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On the incompleteness of the classification of quadratically integrable Hamiltonian systems in the three-dimensional Euclidean space

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Abstract

We present an example of an integrable Hamiltonian system with scalar potential in the three-dimensional Euclidean space whose integrals of motion are quadratic polynomials in the momenta, yet its Hamilton–Jacobi / Schrödinger equation cannot separate in any orthogonal coordinate system. This demonstrates a loophole in the derivation of the list of quadratically integrable Hamiltonian systems in Makarov *et al* (1967 *Nuovo Cimento* A **10** 1061–84) where only separable systems were found, and the need for its revision.

Keywords: integrability, separability, Hamilton–Jacobi equation, natural Hamiltonians

1. Introduction

In their seminal paper [1] Makarov, Smorodinsky, Valiev and Winternitz presented a list of quadratically integrable natural Hamiltonian systems in the three-dimensional Euclidean space and identified them with systems separable in ortogonal coordinate systems, cf [2, 3]. Their result is one of the standard references in the theory of integrable and superintegrable systems and lead to numerous further developments, see e.g. [4, 5] and the review [6] for many others. It was widely accepted as a proof of the equivalence of quadratic integrability and separability

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in Euclidean 3D space, cf [7, 8] or the monograph [9], p 163. (Notice that already Eisenhart in [2] had shown that an analogous equivalence does not hold for systems with integrals that are higher order polynomials in the momenta, like the potentials of [10–12], unless their integrals are reducible to quadratic ones or additional quadratic integrals exist, as in [13].) However, it became forgotten that the derivation of the list in [1] was based on one technical assumption which limits the universality of the above mentioned equivalence. In this short note we shall demonstrate that without that assumption a quadratically integrable however nonseparable 3D natural Hamiltonian system exists and the original derivation needs to be revisited.

2. Review of the original argument and its loophole

Let us review the argument used in [1] (with some streamlining due to later developments) and indicate the point where their analysis becomes incomplete. We consider the natural Hamiltonian for a particle of unit mass moving in the three-dimensional Euclidean space under the influence of the potential $V(\vec{x})$,

$$H = \frac{1}{2}\vec{p}^2 + V(\vec{x}), \tag{1}$$

and assume that it is integrable with a pair of integrals of motion X_1, X_2 which are quadratic polynomials in the momenta (henceforth abbreviated to *quadratic integrals*), with coordinate dependent coefficients. For the sake of simplicity we shall proceed classically; however, the determining equations for the quadratic integrals and their solution are exactly the same in quantum mechanics, assuming total symmetrization of any terms involving noncommuting operators $\hat{x}_a, \hat{p}_a$.

To fix our notation, let us assume that the position vector in the Cartesian coordinates is expressed as $\vec{x} = (x, y, z)$, the canonically conjugated momenta to x, y, z are denoted by p_x, p_y, p_z and the angular momenta are expressed as $l_x = yp_z - zp_y$, $l_y = zp_y - xp_z$ and $l_z = xp_y - yp_x$.

As a consequence of the assumed form of the integrals, the Poisson brackets $\{H, X_1\}_{\text{P.B}}$, $\{H, X_2\}_{\text{P.B}}$ and $\{X_1, X_2\}_{\text{P.B}}$ are third order polynomials in the momenta p_x, p_y, p_z. As the Hamiltonian (1) is an even polynomial in the momenta, the odd and even order terms in the integrals commute with the Hamiltonian (1) independently and the integrals can be without any loss of generality assumed to be even or odd polynomials in the momenta. As any first order integral implies the existence of a second order integral as its square, we can assume that X_1 and X_2 are second order even polynomials in the momenta. The left hand sides of the involutivity conditions

$$\{H, X_1\}_{\text{P.B}} = 0, \quad \{H, X_2\}_{\text{P.B}} = 0, \tag{2}$$

$$\{X_1, X_2\}_{\text{P.B}} = 0 \tag{3}$$

then become third order odd polynomials in the momenta. As the momenta are arbitrary, all their coefficients must vanish.

The third order terms in (2) and (3) are easily solved and imply that the second order terms in X_1 and X_2 must be commuting elements in the universal enveloping algebra of the Euclidean algebra $\mathfrak{U}(\mathfrak{e}_3)$, i.e. quadratic polynomials in the linear and angular momenta. As we may arbitrarily combine the integrals with the Hamiltonian and among themselves, and the systems related by Euclidean transformations are physically equivalent, the leading order terms must belong to any of the classes of three-dimensional Abelian subalgebras consisting of quadratic elements in the universal enveloping algebra of the Euclidean algebra $\mathfrak{U}(\mathfrak{e}_3)$, which were recently classified in [14].

