

FEYNMAN INTEGRAL IN $\mathbb{R}^1 \oplus \mathbb{R}^m$ AND COMPLEX EXPANSION OF ${}_2F_1$

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To Professor Hari Mohan Srivastava on the occasion of his 75th birthday anniversary

ABSTRACT. Closed form expressions are proposed for the Feynman integral

$$I_{D,m}(p, q) = \int \frac{d^m \mathbf{y}}{(2\pi)^m} \int \frac{d^D \mathbf{x}}{(2\pi)^D} \frac{1}{(\mathbf{x} - \mathbf{p}/2)^2 + (\mathbf{y} - \mathbf{q}/2)^4} \frac{1}{(\mathbf{x} + \mathbf{p}/2)^2 + (\mathbf{y} + \mathbf{q}/2)^4} \quad (1)$$

over $d = D + m$ dimensional space with $(\mathbf{x}, \mathbf{y}), (\mathbf{p}, \mathbf{q}) \in \mathbb{R}^D \oplus \mathbb{R}^m$, in the special case $D = 1$. We show that $I_{1,m}(p, q)$ can be expressed in different forms involving real and imaginary parts of the complex variable Gauss hypergeometric function ${}_2F_1$, as well as generalized hypergeometric ${}_2F_2$ and ${}_3F_2$, Horn H_4 and Appell F_2 functions. Several interesting relations are derived between the real and imaginary parts of ${}_2F_1$ and the function H_4 .

Keywords: Feynman integral; Gauss hypergeometric function; Generalized hypergeometric function; Appell functions; Horn functions; Hypergeometric transformation formulae

AMS Subject Classification (2010): 33C05, 33C20, 33C65, 40C10, 40H05, 81Q30.

1. INTRODUCTION

We are interested in calculation of the integral (1) occurring in the field theoretical treatment of m axial Lifshitz points in strongly anisotropic d dimensional systems (see e.g. [1, 2, 3] and references therein).

The integral (1) is over a d dimensional space split into two complementary Euclidean subspaces $\mathbb{R}^D$ and $\mathbb{R}^m$. The co-dimensions D and m are related via $D = d - m$. The integral (1) converges inside of a sector of the (m, d) plane constrained by the border lines $d^*(m) = 4 + m/2$, $d_*(m) = 2 + m/2$, $m = 0$, and $d = m$, see [1, Fig. 1]. The function $I_{1,m}(p, q)$ can be extended outside the convergence region of the integral (1) by means of analytical continuations.

The deviations from the "critical" lines $d^*(m)$ and $d_*(m)$ are defined as $\varepsilon = 4 + m/2 - d$ and $\epsilon = d - 2 - m/2$; hence $\varepsilon = 2 - \epsilon$. When $d = d^*(m)$ the integral (1) diverges at large $\mathbf{x} \in \mathbb{R}^D$ and $\mathbf{y} \in \mathbb{R}^m$, while at $d = d_*(m)$ it diverges for small values of $\mathbf{x}$ and $\mathbf{y}$. Close to marginal dimensions $d^*(m)$ and $d_*(m)$, these divergences manifest themselves as simple poles $\propto 1/\varepsilon$ and $\propto 1/\epsilon$, respectively.

Due to the full rotational symmetry in each of the subspaces $\mathbb{R}^D$ and $\mathbb{R}^m$, the integral $I_{D,m}(p, q) \equiv I(p, q)$ depends on the absolute values p, q of the vectors $\mathbf{p} \in \mathbb{R}^D$, $\mathbf{q} \in \mathbb{R}^m$. Moreover, it is a generalized homogeneous function. This implies that a power of each argument, $p^{-\varepsilon} = p^{-2+\epsilon}$ or $q^{-2\epsilon} = q^{-4+2\epsilon}$, can be scaled out and the function $I(p, q)$ can be represented in two equivalent scaling forms,

$$I(p, q) = p^{-2+\epsilon} I\left(1, \frac{q}{\sqrt{p}}\right) \quad \text{or} \quad I(p, q) = q^{-4+2\epsilon} I\left(\frac{p}{q^2}, 1\right). \quad (2)$$

The special cases of $I(p, q)$ at zero arguments are homogeneous in the sense that

$$I(p, 0) = p^{-2+\epsilon} I(1, 0) \quad \text{and} \quad I(0, q) = q^{-4+2\epsilon} I(0, 1). \quad (3)$$

Thus the constants $I(1,0)$ and $I(0,1)$ give the corresponding asymptotic behaviors of $I(p,q)$ at $p \rightarrow \infty$ and $q \rightarrow \infty$ when the remaining second variable is finite.

For several pairs of m and D , which represent certain separate *points* in the (m,d) plane, $I_{D,m}(p,q)$ has been calculated explicitly [1]:

$$\begin{aligned} I_{2,1}(1,q) &= \frac{\sqrt[4]{w}}{4\sqrt{2}} {}_2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; w\right), \quad w = \frac{4}{(1+q^4)^2(4+q^4)} \quad (\varepsilon = 3/2, \epsilon = 1/2), \\ I_{3,1}(1,q) &= \frac{1}{8\pi\sqrt{2}} \sqrt{\sqrt{4+q^4} - q^2} = \frac{q^{-1}}{4\pi\sqrt{2}} \frac{1}{\sqrt{\sqrt{1+4/q^4} + 1}} \quad (\varepsilon = 1/2, \epsilon = 3/2), \\ I_{1,4}(1,q) &= \frac{1}{32\pi^2} \left[\arctan \frac{2}{q^2(3+q^4)} + \frac{1}{q^2} \ln \frac{1+q^4}{1+q^4/4} \right] \quad (\varepsilon = \epsilon = 1). \end{aligned} \quad (4)$$

The explicit dependence of $I_{D,m}(p,q)$ on p can be reconstructed here by using the homogeneity property (2).

The explicit expressions for $I(1,0)$ and $I(0,1)$ for arbitrary D and m have been derived in [1, Appendices B.1 and B.2]. In the special case $D = 1$ they reduce to

$$I_{1,m}(1,0) = C_1 \frac{2^{-2+\epsilon}}{1-\epsilon} \cos \frac{\pi\epsilon}{2}, \quad I_{1,m}(0,1) = \frac{C_1}{2(1-\epsilon)} \left[\epsilon - \frac{\sqrt{\pi}\epsilon\Gamma(\epsilon)}{\Gamma(-\frac{1}{2}+\epsilon)} \right], \quad (5)$$

where $C_1 = C_1(\epsilon) = 16^{-\epsilon}\pi^{-1-\epsilon}\Gamma(2-\epsilon)/\epsilon$.

