A holographic construction of the Bose-Hubbard model

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November 29, 2015

Abstract

A holographic large-N Bose-Hubbard model has been presented in the paper [1]. The holographic model is a theory with Maxwell fields and charged scalar fields on the AdS_2 hard wall. The lobe-shaped phase structure of the Bose-Hubbard model is realized by using holography. The Mott insulator phase in which bosons are localized on each site is found for large repulsive interaction. In addition to the phase transition between Mott insulator phases with different occupation numbers at zero hopping integral, a holographic phase transition is found at non-zero hopping integral between the Mott insulator phase and a non-homogeneous phase. The perturbations of fields are also analyzed around both the Mott insulator phase and the non-homogeneous phase. Almost zero modes are found in the non-homogeneous phase.

1 Introduction

Translation invariance is broken in real materials including impurities. The DC conductivity is non-zero in real materials including impurities, while it is divergent in the pure system. In lattice theories, the translation invariance is also broken. One of the motivation of my talk is to introduce a (holographic) lattice system in the duality between strongly coupling field theories and the weakly coupling gravity theories, namely, the gauge/gravity correspondence [2]. In the gauge/gravity correspondence and in the probe limit without considering the backreactions [3, 4], holographic lattices of impurities have realized the dimerization transition changing the shape of the Fermi surface [5, 6] and the Kondo effect [7]. The holographic lattices have also been introduced by using periodic functions of the chemical potential [8, 9] and those of scalars [10]. Such a background breaks the translation invariance along the periodic direction.

The Bose-Hubbard model is the effective theory of cold atoms on an optical lattice including the hopping term and short-range repulsive interaction. The Hamiltonian is given by

$$H = -v_{\text{hop}} \sum_{\langle ij \rangle} (b_i^\dagger b_j + c.c.) - \mu \sum_i n_i + \frac{U}{2} \sum_i n_i (n_i - 1), \tag{1.1}$$

where $v_{\rm hop}$ is the hopping integral and the occupation number is defined by $n_i = b_i^\dagger b_i$. The hopping term is important to describe the motion of particles and to analyze the transport. It is known that, at zero temperature and without impurities, only two phases exist in the phase structure of the Bose-Hubbard model [11]. The Mott insulator phase arises when the repulsive interaction is much stronger than the hopping term ($U/v_{\rm hop} \gg 1$). In the Mott insulator phase, bosons are localized on each a site. In the opposite limit $U/v_{\rm hop} \ll 1$, the superfluid phase is favored. In the superfluid phase, $U(1)$ symmetry is broken and bosons are delocalized. Moreover, it is known that the phase structure of the ground state is lobe-shaped. In the $(\mu/U, v_{\rm hop}/U)$ plane, the phase boundary between the Mott insulator phase and the superfluid phase is the lobe-shape whose amplitude decreases as $1/\rho$. The phase structure can be realized by introducing complex fields coupling to b_i and using the meanfield approach. The action in the infinite-range hopping limit becomes

$$S(\psi) = \beta\Big(\frac{1}{2}r(\mu, v_{\rm hop}, T) \mid \psi \mid^2 + u(\mu, T) \mid \psi \mid^4 + O(\mid \psi \mid^6)\Big), \tag{1.2}$$

where $T = 1/\beta$ is the temperature and N is the number of the lattice sites. In the Mott insulator phase, $\langle\psi\rangle = 0$, while $\psi \neq 0$ in the superfluid phase. Thus, ψ is an order parameter of the phase transition. The phase boundary is given by the condition $r(\mu, v_{\rm hop}) = 0$.

There is a general extension of the Bose-Hubbard model to the model of multiple species such as the $SU(N)$ Bose-Hubbard model. The conjecture of our paper is that the $SU(N)$ Bose-Hubbard model is dual to the $2d$ gravity on the AdS_2 hard wall. Actually, the large N limit is powerful and necessary to cause the phase transition in the finite volume system.

In next section, we realize the lobe-shaped phase structure of the Bose-Hubbard model by using the holographic model. We then analyze the spectrum of the perturbations around the cusp point near $v_{\rm hop} = 0$.

