ANALYTICAL PROPERTIES OF FEYNMAN GRAPHS

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(presented by I. T. Todorov)

At the previous Rochester Conference (1960) Eden, Polkinghorne *et al.* reported work devoted to the proof of the Mandelstam representation in any finite order of perturbation theory. As is known, later on it was found that the hopes for proving this were too optimistic. Such proof has not been obtained until now.

In the present report we consider the two following problems.

1. Proof of one-dimensional dispersion relations in any finite order of perturbation theory by the method of majorization of diagrams, developed earlier [1-4] and the derivation of dispersion relations for partial wave amplitudes of the nucleon-nucleon and pion-pion scattering.

2. Derivation of parametrical equations of the surface of singularities of an arbitrary diagram and their application.

A detailed account of these results is given in the preprints of the authors [5, 6].

1. PROOF OF DISPERSION RELATIONS IN PERTURBATION THEORY

In papers [3, 4] it was shown that all strongly connected diagrams of scattering processes involving π-mesons and nucleons are majorized in the Euclidean region by a small number of the simplest diagrams.

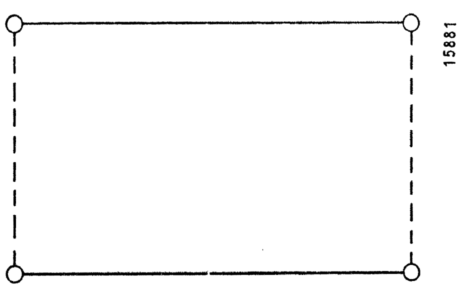

Fig. 1

So, e.g., in the case of the nucleon-nucleon scattering the graph represented in Fig. 1 with different numeration of vertices majorizes all strongly connected diagrams of this process [4, 5]. This diagram gives the maximal domain G_{NN} in the space of the Euclidean external momenta in which the quadratic form of every (strongly connected) diagram of the nucleon-nucleon scattering is negative:

$$Q(\alpha, p) = \sum_{i,j=1}^{3} A_{ij}(\alpha) p_i p_j - \sum_{\nu=1}^{l} \alpha_\nu m_\nu^2 < 0 , \qquad (1.1)$$

for all $\alpha_\nu \geqq 0$ for which $\sum_{\nu=1}^{l} \alpha_\nu > 0$. Here m_ν is the mass on the ν-th line, l is the number of the internal lines of the diagram. It is found that the knowledge of the domain G_{NN} in the Euclidean space allows one to find the domain described by this same inequality (1.1) in the case when the momenta p_i (with real scalar products) belong to a pseudo-Euclidean space with arbitrary signature. For this purpose we decompose vectors p_i into mutually orthogonal Euclidean and anti-Euclidean parts

$$p_i = P_i + Q_i , \qquad P_i Q_j = 0 \qquad (1.2)$$

$$(P_i^2 > 0 ; \; Q_j^2 < 0 ; \; i, j = 1, \ldots, 4)$$

The explicit expressions for the vectors P_i and Q_i depend, generally speaking, on invariants s and t composed of vectors p. If the p_i's lie on the mass shell:

$$p_1^2 = p_2^2 = p_3^2 = p_4^2 = M^2 , \qquad (1.3)$$

then the construction of vectors P_i and Q_i is carried out simply: for this we have to expand the vectors p_i in three orthogonal vectors

$$q_1 = p_1 + p_2\,, \quad q_2 = p_1 + p_3\,, \quad q_3 = p_2 + p_3\,, \quad (1.4)$$

$$(q_1^2 = s,\ q_2^2 = t,\ q_3^2 = u)\,.$$

From the positive definiteness of the matrix $A_{ij}(\alpha)$ it follows that the form $Q(\alpha, P)$ (depending on the P with real scalar products) is smaller than the same form $Q(\alpha, P)$ of the corresponding Euclidean momenta P. This result, obtained in a straightforward way, includes as a special case the substitutions which were used in [1,7,8]. It should be noted that when the momenta p_i are on the mass shell (1.3), the Euclidean vectors P_i are not (the P_i^2 are changing with s and t). Therefore it is essential that in previous papers [2-4] we carried out the majorization of diagrams for arbitrary (variable) squares of external momenta. Thus, we find that the nucleon-nucleon elastic scattering amplitude is expressed in terms of real analytical functions of the scalar products of p_i's (satisfying (1.3)) regular in the triangle

$$G_{NN}\{s, t\}:\ s<4M^2,\ t<4M^2,\ u<4M^2 \qquad (1.5)$$

on the plane $s+t+u = 4M^2$ (M is the nucleon mass, m is the π-meson mass).

