

**PAPER**

A single-particle framework for unitary lattice gauge theory in discrete time

Pablo Arnault^{1,*}  and Christopher Cedzich²¹ Univ. Paris-Saclay, CNRS, ENS Paris-Saclay, INRIA, Laboratoire Méthodes Formelles, 91190 Gif-sur-Yvette, France² Quantum Technology Group, Heinrich Heine Universität Düsseldorf, Universitätsstr. 1, 40225 Düsseldorf, Germany

* Author to whom any correspondence should be addressed.

E-mail: pablo.arnault@inria.fr**Keywords:** quantum walks, lattice gauge theory, quantum automata, quantum field theory**RECEIVED**

13 September 2022

REVISED

1 December 2022

ACCEPTED FOR PUBLICATION

15 December 2022

PUBLISHED

28 December 2022

Original Content from
this work may be used
under the terms of the
[Creative Commons
Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/).

Any further distribution
of this work must
maintain attribution to
the author(s) and the title
of the work, journal
citation and DOI.

**Abstract**

We construct a real-time lattice-gauge-theory (LGT)-type action for a spin-1/2 matter field of a single particle on a $(1 + 1)$ -dimensional spacetime lattice. The framework is based on a discrete-time quantum walk, and is hence inherently unitary and strictly local, i.e. transition amplitudes exactly vanish outside of a lightcone on the lattice. We then provide a lattice Noether's theorem for internal symmetries of this action. We further couple this action to an electromagnetic field by a minimal substitution on the lattice. Finally, we suggest a real-time LGT-type action for the electromagnetic field in arbitrary spacetime dimensions, and derive its classical equations of motion, which are lattice versions of Maxwell's equations.

1. Introduction

Lattice gauge theory (LGT) is a framework used to tackle the non-perturbative regimes of quantum field theories (QFTs) [1]. The basic idea is to formulate the gauge theory under study on a lattice, which furnishes an in-built ultraviolet regulator. LGT was initially introduced by Wilson [2] as a framework accounting successfully for quark confinement; the gauge theory in question was (a simplified version of) quantum chromodynamics (QCD).

There are two fundamental types of LGTs. The traditional one is 'Lagrangian LGT' [2], in which the main object of the theory is an action in discrete spacetime. Lagrangian LGT treats time and space on the same footing, and is in this sense close to Einstein's theory of relativity. However, unitarity is not manifest in Lagrangian LGT³—it has to be proven and does not always hold, see, e.g. [3]. The second type of LGT is 'Hamiltonian LGT' [4, 5], in which the main object of the theory is a Hamiltonian on a spatial lattice while time is usually kept continuous. The pros and cons are therefore reversed with respect to Lagrangian LGT: Hamiltonian LGTs are unitary by construction, but space and time are often treated differently. These differences between Lagrangian and Hamiltonian LGT are actually already present in the continuum: Lagrangian LGT comes from the path-integral approach to QFT, in which Lorentz covariance is manifest but unitarity is not, whereas Hamiltonian LGT comes from the canonical approach to QFT, which is manifestly unitary but not manifestly Lorentz covariant [6].

The standard tool of lattice QCD are Monte Carlo (MC) simulations, which are usually carried out in the Lagrangian formulation with a Wick-rotated spacetime, i.e. in Euclidean spacetime with imaginary time [1]. MC simulations have been successfully used to determine (a) equilibrium properties of QCD, such as the masses of quarks and stable hadrons, hadronic structure-related quantities and non-zero-temperature properties [7], and (b) certain non-equilibrium, i.e. beyond-ground-states properties [8, 9]. Despite its successes, standard LGT with MC simulations fails to give results in several cases, in particular in parameter regimes in which MC simulations encounter a so-called sign problem [7], which can almost certainly not be solved efficiently by (even the most powerful [10]) classical computers with traditional techniques [11].

³ See below in section 2.3 (second paragraph) for an explanation of how unitarity is evaluated in Lagrangian LGTs.

Several techniques exist to overcome the sign problem for certain simplified models (see references in reference [7]), including tensor-network approaches [7], which use a real-time Hamiltonian formulation, where we are going to define just below what ‘real-time’ means.

Let us call ‘real-time LGT’ any LGT framework in which spacetime is not Wick-rotated. Because time is kept real, all such frameworks are expected to be particularly suited to explore the dynamics of the gauge theory in question [12]. Thereby, real-time LGT is one privileged approach to computing non-equilibrium properties of gauge theories. Lately, real-time LGT has entered a new era with the advent of quantum simulation [13] and quantum computation [14], which reduce exponentially the cost in simulating many-body quantum systems. As above, there are two types of real-time LGTs: (a) real-time Hamiltonian LGT, where time is either kept continuous (with the perspective of analog quantum simulations [15–18]) or discretized (with the perspective of digital quantum simulations [19, 20] and quantum algorithms [21–29]), and (b) real-time Lagrangian LGT, where time is discretized but spacetime is not Wick-rotated [30]. Our paper combines both perspectives. Reviews with different focuses on the general topic of quantum-information inspired methods for QFTs can be found in references [12, 31–36]. Several proof-of-principle experiments of quantum simulations of the dynamics of LGTs have already been realized [37] (see also references in [35]).

Among the real-time Hamiltonian-LGT approaches in discrete time, there is one, often not known under the name of LGT, which consists in using quantum cellular automata (QCA) [38–40]. QCA are by-construction unitary evolution operators in discrete spacetime that are *strictly local*, i.e. there is an in-built strict ‘relativistic’ lightcone at the discrete-spacetime level. Results in this field of ‘QCA LGT’ are still preliminary, but promising [41–48]. Let us stress this important point: in the usual discrete-time Hamiltonian-LGT approaches, locality enters merely in the form of an effective lightcone [49] related to Lieb-Robinson bounds, whereas QCA are strictly local by construction. At the level of classical fields, i.e. in the one-particle sector, QCA reduce to so-called discrete-time quantum walks (DQWs) [50]. Understanding DQWs as the building blocks of QCA, one can therefore expect that the field of QCA LGT will benefit in the near future from the numerous results that exist (a) in the one-particle sector, in the free case [51–53], with couplings to Abelian⁴ [54–57] and non-Abelian [58] gauge fields, and with curved spacetimes [54, 59–64], but also (b) in the multiparticle free case [65, 66].

In this manuscript, we further complete the list of achievements in the one-particle sector with the following results:

- (a) We construct a real-time LGT-type action for spin-1/2 matter fields on a $(1+1)$ -dimensional spacetime lattice, that is based on a DQW. This ‘DQW action’ is therefore by-construction unitary—i.e. it delivers unitary equations of motion (EOMs)—, and it treats time and space on the same footing.
- (b) We provide a lattice Noether’s theorem for the internal symmetries of the DQW action. Applying this theorem to the global $U(1)$ symmetry of the DQW action, we find a $U(1)$ -charge current conserved on the lattice.
- (c) We place the particle into an Abelian $U(1)$ gauge field by applying a lattice minimal-coupling scheme to the DQW action.
- (d) For this Abelian $U(1)$ gauge field which generates an electromagnetic field, we suggest a real-time LGT-type action in arbitrary spacetime dimensions, from which we derive the classical EOMs of the gauge field, which are lattice versions of Maxwell’s equations.

Let us give more details on these achievements. We bring together in a unified framework elements from the field of QCA with elements from Lagrangian LGT, and this for spin-1/2 matter fields. In this sense, we extend [47] which considers scalar fields. All is done at the level of classical fields, i.e. the fields are not quantized⁵. Hence, throughout this paper we use ‘fermionic’ synonymously to ‘spin 1/2’.

