International Conference on Non-commutative Geometry and Physics IOP Publishing
Journal of Physics: Conference Series **103** (2008) 012002 doi:10.1088/1742-6596/103/1/012002

Deformation Quantization: a survey

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Abstract. We give an introduction to deformation quantization with emphasis on explicit formulas and sketch some partial results on theory of morphisms and modules in that framework and their relation to Poisson geometry.

Introduction

The theory of deformation quantization has become a large research area covering several algebraic theories like the formal deformation theory of associative algebras and the more recent theory of operades, as well as geometric theories like the theory of symplectic and (more generally) Poisson manifolds), and of physical theories like string theory and noncommutative field theory. The main starting point of deformation qunatization is the seminal article by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in 1978 [5]. In this theory, the noncommutative associative multiplication of operators in quantum mechanics is considered as a formal associative deformation of the pointwise multiplication of the 'algebra of symbols of these operators': in physical terms, this means the 'algebra of classical quantities' which is given by the algebra of all complex-valued $\mathcal{C}^\infty$-functions on a Poisson manifold, i.e the 'phase space' of classical mechanics. The formal parameter is an interpretation of Planck's constant $\hbar$ in convergent situations. The advantage of this method is its universality: according to a theorem by Kontsevich [57] this construction is possible for any Poisson manifold. Moreover, geometric intuition is quite useful in concrete situations since everything is formulated in geometrical terms on a differentiable manifold in contrast to the usual formulation of quantum mechanics where one has to specify a Hilbert space. The price to pay is the fact that complex numbers are replaced by the ring of all formal complex power series whose convergence is a case-by-case study. However, inspite of these difficulties I see the major interest of deformation quantization for a physicist in its rôle as an *asymptotic testing ground* for 'true quantum theories' with admit a reasonable classical limit, i.e. a parameter like Planck's constant $\hbar$ whose asymptotic limit $\hbar \to 0$ will lead in most of the practical applications to a theory like deformation quantization: hence if some concepts *break down* even on the level of deformation quantization we cannot expect them to work in any reasonable quantum theory.

The main objective of this survey is a (hopefully motivating) introduction to this subject: I have not included in detail all the existence and classification proofs which are quite technical. Neither do I speak about the theory of operades which has become the algebraic framework of this theory since Kontsevich. I'd rather would like to underline some motivations from physics, discuss concrete examples and talk -at the end- about the still open theory of the deformation theory of modules, Poisson morphisms, and (symplectic) reduction.

In the first Section, I have given an elementary deduction of several star-product formulas from canonical quantization (or symbol calculus). These formulas motivate Section 2 in which the abstract formal associative deformation theory of associative algebra is sketched. As a by-product, this abstract theory always gives the structure of a Poisson bracket as first order commutator, motivating Section 3 in which a survey on Poisson and symplectic manifolds is given. Section 4 deals with the Definition and general Existence and Equivalence theorems for Deformation quantization on a general Poisson manifold. Some other explicit examples such as T^*S^n and 'fuzzy' $\mathbb{C}P^n$ are discussed in Section 5. The last Section is devoted to some more recent results where I am discussing three algebraic concepts in deformation quantization which all have a physical meaning. Section 6.1 deals with algebra homo*morphisms* of the deformed algebras whose classical limit will become pull-backs with Poisson maps: the quantization of the latter contains the *quantization problem of symmetries and integrable systems.* In Section 6.2 we discuss representations or *modules* of the deformed algebra: apart from possible representations in (formal pre) Hilbert spaces à la GNS, the more general classical limit yields coisotropic maps and *coisotropic* (i.e. first class) submanifolds. This concerns the important physical problem of quantization of constraints. Finally, Section 6.3 deals with *phase space reduction* which is the Leitmotiv of any gauge theory: the quantization problem is to construct a star-product on the reduced space by means of a star-product on the unreduced ('unphysical') space.

1. Canonical Quantization and elementary Star-Products

In order to describe a quantum system it is necessary to know its Hamiltonian operator $\hat{H}$. In practice, the source of inspiration is the Hamiltonian function of the corresponding system of classical mechanics, and any 'reasonable' recipe of translating classical observables to quantum observables is called *quantization*.

According to P.A.M.Dirac, all quantizations should satisfy a *classical limit condition*, i.e. for all classical observables f, g

$$\hat{f}\hat{g} = \widehat{fg} + o(\hbar) \quad (1)$$

$$\hat{f}\hat{g} - \hat{g}\hat{f} = i\hbar\widehat{\{f,g\}} + o(\hbar^2) \quad (2)$$

where $\{f, g\} := \partial f/\partial q \partial g/\partial p - \partial f/\partial p \partial g/\partial q$ denotes the canonical Poisson bracket.

In this section we shall discuss several possible quantizations in 1 degree of freedom which are used in quantum physics.

We recall the differential operators Q (position operator) and P (impulsion operator) in case $n = 1$:

$$(Q\psi)(q) := q\psi(q) \quad (3)$$

$$P := (\hbar/i)\partial/\partial q. \quad (4)$$

In the following sections we denote by $\mathbb{C}[s_1, \ldots, s_N]$ the space of complex polynomials in N variables $s_1, \ldots, s_N$. Moreover the symbol $\mathrm{Diffop}_{poly}(\mathbb{R})$ denotes the space of all differential operators with polynomial coefficients in the space $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$, i.e an element D takes the following general form

$$\sum_{k=0}^{N} f_k \partial^k/\partial q^k \quad (5)$$

where $f_1, \ldots, f_N \in \mathbb{C}[q]$.

Journal of Physics: Conference Series **103** (2008) 012002 doi:10.1088/1742-6596/103/1/012002

1.1. Standard Ordering

We shall consider the the following linear map ρ_s of the space of all complex polynomials in two variables $\mathbb{C}[q,p]$ in the space $\mathrm{Diffop}_{poly}(\mathbb{R})$:

$$\begin{aligned} 1 &\mapsto \rho_s(1) := 1 && (6)\\ q &\mapsto \rho_s(q) := Q && (7)\\ p &\mapsto \rho_s(p) := P && (8)\\ q^m p^n &\mapsto \rho_s(q^m p^n) := Q^m P^n && (9) \end{aligned}$$

Since every differential operator in $\mathrm{Diffop}_{poly}(\mathbb{R})$ takes the form (5) it is obvious that this linear map is a bijection. The principal idea of star-products is to pull back the (noncommutative) associative multiplication of differential operators by the map ρ_s:

Proposition 1.1 *Let f, g be in $\in \mathbb{C}[q,p]$ and $\phi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$. Then*

$$\rho_s(f)\big(\phi\big) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \left.\frac{\partial^r f}{\partial p^r}\right|_{p=0} \frac{\partial^r \phi}{\partial q^r}. \tag{10}$$

Moreover

$$f *_s g := \rho_s^{-1}\big(\rho_s(f)\rho_s(g)\big) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r} \tag{11}$$

is a well-defined associative noncommutative multiplication on the space $\mathbb{C}[q,p]$ which satisfies the classical limit

$$f *_s g = fg - i\hbar \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} + o(\hbar^2).$$

Proof: The proof of eqn (10) is a direct computation on monomials $q^m p^n$. Since $*_s$ is obviously isomorphic to the associative multiplication of differential operators by means of the linear bijection ρ_s it is clear that $*_s$ is also associative. The formula (11) is checked for monomials $f(q,p) = q^a p^b$ and $g(q,p) = q^c p^d$ by observing that $\rho_s(f *_s g) = \rho_s(f)\rho_s(g)$, hence $\rho_s(f) = Q^a P^b$ and $\rho_s(g) = Q^c P^d$ whence we have to bring $Q^a P^b Q^c P^d$ in standard form by the Leibniz rule,

$$\begin{aligned} (P^b Q^c \varphi)(q) &= \left(\frac{\hbar}{\mathrm{i}}\right)^b \frac{\partial^b\big(q^c\varphi(q)\big)}{\partial q^b} = \sum_{r=0}^{b} \left(\frac{\hbar}{\mathrm{i}}\right)^r \binom{b}{r} \frac{c!}{(c-r)!} q^{c-r} \left(\frac{\hbar}{\mathrm{i}}\right)^{b-r} \frac{\partial^{b-r}\big(\varphi(q)\big)}{\partial q^{b-r}} \\ &= \sum_{r=0}^{b} \left(\frac{\hbar}{\mathrm{i}}\right)^r \frac{1}{r!} \left(\frac{c!}{(c-r)!} Q^{c-r} \frac{b!}{(b-r)!} P^{b-r} \varphi\right)(q) \end{aligned}$$

which proves eqn (11). □

Note that for given polynomials f, g the series in $\hbar$ is always a finite sum. Moreover, every term in that series is a *bidifferential operator* $\frac{(1/i)^r}{r!}\frac{\partial^r f}{\partial p^r}\frac{\partial^r g}{\partial q^r}$.

1.2. Weyl-Moyal ordering prescription

From the point of view of physics, standard ordering is not satisfactory: when considering the pre-Hilbert space

$$\mathcal{D}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} | f \text{ is } \mathcal{C}^\infty \text{ and } supp(f) \text{ is compact} \} \tag{12}$$

equipped with the scalar product (for all $\phi, \psi \in \mathcal{D}(\mathbb{R})$):

$$\langle \phi, \psi \rangle := \int dq \; \overline{\phi(q)}\psi(q) \tag{13}$$

we quickly see that the two real-valued functions q and p correspond to *symmetric operators*, i.e. for $A = Q$ or $A = P$

$$\langle \phi, A\psi \rangle = \langle A\phi, \psi \rangle, \tag{14}$$

whereas the real-valued function qp corresponds to the operator QP whose adjoint in $\mathcal{D}(\mathbb{R})$ is equal to $PQ = QP - i\hbar 1$: hence $\rho_s(qp)$ is no longer symmetric which would be necessary to render it a self-adjoint operator in the Hilbert space completion $L^2(\mathbb{R}, dq)$ of $\mathcal{D}(\mathbb{R})$. In order to avoid these problems, the *Weyl-Moyal ordering prescription* had been introduced: this uses a symmetrization of the monomials in Q and P.

We consider the following linear map ρ_w of the space of all complex polynomials of two variables $\mathbb{C}[q, p]$ in the space $\mathrm{Diffop}_{poly}(\mathbb{R})$:

$$1 \mapsto \rho_w(1) := 1 \tag{15}$$

$$q \mapsto \rho_w(q) := Q \tag{16}$$

$$p \mapsto \rho_w(p) := P \tag{17}$$

$$q^m p^n \mapsto \rho_s(q^m p^n) := \frac{1}{(m+n)!} \sum_{\sigma \in S_{m+n}} A_{\sigma(1)} \cdots A_{\sigma(m+n)} \tag{18}$$

where the operators $A_1, \cdots, A_{m+n}$ are given by

$$A_k := \begin{cases} Q & \text{si } 1 \leq k \leq m \\ P & \text{si } m+1 \leq k \leq m+n \end{cases}$$

For example, $\rho_w(qp) = (QP + PQ)/2$ and $\rho_w(q^2 p) = (Q^2 P + QPQ + PQ^2)/3$. By definition, the operators $\rho_w(f)$ are symmetric if f is real because it is easily computed that

$$\rho_w(f)^\dagger = \rho_w(\bar{f})$$

where $A^\dagger$ is the adjoint operator of A in $\big(\mathcal{D}(\mathbb{R})), \langle \;,\; \rangle\big)$.

For two formal parameters α, β (considered as real under complex conjugation) the exponential function $\exp(\alpha q + \beta p)$ is mapped to $\rho_w\big(\exp(\alpha q + \beta p)\big) = \exp(\alpha Q + \beta P)$ because it is easy to see by induction that each power $(\alpha Q + \beta P)^n$ is already symmetrized. Using the fact that $\rho_s\big(\exp(\alpha q + \beta p)\big) = \exp(\alpha Q)\exp(\beta P)$, the fact that $[Q, P] = i\hbar 1$ and the Baker-Campbell-Hausdorff formula, we compute

$$e^{(\alpha Q + \beta P)} = e^{\frac{\hbar \alpha \beta}{2i}} e^{\alpha Q} e^{\beta P}.$$

Since the exponential function $\exp(\alpha q + \beta p)$ is a generating function for all polynomials in q, p one realizes the following fundamental relation standard and Weyl-Moyal ordering:

$$\rho_w(f) = \rho_s(Nf) \tag{19}$$

where the map $N : \mathbb{C}[q, p] \to \mathbb{C}[q, p]$ is defined by

$$N := e^{\frac{\hbar}{2i} \frac{\partial^2}{\partial q \partial p}}. \tag{20}$$

It is clear that N is well-defined and invertible, and one deduces that $\rho_w : \mathbb{C}[q, p] \to \mathrm{Diffop}_{poly}(\mathbb{R})$ is a linear bijection. There is the following analogue of Proposition 1.1:

Proposition 1.2 *Let f, g be in $\mathbb{C}[q,p]$. Then*

$$f *_w g := \rho_w^{-1}\big(\rho_w(f)\rho_w(g)\big) = \sum_{r=0}^{\infty} \frac{(i\hbar/2)^r}{r!} \sum_{a=0}^{r} \binom{r}{a} (-1)^{r-a} \frac{\partial^r f}{\partial q^a p^{r-a}} \frac{\partial^r g}{\partial q^{r-a} p^a} \tag{21}$$

is a well-defined noncommutative associative multiplication on the space $\mathbb{C}[q,p]$ satisfying the classical limit

$$f *_w g = fg + \frac{i\hbar}{2}\{f,g\} + o(\hbar^2).$$

and is isomorphic to $*_s$ via N:

$$N(f *_w g) = (Nf) *_s (Ng).$$

Moreover it is hermitean with respect to pointwise complex conjugation in the following sense

$$\overline{f *_w g} = \overline{g} *_w \overline{f}.$$

Proof: The proof is a direct computation using the operator N. □

Again, it is easy to see that $*_w$ is a series of bidifferential operators.