2 The lobe-shaped phase structure of the holographic Bose-Hubbard model

In this section, we give a review of the holographic construction of large N Bose-Hubbard model [1]. It is conjectured that the large N $SU(N)$ Bose-Hubbard model corresponds to $2d$ gravity on the AdS_2 hard wall geometry. The metric of the AdS_2 hard wall is given by in the unit AdS radius

$$ds^2 = -u^2 dt^2 + \frac{du^2}{u^2}, \tag{2.3}$$

where the hard wall cutoff is put at $u = u_{\rm h}$ ($u \geq u_{\rm h}$). The AdS/CFT correspondence is summarized in the table 1. The occupation number n_i is conjugate to the chemical potential μ. Only the diagonal part of the chemical potential is considered. These are dual to $U(1)^n$ gauge fields A_i, where n corresponds to the number of the lattice site. Besides, the bi-local $b_i^{a\dagger} b_{ja}$ is dual to the hopping integral $v_{\rm hop}$, where $a = 1, \ldots, N$ represent spin indices. These fields are dual to the bi-fundamental matter $\phi_{i,j}$ linking two different nodes of $U(1)^n$ gauge symmetry. Finally, the Coulomb repulsive parameter corresponds to the IR cutoff $u_{\rm h}$. Even if our model is the theory on the finite volume, the phase transition can be captured thanks to the large N limit. We then focus on the 2-site model $n = 1, 2$.

Defining the field strength $f_{(n)\mu\nu} = \partial_\mu a_{(n)\nu} - \partial_\nu a_{(n)\mu}$, the Lagrangian of our holographic model

Table 1: The AdS/CFT correspondence

The large N Bose-Hubbard model side	$2d$ Gravity side
μ (chemical potential) and $b_i^{a\dagger} b_{ia}$ (occupation number)	$a_{t,i}$
$v_{\rm hop}$ (hopping integral) and $b_i^{a\dagger} b_{ja}$ (bi-local operator)	$\Phi_{i,j}$
U	hard wall cut-off $u_{\rm h}$

includes the IR potential and is given by

$$S = S_{\rm gauge} + S_{\rm matter} + S^{\rm IR}_{\rm mixed}, \tag{2.4}$$

$$S_{\rm gauge} = \sum_{n=1}^{2} \int d^2x \sqrt{-g}\Big(-\frac{1}{4} f_{(n)\mu\nu} f^{\mu\nu}_{(n)}\Big), \tag{2.5}$$

$$S_{\rm matter} = -\int d^2x \sqrt{-g}(|\vec{D}\Phi|^2 + M^2 |\Phi|^2), \tag{2.6}$$

$$\begin{aligned} S^{\rm IR}_{\rm mixed} &= -\int_{u=u_{\rm h}} dt u_{\rm h}(2m_{\rm R}^2 |\Phi|^2 + \lambda |\Phi|^4 + \\ &+ \sum_{m,l\geq 1} \lambda_{(m,l)} |\Phi|^{2m} \sum_n (f^{(n)}_\mu f^{(n)\mu})^l + \dots, \end{aligned} \tag{2.7}$$

where $f^{(i)}_\mu$ is the projected field strength in terms of the normal n_μ satisfying $n_\mu n^\mu = 1$ at the IR boundary and $f^{(i)}_\mu \equiv f^{(i)}_{\mu\nu} n^\nu$. Here, the covariant derivative is defined by $D_\mu = \partial_\mu - iqa^{(1)}_\mu + iqa^{(2)}_\mu$ $(\mu = u, t)$.

A few remarks of our holographic construction are as follows: (1) The Lagrangian includes an IR potential and an IR mass [12, 13]. The IR potential and IR mass affect the phase structure of the holographic model keenly and should be present in our bottom-up lagrangian. (2) The cutoff of the AdS_2 hard wall must be larger than $v_{\rm hop}$ ($u_{\rm h}/v_{\rm hop} \gg 1$) because instable modes appear above the energy scale greater than $u_{\rm h}$ [17]. After introducing large cutoff $u_{\rm h}$, small energy can be added to the gravity safely. (3) The Dirac quantization condition should be imposed on charges n_i and magnetic monopole charges [19]. The Dirac quantization condition is non-trivial and appears because of the gauge invariance under the large gauge transformation surviving at the AdS boundary. In the string theory, moreover, the quantization can be understood as the quantization of the coefficient of F1 and D-brane interactions such as $I_{int} \sim n_{\rm F1} \int B^{NSNS}$ $(n_{\rm F1} \in {\rm Z})$ [18].

