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Bounds on photon spheres and shadows of charged black holes in Einstein-Gauss-Bonnet-Maxwell gravity

Liang Ma, H. Lü *

Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, Tianjin 300350, China

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ABSTRACT

We consider spherically symmetric and static charged black holes in Einstein-Gauss-Bonnet-Maxwell gravities in general $D \geq 5$ dimensions and study their photon spheres and black hole shadows. We show that they all satisfy the sequence of inequalities recently proposed relating a black hole's horizon, photon sphere, shadow and its mass.

1. Introduction

Spherically symmetric and static black holes play an important rôle in Einstein's General Relativity owing to their simplicity, not only for their own construction, but also for the analysis of their surrounding geodesic motions. The analysis [1,2] of null geodesics of such a black hole that are asymptotic to the Minkowski spacetime indicates that photons can have a close orbit, forming an photon sphere. For most known exact black hole solutions, there is only one such a photon sphere and it is unstable. There are two classes of photons whose orbits do not cross the photon sphere: those inside will spiral into the horizon and those outside will escape to infinity, surrounding a shadow disk, whose radius, also called optical radius, is the impact parameter of the photons. The research subject was recently boosted by the first picture the photon shadow [3]. (See also, e.g. [4–10].)

For spherically symmetric and static black holes, multiple close orbits can exist even under the stringent dominant energy condition and an explicit example was constructed in Einstein-Maxwell gravity extended with a quasi-topological electromagnetic structure [11]. In this black hole, there exists a stable photon sphere sandwiched between two unstable ones. Thus the trapped photons inside the outer photon sphere can form a photon shield outside the horizon, without falling into the horizon or escaping to infinity.

Recently a sequence of inequalities was proposed relating the radii of the black hole event horizon R_+, the (outer and unstable) photon sphere $R_{\rm ph}$, the black hole shadow $R_{\rm sh}$ [12]

$$\frac{3}{2}R_+ \leq R_{\rm ph} \leq \frac{R_{\rm sh}}{\sqrt{3}} \leq 3M. \tag{1.1}$$

This set includes the well-known Riemann-Penrose inequality $R_+ \leq 2M$ [13], and the inequalities proposed by Hod ($R_{\rm ph} \leq 3M$) [14] and Cvetič, Gibbons and Pope, ($R_{\rm ph} \leq R_{\rm sh}/\sqrt{3}$) [15]. The Riemann-Penrose inequality is considered proven under the dominant energy condition, see e.g. [16]. The other two inequalities can also be proven under the dominant energy condition, together with negative trace of the energy-momentum tensor. However, large number of black holes satisfying at least the null energy condition were examined in [12] and no counterexample was found for (1.1). The whole series of inequalities was recently proven in [17]. A different lower bound for photon sphere was also conjectured in [18], and it was shown [15] to be violated by the Kaluza-Klein dyonic black hole [19,20].

The four-dimensional inequalities (1.1) was generalized to those in higher D-dimensional spacetimes and they become [12]

$$\left(\tfrac{1}{2}(D-1)\right)^{\frac{1}{D-3}} R_+ \leq R_{\rm ph} \leq \sqrt{\tfrac{D-3}{D-1}}\, R_{\rm sh} \leq \left(\tfrac{8\pi M(D-1)}{(D-2)\Omega_{D-2}}\right)^{\frac{1}{D-3}}, \tag{1.2}$$

where Ω_{D-2} is the volume for the unit round $(D-2)$-sphere, namely

* Corresponding author.
E-mail addresses: liangma@tju.edu.cn (L. Ma), mrhonglu@gmail.com (H. Lü).

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$$\Omega_{D-2} = \frac{(2\pi)^{(D-1)/2}}{\Gamma[\frac{1}{2}(D-1)]}\,. \tag{1.3}$$

In particular, it was stated that the Reissner-Nordström black hole in general dimensions was verified to satisfy these inequalities [12]. A sufficient energy condition for the $R_{\rm ph}$-M inequality was established in [21]. The purpose of this paper is not to verify the conjecture with more examples in Einstein gravity in general dimensions. Instead, we shall consider charged black holes in Einstein-Gauss-Bonnet-Maxwell (EGBM) gravities in general $D \ge 5$. This is worth checking for two reasons. On one hand, the theory is beyond Einstein gravity since it involves quadratic curvature invariants. On the other hand, the photon spheres and shadows involve only the geodesic motions of the metrics; therefore, we may regard the Gauss-Bonnet term as matter, in which case, the black holes satisfy the weak energy condition, which makes the verification necessary and nontrivial. The properties of shadows in theories involving the Gauss-Bonnet term coupled to a scalar were also studied in [22,23].

