \frac{X_{a\mu}}{X_{\mu a}} \right), & (3.1.68) \end{aligned}$$

for $q = \exp(-t)$, then the superpotential for the remaining fields, after applying the constraint, reduces to

$$W \; = \; \sum_{i=1}^{n-1} \sum_{a=1}^{k} \exp\left(-Y_{ia}\right) \; + \; \sum_{\mu \neq \nu} X_{\mu\nu} \; + \; \sum_{a=1}^{k} \Pi_a. \qquad (3.1.69)$$

Note that since $\Pi_a$ contains factors of the form $1/X$, the superpotential above has poles where $X_{\mu\nu} = 0$. Such structures can arise after integrating out fields in more nearly conventional Landau-Ginzburg theories, as we shall review in later, and have also appeared in other discussions of two-dimensional (2,2) supersymmetric theories *e.g.* [84–86]. In the next section, we shall argue that physics excludes the loci where $X_{\mu\nu} = 0$, and so at least insofar as our semiclassical analysis of the B-twisted theory is concerned, the presence of poles will not be an issue.

So far we have discussed the untwisted sector of the Weyl orbifold. However, since none of the critical loci land at fixed points of the orbifold, we do not expect any *e.g.* twisted sector contributions, and so for the purposes of this paper, we will omit the possibility of twisted sector contributions.

### Excluded loci

We saw in section 3.1.2 that when the $X$ fields are integrated out, the integration measure is multiplied by a factor proportional to

$$\prod_{\tilde{\mu}} \langle \Sigma, \alpha_{\tilde{\mu}} \rangle \; = \; \prod_{\tilde{\mu}} \left( \sum_a \Sigma_a \alpha^a_{\tilde{\mu}} \right), \tag{3.1.70}$$

which therefore suppresses contributions from vacua such that any $\langle \Sigma, \alpha_{\tilde{\mu}} \rangle$ vanish.

Thus, points where $\langle \Sigma, \alpha_{\mu\nu} \rangle$ vanish, necessarily do not contribute. In (A-twisted) gauge theories in two dimensions, this is a standard and well-known effect: in Coulomb branch computations, one must exclude certain loci. For the case of the Grassmannian, one excludes the loci where any $\sigma_a$s collide, corresponding to the same loci discussed here, and also to loci where there is semiclassically an enhanced nonabelian gauge symmetry. In supersymmetric localization computations, the excluded loci appear in the same fashion – as the vanishing locus of a measure factor in correlation functions (see *e.g.* [94][section 2.2]).

Thus, the loci $\{\langle \sigma, \alpha_{\tilde{\mu}} \rangle = 0\}$ must be excluded. It remains to understand this excluded locus phenomenon from the perspective of the theory containing the $X_{\tilde{\mu}}$ fields, before they are integrated out.

It turns out that a nearly identical argument applies to the theory containing $X_{\tilde{\mu}}$ fields. To understand this fact, we first need to utilize the operator mirror map (3.1.3), which says

$$X_{\tilde{\mu}} \; = \; \sum_a \Sigma_a \alpha^a_{\tilde{\mu}} \; = \; \langle \Sigma, \alpha_{\tilde{\mu}} \rangle. \tag{3.1.71}$$

Thus, the locus where $\langle \Sigma, \alpha_{\tilde{\mu}} \rangle$ vanishes is the same as the locus where $X_{\tilde{\mu}}$ vanishes.

If one tracks through the integration-out, one can begin to see a purely mechanical reason for the zeroes of the measure: the mass of $X_{\tilde{\mu}}$ is proportional to

$$\frac{1}{\langle \sigma, \alpha_{\tilde{\mu}} \rangle}, \tag{3.1.72}$$

and so the $X_{\tilde{\mu}}$ become infinitely massive at the excluded loci. When one computes Hessian factors $H = \partial^2 W$ weighting critical loci in correlation function computations, it turns out that, just as the mass becomes infinite, so too does the Hessian, to which the mass contributes. (Indeed, it is difficult to see how the Hessian could fail to become infinite in such cases.) For example, in section 3.1.3 we will see that the Hessian for the mirror to $G(2,n)$ has a factor

$$\frac{1}{(\Pi_1 - \Pi_2)^2}, \tag{3.1.73}$$

where $\Pi_a$ is mirror to $\sigma_a$. The critical loci where $X_{\tilde{\mu}}$ vanish correspond in this case to the locus where $\Pi_1 \to \Pi_2$, so we see that the Hessian diverges. Since correlation functions are weighted by factors of $1/H$, if $H$ diverges, then the critical locus in question cannot contribute to correlation functions.

More generally, the result above is related by the operator mirror map to results in supersymmetric localizations for Coulomb branch computations with $\sigma$'s. Specifically, we will see later that the operator mirror map relates $\Pi_a$ to $\sigma_a$, so the Hessian above is mirror to

$$H \;\leftrightarrow\; \frac{1}{(\sigma_1 - \sigma_2)^2} \;=\; \frac{1}{\langle \sigma, \alpha_{12} \rangle^2}, \tag{3.1.74}$$

hence $1/H$ is mirror to the measure factor

$$\frac{1}{H} \;\leftrightarrow\; (\sigma_1 - \sigma_2)^2 \;=\; \langle \sigma, \alpha_{12} \rangle^2. \tag{3.1.75}$$

More generally, the operator mirror map directly relates the vanishing measure factors to vanishing $1/H$ factors, and so we see that critical loci along the excluded locus cannot contribute to correlation functions, in both A-twisted theories of $\sigma$'s as well as the proposed mirror, for essentially identical reasons in each case.

So far we have established at a mechanical level that critical loci along the excluded locus (where the $X_{\tilde{\mu}}$ vanish) cannot contribute. Next, we shall outline less mechanical reasons in the physics of the proposed mirror for why this exclusion should take place. This matter will be somewhat subtle, as we shall see, but nevertheless even without computing Hessians one can see several issues with the excluded locus in the mirror theory that would suggest these loci should be excluded.

First, focusing on the $X$ fields, the bosonic potential diverges where any one $X_{\tilde{\mu}}$ vanishes – so generically these points are excluded dynamically. One has to be slightly careful about higher codimension loci, however. Because the superpotential contains ratios of the form $X_{\mu\nu}/X_{\nu\mu}$, if multiple $X_{\tilde{\mu}}$ vanish, then the superpotential has $0/0$ factors, which are ill-defined (as we shall discuss further in later section). Since the critical loci are defined as the loci where $\nabla W$ vanishes, formally the bosonic potential

$$U \;=\; |\nabla W|^2 \tag{3.1.76}$$

also vanishes along critical loci, which appears to say that at higher codimension loci such as critical loci, the bosonic potential will be finite, not infinite, where enough $X_{\tilde{\mu}}$ vanish. That said, in typical examples in this paper, the critical locus consists of isolated points, not a continuum, so working just in critical loci themselves one cannot continuously approach a point where all $X_{\tilde{\mu}}$ vanish, so one cannot reach such points through a limit of critical loci.

The fact that the bosonic potential diverges when any one $X_{\tilde{\mu}}$ vanishes means that the bosonic potential diverges generically when $X_{\tilde{\mu}}$ vanish. As noted above, there are higher-codimension loci where the superpotential and bosonic potential are ambiguous. It is natural to suspect that some regularization of the quantum field theory may effectively 'smooth over' these higher-codimension ambiguities, so that the quantum field theory sees a continuous (and infinite) potential. We will elaborate on this suspicion later.

As this matter is extremely subtle, let us examine it from another perspective. The superpotential is ambiguous at points where all $X_{\tilde{\mu}}$ vanish, but we can still consider limits as one approaches such points. Consider for example the mirror superpotential (3.1.69) for the case of $G(2, n)$. The critical locus equations are

$$\frac{\partial W}{\partial Y_{i1}} = -\exp(-Y_{i1}) + q \left( \prod_{j=1}^{n-1} \exp(+Y_{j1}) \right) \frac{X_{12}}{X_{21}}, \tag{3.1.77}$$

$$\frac{\partial W}{\partial Y_{i2}} = -\exp(-Y_{i2}) + q \left( \prod_{j=1}^{n-1} \exp(+Y_{j2}) \right) \frac{X_{21}}{X_{12}}, \tag{3.1.78}$$

$$\frac{\partial W}{\partial X_{12}} = 1 + q \left( \prod_{j=1}^{n-1} \exp(+Y_{j1}) \right) \frac{1}{X_{21}} - q \left( \prod_{j=1}^{n-1} \exp(+Y_{j2}) \right) \frac{X_{21}}{X_{12}^2}, \tag{3.1.79}$$

$$\frac{\partial W}{\partial X_{21}} = 1 - q \left( \prod_{j=1}^{n-1} \exp(+Y_{j1}) \right) \frac{X_{12}}{X_{21}^2} + q \left( \prod_{j=1}^{n-1} \exp(+Y_{j2}) \right) \frac{1}{X_{12}}. \tag{3.1.80}$$

Because of the ratios in the last two equations, the limit of these equations as one approaches a critical locus point can be a little subtle. Assuming that the first two derivatives vanish, we can rewrite the last two equations in a more convenient form:

$$\frac{\partial W}{\partial X_{12}} = 1 + \frac{\exp(-Y_{i1}) - \exp(-Y_{i2})}{X_{12}}, \tag{3.1.81}$$

$$\frac{\partial W}{\partial X_{21}} = 1 - \frac{\exp(-Y_{i1}) - \exp(-Y_{i2})}{X_{21}}, \tag{3.1.82}$$

for any $i$. If we are approaching a 'typical' critical locus point, not on the proposed excluded locus, then $X_{12,21} \neq 0$ and $\exp(-Y_{i1}) \neq \exp(-Y_{i2})$, so the limits of the derivatives above are well-defined and vanish unambiguously at the critical point, consistent with supersymmetry. Now, consider instead a critical point on the excluded locus, where $X_{\mu\nu}$ vanish and (as we shall see later from *e.g.* the operator mirror map) $\exp(-Y_{i1}) = \exp(-Y_{i2})$. Strictly speaking, the derivatives above are not uniquely defined at this point, as they have a term of the form $0/0$. Suppose we approach this point along a path such that

$$\exp(-Y_{i1}) - \exp(-Y_{i2}) = \alpha X_{12} = -\alpha X_{21}, \tag{3.1.83}$$

for some constant $\alpha$. Then the limit of the derivatives along this path is easily computed to be

$$\lim \frac{\partial W}{\partial X_{12}} \;=\; 1+\alpha \;=\; \lim \frac{\partial W}{\partial X_{21}}. \tag{3.1.84}$$

For most paths of this form, so long as $\alpha \neq -1$, the limit of the derivatives is nonzero, and so appears incompatible with supersymmetry. This is an artifact of the critical locus in the proposed excluded locus; for other critical loci, not in the excluded locus, the limits of these derivatives are well-defined and vanish.

More globally, understanding these excluded loci is one of the motivating factors behind this proposed mirror construction. After all, a condition such as $\sigma_a \neq \sigma_b$ for $a \neq b$ is an example of an open condition, in the sense that it specifies an open set, rather than a closed set. To specify an open condition in physics would seem to require either an integration measure that vanishes at the excluded points, or a potential function that excludes those points. In effect, both arise here: the proposed mirror superpotential describes a bosonic potential that excludes these points, and if we integrate out the pertinent fields to get a theory of just $\sigma$'s, then as we have already seen, the result is an integration measure which vanishes at the points.

In fact, one of the strengths of this proposed mirror is that it gives a purely algebraic way to determine those excluded loci – as the points where the $X_{\mu\nu}$ vanish. Sometimes these A model exclusions have been empirical, see for example [95][footnote 4, p. 26], so in such cases, the analysis here gives one a more systematic means of understanding the A model excluded loci.

Later in this paper we will check in numerous examples beyond Grassmannians that the excluded loci predicted in this fashion by the proposed mirror superpotential, match the excluded loci that are believed to arise on the A model side. In fact, in every example we could find in the literature, the excluded loci determined in gauge theory Coulomb branch analyses match those determined by the loci $\{X_{\tilde{\mu}} = 0\}$.

**Check: Number of vacua**

The Euler characteristic of the Grassmannian $G(k,n)$ is

$$\left( \begin{array}{c} n \\ k \end{array} \right).$$

In this section we will check that the proposed B model mirror has this number of vacua.

The critical locus of the superpotential (3.1.69) is defined by

$$\frac{\partial W}{\partial Y_{ia}} : \qquad \exp\left(-Y_{ia}\right) \;=\; \Pi_a, \tag{3.1.85}$$

$$\frac{\partial W}{\partial X_{\mu\nu}} : \qquad X_{\mu\nu} \;=\; -\Pi_\mu + \Pi_\nu. \tag{3.1.86}$$

Plugging into the definition (3.1.68) of $\Pi_a$, we find

$$\Pi_a \; = \; q \left( \frac{1}{\Pi_a} \right)^{n-1} \left( \prod_{\mu \neq a} \frac{ -\Pi_a + \Pi_\mu }{ -\Pi_\mu + \Pi_a } \right) \; = \; q (-)^{k-1} (\Pi_a)^{1-n},$$

hence

$$(\Pi_a)^n \; = \; (-)^{k-1} q. \tag{3.1.87}$$

As discussed in section 3.1.3, the $X_{\mu\nu}$ do not vanish, which means that the $\Pi_a$ are all distinct. Since the $\Pi_a$ are distinct, and from (3.1.87), each is an $n$th root of $(-)^{k-1}q$, there are therefore

$$n(n-1) \cdots (n-k+1)$$

different vacua, before taking into account the Weyl group orbifold.

Finally, we need to take into account the $S_k$ orbifold. The Weyl group orbifold acts by exchanging $Y_{ia}$ with different values of $a$, hence exchanges different $\Pi_a$. (It also exchanges the $\sigma_a$'s with one another, and interrelates the $X_{\mu\nu}$, though for the moment that is less relevant.) Thus, in the untwisted sector, there are

$$\frac{ n(n-1) \cdots (n-k+1) }{ k! } \; = \; \left( \begin{array}{c} n \\ k \end{array} \right)$$

critical loci or vacua.

The fixed-point locus of the Weyl orbifold lies along loci where some of the $\Pi_a$ coincide; since the critical locus requires all $\Pi_a$ distinct, we see that none of the critical loci can lie at fixed points of the Weyl orbifold group action. As a result, we do not expect any contributions from twisted sectors, as discussed previously in section of “excluded loci.”

Thus, we find that the proposed mirror has

$$\left( \begin{array}{c} n \\ k \end{array} \right)$$

vacua, matching the number of vacua of the original A-twisted GLSM for $G(k,n)$.

In passing, the details of this computation closely match the details of the analogous computation in the A-twisted GLSM for $G(k,n)$, where one counts solutions of $(\sigma_a)^n = (-)^{k-1} q$, subject to the excluded-locus constraint that $\sigma_a \neq \sigma_b$ if $a \neq b$. If one did try to include vacua where the $\Pi_a$ are not distinct, including vacua on the excluded loci, then at minimum the computation would no longer closely match the A model computation, and furthermore (modulo the possibility of extra twisted sector vacua contributing with sufficient signs), it is not at all clear that the resulting Witten index would necessarily match that of the Grassmannian.

**Compare B ring to A ring**

Let us first compare against the operator mirror maps (3.1.2), (3.1.3). For the case of the Grassmannian, these operator mirror maps predict

$$\begin{aligned} \exp\left(-Y_{ia}\right) &= \sum_b \sigma_b \rho^a_{ib} \;=\; \sigma_a, \\ X_{\mu\nu} &= \sum_a \sigma_a \alpha^a_{\mu\nu} \;=\; -\sigma_\mu + \sigma_\nu. \end{aligned}$$

On the critical locus, we computed

$$\begin{aligned} \exp\left(-Y_{ia}\right) &= \Pi_a, \\ X_{\mu\nu} &= -\Pi_\mu + \Pi_\nu. \end{aligned}$$

Thus, we see the operator mirror map is completely consistent with our computations for the critical locus, and in particular, the mirror map identifies

$$\sigma_a \;\leftrightarrow\; \Pi_a. \tag{3.1.88}$$

The equation (3.1.87) above is the mirror of the A model Coulomb branch statement

$$(\sigma_a)^n \;=\; (-)^{k-1} q, \tag{3.1.89}$$

which determines the quantum cohomology ring of the Grassmannian, which is of the form (see *e.g.* [37, 38, 96–100])

$$\mathbb{C}[x_1, \cdots, x_{n-k}] / \langle D_{k+1}, \cdots, D_{n-1}, D_n + (-)^n \tilde{q} \rangle, \tag{3.1.90}$$

for some constant $\tilde{q} \propto q$, where

$$D_m \;=\; \det\left(x_{1+j-i}\right)_{1 \leq i,j \leq m},$$

in conventions in which $x_m = 0$ if $m < 0$ or $m > n - k$, and $x_0 = 1$.

In particular, the chiral ring of this B model theory is finite-dimensional, and matches that of the A-twisted theory. Although the superpotential has poles, in this instance (and for the other theories in this paper), the chiral ring remains finite.

For an explicit derivation of the quantum cohomology ring of the Grassmannian from the Coulomb branch relations (3.1.87), see *e.g.* [94][section 3.3]. In fact, at this point we could stop and observe that the critical loci, defined by the equation above, satisfy the same form as the critical loci of the one-loop effective twisted superpotential on the Coulomb branch in the original A-twisted GLSM, including the orbifold by the Weyl group action, hence the B model shares the quantum cohomology ring of the A model.

Let us briefly outline the idea of how the quantum cohomology ring is derived from the Coulomb branch relations, sketching [94][section 3.3]. First, we identify each $x_i$ with a Schur polynomial $s_\lambda(\sigma)$ in the variables $\sigma_1, \cdots, \sigma_k$, associated to a Young tableau $\lambda$ with $i$ horizontal boxes. These are symmetric polynomials, invariant under the (Weyl-)orbifold group. For example, for $k = 2$,

$$\begin{aligned} x_1 &= s_{\square}(\sigma) \;=\; \sigma_1 + \sigma_2, \\ x_2 &= s_{\square\square}(\sigma) \;=\; \sigma_1^2 + \sigma_1\sigma_2 + \sigma_2^2, \\ x_3 &= s_{\square\square\square}(\sigma) \;=\; \sigma_1^3 + \sigma_1^2\sigma_2 + \sigma_1\sigma_2^2 + \sigma_2^3. \end{aligned}$$

Without using the relation (3.1.89), it is straightforward to verify that $D_m = 0$ for $m > k$, simply as an algebraic consequence of the expressions for $x_i$ in terms of $\sigma_a$'s, and this is the origin of most of the relations in the quantum cohomology ring (3.1.90) in this language.

The relation (3.1.89) modifies relations involving $n$th powers of $\sigma$'s. For example, for $k = 2$, it is straightforward to check that

$$x_4 - x_3\sigma_1 \;=\; (\sigma_2)^4, \tag{3.1.91}$$

again using algebraic properties of the expansions in terms of $\sigma_a$'s. Consider the case $n = 4$, for which we know $x_3 = 0$. The algebraic relation above then implies $x_4 = (\sigma_2)^4 \propto q$, giving the desired relation. Other cases follow similarly.

### Correlation functions in $G(2, n)$

As another consistency test, we will now outline correlation function computations in the proposed B-twisted Landau-Ginzburg orbifold mirror to $G(2, n)$ for various values of $n$, and compare them to results for correlation functions in the original A-twisted gauge theory.

Before wading into the details of the computations, it may be helpful to first very briefly review the analogous computations for the B-twisted mirror to $\mathbb{P}^n$. This is a Landau-Ginzburg model with superpotential of the form

$$W \;=\; \exp(-Y_1) + \cdots + \exp(-Y_n) + q \prod_{i=1}^{n} \exp(+Y_i). \tag{3.1.92}$$

The critical locus is defined by $\exp(-Y_i)^{n+1} = q$ for all $i$. Genus zero correlation functions have the form

$$\langle f \rangle \;=\; \sum_{\text{vacua}} \frac{f}{H}, \tag{3.1.93}$$

where $H$ is the determinant of the matrix of second derivatives of the superpotential $W$, and we identify vacua with the critical locus. In the present case, if we define $X = \exp(-Y_i)$, so that the critical locus is $X^{n+1} = q$, then $H = \det \partial^2 W = (n+1) X^n$. Correlation functions then take the form of a sum over $(n+1)$th roots of unity:

$$\langle X^k \rangle \;\propto\; \sum_{\rm vacua} \frac{X^k}{X^n}, \tag{3.1.94}$$

and so will be nonzero if $k = n + m(n+1)$, corresponding to cases in which the summand is a multiple of $X^{n+1} = q$. For other values of $k$, the sum vanishes, as the corresponding sum over roots of unity vanishes. This result matches the form of genus zero A model correlation functions on $\mathbb{P}^n$, and we will see that computations in the mirror to $G(2,n)$ have a similar flavor.

Now, let us return to the mirror of $G(2,n)$. As stressed previously, since we are computing correlation functions of untwisted sector operators, and critical loci do not overlap orbifold fixed points, the sole effect of the orbifold will be to multiply the correlation function by a factor of $1/|W|$, for $|W|$ the order of the Weyl group.

From the superpotential (3.1.69) (after integrating out $\sigma_1$), we have the following derivatives:

$$\begin{aligned}
\frac{\partial W}{\partial Y_{ia}} &= -\exp\left(-Y_{ia}\right) \;+\; \Pi_a \text{ for } i < n, \\
\frac{\partial W}{\partial X_{\mu\nu}} &= 1 \;+\; \frac{\Pi_\mu}{X_{\mu\nu}} \;-\; \frac{\Pi_\nu}{X_{\mu\nu}} \text{ for } \mu \neq \nu,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 W}{\partial Y_{jb} \partial Y_{ia}} &= \delta_{ij}\delta_{ab} \exp\left(-Y_{ia}\right) \;+\; \delta_{ab} \Pi_a, \\
\frac{\partial^2 W}{\partial X_{\mu\nu} \partial Y_{ia}} &= \delta_{a\mu} \frac{\Pi_\mu}{X_{\mu\nu}} \;-\; \delta_{a\nu} \frac{\Pi_\nu}{X_{\mu\nu}}, \\
\frac{\partial^2 W}{\partial X_{\rho\sigma} \partial X_{\mu\nu}} &= \delta_{\mu\rho}\delta_{\nu\sigma} \frac{-\Pi_\mu + \Pi_\nu}{X_{\rho\sigma} X_{\mu\nu}} \\
& \quad + \left(\delta_{\rho\mu} - \delta_{\sigma\mu}\right) \frac{\Pi_\mu}{X_{\rho\sigma} X_{\mu\nu}} \;-\; \left(\delta_{\rho\nu} - \delta_{\sigma\nu}\right) \frac{\Pi_\nu}{X_{\rho\sigma} X_{\mu\nu}}.
\end{aligned}$$

Clearly, correlation functions will be nontrivial. Let us now specialize to Grassmannians $G(2,n)$. It is straightforward to compute[9]:

- for $G(2,3)$,

$$H \;\equiv\; \det\left(\partial^2 W\right) \;=\; -9 \frac{(\Pi_1)^2 (\Pi_2)^2}{(\Pi_1 - \Pi_2)^2},$$

[9] Some potentially useful identities can be found in *e.g.* [92].