In writing (5) we use a convenient parametrization in terms of $\epsilon = m/2 - 1$ which follows from $\epsilon = D + m/2 - 2$ at $D = 1$. Dividing the original integral (1) by C_1 we define the reduced function $\hat{I}_{1,m}(p,q) := C_1^{-1} I_{1,m}(p,q)$. The constant C_1 contains the both poles in ϵ and in $\varepsilon = 2 - \epsilon$. Thus, the function $\hat{I}_{1,m}(p,q)$ is analytic in both of these variables.

In the sequel we shall calculate the integral (1) along the *line* in the (m,d) plane given by $D = 1$ and m varying in the range from 2 to 6. We shall encounter generalized hypergeometric functions ${}_rF_s$, as well as Horn H_4 and Appell hypergeometric functions of two variables. Thus we need their power series definitions.

The generalized hypergeometric function ${}_rF_s(z)$ with r numerator and s denominator parameters is defined by the series

$${}_rF_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| z \right) = \sum_{n \geq 0} \frac{\prod_{j=1}^r (a_j)_n}{\prod_{k=1}^s (b_k)_n} \frac{z^n}{n!}, \quad (6)$$

where $a_j \in \mathbb{C}; j = \overline{1, r}$ and $b_k \in \mathbb{C} \setminus \mathbb{Z}_0^-; k = \overline{1, s}$. The $(\lambda)_\mu$ is the Pochhammer symbol (or the *shifted factorial*) defined, in terms of Euler's Gamma function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda + \mu)}{\Gamma(\lambda)} = \begin{cases} 1 & \mu = 0; \lambda \in \mathbb{C} \setminus \{0\} \\ \lambda(\lambda+1) \cdots (\lambda+n-1) & \mu = n \in \mathbb{N}; \lambda \in \mathbb{C} \end{cases},$$

and, by convention, $(0)_0 = 1$.

In the case $r = s + 1$ the series (6) converges for $|z| < 1$ and diverges for $|z| > 1$. On the unit circle $|z| = 1$, the sufficient condition for convergence of (6) is $\Re(\sum_{j=1}^s b_j - \sum_{j=1}^r a_j) > 0$. Analytic continuation can be employed for larger z values. When $r \leq s$ the series (6) converges for all finite values of z .

The Appell's hypergeometric functions of two variables involved in the following are defined by double power series expansions [4, 5, 6, 7]

$$F_1(a, b, b'; c; x, y) = \sum_{k \geq 0} \sum_{n \geq 0} \frac{(a)_{k+n} (b)_k (b')_n}{(c)_{k+n}} \frac{x^k}{k!} \frac{y^n}{n!}, \quad |x| < 1, |y| < 1, \quad (7)$$

$$F_2(a, b, b'; c, c'; x, y) = \sum_{k \geq 0} \sum_{n \geq 0} \frac{(a)_{k+n} (b)_k (b')_n}{(c)_k (c')_n} \frac{x^k}{k!} \frac{y^n}{n!}, \quad |x| + |y| < 1, \quad (8)$$

$$F_4(a, b; c, c'; x, y) = \sum_{k \geq 0} \sum_{n \geq 0} \frac{(a)_{k+n} (b)_{k+n}}{(c)_k (c')_n} \frac{x^k}{k!} \frac{y^n}{n!}, \quad \sqrt{|x|} + \sqrt{|y|} < 1. \quad (9)$$

Finally, the Horn function H_4 (see e. g. [7, 8, 9] and their references) is defined by

$$H_4(\alpha, \beta; \gamma, \delta; x, y) = \sum_{k \geq 0} \sum_{n \geq 0} \frac{(\alpha)_{2k+n} (\beta)_n}{(\gamma)_k (\delta)_n} \frac{x^k}{k!} \frac{y^n}{n!}, \quad 2\sqrt{|x|} + |y| < 1. \quad (10)$$

Outside their convergence domains the functions (7)-(10) are defined by analytical continuation.

2. THE INNER INTEGRAL

In (1), the inner D dimensional integral over $\mathbf{x}$ has the form

$$J_D(p; \kappa_1, \kappa_2) = \int_{\mathbf{x}} \frac{1}{(\mathbf{x} - \mathbf{p}/2)^2 + \kappa_1^2} \frac{1}{(\mathbf{x} + \mathbf{p}/2)^2 + \kappa_2^2} = \int_{\mathbf{x}} \frac{1}{x^2 + \kappa_1^2} \frac{1}{(\mathbf{x} + \mathbf{p})^2 + \kappa_2^2}. \quad (11)$$

This is the standard one-loop two-point Feynman integral with arbitrary "masses" κ_1 and κ_2 and external momentum $\mathbf{p}$.¹ As a function suitable for subsequent integration over $\mathbf{y}$ in (1), the integral (11) has been calculated in [10]:

$$J_D(p; \kappa_1, \kappa_2) = \gamma_D \left(\frac{\kappa_1 + \kappa_2}{2} \right)^{D-4} F_1 \left(2 - \frac{D}{2}; 1, \frac{3}{2} - \frac{D}{2}; \frac{3}{2}; \frac{-p^2}{(\kappa_1 + \kappa_2)^2}, \frac{(\kappa_2 - \kappa_1)^2}{(\kappa_1 + \kappa_2)^2} \right). \quad (12)$$

Here F_1 is the Appell function defined in (7),

$$\gamma_D := (4\pi)^{-D/2} \Gamma(2 - D/2), \quad \kappa_1^2 = (\mathbf{y} - \mathbf{q}/2)^4, \quad \text{and} \quad \kappa_2^2 = (\mathbf{y} + \mathbf{q}/2)^4.$$

Hence, for combinations appearing in the arguments of F_1 in (12) we have

$$\kappa_2 - \kappa_1 = 2(\mathbf{y} \cdot \mathbf{q}) \quad \text{and} \quad \kappa_1 + \kappa_2 = 2(y^2 + q^2/4) \quad (13)$$

while $\kappa_1 \kappa_2 = (y^2 + q^2/4)^2 - (\mathbf{y} \cdot \mathbf{q})^2$.