The paper is organized as follows. In section 2, we start with a review of the charged asymptotically flat black holes in EGBM gravities in general $D \ge 5$ dimensions. We then show that the inequalities (1.2) are satisfied by the simpler RN black holes and also the $D = 5$ neutral black hole. We then prove that the inequalities hold for all the static general black holes. We conclude the paper in section 3.

2. Einstein-Gauss-Bonnet-Maxwell gravity

We start with the Lagrangian of EGBM gravity in general D dimensions

$$\begin{aligned}\mathcal{L} &= \sqrt{-g}\big(R - \tfrac14 F^2 + \alpha_{\rm GB} E^{(4)}\big)\,,\\ E^{(4)} &= R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\,.\end{aligned} \tag{2.1}$$

The quadratic Euler integrand is inspired by $\mathcal{N} = 1$ superstring, arising as an α' correction of the string world sheet [24]. The theory admits Minkowski spacetime as its vacuum and ghost-free condition requires that the coupling constant $\alpha_{\rm GB} \ge 0$ [25]. The anti-de Sitter (AdS) vacuum of this theory, on the other hand, has ghostlike graviton modes.

2.1. Charged black holes

The EGBM gravity admits spherically-symmetric and static charged black holes [26,27], given by

$$\begin{aligned} ds_D^2 &= -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{D-2}^2\,,\\ A &= \Phi(r) dt\,, \qquad \Phi = \sqrt{\frac{2(D-2)}{D-3}}\,\frac{q}{r^{D-3}}\,,\\ f &= 1 + \frac{r^2}{2\alpha}\left(1 - \sqrt{1 + \frac{8\alpha\mu}{r^{D-1}} - \frac{4\alpha q^2}{r^{2(D-2)}}}\right),\\ \alpha &= (D-3)(D-4)\alpha_{\rm GB}\,. \end{aligned} \tag{2.2}$$

The mass and electric charge are given by

$$M = \frac{(D-2)\Omega_{D-2}}{8\pi}\mu\,, \quad Q_e = \frac{\sqrt{(D-3)(D-2)}\Omega_{D-2}}{8\sqrt{2}\pi}q\,. \tag{2.3}$$

Here Ω_{D-2} denotes the volume of the unit round S^{D-2}, given by (1.3). Since Q_e/q is some numerical factor, we shall not always distinguish Q_e and q as the electric charge. The neutral solutions were obtained in [25,28]. Note that the spherically-symmetric and static ansatz allows another branch of solution where the square-root term in f takes the opposite sign. The solution then is asymptotic to the AdS spacetime that has ghostlike gravitons [25] and therefore we shall not consider this branch of the solutions in this paper.

For sufficiently large mass, the solution describes a black hole with both inner and outer horizons, $0 \le r_- \le r_+$. We use the notation r_0 to denote a generic horizon, and the corresponding temperature and entropy are

$$\begin{aligned} T &= \frac{(D-3)r_0^2\left(r_0^{2D} - q^2 r_0^6\right) + \alpha(D-5)r_0^{2D}}{4\pi\left(2\alpha + r_0^2\right)}\,,\\ S &= \tfrac14 \Omega_{D-2} r_0^{D-2}\Big(1 + \frac{2\alpha(D-2)}{(D-4)r_0^2}\Big)\,. \end{aligned} \tag{2.4}$$

It is easy to verify that the first law of black hole thermodynamics $dM = TdS + \Phi(r_0)dQ_e$ is satisfied for both inner and outer horizons. For given charge q, there is a smallest horizon radius $r_{\rm ex} > 0$, corresponding to the extremal black hole; it is determined by

$$\mu_{\rm ex} = r_{\rm ex}^{D-3} + \frac{\alpha(D-4)r_{\rm ex}^{D-5}}{D-3}\,, \qquad q^2 = \left(r_{\rm ex}^2 + \frac{D-5}{D-3}\alpha\right) r_{\rm ex}^{2(D-4)}\,. \tag{2.5}$$

For $\mu > \mu_{\rm ex}$, the black holes have two horizons. For the purpose of studying the photon sphere and shadows, which are determined by the metric only, we may regard the Euler integrand $E^{(4)}$ as matter, then from the Einstein gravity point of view, the charged black holes satisfy the weak energy condition. It turns out that $\rho - p_{\rm sphere}$ can be negative, which stops the solution from satisfying the dominant energy condition. Furthermore, the trace of the energy-momentum tensor can be positive.