- for $G(2,4)$,

$$H \;\equiv\; \det\left(\partial^2 W\right) \;=\; -16\frac{(\Pi_1)^3(\Pi_2)^3}{(\Pi_1-\Pi_2)^2},$$

- for $G(2,5)$,

$$H \;\equiv\; \det\left(\partial^2 W\right) \;=\; -25\frac{(\Pi_1)^4(\Pi_2)^4}{(\Pi_1-\Pi_2)^2}.$$

All derivatives above are evaluated on the critical locus. From the results above, we conjecture that for $G(2,n)$ for general $n \geq 3$,

$$H \;\equiv\; \det\left(\partial^2 W\right) \;=\; -n^2\frac{(\Pi_1)^{n-1}(\Pi_2)^{n-1}}{(\Pi_1-\Pi_2)^2}. \tag{3.1.95}$$

Let us compute correlation functions in the mirrors to $G(2,n)$ for $3 \leq n \leq 5$, and compare to the correlation functions computed for the corresponding A-twisted gauge theories in [94]. Note that given the quantum cohomology relations, once we establish that the classical correlation functions match (up to an overall scale), all the remaining correlation functions are guaranteed to match.

Reference [94] considers correlation functions A-twisted gauge theories corresponding to $G(2,3)$ (see [94][section 4.2]), $G(2,4)$ (see [94][section 4.3]), and $G(2,5)$ (see [94][section 4.4]). In each case, for $G(2,n)$, the nonzero classical ($q=0$) correlation functions are

$$\langle \sigma_1^{n-1}\sigma_2^{n-3}\rangle, \;\; \langle \sigma_1^{n-2}\sigma_2^{n-2}\rangle, \;\; \langle \sigma_1^{n-3}\sigma_2^{n-1}\rangle. \tag{3.1.96}$$

All other correlation functions of products of $\sigma$'s of degree $2n-4$ vanish. The three nonzero classical correlation functions are related as

$$\langle \sigma_1^{n-1}\sigma_2^{n-3}\rangle \;=\; \langle \sigma_1^{n-3}\sigma_2^{n-1}\rangle, \;\; \langle \sigma_1^{n-2}\sigma_2^{n-2}\rangle \;=\; -2\langle \sigma_1^{n-1}\sigma_2^{n-3}\rangle \;=\; -2\langle \sigma_1^{n-3}\sigma_2^{n-1}\rangle, \tag{3.1.97}$$

so that

$$\langle\left(\sigma_1^{n-1}\sigma_2^{n-3} \;+\; \sigma_1^{n-2}\sigma_2^{n-2} \;+\; \sigma_1^{n-3}\sigma_2^{n-1}\right)\rangle \;=\; 0. \tag{3.1.98}$$

Although the overall normalization is not essential, in reference [94], we list here the normalized values in the normalization convention of that paper:

$$\langle \sigma_1^{n-2}\sigma_2^{n-2}\rangle \;=\; \frac{2}{2!}, \;\; \langle \sigma_1^{n-1}\sigma_2^{n-3}\rangle \;=\; -\frac{1}{2!} \;=\; \langle \sigma_1^{n-3}\sigma_2^{n-1}\rangle. \tag{3.1.99}$$

Not only will our mirror's correlation functions have the same ratios, in fact their normalized values will be identical.

From the operator mirror map (3.1.88), in the mirror we should make corresponding statements about correlation functions of products of $\Pi_1$ and $\Pi_2$. As explained earlier, correlation functions in the Landau-Ginzburg orbifold mirror to $G(2,n)$ take the form

$$\langle \Pi_1^k \Pi_2^\ell \rangle \;=\; \frac{1}{2!}\sum_{\text{vacua}} \frac{\Pi_1^k\Pi_2^\ell}{H} \;=\; -\frac{1}{2!}\frac{1}{n^2}\sum_{\text{vacua}} \frac{(\Pi_1-\Pi_2)^2\Pi_1^k\Pi_2^\ell}{\Pi_1^{n-1}\Pi_2^{n-2}}. \tag{3.1.100}$$

Note that because of the $(\Pi_1 - \Pi_2)$ factors in the numerator, we no longer need to restrict to vacua described by distinct $\Pi_a$, since cases in which they coincide do not contribute; instead, we can replace the sum over vacua with a sum over two sets of $n$th roots of unity, corresponding (up to scale) with separate solutions for $\Pi_1$ and $\Pi_2$.

For example, let us compute

$$\begin{aligned}
\langle \Pi_1^{n-2}\Pi_2^{n-2}\rangle &= -\frac{1}{2!n^2}\sum_{\text{vacua}}\frac{(\Pi_1-\Pi_2)^2\Pi_1^{n-2}\Pi_2^{n-2}}{\Pi_1^{n-1}\Pi_2^{n-1}}, \\
&= -\frac{1}{2!n^2}\left(-\frac{1}{q}\right)^2\sum_{\text{vacua}}\Pi_1\Pi_2(\Pi_1-\Pi_2)^2\Pi_1^{n-2}\Pi_2^{n-2}, \\
&= -\frac{1}{2!n^2q^2}\sum_{\text{vacua}}\left(\Pi_1^2-2\Pi_1\Pi_2+\Pi_2^2\right)\Pi_1^{n-1}\Pi_2^{n-1}, \\
&= -\frac{1}{2!n^2q^2}\sum_{\text{vacua}}(-2)(\Pi_1\Pi_2)\Pi_1^{n-1}\Pi_2^{n-1}, \\
&= -\frac{1}{2!n^2q^2}\sum_{\text{vacua}}(-2)(-q)^2, \\
&= \frac{2n^2}{2!n^2} = \frac{2}{2!},
\end{aligned}$$

where we have used the relations $\Pi_1^n = -q = \Pi_2^n$. Reasoning similarly, it is straightforward to demonstrate that

$$\langle \Pi_1^{n-1}\Pi_2^{n-3}\rangle = -\frac{n^2}{2!n^2} = -\frac{1}{2!} = \langle \Pi_1^{n-3}\Pi_2^{n-1}\rangle,$$

which immediately obey the analogues of the relations (3.1.97), (3.1.98) for $\Pi_{1,2}$ in place of $\sigma_{1,2}$, and in fact even has the same overall normalization. Using similar reasoning, it is also trivial to verify that all other correlation functions of products of $\Pi$'s of degree $2n-4$ vanish.

Thus, we see that the classical genus zero correlation functions in the proposed B-twisted mirror to $G(2,n)$ match those of the original A-twisted theory, and since the quantum cohomology relations match, we immediately have that all genus zero correlation functions match.

### 3.1.4 Example: Two-Step Flag Manifold

In this section we investigate the flag manifolds, and one can refer to [26] for more examples. To be concrete, let us work out the proposed mirror to a two-step flag manifold, and check that it describes the correct number of vacua.

Consider the two-step flag manifold $F(k_1,k_2,n)$, $k_1 < k_2 < n$, which is described in GLSMs as [40] as a $U(k_1)\times U(k_2)$ gauge theory with

- one set of chiral superfields in the $(\mathbf{k_1}, \overline{\mathbf{k_2}})$ bifundamental representation,
- $n$ chiral superfields in the $(\mathbf{1}, \mathbf{k}_2)$ representation.

(In other words, a representation of a quiver.)

Following our proposal, the mirror is an orbifold of a Landau-Ginzburg model with fields

- $Y_a^\alpha$, $a \in \{1, \cdots, k_1\}$, $\alpha \in \{1, \cdots, k_2\}$, corresponding to the bifundamentals,
- $\tilde{Y}_{i\alpha}$, $i \in \{1, \cdots, n\}$, corresponding to the second set of matter fields,
- $X_{\mu\nu} = \exp(-Z_{\mu\nu})$, $\mu, \nu \in \{1, \cdots k_1\}$, corresponding to the W bosons from the $U(k_1)$,
- $\tilde{X}_{\mu',\nu'} = \exp(-\tilde{Z}_{\mu'\nu'})$, $\mu', \nu' \in \{1, \cdots, k_2\}$, corresponding to the W bosons from the $U(k_2)$,
- $\sigma_h$, $\tilde{\sigma}_{h'}$, $h \in \{1, \cdots, k_1\}$, $h' \in \{1, \cdots, k_2\}$,

and superpotential

$$\begin{aligned} W &= \sum_{h=1}^{k_1} \sigma_h \left( \sum_{a,\alpha} \rho^h_{a\alpha} Y_a^\alpha + \sum_{\mu,\nu} \alpha^h_{\mu\nu} Z_{\mu\nu} - t \right) \\ &\quad + \sum_{h'=1}^{k_2} \tilde{\sigma}_{h'} \left( \sum_{a,\beta} \rho^{h'}_{a\beta} Y_a^\beta + \sum_{i=1}^{n} \rho^{h'}_{i\alpha} \tilde{Y}_{i\alpha} + \sum_{\mu',\nu'} \alpha^{h'}_{\mu'\nu'} \tilde{Z}_{\mu'\nu'} - \tilde{t} \right) \\ &\quad + \sum_{a,\alpha} \exp\left(-Y_a^\alpha\right) + \sum_{i,\alpha} \exp\left(-\tilde{Y}_{i\alpha}\right) + \sum_{\mu,\nu} X_{\mu\nu} + \sum_{\mu',\nu'} \tilde{X}_{\mu',\nu'}. \end{aligned} \quad (3.1.101)$$

In the expression above,

$$\rho^h_{a\alpha} = \delta^h_a, \quad \rho^{h'}_{a\alpha} = -\delta^{h'}_\alpha, \quad \rho^{h'}_{i\alpha} = \delta^{h'}_\alpha, \quad \alpha^h_{\mu\nu} = -\delta^h_\mu + \delta^h_\nu, \quad \alpha^{h'}_{\mu'\nu'} = -\delta^{h'}_{\mu'} + \delta^{h'}_{\nu'}, \quad (3.1.102)$$

so we can rewrite the superpotential as

$$\begin{aligned} W &= \sum_{h=1}^{k_1} \sigma_h \left( \sum_{\alpha} Y_h^\alpha - \sum_{\nu \neq h} (Z_{h\nu} - Z_{\nu h}) - t \right) \\ &\quad + \sum_{h'=1}^{k_2} \tilde{\sigma}_{h'} \left( -\sum_{a} Y_a^{h'} + \sum_{i} \tilde{Y}_{ih'} - \sum_{\nu' \neq h'} (\tilde{Z}_{h'\nu'} - \tilde{Z}_{\nu' h'}) - \tilde{t} \right) \\ &\quad + \sum_{a,\alpha} \exp\left(-Y_a^\alpha\right) + \sum_{i,\alpha} \exp\left(-\tilde{Y}_{i\alpha}\right) + \sum_{\mu,\nu} X_{\mu\nu} + \sum_{\mu',\nu'} \tilde{X}_{\mu'\nu'}. \end{aligned} \quad (3.1.103)$$

For reasons previously discussed, we focus on the untwisted sector of the Weyl group orbifold. Integrating out $\sigma_h$, $\tilde{\sigma}_{h'}$, we get the constraints

$$\begin{aligned}\sum_{\alpha} Y_h^{\alpha} - \sum_{\nu\neq h}(Z_{h\nu} - Z_{\nu h}) &= t, &(3.1.104)\\ -\sum_a Y_a^{h'} + \sum_i \tilde{Y}_{ih'} - \sum_{\nu'\neq h'}\left(\tilde{Z}_{h'\nu'} - \tilde{Z}_{\nu' h'}\right) &= \tilde{t}, &(3.1.105)\end{aligned}$$

which we can solve as

$$\begin{aligned}Y_h^{k_2} &= -\sum_{\alpha=1}^{k_2-1} Y_h^{\alpha} + \sum_{\nu\neq h}(Z_{h\nu} - Z_{\nu h}) + t, &(3.1.106)\\ \tilde{Y}_{nk_2} &= -\sum_{i=1}^{n-1}\tilde{Y}_{ik_2} + \sum_{\nu'\neq k_2}\left(\tilde{Z}_{k_2\nu'} - \tilde{Z}_{\nu' k_2}\right) + \tilde{t}\\ &\quad + \sum_{a=1}^{k_1}\left[-\sum_{\alpha=1}^{k_2-1} Y_a^{\alpha} + \sum_{\nu\neq a}(Z_{a\nu} - Z_{\nu a}) + t\right] &(3.1.107)\end{aligned}$$

and for $h' < k_2$,

$$\tilde{Y}_{nh'} = -\sum_{i=1}^{n-1}\tilde{Y}_{ih'} + \sum_{a=1}^{k_1} Y_a^{h'} + \sum_{\nu'\neq h'}\left(\tilde{Z}_{h'\nu'} - \tilde{Z}_{\nu' h'}\right) + \tilde{t}. \qquad (3.1.108)$$

For later use, define

$$\begin{aligned}\Pi_a &= \exp\left(-Y_a^{k_2}\right), &(3.1.109)\\ &= q\left(\prod_{\alpha=1}^{k_2-1}\exp\left(+Y_a^{\alpha}\right)\right)\left(\prod_{\nu\neq a}\frac{X_{a\nu}}{X_{\nu a}}\right), &(3.1.110)\\ \Gamma_{\alpha} &= \exp\left(-\tilde{Y}_{n\alpha}\right) \text{ for } \alpha < k_2, &(3.1.111)\\ &= \tilde{q}\left(\prod_{i=1}^{n-1}\exp\left(+\tilde{Y}_{i\alpha}\right)\right)\left(\prod_{a=1}^{k_1}\exp\left(-Y_a^{\alpha}\right)\right)\left(\prod_{\nu'\neq\alpha}\frac{\tilde{X}_{\alpha\nu'}}{\tilde{X}_{\nu'\alpha}}\right), &(3.1.112)\\ T &= \exp\left(-\tilde{Y}_{nk_2}\right), &(3.1.113)\\ &= \tilde{q}q^{k_1}\left(\prod_{i=1}^{n-1}\exp\left(+\tilde{Y}_{ik_2}\right)\right)\left(\prod_{\nu'\neq k_2}\frac{\tilde{X}_{k_2\nu'}}{\tilde{X}_{\nu' k_2}}\right)\cdot\\ &\quad\cdot\left(\prod_{a=1}^{k_1}\prod_{\alpha=1}^{k_2-1}\exp\left(+Y_a^{\alpha}\right)\right)\left(\prod_{a=1}^{k_1}\prod_{\nu\neq a}\frac{X_{\nu a}}{X_{a\nu}}\right), &(3.1.114)\end{aligned}$$

for $q = \exp(-t)$, $\tilde{q} = \exp(-\tilde{t})$.

The superpotential then becomes

$$\begin{aligned} W &= \sum_{a=1}^{k_1} \sum_{\alpha=1}^{k_2-1} \exp(-Y_a^\alpha) + \sum_{i=1}^{n-1} \sum_{\alpha=1}^{k_2} \exp\left(-\tilde{Y}_{i\alpha}\right) + \sum_{\mu,\nu} X_{\mu\nu} + \sum_{\mu',\nu'} \tilde{X}_{\mu'\nu'} \\ & \quad + \sum_{a=1}^{k_1} \Pi_a + \sum_{\alpha=1}^{k_2-1} \Gamma_\alpha + T. \end{aligned} \tag{3.1.115}$$

We compute the critical locus as follows:

$$\begin{aligned} \frac{\partial W}{\partial Y_a^\alpha}: &\quad \exp\left(-Y_a^\alpha\right) = \Pi_a - \Gamma_\alpha + T, \\ \frac{\partial W}{\partial \tilde{Y}_{i\alpha}}: &\quad \exp\left(-\tilde{Y}_{i\alpha}\right) = \begin{cases} \Gamma_\alpha, & \alpha \neq k_2, \\ T, & \alpha = k_2, \end{cases} \\ \frac{\partial W}{\partial X_{ab}}: &\quad X_{ab} = -\Pi_a + \Pi_b, \\ \frac{\partial W}{\partial \tilde{X}_{\alpha\beta}}: &\quad \tilde{X}_{\alpha\beta} = \begin{cases} -\Gamma_\alpha + \Gamma_\beta, & \alpha \neq k_2, \beta \neq k_2, \\ +\Gamma_\beta - T, & \alpha = k_2, \beta \neq k_2, \\ -\Gamma_\alpha + T, & \alpha \neq k_2, \beta = k_2. \end{cases} \end{aligned}$$

As discussed earlier in section 3.1.3, we must require that the $X_{ab}$ and $\tilde{X}_{\alpha\beta}$ be nonzero. Also using the fact that $\exp(-Y) \neq 0$, we have

$$\Pi_a \neq \Gamma_\alpha - T, \tag{3.1.116}$$

$$\Gamma_\alpha \neq 0, \tag{3.1.117}$$

$$T \neq 0, \tag{3.1.118}$$

$$\Pi_a \neq \Pi_b \text{ for } a \neq b, \tag{3.1.119}$$

$$\Gamma_\alpha \neq \Gamma_\beta \text{ for } \alpha \neq \beta, \tag{3.1.120}$$

$$\Gamma_\alpha \neq T. \tag{3.1.121}$$

These guarantee that the critical locus does not intersect the fixed point locus of the Weyl orbifold.

On the critical locus, from the definitions we then find

$$\Pi_a = q(-)^{k_1-1} \left( \prod_{\alpha=1}^{k_2-1} \frac{1}{\Pi_a - \Gamma_\alpha + T} \right), \tag{3.1.122}$$

$$(\Gamma_\alpha)^n = \tilde{q}(-)^{k_2-1} \left( \prod_{a=1}^{k_1} (\Pi_a - \Gamma_\alpha + T) \right), \tag{3.1.123}$$

$$T^n = \tilde{q} q^{k_1} (-)^{k_2-1+k_1(k_1-1)} \left( \prod_{a=1}^{k_1} \prod_{\alpha=1}^{k_2-1} \frac{1}{\Pi_a - \Gamma_\alpha + T} \right), \tag{3.1.124}$$

$$= \tilde{q}(-)^{k_2-1} \left( \prod_{a=1}^{k_1} \Pi_a \right), \tag{3.1.125}$$

where the simplification in (3.1.125) was derived using (3.1.122).

It is useful to compare to the operator mirror map. From equations (3.1.2), (3.1.3), we expect that the A and B model variables should be related as

$$\exp\left(-Y_a^\alpha\right) = \sigma_a - \tilde{\sigma}_\alpha, \tag{3.1.126}$$

$$\exp\left(-\tilde{Y}_{i\alpha}\right) = \tilde{\sigma}_\alpha, \tag{3.1.127}$$

$$X_{\mu\nu} = -\sigma_\mu + \sigma_\nu, \tag{3.1.128}$$

$$\tilde{X}_{\mu'\nu'} = -\tilde{\sigma}_{\mu'} + \tilde{\sigma}_{\nu'}. \tag{3.1.129}$$

These relations are consistent with the identities derived for the critical locus above if we identify

$$\sigma_a = \Pi_a + T, \tag{3.1.130}$$

$$\tilde{\sigma}_\alpha = \begin{cases} \Gamma_\alpha & \alpha < k_2, \\ T & \alpha = k_2. \end{cases} \tag{3.1.131}$$

Furthermore, applying the critical locus results (3.1.122), (3.1.123), (3.1.125), we see that

$$\prod_{\alpha=1}^{k_2} (\sigma_a - \tilde{\sigma}_\alpha) = \Pi_a \prod_{\alpha=1}^{k_2-1} (\Pi_\alpha - \Gamma_\alpha + T), \tag{3.1.132}$$

$$= (-)^{k_1-1} q, \tag{3.1.133}$$

$$(\tilde{\sigma}_\alpha)^n = (\Gamma_\alpha)^n \text{ for } \alpha < k_2, \tag{3.1.134}$$

$$= (-)^{k_2-1} \tilde{q} \left( \prod_{a=1}^{k_1} (\Pi_a - \Gamma_\alpha + T) \right) = (-)^{k_2-1} \tilde{q} \prod_{a=1}^{k_1} (\sigma_a - \tilde{\sigma}_\alpha), \tag{3.1.135}$$

$$(\tilde{\sigma}_{k_2})^n = T^n, \tag{3.1.136}$$

$$= (-)^{k_2-1} \tilde{q} \left( \prod_{a=1}^{k_1} \Pi_a \right) = (-)^{k_2-1} \tilde{q} \prod_{a=1}^{k_1} (\sigma_a - \tilde{\sigma}_{k_2}). \tag{3.1.137}$$

Now, let us compare to the A model. The one-loop effective action for $F(k_1, k_2, n)$ on the Coulomb branch was computed in [40][section 5.2], where in the notation of that reference, it was shown that

$$\prod_{\alpha=1}^{k_2} (\Sigma_{1a} - \Sigma_{2\alpha}) \quad = \quad q_1 \text{ for each } a, \tag{3.1.138}$$

$$(\Sigma_{2\alpha})^n \quad = \quad q_2 \prod_{a=1}^{k_1} (\Sigma_{1a} - \Sigma_{2\alpha}) \,. \tag{3.1.139}$$

If we identify $\sigma_a = \Sigma_{1a}$, $\tilde{\sigma}_\alpha = \Sigma_{2\alpha}$, $(-)^{k_1-1}q = q_1$, $(-)^{k_2-1}\tilde{q} = q_2$, then we see that the algebraic equations for the proposed B model mirror match the Coulomb branch relations derived from the A-twisted GLSM, including the Weyl group $S_{k_1} \times S_{k_2}$ orbifold group action which appears both here in the Landau-Ginzburg model and also on the Coulomb branch of the A-twisted GLSM. Since the critical loci here match the critical loci of the one-loop twisted effective superpotential of the original A-twisted GLSM for the flag manifold, the number of vacua of the proposed B model mirror necessarily match those of the A model, and the quantum cohomology ring of the A model matches the relations in the proposed B model mirror.

Let us conclude with a comment on dualities. Flag manifolds have a duality analogous to the duality $G(k, n) \cong G(n - k, n)$ of Grassmannians [40][section 2.4]:

$$F(k_1, k_2, n) \;\cong\; F(n - k_2, n - k_1, n). \tag{3.1.140}$$

In principle, we expect this duality to be realized in the same fashion as the symmetry $G(k, n) \cong G(n - k, n)$, namely as an IR relation between two mirror Landau-Ginzburg orbifolds. For example, from the analysis above for each of the two cases, the two mirrors are guaranteed to have the same Coulomb branch relations and the same number of vacua. One can find more mirror examples for different compact Lie groups in [26, 124].

## 3.2 A Proposal for (0,2) Fano Mirrors

The contents of this section were adapted, with minor modifications, with permission from JHEP, from our publication [27]. In [27], we proposed the mirrors for (0,2) Fanos. Based on the observations in section 2.3.6, we restrict to (0,2) theories obtained by (some) toric deformations of abelian (2,2) GLSMs for Fano spaces, by which we mean physically that we choose $E$'s such that $E_i \propto \phi_i$, where on the (2,2) locus $\phi_i$ is the chiral superfield paired with the Fermi superfield whose superderivative is $E_i$.

In addition, to define a mirror, we also make another choice, namely we pick an invertible[10] $k \times k$ submatrix, of the charge matrix $(Q_i^a)$, which we will denote $S$. The choice of $S$ will

[10] We assume that the charge matrix does indeed have an invertible $k \times k$ submatrix. If not, then the

further constrain the allowed toric deformations – for a given $S$, we only consider some toric deformations. Our mirror will depend upon the choice of $S$, and since different $S$'s will yield different allowed bundle deformations, there need not be a simple coordinate transformation relating results for different choices of $S$ in general. Furthermore, $S$ is only relevant for bundle deformations – it does not enter (2,2) locus computations, and so it has no analogue within [25].

For a given choice of $S$, in the A/2 model, write

$$E_i \;=\; \sum_{a=1}^{k}\sum_{j=1}^{N}\left(\delta_{ij}+B_{ij}\right)Q_j^a\sigma_a\phi_i,$$

where in the expression above, we do not sum over $i$'s. The (0,2) deformations we will consider are encoded in the matrices $B_{ij}$, where $B_{ij}=0$ if $i$ defines a column of the matrix $S$. Note that, at least on its face, this does not describe all possible Euler-sequence-type (0,2) deformations, but only a special subset. We will give a mirror construction for that special subset.