The definition of a real-time LGT-type action S_{DQW} based on a DQW is the first stepping stone in this paper. To the best of our knowledge, the only paper that suggests a discrete-spacetime action for a spin-1/2 matter field based on DQWs is [67]. However, the action in [67] does not relate nicely to usual LGT actions in the sense that it is based on a *one-step* EOM for the matter field ψ , i.e. an EOM of the type $\psi_{j+1} = \mathcal{W}\psi_j$, where $j \in \mathbb{Z}$ labels discrete time and \mathcal{W} is a unitary evolution operator. In contrast, in LGT one usually constructs EOMs with symmetric finite differences; these *two-step* EOMs involve a field ψ at three subsequent time instants ψ_{j-1} , ψ_j and ψ_{j+1} , and therefore require two initial conditions rather than just a

⁴ In references [54–56], various lattice gauge invariances of specific DQW models were reported, but the discrete derivatives in these works are model-dependent and more complicated than the simple finite differences of [57] and of standard LGTs, which are the ones that we use in the present paper.

⁵ In [47] the scalar fields are quantum.

single one. Whether one- or two-step, EOMs in LGT usually do not preserve the unitarity of the continuum model. Building on the construction of [68], we remedy this lack by providing a *unitary* real-time action, that is extremely similar to standard LGT actions in the sense that it is based on a two-step EOM, while being associated with a DQW (hence its unitarity) with a one-step EOM.

The remainder of this paper is organized as follows. We begin with discussing one- and two-step EOMs for spin-1/2 particles in discrete spacetime in section 2, which leads us to the definition of the corresponding DQW action S_{DQW} in section 2.4. In section 3 we prove a lattice Noether's theorem for internal symmetries of a generic real-time action for spin-1/2 particles. In section 4 we couple S_{DQW} to an Abelian U(1) gauge field via a lattice version of minimal substitution. In the last section, section 5, we suggest a real-time LGT-type action for this Abelian U(1) gauge field, in arbitrary spacetime dimensions, and derive the corresponding classical EOMs, which are lattice versions of Maxwell's equations.

2. DQW-based LGT-type action for a classical matter field

2.1. The continuum EOM

Consider in $1 + 1$ dimensions a relativistic classical matter field ψ , with internal components ψ^a where $a = 1, \dots, N$ for some $N \in \mathbb{N}$. The dynamics of ψ is described by the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0, \quad (1)$$

where the γ_μ , $\mu = 0, 1$, act on the internal Hilbert space, and satisfy the Clifford-algebra relations

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad (2)$$

with $[\eta^{\mu\nu}] := \text{diag}(1, -1)$, and where on the right-hand side of equation (2) we omitted for brevity the identity on the internal Hilbert space.

The Dirac equation (1) can be rewritten in the form of a Schrödinger equation,

$$i\partial_t \psi = \mathcal{H}\psi. \quad (3)$$

Here, $\partial_t \equiv \partial_0$ is the partial derivative with respect to time, and we have introduced the Dirac Hamiltonian

$$\mathcal{H} := \alpha^1(-i\partial_1) + m\alpha^0, \quad (4)$$

where $\partial_1 \equiv \partial_{x^1}$ is the partial derivative with respect to the spatial position x^1 , and where we have introduced the operators

$$\alpha^0 := \gamma^0 \quad (5a)$$

$$\alpha^1 := \gamma^0 \gamma^1, \quad (5b)$$

which satisfy the relations

$$\{\alpha^\mu, \alpha^\nu\} = 2\delta^{\mu\nu}, \quad (6)$$

where $[\delta^{\mu\nu}] := \text{diag}(1, 1)$.

In one spatial dimension, it is enough to consider an internal Hilbert space of dimension 2 to find a pair of alpha matrices ($[(\alpha^0)_b^a], [(\alpha^1)_b^a]$) that satisfy equation (6). In that case, the index a of the internal Hilbert space of ψ belongs to $\{1, 2\}$. Unless otherwise mentioned we will work with the abstract objects α^0 and α^1 rather than their matrix representations, such that in particular the dimension of the latter will not play a role. Hence, the notation ' ψ ' is abstract in the internal Hilbert space but 'concrete' in the position Hilbert space.

2.2. The naive discretization

We introduce the $(1 + 1)$ -dimensional spacetime lattice $\mathbb{Z} \times \mathbb{Z}$ and label its sites by the multi-index $n \equiv (j, p)$. Denoting the spacetime-lattice spacing by ϵ , we take j to label time, i.e. we set $t \equiv j\epsilon$, and p to label position, i.e. $x^1 \equiv p\epsilon$. We also define $x \equiv (t, x^1)$, which we use flexibly in the continuum as well as in the discrete, in which case $x \equiv \epsilon n$. Moreover, we write

$$\psi_n := \psi(\epsilon n) \quad (7)$$

for the field ψ evaluated at the lattice site n .

2.2.1. Lattice derivatives

A standard way [1] of discretizing equation (3) in space and time while preserving the Hermiticity of the operators $i\partial_\mu$, $\mu = 0, 1$, and thus that of the Hamiltonian, is to use *symmetric finite differences*, i.e. to perform the substitution

$$i\partial_\mu \longrightarrow id_\mu := \frac{i}{2\epsilon} (\mathcal{T}_\mu^{-1} - \mathcal{T}_\mu). \quad (8)$$

As above, ϵ is the spacetime-lattice spacing, and \mathcal{T}_μ is the translation operator in direction μ , i.e.

$$(\mathcal{T}_\mu \psi)_n = \psi_{n-\hat{\mu}}, \quad (9)$$

where $\hat{\mu}$ is the unit vector in direction μ , compare with reference [57, equation (33)]. Clearly, $\frac{i}{2\epsilon} (\mathcal{T}_\mu^{-1} - \mathcal{T}_\mu)$ is Hermitian because \mathcal{T}_μ is unitary. Moreover, in the continuum limit $\epsilon \rightarrow 0$ we have $d_\mu \rightarrow \partial_\mu$.

For later use we also introduce the left and right lattice derivatives

$$d_\mu^L := \frac{1}{\epsilon} (1 - \mathcal{T}_\mu) \quad (10a)$$

$$d_\mu^R := \frac{1}{\epsilon} (\mathcal{T}_\mu^{-1} - 1), \quad (10b)$$

such that $d_\mu = (d_\mu^L + d_\mu^R)/2$.

2.2.2. Standard-LGT scheme: naive fermions

Discretizing the Dirac equation, (3), with the symmetric finite differences yields

$$\frac{i}{2\epsilon} (\mathcal{T}_0^{-1} - \mathcal{T}_0) \psi = -\alpha^1 \frac{i}{2\epsilon} (\mathcal{T}_1^{-1} - \mathcal{T}_1) \psi + \alpha^0 m \psi. \quad (11)$$

This equation can be rewritten as

$$id_0 \psi = \mathcal{H}^{\text{LGT}} \psi, \quad (12)$$

a scheme which we call that of *naive fermions*, where the lattice Hamiltonian is

$$\mathcal{H}^{\text{LGT}} := \alpha^1 (-id_1) + m\alpha^0. \quad (13)$$

The associated action can be found in reference [1, section 4.1].

A comment must be made at this stage: the lattice Hamiltonian of equation (13) is usually introduced in Hamiltonian LGTs in which time is kept continuous [5, 69], and in those frameworks the role of the Hamiltonian is the usual one, that is, it generates the time evolution. Here, we introduce \mathcal{H}^{LGT} even though we are in discrete time, and we call it a ‘Hamiltonian’ even though it does not generate the time evolution in the usual sense: time evolution is described instead by the lattice EOM of equation (12).

2.3. The DQW discretization

The use of the symmetric lattice derivatives above ensures the Hermiticity of the lattice Hamiltonian \mathcal{H}^{LGT} . Thus, if time was not discretized and time evolution was generated by this lattice Hamiltonian, the scheme would be unitary. However, when also discretizing time, what about unitarity?

In textbook Lagrangian LGT, discrete-time formulations are usually carried out in Euclidean spacetime. In such a framework, unitarity of the model is proven essentially by proving the positivity of the transfer operator⁶ of the system, or alternatively by proving the so-called Osterwalder–Schrader reflection-positivity condition [3].

These properties are far from straightforward to establish: for example, the positivity of the transfer matrix for lattice fermions has only been proven for Wilson fermions with Wilson parameter $r = 1$, and there is no proof for naive fermions, which have $r = 0$ [3], i.e. there is no proof of the positivity of the transfer operator for the scheme of equation (12). Based on 2.3.1 and 2.3.2 below, we believe that this scheme is simply not unitary⁷.