2.2. Photon spheres and shadows

Owing to the spherical symmetry, the null geodesic motions can be easily analysed. For the metric given in (2.2), the radius of the photon sphere is determined by

$$\frac{d}{dr}\left(\frac{f}{r^2}\right)\Big|_{r_{\rm ph}} = 0\,. \tag{2.6}$$

The impact parameter, also called the optical radius or the shadow radius is given by

$$R_{\rm sh} = \frac{r_{\rm ph}}{\sqrt{f(r_{\rm ph})}}\,. \tag{2.7}$$

Note that the form of both radii are independent of the spacetime dimensions. In order to establish (1.2), it is convenient to define

$$\begin{aligned} \mathcal{X} &= \sqrt{\frac{D-1}{D-3}}\frac{R_M}{R_{\rm sh}}\,, \qquad R_M = \left(\frac{8\pi M(D-1)}{(D-2)\Omega_{D-2}}\right)^{\frac{1}{D-3}}\,,\\ \mathcal{Y} &= \sqrt{\frac{D-3}{D-1}}\frac{R_{\rm sh}}{r_{\rm ph}}\,, \qquad \mathcal{Z} = \left(\frac{2}{D-1}\right)^{\frac{1}{D-3}}\frac{r_{\rm ph}}{r_+}\,. \end{aligned} \tag{2.8}$$

Our goal in this paper is to prove

$$\mathcal{X} \ge 1\,, \qquad \mathcal{Y} \ge 1\,, \qquad \mathcal{Z} \ge 1\,, \tag{2.9}$$

for charged black holes in EGBM gravities in general dimensions and therefore verify (1.2).

2.3. RN black holes

We begin with analysing the RN black hole in general dimensions by setting $\alpha = 0$. It was stated in [12] that (1.2) was satisfied by these black holes, but no detail was given. We thus present the proof of this simpler example before we progress to the general solutions. The metric function is

$$f = 1 - \frac{2\mu}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}\,. \tag{2.10}$$

The solution describes a black hole when $q \le \mu$, with the outer horizon radius

$$R_+ = \Big(\mu + \sqrt{\mu^2 - q^2}\Big)^{\frac{1}{D-3}}\,.$$

The photon sphere and black hole shadow radii are

$$R_{\rm ph} = \Big(\tfrac12 (D-1)\mu + \tfrac12\sqrt{(D-1)^2\mu^2 - 4(D-2)q^2}\Big)^{\frac{1}{D-3}}\,,$$
$$R_{\rm sh} = \frac{2^{-\frac{D-1}{2(D-3)}}\Big((D-1)\mu + \sqrt{(D-1)^2\mu^2 - 4(D-2)q^2}\Big)^{\frac{D-2}{D-3}}}{\sqrt{D-3}\sqrt{(D-1)\mu^2 - 2q^2 + \mu\sqrt{(D-1)^2\mu^2 - 4(D-2)q^2}}}\,. \tag{2.11}$$

To verify the inequalities (2.9), it is instructive to introduce a dimensionless parameter λ to replace the charge parameter q:

$$q = \frac{\mu\sqrt{\lambda(\lambda + D - 1)}}{\lambda + D - 2}\,. \tag{2.12}$$

The range $0 \le q \le \mu$ is now mapped to $\lambda \in [0, +\infty]$, with $\lambda = 0$ giving the Schwarzschild black hole, and $\lambda \to +\infty$ yielding the extremal RN black hole. Following the discussion in section 2.2, we find that

$$\mathcal{X} = \frac{(D-1)^{\frac{D-1}{2(D-3)}}(\lambda + D - 2)^{\frac{1}{D-3}}\sqrt{(D-3)\lambda + (D-2)^2}}{(D-2)^{\frac{D-2}{D-3}}(\lambda + D - 1)^{\frac{D-1}{2(D-3)}}}\,,$$
$$\mathcal{Y} = \Big(1 + \frac{\lambda}{(D-1)\big((D-3)\lambda + (D-2)^2\big)}\Big)^{\frac12}\,,$$
$$\mathcal{Z} = \Big(1 + \frac{(D-3)^2\lambda/(D-1)}{2(D-2)\sqrt{(D-3)\lambda + (D-2)^2} + (D-3)\lambda + 2(D-2)^2}\Big)^{\frac{1}{D-3}}\,. \tag{2.13}$$