Then, the mirror can be described by a collection of $\mathbb{C}^\times$-valued fields $Y_i$ (just as on the (2,2) locus, dual to the chiral superfields of the original theory), satisfying the same D-term constraints as on the (2,2) locus, and with (0,2) superpotential

$$\begin{aligned} W \;=\; & \sum_{a=1}^{k}\left[\Upsilon_a\left(\sum_{i=1}^{N}Q_i^aY_i-t_a\right)\;+\;\sum_{i=1}^{N}\Sigma_aQ_i^aF_i\right] \\ & -\mu\sum_i F_i\exp(-Y_i)\;+\;\mu\sum_i F_i\left(\sum_{i_S,j,a}B_{ij}Q_j^a[(S^{-1})^T]_{ai_S}\exp(-Y_{i_S})\right), \end{aligned} \tag{3.2.1}$$

where $i_S$ denotes an index running through the columns of $S$, and where the second term was chosen so that the resulting equations of motion duplicate the chiral ring. (For the moment, we have assumed no twisted masses are present; we will return to twisted masses at the end of this section.)

Now, to do meaningful computations, we must apply the D-term constraints to both $Y_i$'s and $F_i$'s. Applying the D-term constraints to the $F_i$'s to write them in terms of $G_A$'s (*i.e.* integrating out $\Sigma_a$'s), and for simplicity suppressing the $\Upsilon_a$ constraints and setting the mass scale $\mu$ to unity, we have the expression

$$W \;=\; -\sum_{A=1}^{N-k}G_A\left(\sum_i V_i^A\exp(-Y_i)\;+\;\sum_{i_S}D_{i_S}^A\exp(-Y_{i_S})\right), \tag{3.2.2}$$

where

$$D_{i_S}^A \;=\; -\sum_{i,j}\sum_a V_i^AB_{ij}Q_j^a[(S^{-1})^T]_{ai_S}. \tag{3.2.3}$$

theory has at least one free decoupled $U(1)$, and after performing a change of basis to explicitly decouple those $U(1)$'s, our analysis can proceed on the remainder.

Note when $B = 0$, $D = 0$, and the expression for $W$ above immediately reduces to its (2,2) locus form. We will derive this expression for $D$ below.

In this language, the mirror map between A/2- and B/2-model observables is defined by

$$\sum_{a=1}^{k} \sum_{j=1}^{N} (\delta_{ij} + B_{ij})\, Q_j^a \sigma_a \;\leftrightarrow\; \exp(-Y_i) \;=\; e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A). \tag{3.2.4}$$

(Strictly speaking, we will see in examples that these equations define not only the operator mirror map plus some of the chiral ring relations.)

We can derive the operator mirror map above from the superpotential (3.2.1) by taking a derivative with respect to $F_i$, as before. Doing so, one finds

$$Q_i^a \sigma_a - \exp(-Y_i) + \sum_{i_S, j, a} B_{ij} Q_j^a [(S^{-1})^T]_{a i_S} \exp(-Y_{i_S}) \;=\; 0.$$

For $i$ corresponding to columns of $S$, $B_{ij} = 0$, and the expression above simplifies to

$$S_{i_S}^a \sigma_a \;=\; \exp(-Y_{i_S}).$$

Plugging this back in, we find

$$Q_i^a \sigma_a - \exp(-Y_i) + \sum_{j,a} B_{ij} Q_j^a \sigma_a \;=\; 0,$$

which is easily seen to be the operator mirror map (3.2.4).

We can apply the operator mirror map as follows. Recall that the constraints imply

$$\sum_i Q_i^a Y_i \;=\; t_a$$

hence

$$\prod_i \exp(-Q_i^a Y_i) \;=\; \exp(-t_a) \;=\; q_a,$$

hence plugging in the proposed map (3.2.4) above, we have

$$\prod_i \left( \sum_{a=1}^{k} \sum_{j=1}^{N} (\delta_{ij} + B_{ij}) Q_j^a \sigma_a \right)^{Q_i^a} \;=\; q_a,$$

which is the chiral ring relation in the A/2-twisted GLSM.

In passing, to make the method above work, it is important that the determinants appearing in quantum sheaf cohomology relations in *e.g.* [42, 68–70] all factorize. In other words, recall that for a general tangent bundle deformation, the quantum sheaf cohomology ring relations take the form

$$\prod_\alpha (\det M_\alpha)^{Q^a_\alpha} \;=\; q_a,$$

where $\alpha$ denotes a block of chiral fields with the same charges, and $M_\alpha$ encodes the $E$'s, which will mix chiral superfields of the same charges. In order for the operator mirror map construction we have outlined above to work, it is necessary that each $\det M_\alpha$ factorize into a product of factors, one for each matter chiral multiplet. This is ultimately the reason why in this paper we have chosen to focus on 'toric' deformations, in which each $E$'s do not mix different matter chiral multiplets.

Now, in terms of the operator mirror map, let us derive the form of $D$ above in equation (3.2.3). The equations of motion from the superpotential (3.2.2) are given by

$$\frac{\partial W}{\partial G_A} \;=\; \sum_i V_i^A \exp(-Y_i) \;+\; \sum_{i_S} D^A_{i_S} \exp(-Y_{i_S}) \;=\; 0.$$

Now, we plug in the operator mirror map (3.2.4) above to get

$$\sum_i V_i^A \left( \sum_a \sum_j (\delta_{ij} + B_{ij}) Q^a_j \sigma_a \right) \;+\; \sum_{i_S} D^A_{i_S} \left( \sum_a \sum_j (\delta_{i_S j} + B_{i_S j}) Q^a_j \sigma_a \right) \;=\; 0.$$

Using the constraint

$$\sum_i V_i^A Q_i^a \;=\; 0,$$

the first $\delta_{ij}$ term vanishes, and furthermore, since the matrix $B$ is defined to vanish for indices from columns of $S$, we see that in the second term, $B_{i_S j} = 0$, hence the equation above reduces to

$$\sum_{i,j} \sum_a V_i^A B_{ij} Q^a_j \sigma_a \;+\; \sum_{i_S} \sum_a D^A_{i_S} S^a_{i_S} \sigma_a \;=\; 0.$$

Since this should hold for all $\sigma_a$, we have that

$$\sum_{i,j} V_i^A B_{ij} Q^a_j \;+\; \sum_{i_S} D^A_{i_S} S^a_{i_S} \;=\; 0,$$

which can be solved to give expression (3.2.3) for $D$ above.

Thus, the expression for the superpotential (3.2.2) together with the operator mirror map (3.2.4) has equations of motion that duplicate the chiral ring.

In passing, one could also formally try to consider more general cases in which a submatrix $S \subset Q$ is not specified. One might then try to take the expression for the mirror superpotential to be of the form

$$W \;=\; -\sum_{A=1}^{N-k} G_A \left( \sum_i V_i^A \exp(-Y_i) + \sum_i D_i^A \exp(-Y_i) \right),$$

where now the $i$ index on $D$ is allowed to run over all chiral superfields, not just a subset. Following the methods above, one cannot uniquely solve for $D$ – one gets families of possible $D$'s with undetermined coefficients, and we do not know how to argue that the correlation functions match for all such coefficients without restricting to subsets defined by choices $S \subset Q$.

Now, in principle, for (0,2) theories defined by deformations of the (2,2) locus, there is an analogue of twisted masses that one can add to the theory. In the (2,2) case, twisted masses corresponded to replacing a vector multiplet by its vevs, so that only a residue of $\sigma$ survived. In (0,2), by contrast, the vector multiplet does not contain $\sigma$, only the gauge field, gauginos, and auxiliary fields $D$, so we can no longer interpret the twisted mass in terms of replacing a vector multiplet with its vevs.

Instead, we can understand the analogue of a twisted mass in a (0,2) theory corresponding to a deformation of the (2,2) locus in terms of additions to $E_i = \overline{D}_+ \Lambda^i$, for Fermi superfields $\Lambda^i$. In particular, the (2,2) vector multiplet's $\sigma$ field enters GLSMs written in (0,2) superfields as a factor in such $E$'s, so twisted masses enter similarly, as terms of the form

$$E_i \;=\; \tilde{m}_i \phi_i$$

(where as usual we are admitting the possibilty of several toric symmetries, and simply giving each chiral superfield the possibility of its own twisted mass). Such terms are only possible if the (0,2) superpotential has compatible $J$'s, meaning that in order for supersymmetry to hold, one requires $E \cdot J = 0$, as usual. This is a residue of the requirement in the (2,2) theory that twisted masses arise from flavor symmetries.

We have already seen, in section 1.2.3, how (2,2) twisted masses can be represented in the mirror, described in (0,2) superspace. To describe their combination with $E$ deformations is straightforward. Briefly, the (0,2) mirror superpotential takes the form

$$W \;=\; -\sum_{A=1}^{N-k} G_A \left( \sum_i V_i^A \exp(-Y_i) \;+\; \sum_{i_S} D_{i_S}^A \exp(-Y_{i_S}) \right) \;+\; \sum_{i=1}^{N} \sum_{A=1}^{N-k} G_A V_i^A \tilde{m}_i, \tag{3.2.5}$$

with $(D_{i_S}^A)$ defined as in (3.2.3), and the operator mirror map has the form

$$\sum_{a=1}^{k} \sum_{j=1}^{N} (\delta_{ij} + B_{ij})\, Q_j^a \sigma_a + \tilde{m}_i \;\leftrightarrow\; \exp(-Y_i) \;=\; e^{\tilde{t}_i} \prod_{A=1}^{N-k} \exp(-V_i^A \theta_A). \tag{3.2.6}$$

### 3.2.1 Correlation Functions

In this section, we will argue formally that correlation functions in our proposed (0,2) mirrors match those of the original theory. More precisely, we will compare closed-string correlation functions of A- or A/2-twisted GLSM $\sigma$'s to corresponding correlation functions in B- or B/2-twisted Landau-Ginzburg models. (Often, the Landau-Ginzburg mirror will be an orbifold; we will only compare against untwisted sector correlation functions in such orbifolds.) Our computations will focus on genus zero computations, but in (2,2) cases, in principle can be generalized to any genus.

Before doing so, let us first outline in what sense correlation functions match. There are two possibilities:

- First, for special matrices $(V_i^A)$, we will argue that correlation functions match on the nose. In order for this to happen, we will need to require that the determinant of an invertible $k \times k$ submatrix of the charge matrix $Q$, match (up to sign) the determinant of a complementary[11] $(N-k) \times (N-k)$ submatrix of $(V_i^A)$.
- Alternatively, we can always formally rescale some of the $Y_i$s (without introducing or removing orbifolds) to arrange for the determinants above to match, up to sign. In this case, the correlation functions of one theory are isomorphic to those of the other theory, but the numerical factors will not match on the nose. (Instead, the relations between numerical factors will be determined by the rescaling of the $Y_i$s.)

In either event, correlation functions will match.

### (2,2) supersymmetric cases

We will first check that on the (2,2) locus, the ansatz described above (*i.e.* the ansatz of [25]) generates matching correlation functions between the A-twisted GLSM and its B-twisted Landau-Ginzburg model mirror. (See also [83] for an analogous comparison of partition functions.)

First, let us consider correlation functions in an A-twisted GLSM. An exact expression is given for fully massive cases in *e.g.* [41][equ'n (4.77)]:

$$\langle \mathcal{O} \rangle \;=\; \frac{(-)^{N_c}}{|W|} \frac{1}{(-2\pi i)^{\mathrm{rk}\, G}} \sum_{\sigma_P} \mathcal{O} \, \frac{Z_{1-\mathrm{loop}}}{H}$$

where $G$ is the GLSM gauge group, $W$ its Weyl group, $N_c$ its rank,

$$Z_{1-\mathrm{loop}} \;=\; \prod_{i=1}^{N} \left( \sum_{a=1}^{k} Q_i^a \sigma_a \right)^{R(\Phi_i)-1} ,$$

[11] 'Complementary' in this case means that if the $k \times k$ matrix is defined by $i$'s corresponding to certain chiral superfields, then those same chiral superfields cannot appear corresponding to any $i$'s in the $(N-k) \times (N-k)$ submatrix of $(V_i^A)$.

($R(\Phi_i)$ the R-charge, which for simplicity we will assume to vanish,) $\sigma_P$ the vacua, and $H$ is the Hessian of the twisted one-loop effective action, meaning

$$H \;=\; \det\left( \sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right), \tag{3.2.7}$$

using (up to factors) the twisted one-loop effective action in *e.g.* [28][equ'n (3.36)].

Now, up to irrelevant overall factors, there is an essentially identical expression for Landau-Ginzburg correlation functions [78], involving the Hessian of the superpotential rather than $H/Z_{1-\mathrm{loop}}$ above. Therefore, to show that correlation functions match, we will argue that the $H/Z_{1-\mathrm{loop}}$ above, computed for the A-twisted GLSM, matches the Hessian of superpotential derivatives for the mirror Landau-Ginzburg model.

First, since we are only interested in the determinant, we can rotate the charge matrix $(Q_i^a)$ by an element $U \in SL(k,\mathbb{C})$ without changing the determinant:

$$\det\left( \sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right) \mapsto (\det U)^2 \det\left( \sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right) \;=\; \det\left( \sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c} \right).$$

Thus, it will be convenient to rotate the charge matrix to the form[12]

$$Q_i^a = \left( \begin{array}{ccccc} a_1 & & & Q_{k+1}^1 & \cdots & Q_N^1 \\ & \ddots & & \vdots & \ddots & \vdots \\ & & a_k & Q_{k+1}^k & \cdots & Q_N^k \end{array} \right). \tag{3.2.8}$$

Note that for the charge matrix in this form,

$$Z_{1-\mathrm{loop}} \;=\; \left( \prod_{i=1}^k a_i \sigma_i \right)^{-1} \left( \prod_{i=k+1}^N \left( \sum_{a=1}^k Q_i^a \sigma_a \right) \right)^{-1}.$$

To put the charge matrix in this form, write

$$Q \;=\; [S|*] \;=\; (\det S)^{1/k} [S'|*],$$

where $S$ is $k \times k$. Then, multiply in $(S')^{-1}$, to get

$$(S')^{-1} Q \;=\; (\det S)^{1/k} [I|*],$$

which is now diagonal.

[12] As we are not conjugating the charge matrix, but rather multiplying on one side only, it should be possible to arrange for a $k \times k$ submatrix to be diagonal, not just in Jordan normal form.

It is straightforward to compute

$$H = \det\left(\sum_i \frac{Q_i^a Q_i^b}{\sum_c Q_i^c \sigma_c}\right) =$$

$$\det\begin{pmatrix} \frac{a_1}{\sigma_1} + \frac{(Q_{k+1}^1)^2}{Q_{k+1}^c\sigma_c} + \ldots + \frac{(Q_N^1)^2}{Q_N^c\sigma_c} & \frac{Q_{k+1}^1 Q_{k+1}^2}{Q_{k+1}^c\sigma_c} + \ldots + \frac{Q_N^1 Q_N^2}{Q_N^c\sigma_c} & \cdots \\ \frac{Q_{k+1}^2 Q_{k+1}^1}{Q_{k+1}^c\sigma_c} + \ldots \frac{Q_N^2 Q_N^1}{Q_N^c\sigma_c} & \frac{a_2}{\sigma_2} + \frac{(Q_{k+1}^2)^2}{Q_{k+1}^c\sigma_c} + \ldots + \frac{(Q_N^2)^2}{Q_N^c\sigma_c} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \qquad (3.2.9)$$

We define[13] $t_i = a_i/\sigma_i$ and $E_i^a = Q_i^a/\sqrt{\sum_c Q_i^c \sigma_c}$, then the matrix in the above determinant becomes

$$\begin{pmatrix} t_1 + (E_{k+1}^1)^2 + \ldots + (E_N^1)^2 & E_{k+1}^1 E_{k+1}^2 + \ldots + E_N^1 E_N^2 & \cdots \\ E_{k+1}^2 E_{k+1}^1 + \ldots + E_N^2 E_N^1 & t_2 + (E_{k+1}^2)^2 + \ldots + (E_N^2)^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \qquad (3.2.10)$$

When all the $t_i$ vanish, one can straightforwardly see that the matrix above is the product $(E^T)^T E^T$, for matrix $E$

$$E = \begin{pmatrix} E_{k+1}^1 & E_{k+2}^1 & \cdots & E_N^1 \\ E_{k+1}^2 & E_{k+2}^2 & \cdots & E_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ E_{k+1}^k & \cdots & \cdots & E_N^k \end{pmatrix}. \qquad (3.2.11)$$

Using standard results from linear algebra, the generalized characteristic polynomial of matrix (3.2.10), in terms of the variables $t_i$, is given by

$$\sum_{m=0}^{k}\left(\sum_{a_1<\cdots<a_m} t_{a_1}\cdots t_{a_m} \det\left(M_{a_1\cdots a_m}\right)\right), \qquad (3.2.12)$$

where the matrix $M_{a_1\cdots a_m}$ denotes the submatrix of $M = (E^T)^T E^T$ by omitting rows $a_1 \cdots a_m$ and columns $a'_1 \cdots a'_m$ (*i.e.* a principal minor of $M$ of size $k-m$). (In our conventions, the determinant vanishes if $M$ has no entries.) Notice that $M = (E^T)^T E^T$, so the determinant can be written more simply as

$$\det M \;+\; t_1\cdots t_k + \qquad (3.2.13)$$
$$\sum_{m=1}^{k-1}\left(\sum_{a_1<\cdots<a_m} t_{a_1}\cdots t_{a_m}\left(\sum_{i_1<\cdots<i_{N+m-2k}} \left(\det E_{a_1\cdots a_m, i_1\cdots i_{N+m-2k}}\right)^2\right)\right),$$

where $\det E_{a_1\cdots a_m, i_i\cdots i_{N+m-2k}}$ denotes the determinant of the submatrix of $E$ formed by omitting rows $a_1 \cdots a_m$ and columns $i_1 \cdots i_{N+m-2k}$. (We formally require it to be zero when $N + m - 2k < 0$.)

[13] The reader should note that the $t_i$ in this section, defined above, is not related to $t$'s used earlier to describe FI parameters.

Finally, we divide by $Z_{1-\text{loop}}$ to get an expression for $H/Z_{1-\text{loop}}$ where $H$ is the Hessian (3.2.7):

$$\frac{\det M \ + \ t_1 \cdots t_k}{Z_{1-\text{loop}}} + \tag{3.2.14}$$
$$\sum_{m=1}^{k-1} \left( \sum_{a_1 < \cdots < a_m} (a_{a_1})^2 \cdots (a_{a_m})^2 \left( \prod_{i \notin \{a_1, \cdots, a_m\}} \left( \sum_{a=1}^{k} Q_i^a \sigma_a \right) \right) B_{a_1 \cdots a_m} \right),$$

for

$$B_{a_1 \cdots a_m} \quad = \sum_{i_1 < \cdots < i_{N+m-2k}} \left( \det E_{a_1 \cdots a_m, i_1 \cdots i_{N+m-2k}} \right)^2,$$

where $\det M$ vanishes for $N < 2k$. For later use, note that for $N \leq 2k$ we can expand

$$\frac{\det M}{Z_{1-\text{loop}}} \ = \ \left( \prod_{i=1}^{k} a_i \sigma_i \right) \left( \prod_{i \notin \{i_1, \cdots, i_k\}} \left( \sum_{c=1}^{k} Q_{k+i}^c \sigma_c \right) (A_{i_1, \cdots, i_k})^2 \right), \tag{3.2.15}$$

and the terms for $1 \leq m \leq k-1$ are given by

$$a_{a_1}^2 \cdots a_{a_m}^2 \left( \prod_{b \notin \{a_1, \cdots, a_m\}} a_b \sigma_b \right) \left( \prod_{i \notin \{i_{m+1}, \cdots, i_k\}} \left( \sum_{c=1}^{k} Q_{k+i}^c \sigma_c \right) \left( A_{i_{m+1}, \cdots, i_k} \right)^2 \right), \tag{3.2.16}$$

where $A_{i_{m+1}, \cdots, i_k}$ denotes the sum of determinants of all $(k-m) \times (k-m)$ submatrices of the charge matrix $(Q_i^a)$ for values of $i > k$.

Next, we need to compare the ratio $H/Z_{1-\text{loop}}$ above to the analogous Hessian arising in the mirror B-twisted Landau-Ginzburg model. Here, there is a nearly identical computation in which the Hessian we just computed is replaced with the determinant of second derivatives of the mirror superpotential (1.2.6):

$$\begin{aligned} \frac{\partial^2 W}{\partial \theta_A \partial \theta_B} \quad &= \quad - \sum_{i=1}^{N} \left( e^{\tilde{t}_i} \left( \prod_{C=1}^{N-k} \exp(-V_i^C \theta_C) \right) V_i^A V_i^B \right), \\ &= \quad - \sum_{i=1}^{N} \left( \left( \sum_{a=1}^{k} Q_i^a \sigma_a \right) V_i^A V_i^B \right), \end{aligned}$$

using the operator mirror map (1.2.7).