It remains to solve the remaining conditions, which come from linear terms in the momenta in (2) and (3), namely to determine the scalar terms in the integrals X_1 and X_2, denoted by $m_1(\vec{x})$ and $m_2(\vec{x})$ below, and find the restrictions on the potential $V(\vec{x})$ implied by their existence. The conditions coming from (2) are easily solved with respect to the first order derivatives of $m_1(\vec{x})$ and $m_2(\vec{x})$. Substituting these into (3), one arrives at a set of three equations which are homogeneous linear first order PDEs for the potential $V(\vec{x})$, cf (6) below. As the coefficients of $\partial_a V$, $a = x, y, z$ form an antisymmetric 3×3 matrix R, it either has rank 2 or vanishes identically. At this point the authors of [1] stated 'Thus the potential V either satisfies three first-order equations—a case which will be considered separately—or the consistency conditions (39) are satisfied trivially.' and proceeded assuming that the condition (3) vanishes identically. Only under this assumption they arrived at their list of quadratically integrable natural Hamiltonian systems and showed that one by one they precisely match with the separable systems of Eisenhart [2, 3]. (For a more recent analysis, classification and identification of such separable systems see also [15].)

The authors of [1] left several problems to be resolved in the planned Part II of their paper and we can assume that that's where they intended to address the case of the matrix R of rank 2. However, due to external influence of political nature (military occupation of Czechoslovakia by the forces of Soviet Union and its satellites in 1968 and subsequent emigration of P Winternitz to the other side of Iron Curtain) the authors' team split up and the sequel to [1] was never written.

The long forgotten assumption on the rank of the matrix R came back to light recently, when we discussed with P Winternitz the modification of the classification of quadratically integrable systems when linear terms in the momenta are present in the Hamiltonian, cf [16–18]. P Winternitz decided to revisit this question and assigned his student H. Abdul–Reda to work on it, resulting in the Master Thesis [19]. Its author concluded that the case of rank $R = 2$ does not lead to any new system and that it implies that the potential must be invariant under a two-dimensional Abelian subgroup of the Euclidean group, i.e. the system must possess two commuting first order integrals of motion. However, due to P Winternitz's demise in 2021 these results were never submitted for a journal publication and thus never passed validation by an independent peer review. Therefore we decided to investigate the problem of rank $R = 2$ from the perspective of algebraic classification of leading order terms obtained with A. Marchesiello in [14] and, as we shall elucidate in the next section, arrived at the conclusion contradictory to that of H. Abdul–Reda, namely we find that a quadratically integrable nonseparable system does exist.

3. Quadratically integrable nonseparable system

Let us look for quadratically integrable Hamiltonian systems (1) of the form corresponding to the class (c) of [14], namely with the integrals of motion of the form

$$\begin{aligned} X_1 &= l_x^2 + l_y^2 + l_z^2 + 2b\left(l_x p_x - (3a-1)\, l_y p_y - 2 l_z p_z\right) \\ &\quad + 3b^2\left((1-4a)\, p_x^2 - \left(3a^2 - 2a - 1\right) p_y^2 + 2\,(a-1)\, p_z^2\right) + m_1(\vec{x}), \\ X_2 &= a l_y^2 + l_z^2 + 6ab\, l_x p_x + 9ab^2\left(a p_z^2 + p_y^2\right) + m_2(\vec{x}), \quad 0 < a \leqslant \frac{1}{2}, b \in \mathbb{R} - \{0\} \end{aligned} \tag{4}$$

Let us notice that the allowed values for the parameters a and b were chosen to uniquely parametrize the non-equivalent possibilities for the integrals under Euclidean transformations. Choosing $a = 0, 1$ or $b = 0$ would fall into another class ((a) or (b), respectively) in [14], a outside of the range $[0, \frac{1}{2}]$, $a \neq 1$ can be brought to $a \in [0, \frac{1}{2}]$ by a Euclidean transformation and linear combination of the integrals.

As the determinant $\det \frac{\partial(H, X_1, X_2)}{\partial(p_x, p_y, p_z)}$ does not identically vanish for any choice of parameters a and b, the assumed form of the integrals (4) implies functional independence of H, X_1 and X_2 (if they exist).