The inequalities $\mathcal{Y} \ge 1$ and $\mathcal{Z} \ge 1$ are manifest. The inequality $\mathcal{X} \ge 1$ can be established by a numerical plot for given D. As a concrete example, we present $D = 5$ plots of $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in Fig. 1. In general $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ are all monotonically increasing functions of λ. Near the Schwarzschild limit $\lambda \to 0$, we have

$$\{\mathcal{X}, \mathcal{Y}, \mathcal{Z}\} = 1 + \frac{\lambda}{2(D-2)^2}\Big\{\frac{1}{D-3}, \frac{1}{D-1}, \frac{D-3}{2(D-1)}\Big\} + \mathcal{O}(\lambda^2)\,. \tag{2.14}$$

Near the extremal limit $\lambda \to \infty$, we have

$$\mathcal{X} = \frac{(D-2)^{-\frac{D-2}{D-3}}(D-1)^{\frac{D-1}{2(D-3)}}}{2\sqrt{D-3}}\Big(2(D-3) - \frac{1}{\lambda} + \mathcal{O}(\lambda^{-2})\Big)\,,$$
$$\mathcal{Y} = \frac{D-2}{2(D-3)^{3/2}\sqrt{D-1}}\Big(2(D-3) - \frac{1}{\lambda} + \mathcal{O}(\lambda^{-2})\Big).$$
$$\mathcal{Z} = \Big(\frac{2(D-2)}{D-1}\Big)^{\frac{1}{D-3}}\Big(1 - \frac{1}{\sqrt{(D-3)\lambda}} + \mathcal{O}(\lambda^{-1})\Big)\,. \tag{2.15}$$

The leading terms, corresponding to the extremal limit, are all greater than 1 for $D \ge 4$. Intriguingly, they approach one as $D \to \infty$.

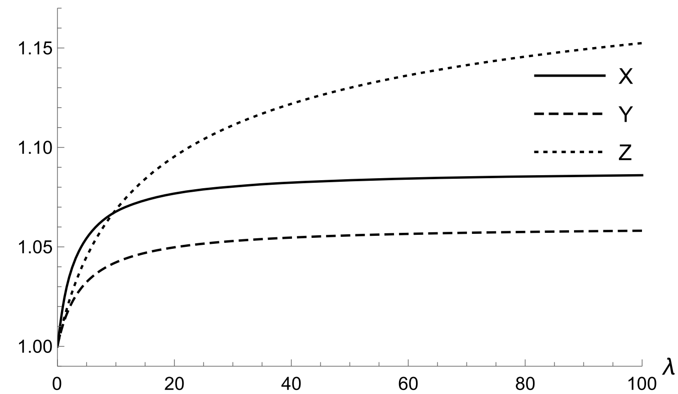

Fig. 1. The $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ as functions of dimensionless parameter λ in $D = 5$. They are monotonically increasing functions as λ runs from $\lambda = 0$ (the Schwarzschild) to $\lambda = \infty$ (the extremal limit).

2.4. $D = 5$ neutral black hole

We now consider the effect on the inequalities by the Gauss-Bonnet term. It is instructive first to examine a simpler example, namely the neutral black hole in five dimensions:

$$f = 1 + \frac{r^2}{2\alpha}\Big(1 - \sqrt{1 + \frac{8\alpha\mu}{r^4}}\Big)\,. \tag{2.16}$$

In this case, there is only one horizon r_+, determined by

$$\mu = \tfrac12 (r_+^2 + \alpha)\,. \tag{2.17}$$

This implies that we must have $\mu > \frac12\alpha$ for the solution to describe a black hole. The radii of photon sphere and shadow are

$$r_{\rm ph} = \Big(8\mu(2\mu - \alpha)\Big)^{\frac14}\,,\qquad R_{\rm sh} = \frac{\alpha\sqrt[4]{8\mu(2\mu-\alpha)}}{\sqrt{2\mu(2\mu-\alpha)} - 2\mu + \alpha}\,. \tag{2.18}$$

It is instructive to define a dimensionless parameter $\beta = \alpha/r_+^2$, we then find

$$\mathcal{X} = \sqrt{1 + \frac{\sqrt{\beta+1} - 1}{\sqrt{\beta+1} + 1}}\,,\qquad \mathcal{Y} = \sqrt{\tfrac12\big(1 + \sqrt{\beta+1}\big)}\,,$$
$$\mathcal{Z} = \sqrt[4]{\beta + 1}\,. \tag{2.19}$$

Since the parameter β runs from 0 to $+\infty$, it is straightforward to see that the inequalities (2.9) are all satisfied, with the saturation occurring at $\beta = 0$, corresponding to the Schwarzschild black hole. The $R_{\rm ph}$-M inequality, corresponding to $\mathcal{X}\mathcal{Y} \ge 1$, was established in [21]. Note that if we relax the ghost-free condition so that $\alpha < 0$ and hence $\beta < 0$, it follows from (2.19) that the reality condition requires that $\beta \ge -1$ and the inequalities (2.9) are no longer true.