Thus, we need to compute

$$\det \left( \sum_i V_i^A V_i^B \sum_c Q_i^c \sigma_c \right),$$

and compare to the ratio $H/Z_{1-\text{loop}}$ from the A model that we computed previously. In principle, the argument here is very similar to the argument just given for the determinant defined by charge matrices. First, using the fact that $V$ has rank $N-k$, inside the determinant we can rotate $V$ to the more convenient form

$$V_i^A = \begin{pmatrix} V_1^1 & \cdots & V_k^1 & \lambda^1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ V_1^k & \cdots & V_k^k & & & \lambda^{(N-k)} \end{pmatrix}. \tag{3.2.17}$$

In fact, we can say more. Given that the $V$ matrix was originally defined to satisfy

$$\sum_i Q_i^a V_i^A \;=\; 0,$$

after the rotation above inside the determinant, the $V$ matrix should in fact have the form

$$V_i^A = \begin{pmatrix} -\frac{\lambda^1 Q_{k+1}^1}{a_1} & \cdots & -\frac{\lambda^1 Q_{k+1}^k}{a_k} & \lambda^1 & & \\ \vdots & \ddots & \vdots & & \ddots & \\ -\frac{\lambda^{(N-k)} Q_N^1}{a_1} & \cdots & -\frac{\lambda^{(N-k)} Q_N^k}{a_k} & & & \lambda^{(N-k)} \end{pmatrix}. \tag{3.2.18}$$

Then, using the more convenient form of $V$ above, we find that we can write the matrix

$$\left(\sum_i V_i^A V_i^B \sum_c Q_i^c \sigma_c\right) = \tag{3.2.19}$$

$$\begin{pmatrix} (\lambda^1)^2\left[\frac{(Q_{k+1}^1)^2\sigma_1}{a_1} + \cdots + \frac{(Q_{k+1}^k)^2\sigma_k}{a_k} + Q_{k+1}^c\sigma_c\right] & \lambda^1\lambda^2\left[\frac{Q_{k+1}^1 Q_{k+2}^1\sigma_1}{a_1} + \cdots + \frac{Q_{k+1}^k Q_{k+2}^k\sigma_k}{a_k}\right] & \cdots \\ \lambda^2\lambda^1\left[\frac{Q_{k+1}^1 Q_{k+2}^1\sigma_1}{a_1} + \cdots + \frac{Q_{k+1}^k Q_{k+2}^k\sigma_k}{a_k}\right] & (\lambda^2)^2\left[\frac{(Q_{k+2}^1)^2\sigma_1}{a_1} + \cdots + \frac{(Q_{k+2}^k)^2\sigma_k}{a_k} + Q_{k+2}^c\sigma_c\right] & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Similarly, we define $s_i = (\lambda^i)^2 Q_{k+i}^c \sigma_c$ and $F_i^a = \lambda^i Q_{k+i}^a \sqrt{\sigma_a/a_a}$ (without summing over the index $a$). Then, the matrix above can be written as

$$\begin{pmatrix} s_1 + (F_1^1)^2 + \cdots + (F_1^k)^2 & F_1^1 F_2^1 + \cdots + F_1^k F_2^k & \cdots \\ F_2^1 F_1^1 + \cdots + F_2^k F_1^k & s_2 + (F_2^1)^2 + \cdots + (F_2^k)^2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.2.20}$$

When all $s_i$ vanish, one can observe that the matrix is the product $F^T F$, for

$$F = \begin{pmatrix} F_1^1 & F_2^1 & \cdots & F_{N-k}^1 \\ F_1^2 & F_2^2 & \cdots & F_{N-k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ F_1^k & F_2^k & \cdots & F_{N-k}^k \end{pmatrix}. \tag{3.2.21}$$

By using the same technique we can show that the determinant of (3.2.19) vanishes for $N > 2k$, and for $N \leq 2k$ is

$$\det(F^T F) \;+\; s_1 \cdots s_{N-k} + \sum_{n=1}^{N-k-1} \left( \sum_{i_1 < \cdots < i_n} (s_{i_1} s_{i_2} \cdots s_{i_n}) \left( \sum_{a_1 < \cdots < a_{2k-N+n}} \left(\det F_{i_1 \cdots i_n, a_1 \cdots a_{2k-N+n}}\right)^2 \right)\right) \tag{3.2.22}$$

For later use, note that

$$\begin{aligned} \det F^T F \;&=\; \sum_{a_1 < \cdots < a_{2k-N}} (\det F_{a_1 \cdots a_{2k-N}})^2, \\ &=\; \left( \prod_{A=1}^{N-k} (\lambda^A)^2 \right) \left( \prod_{b=1}^{k} \frac{\sigma_b}{a_b} \right) \left( \sum_{i_1 < \cdots < i_k} \left( \sum_{a_1, \cdots, a_k} Q^{a_1}_{k+i_1} \cdots Q^{a_k}_{k+i_k} \epsilon_{a_1 \cdots a_k} \right)^2 \right), \end{aligned}$$

where $F_{a_1 \cdots a_{2k-N}}$ denotes the submatrix of $F^a_i$ formed by deleting columns $a_1$ through $a_{2k-N}$. Next, we plug

$$s_{i_j} = (\lambda^{i_j})^2 Q^c_{k+i_j} \sigma_c$$

into equation (3.2.22), and compare equation (3.2.14). First, note that we can expand

$$\frac{t_1 \cdots t_k}{Z_{1-\text{loop}}} \;=\; \left( \prod_{i=1}^{k} a_i^2 \right) \left( \prod_{j=1}^{N} \left( \sum_{a=1}^{k} Q^a_j \sigma_a \right) \right),$$

which matches

$$s_1 \cdots s_{N-k} \;=\; \prod_{A=1}^{N-k} (\lambda^A)^2 \left( \sum_{a=1}^{k} Q^a_{k+A} \sigma_a \right)$$

so long as

$$\prod_{A=1}^{N-k} \lambda^A = \pm \prod_{i=1}^{k} a_i. \tag{3.2.23}$$

Analogous results hold for other terms, as we now verify. First we consider the case $N \geq 2k$. The term $\det M / Z_{1-\text{loop}}$ in the previous determinant corresponds to the term $n = N - 2k$ in the expansion (3.2.22), which is given by

$$\left( \prod_{A=1}^{N-k} \lambda^A \right)^2 \prod_{a=1}^{k} \frac{\sigma_a}{a_a} \prod_{i \notin \{i_1, \cdots, i_k\}} \left( \sum_{c=1}^{k} Q^c_{k+i} \sigma_c \right) (A_{i_1 \cdots i_k})^2,$$

for $A_{i_1 \cdots i_k}$ defined previously. It is easy to verify that this matches equation (3.2.15) for $\det M / Z_{1-\text{loop}}$ so long as condition (3.2.23) is satisfied, just as before. The remaining terms in expansion (3.2.22) for any given $n$ correspond to terms in (3.2.14) with $m$ related by $n = N - 2k + m$, and have the form

$$\left(\prod_{A=1}^{N-k} \lambda^A\right)^2 \prod_{b \notin \{a_1, \cdots, a_m\}} \frac{\sigma_b}{a_b} \left( \prod_{i \notin \{i_{m+1}, \cdots, i_k\}} \left( \sum_{c=1}^{k} Q^c_{k+i} \sigma_c \right) \left(A_{i_{m+1} \cdots i_k}\right)^2 \right),$$

and it is easy to verify that this matches equation (3.2.16) so long as condition (3.2.23) is satisfied, just as before. The reader can now straightforwardly verify that analogous statements hold for the cases $k < N < 2k$, which exhausts all nontrivial possibilities.

Thus, we see that correlation functions will match so long as equation (3.2.23) holds. Furthermore, we can always arrange for equation (3.2.23) to hold. If it does not do so initially, then as discussed at the start of this section, we can perform field redefinitions and rescale $Y_i$s to arrange for it to hold, at the cost of making the isomorphism between the correlation functions of either theory a shade more complicated. For example, the coefficient of

$$\left( \sum_c Q^c_{k+1} \sigma_c \right) \cdots \left( \sum_d Q^d_N \sigma_d \right)$$

in equation (3.2.14) is $(a_1 a_2 \cdots a_k)^2$, and the coefficient of the term of the same order in equation (3.2.22) is $(\lambda^1 \lambda^2 \cdots \lambda^{N-k})^2$. We see that equation (3.2.23) is required for equality hold, and the choice of sign in equation (3.2.23) should not have any physical significance.

So far, we have worked at genus zero, but the same argument also implies that the same closed-string correlation functions match at arbitrary genus. At genus $g$, A-twisted GLSM correlation functions are computed in the same fashion albeit with a factor of $(H/Z_{1-\text{loop}})^{g-1}$ (see *e.g.* [52][section 4], [53][section 5.1]), whereas B-twisted Landau-Ginzburg model correlation functions (in the untwisted sector) are computed with a factor of $(H')^{g-1}$ [78], for $H'$ the determinant of second derivatives of the mirror superpotential. Demonstrating that $H/Z_{1-\text{loop}} = H'$ therefore not only demonstrates that genus zero correlation functions match, but also higher-genus correlation functions. (For (0,2) theories, by contrast, higher genus correlation functions are not yet understood, so there we will only be able to compare genus zero correlation functions.)

Essentially the same argument applies if one adds twisted masses to the theory. One simply makes the substitution

$$\sum_a Q^a_i \sigma_a \to \sum_a Q^a_i \sigma_a + \widetilde{m}_i, \tag{3.2.24}$$

where $\widetilde{m}_i$ is the twisted mass. The details of the proof above are essentially unchanged. Also note that we are free to redefine $\sigma_a$ to $\sigma_a + c_a$, and we can use this to set the first $k$ twisted masses to zero. This leaves $N - k$ twisted masses, consistent with a global flavor symmetry $U(1)^{N-k}$.

The arguments above hold so long as one can integrate out all of the matter Higgs fields, to obtain a pure Coulomb branch. In the (2,2) theory one expects that one should be able to do this if one adds sufficient twisted masses (see *e.g.* [52][section 2.3]). (In particular, adding twisted masses can act as a substitute for going far out along the Coulomb branch, which also makes the matter fields massive.)

### (0,2) supersymmetric cases

We will now generalize the previous argument to (0,2) cases.

Our argument here will be very similar to that given for (2,2) cases. We will compare results for correlation functions in A/2-twisted GLSMs computed with supersymmetric localization to results for correlation functions computed in B/2-twisted (0,2) Landau-Ginzburg models.

First, as before, applying supersymmetric localization to an A/2-twisted GLSM, there is an exact formula for (genus zero) (0,2) correlation functions [42], which has more or less the same form as in the (2,2) case, now involving a Hessian of derivatives of a twisted one-loop (0,2) effective action [68], which takes the form

$$H \;=\; \det\left(\sum_i \frac{\sum_j Q_i^a A_{ij} Q_j^b}{\sum_m A_{im} Q_m^c \sigma_c}\right), \tag{3.2.25}$$

where $A_{ij} = \delta_{ij} + B_{ij}$.

We assume without loss of generality that the invertible $S$ submatrix of the charge matrix corresponds to the first $k$ columns of $Q$. Then, one can show that the determinant (3.2.25) above is equal to

$$\det\begin{pmatrix} \frac{a_1}{\sigma_1} + \frac{Q_{k+1}^1\left(Q_{k+1}^1+\varepsilon_{k+1}^1\right)}{\left(Q_{k+1}^a+\varepsilon_{k+1}^a\right)\sigma_a} + \ldots + \frac{Q_N^1\left(Q_N^1+\varepsilon_N^1\right)}{\left(Q_N^a+\varepsilon_N^a\right)\sigma_a} & \frac{Q_{k+1}^1\left(Q_{k+1}^2+\varepsilon_{k+1}^2\right)}{\left(Q_{k+1}^a+\varepsilon_{k+1}^a\right)\sigma_a} + \ldots \frac{Q_N^1\left(Q_N^2+\varepsilon_N^2\right)}{\left(Q_N^a+\varepsilon_N^a\right)\sigma_a} & \cdots \\ \frac{Q_{k+1}^2\left(Q_{k+1}^1+\varepsilon_{k+1}^1\right)}{\left(Q_{k+1}^a+\varepsilon_{k+1}^a\right)\sigma_a} + \ldots \frac{Q_N^2\left(Q_N^1+\varepsilon_N^1\right)}{\left(Q_N^a+\varepsilon_N^a\right)\sigma_a} & \frac{a_2}{\sigma_2} + \frac{Q_{k+1}^2\left(Q_{k+1}^2+\varepsilon_{k+1}^2\right)}{\left(Q_{k+1}^a+\varepsilon_{k+1}^a\right)\sigma_a} + \ldots + \frac{Q_N^2\left(Q_N^2+\varepsilon_N^2\right)}{(Q_N^a+\varepsilon_N^a)\sigma_a} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.2.26}$$

where $\varepsilon_i^a = \sum_j B_{ij} Q_j^a$.

Now, in the B/2-twisted (0,2) Landau-Ginzburg model, there is an analogous expression for correlation functions [114], involving the Hessian

$$\det \frac{\partial^2 W}{\partial G_A \partial \theta_B}.$$

One can similarly show that the Hessian above is given by (using the (0,2) operator mirror map (3.2.4))

$$\det \begin{pmatrix} (\lambda^1)^2 \left[ \sum_{b=1}^k \frac{Q^b_{k+1}\left(Q^b_{k+1}+\varepsilon^b_{k+1}\right)\sigma_b}{a_b} + S_{k+1} \right] & \lambda^1\lambda^2 \left[ \sum_{b=1}^k \frac{\left(Q^b_{k+1}+\varepsilon^b_{k+1}\right)Q^b_{k+2}\sigma_b}{a_b} \right] & \cdots \\ \lambda^2\lambda^1 \left[ \sum_{b=1}^k \frac{Q^b_{k+1}\left(Q^b_{k+2}+\varepsilon^b_{k+2}\right)\sigma_b}{a_b} \right] & (\lambda^2)^2 \left[ \sum_{b=1}^k \frac{Q^b_{k+2}\left(Q^b_{k+2}+\varepsilon^b_{k+2}\right)\sigma_b}{a_b} + S_{k+2} \right] & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.2.27}$$

where $S_{k+i} = \sum_a \left(Q^a_{k+i} + \varepsilon^a_{k+i}\right)\sigma_a$.

Finally, following the same steps as for (2,2), one can show that the ratio $H/Z_{1-\text{loop}}$ appearing in the A/2-twisted GLSM matches the Hessian appearing in the B/2-twisted Landau-Ginzburg model,

$$\det\left( \sum_i \frac{\sum_j Q^a_i A_{ij} Q^b_j}{\sum_m A_{im} Q^c_m \sigma_c} \right) \left( \prod_{i=1}^k a_i\sigma_i \right) \left( \prod_{j=k+1}^N (Q^a_j + \epsilon^a_j)\sigma_a \right) \;=\; \det \frac{\partial^2 W}{\partial G_A \partial \theta_B},$$

so long as

$$\prod_{i=1}^{N-k} \lambda^i \;=\; \pm \prod_{i=1}^k a_i.$$

(As before, if this does not hold, we can always perform field redefinitions to rescale some of the $Y_i$s and corresponding Fermi fields $F_i$, at the cost of making the isomorphism between correlation functions of either theory slightly more complicated.) Thus, the A/2-twisted GLSM Hessian matches that arising in B/2-twisted Landau-Ginzburg model correlation functions [114]. Since correlation functions in the A/2-twisted GLSM and the B/2-twisted (0,2) Landau-Ginzburg model have essentially the same form, albeit with potential different Hessians, and we have now demonstrated that the Hessians match, it follows that correlation functions match.

### 3.2.2 Examples

So far we have presented formal arguments for a (0,2) mirror defined by a (0,2) Landau-Ginzburg theory with the same chiral ring and correlation functions[14] as the original A/2 theory. In this section we will verify that this proposal reproduces known results in specific examples.

To be specific, we will compare predictions to mirror proposals previously made in [109, 110]. Those papers were originally written by guessing ansatzes for possible mirrors, constrained to match known results on the (2,2) locus and to have the correct correlation functions and chiral ring relations. Here, we will see that the proposal we have presented correctly and

[14] Technically, untwisted sector correlation functions, if the mirror involves an orbifold.

systematically reproduces the results obtained by much more laborious methods in [109,110]. This will implicitly also provide tests that correlation functions and chiral rings do indeed match, as argued formally in the last section.

That said, the systematic proposal of this paper will only apply to special, 'toric' deformations, not all tangent bundle deformations, not even all tangent bundle deformations realizable by Euler-type sequences. Curiously, the terms in the mirrors described in [109, 110] that are not realized are nonlinear in the fields, suggesting that toric deformations are mirror to linear terms. We will not pursue this direction further in this paper, but mention it here for completeness.

$\mathbb{P}^1 \times \mathbb{P}^1$

We use $\mathbb{P}^1 \times \mathbb{P}^1$ as an example to apply our mirror ansatz and apply consistent checks. The charge matrix for this case is

$$Q_i^a = \begin{pmatrix} 1 & 1 & & \\ & & 1 & 1 \end{pmatrix}, \tag{3.2.28}$$

and the dual matrix can be solved as

$$V_i^A = \begin{pmatrix} 1 & -1 & & \\ & & -1 & 1 \end{pmatrix}. \tag{3.2.29}$$

The toric deformation we consider here is

$$E_1 = \sigma\phi_1, \qquad E_2 = \left(\sigma + \epsilon_2\widetilde{\sigma}\right)\phi_2, \qquad E_3 = \widetilde{\sigma}\phi_3, \qquad E_4 = \left(\widetilde{\sigma} + \epsilon_4\sigma\right)\phi_4. \tag{3.2.30}$$

Thus the chiral ring relations of $A/2$-model are

$$\sigma\left(\sigma + \epsilon_2\widetilde{\sigma}\right) = q^1, \qquad \widetilde{\sigma}\left(\widetilde{\sigma} + \epsilon_4\sigma\right) = q^2. \tag{3.2.31}$$

From the $A/2$-model, and following the mirror ansatz we obtain the following Toda dual:

$$W_{eff} = F_1\left(e^{-Y_1} - q^1 e^{Y_1}\right) + F_3\left(e^{-Y_3} - q^2 e^{Y_3}\right) + \epsilon_2 F_1 e^{-Y_3} + \epsilon_4 F_3 e^{-Y_1}. \tag{3.2.32}$$

We define $\Theta$, $\widetilde{\Theta}$ as

$$Y_1 = \Theta, \qquad Y_2 = t_1 - \Theta, \qquad G^1 = F_1 = -F_2, \tag{3.2.33}$$

and

$$Y_3 = \widetilde{\Theta}, \qquad Y_4 = t_2 - \widetilde{\Theta}, \qquad G^2 = -F_3 = F_4. \tag{3.2.34}$$

Let us define the low energy theory in terms of single-valued degrees of freedom $X = e^{-\Theta_1}$ and $\widetilde{X} = e^{-\Theta_3}$. Then you can check that the chiral ring relations are

$$X(X + \epsilon_2\widetilde{X}) = q^1, \qquad \widetilde{X}(\widetilde{X} + \epsilon_4 X) = q^2 \tag{3.2.35}$$

which agree with the $A/2$-model chiral ring relations.

The classical correlation functions of the $A/2$-model are

$$\langle \sigma\sigma \rangle = -\frac{\epsilon_2}{1-\epsilon_2\epsilon_4}, \qquad \langle \sigma\widetilde{\sigma} \rangle = 1, \qquad \langle \widetilde{\sigma}\widetilde{\sigma} \rangle = -\frac{\epsilon_4}{1-\epsilon_2\epsilon_4}. \tag{3.2.36}$$

The Hessian factor in the $B/2$-twisted Landau-Ginzburg model is

$$\det \frac{\partial^2 W_{eff}}{\partial G_A \partial \Theta_B} = \begin{pmatrix} e^{-Y_1} + e^{-Y_2} & \epsilon_2 e^{-Y_3} \\ \epsilon_4 e^{-Y_1} & e^{-Y_3} + e^{-Y_4} \end{pmatrix} = \left( 4X\widetilde{X} + 2\epsilon_2 \widetilde{X}^2 + 2\epsilon_4 X^2 \right), \tag{3.2.37}$$

where we have plugged in $X = e^{-Y_1}$, $\widetilde{X} = e^{-Y_3}$ as well as $X + \epsilon_2\widetilde{X} = e^{-Y_2}$, $\widetilde{X} + \epsilon_4 X = e^{-Y_4}$. The classical correlation functions in this model are

$$\langle X^2 \rangle = -\frac{\epsilon_2}{1-\epsilon_2\epsilon_4}, \qquad \langle X\widetilde{X} \rangle = 1, \qquad \langle \widetilde{X}^2 \rangle = -\frac{\epsilon_4}{1-\epsilon_2\epsilon_4}. \tag{3.2.38}$$

we see that the correlation functions of the B/2-twisted Landau-Ginzburg model match those of the A/2 model, as expected for a mirror.

In this case, we can also consider other possible deformation like [106]

$$E_1 = (\sigma + \epsilon_1\widetilde{\sigma})\,\phi_1, \qquad E_2 = (\sigma + \epsilon_2\widetilde{\sigma})\,\phi_2, \qquad E_3 = \widetilde{\sigma}\phi_3, \qquad E_4 = \widetilde{\sigma}\phi_4. \tag{3.2.39}$$

One can certainly follow the mirror ansatz in this paper, and the result will agree with [106], and furthermore one can prove that the correlation functions are matched exactly between two sides.

$\mathbb{F}_n$

The last examples we consider in this thesis are the Hirzebruch surfaces $\mathbb{F}_n$. For $n > 1$, these are not Fano, but nevertheless one can write down a mirror for the GLSM (which for the non-Fano cases is more properly interpreted as a mirror to a different geometric phase, the UV phase, of the GLSM), as discussed in [110]. The charge matrix of the GLSM is

$$\begin{bmatrix} 1 & 1 & n & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

and deformations of the tangent bundle are described mathematically as the cokernel $\mathcal{E}$ of the short exact sequence

$$0 \longrightarrow \mathcal{O}^2 \stackrel{*}{\longrightarrow} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(n,1) \oplus \mathcal{O}(0,1) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$* = \begin{bmatrix} \tilde{A}x & \tilde{B}x \\ \gamma_1 s & \gamma_2 s \\ \alpha_1 t & \alpha_2 t \end{bmatrix}.$$

In the expression above, $x$ is a two-component vector of homogeneous coordinates of charge $(1,0)$, $s$ is a homogeneous coordinate of charge $(n,1)$, and $t$ is a homogeneous coordinate of charge $(0,1)$, $A$, $B$ are constant $2 \times 2$ matrices, and $\gamma_{1,2}$, $\alpha_{1,2}$ are constants. (In principle, nonlinear deformations are also possible, but as observed previously in *e.g.* [42, 68–70], do not contribute to quantum sheaf cohomology or A/2-model correlation functions, so we omit nonlinear deformations.) The (2,2) locus is given by the special case

$$A = I, \quad B = 0, \quad \gamma_1 = n, \quad \gamma_2 = 1, \alpha_1 = 0, \quad \alpha_2 = 1.$$

We have the same constraints on fields from D terms as on the (2,2) locus, namely

$$Y_1 + Y_2 + nY_s = t_1, \quad Y_s + Y_t = t_2,$$

where $Y_{1,2}$ are dual to the $x$'s, $Y_3$ is dual to $s$, and $Y_4$ is dual to $t$. We can solve them by taking

$$\tilde{t}_1 = 0, \quad \tilde{t}_2 = t_1, \quad \tilde{t}_s = 0, \quad \tilde{t}_t = t_2,$$
$$(V_i^A) = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & -n & 1 & -1 \end{bmatrix},$$

so that

$$Y_1 = \theta, \quad Y_2 = t_1 - \theta - n\tilde{\theta}, \quad G_1 = F_1 = -F_2 - nG_2,$$
$$Y_3 = \tilde{\theta}, \quad Y_4 = t_2 - \tilde{\theta}, \quad G_2 = F_3 = -F_4.$$

**First choice of $S$** We take the submatrix $S \subset Q$ to correspond to the first and third columns of the charge matrix $Q$, *i.e.*

$$S = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

The allowed deformations are

$$(A_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & 1 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} \end{bmatrix}.$$

To find the corresponding elements of $\tilde{A}$, $\tilde{B}$, $\gamma_{1,2}$, $\alpha_{1,2}$, we compare the $E$'s. For the deformations defined by $A_{ij}$,

$$\begin{aligned} E_1 &= \sum_a Q_a^a \sigma_a \phi_1 = \sigma_1 \phi_1, \\ E_2 &= \sum_{j,a} A_{2j} Q_j^a \sigma_a \phi_2, \\ &= \left(A_{21}\sigma_1 + A_{22}\sigma_1 + A_{23}(n\sigma_1 + \sigma_2) + A_{24}\sigma_2\right)\phi_2, \\ E_s &= (n\sigma_1 + \sigma_2)s, \\ E_t &= \left(A_{41}\sigma_1 + A_{42}\sigma_1 + A_{43}(n\sigma_1 + \sigma_2) + A_{44}\sigma_2\right)t, \end{aligned}$$

whereas for the bundle deformation parameters,

$$\begin{aligned}
E_1 &= \left(\tilde{A}_{11}\phi_1 + \tilde{A}_{12}\phi_2\right)\sigma_1 + \left(\tilde{B}_{11}\phi_1 + \tilde{B}_{12}\phi_2\right)\sigma_2, \\
E_2 &= \left(\tilde{A}_{21}\phi_1 + \tilde{A}_{22}\phi_2\right)\sigma_1 + \left(\tilde{B}_{21}\phi_1 + \tilde{B}_{22}\phi_2\right)\sigma_2, \\
E_s &= \gamma_1 s\sigma_1 + \gamma_2 s\sigma_2, \\
E_t &= \alpha_1 t\sigma_1 + \alpha_2 t\sigma_2,
\end{aligned}$$

from which we read off

$$\tilde{A} = \begin{bmatrix} 1 & 0 \\ 0 & A_{21} + A_{22} + nA_{23} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 0 & A_{23} + A_{24} \end{bmatrix},$$

$$a = \det\tilde{A} = A_{21} + A_{22} + nA_{23}, \quad b = \det\tilde{B} = 0, \quad \mu = A_{23} + A_{24},$$

$$\gamma_1 = n, \quad \gamma_2 = 1, \quad \alpha_1 = A_{41} + A_{42} + nA_{43}, \quad \alpha_2 = A_{43} + A_{44}.$$

Next, let us construct the mirror. From formula (3.2.3),

$$(D^A_{i_S}) = \begin{bmatrix} A_{21} + A_{22} - nA_{24} - 1 & A_{23} + A_{24} \\ n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44}) & n(A_{23} + A_{24}) + (A_{43} + A_{44}) - 1 \end{bmatrix},$$

From equation (3.2.2), the proposed mirror superpotential is then

$$\begin{aligned}
W &= -G_1\left(e^{-Y_1} - e^{-Y_2} + (A_{21} + A_{22} - nA_{24} - 1)e^{-Y_1} + (A_{23} + A_{24})e^{-Y_3}\right) \\
&\quad -G_2\left(-ne^{-Y_2} + e^{-Y_3} - e^{-Y_4} + \left(n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44})\right)e^{-Y_1}\right. \\
&\qquad \left. + \left(n(A_{23} + A_{24}) + (A_{43} + A_{44}) - 1\right)e^{-Y_3}\right), \\
&= -G_1\left((A_{21} + A_{22} - nA_{24})X_1 - \frac{q_1}{X_1 X_3^n} + (A_{23} + A_{24})X_3\right) \\
&\quad -G_2\left(-n\frac{q_1}{X_1 X_3^n} + \left(n(A_{23} + A_{24}) + (A_{43} + A_{44})\right)X_3 - \frac{q_2}{X_3}\right. \\
&\qquad \left. + \left(n(A_{21} + A_{22} - nA_{24}) + (A_{41} + A_{42} - nA_{44})\right)X_1\right),
\end{aligned}$$

where $X_i = \exp(-Y_i)$, with operator mirror map (3.2.4)

$$\begin{aligned}
X_1 &\leftrightarrow \sigma_1, \\
X_2 = \frac{q_1}{X_1 X_3^n} &\leftrightarrow (A_{21} + A_{22})\sigma_1 + A_{23}(n\sigma_1 + \sigma_2) + A_{24}\sigma_2, \\
X_3 &\leftrightarrow n\sigma_1 + \sigma_2, \\
X_4 = \frac{q_2}{X_3} &\leftrightarrow (A_{41} + A_{42})\sigma_1 + A_{43}(n\sigma_1 + \sigma_2) + A_{44}\sigma_2.
\end{aligned}$$

Note that the operator mirror map relations for $X_2$, $X_4$ are consequences of the equations of motion $\partial W / \partial G_A = 0$.