Let us recall from [14] that this form of the integrals represents one of the three possibilities for a pair of integrals such that both integrals involve terms quadratic in angular momenta, the other two being classes (a) and (b) therein. The allowed transformations, i.e. linear combinations of the integrals (and the Hamiltonian) and Euclidean transformations were all used in fixing the form of (4), namely

- rotations were used to bring the terms quadratic in angular momenta in X_2 to a diagonal form and to choose the range for the parameter a,
- translations were used to eliminate antisymmetric terms of the form $l_a p_b - l_b p_a$, $a, b = x, y, z$, from X_1,
- linear combinations of X_1 and X_2 were used to put the terms quadratic in angular momenta in X_1 to the simple form $\vec{l}^2$ and to eliminate l_x^2 term from X_2, the addition of the Hamiltonian was used to normalize the terms quadratic in the linear momenta in both X_1 and X_2,

(for a detailed derivation see pages 5–7 in [14]). The other two classes with both integrals quadratic in angular momenta arise for particular choices of the parameters a, b, namely $a = 0$ for class (a) and $b = 0$ for class (b) which on the other hand allow additional terms quadratic in linear momenta. We recall from [14] that class (a) includes pairs of integrals of motion arising from spherical, oblate and prolate spheroidal separation of variables, class (b) corresponds to integrals of the systems separable in conical or ellipsoidal coordinates.

To sum up, we have no available transformations left and the assumed form of the integrals (4) is not equivalent to any other known one, e.g. to the ones known from [1].

As a consequence of the assumed form of the integrals, i.e. the leading order terms of (4) forming an Abelian subalgebra of $\mathfrak{U}(\mathfrak{e}_3)$, the Poisson brackets $\{H, X_1\}_{\mathrm{P.B}}$, $\{H, X_2\}_{\mathrm{P.B}}$ and $\{X_1, X_2\}_{\mathrm{P.B}}$ reduce to first order polynomials in the momenta p_x, p_y, p_z, without zeroth order terms. Separating the conditions (2) into coefficients of p_x, p_y, p_z and solving them with respect to the first order derivatives of $m_1(\vec{x})$ and $m_2(\vec{x})$ we find:

$$
\begin{aligned}
\partial_x m_1(\vec{x}) &= 2\left(3(1-4a)b^2 + y^2 + z^2\right)\partial_x V(\vec{x}) - 2(3abz + xy)\partial_y V(\vec{x}) + 2(3by - xz)\partial_z V(\vec{x}), \\
\partial_y m_1(\vec{x}) &= -2(3abz + xy)\partial_x V(\vec{x}) + 2\left(3\left(1 + 2a - 3a^2\right)b^2 + x^2 + z^2\right)\partial_y V(\vec{x}) \\
&\quad - 2(3b(1-a)x + yz)\partial_z V(\vec{x}), \\
\partial_z m_1(\vec{x}) &= 2(3by - xz)\partial_x V(\vec{x}) - 2(3b(1-a)x + yz)\partial_y V(\vec{x}) \\
&\quad + 2\left(6(a-1)b^2 + x^2 + y^2\right)\partial_z V(\vec{x}), \\
\partial_x m_2(\vec{x}) &= 2\left(az^2 + y^2\right)\partial_x V(\vec{x}) - 2(3abz + xy)\partial_y V(\vec{x}) + 2a(3by - xz)\partial_z V(\vec{x}), \\
\partial_y m_2(\vec{x}) &= -2(3abz + xy)\partial_x V(\vec{x}) + 2\left(9ab^2 + x^2\right)\partial_y V(\vec{x}), \\
\partial_z m_2(\vec{x}) &= 2a(3by - xz)\partial_x V(\vec{x}) + 2a\left(9ab^2 + x^2\right)\partial_z V(\vec{x}).
\end{aligned}
\tag{5}
$$

Their compatibility implies a set of second order linear PDEs for the potential $V(\vec{x})$. On the other hand, substituting (5) into (3) we obtain a set of three first order linear homogeneous PDEs for the potential $V(\vec{x})$. They can be expressed in the matrix form

$$R\cdot\left(\partial_x V(\vec{x}),\partial_y V(\vec{x}),\partial_z V(\vec{x})\right)^{\mathrm{T}}=0 \tag{6}$$

where the matrix R is antisymmetric, $R+R^{\mathrm{T}}=0$, and its independent elements read

$$\begin{aligned}
R_{12}&=(1-a)\,azx^2-6\,(1-a)\,abyx-zy^2a-a^2z^3-9a^2b^2\,(1-a)\,z,\\
R_{13}&=(1-a)\,yx^2+6\,(1-a)\,abzx+y^3+a\left(9\,(a-1)\,b^2+z^2\right)y,\\
R_{23}&=-\,(1-a)^2x^3+(1-a)\left(9a\,(a-1)\,b^2+az^2-y^2\right)x-6\,(1-a)\,abyz.
\end{aligned} \tag{7}$$

As R does not identically vanish for any choice of the parameters a,b and rank of any antisymmetric matrix is even, we are indeed considering the case with $\operatorname{rank} R=2$.