For generic κ_1, κ_2 and integer D , (12) gives the following integrands for the remaining integration over $\mathbf{y}$ in (1) [10, Sec. V], cf. [11, Sec. IV]. Apart from the trivial marginal cases $J_0(p; \kappa_1, \kappa_2) = (\kappa_1 \kappa_2)^{-2}$ and $J_{D \nearrow 4}(p; \kappa_1, \kappa_2) = 1/[8\pi^2(4 - D)] + \mathcal{O}(1)$, these are

$$J_1(p; \kappa_1, \kappa_2) = \frac{\kappa_1 + \kappa_2}{2\kappa_1 \kappa_2} \frac{1}{p^2 + (\kappa_1 + \kappa_2)^2}, \quad (14)$$

$$J_2(p; \kappa_1, \kappa_2) = \frac{1}{2\pi \sqrt{\Delta}} \ln \frac{p^2 + \kappa_1^2 + \kappa_2^2 + \sqrt{\Delta}}{2\kappa_1 \kappa_2}, \quad (15)$$

$$J_3(p; \kappa_1, \kappa_2) = \frac{1}{4\pi p} \arctan \frac{p}{\kappa_1 + \kappa_2}.$$

In (15),

$$\Delta = ((\kappa_1 + \kappa_2)^2 + p^2) ((\kappa_2 - \kappa_1)^2 + p^2) \quad (16)$$

¹Here and throughout the paper we use an obvious short-hand notation $(2\pi)^{-\Omega} \int d^\Omega z \equiv \int_{\mathbf{z}}^{(\Omega)}$, $\Omega \in \{D, m\}$.

is the same as in [12, Eq.(2.16)]; this is the counterpart of the triangle Källen function of [13, Eq.(21)].

In the following, along with (14), we shall use the special case of (12) with $\kappa_1 = 0$ and $\kappa_2 = \kappa$,

$$J_D(p; 0, \kappa) = -\gamma_D \kappa^{D-4} {}_2F_1\left(2 - \frac{D}{2}, 1; \frac{D}{2}; -\frac{p^2}{\kappa^2}\right). \quad (17)$$

The ${}_2F_1$ function here is ubiquitous in the literature on Feynman integrals since, presumably, [14, Eq.(16)] and appears practically in all references to be cited in the next section.

2.1. Formula (12), related references, and a quadratic transformation of F_1 . The explicit formula (12) does not follow from the result [15, Eq.(20)] for the more general integral $J_D(p; \alpha, \beta; m_1, m_2)$ where $\alpha, \beta \geq 1$ denote the (higher) powers of denominators appearing in (11). In the special case $\alpha = \beta = 1$ considered here, the linear combination of the Appell functions F_4 in [15, Eq.(20)] directly reduces to a linear combination of the Gauss hypergeometric functions² ${}_2F_1$. The same happens also to [15, Eq.(21)]. These reductions have been noticed readily in [13].

The elegant approach of functional equations by Tarasov also leads [12, Eqs.(2.18)-(2.22)] to an analytic expression for $J_D(p; m_1, m_2)$ in the form of a weighted sum of $J_D(s_{13}; m_1, 0)$ and $J_D(s_{23}; 0, m_2)$ in agreement with [15, 13] (see (17) for J_D when one of its masses vanishes). The arguments s_{13} and s_{23} are complicated functions of m_1, m_2 involving Δ from (16) along with its square root.

On the other hand, in [17, Eq.(1.6)] and, quite recently, in [18, Eq.(7)] the results for the integral (11) and $J_D(p; \alpha, \beta; m_1, m_2)$, respectively, have been obtained in terms of a single Appell function F_1 , similarly as in (12). However, the arguments of that F_1 functions are directly related to roots $r_{\mp}$ of quadratic polynomials which naturally arise in calculations employing the Feynman parametrization (see e.g [18, Eqs.(7)-(9)]). Thus, the external integration variable $\mathbf{y}$, via masses κ_1 and κ_2 of (11), enters these roots $r_{\mp}$ in a quite complicated manner. There is in fact no chance to do the subsequent $\mathbf{y}$ integration by using Appell functions F_1 with such arguments in the integrand as required by (1).

As an illustration, we quote the result for (11) from [17, Eq.(1.6)]. In our notation

$$J_D(p; \kappa_1, \kappa_2) = \gamma_D \kappa_1^{D-4} F_1\left(1; 2 - \frac{d}{2}, 2 - \frac{d}{2}; 2; \frac{1}{r_-}, \frac{1}{r_+}\right), \quad (18)$$

where $r_{\mp} = (p^2 + \kappa_2^2 - \kappa_1^2 \mp \sqrt{\Delta})/2p^2$, and again $\Delta = (p^2 + \kappa_2^2 - \kappa_1^2)^2 + 4\kappa_1^2 p^2$.

By contrast, the arguments of the F_1 function in (12) have simple and transparent dependencies on $\mathbf{y}$ as can be easily seen from (13). This has been achieved in [10] by taking special care of symmetry of the first integral in (11) and avoiding such standard technical tools as Feynman or Schwinger parametrization. The structure of arguments of the Appell function F_1 in (12) allows its term-by-term integration by using its series definition (7) without producing any spurious divergence. The absolute value p of the external momentum also enters (12) in a non-complicated manner. This suggests an eventual possibility to use this result in higher-loop calculations of standard Feynman integrals.

Apparently, a generalization of the result (12) to $J_D(p; \alpha, \beta; \kappa_1, \kappa_2)$ with higher integer powers of denominators α and β should be possible by differentiating (11) and (12) with respect to κ_1^2 and κ_2^2 and using the differentiation formulae for the Appell function F_1 derived quite recently in [19].

²Similar expressions have been obtained in [16] by avoiding standard techniques, in particular the Feynman parametrization.

At $m = 1$, a term-by-term integration of (12) over y has already been performed in a study of scaling functions at the Lifshitz point in [2, Sec.(5.3)]. The resulting series expansions for $I_{D,1}(p, q)$ (see [2, Eqs.(5.69)-(5.73)]) involve the Clausenian hypergeometric functions ${}_3F_2$ with unit argument and negative integer parameter differences between its numerator and denominator parameters studied very recently by one of the present authors and Srivastava [20]. The series expansion with the same structure can be obtained also in the case of $I_{1,m}(p, q)$ considered throughout the present paper, whereas at $D = 3$ the analytic expression is [21]

$$I_{3,m}(p, q) = 8(16\pi)^{-\epsilon} \Gamma(2 - \epsilon) (4p^2 + q^4)^{-1+\epsilon/2} {}_2F_1\left(1 - \frac{\epsilon}{2}, \frac{\epsilon}{2}; \frac{3}{2}; \frac{4p^2}{4p^2 + q^4}\right),$$

where $\epsilon = m/2 + 1 \in [1, 2[$.