For these deformations, the quantum sheaf cohomology ring is given by [42, 68–70]

$$Q_{(k)} Q_{(s)}^n \; = \; q_1, \quad Q_{(s)} Q_{(t)} \; = \; q_2,$$

where

$$Q_{(k)} \; = \; (A_{21} + A_{22} + nA_{23}) \sigma_1^2 \; + \; (A_{23} + A_{24}) \sigma_1 \sigma_2,$$
$$Q_{(s)} \; = \; n\sigma_1 + \sigma_2, \quad Q_{(t)} \; = \; (A_{41} + A_{42} + nA_{43}) \sigma_1 + (A_{43} + A_{44}) \sigma_2.$$

It is straightforward to check that these relations are implied by the mirror map equations above.

A proposal was made in [110] for the Toda dual to a (GLSM for a) Hirzebruch surface. Briefly, the mirror superpotential had the form

$$W \; = \; -G_1 J_1 - G_2 J_2$$

for [110][section 4.2]

$$J_1 = aX_1 + \mu_{AB}(X_3 - nX_1) + b\frac{(X_3 - nX_1)^2}{X_1} - q_1 X_1^{-1} \left( \gamma_1 X_1 + \gamma_2 (X_3 - nX_1) \right)^{-n}, \tag{3.2.40}$$

$$J_2 = n \left( aX_1 + \mu_{AB}(X_3 - nX_1) + b\frac{(X_3 - nX_1)^2}{X_1} \right) - \frac{nq_1}{X_1 \left( \gamma_1 X_1 + \gamma_2 (X_3 - nX_1) \right)^n} - \frac{q_2}{X_3} + \frac{\left( \gamma_1 X_1 + \gamma_2 (X_3 - nX_1) \right) \left( \alpha_1 X_1 + \alpha_2 (X_3 - nX_1) \right)}{X_3}, \tag{3.2.41}$$

with operator mirror map

$$X_1 \; \leftrightarrow \; \sigma_1, \quad X_3 \; \leftrightarrow \; n\sigma_1 + \sigma_2.$$

It is straightforward to check that the proposal of [110], reviewed above, specializes to our proposal here.

**Second choice of** $S$ Next, consider the case that the submatrix $S \subset Q$ is taken to correspond to the first and fourth columns of $Q$, *i.e.*

$$S \; = \; \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right].$$

The allowed deformations are

$$(A_{ij}) \;=\; \left[ \begin{array}{cccc} 1 & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & 1 \end{array} \right]$$

Proceeding as before, the corresponding $\tilde{A}$, $\tilde{B}$, $\gamma_{1,2}$, $\alpha_{1,2}$ are given by

$$\tilde{A} \;=\; \left[ \begin{array}{cc} 1 & 0 \\ 0 & A_{21}+A_{22}+nA_{23} \end{array} \right], \quad \tilde{B} \;=\; \left[ \begin{array}{cc} 0 & 0 \\ 0 & A_{23}+A_{24} \end{array} \right],$$

$$a = \det \tilde{A} = A_{21}+A_{22}+nA_{23}, \quad b = \det \tilde{B} = 0, \quad \mu = A_{23}+A_{24},$$

$$\gamma_1 = A_{31}+A_{32}+nA_{33}, \quad \gamma_2 = A_{33}+A_{34},$$

$$\alpha_1 = 0, \quad \alpha_2 = 1.$$

Next, let us construct the mirror. From formula (3.2.3),

$$(D^A_{i_S}) \;=\; \left[ \begin{array}{cc} A_{21}+A_{22}+nA_{23}-1 & A_{23}+A_{24} \\ n(A_{21}+A_{22}+nA_{23})-(A_{31}+A_{32}+nA_{33}) & n(A_{23}+A_{24})-(A_{33}+A_{34}-1) \end{array} \right].$$

From equation (3.2.2), the proposed mirror superpotential is then

$$\begin{aligned} W \;=\;& -G_1 \left( e^{-Y_1} - e^{-Y_2} + (A_{21}+A_{22}+nA_{23}-1)e^{-Y_1} + (A_{23}+A_{24})e^{-Y_4} \right) \\ & -G_2 \left( -ne^{-Y_2} + e^{-Y_3} - e^{-Y_4} + \left(n(A_{21}+A_{22}+nA_{23}) - (A_{31}+A_{32}+nA_{33})\right) e^{-Y_1} \right. \\ & \qquad \left. + \left(n(A_{23}+A_{24}) - (A_{33}+A_{34}-1)\right) e^{-Y_4} \right), \\ =\;& -G_1 \left( (A_{21}+A_{22}+nA_{23})X_1 - \frac{q_1}{q_2^n}\frac{X_4^n}{X_1} + (A_{23}+A_{24})X_4 \right) \\ & -G_2 \left( -n\frac{q_1}{q_2^n}\frac{X_4^n}{X_1} + \frac{q_2}{X_4} + \left(n(A_{23}+A_{24}) - (A_{33}+A_{34})\right) X_4 \right. \\ & \qquad \left. + \left(n(A_{21}+A_{22}+nA_{23}) - (A_{31}+A_{32}+nA_{33})\right) X_1 \right), \end{aligned}$$

where $X_i = \exp(-Y_i)$, with operator mirror map (3.2.4)

$$\begin{aligned} X_1 \;&\leftrightarrow\; \sigma_1, \\ X_2 = \frac{q_1}{q_2^n}\frac{X_4^n}{X_1} \;&\leftrightarrow\; (A_{21}+A_{22})\sigma_1 + A_{23}(n\sigma_1+\sigma_2) + A_{24}\sigma_2, \\ X_3 = \frac{q_2}{X_4} \;&\leftrightarrow\; (A_{31}+A_{32})\sigma_1 + A_{33}(n\sigma_1+\sigma_2) + A_{34}\sigma_2, \\ X_4 \;&\leftrightarrow\; \sigma_2. \end{aligned}$$

The operator mirror map relation for $X_2$ is a consequence of $\partial W/\partial G_1 = 0$, and the operator mirror map relation for $X_3$ is a consequence of that plus $\partial W/\partial G_2 = 0$.

A second proposal was made in [110] for the Toda dual to a (GLSM for a) Hirzebruch surface, in which the mirror superpotential had the form

$$W \;=\; -G_1 J_1 - G_2 J_2,$$

for [110][section 4.2]

$$J_1 = \left( a X_1 + \mu_{AB} X_4 + b \frac{X_4^2}{X_1} \right) - \frac{q_1}{q_2^n} \frac{(\alpha_1 X_1 + \alpha_2 X_4)^n}{X_1}, \tag{3.2.42}$$

$$\begin{aligned} J_2 = -n &\left( a X_1 + \mu_{AB} X_4 + b \frac{X_4^2}{X_1} - \frac{q_1}{q_2^n} \frac{(\alpha_1 X_1 + \alpha_2 X_4)^n}{X_1} \right) \\ &+ \left( \alpha_2 \gamma_2 X_4 + \gamma_1 \alpha_1 \frac{X_1^2}{X_4} + (\gamma_1 \alpha_2 + \gamma_2 \alpha_1) X_1 \right) - \frac{q_2}{X_4}, \end{aligned} \tag{3.2.43}$$

with operator mirror map

$$X_1 \;\leftrightarrow\; \sigma_1, \quad X_4 \;\leftrightarrow\; \sigma_2.$$

It is straightforward to check that this proposal of [110] specializes to our proposal. Some other perspectives of (0,2) mirror symmetry can be found [105, 107, 108].

# Chapter 4

# Conclusions and Future Directions

## 4.1 Research Summary

The previous chapters are devoted to detailed discussion of my work, this section I will briefly summarize my work for the conclusion.

**Supersymmetric Localization in GLSMs for Supermanifolds** Supermanifolds have recently been of renewed interest due to progress in superstring perturbation theory [121]. About two decades ago, it was suggested in the literature [55] that A-twisted NLSM correlation functions for certain supermanifolds are equivalent to A-twisted NLSM correlation functions for hypersurfaces in ordinary spaces under certain conditions. We used supersymmetric localization to first show the A-twisted GLSM correlation functions for certain supermanifolds are equivalent to corresponding A-twisted GLSM correlation functions for hypersurfaces. Then, we showed that physical two-sphere partition functions are the same for these two different target spaces, which we conjectured that the map from GLSM parameters to NLSM parameters are the same for them as well [64]. We also found that elliptic genera share similar phenomena, indicating the Witten index and central charges match. Finally, we extended the calculations to (0,2) deformations, and conjectured a (0,2) version of the statement for NLSMs.

**Localization of twisted N=(0,2) gauged linear sigma models in two dimensions** In [42], we use supersymmetric localization to study twisted $\mathcal{N}$=(0,2) GLSMs [16]. For $\mathcal{N}$=(0,2) GLSMs deformed from $\mathcal{N}$=(2,2) GLSMs, we consider the A/2-twisted and formulized the genus-zero correlation functions of certain pseudo topological observables in terms of Jeffrey-Kirwan-Grothendieck residues on Coulomb branches, which generalize the Jeffrey-Kirwan residue prescription relevant for the $\mathcal{N}$=(2,2) locus. We reproduce known results for abelian GLSMs, and can systematically calculate more examples with new formulas that render the quantum sheaf cohomology relations and other properties manifest.

**A proposal for nonabelian mirrors** In [26], we proposed a generalization of the Hori-Vafa mirror construction [25] from abelian (2,2) GLSMs to non-abelian (2,2) GLSMs with connected gauge groups, a potential solution to an old problem. Inspired by the quantum behavior of the non-abelian gauge dynamics, we proposed that the mirror of a nonabelian gauge theory is a Weyl-group orbifold of the Hori-Vafa mirror of an abelian gauge theory containing both matter fields corresponding to those of the original non-abelian theory as well as new matter fields with the same indices as the W bosons of the original theory. We formally showed that topological correlation functions of B-model mirror LGs match the A-model gauge theories' correlation functions in general. We studied two examples, Grassmannians and two-step flag manifolds, verifying in each case that the mirror correctly reproduces details ranging from the number of vacua and correlations functions to quantum cohomology relations. We also studied mirrors of pure gauge theories, comparing and extending claims of [87] for the IR behavior of the original gauge theories. Furthermore, in our paper [124], we studied mirrors to theories with exceptional gauge groups and predicted a similar statement to [87].

**A proposal for (0,2) mirrors of toric varieties** In [27], we proposed an extension of the Hori-Vafa construction [25] of (2,2) GLSM mirrors to (0,2) theories obtained from (2,2) theories by special tangent bundle deformations. Our ansatz can systematically produce the (0,2) mirrors of toric varieties and the results are consistent with existing examples which were produced by laborious guesswork. We also explicitly verified that closed string correlation functions of the original A and A/2 twisted models match those of the mirror B and B/2 twisted Landau-Ginzburg models, extending Hori-Vafa's work in the (2,2) case and providing an important consistency check in the (0,2) case.

## 4.2 Future Work

I will end my thesis by mentioning several directions which are related my previous work.

**3d/2d mirror symmetry** In three dimensions, mirrors to non-abelian theories and Kähler metrics of moduli spaces have been well-explored in the literature [125]. It is therefore natural to ask how three-dimensional mirror symmetry relates to two-dimensional mirror symmetry. Some work relating three-dimensional mirrors to two-dimensional mirrors in abelian cases exists [126]. Our interest lies in extending the relationship to non-abelian mirrors. In passing, some different aspects of the relationship between 3d and 2d mirrors can be found in [87].

**Open string mirror symmetry** My work so far has focused on closed string mirror symmetry, however, open string mirror symmetry has been significantly explored in past from several different directions, see for example [4, 127–131] and the reviews [133, 134]. My starting point for this direction is to understand GLSMs with boundaries first [59], and then use techniques developed in GLSMs and supersymmetric localization methods to uncover some new insights in this direction based on our work [26].

**Correlation functions for $\mathcal{N} = (0, 2)$ theories without (2,2) locus** To get a four-dimensional spacetime with $\mathcal{N} = 1$ SUSY from a heterotic string compactifications requires that the worldsheet has $\mathcal{N} = (0, 2)$ SUSY, and most such theories are not deformations of (2,2) theories [1]. $\mathcal{N} = (0, 2)$ gauged linear sigma models provide a global description of string compactifications [16], however only rare BPS quantities of $\mathcal{N} = (0, 2)$ theories can be calculated exactly, such as elliptic genera [19] and correlation functions for $\mathcal{N} = (0, 2)$ theories with a (2,2) locus [42]. In [42], we provided all of ingredients for calculating correlation functions of $\mathcal{N} = (0, 2)$ theories; however, for general $\mathcal{N} = (0, 2)$ theories, the one-loop determinants do not depend solely upon topological observables. However, based on [132] for Calabi-Yau target spaces, one can define the topological sub-ring for general (0,2) theories, suggesting that supersymmetric localization could apply to general (0,2) theories under some conditions. We first aim to compute correlation functions in some concrete examples, and then try to find hints of what circumstances will allow supersymmetric localization to be applied in other (0,2) theories. Last but not least, I want to mention that any examples we find would also benefit the 6d community, as compactifications of 6d (0,2) theories are typically 2d (0,2) theories without a (2,2) locus.

# Appendices

# Appendix A

# Conventions and One Loop Determinants

The contents of this section were adapted, with minor modifications, with permission from JHEP, from our publication [42].

## A.1 Curved space conventions

Our conventions mostly follow [41, 42, 46], to which we refer for further details. We work on a Riemannian two-manifold with local complex coordinates $z, \bar{z}$, and Hermitian metric:

$$ds^2 = 2g_{z\bar{z}}(z,\bar{z})dzd\bar{z} \ . \tag{A.1.1}$$

We choose the canonical frame

$$e^1 = g^{\frac{1}{4}}dz \ , \quad e^{\bar{1}} = g^{\frac{1}{4}}d\bar{z} \ , \tag{A.1.2}$$

with $\sqrt{g} = 2g_{z\bar{z}}$ by definition. The spin connection is given by

$$\omega_z = -\frac{i}{4}\partial_z \log g \ , \qquad \omega_{\bar{z}} = \frac{i}{4}\partial_{\bar{z}} \log g \ . \tag{A.1.3}$$

The covariant derivative on a field of spin $s \in \frac{1}{2}\mathbb{Z}$ is:

$$D_\mu \varphi_{(s)} = (\partial_\mu - is\omega_\mu)\varphi_{(s)} \ . \tag{A.1.4}$$

We generally write down derivatives in the frame basis as well: $D_1\varphi_{(s)} = e_1^z D_z \varphi_{(s)}$ and $D_{\bar{1}}\varphi_{(s)} = e_{\bar{1}}^{\bar{z}} D_{\bar{z}}\varphi_{(s)}$.

## A.2 Lagrangian on Curved Spaces

In section 2.1.2, we described the GLSM for supermanifolds on flat worldsheets. However, in this paper we also consider GLSMs for supermanifolds on the two-sphere. Since $S^2$ is not flat, the Lagrangian will have curvature correction terms [43, 44, 46]. In this section, we want to write out Lagrangians for GLSMs for supermanifolds on a worldsheet two-sphere. Since the only difference with GLSM for ordinary spaces is the kinetic term for odd chiral superfields (2.1.6), we will only write out $\mathcal{L}_{\text{kin}}^{\text{odd}}$.

First, consider the physical Lagrangian on $S^2$. By solving the supergravity background, one can follow [43] to get the kinetic term for the odd superfield $\tilde{\Phi}$ with vector R-charge $\tilde{R}$ as:[1]

$$\mathcal{L}_{\text{kin}}^{\text{odd}} = D_\mu \bar{\tilde{\phi}} D^\mu \tilde{\phi} + \bar{\tilde{\phi}} \sigma^2 \tilde{\phi} + \bar{\tilde{\phi}} \eta^2 \tilde{\phi} + i \bar{\tilde{\phi}} \tilde{D} \tilde{\phi} + \bar{\tilde{F}} \tilde{F} + \frac{i\tilde{R}}{r} \bar{\tilde{\phi}} \sigma \tilde{\phi} + \frac{\tilde{R}(2-\tilde{R})}{4r^2} \bar{\tilde{\phi}} \tilde{\phi}$$
$$- i \bar{\tilde{\psi}} \gamma^\mu D_\mu \tilde{\psi} + i \bar{\tilde{\psi}} \sigma \tilde{\psi} - \bar{\tilde{\psi}} \gamma_3 \eta \tilde{\psi} + i \bar{\tilde{\psi}} \tilde{\lambda} \tilde{\phi} - i \bar{\tilde{\phi}} \bar{\tilde{\lambda}} \tilde{\psi} - \frac{\tilde{R}}{2r} \bar{\tilde{\psi}} \tilde{\psi}. \quad \text{(A.2.1)}$$

Similarly, we can follow [41] to get the twisted Lagrangian on $S^2$. The kinetic term for odd chiral superfields will have the same form as Eq. (2.35) in [41]. One difference is that the statistical properties for each component field are changed.

## A.3 Elliptic Genera with General R Charges

In this section, we calculate the elliptic genera for more general R-charge assignments, following Appendix A.7. In the same spirit of Section 2.1.3, we focus on comparison of hypersurface and supermanifold.

As an example, we only consider GLSMs for hypersurfaces in $\mathbb{WP}^N_{[Q_1,\ldots,Q_{M+1}]}$ and for $\mathbb{WP}^{N+1|M}_{[Q_1,\ldots,Q_{M+1}|\tilde{Q}]}$. Actually, we only need compare the one-loop determinants for the $P$-field, say $P$ with $U(1)$ charge $-\tilde{Q}$, and that for the odd chiral superfield, say $\Psi$ with $U(1)$ charge $\tilde{Q}$. From appendix A.7, the R-charge for $P$ is $2 - \zeta\tilde{Q}$ and the R-charge for $\Psi$ is $\zeta\tilde{Q}$. Then we have

$$Z_P^{1-\text{loop}} = \frac{\theta_1(q, y^{R_P/2-1} x^{-\tilde{Q}})}{\theta_1(q, y^{R_P/2} x^{-\tilde{Q}})} = \frac{\theta_1(q, y^{-\zeta\tilde{Q}/2} x^{-\tilde{Q}})}{\theta_1(q, y^{1-\zeta\tilde{Q}/2} x^{-\tilde{Q}})},$$
$$Z_\Psi^{1-\text{loop}} = \frac{\theta_1(q, y^{R_\Psi/2} x^{-\tilde{Q}})}{\theta_1(q, y^{R_\Psi/2-1} x^{-\tilde{Q}})} = \frac{\theta_1(q, y^{\zeta\tilde{Q}} x^{-\tilde{Q}})}{\theta_1(q, y^{\zeta\tilde{Q}/2-1} x^{-\tilde{Q}})}.$$

[1] There is another supergravity background used in [44]. These two supergravity backgrounds are claimed to be equivalent to each other as studied in [46]

Then according to the property of $\theta_1$-function, $\theta_1(\tau, x) = -\theta_1(\tau, x^{-1})$, above two one-loop determinants equal to each and so do their elliptic genera. This calculation can be easily generalized to more general cases as in section 2.1.3.

## A.4 One-loop determinants

Consider the gauge theories with a $(2, 2)$ locus of section 2.3. In this Appendix, we compute the one-loop determinant of the matter fields. The one-loop contribution from the $W$-bosons and their superpartners is exactly the same as in [41], to which we refer for further discussions of the gauge sector.

### A.4.1 Matter determinant for $A/2$-twisted GLSM with $(2, 2)$ locus

The matter sector localization is performed with the kinetic terms of the chiral and Fermi multiplets. Placing oneself at a generic point on the Coulomb branch and expanding the Lagrangian at quadratic order in the matter fields, one finds:

$$\mathscr{L}_{\rm loc} = \widetilde{\phi}^I \Delta^{\rm bos}_{IJ} \phi^J \ + \ (\widetilde{\mathcal{B}} \ , \widetilde{\Lambda}_-)^I \Delta^{\rm fer}_{IJ} \begin{pmatrix} \Lambda_- \\ \mathcal{C} \end{pmatrix}^J + i\widetilde{\mathcal{B}}^I Q_I(\widetilde{\lambda})\phi_I + \frac{1}{2}\widetilde{\mathcal{B}}^{\Sigma}_a \widetilde{\phi}^I (\partial_{\widetilde{\sigma}_a} \widetilde{M}_{IJ})\Lambda^J \ , \quad \text{(A.4.1)}$$

with the kinetic operators

$$\Delta^{\rm bos}_{IJ} = -4\delta_{IJ} D_1 D_{\bar{1}} + \widetilde{M}_{IK} M^K{}_J + iQ_I(D) \ , \qquad \Delta^{\rm fer}_{IJ} = \begin{pmatrix} \frac{1}{2}\widetilde{M}_{JI} & 2iD_1 \\ -2iD_{\bar{1}} & 2M_{IJ} \end{pmatrix} \ . \qquad \text{(A.4.2)}$$

Here $M_{IJ}$ was defined in (2.3.9), and $Q_I$ us the gauge charge of $\Phi_I, \Lambda_I$. Since the mixing is limited to the $\gamma$ blocks defined in section 2.3.1, we restrict ourselves to a single block of gauge charge $Q_\gamma$ and effective $R$-charge

$$\mathbf{r}_\gamma = r_\gamma - Q_\gamma(k) \ , \qquad \text{(A.4.3)}$$

in a given flux sector. It is easy to perform the supersymmetric Gaussian integral explicitly. Here we can focus on the case $\widetilde{\lambda} = \widetilde{\mathcal{B}}^{\Sigma} = 0$.