Solving (6) using the method of characteristics we find that $V(\vec{x})$ must be an arbitrary function $v(u)$ of an invariant coordinate u, which can be conveniently chosen as

$$\begin{aligned}
u=&\,(a-1)^2x^4+\left(az^2+y^2\right)^2+2\,(1-a)\,x^2\left(y^2-az^2\right)\\
&+6ab\,(a-1)\left(3\left(\left(x^2-z^2\right)a-x^2+y^2\right)b-4xyz\right)+81a^2\,(1-a)^2\,b^4.
\end{aligned} \tag{8}$$

Substituting $V(\vec{x})=v(u)$ into the compatibility conditions for (5) we find a system of ODEs which reduces to a single equation

$$2u\frac{\mathrm{d}^2v(u)}{\mathrm{d}u^2}=-3\frac{\mathrm{d}v(u)}{\mathrm{d}u}. \tag{9}$$

Ignoring the irrelevant additive constant in the potential, we find the potential determined up to a multiplicative constant w_0,

$$V(\vec{x})=v(u)=\frac{w_0}{\sqrt{u}}, \tag{10}$$

i.e.

$$V(\vec{x})=\frac{w_0}{\sqrt{\begin{array}{l}(a-1)^2x^4+\left(az^2+y^2\right)^2+2\,(1-a)\,x^2\left(y^2-az^2\right)\\+6ab\,(a-1)\left(3\left(\left(x^2-z^2\right)a-x^2+y^2\right)b-4xyz\right)+81a^2\,(1-a)^2\,b^4\end{array}}}. \tag{11}$$

In the next step, the equation (5) determine the scalar terms m_1 and m_2 in the integrals up to irrelevant additive constants. They read

$$\begin{aligned}
m_1(\vec{x})&=2w_0\frac{x^2+y^2+z^2+3b^2\,(1-a)}{\sqrt{u}},\\
m_2(\vec{x})&=w_0\frac{x^2+y^2+a\left(x^2+z^2\right)+9ab^2\,(a+1)}{\sqrt{u}},
\end{aligned} \tag{12}$$

where u is the quartic polynomial in the coordinates introduced in (8).

Let us mention that the polynomial (8) can of course have real roots and thus the potential (10) may blow up in the configuration space. Looking up its roots we find that the polynomial (8) vanishes along the complex hyperplanes

$$\epsilon_1\sqrt{1-a}\mathrm{i}x+\epsilon_1\epsilon_2 y+\epsilon_2\sqrt{a}\mathrm{i}z=3\sqrt{a(1-a)}b, \qquad \epsilon_1,\epsilon_2=\pm 1. \tag{13}$$

As both the parameters a,b and the coordinates x,y,z are real (and $0<a\leqslant\frac{1}{2}$), restricting the hyperplanes (13) to the real domain we see that $V(\vec{x})$ blows up along the two straight lines given by

$$x=-\epsilon_1\epsilon_2\sqrt{\frac{a}{1-a}}z, \quad y=3\epsilon_1\epsilon_2\sqrt{a(1-a)}b, \qquad \epsilon_1,\epsilon_2=\pm 1. \tag{14}$$

As these do not separate $\mathbb{R}^3$ into disconnected domains, everywhere else the potential (10) is a well–defined real function.

If we assume that the parameter w_0 is positive, the singular lines (14) are not dynamically accessible for any initial condition with finite energy. Thus our Hamiltonian system is well-defined in the configuration space defined as $\mathbb{R}^3$ without the two singular lines (14). Whether the singularities are dynamically reachable for negative values of w_0 in finite time we do not know yet.

In order to provide a more intuitive understanding of the potential (10) let us present several of its equipotential surfaces in figure 1.