⁶ A transfer operator is any operator \hat{T} which in the continuum limit coincides with $\exp(-\tau\hat{H})$, where \hat{H} is the Hamiltonian of the system, but which at the discrete level could differ from merely exponentiating \hat{H} . In the discrete, the exact form of \hat{T} is adjusted in order to facilitate the computation of the transition amplitudes $\langle n|\hat{T}|n'\rangle$. For example, for a point particle of mass m in a potential V , a possible suitable definition for the transfer operator is $T := \exp(-\tau V(\hat{x})/2) \exp(-\tau \hat{p}^2/(2m)) \exp(-\tau V(\hat{x})/2)$, where \hat{x} is the position operator, and \hat{p} is the momentum operator. The naive transfer operator is simply the Euclidean version, $\exp(-\tau\hat{H})$, of the one-step evolution operator $\exp(-it\hat{H})$.

⁷ That is to say, more precisely, there exists no underlying unitary one-step scheme that generates naive fermions, see below.

2.3.1. Modification of naive fermions for unitarity in discrete time

First, notice that the scheme of naive fermions, equation (12), is—because of the use of a symmetric lattice derivative for time—a *two-step* scheme, i.e. it needs two initial conditions $\psi_{j=0}$ and $\psi_{j=1}$. As in [68], let us perform in equation (12) the following substitution,

$$\alpha^\mu \longrightarrow \tilde{\alpha}^\mu := \mu_\epsilon \alpha^\mu, \quad \mu = 0, 1, \quad (14)$$

where the prefactor

$$\mu_\epsilon := \frac{1}{\sqrt{1 - (\epsilon m)^2}}, \quad (15)$$

is positive for ϵm small enough. This substitution (14) leads to a new two-step scheme which we call *unitary fermions*,

$$i\tilde{d}_0\psi = \mathcal{H}^{\text{DQW}}\psi, \quad (16)$$

where the lattice Hamiltonian is now

$$\mathcal{H}^{\text{DQW}} := \tilde{\alpha}^1(-id_1) + m\tilde{\alpha}^0. \quad (17)$$

We are going to see in the next section that this new two-step scheme, equation (16), is equivalent to a *unitary one-step* scheme,

$$\mathcal{T}_0^{-1}\psi_j \equiv \psi_{j+1} = \mathcal{W}\psi_j, \quad (18)$$

with properly chosen one-step unitary evolution operator \mathcal{W} and provided that the second initial condition $\psi_{j=1}$ is given precisely by this unitary one-step scheme, that is,

$$\psi_{j=1} = \mathcal{W}\psi_{j=0}. \quad (19)$$

The two schemes are equivalent in the usual sense: if ψ_j satisfies the one-step scheme, then it satisfies the two-step scheme, and vice versa. Note that this equivalence has already been proven in [68]. For the convenience of the reader we review it in the next subsection.

2.3.2. The ‘underlying’ unitary one-step scheme

Let us explicate how the two-step scheme of equation (16) can be obtained from a one-step scheme of the type of equation (18). Since the field ψ in equation (16) satisfies a first-order difference equation in position, we consider the unitary operator

$$\mathcal{W} := W_{-1}\mathcal{T}_1^{-1} + W_{+1}\mathcal{T}_1 + W_0\mathbb{1}. \quad (20)$$

We call the W_i , $i = -1, 0, 1$, *jump operators*. They are, in a basis of the internal Hilbert space, represented by matrices of the dimension of this Hilbert space, here 2×2 , with entries that are at this stage complex numbers⁸. In this sense, the jump operator W_i quantifies ‘how much’ of the wave function is jumping in direction i . Additionally, the set of jump operators has to satisfy some conditions that guarantee unitarity for the operator \mathcal{W} , see reference [68, appendix A], but apart from that they are arbitrary. The operator \mathcal{T}_1 is the translation operator defined in equation (9). We henceforth call \mathcal{W} a *walk operator* in accordance with the literature: it is a unitary evolution operator by one time step which is strictly local.

We define the following *transport operators*,

$$B_\pm := W_1 \pm W_{-1} \quad (21a)$$

$$M := \sum_{i=-1,0,+1} W_i = B_+ + W_0, \quad (21b)$$

which encode transport properties of the scheme and of its continuum limit, as we are going to see below.

⁸ In the general case, and in particular later when we U(1)-gauge the walk in section 4, the entries are functions of the position operator.

Under which conditions on B_+ , B_- and M , can we obtain the two-step scheme of equation (16) from the one-step scheme of equation (18)? Notice first that if ψ satisfies equation (18), then it also satisfies the following two-step scheme,

$$id_0\psi = \mathcal{H}_Q\psi, \quad (22)$$

where

$$\epsilon\mathcal{H}_Q := \frac{i}{2}(\mathcal{W} - \mathcal{W}^\dagger) \quad (23a)$$

$$= A^1(-i\epsilon d_1) + \frac{r}{2}Q(-\mathcal{L}) + \epsilon mA^0, \quad (23b)$$

where we have (a) introduced the lattice Laplacian

$$\mathcal{L} := \mathcal{T}_1^{-1} + \mathcal{T}_1 - 2, \quad (24)$$

(b) forced the appearance of the Wilson parameter r and the ‘mass’ m , and (c) introduced the following operators acting on the internal Hilbert space,

$$A^0 := \frac{i}{2\epsilon m}(M - M^\dagger) \quad (25a)$$

$$A^1 := \frac{1}{2}(B_- + B_-^\dagger) \quad (25b)$$

$$Q := -\frac{i}{2r}(B_+ - B_+^\dagger). \quad (25c)$$

Notice that the two-step local Hamiltonian of a DQW, defined in equation (23a), was already introduced in [70] before appearing in [68].

Now, for $\epsilon\mathcal{H}_Q$ to equal $\epsilon\mathcal{H}^{\text{DQW}}$, we can choose

$$A^0 = \tilde{\alpha}^0 \quad (26a)$$

$$A^1 = \tilde{\alpha}^1 \quad (26b)$$

$$Q = 0, \quad (26c)$$

with $\tilde{\alpha}^\mu$ from equation (14). For the first two equations of equations (26) to hold, we can choose

$$M = \mu_\epsilon(1 - i\epsilon m\alpha^0) \quad (27a)$$

$$B_- = \tilde{\alpha}^1, \quad (27b)$$

and for Q to vanish we must choose

$$B_+ = B_+^\dagger. \quad (28)$$

Note that this implies

$$W_1 - W_1^\dagger = -(W_{-1} - W_{-1}^\dagger). \quad (29)$$

It is easy to show that the choices of equations (27) are compatible with the unitarity constraints involving solely B_- and M that result from $\mathcal{W}^\dagger\mathcal{W} = 1 = \mathcal{W}\mathcal{W}^\dagger$, see [68, appendix A]. A choice for B_+ that satisfies equation (28) and that is compatible with all unitarity constraints involving B_+ [68, appendix A] is ‘simply’

$$B_+ = \mu_\epsilon. \quad (30)$$

In the end, we have found a unitary one-step scheme, namely, equation (18) with the choices of equations (27) and (30), which generates the two-step scheme of *unitary fermions*.