Finally, by comparing the expressions (12) and (18) for the same integral (11) we come to a non-trivial relation between involved Appell functions F_1 . This is the special case of the quadratic transformation [22, Eq.(4.2)] expressed in our notation as

$$F_1\left(a; b, b; 2a; \frac{1}{r_-}, \frac{1}{r_+}\right) = \left(\frac{x+y}{2x}\right)^{-2b} F_1\left(b; a, b - a + \frac{1}{2}; a + \frac{1}{2}; \frac{-1}{(x+y)^2}, \left(\frac{x-y}{x+y}\right)^2\right)$$

where $r_{\mp} = (1 + y^2 - x^2 \pm \sqrt{\Delta})/2$ and $\Delta = (1 + y^2 - x^2)^2 + 4x^2$, taken at $a = 1$ and $b = 2 - D/2$.

Similar transformations are discussed in contemporary mathematical literature: see [23] and its references. We suppose, they should be useful in practical Feynman-integral calculations.

3. THE CASE $D = 1$: SPLITTING $J_1(p; \kappa_1, \kappa_2)$

The simplest way to proceed at $D = 1$ is to split the result (14) of the integration over $\mathbf{x}$ via

$$J_1(p; \kappa_1, \kappa_2) = \frac{1}{2} \left(\frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right) \frac{1}{p^2 + (\kappa_1 + \kappa_2)^2}.$$

Hence we rewrite the original integral $I(p, q)$ as $I_{1,m}(p, q) = \frac{1}{2} (I^- + I^+)$ where

$$I^{\mp} \equiv \int_{\mathbf{y}}^{(m)} \frac{1}{(\mathbf{y} \mp \mathbf{q}/2)^2} \frac{1}{p^2 + 4(y^2 + q^2/4)^2}. \quad (19)$$

The integrals $I^{\mp}$ are independent of the signs at $\mathbf{q}$ and of its direction in $\mathbb{R}^m$. Similarly as in (11) itself, shifting the variable $\mathbf{y} \mp \mathbf{q}/2 \rightarrow \mathbf{y}$, we obtain

$$I_{1,m}(p, q) = \int_{\mathbf{y}}^{(m)} \frac{1}{y^2} \frac{1}{p^2 + 4a^2} \quad \text{with} \quad a := (\mathbf{y} \pm \mathbf{q}/2)^2 + q^2/4. \quad (20)$$

The apparent singularity of the integrand is now shifted to the origin. However, it is integrable because the dimension of the integral over $\mathbf{y}$ is high enough: at $D = 1$ we consider $m \in [2, 6[$.

The simplest possibility to proceed is to factorize the second denominator in (20) via $p^2 + 4a^2 = (2a - ip)(2a + ip)$ and to apply the partial fraction expansion

$$\frac{1}{p^2 + 4a^2} = \frac{1}{4p i} \left[\frac{1}{a - ip/2} - \frac{1}{a + ip/2} \right]. \quad (21)$$

This splits the the integrand in (20) into a difference of two terms and we have

$$I_{1,m}(p, q) = \frac{1}{4p i} (j_- - j_+) = \frac{1}{4p i} (j_- - j_-^*) = \frac{1}{2p} \Im j_- \quad (22)$$

with

$$j_{\mp} \equiv \int_{\mathbf{y}}^{(m)} \frac{1}{y^2} \frac{1}{(\mathbf{y} \pm \mathbf{q}/2)^2 + q^2/4 \mp ip/2}.$$

Now, $j_{\mp}$ can be identified with the second integral of (11) with $\mathbf{x} \mapsto \mathbf{y}$, $D \mapsto m$, $\kappa_1^2 \mapsto 0$, $\kappa_2^2 \mapsto \kappa^2 \mapsto q^2/4 - ip/2$, and $\mathbf{p} \mapsto \pm \mathbf{q}/2$. Hence it is given by (17) with indicated changes of its parameters:

$$j_{\mp} = J_m(q/2; 0, \kappa) = -\frac{16}{(16\pi)^{\frac{m}{2}}} \Gamma\left(1 - \frac{m}{2}\right) (q^2 - i2p)^{\frac{m}{2}-2} {}_2F_1\left(2 - \frac{m}{2}, 1; \frac{m}{2}; \frac{-q^2}{q^2 - i2p}\right).$$

Inserting this result with $m = 2 + 2\epsilon$ into (22) we finish the proof of the

Proposition 1. *For $D = 1$ and real $m \in]2; 6[$, the integral (1) is*

$$I_{1,m}(p, q) = -\frac{1}{p} \frac{\Gamma(-\epsilon)}{2^{1+4\epsilon} \pi^{1+\epsilon}} \Im (q^2 - i2p)^{-1+\epsilon} {}_2F_1\left(1, 1 - \epsilon; 1 + \epsilon; \frac{-q^2}{q^2 - i2p}\right) \quad (23)$$

where $\epsilon = m/2 - 1 \in]0; 2[$.

This is one of the main results of the present article. Expressed in terms of ${}_2F_1$, it provides an analytical continuation for the function $I_{1,m}(p, q)$ outside of the convergence region of the integral (1) at $D = 1$.

At $q = 0$, (23) reduces to

$$\hat{I}_{1,m}(p, 0) = C_1 \frac{(2p)^{-2+\epsilon}}{1 - \epsilon} \Im (-i)^{-1+\epsilon} = C_1 \frac{(2p)^{-2+\epsilon}}{1 - \epsilon} \cos \frac{\pi\epsilon}{2},$$

in agreement with (3) and (5).

As $p \rightarrow 0$, the imaginary part of the complex function in (23) vanishes. To obtain the correct result for $I_{1,m}(0, q)$ one has to expand it to first order in p . This cancels the overall factor p^{-1} in (23) yielding $I_{1,m}(0, q) = q^{-4+2\epsilon} I_{1,m}(0, 1)$ with $I_{1,m}(0, 1)$ given in (5).

3.1. Special cases. Let us consider the implications of the formula (23) at integer values of $m \in [2; 6]$. To simplify the exposition we consider the reduced function

$$\hat{I}_{1,m}(p, q) = C_1^{-1} I_{1,m}(p, q) = \Im \frac{(q^2 - i2p)^{\epsilon-1}}{2(1 - \epsilon)p} {}_2F_1\left(1, 1 - \epsilon; 1 + \epsilon; \frac{-q^2}{q^2 - i2p}\right), \quad (24)$$

and introduce the short-hands $z := q^2 - i2p$ and $y := -q^2/z$.