## A.5 Operator determinants in A/2 deformations of (2,2) theories

In this section, we will compute operator determinants and their ratios for (0,2) (A/2 twisted) GLSM's obtained by deforming (2,2) theories off of the (2,2) locus.

We shall group (0,2) chiral and Fermi superfields originating from (2,2) chirals according to their gauge charges. Groups of fields with the same charges can be mixed into one another with the $E$ deformations, defined by

$$\overline{D}_+ \Lambda^i \;=\; E^i(\phi).$$

In this appendix we will compute $Z^{1-\mathrm{loop}}$ for a set of (0,2) chiral and Fermi superfields all of the same charges, obtained as deformations of a set of (2,2) chiral superfields of the same charges. These can be multiplied together in the obvious way to get $Z^{1-\mathrm{loop}}$ for sets of fields with different charges, as we discuss in examples in the main text.

We will localize around the background

$$\lambda = 0, \quad D = iF_{12}, \quad [\sigma, \overline{\sigma}] = 0, \quad D_{\overline{z}}\sigma = 0.$$

Our computations will take place on the Coulomb branch given above. In nonabelian cases, we will work with a set of commuting $\sigma$ fields providing coordinates on the Coulomb branch, but as they commute, we can treat them just as if the gauge group were abelian. Thus, we are able to treat abelian and nonabelian cases identically in what follows.

Our conventions will largely follow [53].

### A.5.1 Bosonic operator determinant

We will begin by examining the bosons in the theory. The kinetic terms for the bosons are standard, and the potential terms are of the form

$$|E^i(\phi)|^2$$

where we will assume linear deformations of the form

$$E^i(\phi) \;=\; \sum_a A^i_{aj} \sigma^a \phi^j$$

for $\sigma^a$'s the adjoint-valued (neutral in abelian theories) fields in the (2,2) gauge multiplet (here, a (0,2) chiral multiplet), $\phi^i$'s a given set of bosons of identical charges, and $A^i_{aj}$ a set of constant matrices that mix the bosons. Let $N$ denote the number of complex bosons of the same charge. It will be convenient to define

$$E^i_j(\phi) \;=\; \sum_a A^i_{aj} \sigma^a$$

so that

$$E^i(\phi) \;=\; E^i_j \phi^j.$$

Expanding out the bosonic potential terms, we have

$$
\begin{aligned}
|E^i(\phi)|^2 &= \sum_i \left( \sum_j |E^i_j|^2 |\phi^j|^2 \right) \\
&\quad + \sum_{i \neq j} \left( \sum_k (E^k_i)^* (E^k_j) \right) \overline{\phi}^{\bar{\imath}} \phi^j .
\end{aligned}
$$

Since the potential terms mix the various $\phi^i$, we need to consider a matrix of operators for all the $\phi^i$ simultaneously. Thus, we define $\mathcal{O}_\phi$ to be the operator such that the boson kinetic and potential terms are encoded by

$$\int (\vec{\phi})^\dagger \mathcal{O}_\phi \vec{\phi}$$

where

$$(\vec{\phi})^T = (\phi^1, \phi^2, \cdots)$$

and the operator $\mathcal{O}_\phi$ is

$$
\begin{bmatrix}
t^2 + |E^1_1|^2 + \cdots + |E^N_1|^2 & (E^1_1)^* E^1_2 + \cdots + (E^N_1)^* E^N_2 & \cdots & (E^1_1)^* E^1_N + \cdots + (E^N_1)^* E^N_N \\
E^1_1 (E^1_2)^* + \cdots + E^N_1 (E^N_2)^* & t^2 + |E^1_2|^2 + \cdots + |E^N_2|^2 & \cdots & (E^1_2)^* E^1_N + \cdots + (E^N_2)^* E^N_N \\
\vdots & \vdots & \ddots & \vdots \\
(E^1_N)^* E^1_1 + \cdots + (E^N_N)^* E^N_1 & (E^1_N)^* E^1_2 + \cdots + (E^N_N)^* E^N_2 & \cdots & t^2 + |E^1_N|^2 + \cdots + |E^N_N|^2
\end{bmatrix},
$$

where $t^2$ represents $-D^2 + i\rho(D) + ir/2R^2$, for $\rho$ the gauge representation, $r$ the twisting of the boson[2], and

$$\rho(D) = \rho(i\mathfrak{m}/2R^2) + \rho(D_0).$$

Now, we need to compute the determinant of the operator above. We will begin by formally computing the determinant for a single mode, then expanding across all modes.

If we think of $t^2$ as $-\lambda$ for $\lambda$ an eigenvalue, the determinant of $\mathcal{O}_\phi$ above is the same as the characteristic polynomial of the matrix obtained by setting $t$ to 0, which one can straightforwardly see is the product $(E^T)^\dagger E^T$, for

$$
E = \begin{bmatrix}
E^1_1 & E^2_1 & \cdots & E^N_1 \\
E^1_2 & E^2_2 & \cdots & E^N_2 \\
\vdots & \vdots & \ddots & \vdots \\
E^1_N & E^2_N & \cdots & E^N_N
\end{bmatrix}.
$$

[2] This term should be carefully distinguished from curvature-dependent terms that arise in (2,2). Because we are putting a topological field theory on $S^2$, we do not need to add curvature-dependent terms to the action; however, we are twisting a boson, and this term arises in the supersymmetric action.

Using standard results from linear algebra, the characteristic polynomial, in terms of $t^2$ rather than $-\lambda$, is given by

$$\sum_{k=0}^{N} t^{2(N-k)} \mathrm{Tr} \wedge^k ((E^T)^\dagger E^T)$$

where for any matrix $M$, $\wedge^k M$ denotes a matrix encoding principal minors[3] of $M$ of size $k$. Furthermore, since it is a product, the principal minors of $(E^T)^\dagger E^T$ are the same as norm squares of principal minors of $E$.

If we let $\tilde{E}_{i_1,\cdots,i_k,j_1,\cdots,j_k}$ denote the determinant of the submatrix of $E$ formed by omitting rows $i_1 \cdots i_k$ and columns $j_1 \cdots j_k$ (*i.e.* a principal minor of $E$ of size $N-k$), then the characteristic polynomial above, *i.e.* the determinant of the bosonic operator matrix, can be written in the form

$$\sum_{k=0}^{N} t^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right).$$

So far we have discussed the operator determinant for a single mode. Let us now assemble the complete result. We enumerate each boson mode as a tensor spherical harmonic $Y^s_{j,j_3}$, which has $D^2$ eigenvalues [53]

$$-D^2 Y^s_{j,j_3} \;=\; \frac{j(j+1)-s^2}{R^2} \, Y^s_{j,j_3}.$$

Define $b = \rho(\mathfrak{m}) + r$, for $r$ the twisting of the boson and $\rho$ the charge of the matter field, then $s = b/2$, and the angular momentum is $j = (|b|/2) + n$, $n \geq 0$, and we have

$$-D^2 Y^s_{j,j_3} \;=\; \frac{2n(n+1)+(2n+1)|b|}{2R^2} \, Y^s_{j,j_3}.$$

Putting this together, we find that the complex bosonic operator determinant, taking into account the $2j+1 = 2n+|b|+1$ multiplicities, is given by

$$\det \mathcal{O}_\phi \;=\; \prod_{n\geq 0} \left[ \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right) \right]^{2n+|b|+1} \quad \text{(A.5.1)}$$

for

$$t_n \;=\; \frac{2n(n+1)+(2n+1)|b|-b}{2R^2}.$$

The last term, namely $-b/2R^2$, arises from

$$\rho(D) \;=\; \rho(i\mathfrak{m}/2R^2) + \rho(D_0).$$

in the Lagrangian terms $-D^2 + i\rho(D) + ir/(2R^2)$. In the expression above, we take $D_0 = 0$.

[3] Recall a principal minor is the determinant of a submatrix formed by omitting rows and columns.

### A.5.2 Fermion operator determinant

Next, we consider the Fermi fields. In (0,2) supermultiplets, there are two contributions: right-moving fermions $\psi_+$ in the chiral superfields, and left-moving fermions $\psi_-$ in the Fermi superfields. In general in (0,2) theories, one would want to consider them separately; however, in the present case, since we are describing deformations off the (2,2) locus, it is more convenient to combine the left- and right-moving fermions into a single operator. This is possible because in this theory, the left- and right-moving fermions couple to the same bundles[4], and it is necessary because the $E$ interactions mix the left- and right-moving fermions. Specifically, because of the $E$ potentials, we have interaction terms[5]

$$\overline{\psi}^i_- \psi^j_+ E^j_i \ + \ \overline{\psi}^j_+ \psi^i_- (E^j_i)^*. \tag{A.5.2}$$

The resulting operator for the left- and right-moving fermions $\mathcal{O}_\psi$ is given as

$$\int \vec{\overline{\psi}} \mathcal{O}_\psi \vec{\psi}$$

where

$$(\vec{\psi})^T \ = \ \left( \psi^1_+, \psi^1_-, \psi^2_+, \psi^2_-, \cdots \right),$$
$$\vec{\overline{\psi}} \ = \ \vec{\psi}^\dagger \gamma^0 \ = \ \left( \overline{\psi}^1_-, \overline{\psi}^1_+, \overline{\psi}^2_-, \overline{\psi}^2_+, \cdots \right),$$

and $\mathcal{O}_\psi$ is described by a $2N \times 2N$ matrix of the form

$$\mathcal{O}_\psi \ = \ \left[ \begin{array}{ccccccc} E^1_1 & (2/R)D_+ & E^2_1 & 0 & \cdots & E^N_1 & 0 \\ (2/R)D_- & (E^1_1)^* & 0 & (E^1_2)^* & \cdots & 0 & (E^1_N)^* \\ E^1_2 & 0 & E^2_2 & (2/R)D_+ & \cdots & E^N_2 & 0 \\ 0 & (E^2_1)^* & (2/R)D_- & (E^2_2)^* & \cdots & 0 & (E^2_N)^* \\ \vdots & & & & \ddots & & \vdots \\ E^1_N & 0 & E^2_N & 0 & \cdots & E^N_N & (2/R)D_+ \\ 0 & (E^N_1)^* & 0 & (E^N_2)^* & \cdots & (2/R)D_- & (E^N_N)^* \end{array} \right].$$

Since the eigenvalues of $D_+$ and $D_-$ are minus one another, it will often be convenient to

[4] Although the gauge bundle in the corresponding low-energy nonlinear sigma model is different from the tangent bundle, in the UV GLSM, the left-moving fermions couple to the same bundle as the right-moving fermions, and the different gauge bundle is realized via interactions and RG flow.

[5] Our notation is unfortunately slightly confusing. In [16][eqn. (6.18)] the interaction terms are of the form $\overline{\psi}_- \psi_+ (\partial E / \partial \phi)$ instead of $\overline{\psi}_- \psi_+ E$, but the $E$'s we have defined above, which depend upon $\sigma$ and $\widetilde{\sigma}$ but not $\phi$, are the derivatives of the $E$'s appearing in [16][section 6].

represent the matrix above as

$$\mathcal{O}(t) \;=\; \left[ \begin{array}{ccccccc} E_1^1 & +t & E_1^2 & 0 & \cdots & E_1^N & 0 \\ -t & (E_1^1)^* & 0 & (E_2^1)^* & \cdots & 0 & (E_N^1)^* \\ E_2^1 & 0 & E_2^2 & +t & \cdots & E_2^N & 0 \\ 0 & (E_1^2)^* & -t & (E_2^2)^* & \cdots & 0 & (E_N^2)^* \\ \vdots & & & & \ddots & & \vdots \\ E_N^1 & 0 & E_N^2 & 0 & \cdots & E_N^N & +t \\ 0 & (E_1^N)^* & 0 & (E_2^N)^* & \cdots & -t & (E_N^N)^* \end{array} \right]$$

where $t$ represents the eigenvalues of $(2/R)D_+$, and $-t$ represents the eigenvalues of $(2/R)D_-$.

Note that for $t = 0$, the nonzero entries of the matrix $(\mathcal{O}(t)_{ij})$ have the property that $i$, $j$ are either both even or both odd, and the entries with $t$'s are all one step above or below the diagonal: for $j$ odd, $\mathcal{O}(t)_{j+1,j} = -t$, and for $j$ even, $\mathcal{O}(t)_{j-1,j} = +t$.

Now, we wish to compute $\det \mathcal{O}(t)$. This will be a polynomial in $t$ of degree $4N$. We will compute the coefficients in that polynomial separately.

First, let us consider the $t$-independent term in that polynomial. This can be computed as $\det \mathcal{O}(t)$ for $t = 0$. We will show that in this case,

$$\det \mathcal{O}(0) \;=\; |\det E|^2.$$

More generally, consider a $2n \times 2n$ matrix $C$ built by interweaving two $n \times n$ matrices $A$, $B$, in the form

$$(C_{ij}) \;=\; \left[ \begin{array}{ccccccc} A_{11} & 0 & A_{12} & 0 & A_{13} & 0 & \cdots \\ 0 & B_{11} & 0 & B_{12} & 0 & B_{13} & \cdots \\ A_{21} & 0 & A_{22} & 0 & A_{23} & 0 & \cdots \\ 0 & B_{21} & 0 & B_{22} & 0 & B_{23} & \cdots \\ \vdots & & & & & & \ddots \end{array} \right].$$

Now,

$$\begin{aligned} \det C &= \epsilon_{i_1 \cdots i_{2n}} C_{i_1 1} \cdots C_{i_{2n} n}, \\ &= \sum_{j_1=1}^n \sum_{k_1=1}^n \cdots \sum_{k_n=1}^n \epsilon_{(2j_1-1)(2k_1)(2j_2-1)(2k_2)\cdots(2k_n)} A_{j_1 1} B_{k_1 1} A_{j_2 2} B_{k_2 2} \cdots B_{k_n n}, \\ &= \sum_{j_1=1}^n \sum_{k_1=1}^n \cdots \sum_{k_n=1}^n \epsilon_{j_1 \cdots j_n} \epsilon_{k_1 \cdots k_n} A_{j_1 1} B_{k_1 1} A_{j_2 2} B_{k_2 2} \cdots B_{k_n n}, \\ &= (\det A)(\det B). \end{aligned}$$

For our current matrix $\mathcal{O}(t)$, when $t = 0$ it can be written as an interweaving of $E$ and $E^\dagger$, so using the lemma above we immediately have that

$$\det \mathcal{O}(0) \;=\; |\det E|^2.$$

Now, consider terms in the expansion of $\det \mathcal{O}(t)$ that are linear in $t$. Such terms arise from terms in the determinant of the form

$$(\mathcal{O}(t)_{j\pm 1,j})(2N-1 \text{ factors of the form } \mathcal{O}(t)_{\text{odd,odd}} \text{ or } \mathcal{O}(t)_{\text{even,even}}).$$

However, when computing the determinant, the total number of odd indices should be the same as the total number of even indices, and the $\mathcal{O}(t)_{j\pm 1,j}$ factor creates an imbalance. Therefore there are no terms of this form. In fact, the same argument implies that there can be no terms with odd powers of $t$.

What remains is to compute the coefficients of nonzero even powers of $t$. For the same imbalance reasons as above, all contributions to even powers of $t$ must arise from terms involving the same number of factors of $-t$ (from below the diagonal) as $+t$ (from above the diagonal). With that in mind, the coefficient of $t^{2k}$ will be a sum of terms of each of which is a determinant of two interweaved matrices, each formed from $E$ and $E^\dagger$ by omitting $k$ rows and columns. In other words, the coefficient of $t^{2k}$ is given by

$$\sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2$$

where $\tilde{E}_{i_1\cdots i_k, j_1\cdots j_k}$ denotes the determinant of the submatrix of $E$ formed by omitting rows $\{i_1, \cdots, i_k\}$ and columns $\{j_1, \cdots, j_k\}$, *i.e.* a principal minor of $E$. (It can be shown that the signs are all positive, which we leave as an exercise for the reader.) The sum above is the same as the sum of norm squares of the size $N-k$ principal minors of $E$.

As a special case, in the case $N=2$, it is straightforward to compute that

$$\det \mathcal{O}(t) \;=\; t^4 \;+\; t^2 \left( |E_1^1|^2 + |E_2^1|^2 + |E_1^2|^2 + |E_2^2|^2 \right) \;+\; |\det E|^2.$$

The coefficient of $t^2$, namely

$$|E_1^1|^2 + |E_2^1|^2 + |E_1^2|^2 + |E_2^2|^2$$

is the sum of the norm squares of the determinants of the $1\times 1$ matrices formed by omitting all possible pairs of rows and columns from $E$. The coefficient of $t^4$, namely 1, is the sum of the norm squares of the determinants of the $0\times 0$ matrices formed by omitting all possible quadruples of rows and columns, leaving nothing. Finally, note that the last, $t$-independent, term is of the same form, but involving no subtractions of any rows or columns.

Thus, we find that the general expression is of the form

$$\det \mathcal{O}(t) \;=\; \sum_{k=0}^{N} t^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right).$$

Note that we obtained the same expression for the determinant of the boson operator matrix, which means that most oscillator modes will cancel, exactly as one would expect in a topological field theory.

So far we have just computed the determinant for a single mode, and have not considered special cases. Next, we consider special cases and take into account mode multiplicities.

First, we recall that a Dirac spinor on $S^2$ has generically two components:

$$Y^{-b/2}_{j,j_3} \text{ for } j \geq \left| \frac{b}{2} \right|, \qquad Y^{-b/2-1}_{j,j_3} \text{ for } j \geq \left| \frac{b}{2} + 1 \right|.$$

We expand $\vec{\psi}$ in eigenmodes as

$$(\vec{\psi})^T \; = \; \left( Y^{-b/2}_{j,j_3}, Y^{-b/2-1}_{j,j_3}, Y^{-b/2}_{j,j_3}, Y^{-b/2-1}_{j,j_3}, \cdots \right).$$

(The pairs of modes are forced to have matching spins and other quantum numbers by virtue of the existence of off-diagonal interactions.)

Now, we distinguish various cases. First consider the case $b \leq -2$. At $j = -(b/2) - 1$, only the $(2j + 1 = -b - 1)$ left-moving modes exist, while for $j = -(b/2) - 1 + n$, and $n \geq 1$, both modes survive. In the special case $j = -(b/2) - 1$, the fermion operator reduces to

$$\left[ \begin{array}{cccc} E^1_1 & E^2_1 & E^3_1 & \cdots \\ E^1_2 & E^2_2 & E^3_2 & \cdots \\ E^1_3 & E^2_3 & E^3_3 & \cdots \\ \vdots & & & \ddots \end{array} \right]$$

which is the matrix $E$. Thus, when $b \leq -2$, the entire fermion operator determinant, taking into account mode multiplicities, takes the form

$$(\det E)^{-b-1} \prod_{n \geq 1} \left( \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{E}_{i_1 \cdots i_k, j_1 \cdots j_k} \right|^2 \right) \right)^{2n-b-1}$$

where

$$t_n \; = \; \frac{n(n + |b+1|)}{R^2}$$

(and using the fact that for $n \geq 1$, $2j + 1 = 2n - b - 1$).

Next, consider the special case $b = -1$, in which $j = 1/2$. In this case, both left- and right-moving modes exist, of multiplicity $2j + 1$ for $j = (1/2) + n$, for $n \geq 0$. Thus, taking into account mode multiplicities, the entire fermion determinant is given by

$$\prod_{n \geq 0} \left( \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{E}_{i_1 \cdots i_k, j_1 \cdots j_k} \right|^2 \right) \right)^{2n+2}$$

where

$$t_n \; = \; \frac{(n+1)^2}{R^2}.$$

Next, consider the case $b \geq 0$. At $j = b/2$, only the right-moving modes survive, of multiplicity $2j+1 = b+1$. In this case, the fermion operator reduces to

$$\begin{bmatrix} (E_1^1)^* & (E_2^1)^* & (E_3^1)^* & \cdots \\ (E_1^2)^* & (E_2^2)^* & (E_3^2)^* & \cdots \\ (E_1^3)^* & (E_2^3)^* & (E_3^3)^* & \cdots \\ \vdots & & & \ddots \end{bmatrix}$$

which is the matrix $E^\dagger$. For $j = (b/2) + n$ for $n \geq 1$, both modes exist. Thus, for $b \geq 0$, the entire fermion operator determinant, taking into account mode multiplicites, takes the form

$$(\det E^\dagger)^{b+1} \prod_{n\geq 1} \left[ \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right) \right]^{2n+b+1}$$

where

$$t_n \; = \; \frac{n(n+|b+1|)}{R^2}.$$

Putting this together, an expression for the fermion determinant valid for all $b$ is given by

$$(S(\det E))^{|b+1|} \prod_{n\geq 1} \left[ \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{E}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right) \right]^{2n+|b+1|} \tag{A.5.3}$$

where

$$t_n \; = \; \frac{n(n+|b+1|)}{R^2}$$

where $S$ is the identity for $b \leq -2$ and complex conjugation for $b \geq 0$.

### A.5.3 Matter determinant ratios

Now, let us assemble $Z^{1-\text{loop}}$. This is the ratio of the fermionic to bosonic operator determinants.

First, consider the case $b \geq 0$. Here, the $t_n$'s appearing in the bosonic and fermionic determinants match:

$$t_n \; = \; \frac{n(n+|b+1|)}{R^2} \; = \; \frac{2n(n+1)+(2n+1)|b|-b}{2R^2},$$

so most of the factors in the infinite products for $\det \mathcal{O}_\psi$, $\det \mathcal{O}_\phi$ cancel out. What remains from the fermionic determinant $\det \mathcal{O}_\psi$ is the factor

$$(S(\det E))^{|b+1|} \; = \; (\det E^\dagger)^{b+1},$$

and the remaining uncancelled part of the bosonic determinant $\det \mathcal{O}_\phi$ is the $n = 0$ factor in the infinite product,

$$\left[ \sum_{k=0}^{N} t_0^{2k} \left( \sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{E}_{i_1 \cdots i_k, j_1 \cdots j_k} \right|^2 \right) \right]^{2(0)+|b|+1} = (|\det E|^2)^{b+1},$$

using the fact that $t_0 = 0$. Putting this together, the ratio of bosonic and fermionic determinants can easily be shown to be

$$\frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi} = \frac{(\det E^\dagger)^{b+1}}{(|\det E|^2)^{b+1}} = \left( \frac{1}{\det E} \right)^{b+1}.$$

Now, let us consider the case that $b < 0$. Here, we use the fact that the $t_n$'s appearing in fermion determinants differ from the $t_n$'s appearing in bosonic determinants by a shift of $n$:

$$\text{fermion } t_n = \frac{n(n-b-1)}{R^2} = \text{boson } t_{n-1}$$

to cancel all of the factors in the infinite product appearing in the bosonic determinant $\det \mathcal{O}_\phi$, against all of the factors in the infinite product appearing in the fermionic determinant $\det \mathcal{O}_\psi$, leaving just the overall multiplicative factor of the fermionic determinant:

$$\frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi} = (S(\det E))^{|b+1|} = (\det E)^{-b-1}.$$

In any event, for any value of $b$, we get the same result:

$$Z^{1-\text{loop}} = \left( \frac{1}{\det E} \right)^{b+1}.$$

In passing, we should remark on the fact that the result above is independent of the radius $R$ of $S^2$ – a standard property for a topological field theory, but more noteworthy in these twisted (0,2) theories. One can easily check that one-loop determinants agree with (2,2 theories when E is the identity matrix.