Since the black hole entropies in higher-order gravities are no longer simply a quarter of the area of the horizon. The R_+ and M relation in (1.2) is no longer related to the Penrose entropy conjecture. The black hole entropy can be obtained from the Wald entropy formula, yielding

$$S = \tfrac14 \Omega_3\Big(r_+^3 + 6\alpha r_+\Big)\,. \tag{2.20}$$

The mass and entropy relation now becomes

$$\frac{256\pi^3 M^3}{27\Omega_3 S^2} = \frac{(1+\beta)^3}{(1+6\beta)^2}\,. \tag{2.21}$$

Thus for small but non-vanishing β, the Penrose conjecture is violated, but it is restored for sufficiently large β. We may define the effective radius associated with the entropy by $S = \frac{1}{4}\Omega_3(\bar{R}_+^S)^3$, and we have

$$\bar{R}_+^S = (r_+^3 + 6\alpha r_+)^{\frac{1}{3}} . \tag{2.22}$$

We then have

$$\frac{R_{\rm ph}}{\sqrt{2}\,\bar{R}_+^S} = \frac{\sqrt[4]{\beta+1}}{\sqrt[3]{6\beta+1}} . \tag{2.23}$$

Intriguingly, this ratio is a monotonically decreasing function of β. In other words, $R_{\rm ph}$ can be smaller than $\bar{R}_+^S$, a clear indication that R_+ is a better size parameter than $\bar{R}_+^S$.

We see that the $D = 5$ example is particularly simple and the series of inequalities (1.2) can be established analytically. We find that for general higher dimensions, the proof of (1.2) is not made much simpler by turning off the Maxwell field and we therefore prove the series in the next with the Maxwell field included.

2.5. General black holes

The reason we can easily prove the identities in the previous subsections was that we could analytically solve the photon sphere equation (2.6) for the photon sphere radius in the RN black holes or the $D = 5$ neutral black hole. This turns out not be possible for the general black holes. We shall adopt the technique developed in [12] to prove the inequalities. We first prove $Z \ge 1$ and then use this inequality to prove $\mathcal{X} \ge 1$ and $\mathcal{Y} \ge 1$. We define a function $W(r)$ as

$$W(r) = \sqrt{1 + \frac{8\alpha\mu}{r^{D-1}} - \frac{4\alpha q^2}{r^{2(D-2)}}}\,\Big(\frac{f}{r^2}\Big)' . \tag{2.24}$$

The photon sphere is located at $r_{\rm ph}$, which is the largest root of W. It can be easily seen that as $r \to \infty$, W is negative with

$$W = -\frac{2}{r^3} + \frac{2(D-1)\mu}{r^D} + \cdots \tag{2.25}$$

Since $r_{\rm ph}$ is the largest root, it follows that for any r with $W(r) > 0$, then we must have $r < r_{\rm ph}$. We thus define

$$\rho = (\tfrac{1}{2}(D-1))^{\frac{1}{D-3}}\, r_+ , \tag{2.26}$$

and we find

$$\begin{aligned} W(\rho) &= U - \sqrt{V} ,\\ U &= \frac{2}{\rho^3} + \frac{2^{\frac{D-5}{D-3}}(D-1)^{\frac{2}{D-3}}\alpha}{\rho^5} + \frac{(D-3)^2 q^2}{2\rho^{2D-3}} ,\\ V &= \frac{4}{\rho^6} + \frac{32\alpha}{(D-1)\rho^8} + \frac{2^{\frac{5D-17}{D-3}}\alpha^2}{(D-1)^{\frac{D-5}{D-3}}\rho^{10}} + \frac{8(D-3)\alpha q^2}{\rho^{2(D+1)}} . \end{aligned} \tag{2.27}$$

Note that in the above, we have expressed μ in terms r_+ and hence ρ. It is clear that both (U, V) are positive and further more, it is quite straightforward to prove that $U^2 - V \ge 0$ for $\rho > 0$. We therefore demonstrate that $W(\rho) > 0$. It follows that $r_{\rm ph} \ge \rho$, proving that $\mathcal{Z} \ge 1$. It is clear that this lower bound of the photon sphere holds for the case of $q = 0$.