The system with the potential (11) is an integrable system with the integrals of the form (4) where the functions m_1 and m_2 are specified by (12). Looking for another hypothetical integral at most quadratic in the momenta we can assume without any loss of generality, as discussed in section 2[1], that it is an even quadratic polynomial in the momenta

$$X_3=\sum_{a,b=x,y,z}K^{ab}(\vec{x})\,p_a p_b+m_3(\vec{x}), \qquad K^{ba}(\vec{x})=K^{ab}(\vec{x}). \tag{15}$$

The Poisson bracket $\{H,X_3\}_{\mathrm{P.B.}}$ now becomes an odd third order polynomial in p_x,p_y,p_z. The vanishing of the cubic terms implies the conditions

$$\partial_a K^{bc}+\partial_b K^{ac}+\partial_c K^{ab}=0, \quad a,b=x,y,z \tag{16}$$

which are equivalent to the statement that K^{ab} is a Killing tensor for the flat, Euclidean metric $g_{ab}=\delta_{ab}$, thus $\sum_{a,b=x,y,z}K^{ab}(\vec{x})p_a p_b$ is a quadratic polynomial in the linear and angular momenta. Taking into account the identity

$$\vec{p}\cdot\vec{l}=0 \tag{17}$$

one expresses the tensor K in terms of 20 constants α_k (see e.g. [15]). The remaining terms in $\{H,X_3\}_{\mathrm{P.B.}}$ are linear in p_x,p_y,p_z and equating them to zero one finds three equations determining the first order derivatives $\partial_x m_3(\vec{x})$, $\partial_y m_3(\vec{x})$, $\partial_z m_3(\vec{x})$ in terms of the tensor K, the potential $V(\vec{x})$ and their derivatives, as in equation (5). Taking their compatibility conditions, i.e. substituting for $\partial_x m_3(\vec{x})$, $\partial_y m_3(\vec{x})$ and $\partial_z m_3(\vec{x})$ in

$$\partial_a(\partial_b m_3(\vec{x}))=\partial_b(\partial_a m_3(\vec{x})) \tag{18}$$

and substituting the explicit form of the potential (11), one finds a set of algebraic constraints on the constants α_k defining the tensor K, w_0, a and b, which must be satisfied for all the

[1] Notice that any linear integral would imply existence of a quadratic integral as its square. The nonexistence of first order integrals, or equivalently of Killing vectors leaving the potential (11) invariant, can be deduced also from the shape of equipotential surfaces in figure 1, as they are obviously not invariant under any Euclidean transformation.