- At $m = 2$ we have $\epsilon = 0$, the equation (24) reduces to

$$\hat{I}_{1,2}(p, q) = \frac{1}{2p} \Im z^{-1} {}_2F_1(1, 1; 1; y) = \frac{1}{2p} \Im \frac{z^{-1}}{1 - y} = \frac{1}{4} \frac{1}{p^2 + q^4},$$

and hence

$$I_{1,2+0}(p, q) = \frac{1}{4\pi\epsilon} \frac{1}{p^2 + q^4} + \mathcal{O}(1).$$

- At $m = 3$, $\epsilon = 1/2$ and we have

$$\hat{I}_{1,3}(p, q) = p^{-1} \Im z^{-1/2} {}_2F_1\left(1, \frac{1}{2}; \frac{3}{2}; y\right) = -\frac{1}{4pq} \ln \left| \frac{\sqrt{z} + iq}{\sqrt{z} - iq} \right|^2.$$

Some straightforward algebra leads to

$$\hat{I}_{1,3}(p, q) = -\frac{1}{4pq} \ln \frac{q^2 \sqrt{4p^2 + q^4} + p^2 - pq\sqrt{2} \sqrt{\sqrt{4p^2 + q^4} + q^2}}{p^2 + q^4}. \quad (25)$$

- At $m = 4$ (or $\epsilon = 1$) the result quoted in (4) has been obtained by different means and written down, without derivation, in [1, Eq.(68)]. Now we obtain it from (24). Here, as in the case of $p = 0$, the imaginary part of the function on the right vanishes at $\epsilon = 1$ and we need to calculate

$$\hat{I}_{1,4}(p, q) = \frac{1}{2p} \lim_{\alpha \rightarrow 0} \alpha^{-1} \Im z^{-\alpha} {}_2F_1(\alpha, 1; 2 - \alpha; y) \quad \text{with} \quad \alpha := 1 - \epsilon.$$

Taking into account that $(\alpha)_n = \alpha \cdot (n-1)! + \mathcal{O}(\alpha^2)$ and proceeding in the standard way we obtain³

$$\hat{I}_{1,4}(p, q) = \frac{1}{2p} \Im \left[\frac{1-y}{y} \ln(1-y) - \ln z \right] = \frac{1}{2p} \left[\frac{p}{q^2} \ln |1-y|^2 - (2\varphi_1 + \varphi) \right] \quad (26)$$

where $\varphi_1 = \text{Arg}(1-y) = \arctan[pq^2/(2p^2 + q^4)]$ and $\varphi = \text{Arg} z = -\arctan[2p/q^2]$ are the principal arguments of $1-y$ and z . Applying repeatedly the formula (see e.g. [25, Sec. I.3.5])

$$\arctan x - \arctan y = \arctan \frac{x-y}{1+xy}, \quad xy > -1,$$

to the sum of arguments $2\varphi_1 + \varphi$ in (26) we obtain

$$\hat{I}_{1,4}(p, q) = \frac{1}{2p} \left[\frac{p}{q^2} \ln \frac{p^2 + q^4}{p^2 + q^4/4} + \arctan \frac{2p^3}{q^2(3p^2 + q^4)} \right]. \quad (27)$$

With $C_1(\epsilon = 1) = 1/(16\pi^2)$ and $p = 1$, this matches the result (4).

- At $m = 5$, $\epsilon = 3/2$ and the equation (24) simplifies to

$$\begin{aligned} \hat{I}_{1,5}(p, q) &= -p^{-1} \Im z^{1/2} {}_2F_1\left(1, -\frac{1}{2}; \frac{5}{2}; y\right) \\ &= -p^{-1} \Im z^{1/2} \frac{3}{8y} \left[1 + y - \frac{(1-y)^2}{2\sqrt{y}} \ln \frac{1+\sqrt{y}}{1-\sqrt{y}} \right]. \end{aligned}$$

This can be written, for example, as

$$\hat{I}_{1,5}(p, q) = -\frac{3}{4q} \Re \left[\sqrt{\zeta} - \frac{1}{u} (1-ui)^2 \ln \frac{\sqrt{\zeta} + i}{\sqrt{\zeta} - i} \right], \quad (28)$$

which is the scaling form implied by the second expression in (2) with $p/q^2 \mapsto u$ and $\zeta = 1 - 2ui$. Now, (28) can be re-expressed in a form similar to (25), (27).

- Finally, at the marginal value $m = 6$ (or $\epsilon = 2$) we obtain

$$\hat{I}_{1,6}(p, q) = -\frac{1}{2p} \Im z {}_2F_1\left(-1, 1; 3; y\right) = -\frac{1}{2p} \Im z \left(1 - \frac{y}{3}\right) = 1.$$

Hence

$$I_{1,6-0}(p, q) = \frac{\Gamma(2-\epsilon)}{512\pi^3} = \frac{1}{512\pi^3\epsilon} + \mathcal{O}(1) \quad \text{with} \quad \epsilon = 2 - \epsilon.$$

Here, the $1/\epsilon$ contribution matches the pole terms of both $I_{1,6-2\epsilon}(1, 0)$ and $I_{1,6-2\epsilon}(0, 1)$ from (5) when $\epsilon \rightarrow 0$.

Remark 3.1. By splitting the denominator of the integrand in (20) we also expressed the sum $p^2 + q^4/4$ originally present in (19) as the product $(q^2/2 - ip)(q^2/2 + ip)$. The resulting factors landed in the arguments of functions in the final result of this section, (23). Most important here is the complex-argument Gauss hypergeometric function ${}_2F_1(z)$. In the previous paragraphs we sketched its complex expansions in several simplifying cases where the parameter m is integer. As expected,

³It is worth mentioning that the Mathematica package HypExp [24] allows to expand arbitrary ${}_rF_{r-1}$ hypergeometric functions to any order in a small quantity around integer parameters.

the original combination $p^2 + q^4/4$ is reconstructed in (25), (27), (28). The same expression is also present in evaluations related to the special case $m = 1$ with $D = 2$ and $D = 3$ quoted in the Introduction. $\square$

It is natural to ask here whether we can do a similar calculation in generic case given by (23). That is, can we do the complex expansion of a Gauss function ${}_2F_1$ by finding explicit expressions for its real and imaginary parts? We address this general question in the following section.