1.3. Wick ordering

There is a third quantization related to the harmonic oscillator which is very often used in quantum field theory: firstly, one forms the following complex variable

$$z := q + ip. \tag{22}$$

On the complex vector space

$$\bar{\mathcal{O}}(\mathbb{C}) := \{\phi : \mathbb{C} \to \mathbb{C} | \phi \text{ antiholomorphic } \}$$

one then defines the following scalar product

$$\langle \phi, \psi \rangle := \frac{1}{4\pi\hbar} \int dz d\bar{z} e^{-\frac{|z|^2}{2\hbar}} \overline{\phi(\bar{z})} \psi(\bar{z}) \tag{23}$$

(which may still diverge), and finally the pre-Hilbert space of all square integrable antiholomorphic functions

$$\mathcal{H} := \{\phi \in \bar{\mathcal{O}}(\mathbb{C}) | \langle \phi, \phi \rangle < \infty\}, \tag{24}$$

which embeds as a closed subspace of the big Hilbert space $L^2\big(\mathbb{R}^2, e^{-(q^2+p^2)/(2\hbar)} dqdp/(2\pi\hbar)\big)$ and is therefore already a Hilbert space. The subspace of all polynomials in the variable $\bar{z}$, $\mathbb{C}[\bar{z}]$, is a dense subspace of $\mathcal{H}$. Partial integration yields the fact that the operator A which mutiplies by the variable z (in the big pre-Hilbert space) induces the *annihilation operator*

$$A := 2\hbar \frac{\partial}{\partial \bar{z}} \tag{25}$$

on $\mathbb{C}[\bar{z}]$. By a second partial integration in $\mathbb{C}[\bar{z}]$ we see that its adjoint $A^\dagger$ (*the creation operator* is the operator

$$(A^\dagger \phi)(\bar{z}) := \bar{z}\phi(\bar{z}). \tag{26}$$

It follows that we can –in a manner completely analogous to standard ordering– consider the following linear map ρ_{wick} of the space of all complex polynomials in two variables $\mathbb{C}[z,\bar{z}]$ in the space $\mathrm{Diffop}_{poly}(\bar{z})$ of all differential operators having polynomial coefficients and acting in the space of polynomials $\mathbb{C}[\bar{z}]$:

$$1 \mapsto \rho_{wick}(1) := 1 \tag{27}$$
$$z \mapsto \rho_{wick}(z) := A \tag{28}$$
$$\bar{z} \mapsto \rho_{wick}(\bar{z}) := A^\dagger \tag{29}$$
$$\bar{z}^m z^n \mapsto \rho_{wick}(\bar{z}^m z^n) := A^{\dagger m} A^n \tag{30}$$

It is obvious that this linear map is a bijection.

Proposition 1.3 *Let f, g be in $\mathbb{C}[q,p]$ and $\phi \in \mathbb{C}[\bar{z}]$. Then*

$$\rho_{wick}(f)(\phi) = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \left.\frac{\partial^r f}{\partial z^r}\right|_{z=0} \frac{\partial^r \phi}{\partial \bar{z}^r}.$$

Moreover

$$f *_{wick} g := \rho_{wick}^{-1}\big(\rho_{wick}(f)\rho_{wick}(g)\big) = \sum_{r=0}^{\infty} \frac{(2\hbar)^r}{r!} \frac{\partial^r f}{\partial z^r} \frac{\partial^r g}{\partial \bar{z}^r}$$

is a well-defined noncommutative associative multiplication on the space $\mathbb{C}[q,p]$ satisfying the classical limit

$$f *_{wick} g = fg + 2\hbar \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} + o(\hbar^2).$$

and the hermitean property with respect to pointwise complex conjugation

$$\overline{f *_{wick} g} = \overline{g} *_{wick} \overline{f}.$$

Proof: The proof is completely analogous to the proof of Proposition 1.1. Note that

$$\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p} = \frac{2}{i} \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}.$$

□

As for the relation between standard ordering and Weyl ordering there is also an analogue of the operator N N (37): one defines

$$\Delta' := \frac{\partial^2}{\partial q^2} + \frac{\partial^2}{\partial p^2}$$

and

$$N' := e^{\frac{\hbar}{4}\Delta'}.$$

Then for all $f, g \in \mathbb{C}[z,\bar{z}]$:

$$N'(f *_w g) = (N'f) *_{wick} (N'g).$$

Remark: The seemingly bizarre use of antiholomorphic instead of holomorphic functions is a quantum mechanical tradition: the creation operators (i.e. increase of the degree of the polynomial) is historically related to $A^\dagger$.

1.4. Multidifferential operators and their standard symbols in $\mathbb{R}^n$

Before going further it is useful to note the well-known definition of multidifferential operators on a manifold since we shall encounter them several times.

Let M be an n-dimensional differentiable manifold. Let $\big(U, \phi = (x_1, \ldots, x_n)\big)$ be a chart. Recall that a multi-index $I = (i_1, \ldots, i_n)$ is an element of $\mathbb{N}^n$ with $|I| := i_1 + \cdots i_n$, and we denote by

$$\partial_I := \frac{\partial^{i_1+\cdots+i_n}}{(\partial x_1)^{i_1} \cdots (\partial x_n)^{i_n}}$$

the usual abbreviation for iterated partial derivatives. For a vector $y = (y_1, \ldots, y_n)$ the expression y_I is short for the monomial $(y_1)^{i_1} \cdots (y_n)^{i_n}$. Recall that a *differential operator* D of order N is a $\mathbb{C}$-linear map $\mathcal{C}^\infty(M, \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ such that in each chart $\big(U, \phi = (x_1, \ldots, x_n)\big)$ the operator takes the local form ($f \in \mathcal{C}^\infty M, \mathbb{C}$):

$$\mathsf{D}(f)|_U = \sum_{I \in \mathbb{N}^n, |I| \leq N} \mathsf{D}^I \partial_I (f|_U) \tag{31}$$

where for each multi-index I the function $\mathsf{D}^I : U \to \mathbb{C}$ is $\mathcal{C}^\infty$. It is clear that the composition $\mathsf{D}_1\mathsf{D}_2 := \mathsf{D}_1 \circ \mathsf{D}_2$ of two differential operators D_1 and D_2 is again a differential operator. Examples are of course the position and momentum operators Q_k (see eqn (3)) and P_l (see eqn (4)).

More generally, a *multidifferential operator of rank* k or a k-differential operator is a $\mathbb{C}$-k-multilinear map $\mathsf{D} : \mathcal{C}^\infty(M, \mathbb{C}) \times \cdots \times \mathcal{C}^\infty(M, \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ such that there is an integer N such that in each chart $\big(U, \phi = (x_1, \ldots, x_n)\big)$ the operator takes the local form $(f_1, \ldots, f_k \in \mathcal{C}^\infty(M, \mathbb{C}))$:

$$\mathsf{D}(f_1, \ldots, f_k)|_U = \sum_{I_1, \ldots, I_k \in \mathbb{N}^n, |I_1|, \ldots, |I_k| \leq N} \mathsf{D}^{I_1, \ldots, I_k} \partial_{I_1}(f_1|_U) \cdots \partial_{I_k}(f_k|_U) \tag{32}$$

where for each k-tuple of multi-indices $(I_1, \ldots, I_k)$ the function $\mathsf{D}^{I_1, \ldots, I_k} : U \to \mathbb{C}$ is $\mathcal{C}^\infty$. The Poisson bracket $\{\ ,\ \}$ associated to a Poisson structure P on a manifold M is an example of a 2-differential or bidifferential operator.

Multidifferential operators can be composed in the following way: if D_1 is k-differential and if D_2 is l-differential and i is an integer such that $1 \leq i \leq k$, then (for all $f_1, \ldots, f_{k+l-1} \in \mathcal{C}^\infty(M, \mathbb{C})$)

$$(\mathsf{D}_1 \circ_i \mathsf{D}_2)(f_1, \ldots, f_{k+l-1}) := \mathsf{D}_1\big(f_1, \ldots, f_{i-1}, \mathsf{D}_2(f_i, \ldots, f_{l+i-1}), f_{l+1}, \ldots f_{k+l-1}\big)$$

is a $k + l - 1$-differential operator.

We shall write $\mathrm{Diffop}(M)$ for the space of all differential operators acting on $\mathcal{C}^\infty(M, \mathbb{C})$ and $\mathrm{Diffop}^k(M)$ for the space of all k-differential operators acting on $\mathcal{C}^\infty(M, \mathbb{C})^{\times k}$. For $M = U$ an open subset of $\mathbb{R}^n$ the subspace $\mathrm{Diffop}_{poly}(M)$ is important where the functions D^I are polynomials in $\mathbb{R}^n$.

For $M = U$ being an open set in $\mathbb{R}^n$ multidifferential operators can equivalently be described by their *standard symbols*: for $\alpha \in \mathbb{R}^{n*}$ define the exponential function associated to α by

$$e_\alpha(x) := e^{\langle \alpha, x \rangle}.$$

For a given k-differential operator D on U given in the form (32) its *standard symbol* $\check{\mathsf{D}}$ is a $\mathcal{C}^\infty$-function of $U \times \mathbb{R}^{n*} \times \cdots \times \mathbb{R}^{n*}$ into the complex numbers defined by

$$\begin{aligned} \check{\mathsf{D}}(x, \alpha_1, \ldots, \alpha_k) &:= e^{-\langle \alpha_1 + \cdots \alpha_k, x \rangle} \mathsf{D}\big(e_{\alpha_1}, \ldots, e_{\alpha_k}\big)(x) \\ &= \sum_{I_1, \ldots, I_k \in \mathbb{N}^n, |I_1|, \ldots, |I_k| \leq N} \mathsf{D}^{I_1, \ldots, I_k}(x) \alpha_{1 I_1} \cdots \alpha_{k I_k} \end{aligned} \tag{33}$$

Since the standard symbol is just 'replacing partial derivatives ∂_{I_k} by the monomial α_{kI_k}' it is obvious that the space of all k-differential operators on $U \subset \mathbb{R}^n$ is in bijection with the space of all $\mathcal{C}^\infty$-functions $F : U \times \mathbb{R}^{n*} \times \cdots \times \mathbb{R}^{n*} \to \mathbb{C}$ which are polynomial in the nk variables $\alpha_1, \ldots, \alpha_k$. Hence we have the following

Lemma 1.1 *Each k-differential operator D in an open set U of $\mathbb{R}^n$ is uniquely determined by its standard symbol $\check{\mathsf{D}}$ or, equivalently, by its values on exponential functions.*

1.5. Quantization in $\mathbb{R}^n$

By using exponential maps, it is easy to check the following generalizations of the standard ordered star-product $*_s$, the Moyal-Weyl star-product $*_w$, and the Wick star-product $*_w$: Here we consider the space of polynomials $\mathbb{C}[q,p] := \mathbb{C}[q_1, \ldots, q_n, p_1, \ldots, p_n] \ni f, g$:

$$f *_s g = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \sum_{k_1,\ldots,k_r=1}^{n} \frac{\partial^r f}{\partial p_{k_1} \cdots \partial p_{k_r}} \frac{\partial^r g}{\partial q_{k_1} \cdots \partial q_{k_r}} \tag{34}$$

with standard-ordered representation

$$\rho_s(f)(\phi) = \sum_{r=0}^{\infty} \frac{(\hbar/i)^r}{r!} \sum_{k_1,\ldots,k_r=1}^{n} \frac{\partial^r f}{\partial p_{k_1} \cdots \partial p_{k_r}}\Big|_{p=0} \frac{\partial^r \phi}{\partial q_{k_1} \cdots \partial q_{k_r}}. \tag{35}$$

Upon writing the Poisson bracket $\sum_{k=1}^n \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - (f \leftrightarrow g)$ as $\sum_{k,l=1}^{2n} P^{kl} \frac{\partial f}{\partial x_k}\frac{\partial g}{\partial x_l}$ with $(q,p) = x$ we have

$$f *_w g \;=\; \sum_{r=0}^{\infty} \frac{(i\hbar/2)^r}{r!} \sum_{k_1,\ldots,k_r,l_1,\ldots,l_r=1}^{2n} P^{k_1 l_1} \cdots P^{k_r l_r} \frac{\partial^r f}{\partial x_{k_1} \cdots \partial x_{k_r}} \frac{\partial^r g}{\partial x_{l_1} \cdots \partial x_{l_r}}. \tag{36}$$

The operator N being

$$N := e^{\frac{\hbar}{2i} \sum_{j=1}^n \frac{\partial^2}{\partial q_j \partial p_j}}. \tag{37}$$

one has

$$N(f *_w g) = (Nf) *_s (Ng), \tag{38}$$

and there is the Weyl-Moyal representation

$$\rho_w(f) := \rho_s(Nf). \tag{39}$$

Moreover, by defining complex coordinates $z_j := q_j + ip_j$ $(1 \le j \le n)$ and we get the *Wick star-product* in $\mathbb{C}^n = \mathbb{R}^{2n}$

$$f *_{wick} g := \sum_{r=0}^{\infty} \frac{(2\lambda)^r}{r!} \sum_{k_1,\ldots,k_r=1}^{n} \frac{\partial^r f}{\partial z_{k_1} \cdots \partial z_{k_r}} \frac{\partial^r g}{\partial \bar{z}_{k_1} \cdots \partial \bar{z}_{k_r}}. \tag{40}$$

which can be seen as a finite-dimensional version of the Wick ordering for quantum field theoretic observable algebras.

2. Formal Deformations

If we want to replace the polynomials in the preceding section by smooth complex-valued functions, we immediately see that formulas like eqns (34) and (36) do no longer converge in general. To make them well-defined it is useful to replace the real number $\hbar$ by a formal parameter λ.