Let us as a sum-up explicitly write this scheme of *one-step unitary fermions*:

$$\psi_{j+1} = \mathcal{W}_{\text{Dirac}}\psi_j, \quad (31)$$

with

$$\mathcal{W}_{\text{Dirac}} := \mu_\epsilon \left[\frac{1}{2}(1 - \alpha^1)\mathcal{T}^{-1} + \frac{1}{2}(1 + \alpha^1)\mathcal{T} - i\epsilon m\alpha^0 \right]. \quad (32)$$

2.4. The lattice action of the matter field

The only difference between naive fermions and unitary fermions is the prefactor μ_ϵ appearing in the modified α operators and hence also in the following modified γ operators,

$$\tilde{\gamma}^0 := (\tilde{\alpha}^0)^{-1} = \frac{\alpha^0}{\mu_\epsilon} = \frac{\gamma^0}{\mu_\epsilon} \quad (33a)$$

$$\tilde{\gamma}^1 := (\tilde{\alpha}^0)^{-1} \tilde{\alpha}^1 = \gamma^1. \quad (33b)$$

It turns out that one can obtain a valid lattice action for unitary fermions by performing the substitution $\gamma^\mu \rightarrow \tilde{\gamma}^\mu$ in the, e.g. *asymmetric* lattice action of naive fermions. This results in

$$S_{\text{DQW}}^{\text{asym.}} := \epsilon^2 \sum_n \bar{\psi}_n [(i\tilde{\gamma}^\mu d_\mu - m) \psi]_n, \quad (34)$$

with

$$\bar{\psi}_n := \psi_n^\dagger (\tilde{\gamma}^0)^{-1}. \quad (35)$$

In appendix A, we show how extremalizing this action, equation (34), indeed yields the correct EOM of unitary fermions, equation (16).

We call the action in equation (34) ‘asymmetric’, since the lattice derivative only acts to the right. As a consequence, it is in general not a real but a complex number. The variational problem is usually conceived for a real action. Yet, as we show in appendix B, this asymmetric action $S_{\text{DQW}}^{\text{asym.}}$ is equal (up to boundary terms) to the following *symmetric* and therefore real-valued action,

$$S_{\text{DQW}} := \epsilon \frac{i}{2} \sum_n \bar{\psi}_n \tilde{\gamma}^\mu \psi_{n+\hat{\mu}} + \text{H. c.} - \epsilon \sum_n \epsilon m \bar{\psi}_n \psi_n. \quad (36)$$

One can show that extremalizing the symmetric action in equation (36) yields equation (16): this is proven in a more general case in appendix E, where we obtain the Euler–Lagrange equations from a generic real action. That being said, note that boundary terms need not be taken into account in variational problems that determine the EOMs⁹; hence, since the symmetric and the asymmetric actions only differ by boundary terms, a proof with one of the two (for the type of variational problem mentioned) yields the corresponding result for the other one.

3. Noether’s theorem in discrete spacetime for internal symmetries

In section 3, we derive a lattice Noether’s theorem for internal symmetries. By ‘internal symmetry’ we mean, as usual, that the corresponding transformation acts only on the internal Hilbert space of the system, and does not affect the spacetime coordinates.

3.1. The framework

Consider an action being the sum, over the lattice sites, of a Lagrangian density which is function of (a) the fields ψ^a , $a = 1, 2$, and (b) their shifts in time and space, that is,

$$S_{\text{F}} := \sum_n \mathcal{L}(\psi_n, \psi_{n+\hat{\mu}}, \psi_n^\dagger, \psi_{n+\hat{\mu}}^\dagger), \quad (37)$$

where ‘F’ stands for ‘fermionic’ and where we consider, for the sake of correctness, a symmetric and therefore real-valued action¹⁰. Let us be fully precise on our notations: here and below, $\phi_n = \psi_n, \psi_n^\dagger$ subsumes the family $(\phi_n^a)_a$ and, analogously, $\phi_{n+\hat{\mu}}^a$ subsumes the family $(\phi_{n+\hat{\mu}}^a)_\mu$.

The lattice Noether’s theorem that we are going to derive is inspired by the general one for usual continuum field theories in [71]. Consider an arbitrary transformation, $\psi_n^a \rightarrow (\psi_n^a)'$:= $f^a((\psi_n^b)_b, \alpha) \equiv f^a(\psi_n, \alpha)$, of the field ψ , acting solely on its internal Hilbert space, where α denotes a family of real parameters¹¹. In the present work, we will only consider the case of a global U(1) transformation, for

⁹ This is also true in the continuum. However, boundary terms *do* need to be taken into account in variational problems of the type of Noether’s theorem, see section 3 and appendix D for present discrete setting.

¹⁰ That being said, it turns out that in the concrete case $S_{\text{F}} = S_{\text{DQW}}$ all the computations that we are going to carry out can be carried out with $S_{\text{DQW}}^{\text{asym.}}$ instead of S_{DQW} and still deliver the same results, at least in the current case of the U(1)-charge current. This observation is also a feature of the continuum theory. Notice that when two actions which differ by boundary terms are real-valued, the Noether currents are in general different.

¹¹ To consider transformations acting also on the external Hilbert space, one would have to supplement the transformation in equation (38) by some coordinate transformation, see [71].

which α reduces to a single real parameter. The generalization to a larger family of α 's poses no major difficulty, but it renders the derivation more cumbersome and blurs its important aspects. Although the single application we know for α being a single real parameter is that of a global U(1) symmetry, we still present the proof for a general $f^a(\psi_n, \alpha)$ since this renders its generalization to a larger family of α 's easier. The transformed state is collectively given by

$$\psi'_n := f(\psi_n, \alpha). \quad (38)$$

We parametrize the transformation f such that

$$f(\psi_n, \alpha = 0) = \psi_n. \quad (39)$$

Finally, we assume f to be differentiable and expand it to first order in the small parameter $\delta\alpha$ as

$$\psi'_n = f(\psi_n, \delta\alpha) = f(\psi_n, 0) + \left. \frac{\partial f}{\partial \alpha} \right|_{(\psi_n, 0)} \delta\alpha + O(\delta\alpha^2). \quad (40)$$

Taking into account equation (39) and omitting higher-order terms this gives

$$\psi'_n = \psi_n + C_n \delta\alpha, \quad (41)$$

where

$$C_n := \left. \frac{\partial f}{\partial \alpha} \right|_{(\psi_n, 0)}. \quad (42)$$

In appendix C, we show how extremalizing the action in equation (37) yields an Euler–Lagrange equation for ψ .

3.2. The Noether theorem

The precise statement of our lattice Noether's theorem for internal symmetries is the following: if the generic action S_F of equation (37) is invariant under the transformation in equation (41), i.e. if equation (41) is a(n) (internal) symmetry of the action, then the Noether current J associated to the internal symmetry, and defined component-wise at site n by

$$J_n^\mu := \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_n C_{n+\hat{\mu}} + C_{n+\hat{\mu}}^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_n, \quad (43)$$

$\mu = 0, 1$, is conserved on the lattice (for a field ψ that satisfies the Euler–Lagrange equation). That is, the lattice (one-step) $(1+1)$ -divergence of J vanishes, i.e.

$$d_\mu^L J^\mu = 0, \quad (44)$$

where d_μ^L is the left lattice derivative defined in equation (10a). We prove this theorem in appendix D.

3.3. Example: U(1) symmetry and charge conservation

Let us apply the general lattice Noether's theorem for internal symmetries of the preceding subsection, to the global U(1) symmetry of the (symmetric) DQW action, equation (36). The corresponding transformation is

$$f(\psi_n, \alpha) := e^{i\alpha} \psi_n, \quad (45)$$

so that the C_n in equation (41) is

$$C_n = i\psi_n. \quad (46)$$

The computation of the Noether current of equation (43), which we call in the present case U(1)-charge current, delivers here, denoting it by $-J_{U(1)}$ ¹²,

$$(J_{U(1)}^\mu)_n := \frac{\epsilon}{2} \left(\bar{\psi}_n \tilde{\gamma}^\mu \psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}} \tilde{\gamma}^\mu \psi_n \right), \quad (47)$$

¹² The minus sign is to match with the most-used notations in the continuum limit.

where the $\tilde{\gamma}^\mu$'s have been defined in equations (33), and $\bar{\psi}_n$ in equation (35). The continuum limit of $(J_{U(1)}^\mu)_n$ is trivially the well-known Dirac charge current (divide equation (47) by ϵ and let $\epsilon \rightarrow 0$),

$$J_{\text{Dirac}}^\mu(x) := \bar{\psi}(x)\gamma^\mu\psi(x), \quad (48)$$

where, as usual, $\bar{\psi} := \psi^\dagger\gamma^0$.