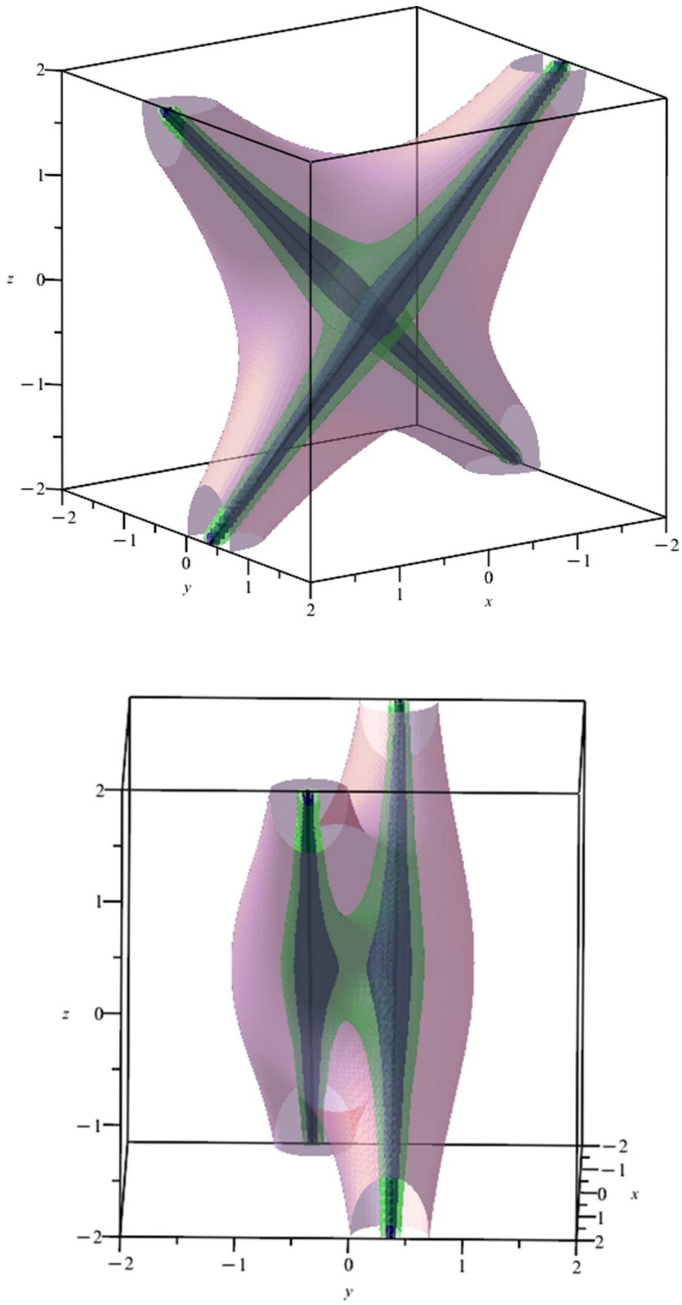

Figure 1. Equipotential surfaces of (10) with the parameters $a = \frac{1}{4}$, $b = 1$, $w_0 = 1$ for the values $V(x,y,z) = 8$, $V(x,y,z) = 4$ and $V(x,y,z) = 1$, viewed from two different directions.

values of the coordinates x, y, z. We find that these constraints are linear in the constants α_k and polynomial in the coordinates x, y, z, thus coefficients of various monomials in x, y, z must vanish independently. Assuming that $w_0 \neq 0$ and a, b are chosen in the prescribed range, we find that any such integral of motion must be a linear combination of H, X_1 and X_2, i.e. the system (1) with the potential (11) does not possess any other integrals of motion linear or quadratic in the momenta. Considering that the classification of the possible leading order terms of the pairs of commuting integrals in [14]

- splits the admissible leading order structures into nonequivalent and disjoint classes, and
- all the known quadratically integrable and separable systems of [1] possess pairs of commuting integrals corresponding to other classes than (c) of [14],

we see that the considered system cannot be transformed using Euclidean transformations to any of the quadratically integrable and separable systems of [1]. Therefore, it provides an example of a quadratically integrable yet not separable natural Hamiltonian system.

While discussing a preliminary version of this result with G. Rastelli, he hinted at an alternative, more straightforward argument leading to the same conclusion, using a theorem due to Eisenhart [2] (see also [20–22] and the monograph [23], section 4.3.1, for its recent formulations and discussion). That theorem restricted to our setting states that the system described by the Hamiltonian (1) is separable in an orthogonal coordinate system if and only if two Killing tensors K_1 and K_2 exist such that

- they are in involution, i.e. $\left\{\sum_{a,b} K_1^{ab} p_a p_b, \sum_{a,b} K_2^{cd} p_c p_d\right\}_{\mathrm{P.B.}} = 0$,
- if interpreted as (1,1)–tensors by lowering one of their indices by the metric (in our case Cartesian, i.e. $K_{ab} = K_a{}^b = K^{ab}$), the Killing tensors K_1 and K_2 possess a basis of common eigenforms, in other words they commute as linear operators on the cotangent space in every point of the configuration space,
- and the equation

$$\mathrm{d}\left(K_k \cdot \mathrm{d}V\right) = \mathrm{d}\left(\sum_{a,b} \left(K_k\right)_a{}^b \partial_b V(\vec{x})\, \mathrm{d}x^a\right) = 0, \quad k = 1,2. \tag{19}$$

holds.

We notice that the first order equations coming from the Poisson bracket $\{H, X_k\}_{\mathrm{P.B.}} = 0$ can be written in the language of differential forms as

$$\mathrm{d}m = 2K \cdot \mathrm{d}V; \tag{20}$$

thus the existence of the function m is (locally) equivalent to the closedness of the 1–form $K \cdot \mathrm{d}V$, i.e. the equation (19) (which in turn is equivalent to (18)). However the commutativity of the Killing tensors needed for the existence of separable coordinates is imposed in addition to the (leading order) involutivity condition $\left\{K_1^{ab} p_a p_b, K_2^{cd} p_c p_d\right\}_{\mathrm{P.B.}} = 0$ required for the integrability; notice that the involutivity condition involves only derivatives of the Killing tensors, whereas the condition $[K_1, K_2] = 0$ involves the Killing tensors themselves.