Then using the fact that $Q(\alpha, p)$ is a linear function of the invariants s and t, it can be shown by the Wu method [8] that the NN-scattering amplitude is analytical [(*)] also in a complex neighbourhood $\widetilde{G}_{NN}$ of the real domain G_{NN}. A point ($s = s'+is''$, $t = t'+it''$) lies in the domain $\widetilde{G}_{NN}$, if it is possible to find such real λ, so that the point ($s_\lambda = s'+\lambda s''$, $t_\lambda = t'+\lambda t''$) lies in the (real) domain G_{NN}. These analytical properties of the NN-scattering amplitude are sufficient to prove one-dimensional dispersion relations for fixed $t(-4m^2<t<4m^2)$ as well as for fixed s ($4(M^2-m^2)<s<4M^2$) and also for (any) fixed $\cos\theta$ (θ is the scattering angle in c.m.s.). From dispersion relations for fixed $\cos\theta$ we can deduce dispersion relations for the partial wave NN-scattering amplitudes. This derivation is more direct and conceptually simpler than that obtained in [9,10] which is based on a rather non-trivial integral representation of the scattering amplitude suggested in [9] [(**)].

In the case of meson-nucleon scattering it is possible to prove dispersion relations for fixed t or s too, while dispersion relations for the square of three-dimensional momenta in c.m.s. for fixed $\cos\theta$ are established only in the interval

$$1-2\left(\frac{m}{M}\right)^2 \leqq \cos\theta \leqq 1 \qquad (1.6)$$

and this is not sufficient to obtain dispersion relations for the πN-scattering partial waves.

2. SURFACE OF SINGULARITIES FOR AN ARBITRARY DIAGRAM

In formulating and proving the Symanzik theorem on the majorization of diagrams and its generalization [3,4] we have characterised each diagram by an incidence matrix $E = (\varepsilon_{i\nu})$ [(†)] ($\varepsilon_{i\nu} = 1$, if the line ν goes out from the vertex i; $\varepsilon_{i\nu} = -1$, if the line ν enters the vertex i, $\varepsilon_{i\nu} = 0$, if the vertex i does not lie on the line ν). The norm of an arbitrary diagram in the coordinate space (in terms of which the Symanzik theorem is formulated) has the form:

$$L(x) = \sum_{\nu=1}^{l} m_\nu |R_\nu|\,, \qquad (2.1)$$

where

$$R_\nu = \sum_{i=1}^{n} \varepsilon_{i\nu} x_i \qquad (2.2)$$

(n is the number of the diagram vertices, x_i are four-dimensional vectors).

It turns out that the technique of the incidence matrix and of the norm (2.1) is a convenient tool, not only for the comparison of the quadratic forms of different diagrams (i.e. in the majorization procedure) but also for the description of the singularities of a

(*) Speaking about the analytical properties of the amplitude, we have in mind the analytical properties of any partial sum of the perturbation series representing this amplitude.

(**) This representation may be proved by the majorization method (it follows easily from the fact that $Q(\alpha, p) < 0$ for all strongly connected graphs in the domain (1.5)). In the equal mass case (the $\pi\pi$ scattering) a derivation of the spectral representation for partial wave amplitudes is given also in ref. 13, (but without proof of (1.5)).

(†) The incidence matrix was introduced by H. Poincaré in 1901 and is widely used in topology.

given diagram. Following Landau [11] we call the points p for which the set of equations

$$\frac{\partial Q(\alpha, p)}{\partial \alpha_1} = 0, \quad \dots \quad, \frac{\partial Q(\alpha, p)}{\partial \alpha_l} = 0, \qquad (2.3)$$

has a solution $(\alpha_1, \dots, \alpha_l)$ with all non-zero α_ν proper singularities of a graph. We have shown [6] that the surface of proper singularities of a graph is given by the set of parametrical equations

$$p_i = \frac{\partial L}{\partial x_i} = \sum_{\nu=1}^{l} \varepsilon_{i\nu} m_\nu \frac{R_\nu}{|R_\nu|}, \qquad i = 1, \dots, n \qquad (2.4)$$

where $L(x)$ and R_ν are given by Eqs. (2.1) and (2.2). Eqs. (2.4) are applicable not only for any real x, but also for complex x (according to [12]). However, there remains still the important question of finding an algebraic criterion which allows to decide whether or not a complex point p of the surface (2.4) is a singularity on the main Riemann sheet of the scattering amplitude (i.e. on the so-called " physical " sheet).