4. COMPLEX EXPANSION OF ${}_2F_1(a, b; c; z)$

Our aim in this section is to derive explicit expressions for real and imaginary parts of the Gauss hypergeometric function with real parameters and complex argument. We shall do it in different ways by producing several series representations and a Laplace-type integral representation. We hope that our results are of independent interest and can be considered as a contribution to the theory of special functions.

Consider ${}_2F_1(a, b; c; z)$ with real parameters a, b , and c and the complex argument z . We define its real and imaginary parts via

$$X + iY := \Re {}_2F_1(a, b; c; z) + i \Im {}_2F_1(a, b; c; z)$$

where $z = x + iy = |z|e^{i\varphi} \in \mathbb{C}$, finite $x, y \in \mathbb{R} \setminus \{0\}$, $|z| = \sqrt{x^2 + y^2}$, and $\varphi = \arctan(y/x)$. Let us start with the series representations for X and Y .

Proposition 2. *For all $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and for all $|z| < 1$, or for $c - a - b > 0$ when $|z| = 1$, we have*

$$X = \sum_{k \geq 0} \frac{(a)_k (b)_k}{(c)_k k!} \left(\frac{|z|^2}{x} \right)^k {}_2F_1\left(\frac{k}{2}, \frac{k+1}{2}; \frac{1}{2}; -\frac{y^2}{x^2}\right), \quad (29)$$

$$Y = y \frac{|z|^2}{x^2} \frac{ab}{c} \sum_{k \geq 0} \frac{(a+1)_k (b+1)_k}{(c+1)_k k!} \left(\frac{|z|^2}{x} \right)^k {}_2F_1\left(\frac{k+2}{2}, \frac{k+3}{2}; \frac{3}{2}; -\frac{y^2}{x^2}\right). \quad (30)$$

Proof. Let $|y| \leq |x|$ (otherwise we consider $w = iz$). Using the polar form of z , from the series definition (6) for ${}_2F_1$ we read off

$$X = \sum_{k \geq 0} \frac{(a)_k (b)_k |z|^k}{(c)_k k!} \cos\left(k \arctan \frac{y}{x}\right), \quad Y = \sum_{k \geq 0} \frac{(a)_k (b)_k |z|^k}{(c)_k k!} \sin\left(k \arctan \frac{y}{x}\right). \quad (31)$$

Using the entry [26, 7.3.3.1] with $a \mapsto k/2$ transforms immediately (31) to (29). By [26, 7.3.3.2] with $a \mapsto (k+1)/2$ we obtain

$$Y = \frac{y}{x} \sum_{k \geq 1} \frac{(a)_k (b)_k}{(c)_k (k-1)!} \left(\frac{|z|^2}{x} \right)^k {}_2F_1\left(\frac{k+1}{2}, \frac{k}{2} + 1; \frac{3}{2}; -\frac{y^2}{x^2}\right).$$

Shifting here the summation index by 1 leads to the asserted formula (30). $\square$

In (29) and (30) we obtained X and Y in the form of series expansions in powers of $|z|^2/x$, while the ratios $-y^2/x^2$ appear as arguments of ${}_2F_1$ functions in expansion coefficients. In the next proposition we exchange the roles of $|z|^2/x$ and $-y^2/x^2$. This time the Clausenian function ${}_3F_2(|z|^2/x)$ will give the expansion coefficients at powers of $-y^2/x^2$.

Proposition 3. *For all $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \mathbb{Z}_0^-$ and for all $|z| < 1$, or for $c - a - b > 0$ when $|z| = 1$, we have*

$$X = 1 + \frac{|z|^2}{x} \frac{ab}{c} \sum_{k \geq 0} {}_3F_2 \left(\begin{matrix} a+1, b+1, 2k+1 \\ c+1, 2 \end{matrix}; \frac{|z|^2}{x} \right) \left(-\frac{y^2}{x^2} \right)^k, \quad (32)$$

$$Y = y \frac{|z|^2}{x^2} \frac{ab}{c} \sum_{k \geq 0} {}_3F_2 \left(\begin{matrix} a+1, b+1, 2k+2 \\ c+1, 2 \end{matrix}; \frac{|z|^2}{x} \right) \left(-\frac{y^2}{x^2} \right)^k. \quad (33)$$

Proof. Under the same conditions as in Proposition 2, let us write $z = |z|^2/z^*$ where the asterisk denotes complex conjugate. Hence

$$z^n = \frac{|z|^{2n}}{x^n} \left(1 - i \frac{y}{x} \right)^{-n} = \frac{|z|^{2n}}{x^n} {}_1F_0 \left(n; -; i \frac{y}{x} \right).$$

Then, by suitable series transformations including an exchange of the summation order,

$$\begin{aligned} {}_2F_1(a, b; c; z) &= 1 + \sum_{n \geq 1} \frac{(a)_n (b)_n}{(c)_n n!} \frac{|z|^{2n}}{x^n} \sum_{k \geq 0} \frac{\Gamma(n+k)}{\Gamma(n) k!} i^k \left(\frac{y}{x} \right)^k \\ &= {}_2F_1(|z|^2/x) + \frac{|z|^2}{x} \frac{ab}{c} \sum_{k \geq 1} i^k \left(\frac{y}{x} \right)^k \sum_{n \geq 0} \frac{(a+1)_n (b+1)_n (k+1)_n}{(c+1)_n (2)_n n!} \frac{|z|^{2n}}{x^n} \\ &= {}_2F_1(|z|^2/x) + \frac{|z|^2}{x} \frac{ab}{c} \sum_{k \geq 1} i^k \left(\frac{y}{x} \right)^k {}_3F_2 \left(\begin{matrix} a+1, b+1, k+1 \\ c+1, 2 \end{matrix}; \frac{|z|^2}{x} \right). \end{aligned} \quad (34)$$

Taking into account that the last ${}_3F_2$ function simplifies at $k = 0$, we end up with

$${}_2F_1(a, b; c; z) = 1 + \frac{|z|^2}{x} \frac{ab}{c} \sum_{k \geq 0} i^k \left(\frac{y}{x} \right)^k {}_3F_2 \left(\begin{matrix} a+1, b+1, k+1 \\ c+1, 2 \end{matrix}; \frac{|z|^2}{x} \right). \quad (35)$$

Separating sums over the odd and even k we come directly to (32) and (33). $\square$

The next proposition gives Laplace-type integral representations for X and Y .