In appendix E we show how the lattice conservation equation satisfied by the U(1)-charge current in virtue of our lattice Noether's theorem, namely,

$$d_\mu^L J_{U(1)}^\mu = 0, \quad (49)$$

can be obtained, as in the continuum, from the EOM, either the one-step or directly the two-step.

4. The U(1)-gauged matter-field action and EOMs

In section 4, we are going to modify the matter-field action, equation (36), in order to account for a coupling of the matter field ψ to an Abelian U(1) lattice gauge field A which is the gauge field of the continuum evaluated on the lattice. This gauge field is external for now, but it will become dynamical in the section 5.

4.1. The gauging procedure

The usual gauging procedure for lattice systems is well-known for LGTs [1] and for DQWs [57]: it consists in performing the following lattice *minimal-coupling substitutions*,

$$\mathcal{T}_\mu \longrightarrow \mathcal{T}'_\mu := \mathcal{T}_\mu e^{-iq\epsilon A_\mu}, \quad (50)$$

where q is the charge of the matter field ψ , and $A := (A_\mu)_{\mu=0,1}$ is the spacetime-dependent U(1) gauge field that we couple ψ to. Applying the modified translation operators \mathcal{T}'_μ to the matter field ψ at n yields

$$(\mathcal{T}_\mu e^{-iq\epsilon A_\mu} \psi)_n = e^{-iq\epsilon(A_\mu)_{n-\hat{\mu}}} \psi_{n-\hat{\mu}}. \quad (51)$$

The inverse of the modified translation operators,

$$(\mathcal{T}'_\mu)^\dagger = e^{iq\epsilon A_\mu} \mathcal{T}_\mu^{-1}, \quad (52)$$

accordingly act as

$$(e^{iq\epsilon A_\mu} \mathcal{T}_\mu^{-1} \psi)_n = e^{iq\epsilon(A_\mu)_n} \psi_{n+\hat{\mu}}. \quad (53)$$

4.2. The gauged action and two-step EOM

To couple ψ to the gauge field A_μ , we first rewrite the matter-field action, equation (36), as

$$S_{\text{DQW}} = \epsilon \frac{i}{2} \sum_n \bar{\psi}_n \tilde{\gamma}^\mu (\mathcal{T}_\mu^{-1} \psi)_n + \text{H.c.} - \epsilon \sum_n \epsilon m \bar{\psi}_n \psi_n. \quad (54)$$

Performing the substitution (52) (and (50) in the Hermitian-conjugate term) on this action, we end up with the 'gauged' action,

$$S_{\text{DQW}}^g = \epsilon \frac{i}{2} \sum_n \bar{\psi}_n \tilde{\gamma}^\mu e^{iq\epsilon(A_\mu)_n} \psi_{n+\hat{\mu}} + \text{H.c.} - \epsilon \sum_n \epsilon m \bar{\psi}_n \psi_n. \quad (55)$$

This gauged action is exactly that of standard LGT, see reference [1, chapter 5], up to substituting $\gamma^\mu \longrightarrow \tilde{\gamma}^\mu$ and $\bar{\psi} \longrightarrow \bar{\psi}$. It is invariant under the following gauge transformation,

$$\psi_n \longrightarrow \psi'_n := G_n \psi_n := e^{iq\varphi_n} \psi_n \quad (56a)$$

$$(A_\mu)_n \longrightarrow (A'_\mu)_n := (A_\mu)_n - d_\mu^R \varphi|_n, \quad (56b)$$

where d_μ^R is the right lattice derivative defined in equation (10b), and φ_n is an arbitrary spacetime-dependent field.

Writing the Euler–Lagrange equation,

$$\left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_{n-\hat{\mu}} = 0, \quad (57)$$

derived in appendix C, for the action $S_F = S_{\text{DQW}}^g$, results in the following two-step EOM which we call *U(1)-gauged unitary fermions*,

$$\tilde{\gamma}^\mu \frac{i}{2} \left(e^{iq\epsilon(A_\mu)_n} \psi_{n+\hat{\mu}} - e^{-iq\epsilon(A_\mu)_{n-\hat{\mu}}} \psi_{n-\hat{\mu}} \right) - \epsilon m \psi_n = 0, \quad (58)$$

which can easily be shown to be exactly the EOM that we obtain if we directly gauge the EOM of unitary fermions, equation (16). This EOM is invariant under the Abelian U(1) gauge transformation (56).

4.3. About gauging the one-step EOM

In equation (36) we have considered an action that delivers the *two-step* scheme, equation (16). When coupling the matter field to a gauge field, it is therefore natural to apply the gauging procedure (50) and (52) to the two-step scheme, as done in the previous subsection. Yet, the unitarity of our model relies on the fact that—without gauge fields—we can find a unitary one-step scheme, equation (31), that generates the two-step scheme. This immediately leads to the question whether there is any unitary one-step scheme that generates the *gauged* two-step scheme. A natural follow-up question is then: is the gauged two-step scheme (58) generated by the gauged version of the one-step scheme? Equivalently, do the operations ‘generating a two-step scheme’ and ‘gauging’ commute? We are going to answer both questions in the affirmative, but only under a certain condition on the gauge field.

4.3.1. Gauging the unitary one-step scheme

The gauging of one-step discrete-spacetime systems is described in [57], and is in essence the same as that known in LGT: it consists in substituting translation operators as in (50) and (52). Thus, gauging the one-step EOM of equation (18) gives

$$e^{iq\epsilon(A_0)_j} \psi_{j+1} = (\mathcal{W}_g)_j \psi_j, \quad (59)$$

where $(\mathcal{W}_g)_j$ is the gauged walk operator of equation (20) evaluated at time j , i.e.

$$(\mathcal{W}_g)_j := W_{-1} e^{iq\epsilon(A_1)_j} \mathcal{T}_1^{-1} + W_1 \mathcal{T}_1 e^{-iq\epsilon(A_1)_j} + W_0 \mathbb{1}. \quad (60)$$

Moreover, the appearance of the gauge field on the left-hand side of equation (59) stems from gauging \mathcal{T}_0^{-1} in equation (18). One can verify that the unitarity conditions on $e^{-iq\epsilon(A_0)_j} (\mathcal{W}_g)_j$ are the same as those for the ungauged scheme (see reference [68, appendix A]).

4.3.2. Recovering the gauged two-step scheme

Multiplying equation (59) on the left by $(\mathcal{W}_g^\dagger)_j \equiv ((\mathcal{W}_g)_j)^\dagger$ and shifting indices as $j \rightarrow j-1$, we obtain

$$\psi_{j-1} = (\mathcal{W}_g^\dagger)_{j-1} e^{iq\epsilon(A_0)_{j-1}} \psi_j. \quad (61)$$

Now, multiplying on the left by $e^{-iq\epsilon(A_0)_{j-1}}$, we obtain

$$e^{-iq\epsilon(A_0)_{j-1}} \psi_{j-1} = e^{-iq\epsilon(A_0)_{j-1}} (\mathcal{W}_g^\dagger)_{j-1} e^{iq\epsilon(A_0)_{j-1}} \psi_j. \quad (62)$$

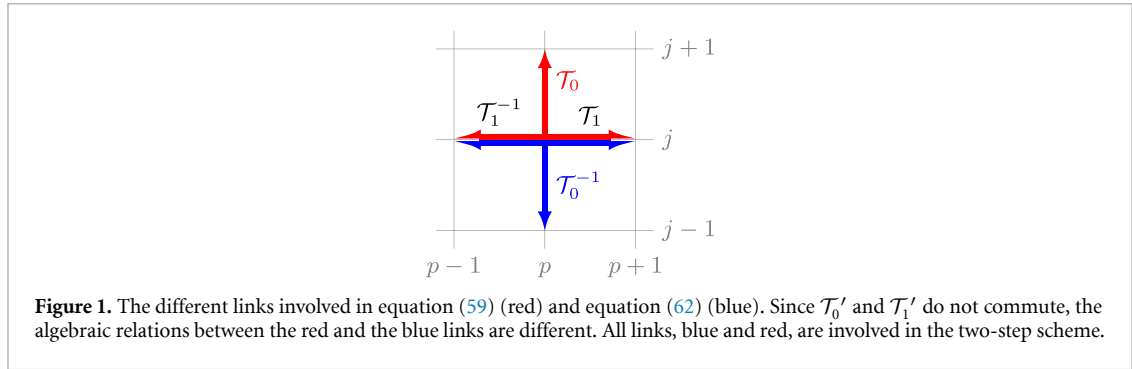
Note that this is different from simply evolving ψ backwards in time by the inverse of \mathcal{W}_g , i.e. in general $e^{-iq\epsilon(A_0)_{j-1}} (\mathcal{W}_g^\dagger)_{j-1} e^{iq\epsilon(A_0)_{j-1}} \neq (\mathcal{W}_g^\dagger)_{j-1}$. This is due to the presence of the gauge fields, i.e. the fact that \mathcal{T}_0' and \mathcal{T}_1' do not commute, see also figure 1.