### A.5.4 Alternative computation

In this section we briefly outline an alternative way to arrive at the same result for matter determinant ratios. Specifically, the fermion matrix $\mathcal{O}_\psi(t)$ can, after a coordinate transformation, be rearranged into the block form

$$\begin{bmatrix} E & -tI \\ tI & E^\dagger \end{bmatrix}.$$

Now, for any matrix of the form

$$\left[ \begin{array}{cc} A & B \\ C & D \end{array} \right],$$

where $A$, $B$, $C$, $D$ are square matrices of the same size, and $CD = DC$, then

$$\det \left[ \begin{array}{cc} A & B \\ C & D \end{array} \right] \; = \; \det(AD - BC).$$

In the present case, this implies

$$\det \mathcal{O}_{\psi}(t) \; = \; \det(EE^{\dagger} + t^2 I) \; = \; \det \mathcal{O}_{\phi},$$

and so we see that the fermion and boson operator determinants, for any one fixed mode, cancel one another. Of course, to complete the verification in this language, one would also need to check various special cases, as was done above.

### A.5.5 Gauge fields

Contributions from gauge multiplets in theories obtained by deforming off the (2,2) locus should be identical[6] to those in the (2,2) theory, so we can be brief.

In the conventions of *e.g.* [53][section 5.1], the contribution from the gauge multiplet (here involving both a (0,2) gauge multiplet plus adjoint-valued (0,2) chiral multiplets, typically denoted $\sigma$, so as to fill out the matter content of a (2,2) gauge multiplet) is given by

$$Z^{1-\text{loop}}_{\text{gauge}} \; = \; (-)^{\sum_{\alpha > 0} \alpha(\mathfrak{m})} \prod_{\alpha \in G} \alpha(\sigma) (d\sigma)^r$$

for $r$ the rank of $G$. As we are working along the Coulomb branch, we take $\sigma$'s lying in the maximal torus of $G$ to be the integration parameters. For example, if $G = U(1)^k$, then the contribution above is merely

$$(d\sigma)^k \; = \; d\sigma_1 \cdots d\sigma_k,$$

and if $G = U(k)$, then the contribution above takes the form

$$\prod_{\alpha \neq \beta} (\sigma_{\alpha} - \sigma_{\beta}) (d\sigma)^k.$$

[6] To be slightly careful, the $E$'s can also contribute to interaction terms between the fermions and the gauginos; however, because we localize on vanishing gauginos, those interaction terms are not relevant here.

## A.6 Operator determinants in B/2 deformations of cotangent bundles

Dual to A/2 twists of deformations of (2,2) theories are B/2 twists of deformations of cotangent bundles. The reason for that relationship is discussed in [113]; here, we will briefly focus on computing operator determinants, utilizing the results of appendix A.5 for brevity.

To begin, let us outline the GLSM's whose B/2 twists will be dual to A/2 deformations of tangent bundles. Suppose a toric variety is described mathematically as a quotient $\mathbb{C}^n//(\mathbb{C}^\times)^r$, which is to say that the GLSM has $n$ chiral superfields and $r$ $U(1)$ gauge fields. A deformation $\mathcal{E}$ of the cotangent bundle of a toric variety is described mathematically by

$$0 \longrightarrow \mathcal{E} \longrightarrow \oplus_\alpha \mathcal{O}(-\vec{q}_\alpha) \stackrel{\tilde{J}^a_\alpha}{\longrightarrow} \mathcal{O}^r \longrightarrow 0$$

and physically in (0,2) superspace language by a set of Fermi superfields $\Lambda^\alpha$ (of charge vectors $-\vec{q}_\alpha$, opposite to the charges of the chirals $\phi^\alpha$ describing the homogeneous coordinates), obeying

$$\overline{D}_+ \Lambda^\alpha \; = \; 0,$$

and a set of $r$ neutral chiral superfields $p_a$, along with a (0,2) superpotential

$$W \; = \; \Lambda^\alpha J_\alpha$$

where each $J_\alpha$ is a holomorphic function of the chiral superfields, including the $p_a$, related to the map in the short exact sequence above as

$$J_\alpha \; = \; p_a \tilde{J}^a_\alpha.$$

To help make this more clear, consider the example of $\mathbb{P}^1 \times \mathbb{P}^1$ with a deformation of the cotangent bundle. The corresponding GLSM is a $U(1)^2$ gauge theory with (0,2) chiral superfields $x_i$, $\tilde{x}_i$, of charges $(1,0)$, $(0,1)$, respectively, corresponding to homogeneous coordinates on the base space; (0,2) Fermi superfields $\Lambda^i$, $\tilde{\Lambda}^i$, of charges $(-1,0)$, $(0,-1)$, respectively, corresponding to part of the bundle; and two neutral (0,2) chiral superfields $p_a$. This theory has a (0,2) superpotential of the form

$$W \; = \; \Lambda^i (p_1 A x + p_2 B x)_i \; + \; \tilde{\Lambda}^i (p_1 C \tilde{x} + p_2 D \tilde{x})_i$$

where $A$, $B$, $C$, $D$ are four $2 \times 2$ matrices. The cotangent bundle itself is described by taking $A = D = I_{2\times 2}$, and $B = C = 0$. (It is no accident that this is the same data defining a deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$.) In the language above,

$$J_i \; = \; (p_1 A x + p_2 B x)_i, \quad \tilde{J}_i \; = \; (p_1 C \tilde{x} + p_2 D \tilde{x})_i$$

(where here $\tilde{J}$ indicates other components of $J_\alpha$, rather than components of $\tilde{J}^a_\alpha$).

Now, let us compute the bosonic interactions, in order to compute the bosonic operator determinant. From the superpotential above, after integrating out the auxiliary fields we have interaction terms

$$\sum_\alpha |J_\alpha|^2 \;=\; \sum_\alpha |p_a \tilde{J}^a_\alpha|^2.$$

We will only consider 'linear' deformations, which means that each $J_\alpha$ will be of the form

$$J_\alpha \;=\; p_a B^a_{\alpha\beta}\phi^\beta.$$

Define

$$F_{\alpha\beta} \;=\; p_a B^a_{\alpha\beta}$$

so that

$$J_\alpha \;=\; F_{\alpha\beta}\phi^\beta.$$

Thus, the bosonic potential terms are of the form

$$\sum_\alpha |J_\alpha|^2 \;=\; \sum_\alpha \left( \sum_\beta |F_{\alpha\beta}|^2 |\phi^\beta|^2 \right) \;+\; \sum_{\beta\neq\gamma} \left( \sum_\alpha F^*_{\alpha\gamma} F_{\alpha\beta} \right) (\phi^\gamma)^* \phi^\beta.$$

The boson kinetic and potential terms are then encoded by

$$\int (\vec{\phi})^\dagger \mathcal{O}_\phi \vec{\phi}$$

where

$$(\vec{\phi})^T \;=\; \left( \phi^1, \phi^2, \cdots \right),$$

and the operator $\mathcal{O}_\phi$ is

$$\left[ \begin{array}{cccc} t^2 + |F_{11}|^2 + \cdots + |F_{N1}|^2 & (F_{11})^* F_{12} + \cdots + (F_{N1})^* F_{N2} & \cdots & (F_{11})^* F_{1N} + \cdots + (F_{N1})^* F_{NN} \\ (F_{12})^* F_{11} + \cdots + (F_{N2})^* F_{N1} & t^2 + |F_{12}|^2 + \cdots + |F_{N2}|^2 & \cdots & (F_{12})^* F_{1N} + \cdots + (F_{N2})^* F_{NN} \\ \vdots & \vdots & \ddots & \vdots \\ (F_{1N})^* F_{11} + \cdots + (F_{NN})^* F_{N1} & (F_{1N})^* F_{12} + \cdots + (F_{NN})^* F_{N2} & \cdots & t^2 + |F_{1N}|^2 + \cdots + |F_{NN}|^2 \end{array} \right]$$

where $t^2$ represents $-D^2 + i\rho(D)$, for $\rho$ the gauge representation, and

$$\rho(D) \;=\; \rho(i\mathfrak{m}/2R^2) + \rho(D_0).$$

The operator $\mathcal{O}_\phi$ is essentially identical to the bosonic operator appearing in the A/2 discussion, so we can apply the same analysis as for the A/2 examples previously discussed. One ultimately finds, from equation (A.5.1), that

$$\det \mathcal{O}_\phi \;=\; \prod_{n\geq 0} \left[ \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1<i_2<\cdots<i_k, j_1<j_2<\cdots<j_k} \left| \tilde{F}_{i_1\cdots i_k, j_1\cdots j_k} \right|^2 \right) \right]^{2n+|b|+1} \tag{A.6.1}$$

for

$$t_n \;=\; \frac{2n(n+1)+(2n+1)|b|-b}{2R^2}.$$

Now, let us compute the fermion operator determinant.

Returning to the general case above, the (0,2) superpotential yields the following Yukawa-type interaction terms:

$$\lambda_-^\alpha \psi_{+pa} \tilde{F}_\alpha^a \;+\; \lambda_-^\alpha p_a \frac{\partial \tilde{F}_\alpha^a}{\partial \phi^\beta} \psi_+^\beta \;+\; \text{c.c.}$$

Although this theory no longer has $\sigma$ fields, *i.e.* (0,2) chiral multiplets which become part of the vector multiplets on the (2,2) locus, the $p_a$ play an analogous role. In particular, instead of a Coulomb branch formed from $\sigma$ vevs, in this theory we have an analogous gauge-theoretic branch formed from $p$ vevs. Thus, our analysis will closely follow the A/2 case.

Working on the $p$ branch, and expanding about constant $p$ vevs, the pertinent Yukawa interaction terms are given by

$$\lambda_-^\alpha \psi_+^\beta F_{\alpha\beta} \;+\; \overline{\lambda}_-^\alpha \overline{\psi}_+^\beta (F_{\alpha\beta})^* \tag{A.6.2}$$

where

$$F_{\alpha\beta} \;=\; p_a \frac{\partial \tilde{J}_\alpha^a}{\partial \phi^\beta}.$$

Note that the Yukawa couplings above are extremely similar to those given in the previous A/2 twisting in equation (A.5.2).

Briefly, much of the analysis of appendix A.5 carries over to this case, albeit for computations on the "$p$ branch" rather than the usual Coulomb branch of $\sigma$ vevs. For example, the bosonic operator determinant has the same form as before.

For the fermionic operator determinants, we need to work slightly harder. As before, since the interaction terms mix fermions of different chiralities, we shall work with an operator spanning both. Unlike the previous case, each $\lambda_-$ has the opposite gauge charge from $\psi_+$, so for reasons of consistency, we will use a slightly different vector of fermions, of the form

$$\vec{\psi}^T \;=\; \left( \psi_+^1, \overline{\lambda}_-^1, \psi_+^2, \overline{\lambda}_-^2, \cdots \right),$$

so that

$$\overline{\vec{\psi}} \;=\; \vec{\psi}^\dagger \gamma^0 \;=\; \left( \lambda_-^1, \overline{\psi}_+^1, \lambda_-^2, \overline{\psi}_+^2, \cdots \right),$$

so that if we write the quadratic fermion terms as

$$\int \overline{\vec{\psi}} \mathcal{O}_\psi \vec{\psi}$$

then

$$\mathcal{O}_\psi = \begin{bmatrix} F_{11} & (2/R)D_+ & F_{12} & 0 & \cdots & F_{1N} & 0 \\ (2/R)D_- & (F_{11})^* & 0 & (F_{21})^* & \cdots & 0 & (F_{N1})^* \\ F_{21} & 0 & F_{22} & (2/R)D_+ & \cdots & F_{2N} & 0 \\ 0 & (F_{12})^* & (2/R)D_- & (F_{22})^* & \cdots & 0 & (F_{N2})^* \\ \vdots & & & & \ddots & & \vdots \\ F_{N1} & 0 & F_{N2} & 0 & \cdots & F_{NN} & (2/R)D_+ \\ 0 & (F_{1N})^* & 0 & (F_{2N})^* & \cdots & (2/R)D_- & (F_{NN})^* \end{bmatrix}$$

which is clearly isomorphic to the fermion operator appearing in A/2 discussions in section A.5.2.

Expanding the $\vec{\psi}$ in modes as in the A/2 section (with the different twisting implicitly taken into account in the fact that the $\vec{\psi}$ here involves a different set of fermions), we can immediately recover an expression for the fermion determinant from equation (A.5.3):

$$(S(\det F))^{|b+1|} \prod_{n \geq 1} \left[ \sum_{k=0}^{N} t_n^{2k} \left( \sum_{i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_k} \left| \tilde{F}_{i_1 \cdots i_k, j_1 \cdots j_k} \right|^2 \right) \right]^{2n+|b+1|} \tag{A.6.3}$$

where

$$t_n = \frac{n(n+|b+1|)}{R^2},$$

$F$ the matrix given by

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & \cdots \\ F_{21} & F_{22} & F_{23} & \cdots \\ F_{31} & F_{32} & F_{33} & \cdots \\ \vdots & & & \ddots \end{bmatrix},$$

and where $S$ is the identity for $b \leq -2$ and complex conjugation for $b \geq 0$.

Now that we have computed the (one-loop) operator determinants for bosons and fermions, we can now compute

$$Z^{1-\text{loop}} = \frac{\det \mathcal{O}_\psi}{\det \mathcal{O}_\phi} = \left( \frac{1}{\det F} \right)^{b+1}$$

which is precisely dual to the result for the A/2 models describing deformations of the tangent bundle.

## A.7 Vector R-charges

In this section, we will discuss the assignment of R-charges to chiral superfields in physical models, especially for odd chiral superfields.

For A-twisted models without superpotential (e.g. without $P$-fields), we always assign vanishing R-charges to chiral superfields $\Phi_i$'s. If the superpotential is nonzero, then it must have total R-charge two, so one must assign nonzero R-charges to some of the chiral superfields.

First consider all chiral superfield $\Phi_i$ are charged under only one $U(1)$ gauge symmetry. We can mix $U(1)_R$ with this $U(1)$ to get a new $U(1)'_R$ R-symmetry [60, 122]:

$$U(1)'_R = U(1)_R + \zeta U(1),$$

where $\zeta$ is the deformation parameter. After mixing, the new $U(1)$ R-charge is

$$R'_i = R_i + \zeta Q_i.$$

If starting with $R_i = 0$, we can continuously deform it to be $R'_i = \zeta Q_i$ as the new R-charge. Therefore, nonzero R-charges assigned to (even) chiral superfields should be proportional to their weights. For convenience, we will denote $R'_i$ also as $R_i$ following without causing any confusion. Thus, the R-charges are assigned to be:

$$R_i = \zeta Q_i.$$

Now consider the $P$ field, in the superpotential $W = PG(\Phi)$, where $G(\Phi)$ is a degree $d$ polynomial in $\Phi_i$'s.

$$d = \sum_i n_i Q_i,$$

for a set of integers $\{n_i\}$ and $n_i$ comes from the power of $\Phi_i$ in one term of the (quasi-)homogeneous polynomial $G$. Then the $U(1)$ charge for this $P$-field should be $-d$. To guarantee $R_W = 2$, we need to assign the $P$ field R-charge:

$$R_P = 2 - \sum_i n_i R_i = 2 - \zeta \sum_i n_i Q_i = 2 - \zeta d.$$

In the above, when $\zeta = 0$, it agrees with the assignments in A-twisted models.

In the toric supermanifold case, odd chiral superfields and even chiral superfields share the same $U(1)$ gauge, and so we should assign R charges to those odd chiral superfields by:

$$\tilde{R}_\mu = \zeta \tilde{Q}_\mu.$$

Specifically, if we consider A-twisted theories, R charges should be assigned as

$$R_i = 0, \quad \text{and} \quad \tilde{R}_\mu = 0.$$

These computations can be generalized to multiple $U(1)$'s.

# Appendix B

# Brief Notes on the (2,2) Mirror Ansatz

The contents of this section were adapted, with minor modifications, with permission from JHEP, from our publication [27].

In this appendix we will briefly outline how symmetries and the operator mirror map partially determine the exponential terms in the (2,2) GLSM mirror superpotential. Suppose we have not derived the instanton-generated terms, and only have an ansatz for the mirror superpotential of the form

$$W \;=\; \sum_{a=1}^{k} \Sigma_a \left( \sum_{i=1}^{N} Q_i^a Y_i - t_a \right) \;+\; g(Y_i), \tag{B.0.1}$$

for some unknown function $g(Y_i)$. (Requiring R-charges match only fixes terms $\exp(-Y_i)$ up to an R-invariant function.) Instead of deriving $g$ from a direct instanton computation in the A-twisted theory, we outline here how the same result could be obtained using other properties of the theory.

Now, previously we derived the operator mirror map (1.2.7) from the form of the mirror superpotential, but one can outline an independent justification, and then use it to demonstrate the form of $g$. To see this, use the relation [25][equ'n (3.17)]

$$Y + \overline{Y} \;=\; 2\overline{\Phi} e^{2QV} \Phi,$$

which implies component relations [25][equ'ns (3.20), (3.21)]

$$\chi_+ = 2\overline{\psi}_+ \phi, \quad \overline{\chi}_- = -2\phi^\dagger \psi_-,$$

for $\chi$ the superpartners of $Y$ and $\psi$ the superpartners of $\phi$. From [16][equ'n (2.19)], the equations of motion of $\overline{\sigma}_a$ (in the limit $e^2 \to \infty$, so that the kinetic terms drop out) are

$$\left( \sum_{i=1}^{N} Q_i^a |\phi_i|^2 \right) \sigma_a \;\propto\; \sum_{i=1}^{N} \overline{\psi}_{+i} \psi_{-i}.$$

If we add twisted masses so that only one $\phi$ field is light, then this becomes

$$Q_i^a \sigma_a + \tilde{m}_i \;\propto\; \frac{\overline{\psi}_{+i} \psi_{-i}}{|\phi_i|^2} \;\propto\; \chi_{+i} \overline{\chi}_{-i}.$$

Now, the $\chi_+ \overline{\chi}_-$ could come from a $Y^2$, but that has the wrong R charge to make the expression sensible. However, $\exp(-Y_i)$ has the correct R charge and contains $Y^2$, so up to overall factors, which can be reabsorbed into field redefinitions, this suggests

$$Q_i^a \sigma_a + \tilde{m}_i \;=\; \exp(-Y_i),$$

which is the operator mirror map (1.2.7). (Granted, we are again using axial R-charges, but here since we know some components, there is less ambiguity.)

Returning to the ansatz (B.0.1), we can now determine the function $g$. The equations of motion from the superpotential above imply

$$\begin{aligned} \frac{\partial W}{\partial Y_i} &= Q_i^a \sigma_a \;+\; \frac{\partial g}{\partial Y_i}, \\ &= 0, \end{aligned}$$

and the operator mirror map implies

$$Q_i^a \sigma_a + \tilde{m}_a \;=\; \exp(-Y_i),$$

hence

$$\frac{\partial g}{\partial Y_i} \;=\; -Q_i^a \sigma_a \;=\; \tilde{m}_i - \exp(-Y_i),$$

hence

$$g(Y_i) \;=\; \tilde{m}_i Y_i + \sum_i \exp(-Y_i),$$

up to an irrelevant additive constant.

It is tempting to apply the same methods to (0,2) theories. Unfortunately, the decreased symmetry leads to multiple possible potential (0,2) mirror superpotentials, derived from applying the operator mirror map in different ways, which must be independently tested against chiral rings and correlation functions.

# Bibliography


[1] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, "Vacuum Configurations for Superstrings," Nucl. Phys. B **258**, 46 (1985). (Cited on pages 1 and 116.)

[2] L. B. Anderson and M. Karkheiran, "TASI Lectures on Geometric Tools for String Compactifications," PoS TASI **2017**, 013 (2018) [arXiv:1804.08792 [hep-th]]. (Cited on page 1.)

[3] E. Witten, "String theory dynamics in various dimensions," Nucl. Phys. B **443**, 85 (1995) [hep-th/9503124]. (Cited on page 1.)

[4] A. Strominger, S. T. Yau and E. Zaslow, "Mirror symmetry is T duality," Nucl. Phys. B **479**, 243 (1996) [hep-th/9606040]. (Cited on pages 1 and 116.)

[5] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and Exceptional Branes in F-theory - I," JHEP **0901**, 058 (2009) [arXiv:0802.3391 [hep-th]]. (Cited on page 2.)

[6] C. Beasley, J. J. Heckman and C. Vafa, "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions," JHEP **0901**, 059 (2009) [arXiv:0806.0102 [hep-th]]. (Cited on page 2.)

[7] L. B. Anderson, J. Gray, A. Lukas and E. Palti, "Two Hundred Heterotic Standard Models on Smooth Calabi-Yau Threefolds," Phys. Rev. D **84**, 106005 (2011) [arXiv:1106.4804 [hep-th]] (Cited on page 2.)

[8] L. B. Anderson, J. Gray, A. Lukas and E. Palti, "Heterotic Line Bundle Standard Models," JHEP **1206**, 113 (2012) [arXiv:1202.1757 [hep-th]]. (Cited on page 2.)

[9] M. Duncan, W. Gu, Y. H. He and D. Zhou, "The Statistics of Vacuum Geometry," JHEP **1406**, 042 (2014) [arXiv:1402.3162 [hep-th]]. (Cited on page 2.)

[10] Y. H. He, "The Calabi-Yau Landscape: from Geometry, to Physics, to Machine-Learning," arXiv:1812.02893 [hep-th]. (Cited on page 2.)

[11] C. M. Hull and E. Witten, "Supersymmetric Sigma Models and the Heterotic String," Phys. Lett. B **160**, 398 (1985) [Phys. Lett. **160B**, 398 (1985)]. (Cited on page 2.)

[12] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, "Mirror symmetry," (Cited on pages 30 and 33.)

[13] E. Witten, "Topological Sigma Models," Commun. Math. Phys. **118**, 411 (1988). (Cited on pages 2 and 4.)

[14] D. Gepner, "Space-Time Supersymmetry in Compactified String Theory and Superconformal Models," Nucl. Phys. B **296**, 757 (1988). (Cited on page 2.)

[15] B. R. Greene, C. Vafa and N. P. Warner, "Calabi-Yau Manifolds and Renormalization Group Flows," Nucl. Phys. B **324**, 371 (1989). (Cited on pages 2 and 9.)

[16] E. Witten, "Phases of N=2 theories in two-dimensions," Nucl. Phys. **B403** (1993) 159-222 [AMS/IP Stud. Adv. Math. **1** (1996) 143-211] `hep-th/9301042`. (Cited on pages 2, 3, 9, 11, 18, 22, 27, 37, 114, 116, 124, and 136.)

[17] P. S. Aspinwall, B. R. Greene and D. R. Morrison, "Calabi-Yau moduli space, mirror manifolds and space-time topology change in string theory," Nucl. Phys. B **416**, 414 (1994) [AMS/IP Stud. Adv. Math. **1**, 213 (1996)] (Cited on pages 2 and 9.)

[18] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," Commun. Math. Phys. **313**, 71 (2012) [arXiv:0712.2824 [hep-th]]. (Cited on page 2.)

[19] V. Pestun *et al.*, "Localization techniques in quantum field theories," J. Phys. A **50**, no. 44, 440301 (2017) [arXiv:1608.02952 [hep-th]]. (Cited on pages 2 and 116.)

[20] D. R. Morrison and M. R. Plesser, "Towards mirror symmetry as duality for two-dimensional abelian gauge theories," Nucl. Phys. Proc. Suppl. **46**, 177 (1996) [hep-th/9508107]. (Cited on pages 2 and 66.)

[21] V. Batyrev, "Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties," J. Algebraic Geom. **3** (1994) 493-535. (Cited on page 2.)

[22] L. Borisov, "Towards the mirror symmetry for Calabi-Yau complete intersections in toric Fano varieties," `alg-geom/9310001`. (Cited on page 2.)