Proposition 4. *Under the same conditions as in Proposition 2, we have*

$$\begin{aligned} X &= 1 + |z|^2 \frac{ab}{c} \int_0^\infty e^{-xt} \cos(yt) {}_2F_2 \left(\begin{matrix} a+1, b+1 \\ c+1, 2 \end{matrix}; |z|^2 t \right) dt, \\ Y &= |z|^2 \frac{ab}{c} \int_0^\infty e^{-xt} \sin(yt) {}_2F_2 \left(\begin{matrix} a+1, b+1 \\ c+1, 2 \end{matrix}; |z|^2 t \right) dt. \end{aligned}$$

Proof. Taking into account the integral representation [27, p.115, Exersize 11]

$${}_3F_2 \left(\begin{matrix} a, b, c \\ f, g \end{matrix}; w \right) = \frac{1}{\Gamma(c)} \int_0^\infty e^{-t} t^{c-1} {}_2F_2 \left(\begin{matrix} a, b \\ f, g \end{matrix}; wt \right) dt \quad (\operatorname{Re}(c) > 0)$$

with $c = k+1$, we deduce from the series (35) the both statements. $\square$

5. GAUSS ${}_2F_1$ versus HORN H_4

In this section we shall consider a special situation in which one of the numerator parameters of ${}_2F_1(a, b; c; z)$, say b , equals to 1. We shall show that in this case X and Y can be expressed in terms of the Horn function of two variables H_4 defined by the double series in (10).

Before proceeding we recall that the Horn function H_4 can be expressed as a single power series involving ${}_2F_1$ [28, Eq. (3.9)]:

$$H_4(\alpha, \beta; \gamma, \delta; s, t) = \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{(\delta)_n} {}_2F_1\left(\frac{n+\alpha}{2}, \frac{n+\alpha+1}{2}; \gamma; 4s\right) \frac{t^n}{n!}. \quad (36)$$

Indeed, using the Legendre duplication formula $\Gamma(z) \Gamma(z + \frac{1}{2}) = \sqrt{\pi} 2^{1-2z} \Gamma(2z)$ ($\Re(z) > 0$) in the right-hand side we reproduce the double-series definition (10). The parameter space for (36) is $\alpha, \beta \in \mathbb{C}; \gamma, \delta \in \mathbb{C} \setminus \mathbb{Z}_-^0$, while the convergence domain, according to Horn's theorem [9, p. 56 *et seq.*], is given by $\{(s, t) : 2\sqrt{|s|} + |t| < 1\}$. We note also an alternative form of writing H_4 in terms of ${}_2F_1$,

$$H_4(\alpha, \beta; \gamma, \delta; s, t) = \sum_{n \geq 0} \frac{(\alpha)_{2n}}{(\gamma)_n} {}_2F_1(\alpha + 2n, \beta; \delta; t) \frac{s^n}{n!}. \quad (37)$$

Let us also recall [28, Eq. (3.2)] that the Horn function H_4 is expressible in terms of the the Appell function F_2 (see (8)) *via*

$$H_4(\alpha, \beta; \gamma, \delta; s, t) = (1 + 2\sqrt{s})^{-\alpha} F_2\left(\alpha, \beta, \gamma - \frac{1}{2}; \delta, 2\gamma - 1; \frac{t}{1 + 2\sqrt{s}}, \frac{4\sqrt{s}}{1 + 2\sqrt{s}}\right). \quad (38)$$

Moreover, when the parameters of the Horn function H_4 are related by $\beta = \alpha - \gamma + 1$, we find its expression in terms of the Appell function F_4 (see (9)): ⁴

$$\begin{aligned} & H_4(\alpha, \beta; \alpha - \beta + 1, \delta; s, t) \\ &= \left(\frac{1 + \sqrt{1 - 4s}}{2}\right)^{-\alpha} F_4\left(\alpha, \beta; \delta, \alpha - \beta + 1; \frac{2t}{1 + \sqrt{1 - 4s}}, \frac{1 - \sqrt{1 - 4s}}{1 + \sqrt{1 - 4s}}\right). \end{aligned}$$

This can be proved by employing the quadratic transformation [26, 7.3.1.98] to the Gauss function in (36). Note that the relation of the parameters $\gamma = \alpha - \beta + 1$ in F_4 is typical for another quadratic transformation of Gauss functions, [26, 7.3.1.54].

5.1. Real and imaginary parts of ${}_2F_1(a, 1; c; z)$. Let us return to our discussion of the complex expansion of the Gauss function ${}_2F_1(a, b; c; z)$. Comparing its real and imaginary parts (29) and (30) with the series definition of the Horn function H_4 (36) we conclude that

$$\Re {}_2F_1(a, 1; c; z) = 1 + \frac{a}{c} \frac{|z|^2}{x} H_4\left(1, a + 1; \frac{1}{2}, c + 1; -\frac{y^2}{4x^2}, \frac{|z|^2}{x}\right), \quad (39)$$

$$\Im {}_2F_1(a, 1; c; z) = \frac{a}{c} \frac{y|z|^2}{x^2} H_4\left(2, a + 1; \frac{3}{2}, c + 1; -\frac{y^2}{4x^2}, \frac{|z|^2}{x}\right). \quad (40)$$

This implies that the result (23) for the integral $I_{1,m}(p, q)$ is expressible as a linear combination of Horn functions H_4 from (39) and (40) with $a=1-\epsilon$, $c=1+\epsilon$, and

$$z = -\frac{q^2}{q^2 - i2p} = -\frac{q^4}{q^4 + 4p^2} - i \frac{2pq^2}{q^4 + 4p^2} := x + iy, \quad |z|^2 = \frac{q^4}{q^4 + 4p^2}.$$

Namely, we have to deal with

$$\begin{aligned} \Re {}_2F_1\left(1 - \epsilon, 1; 1 + \epsilon; \frac{-q^2}{q^2 - i2p}\right) &= 1 - \frac{1 - \epsilon}{1 + \epsilon} H_4\left(1, 2 - \epsilon, \frac{1}{2}, 2 + \epsilon; -\frac{p^2}{q^4}, -1\right) \\ \Im {}_2F_1\left(1 - \epsilon, 1; 1 + \epsilon; \frac{-q^2}{q^2 - i2p}\right) &= -2 \frac{1 - \epsilon}{1 + \epsilon} \frac{p}{q^2} H_4\left(2, 2 - \epsilon, \frac{3}{2}, 2 + \epsilon; -\frac{p^2}{q^4}, -1\right). \end{aligned}$$

⁴A more special relation [28, Eq.(3.4)] is misprinted: The both parameters γ and $\sigma + b'$ of H_4 should be replaced there by $(\alpha + 1)/2$.