Journal of Physics: Conference Series **103** (2008) 012002 doi:10.1088/1742-6596/103/1/012002

2.1. Formal Power Series

In this section I shall review several elementary notions about formal power series which I shall need later on, for more details and proofs see e.g the book by Ruiz [70]. Let R be a ring (always with unit element, for instance a field) and M a left module over R (for example an R-vector space in case R is a field). We shall write a map $a : \mathbb{N} \to M$ in the form of a *formal power series with coefficients in M*

$$a =: \sum_{r=0}^{\infty} \lambda^r a_r$$

where $a_r := a(r)$ is called the *rth component of a*, and the symbol λ is called the *formal parameter*. The set of all formal power series with coefficients in M is denoted by $M[[\lambda]]$. The sets $M[[\lambda]]$ and $R[[\lambda]]$ are abelian groups in the canonical way, i.e. for $b = \sum_{r=0}^{\infty} \lambda^r b_r$ where $b_r \in M$:

$$a + b := \sum_{r=0}^{\infty} \lambda^r (a_r + b_r).$$

Furthermore, $R[[\lambda]]$ carries the structure of a ring via ($\alpha = \sum_{r=0}^{\infty} \lambda^r \alpha_r, \beta = \sum_{r=0}^{\infty} \lambda^r \beta_r$, $\alpha_r, \beta_r \in R$)

$$\alpha\beta := \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \alpha_s \beta_{r-s}$$

and $M[[\lambda]]$ becomes a left $R[[\lambda]]$-module via

$$\alpha b := \sum_{r=0}^{\infty} \lambda^r \sum_{s=0}^{r} \alpha_s b_{r-s}.$$

The *order* of a formal power series a, $o(a)$, is defined as the minimum of the set of all nonnegative integers r such that $a_r \neq 0$ in case $a \neq 0$ and is defined to be $+\infty$ if $a = 0$. It can be shown that the function

$$d : M[[\lambda]] \times M[[\lambda]] \to \mathbb{R} : (a, b) \mapsto d(a, b) := \begin{cases} 2^{-o(a-b)} & \text{if } a \neq b \\ 0 & \text{if } a = b \end{cases}$$

defines a metric on $M[[\lambda]]$, and $(M[[\lambda]], d)$ becomes a complete metric space. The induced topology is called the *λ-adic topology* of $M[[\lambda]]$.

The following lemma is very important since multilinear maps over $\mathbb{C}[[\lambda]]$ always reduce to formal power series of multilinear maps over $\mathbb{C}$ which will be applied to deformed multiplications:

Lemma 2.1 *Let R be a commutative ring, $M, M_1, \ldots, M_k$ R-modules, and $\Phi : M_1[[\lambda]] \times \cdots \times M_k[[\lambda]] \to M[[\lambda]]$ a $R[[\lambda]]$-multilinear map.*
Then for each nonnegative integer r there is a unique R-multilinear map $\Phi_r : M_1 \times \cdots \times M_k \to M$ such that

$$\Phi(a_{(1)}, \ldots, a_{(k)}) = \sum_{r=0}^{\infty} \lambda^r \sum_{\substack{0 \leq s, r_1, \ldots, r_k \leq r \\ s + r_1 + \cdots + r_k = r}} \Phi_s(a_{(1)r_1}, \ldots, a_{(k)r_k}) \tag{41}$$

for all $a_{(i)} = \sum_{r_i=0}^{\infty} \lambda^{r_i} a_{(i)r_i} \in M_i[[\lambda]]$, $1 \leq i \leq k$.

2.2. Formal Deformations of Associative Algebras
Let $(\mathcal{A}_0, \mu_0)$ be an associative algebra with unit 1 over a commutative ring R.

Definition 2.2 *A formal associative deformation of the associative algebra with unit* 1, $(\mathcal{A}_0, \mu_0)$ *is given by a sequence of* R*-bilinear maps* $\mu_1, \mu_2, ... : \mathcal{A}_0 \times \mathcal{A}_0 \to \mathcal{A}_0$ *such that:*

(i)

$$\sum_{s=0}^{r} \big(\mu_s\big(\mu_{r-s}(a,b),c\big) - \mu_s\big(a, \mu_{r-s}(b,c)\big)\big) = 0 \tag{42}$$

for all $r \in \mathbb{N}$ *and* $a, b, c \in \mathcal{A}_0$.

(ii) $\mu_r(1, a) = 0 = \mu_r(a, 1)$ *for all* $r \in \mathbb{N}$, $r \geq 1$ *and* $a \in \mathcal{A}_0$.

It is not hard to see that the formulas (34) and (36) define a formal associative deformation of the algebra $\mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})$ when $\hbar$ is replaced by λ.
The following Proposition is obvious:

Proposition 2.1 *The space* $\mathcal{A} := \mathcal{A}_0[[\lambda]]$ *equipped with the* $R[[\lambda]]$*-bilinear multiplication* $\mu := \sum_{r=0}^{\infty} \lambda^r \mu_r$, *i.e.*

$$\mu(a, b) := \sum_{r=0}^{\infty} \lambda^r \sum_{s+t+u=0} \mu_s(a_t, b_u)$$

for all $a = \sum_{t=0}^{\infty} \lambda^t a_t$ *and* $b = \sum_{u=0}^{\infty} \lambda^u b_u$ *in* $\mathcal{A}$, *is an associative algebra over the ring* $R[[\lambda]]$.

For the case $r = 1$ of eqn (42) we get (writing $\mu_0(a, b) =: ab$):

$$0 = a\mu_1(b, c) - \mu_1(ab, c) + \mu_1(a, bc) - \mu_1(a, b)c =: (\delta_H \mu_1)\big(a, b, c\big)$$

where δ_H is the *Hochschild coboundary operator* defined on the R-module

$$CH(\mathcal{A}_0, \mathcal{A}_0) := \bigoplus_{k=0}^{\infty} CH^k(\mathcal{A}_0, \mathcal{A}_0) := \bigoplus_{k=0}^{\infty} Hom_R(\mathcal{A}_0 \otimes_R \cdots \otimes_R \mathcal{A}_0, \mathcal{A}_0)$$

(the space of *Hochschild cochains*) by

$$\begin{aligned}(\delta_H f)(a_1 \otimes \cdots \otimes a_{k+1}) &:= a_1 f(a_2 \otimes \cdots \otimes a_{k+1}) \\ &+ \sum_{r=1}^{k} (-1)^r f(a_1 \otimes \cdots \otimes a_{r-1} \otimes a_r a_{r+1} \otimes \cdots \otimes a_{k+1}) \\ &+ (-1)^{k+1} f(a_1 \otimes \cdots \otimes a_k) a_{k+1}\end{aligned}$$

It is well-known that $\delta_H^2 = 0$, hence that operator defines a cohomology theory called the *Hochschild cohomology*:

$$\begin{aligned} ZH^k(\mathcal{A}_0, \mathcal{A}_0) &:= Ker\big(\delta_H : CH^k(\mathcal{A}_0, \mathcal{A}_0) \to CH^{k+1}(\mathcal{A}_0, \mathcal{A}_0)\big) \\ BH^k(\mathcal{A}_0, \mathcal{A}_0) &:= Im\big(\delta_H : CH^{k-1}(\mathcal{A}_0, \mathcal{A}_0) \to CH^k(\mathcal{A}_0, \mathcal{A}_0)\big) \\ HH^k(\mathcal{A}_0, \mathcal{A}_0) &:= ZH^k(\mathcal{A}_0, \mathcal{A}_0)/BH^k(\mathcal{A}_0, \mathcal{A}_0) \end{aligned}$$

The elements of $ZH^k(\mathcal{A}_0, \mathcal{A}_0)$ are called *Hochschild* k*-cocycles* of $\mathcal{A}_0$, the elements of $BH^k(\mathcal{A}_0, \mathcal{A}_0)$ are called *Hochschild* k*-coboundaries* of $\mathcal{A}_0$, and $HH^k(\mathcal{A}_0, \mathcal{A}_0)$ is called the k^{th} *Hochschild cohomology group* of $\mathcal{A}_0$ (with values in $\mathcal{A}_0$).

It follows that for any formal deformation the term μ_1 is always a Hochschild 2-cocycle. In the more general case where μ is not necessarily associative it is easily computed that the *associator* of μ

$$A(a,b,c) := \mu\big(\mu(a,b),c\big) - \mu\big(a,\mu(b,c)\big)$$

satisfies the following identity:

$$\begin{aligned} 0 \;=\; & \mu\big(a, A(b,c,d)\big) - A\big(\mu(a,b),c,d\big) + A\big(a,\mu(b,c),d\big) \\ & -A\big(a,b,\mu(c,d)\big) + \mu\big(A(a,b,c),d\big) \end{aligned}$$

for all $a,b,c,d \in \mathcal{A}_0$. For an associative formal deformation we demand that $A = \sum_{r=0}^{\infty} \lambda^r A_r = 0$. Let us suppose that the components $A_0, A_1, \ldots, A_k$ are already zero. Thanks to the preceding identity we get at order $r+1$ of λ:

$$\delta_H A_{r+1} = 0.$$

Since

$$A_{r+1} = \delta_H \mu_{r+1} + A'_{r+1}$$

where the rest A'_{r+1} contains only the terms $\mu_0, \cdots, \mu_r$ it follows that

$$\begin{array}{lccc} \text{We have :} & \delta_H A'_{r+1} = 0 & \Longrightarrow & A'_{r+1} \;\in\; Z^3(\mathcal{A}_0, \mathcal{A}_0) \\ \text{We want :} & A'_{r+1} \stackrel{!}{=} -\delta_H \mu_{r+1} & \Longrightarrow & A'_{r+1} \;\stackrel{!}{\in}\; B^3(\mathcal{A}_0, \mathcal{A}_0) \end{array}$$

Consequently the *recursive obstructions* to continue the construction of the term μ_{r+1} of a formal associative deformation of μ_0 (where $\mu_1, \ldots, \mu_r$ are already chosen) are contained at each stage r in

$$HH^3(\mathcal{A}_0, \mathcal{A}_0).$$

For the very important particular case where $\mathcal{A}_0$ is given by $\mathcal{A}_0 = C^\infty(M, \mathbb{C})$ (equipped with μ_0 equal to pointwise multiplication) one usually considers Hochschild cochains given by multidifferential operators: This subspace of the complex of Hochschild cochains which we shall write in the form $CH_{\mathrm{diff}}\big(C^\infty(M,\mathbb{C}), C^\infty(M,\mathbb{C})\big)$ is a subcomplex with respect to the Hochschild coboundary. Its cohomology is called the *differential Hochschild cohomology of* $\mathcal{C}^\infty(M,\mathbb{C})$, and we shall write it as $HH_{\mathrm{diff}}\big(C^\infty(M,\mathbb{C}), C^\infty(M,\mathbb{C})\big)$. The computation of this cohomology originates in the article by Hochschild-Kostant-Rosenberg [54] (for polynomials), has been generalized to the smooth case by Vey [73], Cahen-DeWilde-Gutt [21], Cahen-Gutt [22], and DeWilde-Lecomte [32], and gives nothing but the space of multivector fields:

Theorem 2.3

$$\forall\, k \in \mathbb{N} : \quad HH^k_{\mathrm{diff}}\big(C^\infty(M,\mathbb{C}), C^\infty(M,\mathbb{C})\big) \cong \Gamma(M, \Lambda^k TM).$$

A generalization of this result had been obtained by A.Connes en 1985 (see [28], p.207-210) who has replaced differential cochains by cochains which are continuous with respect to the standard Fréchet topology of this space. Pflaum [69] and Nadaud [61] have shown that one may drop Connes' hypotheses that the Euler characteristic of the manifold is zero. In these cases the resulting Hochschild cohomology is isomorphic to the right hand side of the HKR-Theorem 2.3, i.e. the space of all smooth multivector fields $\Gamma(\Lambda TM)$.

2.3. Poisson brackets in a general algebraic context

Proposition 2.2 *If* $(\mathcal{A}, \mu_0)$ *is a* commutative *associative algebra and* $\mathsf{C} = \sum_{r=0}^{\infty} \lambda^r \mu_r$ *is a formal associative deformation then it turns out that*

$$\{f, g\} := \mu_1(f, g) - \mu_1(g, f) \quad \forall\ f, g \in \mathcal{A}$$

defines a Poisson bracket on $\mathcal{A}$*, i.e. a Lie bracket which satisfies the* Leibniz rule*:*

$$\{f, gh\} = \{f, g\}h + \{f, h\}g \quad \forall\ f, g, h \in \mathcal{A}.$$

Proof: Indeed, the Jacobi identity for the bracket $\{\ ,\ \}$ follows from total antisymmetrization of the associativity condition of μ at order 2. In order to obtain the Leibniz identity one takes the associativity condition of μ at order 1,

$$0 = -f\mathsf{C}_1(g, h) + \mathsf{C}_1(fg, h) - \mathsf{C}_1(f, gh) + \mathsf{C}_1(f, g)h,$$

and adds to it the same identity with f and h interchanged, which gives

$$0 = -f\{g, h\} + \{fg, h\} - \{f, gh\} + \{f, g\}h$$

(showing that $\delta_H\{\ ,\ \} = 0$). Adding to this identity the one with g and h interchanged and subtracting the one with f and g interchanged yields (twice) the Leibniz rule. □

This rather general simple fact means that if Poisson structures in classical mechanics are replaced by more general mathematical objects, their quantization can no longer lead to *associative* quantum observable algebras.