The EOMs are derived from (2.4) in the radial gauge $a^{(m)}_u = 0$ as

$$\begin{aligned} &(u^2\Phi')' - M^2\Phi + \frac{q^2}{u^2}(a^{(1)}_t - a^{(2)}_t)^2 \Phi = 0, \\ &a^{(m)\prime\prime}_t - \frac{2q^2 |\Phi|^2}{u^2}(a^{(m)}_t - a^{(m+1)}_t) = 0, \end{aligned} \tag{2.8}$$

Let us consider the $M = 0$ case. In general, there are two cases of solutions classified by the difference of the gauge fields. First, the homogeneous case corresponds to the ansatz of equal gauge fields $a^{(1)}_t = a^{(2)}_t$. In the homogeneous case, the non-linear interaction vanishes in (2.8) and the analytic solutions to the EOMs are known. At the hard wall $u = u_{\rm h}$, the Dirichlet type boundary condition is imposed on fields. The homogeneous phase corresponds to the Mott insulator phase since gauge fields are equal and the occupation number is equal in each a site. Secondly, the non-homogeneous phase corresponds to the

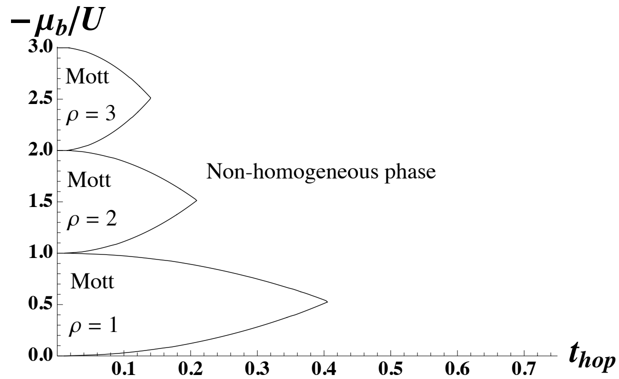

Figure 1: The figure is taken from [1]. The lobe-shaped phase structure of our holographic model with $t_{\rm hop} = w_{\rm hop}$. Parameters are chosen as $u_{\rm h} = 40,\ \lambda = 1,\ m_{\rm R} = 0,\ \lambda_{(1,1)} = -3/2$. Using the occupation number in the Mott insulator phase, the amplitude of the lobe (an edge of the phase boundary) decreases as $1/\rho$.

ansatz of the unequal gauge fields $a_t^{(1)} \neq a_t^{(2)}$. The boundary condition at the hard wall is not unique in the non-homogeneous phase. The (bottom-up) free type boundary condition is chosen for the bi-fundamental matter. The occupation number is usually different in each a site in the non-homogeneous phase. In the superfluid phase, the occupation number is also different in each a site. So, the non-homogeneous phase should correspond to the superfluid phase arising in the limit of the large kinetic energy. The non-homogeneous phase looks similar to the sector of the axial vector in the hard wall AdS/QCD model [14] which has the same matter content in the gravity dual. Two different solutions are consistent with only two phases in the Bose-Hubbard model at zero temperature and without impurities.

Performing the Wick rotation to the Euclidean signature, the free energy is evaluated using the holographically renormalized action. By comparing the free energy between (non)-homogeneous phases, the lobe-shaped structure of our holographic model can be realized in the large N limit in Fig. 1. When $v_{\rm hop}/U$ is large, the non-homogeneous phase is favored thermodynamically. In the different limit $v_{\rm hop}/U \ll 1$, the homogeneous phase is favored. The phase boundary of these two phases has the shape of the lobe. As seen in the Fig. 1, the decreasing amplitude of the lobe is also found. The edge of the lobe behaves like $v_{\rm hop}|_{\rm edge} = 1/\rho$, where ρ is the occupation number in the Mott insulator phase surrounded by the boundary. This edge also becomes the particle-hole symmetric point where three phases with different occupation numbers are degenerate.