In order to demonstrate that $\mathcal{X} \ge 1$ and $\mathcal{Y} \ge 1$, we find it is useful to express μ in terms of the photon sphere radius $r_{\rm ph}$ by solving (2.6). We have

$$\begin{aligned} \mu &= \frac{4\alpha r_{\rm ph}^{D-5}}{(D-1)^2} + \frac{(D-2)q^2}{(D-1)r_{\rm ph}^{D-3}}\\ &\quad + \sqrt{\frac{r_{\rm ph}^{2(D-3)}}{(D-1)^2} + \frac{16\alpha^2 r_{\rm ph}^{2(D-5)}}{(D-1)^4} + \frac{4(D-3)\alpha q^2}{(D-1)^3 r_{\rm ph}^2}} . \end{aligned} \tag{2.28}$$

The shadow radius is now given by

$$R_{\rm sh} = \sqrt{\frac{2\alpha r_{\rm ph}^{2(D+1)}}{(D-2)q^2 r_{\rm ph}^8 - (D-1)\mu r_{\rm ph}^{D+5} + r_{\rm ph}^{2D}\left(2\alpha + r_{\rm ph}^2\right)}} , \tag{2.29}$$

implying that

$$\begin{aligned} \mathcal{X}^2 &= \frac{(D-1)^{\frac{D-1}{D-3}}\mu^{\frac{2}{D-3}}\left((D-2)q^2 r_{\rm ph}^8 - (D-1)\mu r_{\rm ph}^{D+5} + r_{\rm ph}^{2D}\left(2\alpha + r_{\rm ph}^2\right)\right)}{2\alpha(D-3)r_{\rm ph}^{2(D+1)}} ,\\ \mathcal{Y}^2 &= \frac{2\alpha(D-3)r_{\rm ph}^{2D}}{(D-1)\left((D-2)q^2 r_{\rm ph}^8 - (D-1)\mu r_{\rm ph}^{D+5} + r^{2D}\left(2\alpha + r_{\rm ph}^2\right)\right)} . \end{aligned} \tag{2.30}$$

The trick now is to make use of $\mathcal{Z} \ge 1$, which implies

$$r_{\rm ph} \ge (\tfrac{1}{2}(D-1))^{\frac{1}{D-3}}\, r_+ \ge (\tfrac{1}{2}(D-1))^{\frac{1}{D-3}}\, r_{\rm ex} . \tag{2.31}$$

The second inequality holds because $r_{\rm ex}$ is the (smallest) horizon radius of the extremal black hole for given charge q, determined by (2.5). The $q = 0$ zero limit is achieved by taking $r_{\rm ex} \to 0$.

We can now define two dimensionless parameters $\beta \ge 0$ and $\gamma > 1$, defined by

$$\alpha = \beta r_{\rm ex}^2 , \qquad r_{\rm ph} = (\tfrac{1}{2}(D-1))^{\frac{1}{D-3}}\gamma r_{\rm ex} . \tag{2.32}$$

Note that the lower bound for γ is bigger than 1, but for our purpose, it is sufficient to assume $\gamma > 1$. Substituting α and $r_{\rm ph}$ into (2.30), and we find that both $\mathcal{X}$ and $\mathcal{Y}$ are functions of the dimensionless quantities (β, γ) only, with the dimensionful parameter $r_{\rm ex}$ dropped out. In $D = 5$, the expressions are quite simple and they are manifestly no smaller than 1:

$$\begin{aligned} \mathcal{X}^2 &= 1 + \frac{(4\gamma^4 - 3)\left(\sqrt{\beta^2\gamma^2 + \beta + 4\gamma^6} - 2\gamma^3\right) + \beta\gamma}{4\beta\gamma^5} \ge 1 ,\\ \mathcal{Y}^2 &= 1 + \frac{1 + \gamma\left(\sqrt{\beta^2\gamma^2 + \beta + 4\gamma^6} + \beta\gamma - 2\gamma^3\right)}{4\gamma^4 - 1} \ge 1 . \end{aligned} \tag{2.33}$$