In the next section we shall remove the constraint $b = 1$ in considering further relations between the functions ${}_2F_1$ and H_4 .

5.2. ${}_2F_1$ with arbitrary parameters. Let us reverse the problem discussed before and start now from the Horn functions H_4 with parameters $\gamma \in \{\frac{1}{2}, \frac{3}{2}\}$. Such consideration leads us to the

Proposition 5. *Under the same assumptions on the parameters as above, $x \in \mathbb{R}_+$, and $y \in \mathbb{R}$, the following relations hold:*

$$H_4\left(\alpha, \beta; \frac{1}{2}, \delta; -x, y\right) = \Re (1 \pm i2\sqrt{x})^{-\alpha} {}_2F_1\left(\alpha, \beta; \delta; \frac{y}{1 \pm i2\sqrt{x}}\right), \quad (41)$$

$$H_4\left(\alpha, \beta; \frac{3}{2}, \delta; -x, y\right) = \frac{\mp 1}{2(\alpha-1)\sqrt{x}} \Im (1 \pm i2\sqrt{x})^{1-\alpha} {}_2F_1\left(\alpha-1, \beta; \delta; \frac{y}{1 \pm i2\sqrt{x}}\right).$$

Proof. Consider (36) with $\gamma = 1/2$ and $\gamma = 3/2$, negative $s \mapsto -x$, and $t \mapsto y$. Thus, the Gaussian function on the right is given by ${}_2F_1(\mu, \mu + \frac{1}{2}; \frac{1}{2}; -4x)$ or ${}_2F_1(\mu, \mu + \frac{1}{2}; \frac{3}{2}; -4x)$ with $\mu \mapsto (\alpha + n)/2$. For these two cases the known formulae [26, 7.3.1.106 and 7.3.1.107] read

$${}_2F_1\left(\mu, \mu + \frac{1}{2}; \frac{1}{2}; -4x\right) = \frac{1}{2} [(1 - i2\sqrt{x})^{-2\mu} + (1 + i2\sqrt{x})^{-2\mu}], \quad (42)$$

$${}_2F_1\left(\mu, \mu + \frac{1}{2}; \frac{3}{2}; -4x\right) = \frac{[(1 - i2\sqrt{x})^{1-2\mu} - (1 + i2\sqrt{x})^{1-2\mu}]}{i4\sqrt{x}(2\mu - 1)}. \quad (43)$$

We see that the square brackets in the last two equations contain the sum and the difference of certain complex conjugate values thus giving for these values their real and imaginary parts. Keeping this in mind, we substitute the right-hand sides of (42) and (43) into (36). Remembering that $\mu = (\alpha + n)/2$ we sum up the resulting series over n and obtain the asserted formulae. While the summation leading to (41) is straightforward, in deriving the following result we had to take into account the relation $(\alpha)_n/(\alpha + n - 1) = (\alpha - 1)_n/(\alpha - 1)$. $\square$

The results of the Proposition 5.1 are more general than that of (39) and (40). Here the previous constraint $b = 1$ is removed, and the involved Gauss hypergeometric functions ${}_2F_1$ are considered with generic real parameters.

Corollary 5.1. *The result (23) for $I_{1,m}(p, q)$ can be expressed in terms of a single Horn function H_4 :*

$$I_{1,m}(p, q) = \frac{\Gamma(2 - \epsilon)}{16^\epsilon \pi^{1+\epsilon} \epsilon} q^{-4+2\epsilon} H_4\left(2 - \epsilon, 1; \frac{3}{2}, 1 + \epsilon; \frac{-p^2}{q^4}, -1\right).$$

The constant in front of the last formula is just $C_1(\epsilon)$ defined in (5). In view of the relation (38) between H_4 and F_2 , this result, as well as (39) and (40), can be expressed in terms of the Appell function F_2 (see (8)).

Though very compact and elegant, we must admit that such representations for the function $I_{1,m}(p, q)$ in terms of the Horn function H_4 or the Appell function F_2 are mainly of academic interest from the mathematical point of view. Their practical implications seem to be at the moment rather obscure.

In concluding, we show the following relation of the formula [26, 6.8.1.17] to our calculations. We take this entry at $p = q = 1$ with $t \mapsto it$ and rewrite it as

$$(1 - it)^{-a} {}_2F_1\left(a, b; c; \frac{s}{1 - it}\right) = \sum_{k \geq 0} \frac{(\alpha)_k}{k!} (it)^k {}_2F_1(a + k, b; c; s) \quad (44)$$

where $|t| < 1$, $|s| < 1$. The series on the right-hand side is similar by its structure to that of (34). Again, its real and imaginary parts are given by partial sums over even and odd summation indices k , respectively.

Let us express the argument of the Gauss function on the right of (44) as

$$z = x + iy \quad \text{with} \quad x := \frac{s}{1+t^2} \quad \text{and} \quad y := \frac{ts}{1+t^2}.$$

This implies that $t = y/x$ and $s = |z|^2/x$ with $|z|^2 = x^2 + y^2$ as usual. Thus, for the real part of (44) we obtain

$$\Re z^a {}_2F_1(a, b; c; z) = \left(\frac{|z|^2}{x}\right)^a \sum_{k \geq 0} \frac{(\alpha)_{2k}}{(\frac{1}{2})_k k!} \left(\frac{-y^2}{4x^2}\right)^k {}_2F_1\left(a + 2k, b; c; \frac{|z|^2}{x}\right)$$

where we took into account that $(2k)! = 4^k (\frac{1}{2})_k k!$. The last series matches the right-hand side of the definition (37) of the Horn function H_4 in terms of ${}_2F_1$. Hence we have

$$\Re z^a {}_2F_1(a, b; c; z) = \left(\frac{|z|^2}{x}\right)^a H_4\left(a, b; \frac{1}{2}, c; \frac{-y^2}{4x^2}, \frac{|z|^2}{x}\right)$$

which is equivalent to (41). An analogous expression can be obtained for the imaginary part of $z^a {}_2F_1(a, b; c; z)$.

ACKNOWLEDGEMENTS

The authors are grateful to the anonymous referee for valuable comments and suggestions, which led to the substantial improvement of the article, and to Professor Yu. A. Brychkov for drawing our attention to the entry [26, 6.8.1.17] in connection with our work.

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