[23] V. Batyrev, L. Borisov, "Dual cones and mirror symmetry for generalized Calabi-Yau manifolds, `alg-geom/9402002`. (Cited on pages 2 and 57.)

[24] T. Maxfield, D. R. Morrison and M. R. Plesser, "Mirror Symmetry and Partition Functions," arXiv:1902.05552 [hep-th]. (Cited on page 2.)

[25] K. Hori and C. Vafa, "Mirror symmetry," `hep-th/0002222`. (Cited on pages ii, iii, 2, 5, 7, 20, 21, 23, 40, 57, 58, 59, 60, 62, 63, 64, 65, 66, 68, 77, 94, 98, 115, and 136.)

[26] W. Gu and E. Sharpe, "A proposal for nonabelian mirrors," arXiv:1806.04678 [hep-th]. (Cited on pages 2, 57, 88, 93, 115, and 116.)

[27] W. Gu and E. Sharpe, "A proposal for (0,2) mirrors of toric varieties," JHEP **1711** (2017) 112, arXiv:1707.05274. (Cited on pages 2, 20, 57, 61, 73, 93, 115, and 136.)

[28] D. R. Morrison and M. R. Plesser, "Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties," Nucl. Phys. B **440** (1995) 279-354, hep-th/9412236. (Cited on pages 3, 7, 27, 28, 29, 30, 37, and 99.)

[29] A. Caldararu, J. Distler, S. Hellerman, T. Pantev and E. Sharpe, "Non-birational twisted derived equivalences in abelian GLSMs," Commun. Math. Phys. **294** (2010) 605-645, arXiv:0709.3855. (Cited on pages 9 and 61.)

[30] T. Pantev and E. Sharpe, "Notes on gauging noneffective group actions," hep-th/0502027. (Cited on pages 9 and 61.)

[31] T. Pantev and E. Sharpe, "GLSM's for gerbes (and other toric stacks)," Adv. Theor. Math. Phys. **10** (2006) 77-121, hep-th/0502053. (Cited on pages 9 and 61.)

[32] T. Pantev and E. Sharpe, "String compactifications on Calabi-Yau stacks," Nucl. Phys. B **733** (2006) 233-296, hep-th/0502044. (Cited on page 9.)

[33] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe and M. Ando, "Cluster decomposition, T-duality, and gerby CFT's," Adv. Theor. Math. Phys. **11** (2007) 751-818, hep-th/0606034. (Cited on page 61.)

[34] M. Bertolini and M. R. Plesser, "(0,2) hybrid models," JHEP **1809**, 067 (2018) [arXiv:1712.04976 [hep-th]] (Cited on pages 9 and 61.)

[35] M. Bertolini and M. Romo, "Aspects of (2,2) and (0,2) hybrid models," arXiv:1801.04100 [hep-th]. (Cited on pages 9 and 61.)

[36] Z. Chen, T. Pantev and E. Sharpe, "Landau–Ginzburg models for certain fiber products with curves," J. Geom. Phys. **137**, 204 (2019) [arXiv:1806.01283 [hep-th]] (Cited on page 9.)

[37] E. Witten, "The Verlinde algebra and the cohomology of the Grassmannian," In *Cambridge 1993, Geometry, topology, and physics* 357-422 [hep-th/9312104]. (Cited on pages 77 and 84.)

[38] K. Hori and D. Tong, "Aspects of non-Abelian gauge dynamics in two-dimensional N=(2,2) theories," JHEP **0705** (2007) 079, hep-th/0609032. (Cited on page 84.)

[39] K. Hori, "Duality in two-dimensional (2,2) supersymmetric non-Abelian gauge theories," JHEP **1310** (2013) 121, arXiv:1104.2853. (Cited on pages 27 and 59.)

[40] R. Donagi and E. Sharpe, "GLSM's for partial flag manifolds," J. Geom. Phys. **58** (2008) 1662-1692, arXiv:0704.1761. (Cited on pages 88 and 93.)

[41] C. Closset, S. Cremonesi and D. S. Park, "The equivariant A-twist and gauged linear sigma models on the two-sphere," JHEP **1506** (2015) 076, arXiv:1504.06308. (Cited on pages 11, 14, 27, 34, 35, 37, 43, 45, 46, 47, 50, 51, 67, 68, 73, 98, 118, 119, and 120.)

[42] C. Closset, W. Gu, B. Jia and E. Sharpe, "Localization of twisted $\mathcal{N} = (0,\ 2)$ gauged linear sigma models in two dimensions," JHEP **1603** (2016) 070, arXiv:1512.08058. (Cited on pages 10, 26, 63, 73, 95, 105, 109, 111, 114, 116, and 118.)

[43] F. Benini and S. Cremonesi, "Partition functions of $\mathcal{N} = (2, 2)$ gauge theories on $S^2$ and vortices," Commun. Math. Phys. **334** (2015) 1483-1527, arXiv:1206.2356. (Cited on pages 65 and 119.)

[44] N. Doroud, J. Gomis, B. Le Floch and S. Lee, "Exact results in D=2 supersymmetric gauge theories," JHEP **1305** (2013) 093, arXiv:1206.2606. (Cited on pages 65 and 119.)

[45] A. Adams, H. Jockers, V. Kumar and J. M. Lapan, "N=1 Sigma Models in $AdS_4$," JHEP **1112**, 042 (2011) [arXiv:1104.3155 [hep-th]]. (Cited on page 10.)

[46] C. Closset and S. Cremonesi, "Comments on $\mathcal{N} = (2,\ 2)$ supersymmetry on two-manifolds," JHEP **1407**, 075 (2014) (Cited on pages 10, 15, 118, and 119.)

[47] C. Closset, T. T. Dumitrescu, G. Festuccia and Z. Komargodski, "Supersymmetric Field Theories on Three-Manifolds," JHEP **1305**, 017 (2013) (Cited on page 10.)

[48] T. T. Dumitrescu and N. Seiberg, "Supercurrents and Brane Currents in Diverse Dimensions," JHEP **1107**, 095 (2011) (Cited on pages 10 and 18.)

[49] S. Sethi, "Supermanifolds, rigid manifolds and mirror symmetry," Nucl. Phys. B **430** (1994) 31-50 [AMS/IP Stud. Adv. Math. **1** (1996) 793-815] [hep-th/9404186]. (Cited on pages 26, 29, 32, 37, 38, and 39.)

[50] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," JHEP **1106**, 114 (2011) [arXiv:1105.0689 [hep-th]]. (Cited on page 10.)

[51] J. Gomis, P. S. Hsin, Z. Komargodski, A. Schwimmer, N. Seiberg and S. Theisen, "Anomalies, Conformal Manifolds, and Spheres," JHEP **1603**, 022 (2016) [arXiv:1509.08511 [hep-th]]. (Cited on page 10.)

[52] N. A. Nekrasov and S. L. Shatashvili, "Bethe/Gauge correspondence on curved spaces," JHEP **1501** (2015) 100, arXiv:1405.6046. (Cited on pages 22, 34, 73, 104, and 105.)

[53] F. Benini, A. Zaffaroni, "A topologically twisted index for three-dimensional supersymmetric theories," JHEP **1507** (2015) 127, `arXiv:1504.03698`. (Cited on pages 22, 34, 35, 52, 104, 121, 123, and 130.)

[54] N. A. Nekrasov and S. L. Shatashvili, "Supersymmetric vacua and Bethe ansatz," Nucl. Phys. Proc. Suppl. **192-193**, 91 (2009) [arXiv:0901.4744 [hep-th]]. (Cited on page 27.)

[55] A. S. Schwarz, "Sigma models having supermanifolds as target spaces," Lett. Math. Phys. **38** (1996) 91-96 [hep-th/9506070]. (Cited on pages 26, 29, 30, 32, 37, 38, 39, and 114.)

[56] E. Witten, "Notes On Supermanifolds and Integration," arXiv:1209.2199 [hep-th]. (Cited on page 29.)

[57] S. Seki and K. Sugiyama, "Gauged linear sigma model on supermanifold," hep-th/0503074. (Cited on pages 30, 31, 32, and 34.)

[58] F. Benini and A. Zaffaroni, "Supersymmetric partition functions on Riemann surfaces," Proc. Symp. Pure Math. **96**, 13 (2017) [arXiv:1605.06120 [hep-th]]. (Cited on page 34.)

[59] M. Herbst, K. Hori and D. Page, "Phases Of N=2 Theories In 1+1 Dimensions With Boundary," arXiv:0803.2045 [hep-th]. (Cited on pages 34 and 116.)

[60] F. Benini, R. Eager, K. Hori and Y. Tachikawa, "Elliptic genera of two-dimensional $\mathcal{N}$ = 2 gauge theories with rank-one gauge groups," Lett. Math. Phys. **104**, 465 (2014) [arXiv:1305.0533 [hep-th]]. (Cited on pages 35, 36, 37, 39, 41, and 135.)

[61] E. Witten, "On the Landau-Ginzburg description of N=2 minimal models," Int. J. Mod. Phys. A **9**, 4783 (1994) [hep-th/9304026]. (Cited on page 35.)

[62] F. Benini, R. Eager, K. Hori and Y. Tachikawa, "Elliptic Genera of 2d $\mathcal{N} = 2$ Gauge Theories," Commun. Math. Phys. **333**, no. 3, 1241 (2015) [arXiv:1308.4896 [hep-th]]. (Cited on pages 35, 41, 47, 48, and 51.)

[63] A. Gadde and S. Gukov, "2d Index and Surface operators," JHEP **1403**, 080 (2014) [arXiv:1305.0266 [hep-th]] (Cited on pages 35 and 41.)

[64] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison and M. Romo, "Two-sphere partition functions and Gromov-Witten invariants," Commun. Math. Phys. **325** (2014) 1139-1170, `arXiv:1208.6244`. (Cited on pages 37 and 114.)

[65] W. Gu and H. Zou, "Supersymmetric Localization in GLSMs for Supermanifolds," arXiv:1807.05455 [hep-th]. (Cited on pages 26 and 39.)

[66] J. Halverson, V. Kumar and D. R. Morrison, "New methods for characterizing phases of 2D supersymmetric gauge theories," JHEP **1309** (2013) 143, `arXiv:1305.3278`. (Cited on pages 40, 41, and 65.)

[67] Ciocan-Fontanine, Ionuţ; Kim, Bumsig; Sabbah, Claude, "The Abelian/Nonabelian correspondence and Frobenius manifolds," Inventiones mathematicae, vol. 171, issue 2, pp. 301-343 [arXiv:math/0610265 [math.AG]]. (Cited on page 40.)

[68] J. McOrist, I. Melnikov, "Summing the instantons in half-twisted linear sigma models," JHEP **0902** (2009) 026, `arXiv:0810.0012`. (Cited on pages 51, 53, 95, 105, 109, and 111.)

[69] R. Donagi, J. Guffin, S. Katz, E. Sharpe, "A mathematical theory of quantum sheaf cohomology," Asian J. Math. **18** (2014) 387-418, `arXiv:1110.3751`. (Cited on pages 51, 53, 95, 109, and 111.)

[70] R. Donagi, J. Guffin, S. Katz, E. Sharpe, "Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties," Adv. Theor. Math. Phys. **17** (2013) 1255-1301, `arXiv:1110.3752`. (Cited on pages 51, 52, 53, 95, 109, and 111.)

[71] N. Jacobson, *Lie algebras*, Dover, New York, 1962. (Cited on page 60.)

[72] W. Fulton, J. Harris, *Representation theory: a first course*, Graduate texts in mathematics 129, Springer-Verlag, New York, 1991. (Cited on page 60.)

[73] E. Sharpe, "Notes on certain other (0,2) correlation functions," Adv. Theor. Math. Phys. **13** (2009) 33-70, `hep-th/0605005`. (Cited on pages 60, 78, and 131.)

[74] Y. Tachikawa, "N=2 supersymmetric dynamics for pedestrians," Lect. Notes Phys. **890** (2014), `arXiv:1312.2684`. (Cited on page 61.)

[75] K. A. Intriligator and C. Vafa, "Landau-Ginzburg orbifolds," Nucl. Phys. B **339** (1990) 95-120. (Cited on page 61.)

[76] S. Kachru and E. Witten, "Computing the complete massless spectrum of a Landau-Ginzburg orbifold," Nucl. Phys. B **407** (1993) 637-666, `hep-th/9307038`. (Cited on page 61.)

[77] J. Guffin and E. Sharpe, "A-twisted Landau-Ginzburg models," J. Geom. Phys. **59** (2009) 1547-1580, `arXiv:0801.3836`. (Cited on pages 27, 61, 63, and 70.)

[78] C. Vafa, "Topological Landau-Ginzburg models," Mod. Phys. Lett. A **6** (1991) 337-346. (Cited on pages 61, 70, 99, and 104.)

[79] T. Eguchi, K. Hori, C.-S. Xiong, "Gravitational quantum cohomology," Int. J. Mod. Phys. **A12** (1997) 1743-1782, `hep-th/9605225`. (Cited on page 77.)

[80] K. Rietsch, "A mirror symmetry construction for $qH_T^*(G/P)_q$," Adv. Math. **217** (2008) 2401-2442, `math/0511124`. (Cited on page 77.)

[81] R. Marsh, K. Rietsch, "The B-model connection and mirror symmetry for Grassmannians," arXiv: 1307.1085. (Cited on page 77.)

[82] K. Rietsch, L. Williams, "Newton-Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians," arXiv: 1712.00447. (Cited on page 77.)

[83] J. Gomis and S. Lee, "Exact Kahler potential from gauge theory and mirror symmetry," JHEP **1304** (2013) 019, arXiv:1210.6022. (Cited on pages 65, 77, and 98.)

[84] P. Berglund and T. Hubsch, "On Calabi-Yau generalized complete intersections from Hirzebruch varieties and novel K3-fibrations," arXiv:1606.07420. (Cited on page 78.)

[85] P. Berglund and T. Hubsch, "A generalized construction of Calabi-Yau models and mirror symmetry," SciPost Phys. **4** (2018) 009, arXiv:1611.10300. (Cited on page 78.)

[86] L. B. Anderson, F. Apruzzi, X. Gao, J. Gray and S. J. Lee, "A new construction of Calabi–Yau manifolds: generalized CICYs," Nucl. Phys. B **906** (2016) 441-496, arXiv:1507.03235. (Cited on page 78.)

[87] O. Aharony, S. S. Razamat, N. Seiberg and B. Willett, "The long flow to freedom," JHEP **1702** (2017) 056, arXiv:1611.02763. (Cited on pages 68 and 115.)

[88] E. Sharpe, "Decomposition in diverse dimensions," Phys. Rev. D **90** (2014) 025030, arXiv:1404.3986. (Cited on page 61.)

[89] K. Hori and M. Romo, "Exact results in two-dimensional (2,2) supersymmetric gauge theories with boundary," arXiv:1308.2438. (Cited on pages 27, 63, 64, and 71.)

[90] N. Cabo Bizet, A. Martínez-Merino, L. A. Pan, "Non abelian T-duality in gauged linear sigma models," JHEP **1804** (2018) 054, arXiv:1711.08491. (Cited on page 68.)

[91] J. Guffin and S. Katz, "Deformed Quantum Cohomology and (0,2) Mirror Symmetry," JHEP **1008**, 109 (2010) [arXiv:0710.2354 [hep-th]]. (Cited on page 54.)

[92] P. Powell, "Calculating determinants of block matrices," arXiv:1112.4379. (Cited on pages 75, 76, and 86.)

[93] F. Hirzebruch, T. Höfer, "On the Euler number of an orbifold," Math. Ann. **286** (1990) 255-260. (Cited on page 78.)

[94] J. Guo, Z. Lu and E. Sharpe, "Quantum sheaf cohomology on Grassmannians," Commun. Math. Phys. **352** (2017) 135-184, arXiv:1512.08586. (Cited on pages 79, 84, 85, and 87.)

[95] K. Hori and J. Knapp, "A pair of Calabi-Yau manifolds from a two parameter non-Abelian gauged linear sigma model," arXiv:1612.06214. (Cited on page 82.)

[96] A. S. Buch, “Quantum cohomology of Grassmannians”, Compositio Math. **137** (2003) 227-235, `math/0106268`. (Cited on page 84.)

[97] A. S. Buch, A. Kresch, H. Tamvakis, “Gromov-Witten invariants on Grassmannians”, J. Amer. Math. Soc. **16** (2003) 901-915, `math/0306388`. (Cited on page 84.)

[98] H. Tamvakis, “Gromov-Witten invariants and quantum cohomology of Grassmannians,” pp. 271-297 in *Topics in cohomological studies of algebraic varieties*, Trends Math., Birkhäuser, Basel, 2005, `math/0306415`. (Cited on page 84.)

[99] A. Bertram, “Quantum Schubert calculus,” Adv. Math. **128** (1997) 289-305, `alg-geom/9410024`. (Cited on page 84.)

[100] B. Siebert, G. Tian, “On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator,” Asian J. Math. **1** (1997) 679-695, `alg-geom/9403010`. (Cited on page 84.)

[101] M. Aganagic and C. Vafa, “Mirror symmetry and supermanifolds,” Adv. Theor. Math. Phys. **8** (2004) 939-954, `hep-th/0403192`. (Cited on page 31.)

[102] Z. Chen, E. Sharpe and R. Wu, “Toda-like (0,2) mirrors to products of projective spaces,” JHEP **1608** (2016) 093, `arXiv:1603.09634`. (Cited on pages 106 and 107.)

[103] Z. Chen, J. Guo, E. Sharpe and R. Wu, “More Toda-like (0,2) mirrors,” JHEP **1708** (2017) 079, `arXiv:1705.08472`. (Cited on pages 106, 107, 108, 111, and 113.)

[104] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, New York, 1978. (Cited on page 46.)

[105] R. Blumenhagen, R. Schimmrigk, A. Wisskirchen, “(0,2) mirror symmetry,” Nucl. Phys. **B486** (1997) 598-628, `hep-th/9609167`. (Cited on page 113.)

[106] A. Adams, A. Basu and S. Sethi, “(0,2) duality,” Adv. Theor. Math. Phys. **7** (2003) 865-950, `hep-th/0309226`. (Cited on pages 55 and 108.)

[107] I. Melnikov, R. Plesser, “A (0,2) mirror map,” JHEP **1102** (2011) 001, `arXiv:1003.1303`. (Cited on page 113.)

[108] I. Melnikov, S. Sethi and E. Sharpe, “Recent developments in (0,2) mirror symmetry,” SIGMA **8** (2012) 068, `arXiv:1209.1134`. (Cited on page 113.)

[109] Z. Chen, E. Sharpe and R. Wu, “Toda-like (0,2) mirrors to products of projective spaces,” JHEP **1608** (2016) 093, `arXiv:1603.09634`. (Cited on pages 106 and 107.)

[110] Z. Chen, J. Guo, E. Sharpe, R. Wu, “More Toda-like (0,2) mirrors,” `arXiv:1705.08472`. (Cited on pages 106, 107, 108, 111, and 113.)

[111] S. Katz, E. Sharpe, “Notes on certain (0,2) correlation functions,” Comm. Math. Phys. **262** (2006) 611-644, `hep-th/0406226`. (Cited on page 42.)

[112] A. Adams, J. Distler, M. Ernebjerg, “Topological heterotic rings,” Adv. Theor. Math. Phys. **10** (2006) 657-682, `hep-th/0506263`. (Cited on page 116.)

[113] E. Sharpe, “Notes on certain other (0,2) correlation functions,” Adv. Theor. Math. Phys. **13** (2009) 33-70, `hep-th/0605005`. (Cited on pages 60, 78, and 131.)

[114] I. Melnikov, S. Sethi, “Half-twisted (0,2) Landau-Ginzburg models,” JHEP **0803** (2008) 040, `arXiv:0712.1058`. (Cited on pages 105 and 106.)

[115] J. McOrist, I. Melnikov, “Half-twisted correlators from the Coulomb branch,” JHEP **0804** (2008) 071, `arXiv:0712.3272`. (Cited on page 44.)

[116] M. Kreuzer, J. McOrist, I. Melnikov, R. Plesser, “(0,2) deformations of linear sigma models,” JHEP **1107** (2011) 044, `arXiv:1001.2104`. (Cited on page 54.)

[117] J. McOrist, “The revival of (0,2) linear sigma models,” Int. J. Mod. Phys. **A26** (2011) 1-41, `arXiv:1010.4667`. (Cited on page 42.)

[118] J. Guffin, “Quantum sheaf cohomology, a precis,” Matemática Contemporânea **41** (2012) 17-26, `arXiv:1101.1305`. (Cited on page 42.)

[119] R. Donagi, Z. Lu, I. Melnikov, “Global aspects of (0,2) moduli space: toric varieties and tangent bundles,” Comm. Math. Phys. **338** (2015) 1197-1232, `arXiv:1409.4353`. (Cited on page 54.)

[120] J. Guo, Z. Lu, E. Sharpe, “Classical sheaf cohomology rings on Grassmannians,” `arXiv:1605.01410`, to appear in J. Algebra. (Cited on page 42.)

[121] E. Witten, “Superstring Perturbation Theory Revisited,” arXiv:1209.5461 [hep-th]. (Cited on page 114.)

[122] F. Benini and B. Le Floch, “Supersymmetric localization in two dimensions,” J. Phys. A **50**, no. 44, 443003 (2017) [arXiv:1608.02955 [hep-th]]. (Cited on page 135.)

[123] J. Guffin and E. Sharpe, “A-twisted Landau-Ginzburg models,” J. Geom. Phys. **59** (2009) 1547 [arXiv:0801.3836 [hep-th]]. (Cited on pages 27, 61, 63, and 70.)

[124] Z. Chen, W. Gu, H. Parsian and E. Sharpe, “Two-dimensional supersymmetric gauge theories with exceptional gauge groups,” arXiv:1808.04070 [hep-th]. (Cited on pages 93 and 115.)

[125] K. A. Intriligator and N. Seiberg, “Mirror symmetry in three-dimensional gauge theories,” Phys. Lett. B **387**, 513 (1996) [hep-th/9607207]. (Cited on page 115.)

[126] O. Aharony, S. S. Razamat and B. Willett, "From 3d duality to 2d duality," JHEP **1711**, 090 (2017) [arXiv:1710.00926 [hep-th]]. (Cited on page 115.)

[127] K. Hori, A. Iqbal and C. Vafa, "D-branes and mirror symmetry," hep-th/0005247. (Cited on page 116.)

[128] E. R. Sharpe, "D-branes, derived categories, and Grothendieck groups," Nucl. Phys. B **561**, 433 (1999) [hep-th/9902116]. (Cited on page 116.)

[129] M. R. Douglas, "D-branes, categories and N=1 supersymmetry," J. Math. Phys. **42**, 2818 (2001) [hep-th/0011017]. (Cited on page 116.)

[130] S. H. Katz and E. Sharpe, "D-branes, open string vertex operators, and Ext groups," Adv. Theor. Math. Phys. **6**, 979 (2003) [hep-th/0208104]. (Cited on page 116.)

[131] R. Donagi, S. Katz and E. Sharpe, "Spectra of D-branes with higgs vevs," Adv. Theor. Math. Phys. **8**, no. 5, 813 (2004) [hep-th/0309270]. (Cited on page 116.)

[132] A. Adams, J. Distler and M. Ernebjerg, "Topological heterotic rings," Adv. Theor. Math. Phys. **10**, no. 5, 657 (2006) [hep-th/0506263]. (Cited on page 116.)

[133] E. Sharpe, "Lectures on D-branes and sheaves," hep-th/0307245. (Cited on page 116.)

[134] P. S. Aspinwall, "D-branes on Calabi-Yau manifolds," hep-th/0403166. (Cited on page 116.)