For general dimensions, the expressions are much more complicated, we find

$$\begin{aligned} \mathcal{X} &= 2^{\frac{2(D-2)}{(D-3)^2}}(D-1)^{\frac{1-D}{(D-3)^2}}\left(\frac{\beta}{\gamma^2}\right)^{\frac{1}{D-3}}\left(C_1 - \frac{2\sqrt{C_2}}{D-3}\right)^{\frac{1}{2}}\\ &\quad\times\left(C_3 + \sqrt{C_2}\right)^{\frac{1}{D-3}} ,\\ \mathcal{Y} &= \left(C_1 - \frac{2\sqrt{C_2}}{D-3}\right)^{-\frac{1}{2}} , \end{aligned} \tag{2.34}$$

where

$$\begin{aligned} C_1 &= 1 + \frac{\left(\frac{1}{2}(D-1)\right)^{\frac{D-1}{D-3}}\gamma^2}{(D-3)\beta} ,\\ C_2 &= 1 + \frac{(D-1)^{\frac{5-D}{D-3}}(D-3+(D-5)\beta)}{2^{\frac{2}{D-3}}\beta\gamma^{2(D-4)}} + \frac{(D-1)^{\frac{2(D-1)}{D-3}}\gamma^4}{2^{\frac{4(D-2)}{D-3}}\beta^2} ,\\ C_3 &= 1 + \frac{(D-2)(D-1)^{\frac{5-D}{D-3}}(D-3+(D-5)\beta)}{(D-3)2^{\frac{2}{D-3}}\beta\gamma^{2(D-4)}} . \end{aligned} \tag{2.35}$$

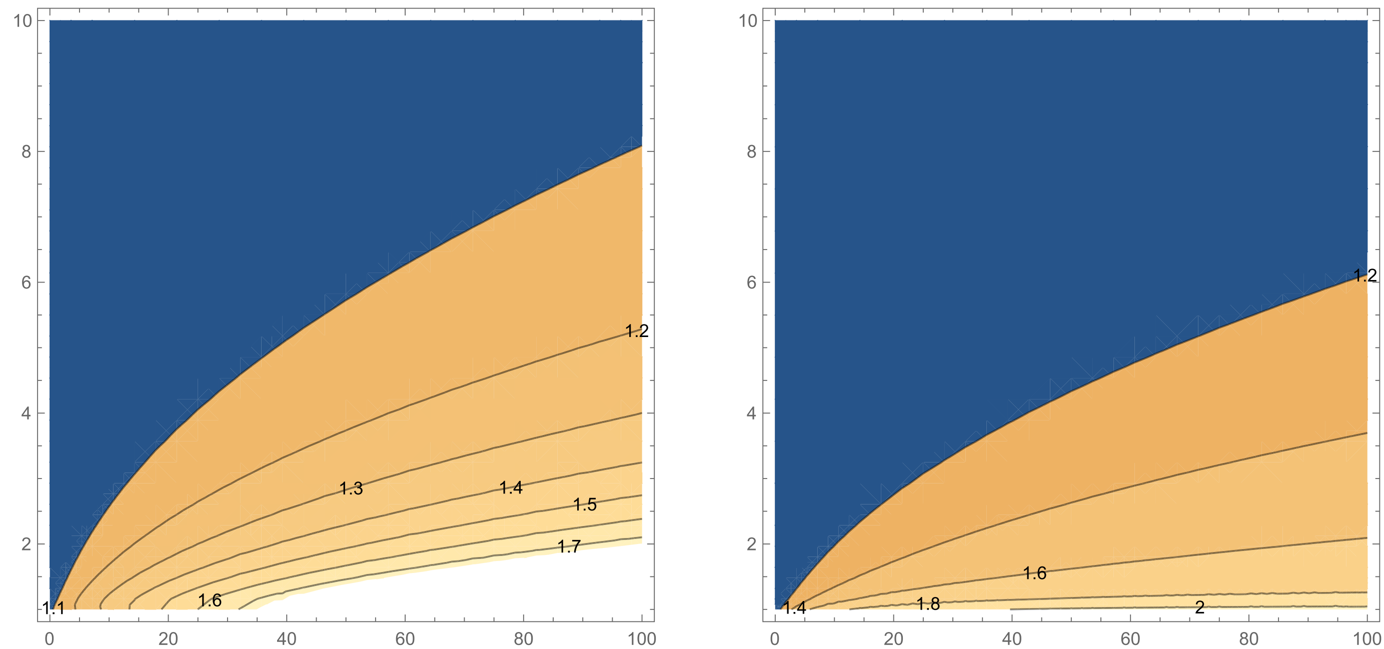

Fig. 2. Contour plots of $(\mathcal{X},\mathcal{Y})$ for $0\le\beta\le 100$, $0\le\gamma\le 10$ for $D=6$. The left and right are for $\mathcal{X}$ and $\mathcal{Y}$ respectively. The values in the blue regions lie within $[1,1.1]$ and $[1,1.2]$ respectively and these values increase as the colour becomes lighter. There are no regions in $\gamma>1$ and $\beta\ge 0$ that have values of $\mathcal{X}$ and $\mathcal{Y}$ less than 1.