2.4. Gerstenhaber's formula

The explicit formulas for the associative multiplications $*_s$ and $*_w$ have a common algebraic feature: the following rather useful theorem is due to M.Gerstenhaber [46], p.13, Thm.8:

Theorem 2.4 *Let* (A, μ_0) *be an associative algebra with unit* 1 *over a commutative ring* k *which contains the rationals* $\mathbb{Q}$ *where* $\mu_0 : A \otimes A \to A$ *denotes the (not necessarily commutative) multiplication of* A*. Let* $D_1, \ldots, D_n, E_1, \ldots,\ E_n$ $2n$ *derivations of* (A, μ_0) *which all pairwise commute, i.e.* $D_k \circ \mu_0 = \mu_0 \circ (D_k \otimes 1 + 1 \otimes D_k)$*,* $E_l \circ \mu_0 = \mu_0 \circ (E_l \otimes 1 + 1 \otimes E_l)$*,* $D_k \circ D_l = D_l \circ D_k$*,* $D_k \circ E_l = E_l \circ D_k$ *and* $E_k \circ E_l = E_l \circ E_k$ *for all integers* $1 \leq k, l \leq n$*. Let* $r := \sum_{k=1}^{n} D_k \otimes E_k$*. Then on the* $\mathsf{k}[[\lambda]]$*-module* $A[[\lambda]]$ *there is a* $\mathsf{k}[[\lambda]]$*-bilinear multiplication* μ *defined by*

$$\mu := \mu_0 \circ e^{\lambda r} \tag{43}$$

which deforms μ_0 *with unit element* 1*.*

Proof: The following elegant reasoning has been found by A.Dimakis and F.Müller-Heussen in [35] for a particular case: one defines the following three linear maps: $A \otimes A \otimes A \to A \otimes A \otimes A$ (where $\mathbf{1}$ denotes the identitiy map $A \to A$) $r_{12} := r \otimes \mathbf{1}$, $r_{23} := \mathbf{1} \otimes r$ and $r_{13} := \sum_{k=1}^{n} D_k \otimes \mathbf{1} \otimes E_k$. Since the derivations commute we have $[r_{12}, r_{13}] = 0$, $[r_{12}, r_{23}] = 0$ and $[r_{13}, r_{23}] = 0$. Thanks to the derivation identity it follows that

$$\begin{aligned} r \circ (\mu_0 \otimes \mathbf{1}) &= (\mu_0 \otimes \mathbf{1}) \circ (r_{13} + r_{23}) \quad \text{and} \\ r \circ (\mathbf{1} \otimes \mu_0) &= (\mathbf{1} \otimes \mu_0) \circ (r_{12} + r_{13}), \end{aligned}$$

hence

$$\begin{aligned} e^{\lambda r} \circ (\mu_0 \otimes \mathbf{1}) &= (\mu_0 \otimes \mathbf{1}) \circ e^{(r_{13} + r_{23})} \quad \text{and} \\ e^{\lambda r} \circ (\mathbf{1} \otimes \mu_0) &= (\mathbf{1} \otimes \mu_0) \circ e^{(r_{12} + r_{13})}, \end{aligned}$$

therefore, as the r_{ij} commute:

$$\mu \circ (\mu \otimes \mathbf{1}) = \mu_0 \circ e^{\lambda r} \circ (\mu_0 \otimes \mathbf{1}) \circ e^{\lambda r_{12}} = \mu_0 \circ (\mu_0 \otimes \mathbf{1}) \circ e^{\lambda(r_{12}+r_{13}+r_{23})}.$$

Analogously:

$$\mu \circ (\mathbf{1} \otimes \mu) = \mu_0 \circ e^{\lambda r} \circ (\mathbf{1} \otimes \mu_0) \circ e^{\lambda r_{23}} = \mu_0 \circ (\mathbf{1} \otimes \mu_0) \circ e^{\lambda(r_{12}+r_{13}+r_{23})}.$$

Since μ_0 is associative we get $\mu_0 \circ (\mu_0 \otimes \mathbf{1}) = \mu_0 \circ (\mathbf{1} \otimes \mu_0)$, whence the associativity of the multiplication μ. Moreover, since any derivation vanishes on 1 it follows that 1 is still the unit element of μ. □

It is easily seen that the multiplications $*_s$, $*_w$, are particular cases of Gerstenhaber's formula if the real number $\hbar$ is replaced by the formal parameter λ, and if we set

$$r_s := \frac{1}{i}\sum_{k=1}^{n} \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k} \tag{44}$$

$$r_w := \frac{i}{2}\sum_{k=1}^{n} \left(\frac{\partial}{\partial q_k} \otimes \frac{\partial}{\partial p_k} - \frac{\partial}{\partial p_k} \otimes \frac{\partial}{\partial q_k} \right) \tag{45}$$

3. Poisson Geometry

3.1. Poisson Manifolds

We have seen in Proposition 2.2 that the first order commutator of a formal associative deformation of commutative algebra A always gives rise to a Poisson bracket. In this Section we shall deal with the important case where the undeformed algebra is the space of smooth complex valued functions, $\mathcal{C}^\infty(M, \mathbb{C})$ on a smooth mannifold M.

Let $\{\ ,\ \}$ be a Poisson bracket on $A := \mathcal{C}^\infty(M, \mathbb{C})$. For any $g \in A$ let X_g denote the linear map $A \to A$ defined by $f \mapsto X_g(f) := \{f, g\}$. By the Leibniz rule each X_g is a derivation on A, i.e. $X_g(ff') = X_g(f)f' + fX_g(f')$ for all $f, f' \in A$. It is a classical fact (see e.g. [1]) that to each derivation there corresponds a unique vector field on M, also denoted by X_g, such that $X_g(f)$ is given by the Lie derivative of f with respect to X_g. In other words, in each local chart $(U, x = (x^1, \ldots, x^n))$ on M we can write $X_g(f) = \sum_{i=1}^n X_g^i(\partial f/\partial x^i)$ for certain complex valued smooth functions X_g^i on the chart domain U. The vector field X_g is called the *Hamiltonian vector field* associated to g. Note that the Jacobi identity of the Poisson bracket implies

$$[X_f, X_g] = -X_{\{f,g\}} \quad \forall\ f, g \in A.$$

Using the fact that the linear map $g \mapsto X_g$ is also a derivation in the sense that $X_{gg'} = g'X_g + gX_{g'}$ for all $g, g' \in A$, a slight generalization of the above classical argument shows that the Poisson bracket uniquely determines a so-called *bivector field*, i.e. a smooth section P in the bundle $\Lambda^2 TM^{\mathbb{C}}$. In a local chart $\big(U, (x^1, \ldots, x^n)\big)$ this bivector field P takes the form

$$P = \frac{1}{2}\sum_{i,j=1}^{n} P^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \quad \text{hence} \quad \{f, g\} = \sum_{i,j=1}^{n} P^{ij} \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}. \tag{46}$$

for all $f, g \in A$ where (P^{ij}) is an antisymmetric matrix of smooth complex-valued functions on U. We shall henceforth restrict to the case where P is real. The *rank of a bivector field* P in $x \in M$ is defined by the rank of the antisymmetric matrix P_x^{ij} in an arbitrary chart.

On the other hand, each bivector field P can canonically be considered as an antisymmetric bilinear form on the cotangent bundle T^*M by using the natural pairing: let α, β two 1-forms on M

$$P(\alpha,\beta) = i_\beta i_\alpha P = \sum_{i,j=1}^{n} P^{ij}\alpha_i\beta_j, \tag{47}$$

and one can thus define the antisymmetric bracket

$$\{f,g\} := \{f,g\}_P := P(df,dg) = df\ X_g = -dg\ X_f. \tag{48}$$

for any $f, g \in A$ which automatically satisfies the Leibniz rule. The Jacobi identity for such a bracket, however, is not necessarily valid and has to be demanded: it is easily seen that in co-ordinates this leads to the following quadratic PDE on P^{ij}:

$$0 \stackrel{!}{=} \sum_{r=1}^{n}\left(P^{ir}\frac{\partial P^{jk}}{\partial x^r} + P^{jr}\frac{\partial P^{ki}}{\partial x^r} + P^{kr}\frac{\partial P^{ij}}{\partial x^r}\right). \tag{49}$$

As a matter of fact, the above condition behaves well under co-ordinate changes and is a special case of the so-called *Schouten bracket* $[\ ,\ ]_s$ which is a graded Lie bracket on the space of the so-called *multivector fields* (or *polyvector fields*), i.e. $\Gamma^\infty(M, \Lambda TM)$, and the above condition reads $[P,P]_s = 0$, see [72] for details.

The above condition (49) implies the simplest example of a Poisson manifold, i.e. any open set M of $\mathbb{R}^n$ equipped with a constant antisymmetric matrix P^{ij}.

A bivector field P on a manifold M satisfying (49) is called a *Poisson structure*, and the pair (M,P) is called a *Poisson manifold*. On such a manifold one can always define *Hamiltonian mechanics* by associating to each smooth real-valued function H on M its Hamiltonian vector field X_H which gives rise to a dynamical system, i.e. the first order ODE

$$\dot{x} = X_H(x) \quad \text{and} \quad x(0) = x_0 \in M, \tag{50}$$

the so-called *Hamiltonian equations of motion*. Since for any $f \in A$ one has $d(f(x(t)))/dt = (X_H(f))(t) = \{f,H\}(x(t))$ it follows that f is a conserved quantity for H iff $\{f,H\} = 0$, hence in particular H is always a conserved quantity (conservation of energy).

The above condition (49) implies the simplest example of a Poisson manifold, i.e. any open set M of $\mathbb{R}^n$ equipped with a constant antisymmetric matrix P^{ij}. In contrast to that the local structure of a Poisson manifold can be very complicated, and the rank of the bivector field may change over the manifold. On the other hand there are no topological restrictions for a manifold of dimension $n \geq 2$ to admit a nonzero Poisson structure: in fact, by choosing in the image of a chart domain n commuting vector fields $X_1,\ldots,X_n$ which are independent at the origin and have compact support (exercise: construct them!) and by pulling them back to the manifold, one can define P as a sum of terms of the form $X_i \wedge X_j$. Moreover it is clear that on any 2 dimensional manifold any bivector field is automatically Poisson.

3.1.1. The dual of a Lie algebra Apart from constant Poisson structures, the following example of the so-called *linear Poisson structure* is the most important: Let $(\mathfrak{g}, [\ ,\ ])$ be an n-dimensional real Lie algebra and $M := \mathfrak{g}^*$ its dual space. Let $e_1,\ldots,e_n$ be a base of $\mathfrak{g}$, let $e^1,\ldots,e^n$ be the dual base, and $c^k_{lm} := e^k([e_l,e_m])$ the structure constants of $\mathfrak{g}$.

Then for all $\xi \in \mathfrak{g}^*$ one defines on M the *linear Poisson structure* corresponding to $[\ ,\ ]$:

$$P_{\mathfrak{g}}(\xi) := \xi([\ ,\ ]) = \frac{1}{2}\sum_{k,l,m=1}^{n} \xi_k c^k_{lm}\frac{\partial}{\partial \xi_k} \wedge \frac{\partial}{\partial \xi_l}. \tag{51}$$

The Jacobi identity for this Poisson structure is a direct consequence of the Jacobi identity for the Lie bracket $[\ ,\ ]$ of $\mathfrak{g}$.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(3) \cong \mathbb{R}^3$ of all real 3×3 antisymmetric matrices with the bracket $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$ coming from the vector product is an important example for the dynamics of a *freely spinning top*: let Θ be a positive definite 3×3-matrix (the inertia tensor), and $H : \mathbb{R}^3 \to \mathbb{R}$ the real valued function $H(L) = \frac{1}{2}\sum_{i,j=1}^{3} L_i(\Theta^{-1})^{ij} L_j$. Then the dynamical system corresponding to H is the *Euler equation of a freely spinning top*

$$\frac{dL}{dt} = [\Theta^{-1}L, L]$$

where L is the angular momentum and $\Theta^{-1}L$ the angular velocity of the top.

3.2. Symplectic manifolds

Symplectic manifolds are the most important examples of Poisson manifolds, see the book [1] for an excellent introduction.

A Poisson manifold (M, P) such that the bivector field P is an invertible antisymmetric matrix at each point $x \in M$ in some co-ordinate chart is called a *symplectic manifold*. The natural inverse of P is a 2-form $\omega \in \Gamma^2(M, \Lambda^2 T^*M) = \Omega^2(M)$: we adopt the convention that $\sum_{r=1}^{n} \omega_{ir} P^{jr} = \delta_i^j$. It is an easy exercise that the Jacobi identity (49) is equivalent to the linear PDE

$$d\omega = 0 \quad \text{or} \quad 0 = \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{ki}}{\partial x^j}. \tag{52}$$

Therefore symplectic manifolds are denoted (M, ω) with ω a nondegenerate closed 2-form.

Unlike general Poisson manifolds, symplectic manifolds have a very simple local structure ensured by *Darboux's Theorem*: around each point of M there always exists so-called canonical or Darboux co-ordiantes $(q, p) = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ in which the symplectic form takes the constant form

$$\omega := \sum_{i=1}^{n} dq^i \wedge dp_i. \tag{53}$$

See e.g. [1], p.175, Thm 3.2.2. for a proof. In particular, symplectic manifolds are always evendimensional, and the Poisson bracket takes the usual form $\sum_{i=1}^{n} \partial f/\partial q^i \partial g/\partial p_i - (f \leftrightarrow g)$.

Simple examples of symplectic manifolds are $\mathbb{R}^{2n}$ equipped with the constant 2-form (53). Moreover, any oriented 2-dimensional manifold (a so-called oriented *Riemann surface*) carries a volume form and is hence symplectic. Unlike general Poisson manifolds there are topological obstructions for compact manifolds to admit a symplectic form ω: in the de Rham cohomology, the class of each of the following closed $2k$-forms $\omega^{\wedge k}$, $1 \leq k \leq n := \dim M/2$ has to be non zero: in fact, if there was a $k-1$-form θ with $\omega^{\wedge k} = d\theta$, then the volume form $\omega^{\wedge n}$ would be equal to $d\theta \wedge \omega^{\wedge(n-k)} = d(\theta \wedge \omega^{\wedge(n-k)})$ which would be absurd since the total volume $\int_M \omega^{\wedge n}$ of M would be zero by Stokes's Theorem. For example, the spheres S^{2n} do not admit any symplectic structure for all $n \geq 2$.

A general Poisson manifold is known to be foliated by local symplectic submanifolds, the so-called *symplectic leaves*, see [72] for details, whose dimension is in general non constant.