Subtracting equation (62) from equation (59) and multiplying by $i/2$ yields the EOM

$$\begin{aligned} & \frac{i}{2} \left(e^{iq\epsilon(A_0)_j} \psi_{j+1} - e^{-iq\epsilon(A_0)_{j-1}} \psi_{j-1} \right) \\ &= \frac{i}{2} \left((\mathcal{W}_g)_j - e^{-iq\epsilon(A_0)_{j-1}} (\mathcal{W}_g^\dagger)_{j-1} e^{iq\epsilon(A_0)_{j-1}} \right) \psi_j, \end{aligned} \quad (63)$$

which is invariant under the U(1) gauge transformation (56).

The left-hand side of equation (63) is indeed the gauged version of the left-hand side of the (generic) two-step scheme, equation (22). However, the right-hand side is *not* the gauged version of the lattice Hamiltonian operator defined directly at the two-step level, i.e. $(\mathcal{H}_g)_j := \frac{i}{2} [(\mathcal{W}_g)_j - ((\mathcal{W}_g^\dagger)_g)_{j-1}]$, because in general $e^{-iq\epsilon(A_0)_{j-1}} (\mathcal{W}_g^\dagger)_{j-1} e^{iq\epsilon(A_0)_{j-1}} \neq ((\mathcal{W}_g^\dagger)_g)_{j-1}$. So, the lesson to take away is that applying the gauging procedure to a one-step scheme and then deducing from it a two-step scheme, equation (63), does not lead to the same two-step scheme as that obtained from applying the gauging procedure directly to the two-step scheme, equation (58).



What we can say is that $((\mathcal{W}^\dagger)_g)_j = (\mathcal{W}^\dagger)_j$, which holds even in the non-Abelian case (this is easy to show). Given the previous identity, a sufficient condition for the gauging of the one-step scheme to produce as a two-step scheme the directly gauged version of the two-step scheme, i.e. a sufficient condition to have $e^{-iq\epsilon(A_0)_{j-1}}(\mathcal{W}^\dagger)_{j-1}e^{iq\epsilon(A_0)_{j-1}} = (\mathcal{W}^\dagger)_{j-1}$, is that $(A_0)_n$ be independent of the spatial position; one may also take the well-known, stronger condition $A_0 = 0$, known as the temporal gauge.

5. The action of the gauge field

In this section 5, we consider a (3 + 1)-dimensional spacetime unless otherwise mentioned.

5.1. Link variables

In section 4 above, we introduced a lattice gauge field $(A_\mu)_n$ that appears only in the form of the exponential

$$(U_\mu)_n := e^{iq\epsilon(A_\mu)_n}. \tag{64}$$

As in the continuum, the gauge field $(A_\mu)_n$ depends on a lattice site n and a direction μ . On the lattice, this can be rephrased as a dependence on two neighbouring lattice sites n and $n + \hat{\mu}$. As in the LGT literature, we therefore call $(U_\mu)_n$ a *link variable*. This link variable is conventionally associated with a hopping from n to $n + \hat{\mu}$, and we stick to this convention here. Accordingly, we denote the link variable by $U_{n,n+\hat{\mu}}$ [1], and its inverse by $U_{n+\hat{\mu},n} := (U^\dagger_\mu)_n$.

5.2. Lattice field strength

The gauge transformation of the gauge field, equation (56b), reads, at the level of the link variable,

$$(U_\mu)_n \longrightarrow (U'_\mu)_n := G_n(U_\mu)_n G_{n+\hat{\mu}}^{-1}, \tag{65}$$

where $G_n := e^{iq\varphi_n}$ as in the transformation (56a). The gauge transformation of the adjoint link variable is

$$\left(U^\dagger_\mu \right)_n \longrightarrow \left(U'^\dagger_\mu \right)_n := G_{n+\hat{\mu}} \left(U^\dagger_\mu \right)_n G_n^{-1}. \tag{66}$$

In the continuum theory, the gauge-field action, that determines the dynamics of the gauge field, is constructed from the so-called field strength $F_{\mu\nu}$. In the Abelian case, $F_{\mu\nu}$ is given by

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{67}$$

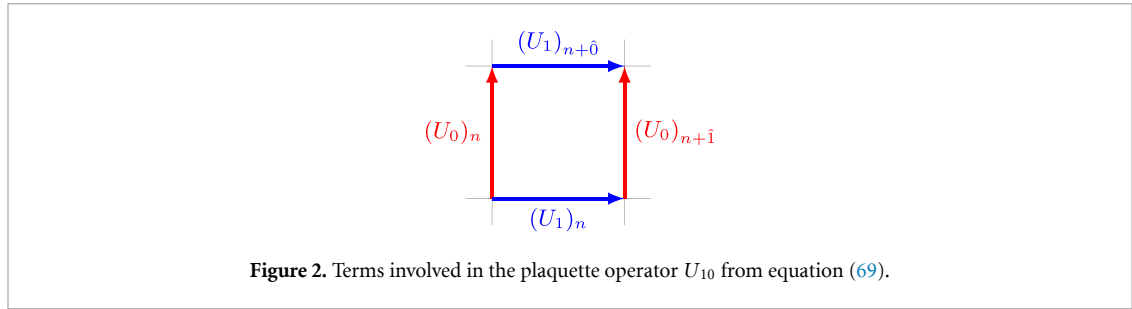
It is gauge invariant and determines the continuum gauge-field action S_G^{cont} . via

$$S_G^{\text{cont}} := -\frac{1}{4} \int d^4x F_{\mu\nu}(x) F^{\mu\nu}(x). \tag{68}$$

To build an action for the gauge field in discrete spacetime, we follow standard LGT and also [57]. Multiplying the link variables along the smallest possible path on the lattice yields the so-called *plaquette* operator,

$$\left(U_{\mu\nu} \right)_n := (U_\mu)_n (U_\nu)_{n+\hat{\mu}} (U^\dagger_\mu)_{n+\hat{\nu}} (U^\dagger_\nu)_n. \tag{69}$$

When viewing the link variable as a directed quantity the plaquette operator reads $(U_{\mu\nu})_n = U_{n,n+\hat{\mu}} U_{n+\hat{\mu},n+\hat{\mu}+\hat{\nu}} U_{n+\hat{\mu}+\hat{\nu},n+\hat{\nu}} U_{n+\hat{\nu},n}$, where the indices follow a path which is an elementary square in the $\mu\nu$ plane on the lattice, see figure 2.



This plaquette operator is in general gauge covariant, i.e. its transformation law under a change of gauge is

$$(U_{\mu\nu})_n \longrightarrow (U'_{\mu\nu})_n = G_n(U_{\mu\nu})_n G_n^{-1}. \tag{70}$$

In the Abelian case this implies that $U_{\mu\nu}$ is gauge invariant, i.e.

$$(U'_{\mu\nu})_n = (U_{\mu\nu})_n. \tag{71}$$

Inserting the definition of the link variables, equation (64), into equation (69), we can express the Abelian plaquette operator also as

$$(U_{\mu\nu})_n = e^{iq\epsilon^2(F_{\mu\nu})_n}, \tag{72}$$

where $(F_{\mu\nu})_n$ is the lattice field strength

$$(F_{\mu\nu})_n = d_\mu^R A_\nu|_n - d_\nu^R A_\mu|_n, \tag{73}$$

where we recall that d_μ^R is the right lattice derivative, defined in equation (10b).