Eqs. (2.4) point not only to a connection between the technique used by Symanzik [1] and the Landau equations [11] but are also a convenient tool for finding the graph singularities. The main advantage of Eqs. (2.4) compared to the initial Eqs. (2.3) is that Eqs. (2.4) allow us to express all momenta on the surface of singularities in terms of a minimal number of independent vectors.

We illustrate the efficiency of Eqs. (2.4) by a non-trivial example of the fourth order scattering diagram

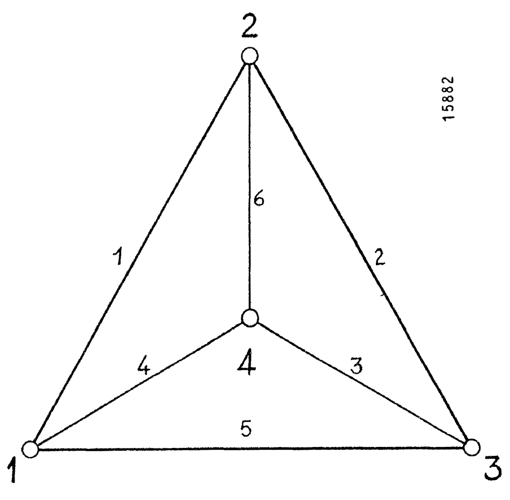

Fig. 2

((Fig. 2) all vertices in it are external). We assume that all masses of the internal lines of the diagram are equal to unity and that the squares of the external momenta are equal to each other (1.3). From (2.4) and (1.3) it follows that (if we assume $x_4 = 0$) the vectors

$$\begin{aligned} \xi_1 &= \tfrac{1}{2}(x_1 + x_2 - x_3), \\ \xi_2 &= \tfrac{1}{2}(x_1 - x_2 + x_3), \\ \xi_3 &= \tfrac{1}{2}(-x_1 + x_2 + x_3) \end{aligned} \qquad (2.5)$$

are mutually orthogonal and that the modules of the vectors (2.2) are

$$\begin{aligned} \lambda_1 &= |R_1| = |R_3| = \sqrt{\xi_2^2 + \xi_3^2}, \\ \lambda_2 &= |R_5| = |R_6| = \sqrt{\xi_1^2 + \xi_3^2}, \\ \lambda_3 &= |R_2| = |R_4| = \sqrt{\xi_1^2 + \xi_2^2}. \end{aligned} \qquad (2.6)$$

By expressing the orthogonal vectors q_i (1.4) by the ξ_j with the help of (2.4) and taking the square of the equalities obtained, we find the parametrical equations of the surface of singularities of the diagram of Fig. 2 in invariant variables:

$$\begin{aligned} s &= 2(\lambda_2^2 + \lambda_3^2 - \lambda_1^2)\left(\frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)^2, \\ t &= 2(\lambda_1^2 + \lambda_3^2 - \lambda_2^2)\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_3}\right)^2, \\ u &= 2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2)\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)^2. \end{aligned} \qquad (2.7)$$

The right-hand sides of (2.7) are homogeneous functions (of degree zero) of the parameters λ, therefore it is possible to introduce two independent parameters only, making use e.g. of the normalization $\lambda_1 + \lambda_2 + \lambda_3 = 1$. In the particular case, when $M = 1$ (see (1.3)) the parameters λ can be excluded. We obtain the equation for the singularity curve in the form:

$$s^{1/3} + t^{1/3} + u^{1/3} = 16^{1/3}, \qquad s + t + u = 4 \qquad (2.8)$$

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COSMOLOGICAL INTERPRETATION OF THE MASS SPECTRUM OF ELEMENTARY PARTICLES

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Ten years ago Nambu [1] proposed a mass formula on the basis of contemporary information about new particles. According to this formula the mass of any particle except the massless photon and neutrino is well approximated by

$$m = n \cdot 137\, m_e\,, \qquad (1)$$

where m_e is the electron mass and n is an integral or a half integral number according to whether the particle concerned is a boson or a fermion, and the mysterious number 137 may be equated to $\hbar c/e^2$. Although the meaning of this formula is not understood, it has been found to agree fairly well with observed masses of elementary particles.

Some attempts have been put forward to explain this formula within the framework of atomic physics or particle physics. However, physics in the current sense is devoted to finding out some relations between phenomena or laws governing the world of matter. In doing so the existence of matter and of its fundamental properties is implicitly assumed. More specifically, the fundamental physical constants are assumed to be given and so are the particles, or things which have energies, momenta, charges and other well defined properties if one is reluctant to use the term particle. Therefore, physics does not seem to be suitable to explain such things which are to be assumed. On the other hand, there is a branch of science which aims at understanding the existence