3.2.1. Cotangent bundles Let Q be a differentiable manifold, T^*Q its cotangent bundle, and $\tau_Q^* : T^*Q \to Q$ the canonical bundle projection. *The canonical* 1*-form* θ_0 on the manifold T^*Q is defined in the following manner: let $q \in Q$, $\alpha \in T_qQ^*$, and $W_\alpha \in T_\alpha T^*Q$, then

$$\theta_0(\alpha)\big(W_\alpha\big) := \alpha\big(T_\alpha \tau_Q^*\, W_\alpha\big). \tag{54}$$

Let $\big((U,(q^1,\ldots,q^n)\big)$ be a chart of Q, and $\big(T^*U,(q^1,\ldots,q^n,p_1,\ldots,p_n)\big)$ the corresponding canonical chart of T^*Q (i.e. $q^k(\alpha):=q^k\big(\tau_Q^*(\alpha)\big)$ and $p_l(\alpha):=\alpha(\partial/\partial q^l)$), then θ_0 takes the form

$$\theta_0 := \sum_{k=1} p_k dq^k \tag{55}$$

whence the fact that the *canonical 2-form*

$$\omega_0 := -d\theta_0 \tag{56}$$

is nondegenerate, hence a symplectic form on T^*Q. The cotangent bundles generalize the phase spaces in physics where Q is a configuration space and the fibres represent the conjugate momenta.

3.2.2. Complex Projective Space Apart from the tori of even dimension, the complex projective spaces are the simplest compact symplectic manifolds:

Consider the complex manifold $\mathbb{C}^{n+1}\setminus\{0\}$ equipped with complex coordinates $z:=(z_1:=q_1+ip_1,\ldots,z_{n+1}:=q_{n+1}+ip_{n+1})$ and with the standard symplectic form

$$\omega_0 := \frac{i}{2}\sum_{k=1}^{n+1} dz_k\wedge d\bar{z}_k = \sum_{k=1}^{n+1} dq_k\wedge dp_k. \tag{57}$$

Complex projective space $\mathbb{C}P^n$ is defined by the following equivalence relation

$$z\sim z' \quad \text{iff} \quad \exists\alpha\in\mathbb{C}\setminus\{0\} \text{ such that } z'=\alpha z, \tag{58}$$

and $\mathbb{C}P^n := \mathbb{C}^{n+1}\setminus\{0\}/\sim$. Let

$$\pi:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{C}P^n : z\mapsto[z] \tag{59}$$

be the canonical projection whose fibres obviously are the complex lines in $\mathbb{C}^{n+1}\setminus\{0\}$ passing through the origin. There are $n+1$ complex charts (U_k,v) defined by

$$\begin{aligned} U_k &:= \{[z]\in\mathbb{C}P^n | z_k\neq 0\} \\ v &:= \left(v_1:=\frac{z_1}{z_k},\ldots,v_{k-1}:=\frac{z_{k-1}}{z_k},v_{k+1}:=\frac{z_{k+1}}{z_k},\ldots,v_{n+1}:=\frac{z_{n+1}}{z_k}\right). \end{aligned}$$

The *Fubini-Study 2-form* ω is defined in each chart (U_k,v) (where we set $|v|^2:=\sum_{\substack{l=1,\\ l\neq k}}^{n+1}|v_l|^2$):

$$\omega|_{U_k} := \frac{i}{2(1+|v|^2)}\left(\sum_{\substack{l=1\\ l\neq k}}^{n+1} dv_l\wedge d\bar{v}_l - \frac{1}{(1+|v|^2)}\sum_{\substack{l,l'=1\\ l,l'\neq k}}^{n+1}\bar{v}_l dv_l\wedge v_{l'}d\bar{v}_{l'}\right) \tag{60}$$

It can be shown that these locally defined closed two-forms $\omega|_{U_k}$ are well-behaved under the change of charts and thus define a global 2-form ω. Moreover, the map $\Phi:\mathbb{C}P^1\to S^2$

$$[z_1,z_2]\mapsto\frac{1}{|z_1|^2+|z_2|^2}\left(z_1z_2+\bar{z}_1\bar{z}_2,-i(z_1z_2-\bar{z}_1\bar{z}_2),|z_1|^2-|z_2|^2\right)$$

is easily computed to be a diffeomorphism.

4. Star-products

In the preceding chapter we have seen that one can construct noncommutative or “quantum” associative multiplications $*$ on $\mathbb{C}[[\lambda]]$ and even on $\mathcal{C}^\infty(\mathbb{R}^{2n},\mathbb{C})$ by using symbol calculus, i.e. by using a linear bijiection between $\mathbb{C}[q,p]$ and an already given associative algebra, namely the algebra of all differential operators with polynomial coefficients acting on $\mathcal{C}^\infty(\mathbb{R}^n,\mathbb{C})$.

The principal idea of star-products is to construct such an associative multiplication $*$ directly on the space of classical observables, i.e. on the function space $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ (where M is a given Poisson manifold) without a priori referring to a ‘representation’ in a differential or operator algebra: for most of the Poisson manifolds it is not at all clear how such a differential operator algebra could be chosen. From the point of view of physics this means that the construction of the quantum system starts with the observable algebra (unlike the classical approach), whereas the construction of the Hilbert space is postponed.

4.1. Definition

The following definition had been given by F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer in 1978 [5]:

Definition 4.1 *Let (M,P) be a Poisson manifold. The structure of a* star-product *on M or a* deformation quantization *on M is defined by the following sequence of $\mathbb{C}$-bilinear maps*

$$\mathsf{C}_r : \mathcal{C}^\infty(M,\mathbb{C}) \times \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$$

for all $r \in \mathbb{N}$ subject to the following conditions ($f,g,h \in \mathcal{C}^\infty(M,\mathbb{C})$):

(i) Every C_r is a bidifferential operator

(ii) $\mathsf{C}_0(f,g) = fg$ (classical limit).

(iii) $\mathsf{C}_1(f,g) - \mathsf{C}_1(g,f) = i\{f,g\} := iP(df,dg)$ (classical limit).

(iv) $\mathsf{C}_r(1,g) = 0 = \mathsf{C}_r(f,1)$ for all $r \geq 1$ (the constant function 1 remains a unit element).

(v) $\sum_{s=0}^r \big(\mathsf{C}_s(\mathsf{C}_{r-s}(f,g),h\big) = \sum_{s=0}^r \big(\mathsf{C}_s(f,\mathsf{C}_{r-s}(g,h)\big)$ for all $r \in \mathbb{N}$ (associativity).

The formal series

$$* := \sum_{r=0}^\infty \lambda^r \mathsf{C}_r$$

is called a star-product *on M.*

Furthermore, if for all $r \in \mathbb{N}$ and $f,g \in \mathcal{C}^\infty(M,\mathbb{C})$

$$\overline{\mathsf{C}_r(f,g)} = \mathsf{C}_r(\bar{g},\bar{f}) \tag{61}$$

(where $^-$ denotes pointwise complex conjugation) the star-product is called symmetric *or* hermitian.

For example, $*_w$ and $*_{wick}$ are hermitean, $*_s$ is not.

The following corollary is obvious:

Corollary 4.1 *Let $*$ be a star-product on the Poisson manifold (M,P). Then the $\mathbb{C}[[\lambda]]$-module $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ becomes an associative algebra over the ring $\mathbb{C}[[\lambda]]$ via ($F = \sum_{r=0}^\infty \lambda^r F_r, G = \sum_{r=0}^\infty \lambda^r G_r \in \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$)*

$$F * G := \sum_{r=0}^\infty \lambda^r \sum_{s+t+u=0} \mathsf{C}_s(F_t,G_u).$$

If moreover the star-product is hermitean, then the pointwise complex conjugation $^-$ becomes an (antilinear) antiautomorphism of the associative algebra $(\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]],*)$, i.e.:

$$\overline{F*G} = \overline{G} * \overline{F}.$$

We list further properties of star-products:

Definition 4.2 *(i) If the order N_r of the bidifferential operator C_r is always equal to r the star-product is called* natural *by S.Gutt and J.Rawnsley.*

(ii) A hermitean star-product is called of Weyl-Moyal type *iff*

$$\mathsf{C}_r(g,f) = (-1)^r \mathsf{C}_r(f,g) \quad forall\ r \geq 0.$$

For example, $*_s$ and $*_w$ are natural, and $*_w$ is of Weyl-Moyal type.

For two star-products $*$ and $*'$ there is the following notion of formal isomorphy which we had already encountered for $*_s$ and $*_w$:

Definition 4.3 *Let (M,P) be a Poisson manifold and $*, *'$ two star-products. They are called* equivalent *($* \sim *'$) iff there is a formal series of linear maps, called an* equivalence transformation

$$S = id + \sum_{r=1}^{\infty} \lambda^r S_r$$

(where each $S_r : \mathcal{C}^\infty(M,\mathbb{C}) \to \mathcal{C}^\infty(M,\mathbb{C})$ is $\mathbb{C}$-linear) such that

$$F *' G = S^{-1}\big((SF) * (SG)\big)$$

for all $F, G \in \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$.

It can be shown using the computations of Hochschild cohomology that all the S_r $(r \geq 1)$ are necessarily differential operators vanishing on the constants.

Since the operator series N (see eqn (37)) takes the form $id + \frac{\lambda}{2i}\frac{\partial}{\partial q}\frac{\partial}{\partial p} + o(\lambda^2)$, it defines an equivalence transformation between the star-products $*_w$ and $*_s$.

4.2. Existence

4.2.1. Symplectic manifolds After some important results for special cases (like symplectic manifolds whose third de Rham cohomology group vanishes [62] and cotangent bundles of parallelizable manifolds [22]) the first complete existence result had been shown by M.DeWilde and P.Lecomte in 1983, [33]:

Theorem 4.4 (DeWilde-Lecomte 1983) *On every symplectic manifold (M,ω) there is a star-product.*

The proof was based on one hand on explicit computations of the differential Hochschild cohomology of the commutative associative algebra $\mathcal{C}^\infty(M,\mathbb{C})$ and on the second and third Chevalley-Eilenberg cohomology of the Lie algebra $\mathcal{C}^\infty(M,\mathbb{C})$ equipped with the Poisson bracket, see [50], [31], and for a survey [34]. On the other hand, an important ingredient had been a a local homogeneity argument based on a generalization of the Euler field of a cotangent bundle which has already occurred in [22].

Independently of this result, B.Fedosov had given a proof of Theorem 4.4 in 1985, [39]. His method is remarkable since it rather uses symplectic connections than local charts: therefore his proof allows to construct the bidifferential operators directly in tensorial terms, which sometimes is more adapted to the implementation of symmetries. Yet another existence proof had been given by H.Omori, Y.Maeda, and A.Yoshioka in 1991, see [66]: here local Weyl-type star-products (isomorphic to $*_w$ on $\mathbb{R}^{2n}$) are glued together by means of cocycles of equivalence transformations.

4.2.2. Poisson manifolds The main obstacle to translate even locally the methods of the previous section to a general Poisson manifold was the fact that the local structure of a Poisson manifold can still be very complicated (unlike the symplectic situation where Darboux's theorem holds), and that there is in general no connection in the tangent bundle leaving invariant the Poisson structure: if this is the case the Poisson structure must have constant rank, a case Fedosov also dealt with. In the deformation quantization community it came out as a big sensational surprise in 1997 when the following result was announced by Maxim Kontsevich:

Theorem 4.5 (Kontsevitch 1997) *On every Poisson manifold (M, P) there exists a star-product.*

For the algebraic framework of operades, L_∞-structures, and formality, see the original article [57] and the article [3] which unzips some details of the proof. Kontsevich gave the result first for $M = \mathbb{R}^n$ in terms of an explicit formula, see the discussion further down, and sketched its globalization to manifolds. A.Cattaneo and G.Felder (see [24]) have retraced the quantum field theoretical roots of Kontsevich's construction in the physical theory of Poisson-Sigma models due to P.Schaller and T.Strobl (see [71]), and have given a more explicit globalization of Kontsevich's result à la Fedosov in [25] (with Tomasini). V.Dolgushev has globalized the formality map, see [37].

For the Poisson manifold $(\mathbb{R}^n, \frac{1}{2}\sum_{a,b=1}^n P^{ab}\partial_a \wedge \partial_b)$ Kontsevitch uses the following Ansatz for the bidifferential operators C_r of the star-product: let f, g be in $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{C})$, let $2r = n_1 + \cdots + n_r + M + N$ be a partition of the nonnegative integer $2r$ as a sum of nonnegative integers, and let σ be a permutation of $\{1, 2, \ldots, 2r\}$. Let us denote $\big((n_1, \ldots, n_r, M, N), \sigma\big)$ by Γ_r, and one defines the bidifferential operator

$$\begin{aligned}\mathsf{C}_{\Gamma_r}(f,g) \quad := \quad \sum_{a_1,\ldots,a_{2r}=1}^{n} \Bigg(& \frac{\partial^{n_1} P^{a_{\sigma(1)}a_{\sigma(2)}}}{\partial x_{a_1}\cdots\partial x_{a_{n_1}}} \cdots \frac{\partial^{n_r} P^{a_{\sigma(2r-1)}a_{\sigma(2r)}}}{\partial x_{a_{n_1+\cdots+n_{r-1}+1}}\cdots\partial x_{a_{n_1+\cdots+n_r}}} \\ & \frac{\partial^M f}{\partial x_{a_{n_1+\cdots+n_r+1}}\cdots\partial x_{a_{n_1+\cdots+n_r+M}}} \; \frac{\partial^N g}{\partial x_{a_{n_1+\cdots+n_r+M+1}}\cdots\partial x_{a_{2r}}}\Bigg).\end{aligned} \tag{62}$$

The bidifferential operator C_r is obtained by a particular real linear combination of the preceding operators parametrised by all possible Γ_r with weights w_{Γ_r} which are at the heart of Kontsevich's construction: he represents the Γ_r by graphs having $r+2$ vertices (corresponding to r Poisson structures and two functions) and $2r$ edges (corresponding to $2r$ partial derivatives) in the upper halfplane, and the coefficients w_{Γ_r} of the linear combination are obtained by a multiple integration related to the geometric image of the graph:

$$\mathsf{C}_r(f,g) = \sum_{\Gamma_r} w_{\Gamma_r}\mathsf{C}_{\Gamma_r}(f,g). \tag{63}$$

4.3. Equivalence

4.3.1. Symplectic manifolds A couple of years before the first existence proof, the above computations of Hochschild and Chevalley-Eilenberg cohomology had made it clear that recursive obstructions to equivalence lie in the second de Rham cohomology of the underlying symplectic manifold. On the other hand, the above-mentioned existence proofs by DeWilde-Lecomte [33] and by Fedosov [39] already included second de Rham cohomology classes. In the papers by P.Deligne [29], R.Nest-B.Tsygan [63, 64] and M.Bertelson-M.Cahen-S.Gutt [7] a somewhat canonical parametrization of the equivalence classes has been found:

Theorem 4.6 *Let (M,ω) be a symplectic manifold. Hence the equivalence classes of star-products on (M,ω) are in bijection with the formal power series having coefficients in $H^2_{\mathrm{dR}}(M)$, the second cohomology group of the manifold M.*

The above bijection is given explicitly (in terms of Čech-cohomology based on an atlas consisting of Darboux charts) and is called the *Deligne class* $[*]$ of $*$. For an excellent review of these things including a very simple existence proof of symplectic star-products, see [52]. Moreover, N.Neumaier has shown that the series of closed 2 forms occurring in the Fedosov construction [39] coincide with the representatives of the Deligne class, see [65].