5.3. Lattice gauge-field action

We now have to construct from the lattice gauge-invariant quantity $(U_{\mu\nu})_n$ an action that has the correct continuum limit: we show in appendix F that the lattice action

$$S_G := S_G^{\text{time}} + S_G^{\text{space}}, \tag{74}$$

where

$$S_G^{\text{time}} := \frac{1}{q^2} \sum_n \sum_l \left[1 - \frac{1}{2} [(U_{0l})_n + (U_{0l})_n^\dagger] \right] \tag{75a}$$

$$S_G^{\text{space}} := \frac{1}{q^2} \sum_n \sum_{\substack{k,l \\ k < l}} \left[\frac{1}{2} [(U_{kl})_n + (U_{kl})_n^\dagger] - 1 \right], \tag{75a}$$

has the correct continuum limit S_G^{cont} [6], which can be seen from expanding S_G in ϵ and rewriting it as

$$S_G = -\frac{1}{4} \sum_n \epsilon^4 (F_{\mu\nu})_n (F^{\mu\nu})_n + O(\epsilon^6). \tag{76}$$

In 1 + 1 dimensions, S_G^{space} trivially vanishes. Note that S_G^{time} is a purely electric term, i.e. it involves only the lattice electric field [72, 73] with components $(F_{0l})_n$, while S_G^{space} is a purely magnetic term, i.e. it involves only the lattice magnetic field with components $(F_{kl})_n$. Note that the suggested action, equation (74), is nothing but a real-time version of the Euclidean one in equation (5.21) of [1].

5.4. The classical lattice dynamics of the gauge field

In this section 5.4, we consider S_G in a $(1 + 1)$ -dimensional spacetime, so that in equations (75) one must replace $1/q^2$ by $1/(q^2\epsilon^2)$. Still, we continue to write the magnetic terms in sections 5.4.2 and 5.4.3 as if we were in $3 + 1$ dimensions, in order to make the equations look more general. For these equations to actually make sense in more than $1 + 1$ dimensions one would have to find a $(3 + 1)$ -dimensional version of S_{DQW}^g , which would give sense to a $(3 + 1)$ -dimensional $U(1)$ -charge current $(J_{U(1)}^g)_n^\mu$ below; this has not been done in the present paper. Alternatively, one can assume to have found a $(3 + 1)$ -dimensional version of S_{DQW}^g , which gives sense to a $(3 + 1)$ -dimensional $U(1)$ -charge current $(J_{U(1)}^g)_n^\mu$ below, and then consider a $(3 + 1)$ -dimensional version of S_G ; note however that this leads to additional ϵ factors in the right-hand sides of equations (81) and (82).

5.4.1. Lattice Euler–Lagrange equations for the gauge field

Consider the total action

$$S := S_{\text{DQW}}^g + S_G, \tag{77}$$

where S_{DQW}^g is given in equation (55) and S_G in equation (74). We may be tempted to consider S as a function of the link variables (and their translates) rather than of the $(A_\mu)_n$'s (and their discrete derivatives), but then we would also have to take into account the adjoints of the link variables. If, instead, we take S as a function of the $(A_\mu)_n$'s no such question arises. Hence, we consider S as the following action functional,

$$S = \sum_n \mathcal{L} \left(((A_\mu)_n)_{\mu=0,1}, (d_\nu^R A_\mu|_n)_{\mu,\nu=0,1} \right). \tag{78}$$

Extremalizing this action, equation (78), leads to the following Euler–Lagrange EOMs for the gauge field A_μ ,

$$\left. \frac{\partial \mathcal{L}}{\partial (A_\mu)_n} \right|_n - d_\nu^L \left. \frac{\partial \mathcal{L}}{\partial d_\nu^R A_\mu|_n} \right|_n = 0, \tag{79}$$

where we recall that d_μ^L and d_μ^R are the left and right lattice derivatives, defined in equations (10).

5.4.2. Evaluating the lattice Euler–Lagrange equations

The first term of equation (79) is easily computed, and yields

$$\left. \frac{\partial \mathcal{L}}{\partial (A_\mu)_n} \right|_n = -q\epsilon (J_{U(1)}^g)_n^\mu, \tag{80}$$

where

$$(J_{U(1)}^g)_n^\mu := \frac{\epsilon}{2} \left(\bar{\psi}_n \tilde{\gamma}^\mu e^{iq\epsilon(A_\mu)_n} \psi_{n+\hat{\mu}} + \bar{\psi}_{n+\hat{\mu}} e^{-iq\epsilon(A_\mu)_n} \tilde{\gamma}^\mu \psi_n \right), \tag{81}$$

which coincides with the gauged version of the Noether $U(1)$ -charge current given in equation (47).

To compute the second term of the Euler–Lagrange equations, we need to distinguish between $\mu = 0$ and $\mu = l \neq 0$. After a few lines of computation for $\mu = 0$, remembering that $F_{\nu 0} = -F^{\nu 0}$, another few lines for $\nu = 0$, and another few lines for $\nu = k$ and $\mu = l$, remembering that $F_{kl} = F^{kl}$, we combine all three formulae into

$$- d_\nu^L \left. \frac{\partial \mathcal{L}}{\partial d_\nu^R A_\mu|_n} \right|_n = d_\nu^L \left[\frac{1}{q} \sin(q\epsilon^2 F_n^{\nu\mu}) \right]. \tag{82}$$

Inserting equations (80) and (82) into equation (79), we obtain the following EOMs for the gauge field,

$$d_\nu^L \left[\frac{1}{q} \sin(q\epsilon^2 F_n^{\nu\mu}) \right] = q\epsilon (J_{U(1)}^g)_n^\mu. \tag{83}$$

5.4.3. The two inhomogeneous lattice Maxwell equations

Let us show that the EOMs for the gauge field, equation (83), correspond to the two inhomogeneous lattice Maxwell equations. For $\mu = 0$, equation (83) gives

$$d_k^l \left[\frac{1}{q} \sin(q\epsilon^2 E_n^k) \right] = q\epsilon (J_{\text{DQW}}^g)_n^0, \quad (84)$$

where

$$E_n^k := F_n^{k0}, \quad (85)$$

is the lattice electric field. Equation (84) is a lattice version of Maxwell–Gauss' equation, which is a constraint rather than a dynamical equation: dividing by ϵ^2 on both sides and taking the limit $\epsilon \rightarrow 0$ indeed yields

$$\partial_k E^k = qJ_{\text{Dirac}}^0. \quad (86)$$

For $\mu = l$, equation (83) becomes

$$d_0^l \left[\frac{1}{q} \sin(q\epsilon^2 F_n^{0l}) \right] + d_k^l \left[\frac{1}{q} \sin(q\epsilon^2 F_n^{kl}) \right] = q\epsilon (J_{\text{DQW}}^g)_n^l, \quad (87)$$

that is, taking into account

$$F_n^{kl} \equiv -\varepsilon^{kl} B_n^m, \quad (88)$$

where B_n^m is the lattice magnetic field, and swapping k and l in ε^{kl} (so that we pick up a minus sign),

$$-d_0^l \left[\frac{1}{q} \sin(q\epsilon^2 E_n^l) \right] + \varepsilon^{lk} d_k^l \left[\frac{1}{q} \sin(q\epsilon^2 B_n^m) \right] = q\epsilon (J_{\text{DQW}}^g)_n^l. \quad (89)$$

This is a lattice version of Maxwell–Ampère's equation, which can be seen by taking in equation (89) the limit $\epsilon \rightarrow 0$ after having divided by ϵ^2 on both sides, which yields

$$-\partial_0 E^l + \varepsilon^{lk} \partial_k B^m = qJ_{\text{Dirac}}^l. \quad (90)$$

The convergence of these two lattice Maxwell equations, equations (84) and (89), has been proven in [74].