Having expressed $\mathcal{X},\mathcal{Y}$ analytically in terms of two dimensionless parameters (β,γ), it is then straightforward to use numerical plots to show that $\mathcal{X}\ge 1$ and $\mathcal{Y}\ge 1$, for any given dimension $D\ge 5$. (Unlike the case of $\mathcal{Z}\ge 1$, a clever analytical proof is unlikely in general D, and we present contour plots of $(\mathcal{X},\mathcal{Y})$ as functions of $\beta\ge 0$ and $\gamma>1$ for $D=6$ in Fig. 2.) In particular, when $\beta=0$, corresponding to the RN black hole in general dimensions, we have

$$\mathcal{X}=\frac{2}{(D-1)\gamma^{D-3}}\left(1+\frac{4(D-2)}{(D-1)^2\gamma^{2(D-3)}}\right)^{\frac{1}{D-3}}\times\sqrt{\frac{1}{4}(D-1)^2\gamma^{2(D-3)}-1}\,,$$
$$\mathcal{Y}=\sqrt{1+\frac{1}{\frac{1}{4}(D-1)^2\gamma^{2(D-3)}-1}}\,,\qquad(2.36)$$

where γ is related to λ in (2.13) by

$$\gamma^{2(D-3)}=\frac{4(D-2)^2(\lambda+D-1)}{(D-1)^2\lambda}\,.\qquad(2.37)$$

In the limit of $\beta\to\infty$, $\mathcal{X}$ is positive and divergent at the order $\beta^{\frac{1}{D-3}}$ for $D\ge 6$ and

$$\mathcal{Y}=\left(1-\frac{2}{D-3}\sqrt{1+4^{\frac{1}{3-D}}(D-5)(D-1)^{-\frac{D-5}{D-3}}\gamma^{2(4-D)}}\right)^{-\frac{1}{2}}\,.\qquad(2.38)$$

It follows from the second equation in (2.32) that the Schwarzschild black hole limit is achieved by taking $\gamma\to\infty$. For large γ, we have

$$\mathcal{X}\sim\mathcal{Y}=1+\frac{2^{\frac{D-1}{D-3}}(D-1)^{\frac{1-D}{D-3}}\beta}{(D-3)\gamma^2}+\cdots\,.\qquad(2.39)$$

For the neutral black hole in the general Einstein-Guass-Bonnet theory, the limit is somewhat more subtle, since we now have $r_{\rm ex}\to 0$. It follows from (2.32), we define

$$\beta=\tilde\beta\,(\tfrac{1}{2}(D-1))^{\frac{2}{D-3}}\gamma^2\,,\qquad(2.40)$$

such that $\alpha=\tilde\beta r_{\rm ph}^2$. We now take the limit $\beta\to\infty$ and $\gamma\to\infty$ while keeping $\tilde\beta$ positive and finite. We find

$$\mathcal{X}^2=\frac{1}{\mathcal{Y}^2}\left(\sqrt{\frac{16\tilde\beta^2}{(D-1)^2}+1}+\frac{4\tilde\beta}{D-1}\right)^{\frac{2}{D-3}}\,,$$
$$\mathcal{Y}^2=\frac{2\tilde\beta(D-3)}{D-1+2(D-3)\tilde\beta-\sqrt{(D-1)^2+16\tilde\beta^2}}\,.\qquad(2.41)$$

It is straightforward to show that $\mathcal{X}\ge 1$ and $\mathcal{Y}\ge 1$ for positive $\tilde\beta$. The inequalities are saturated when $\tilde\beta=0$, corresponding to the Schwarzschild black hole in Einstein gravity. The result reduces to (2.19) when $D=5$ with $\tilde\beta=\beta/(2\sqrt{\beta+1})$.

3. Conclusions

In this paper, we considered charged static black holes in EGBM gravities in general dimensions. These black holes are spherically symmetric and asymptotic to Minkowski spacetimes. From the view of Einstein gravity, these black holes satisfy the weak energy condition, provided that the Gauss-Bonnet coupling is nonnegative, which also ensures that the perturbation is free of ghost excitations. There exists an unstable photon sphere outside the horizon, giving rise to the edge of a shadow disk for an observer at infinity. We found the radii of the horizon, photon sphere and shadow disk satisfy the sequence of inequalities (1.2), conjectured for the black holes in Einstein gravity. The robustness of this sequence calls for a better understanding of the underlying condition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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