4.3.2. Poisson manifolds In the case of a Poisson manifold the classification result proved to be much more difficult and had also been done by M.Kontsevich [57]:

The Schouten bracket $[\ ,\ ]_S$ (see [72]) can be extended to the space of formal multivector fields $\Gamma(M,\Lambda TM)[[\lambda]]$ in the obvious $\mathbb{C}[[\lambda]]$-bilinear manner

$$[\sum_{r=0}^{\infty}\lambda^r P_r, \sum_{t=0}^{\infty}\lambda^t Q_t]_s := \sum_{r=0}^{\infty}\lambda^r\sum_{t=0}^{r}[P_t, Q_{r-t}]_s$$

where it satisfies the graded Leibniz and Jacobi identities. A *formal Poisson structure* P on a differentiable manifold M is a formal power series $P=\sum_{r=0}^{\infty}\lambda^r P_r$ of bivector fields with $[P,P]_s = 0$. Moreover, any *formal vector field* X acts on the formal bivector fields by Lie derivative $L_X P := [X,P]_s$. Finally, two formal Poisson structures P and P' are called *formally diffeomeorphic* iff there exists a formal vector field X such that

$$P' = e^{\lambda L_X}(P).$$

This is an equivalence relation which can be seen by using the Baker-Campbell-Hausdorff formula. By means of these structures the set of equivalence classes of star-products is described as follows:

Theorem 4.7 (Kontsevitch 1997) *Let (M,P_0) be a Poisson manifold. Then the equivalence classes of star-products on (M,P_0) are in bijection with the formal diffeomorphism classes of formal Poisson structures whose zeroth order term is equal to P_0.*

5. Explicit Examples

5.1. Cotangent bundle of S^n

This example is due to F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz and D. Sternheimer [5]:

Consider the symplectic manifold $M' := T^*(\mathbb{R}^{n+1}\setminus\{0\}) = (\mathbb{R}^{n+1}\setminus\{0\})\times\mathbb{R}^{n+1}$ equipped –as an open subset of $\mathbb{R}^{2n}$– with canonical Darboux coordinates (q,p) and the canoncial symplectic form $\sum_{k=1}^{n+1} dq_k\wedge dp_k$. The following two functions $H_1(q,p) := \sum_{k=1}^{n+1} q_k p_k =: q\cdot p$ and $H_2(q,p) := \sum_{k=1}^{n+1}(q_k)^2 =: |q|^2$ span the two-dimensional non-abelian Lie algebra (with respect to the Poisson bracket), i.e. $\{H_1,H_2\} = -2H_2$. Moreover the Hamiltonian flows of H_1 and H_2 take the form $\Phi^1_s(q,p) = (e^s q, e^{-s}p)$ and $\Phi^2_t(q,p) = (q, p-2tq)$, and generate the action of the two-dimensional non-abelian Lie group

$$G := \{(\alpha,t)\in\mathbb{R}^2 \mid \alpha > 0\} \tag{64}$$

on M' given by $(\alpha,t).(q,p) := (\alpha q, -2tq+\alpha^{-1}p)$. Let T^*S^n be defined

$$M := T^*S^n := \{(q,p)\in M' \mid q\cdot p = 0 \text{ and } |q|^2 = 1\}. \tag{65}$$

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It is easily seen that this definition gives the tangent bundle of the n-sphere S^n which is isomorphic to its cotangent bundle via the canonical ('round') Riemannian metric on S^n. There is a projection

$$\pi : M' \to M : (q,p) \mapsto \left(\frac{q}{|q|}, |q|p - \frac{q\cdot p}{|q|}q\right) \tag{66}$$

which clearly is a surjective submersion. The fibres of that projection are the orbits of the group G. Therefore there is the following

Lemma 5.1 *Let F be in $\mathcal{C}^\infty(M',\mathbb{C})$. Then there is a function $f \in \mathcal{C}^\infty(M,\mathbb{C})$ such that $F = f\circ\pi$ if and only if F is G-invariant, i.e. $F\big((\alpha,t).(q,p)\big) = F\big(q,p\big)$.*

Since G is connected it follows that F is G-invariant iff

$$\{F, H_1\} = 0 = \{F, H_2\}. \tag{67}$$

Using the Moyal-Weyl star-product $*_w$ (36) on M' it is easily seen that for each quadratic polynomial F and every function $\tilde{F} \in \mathcal{C}^\infty(M',\mathbb{C})$ there is the following important formula

$$F *_w \tilde{F} - \tilde{F} *_w F = i\lambda\{F, \tilde{F}\}, \tag{68}$$

i.e. the higher order terms in the star-product commutator vanish. If this formula is applied to $F = H_1$ or $F = H_2$ one directly sees –using (67)– that a function H is G-invariant iff it commutes with H_1 and H_2 with respect to $*_w$. It follows that the space of all G-invariant functions is an associative subalgebra of $(\mathcal{C}^\infty(M',\mathbb{C})[[\lambda]], *_w)$. Consequently

Theorem 5.2 *There exists a star-product $*_{BFFLS}$ on M for which one has the following explicit formula:*

$$f *_{BFFLS} g\big(\pi(q,p)\big) = (\pi^* f) *_w (\pi^* g)(q,p).$$

5.2. Complex projective space

The following explicit formula described further down for a star-product on complex projective space $\mathbb{C}P^n$ has been found in [18] where one may find the details of its proof.

Let

$$\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n \tag{69}$$

be the canonical projection whose fibres are the complex lines in $\mathbb{C}^{n+1} \setminus \{0\}$ passing through the origin. As in the preceding example, the fibres are obtained by the action of a two-dimensional Lie group, namely the multiplicative group of all non zero complex numbers, $\mathbb{C} \setminus \{0\}$. Unfortunately this group does no longer preserve the star-product of Wick type on $\mathbb{C}^{n+1} \setminus \{0\}$ which renders the deduction more difficult. By means of the complex coordinates $z := (z_1, \ldots, z_{n+1})$ on $\mathbb{C}^{n+1} \setminus \{0\}$ we denote the square of the Euclidean distance to the origin by

$$x := \sum_{k=1}^{n+1} |z_k|^2. \tag{70}$$

By modifying the usual star-product of Wick type on $\mathbb{C}^{n+1} \setminus \{0\}$ we get the following

Theorem 5.3 *Let f, g be in $\mathcal{C}^\infty(\mathbb{C}P^n,\mathbb{C})$. Then the following fomula gives a star-product of Wick type on the Kähler manifold $\mathbb{C}P^n$:*

$$\begin{aligned}\pi^*(f*g)(z) \quad &:= \quad \pi^*(fg)(z) \\ &+ \sum_{r=1}^{\infty} \frac{(2\lambda)^r}{r!} \frac{x^r}{(1+\lambda)\cdots(1+r\lambda)} \sum_{k_1,\ldots,k_r=1}^{n+1} \frac{\partial^r \pi^* f}{\partial z_{k_1}\cdots\partial z_{k_r}}(z) \frac{\partial^r \pi^* g}{\partial \bar{z}_{k_1}\cdots\partial \bar{z}_{k_r}}(z).\end{aligned}$$

In[17] we have shown that this star-product converges on all representative functions of the canonical action of the unitary group $U(n+1)$ for certain real values of the parameter λ.

5.3. The Gutt star-product on the dual space of a Lie algebra
This very important explicit example has been found independently by S.Gutt [51] and by V.Drinfel'd [38] in 1983:

Let $(\mathfrak{g}, [\ ,\ ])$ be an n-dimensional real Lie algebra and $M := \mathfrak{g}^*$ its dual space. Let $e_1, \ldots, e_n$ be a base of $\mathfrak{g}$, let $e^1, \ldots, e^n$ be the dual base, and $c^k_{lm} := e^k([e_l, e_m])$ the structure constants of $\mathfrak{g}$. Here we shall use a formal parameter ν. Let $H : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}[[\nu]]$ be the formal group law by *Baker-Campbell-Hausdorff*:

$$\begin{aligned} H(x,y) &:= x + y \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \sum_{\substack{k_1,\ldots,k_n \geq 0 \\ l_1,\ldots,l_n \geq 0 \\ k_i + l_i \geq 1}} \nu^{\sum_{i=1}^n k_i + l_i} \frac{\big(ad(x)\big)^{k_1}\big(ad(y)\big)^{l_1} \cdots \big(ad(x)\big)^{k_n}\big(ad(y)\big)^{l_n}}{(k_1 + \cdots + k_n + 1)k_1! \cdots k_n! l_1! \cdots l_n!} x \end{aligned} \tag{71}$$

where $\mathfrak{g}[[\nu]]$ is the space of all formal power series with coefficients in $\mathfrak{g}$, see the next Section for details. It is easily seen that one can extend H to $\mathfrak{g}[[\nu]] \times \mathfrak{g}[[\nu]]$. By its definition, H is equal to the logarithm of a product of two exponential functions in the completed universal envelopping algebra of $\mathfrak{g}$, i.e.

$$e^{\nu x} e^{\nu y} = e^{\nu H(x,y)}.$$

whence

$$H\big(H(x,y), z\big) = H\big(x, H(y,z)\big) \quad \forall x, y, z \in \mathfrak{g}. \tag{72}$$

The standard symbol for the star-product $*$ on $\mathfrak{g}^*$ is defined as follows ($x, y \in \mathfrak{g} \cong \mathfrak{g}^{**}$):

$$e_x * e_y := e_{H(x,y)} \tag{73}$$

Since it is obvious using (71) that $H(x,y) - x - y$ is a multiple of ν, we get that the standard symbol of $*$ is a formal power series in ν. Moreover for each power of ν there is only a finite number of summands in $H(x,y)$ (71): this implies that the standard symbol of $*$ is polynomial in (x,y) for each power of ν. Therefore the formula (73) is well-defined. The star-product is associative because ($x, y, z \in \mathfrak{g}$)

$$\begin{aligned} (e_x * e_y) * e_z = e_{H(x,y)} * e_z &= e_{H\left(H(x,y),z\right)} \\ = e_{H\left(x,H(y,z)\right)} &= e_x * e_{H(y,z)} = e_x * (e_y * e_z) \end{aligned}$$

thanks to (72). Hence the two tridifferential operators defined by their standard symbols $(e_x * e_y) * e_z$ et $e_x * (e_y * e_z)$ coincide on exponential functions, hence they are equal thanks to Lemma 1.1. The formal power series of the standard symbol of $*$ has the following terms of order zero and of order one:

$$\check{*}(\xi, x, y) = e^{\xi(H(x,y)-x-y)} = 1 + \frac{\nu}{2}\xi([x,y]) + o(\nu^2)$$

whence the classical limit of $*$ is readily deduced. Finally, the Euler like operator

$$\nu\frac{\partial}{\partial\nu} + \sum_{k=1}^{n} \xi_k \frac{\partial}{\partial\xi_k}$$

counting the sum of the degree in ν and the degree of homogeneous polynomial functions on $\mathfrak{g}^*$ is a derivation of $*$ by (71), hence the bidifferential operator C_r de $*$ (which is of degree r in

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ν) has at most r partial derivatives with respect to ξ distributed on the two functions f and g. It follows that $f * g$ is a polynomial in ν if f and g are polynomials on $\mathfrak{g}^*$. Therefore it is legal to set $\nu = 1$ on polynomials. This latter complex associative algebra is isomorphic to the complexified universal envelopping algebra $U\mathfrak{g}$ of the real Lie algebra $\mathfrak{g}$ (see [51]). The preceding discussion gives the following

Theorem 5.4 *Let $\mathfrak{g}$ ba a finite-dimensional real Lie algebra. Then there exists a star-product $*$ defined by (71) on the Poisson manifold $(\mathfrak{g}^*, P_{\mathfrak{g}})$ which converges (in ν) on the subspace of polynomial functions on $\mathfrak{g}^*$ where the multilplication is isomorphic to the complexified envelopping algebra of $\mathfrak{g}$.*
In particular, for $\xi, \eta \in \mathfrak{g}$ it follows that the $\mathbb{C}[[\lambda]]$-module of all linear functions $\tilde{\xi}$ and $\tilde{\eta}$ defined on $\mathfrak{g}^*$ by $\tilde{\xi}(\alpha) := \langle \alpha, \xi \rangle$ forms a Lie subalgebra, i.e.

$$\tilde{\xi} * \tilde{\eta} - \tilde{\eta} * \tilde{\xi} = i\lambda \widetilde{[\xi, \eta]}.$$

5.4. The dual space of an associative algebra
This example is due to the author in this context. Let A be an n-dimensional real associative algebra. If A^- denotes the Lie algebra whose underlying vector space is A equipped with the commutator of the associative multiplication one can repeat the preceding Baker-Campbell-Hausdorff star-product. However, the associative structure of A allows for a much more explicit description: in a base $e_1, \ldots, e_n$ of A (where the dual base is denoted by $e^1, \ldots, e^n$) the structure constants of the multiplication are expressed as follows:

$$m^i_{jk} := e^i(e_j e_k) \in \mathbb{R}. \tag{74}$$

Then we can define the following star-product $*$ on A^*: $(f, g \in \mathcal{C}^\infty(A^*, \mathbb{C}))$:

$$f * g\ (\xi) := \sum_{r=0}^{\infty} \frac{\nu^r}{r!} \sum_{\substack{1 \le i_1, \ldots, i_r \le n \\ 1 \le j_1, \ldots, j_r \le n \\ 1 \le k_1, \ldots, k_r \le n}} m^{i_1}_{j_1 k_1} \cdots m^{i_r}_{j_r k_r} \xi_{i_1} \cdots \xi_{i_r} \frac{\partial^r f}{\partial \xi_{j_1} \cdots \partial \xi_{j_r}}(\xi) \frac{\partial^r g}{\partial \xi_{k_1} \cdots \partial \xi_{k_r}}(\xi) \tag{75}$$