The Killing tensors K_1 and K_2 corresponding to the integrals X_1 and X_2 of the form (4) are easily found from the coefficients of p_x, p_y, p_z as in equation (15), namely

$$K_1 = \begin{pmatrix} y^2 + z^2 + 3(1-4a)b^2 & -3abz - xy & 3by - xz \\ -3abz - xy & x^2 + z^2 + 3\left(1 + 2a - 3a^2\right)b^2 & 3(a-1)bx - yz \\ 3by - xz & 3(a-1)bx - yz & x^2 + y^2 + 6(a-1)b^2 \end{pmatrix} \tag{21}$$

and

$$K_2 = \begin{pmatrix} az^2+y^2 & -3abz-xy & a(3by-xz) \\ -3abz-xy & x^2+9ab^2 & 0 \\ a(3by-xz) & 0 & ax^2+9a^2b^2 \end{pmatrix}. \tag{22}$$

The commutator $[K_1, K_2]$ of the symmetric matrices K_1, K_2 equals the antisymmetric matrix

$$[K_1,K_2] = 3b \begin{pmatrix} 0 & -a\left(-az^3+(1-a)x^2z-y^2z+6(a-1)bxy+9a(a-1)b^2z\right) & (a-1)x^2y-ayz^2-y^3+6a(a-1)bxz+9a(1-a)b^2y \\ \dots & 0 & (a-1)\left(9a(a-1)b^2x-6abyz+(a-1)x^3+axz^2-xy^2\right) \\ \dots & \dots & 0 \end{pmatrix}. \tag{23}$$

In our allowed range of the parameters $0 < a \leqslant \frac{1}{2}$ and $b \neq 0$ for the integrals belonging to the class (c) of [14] the two Killing tensors do not commute and thus do not possess a basis of common eigenforms. As no quadratic integrals of motion other than X_1, X_2 and H and thus no Killing tensors other than linear combinations of K_1, K_2 and $\mathbf{1}$ (corresponding to the Hamiltonian H itself) are allowed by the potential (11), no characteristic Killing tensor of [15, 22] exists for our potential and we again conclude that the system with the potential (11) can not separate in any orthogonal coordinate system.

For the sake of completeness, let us consider non-orthogonal separation of variables in the Hamilton–Jacobi equation. Due to theorem 5 in [22] such a separation would require the existence of at least one Killing vector k^a such that the corresponding function $F = \sum_{a=x,y,z} k^a p_a$ is an integral of motion. As the potential (11) does not possess any integrals of motion first order in the momenta, we can conclude that it does not allow non–orthogonal separation either.

For consistency we may also investigate the limits of the parameters which correspond to transition to other classes of [14], namely (a) and (b). In the case $b = 0$ the leading order terms of the integrals belong to the class (b), the commutator (23) vanishes and the system correspondingly separates in the conical coordinates

$$x = \frac{r\theta\lambda}{\sqrt{a}}, \quad y = r\sqrt{\frac{(\theta^2-a)(a-\lambda^2)}{a(1-a)}}, \quad z = r\sqrt{\frac{(1-\theta^2)(1-\lambda^2)}{(1-a)}}, \tag{24}$$

in which the potential (11) reads

$$V(x,y,z) = \frac{w_0}{r^2(\theta^2-\lambda^2)}. \tag{25}$$

If $a = 0$ which implies that the integrals X_1 and X_2 are of the form (a) of [14], the commutator (23) does not vanish. However, in this case the potential (11) depends only on $u = x^2 + y^2$, thus the integral X_2 reduces to a linear one, namely the third component of the angular momentum l_z, and an additional functionally independent integral p_z exists. Their combinations with the known integrals X_1 and H allow to construct new pairs of quadratic integrals responsible for the separation of variables in the Hamilton–Jacobi equation in the spherical ($X_1 = l_x^2 + l_y^2 + l_z^2 + 2\frac{w_0 z^2}{x^2+y^2}$, $X_2 = l_z^2$) and cylindrical ($X_1 = p_z^2$, $x_2 = l_z^2$) coordinates.

4. Conclusions

The purpose of this paper is to bring the research community's attention to the forgotten assumption in [1] and explicitly demonstrate that the statement on the equivalence of quadratic integrability and separability in 3D Euclidean space does not hold if that assumption is violated, arriving at a new quadratically integrable yet not separable system with the potential (10). Let us recall that as the determining equations for the quadratic integrals of motion of natural Hamiltonian systems (1) in classical and quantum mechanics coincide, also the quantum system with the potential (10) is integrable with integrals of motion quadratic in momenta and its Schrödinger equation does not separate in any orthogonal coordinate system, i.e. the potential can not be expressed in any of the forms of [1, 3].

As we have already mentioned in section 2, P Winternitz, one of the original authors of [1], learned about the possible loophole in [1] and was aiming to fill it in with his student in the thesis [19]. Our system should have been found there in section 2.3.1, as it is of the form closely resembling (2.3.5) therein (up to typos), but it was not. As we have seen above, its existence invalidates the two conclusions of [19], namely that the case of rank $R = 2$ does not lead to any new integrable systems and that it implies that any integrable system with rank $R = 2$ must possess two commuting first order integrals of motion.