For non-zero hopping $v_{\rm hop} \neq 0$, symmetry breaking $U(1)^2 \to U(1)$ happens. This symmetry breaking becomes spontaneous symmetry breaking near the cusp point of the μ_b axis. [1] The second order phase transition happens as the function of $v_{\rm hop}$ at the cusp point. Far from the cusp point of the μ_b axis, symmetry breaking becomes explicit breaking. The phase transition becomes a large N first order phase transition in general. The order parameter of the phase transition becomes $\delta n = n_1 - n_2$ or the effective hopping defined by $dF/dv_{\rm hop} \sim \langle b_i^\dagger b_j + c.c. \rangle$. Remind that this effective hopping is not equal to the long-range correlations in the superfluid phase of the field theory side but equal to the VEV of the bi-local between nearest neighbor sites. Nevertheless, $dF/dv_{\rm hop}$ can be understood as the order parameter of the

[1] As the function of μ, on the other hand, the phase transition is of the first order.

phase transition in the large N limit.

The effective hopping in the gravity dual has some interesting properties. In the Mott insulator phase and in the $v_{\rm hop} \to 0$ limit, the effective hopping behaves as $dF/dv_{\rm hop} \propto v_{\rm hop}$ similar to those of the $SU(N)$ Bose-Hubbard model. By introducing non-zero bulk mass M, actually, the effective hopping can be fitted to the numerical calculation in the $SU(N)$ Bose-Hubbard model side at small hopping limit [15]. In the non-homogeneous phase, the behavior is affected by the non-linear interaction with the gauge potential for non-zero q. When q is small, the effective hopping should be similar to those of the Bose-Hubbard model since the interactions with a large N CFT become small for small q.

To confirm that our non-homogeneous phase corresponds to the superfluid phase, zero modes should be found around the second order phase transition point because a massless Goldstone mode is expected to appear at the critical point. So, it is interesting to analyze the perturbation around the cusp point in both the (non)-homogeneous phases. The holographic two point function can be computed from the perturbation of the bi-fundamental scalar around the background solution. In the homogeneous phase, the spectrum obtained from the perturbation of the bi-fundamental scalar is always gapped (see also [16]). The gap is given by

$$\Delta = \pi U n \quad (n\text{: a positive integer}), \tag{2.9}$$

where $U = u_{\rm h}$. The gap proportional to U is an expected result in the Mott insulator phase. On the other hand, a zero mode appears in the spectrum in the non-homogeneous phase. The presence of the zero mode could be connected to the spontaneous symmetry breaking $U(1)^2 \to U(1)$.

3 Conclusion

We realized the lobe-shape of the phase structure of the Bose-Hubbard model. The Mott/non-homogeneous phase transition was of the 1st order except for the cusp point by using $dF/dv_{\rm hop}$ or the difference of the occupation number as the order parameter. The phase transition became of the 2nd order at the cusp point near $v_{\rm hop} \sim 0$, where a zero mode arised from the spectrum of the perturbation.

A top down construction can be proposed in terms of a D3/D5/D7 system. The D3/D7 brane configuration is based on the paper [20]. In the gravity dual, N D3-branes are replaced by the AdS_5 soliton. Non-Abelian $U(n_F)$ D5-branes are embedded and wrapped on asymptotic $AdS_2 \times S^4$ in AdS_5 soliton $\times S^5$, where n_F corresponds to the number of the lattice of the effective theory. After letting a transverse scalar have the VEV, $U(n_F)$ symmetry is broken to $U(1)^{n_F}$. The diagonal elements of the adjoint field of $U(n_F)$ and the off-diagonal elements of it can then be interpreted as $U(1)^{n_F}$ gauge fields and the bi-fundamental matter like matter contents of our bottom-up model, respectively.

Acknowledgement: M.F. would like to thank M. Tezuka for providing numerical data of the $SU(N)$ Bose-Hubbard model. M.F. is partially supported by the grants NSF-PHY-1521045 and NSF-PHY-1214341. S.H. is supported by the Harvard University Golub Fellowship in the Physical Sciences. The work of A. K. is in part supported by the U.S. Department of Energy under Grant number DE-SC0011637. This work of R.M. was supported by World Premier International Research Center Initiative (WPI), MEXT, Japan. N.M.P. acknowledges support from an NSF Graduate Research Fellowship.

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