In order to check associativity we compute the standard symbol of $*$ $(x, y, z \in A \cong A^{**})$:

$$e_x * e_y = e_{x+y+\nu xy}$$

and therefore

$$\begin{aligned} (e_x * e_y) * e_z = e_{x+y+\nu xy} * e_z &= e_{x+y+z+\nu(xy+xz+yz)+\nu^2 xyz} \quad \text{and} \\ e_x * (e_y * e_z) = e_x * e_{y+z+\nu yz} &= e_{x+y+z+\nu(xy+xz+yz)+\nu^2 xyz}, \end{aligned}$$

which shows associativity. Moreover, we can prove the following

Theorem 5.5 *Let A be a finite-dimensional real associative algebra. Then formula (75) defines a star-product $*$ on the Poisson manifold A^* (equipped with the linear Poisson structure (51) for the Lie bracket $(x, y) \mapsto [x, y] := xy - yx$) which converges (with respect to ν) on the subspace of all polynomial functions on A^*. The multiplication $*$ for $\nu = 1$ is isomorphic to the complexified universal envelopping algebra of the Lie algebra A^-.*

The restriction of this multiplication to the polynomial functions, hence to the symmetric algebra $\mathcal{S}A$, generalizes to infinite-dimensional algebras: here A can be seen as the 'one-particle observable algebra', and $(\mathcal{S}A, *)$ is a 'many boson observable algebra' preserving particle number in quantum mechanics. The above polynomial construction is extendable to infinite-dimensional associative algebras A.

6. Morphisms, modules, and reduction of star-products

In this section I shall review some results on some still open questions concerning algebra and Poisson morphisms, left modules (i.e. representations) and coisotropic (first class) submanifolds and reduction. For details on these results, see [8]. In the following, (M, P) always denotes a given Poisson manifold. For more details on Poisson manifolds and interesting maps merging into them, see also [72], [59], [76], [11], [23], [49], [53], [58], [75].

6.1. Morphisms

Let (M, P) and (M', P') be two Poisson manifolds, and let $\phi : M \to M'$ be a smooth map. Its *pull-back* $\phi^* : \mathcal{C}^\infty(M', \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ defined in the usual way by $\phi^*(f') := f \circ \phi$ is well-known to be a *morphism of associative algebras*. Moreover, ϕ is called a *Poisson map* iff for all $m \in M$ we have $(T_m\phi \otimes T_m\phi)P_m = P'_{\phi(m)}$. This is equivalent to saying that the pull-back ϕ^* is a *morphism of Poisson algebras*, i.e. $\phi^*\{f', g'\}_{P'} = \{\phi^* f', \phi^* g'\}_P$ for all $f', g' \in \mathcal{C}^\infty(M, \mathbb{C})$.

Examples for Poisson maps are given by all *symplectomorphims*, i.e. diffeomophisms ϕ between two symplectic manifolds (M, ω) and (M', ω') such that $\phi^*\omega' = \omega$. Another very important class of examples is provided by *(Lie algebra) momentum maps*: these are Poisson maps $J : (M, P) \to (\mathfrak{g}^*, P_\mathfrak{g})$ where $(\mathfrak{g}, [\ ,\ ]$ is a finite-dimensional real Lie algebra. The more familiar equivalent definition is given by

$$\{\langle J, \xi\rangle, \langle J, \eta\rangle\} = \langle J, [\xi, \eta]\rangle \qquad \text{for all} \quad \xi, \eta \in \mathfrak{g}, \tag{76}$$

which means that the Hamiltonian vector fields $X_{\langle J,\xi\rangle}$ define a left Lie algebra Poisson action of $\mathfrak{g}$ on M. Souriau's original definition supposed that this Lie algebra action comes from a left Lie group action $G \times M \to M$ (where $\mathfrak{g}$ is the Lie algebra of G) and in addition J is equivariant (i.e. $J(gm) = Ad^*(g)J(m)$ for all $g \in G$ and $m \in M$). A particular important case of a Lie algebra momentum map is given by a symplectic manifold (M, ω), an abelian Lie algebra whose dimension is one half of the dimension of M such that J is a submersion almost everywhere on M: this is called a *completely integrable system* (in the sense of Liouville).

Furthermore, Poisson maps between symplectic manifolds $\phi : (M, \omega) \to (M', \omega')$ take a very simple form: these are submersions whose kernel bundle $Ker\ T\phi \subset TM$ is symplectic, whose ω-orthogonal bundle $E := Ker\ T\phi^\omega \subset TM$ is integrable, and such that $\phi^*\omega'$ and ω coincide on $E \times_M E$. Locally, ϕ is the projection on the local leaves of E along the fibres of ϕ.

Now let $*$ be a star-product on (M, P) and $*'$ be a star-product on (M', P'). A *star-product morphism* $\Phi : \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ is defined to be a morphism of associative algebras over $\mathbb{C}[[\lambda]]$, i.e. a sequence of $\mathbb{C}$-linear maps $\Phi_r : \mathcal{C}^\infty(M', \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ with $\Phi = \sum_{r=0}^\infty \lambda^r \Phi_r$ such that

$$\Phi(f' *' g') = \Phi(f') * \Phi(g') \quad \text{for all} \quad f', g' \in \mathcal{C}^\infty(M', \mathbb{C})[[\lambda]]. \tag{77}$$

By inspection of this condition one finds that Φ_0 must be equal to the pull-back ϕ^* of a Poisson map $\phi : (M, P) \to (M', P')$. Here *Milnor's exercise* (see e.g. [56], p. 301, Cor. 35.9) is crucial which states that every algebra morphism $\mathcal{C}^\infty(M', \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ is the pull-back of a $\mathcal{C}^\infty$-map $M \to M'$. S.Gutt and J.Rawnsley found out that the higher order terms of Φ have to be *differential operators along* ϕ, i.e. linear maps $\mathsf{D} : \mathcal{C}^\infty(M', \mathbb{C}) \to \mathcal{C}^\infty(M, \mathbb{C})$ which in a charts $\big(U, (x_1, \ldots, x_n)\big)$ of M and $\big(U', (x'_1, \ldots, x'_{n'})\big)$ of M' take the local form

$$\mathsf{D}(f')|_{U\cap\phi^{-1}(U')} = \sum_{I\in\mathbb{N}^{n'}, |I|\le N} \mathsf{D}^I \phi^*\big(\partial'_I(f'|_{U'})\big)$$

where D^I is a $\mathbb{C}$-valued $\mathcal{C}^\infty$-function on the open set U.

The problem of the *quantization of a given Poisson map* ϕ is still open, i.e. to find star-products $*$ on (M,P) and $*'$ on (M',P') and differential operators Φ_r along ϕ for $r \geq 1$ such that $\Phi := \phi^* + \sum_{r=1}^{\infty} \lambda\Phi_r$ satisfies (77).

For certain momentum maps J on symplectic manifolds whose Hamiltonian vector fields $X_{\langle J,\xi\rangle}$ all preserve any connection ∇ in the tangent bundle (e.g. for proper Hamiltonian Lie group actions such as for compact Lie groups) B.Fedosov [41] showed the existence of a *strongly invariant star-product* $*$ i.e. for which

$$\langle J,\xi\rangle * f - f * \langle J,\xi\rangle = i\lambda\{\langle J,\xi\rangle, f\} \quad \text{for all} \quad f \in \mathcal{C}^\infty(M,\mathbb{C}),\ \xi \in \mathfrak{g}$$

(see also [2]) for the definition) which implies the quantum version of (76) for $f = \langle J,\eta\rangle$, i.e

$$\langle J,\xi\rangle * \langle J,\eta\rangle - \langle J,\eta\rangle * \langle J,\xi\rangle = i\lambda\langle J,[\xi,\eta]\rangle \qquad \text{for all} \quad \xi,\eta \in \mathfrak{g}, \tag{78}$$

and thus the quantization of J where $\mathfrak{g}^*$ carries the BCH-star-product, see also [78] for definitions and some cohomological statements of *quantum moment maps.* A star-product $*$ satisfying (78) has been called *covariant* by [2]. This can be seen as a *quantization of a symmetry.*
Many classically integrable systems such as the Toda chain and the Calogero-Moser systems are also quantum integrable in the above sense, see e.g. [12].

For the above Poisson maps ϕ between symplectic manifolds $(M,\omega) \to (M',\omega')$ I have found a partial answer to the quantization problem by a Fedosov-type analysis: the ω-orthogonal bundle E to the kernel bundle *ker* $T\phi$ is integrable and gives thus rise to a a regular foliation of M. There is a differential topological invariant attached to this foliation, the so-called *Atiyah-Molino class* $\kappa(M,E)$, see [8, p.85-91]. I proved in [8] that there is a total obstruction at order 3 in λ to the existence of a quantization related to a quadratic expression in the Atiyah-Molino class which has to be matched by the first two Deligne classes, see [8, Thm.5.3]. However, I do not know whether this obstruction is geometrically realized in a counterexample. On the other hand, if the Atiyah-Molino class vanishes, and if the Deligne classes of the two star-products are sufficiently ϕ-related I could prove that such a quantization of ϕ always exists, see [8, Thm.5.4].

6.2. Modules

Given a class of associative algebras, a natural algebraic question concerns its *(left) modules* or its *representations*: in the context of deformation quantization this means that one tries to find a linear map ρ which sends an element $f \in \big(\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]], *\big)$ to a 'suitable' $\mathbb{C}[[\lambda]]$-linear operator $\rho(f)$ in a $\mathbb{C}[[\lambda]]$-module $\mathcal{M}$ such that for all $f,g \in \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ one has the representation identity

$$\rho(f * g) = \rho(f) \circ \rho(g) \quad \text{and} \quad \rho(1) = \mathrm{id}_{\mathcal{M}}. \tag{79}$$

For a physicist, this is not an artificial problem since (s)he is for instance interested in the question whether a given star-product algebra can be represented in some 'Hilbert space'.

In contrast to the morphisms the variety of possible modules seems to be very large: we shall henceforth restrict ourselves to a subclass where $\mathcal{M} = \mathcal{C}^\infty(C,\mathbb{C})[[\lambda]]$ for a given differentiable manifold C and where each $\rho(f)$ is a formal power series of differential operators on C. For generalizations to sections of vector bundles over C, see [8].

A simple example is $C = M$, i.e. $\mathcal{M} = \mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ where the algebra $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ acts on itself by left multiplications given by the star-product $*$ itself. Clearly $\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]]$ acts equally well on a finite direct sum of copies of itself which would correspond to the space of smooth sections of a trivial vector bundle over M. B.Fedosov generalized this representation to the space of sections of any vector bundle E over M by first realizing E as the subbundle of a trivial vector bundle of rank N given by the image of a projection-valued function $\mathsf{P} = \big(\mathsf{P}_{ij}\big)_{1\leq i,j\leq N}$

over M (which has constant rank N) and by deforming P to a projection $\hat{\mathsf{P}}$ with respect to the star-product, i.e. $\sum_{j=1}^{N} \hat{\mathsf{P}}_{ij} * \hat{\mathsf{P}}_{jk} = \hat{\mathsf{P}}_{ik}$ for which he has the explicit formula

$$\hat{\mathsf{P}} = \frac{1}{2}\mathbf{1} + \left(\mathsf{P} - \frac{1}{2}\mathbf{1}\right) * \big(\mathbf{1} + 4(\mathsf{P} * \mathsf{P} - \mathsf{P})\big)^{-1/2}$$

where the square root is well-defined by its formal Taylor series (since $\mathsf{P} * \mathsf{P} - \mathsf{P}$ is proportional to λ), see [41, p.120, eqn(4.1.16)]. Thereby all the finitely generated *projective modules* of $\mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ are obtained. See Fedosov's book [41] for his treatment of K-theory and index theorems. See also the works of H.Bursztyn and S.Waldmann on Morita theory [19], [20].

A second class of examples is given by *symbol calculus*, the generalization of canonical quantization: the maps ρ_s and ρ_w –extended to formal power series– define modules of $\mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{C})[[\lambda]]$ equipped with the standard ordered star-product $*_s$ or with the Moyal-Weyl star-product $*_w$, respectively, on $C = \mathbb{R}^n$. In these examples one can replace the symplectic manifold $\mathbb{R}^{2n}$ with the canonical symplectic form by the cotangent bundle T^*C of any n-dimensional manifold C (see section 3.2.1) equipped with the canonical 2-form (56): fixing a covariant derivative ∇ in the tangent bundle of C one can assign to any $\mathcal{C}^\infty$-function F on T^*C which is of bounded polynomial degree in the fibres a differential operator $\rho_s(F)$ by a generalization of formula (35) where the partial derivatives $\partial/\partial p_i$ have to be replaced by the natural *fibre derivatives* in the direction of the fibres of T^*C, where the evaluation at $p = 0$ is interpreted as the restriction of the result to the zero-section C in T^*C whereas the partial derivatives $\frac{\partial^r \phi}{\partial q_{k_1} \cdots \partial q_{k_r}}$ are to be replaced by the r-fold symmetrized iteration of the covariant derivative ∇. This prescription defines a star-product $*_s$ on T^*C represented on the functions on C, see e.g. [14], [13], [9] where also analogues of Moyal-Weyl star-products $*_w$, their representations ρ_w, and the equivalence transformation N (37) can be found.