It is not yet known whether our system (10) is the sole exception to the statements in [1, 19] or whether other quadratically integrable nonseparable systems in Euclidean 3D space do exist. Thus, a complete re-derivation of the list of quadratically integrable systems based on the classification of the leading order terms in [14] is currently under way and we expect to report on it in not too distant future. We also intend to study the presented system (11) in more detail, e.g. to attempt to find its (generalized) action-angle variables (notice that the shape of the equipotential surfaces as in figure 1 implies noncompact level sets of the Hamiltonian), as well as to analyse its quantum counterpart.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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References

[1] Makarov A A, Smorodinsky Y A, Valiev K and Winternitz P 1967 A systematic search for nonrelativistic systems with dynamical symmetries *Nuovo Cimento* A **10** 1061–84
[2] Eisenhart L P 1934 Separable systems of Stackel *Ann. Math.* **35** 284–305

[3] Eisenhart L P 1948 Enumeration of potentials for which one-particle Schrödinger equations are separable *Phys. Rev.* **74** 87–89
[4] Evans N W 1990 Superintegrability in classical mechanics *Phys. Rev.* A **41** 5666–76
[5] Cleary P W 1990 Nonexistence and existence of various order integrals for two- and three-dimensional polynomial potentials *J. Math. Phys.* **31** 1351–5
[6] Miller Jr W, Post S and Winternitz P 2013 Classical and quantum superintegrability with applications *J. Phys. A: Math. Theor.* **46** 423001
[7] Datta Majumdar S and Englefield M J 1977 Third-order constants of motion in quantum mechanics *Int. J. Theor. Phys.* **16** 829–35
[8] Bérubé J and Winternitz P 2004 Integrable and superintegrable quantum systems in a magnetic field *J. Math. Phys.* **45** 1959–73
[9] Miller Jr W 1977 *Symmetry and separation of variables Encyclopedia of Mathematics and its Applications* vol 4 (Addison-Wesley Publishing Co.)
[10] Drach J 1935 Sur l'intégration logique et sur la transformation des équations de la dynamique à deux variables: Forces conservatives. Intégrales cubiques *C. R. Acad. Sci., Paris* **200** 599–602
[11] Cariglia M, Gibbons G W, van Holten J-W, Horvathy P A, Kosiński P and Zhang P-M 2014 Killing tensors and canonical geometry *Class. Quantum Grav.* **31** 10
[12] Dunajski M, Maciejewski A J and Przybylska M 2024 Integrability of quantum dots *Phys. Lett.* B **859** 139097
[13] Rañada M F 1997 Superintegrable $n = 2$ systems, quadratic constants of motion and potentials of Drach *J. Math. Phys.* **38** 4165–78
[14] Marchesiello A and Šnobl L 2022 Pairs of commuting quadratic elements in the universal enveloping algebra of Euclidean algebra and integrals of motion *J. Phys. A: Math. Theor.* **55** 145203
[15] Horwood J T, McLenaghan R G and Smirnov R G 2005 Invariant classification of orthogonally separable Hamiltonian systems in Euclidean space *Commun. Math. Phys.* **259** 679–709
[16] Marchesiello A, Šnobl L and Winternitz P 2015 Three-dimensional superintegrable systems in a static electromagnetic field *J. Phys. A: Math. Theor.* **48** 395206
[17] Marchesiello A and Šnobl L 2017 Superintegrable 3D systems in a magnetic field corresponding to Cartesian separation of variables *J. Phys. A: Math. Theor.* **50** 245202
[18] Marchesiello A, Šnobl L and Winternitz P 2018 Spherical type integrable classical systems in a magnetic field *J. Phys. A: Math. Theor.* **51** 135205
[19] Abdul-Reda H 2019 Intégrabilité et superintégrabilité de deuxième ordre dans l'espace euclidien tridimensionel *Master's Thesis* Université de Montréal (available at: https://hdl.handle.net/1866/23971)
[20] Kalnins E G and Miller Jr W 1980 Killing tensors and variable separation for Hamilton-Jacobi and Helmholtz equations *SIAM J. Math. Anal.* **11** 1011–26
[21] Benenti S, Chanu C and Rastelli G 2000 The super-separability of the three-body inverse-square Calogero system *J. Math. Phys.* **41** 4654–78
[22] Benenti S 1997 Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation *J. Math. Phys.* **38** 6578–602
[23] Kalnins E G, Kress J M and Miller W Jr 2018 *Separation of Variables and Superintegrability* (IOP Expanding Physics) (IOP Publishing)