Encouraged by the algebraic quantum field theorist K.Fredenhagen, S.Waldmann and I have formulated some the above representations in a *"GNS fashion"*, see [16]: in case the star-product $*$ is hermitean (see eqn (61)) the algebra $\mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ obviously has an involution by pointwise complex conjugation (in the ring $\mathbb{C}[[\lambda]]$). The subring $\mathbb{R}[[\lambda]]$ is known to have a non-archimedean *ring ordering* defined by

$$\alpha = \sum_{r=0}^{\infty} \lambda^r \alpha_r \quad \begin{cases} > 0 & \text{if } \alpha \neq 0 \text{ and } \alpha_{o(\alpha)} > 0 \\[2ex] < 0 & \text{if } \alpha \neq 0 \text{ and } \alpha_{o(\alpha)} < 0 \end{cases} \tag{80}$$

which means that the strictly positive elements are closed under addition and multiplication. This has allowed us to speak of *positive linear functionals* $\Omega : \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]] \to \mathbb{C}[[\lambda]]$ by declaring that for all $f \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ the complex power series $\Omega(\overline{f} * f)$ is a real power series, and is nonnegative w.r.t. the above ring ordering. Then the algebraic part of the GNS machinery works: thanks to the *Cauchy-Schwartz inequality* $\overline{\Omega(\overline{f} * g)}\ \Omega(\overline{f} * g) \leq \Omega(\overline{f} * f)\ \Omega(\overline{g} * g)$ it is shown that the *Gel'fand ideal* $\mathcal{I}_\Omega := \{f \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]] \mid \Omega(\overline{f} * f) = 0\}$ is a left ideal in $\mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$, and hence the quotient $\mathbb{C}[[\lambda]]$-module $\mathcal{H}_\Omega := \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]/\mathcal{I}_\Omega$ is a left module of the algebra $\mathcal{C}^\infty(M, \mathbb{C})$ which has a canonical scalar product $\langle \psi_f, \psi_g \rangle := \Omega(\overline{f} * g)$ where ψ_f denotes the class of $f \in \mathcal{C}^\infty(M, \mathbb{C})[[\lambda]]$ in $\mathcal{H}_\Omega$. This so-called *formal GNS-representation* respects the scalar product of the *formal pre-Hilbert space* $\mathcal{H}_\Omega$, i.e. the adjoint operator of f corresponds to $\overline{f}$. We found that all relevant representations in quantum mechanics such as the Schrödinger representation and the WKB representations can be obtained as formal GNS representations where the linear functional is an integration over configuration space C of T^*C or, more generally, a projectable lagrangian submanifold L of T^*C, see also [13] and [9] for details.

Returning to the general case of a differential operator representation of the algebra $(\mathcal{C}^\infty(M,\mathbb{C})[[\lambda]],*)$ on $\mathcal{C}^\infty(C,\mathbb{C})[[\lambda]]$ an order-by-order analysis of eqn (79) shows that the map ρ_0 must be a differential operator of order zero, whence there is a smooth map $\iota : C \to M$ with $\rho_0(f)\varphi = (\iota^* f)\varphi$, see [8, p.44, Prop.2.3] for details. By induction we have shown that all the higher order bidifferential operators ρ_r are *bidifferential operators* $\mathcal{C}^\infty(M,\mathbb{C}) \times \mathcal{C}^\infty(C,\mathbb{C}) \to \mathcal{C}^\infty(C,\mathbb{C})$ *along* id_c *and* ι in the sense that in local charts $\big(U,(x_1,\ldots,x_n)\big)$ of M and $\big(U',(x'_1,\ldots,x'_{n'})\big)$ we have

$$\rho_r(f,\phi)|_{U'\cap\iota^{-1}(U)} = \sum_{I\in\mathbb{N}^n,|I|\leq N}\ \sum_{I'\in\mathbb{N}^{n'},|J|\leq N'} \rho_r^{II'}\ (\iota^*\partial_I f|_U)\partial'_{I'}\varphi|_{U'}$$

where $\rho_r^{II'}$ are $\mathcal{C}^\infty$-functions on U'. An important notion is the *vanishing ideal* $\mathcal{I} := \{g \in \mathcal{C}^\infty(M,\mathbb{C}) \mid \iota^* g = 0\}$ of ι. It is an ideal of the commutative algebra $\mathcal{C}^\infty(M,\mathbb{C})$. At order λ^1 of (79) it can be seen that $\mathcal{I}$ is a Poisson subalgebra of $(\mathcal{C}^\infty(M,\mathbb{C}),\{\ ,\ \})$. This last statement is the definition of a so-called *coisotropic map*: if ι is an embedding of C as a smooth submanifold of M it is equivalent to $E_c := i_\alpha(P_c) \in T_cC$ for all $c \in C$ and for all $\alpha \in T_cM^*$ vanishing on T_cC. For symplectic manifolds this means that for all $c \in C$ the tangent space $E_c = T_cC^\omega \subset T_cC$. The equation $[P,P]_s = 0$ implies that the distribution $c \mapsto E_c$ is involutive and allows for an in general *singular foliation* $\mathcal{F}$ of C (by Frobenius' theorem, see e.g. [56], p. 28, Thm.3.25.) i.e. that for each $c \in C$ the tangent space of the leaf through c at c is given by E_c, see e.g. [72], p.99, Prop.7.6 for a proof. Physicists call coisotropic submanifolds *first-class constraint surfaces* and the leaves *gauge orbits*. For example, any submanifold of codimension 1 is automatically coistropic. The foliation is regular (i.e. $\cup_{c\in C}E_c$ is a subbundle of the tangent bundle TC of C) in the symplectic case: the quotient space of all leaves, $C/\mathcal{F}$, is called the *reduced phase space* of (M,ω) which we shall come to in the next section.

The *quantization problem for coisotropic submanifolds* $\iota : C \to M$ of a Poisson manifold is the inverse problem to find a representation ρ of a given star-product $*$ on (M,P) on the space $\mathcal{C}^\infty(C,\mathbb{C})[[\lambda]]$ such that $\rho_0(f)(\varphi) = (\iota^* f)\varphi$. Up to now, in complete generality, this is an open problem.

However, there are the following partial results: We have already seen that symbol calculus provides representations of certain star-products on T^*C on the base space C. More generally, as Alan Weinstein has shown (see [75]), if $C \subset M$ is a *Lagrangian submanifold* of the symplectic manifold (M,ω), i.e. $T_cC^\omega = T_cC$ for all $c \in C$, then there is a tubular neighbourhood around C which is symplectomorphic to a tubular neighbourhood of the zero-section of T^*C. Using symbol calculus there always exist representations iff $\iota^*[*] = 0$, e.g. if the Deligne class of $*$ vanishes, see [8, p.61, Thm.3.3]. More generally, if C is coisotropic in a symplectic manifold and if the Atiyah-Molino class of the foliation of C vanishes (for example if the reduced phase-space $C/\mathcal{F}$ is a smooth manifold), and if $\iota^*[*] = 0$, then there are star-product representations on the functions on C, see [8, p.115, Thm.5.6]. Here, it turned out to be useful to look at those star-products for which the vanishing ideal $\mathcal{I}[[\lambda]]$ is a *left ideal* in the deformed algebra (so-called *adapted star-products*). On the other hand, a certain quadratic expression in the Atiyah-Molino class is a total obstruction to representability, see [8, p.91, Thm.4.6]. As in the case of morphisms, I do not know of any counterexample where these obstructions are realized. As P.Glößner has shown in his thesis, see [48], star-product representations always exist for codimension 1 submanifolds. An interesting construction using a generalization of Kontsevich's graph method was done by Cattaneo and Felder who succeeded in finding representations for certain in general non symplectic Poisson structures in $\mathbb{R}^n$ with $\mathbb{R}^k$ as co-isotropic submanifold, see [26] and [27].

6.3. Reduction

Let $\iota : C \to M$ be a coisotropic (first-class) submanifold of a symplectic manifold (M, ω). We have seen that the subbundle $E = \cup_{c \in C} E_c$ with $E_c = T_c C^\omega = \{v \in T_c M \mid \omega(v, w) = 0 \;\; \forall\, w \in T_c C\}$ is integrable (since $d\omega = 0$), and thus C is automatically equipped with a foliation $\mathcal{F}$ such that each E_c is the tangent space of the leaf passing through c. If the space of all leaves, i.e. the *reduced space* $M_r := C/\mathcal{F}$ is a smooth manifold such that the canonical projection $\pi : C \to C/\mathcal{F}$ is a submersion, then it is known to be equipped with a natural symplectic form ω_r which is defined by

$$\iota^* \omega =: \pi^* \omega_r,$$

see e.g. [1], p. 416, Thm. 5.3.23, for a proof.

An important particular case is obtained by a moment map $J : M \to \mathfrak{g}^*$ coming from a Hamiltonian Lie group action $G \times M \to M$ (where $\mathfrak{g}$ is the Lie algebra of G) for which 0 is a regular value and whose inverse image $C := J^{-1}(0)$ is not empty. In this case, C is coisotropic, and the reduced space –in case it exists as a smooth manifold– is given by the orbit space of the group action of G on C (in case G is connected this amounts to the same as the definition above, for nonconnected G the quotient space is still symplectic). In this extremely important and useful situation one speaks of *Marsden-Weinstein reduction*, see [60].

For example, complex projective space (see also section 3.2.2) is obtained as a reduced symplectic manifold of $(M = \mathbb{R}^{2n+2}, \omega = \sum_{k=1}^{n+1} dq_k \wedge dp_k)$ by means of the moment map $J(q,p) := \frac{\sum_{k=1}^{n+1}(q_k^2 + p_k^2)}{2} - \frac{1}{2}$ for the usual circle ($= U(1)$)-action on $\mathbb{C}^{n+1} = \mathbb{R}^{2n+2}$. The canonical projection of $C = S^{2n+1}$ on $\mathbb{C}P(n)$ is called the *Hopf fibration*.

The physicist's interest in reduced phase spaces is the fact that many interesting situations (as e.g. all gauge theories) are formulated on a phase space M which is simple, but 'too big', i.e. contains superfluous degrees of freedom. The 'true physical phase space' is obtained by imposing *constraints on* M, that is restriction to C, and then passing to the space of 'gauge orbits', i.e. to the reduced space M_r.

It is therefore interesting to know how *star-products* $*$ *on* M *reduce* to star-products $*_r$ on M_r. For the Marsden-Weinstein situation there have been results of B.Fedosov [42], and for an account of the BRST theory in [11].

For the general case, I could give a fairly complete positive answer in [8]: the main idea is to construct a star-representation ρ of $*$ on $\mathcal{C}^\infty(C, \mathbb{C})[[\lambda]]$ and to arrange things that the reduced algebra is isomorphic to the *commutant* of this representation, i.e. the space of all those linear operators $\mathsf{D} : \mathcal{C}^\infty(C, \mathbb{C})[[\lambda]] \to \mathcal{C}^\infty(C, \mathbb{C})[[\lambda]]$ which commute with all the representors: $\rho(f) \circ \mathsf{D} = \mathsf{D} \circ \rho(f)$. This also transforms $\mathcal{C}^\infty(C, \mathbb{C})[[\lambda]]$ into a bimodule for the deformed algebra and its commutant:

Theorem 6.1 *Let $\iota : C \to M$ be a connected coisotropic submanifold of a symplectic manifold (M, ω) such that the reduced phase space $\pi : C \to M_{\mathrm{red}}$ exists. Let $*$ be a star-product on M. Then the following conditions are equivalent:*

(i) $\iota^*[*]$ is basic, i.e. there is a class β on M_{red} with $\iota^*[*] = \pi^* \beta$.

(ii) There is a star-product $*_r$ on M_{red} such that $\mathcal{C}^\infty(C, \mathbb{C})[[\lambda]]$ becomes a $* - *_r$ or a $*_r - *$-bimodule.

If one of these conditions is satisfied then

- *$*$ is representable.*
- *$\iota^*[*] = \pi^*[*_r]$.*
- *$\big(\mathcal{C}^\infty(M_r, \mathbb{C})[[\lambda]], *_r\big)$ is the commutant of the algebra $\big(\mathcal{C}^\infty(M, \mathbb{C})[[\lambda]], *\big)$ acting on $\mathcal{C}^\infty(C, \mathbb{C})[[\lambda]]$.*

- *The isomorphism classes of $* - *_r$-bimodule-structures on $\mathcal{C}^\infty(C,\mathbb{C})[[\lambda]]$ are in bijection to the following deRham cohomology groups*

$$\lambda H^{1\prime}_{dR}(C,\mathbb{C}) \oplus \lambda^2 H^1_{dR}(C,\mathbb{C})[[\lambda]]$$

where the $\,'$ means quotienting by $2\pi i$ times the integer classes.

The idea of the proof (see [8, p.69,Thm.3.5] for details) is to use Weinstein's theorem of embedding C as a Lagrangian submanifold of $M \times \overline{M_r}$ by the canonical map $c \mapsto \big(\iota(c), \pi(c)\big)$, and to use symbol calculus in a tubular neighbourhood of C in $M \times \overline{M_r}$ (which looks like T^*C): Hence the algebra $\big(\mathcal{C}^\infty(M \times \overline{M_r}, \mathbb{C})[[\lambda]], * \otimes *_r^{\mathrm{opp}}\big)$ acts on $\mathcal{C}^\infty(C,\mathbb{C})[[\lambda]]$ iff the condition of the Deligne classes is as above. The tensor product of the star-products $* \otimes *_r^{\mathrm{opp}}$ is locally defined by

$$F * \otimes *_r^{\mathrm{opp}}\, G(x,y) := \sum_{s=0}^{\infty} \lambda^s \sum_{a=0}^{s} \mathsf{C}_s^{IJ}(x) \mathsf{C}'^{I'J'}_{a-s}(y) (\partial_I \partial'_{J'} F)(x,y)(\partial_J \partial'_{I'} G)(x,y)$$

Thereby one gets a representation of the tensor product $* \otimes *_r^{\mathrm{opp}}$ on C which gives the bimodule structure.

Acknowledgments
I'd like to thank all the organizers of the conference in Orsay in April 2007 for the invitation, for the friendly *accueil*, and for the perfect organization.

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