5.4.4. The two inhomogeneous lattice Maxwell equations in (1 + 1)-dimensional spacetime

In (1 + 1)-dimensional spacetime, the lattice Maxwell–Gauss constraint, equation (84), reduces to

$$d_1^l \left[\frac{1}{q} \sin(q\epsilon^2 E_n^1) \right] = q\epsilon (J_{\text{DQW}}^g)_n^0, \quad (91)$$

with continuum limit

$$\partial_1 E^1 = qJ_{\text{Dirac}}^0, \quad (92)$$

and the lattice Maxwell–Ampère equation, equation (89), reduces to

$$-d_0^l \left[\frac{1}{q} \sin(q\epsilon^2 (E^1)_n) \right] = q\epsilon (J_{\text{DQW}}^g)_n^1, \quad (93)$$

with continuum limit

$$-\partial_0 E^1 = qJ_{\text{Dirac}}^1. \quad (94)$$

Finally, the homogeneous Maxwell equations, namely, Maxwell–Thompson's and Maxwell–Faraday's equations, are not relevant in one spatial dimension.

6. Conclusions and discussion

Let us sum up the main achievements of this work. First, we have constructed a discrete-spacetime action S_{DQW} for a spin-1/2 matter field, with the following properties: it is (a) in real time, (b) based on a two-step EOM and therefore extremely similar to usual actions of LGT, (c) associated to a unitary (classical-fields) one-step scheme, i.e. a DQW, which in the continuum limit yields the Dirac equation. Second, we have derived a lattice Noether's theorem for internal symmetries of S_{DQW} . More precisely, we have proven the theorem for an internal symmetry depending on a single real parameter α , but believe that the generalization to a larger parameter family poses no major difficulty. We have applied the Noether's theorem to the global U(1) symmetry of S_{DQW} and have obtained a conserved current which in the continuum coincides with the usual Dirac charge current. Third, we have coupled S_{DQW} via a minimal coupling on the lattice to an Abelian U(1) gauge field. Although S_{DQW} is based on a two-step EOM, unitarity relies on the fact that this two-step EOM is associated to a one-step EOM. We have thus explored the gauging of the one-step EOM as in [57]. This has led us to the observation that gauging directly the two-step EOM does not yield the same EOM as that obtained by first gauging the one-step EOM and then constructing a two-step EOM from it; the two procedures are equivalent only if the temporal component of the gauge field, A_0 , is independent of space. Finally, we have suggested a real-time LGT-type action for the Abelian U(1) gauge field, from which we have derived the classical EOMs of the gauge field, which are lattice versions of Maxwell's equations.

A first question that remains unanswered is the following: Is it true that there exist no underlying unitary scheme for naive fermions? While we believe this to be the case, we do not have a proof. A second topic that is unaddressed in this work is that of external (i.e. spacetime) symmetries of the action: Can one define such symmetries and derive associated Noether's theorems? references [75, 76] partially address this problem in the realm of DQWs. Also, we did not address quantized fields: What multi-particle concepts from the QCA LGT in references [43–45, 47] should we import into our Lagrangian framework in order to build a fully fledged action-based LGT that respects strict locality and describes fermionic matter fields?

Finally, the issue of fermion doubling has not been addressed, because it is still unsolved for the following reasons. In [68] a fermion-doubling issue is solved for the two-step 'Hamiltonian'. But, although such a two-step Hamiltonian would indeed be subject to a fermion-doubling issue if the scheme was in continuous time, in [68] time is discrete. In addition, the two-step scheme is generated by a one-step scheme which does *not* exhibit a fermion-doubling issue [1]. One should hence examine whether the fact that the two-step Hamiltonian of [68] has a fermion-doubling issue is actually relevant if time is discrete and, moreover, if the generating one-step scheme does not have this issue. Now, all that being said, in the present framework the matter-field action S_{DQW} , that is defined with the two-step Hamiltonian, is extremely similar to usual LGT fermionic actions [3], which suggests that a fermion-doubling issue *does* arise. To sum up: if S_{DQW} does not exhibit fermion doubling, then there is nothing to be done, and if it does, then modifying the two-step Hamiltonian as in [68] should remove at least spatial doublers. Finally, if there are temporal doublers in S_{DQW} , then some fix must be made in order to remove them without breaking the unitarity of the model.

Data availability statement

No new data were created or analysed in this study.

Acknowledgment

C Cedzich was supported in part by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under the grant number 441423094. The authors thank Pablo Arrighi for his professional support.

Appendix A. Variational principle for unitary fermions

For brevity, in this appendix we write the Hamiltonian of equation (17) as

$$\mathcal{H}^{\text{DQW}} = \mathcal{H}. \quad (\text{A1})$$

For the sake of pedagogy, we are going to consider here the variational problem for the EOMs, with the non-real action, i.e. that of equation (34), because with it one gets a good feeling of what is going on, but below in appendix C we will also consider this variational problem on the generic real action of equation (37) (a particular case of which is the real action of equation (36)).

The action of equation (34) can be rewritten in the following Hamiltonian form,

$$S_{\text{DQW}}^{\text{asym.}} = \epsilon \sum_j \langle \psi_j | id_0 \psi |_j - \mathcal{H} \psi_j \rangle \quad (\text{A2a})$$

$$\equiv \epsilon^2 \sum_j \sum_p \psi_{j,p}^\dagger \left[id_0 \psi |_{j,p} - (\mathcal{H} \psi_j)_p \right], \quad (\text{A2b})$$

where d_0 is the symmetric lattice derivative defined in equation (8), and where we have introduced the Hermitian product

$$\langle \psi_j | \phi_j \rangle := \epsilon \sum_p \psi_{j,p}^\dagger \phi_{j,p}. \quad (\text{A3})$$

We remark that here and below \mathcal{H} could depend on time, but we omit the corresponding index j to keep the notation light.

The action equation (A2a) is a functional of the following real independent fields, $\text{Re}(\psi^a)$ and $\text{Im}(\psi^a)$, where $a = 1, 2$ labels the internal components of ψ . In the continuum, one can show that variational problems for a Dirac-field action can be equivalently solved by considering this action as a functional of the following complex independent fields, ψ^a and $(\psi^a)^*$, $a = 1, 2$. We assume that this feature is robust to the discretization. Moreover, since the discretization is the same for all internal components of the field, for convenience we treat the fields collectively (see section 3.1) as ψ and ψ^\dagger , and therefore write

$$S_{\text{DQW}}^{\text{asym.}} \equiv \mathcal{S}[\psi, \psi^\dagger]. \quad (\text{A4})$$

Let ψ be a field that extremalizes $S_{\text{DQW}}^{\text{asym.}}$, and consider an arbitrary variation $\delta\psi$ of ψ . Let us compute the variation of the action under the variations $\delta\psi$ and $\delta\psi^\dagger$, namely,

$$\delta S := \mathcal{S}[\psi + \delta\psi, \psi^\dagger + \delta\psi^\dagger] - \mathcal{S}[\psi, \psi^\dagger]. \quad (\text{A5})$$

In variational problems like this, one can usually show that as long as the action is linear in its arguments, one can keep in δS only the first-order terms in the variations $\delta\psi$ and $\delta\psi^\dagger$, which we are going to assume and which here delivers,

$$\begin{aligned} \delta S &= \epsilon^2 \sum_{j,p} \delta\psi_{j,p}^\dagger \left[\frac{i}{2\epsilon} (\psi_{j+1,p} - \psi_{j-1,p}) - (\mathcal{H} \psi_j)_p \right] \\ &\quad + \epsilon^2 \sum_{j,p} \psi_{j,p}^\dagger \left[\frac{i}{2\epsilon} (\delta\psi_{j+1,p} - \delta\psi_{j-1,p}) - (\mathcal{H} \delta\psi_j)_p \right] \end{aligned} \quad (\text{A6})$$

$$=: \delta S_1 + \delta S_2, \quad (\text{A7})$$

where

$$\delta S_1 := \epsilon \sum_j \left\langle \delta\psi_j \left| \frac{i}{2\epsilon} (\psi_{j+1,p} - \psi_{j-1,p}) - \mathcal{H} \psi_j \right. \right\rangle \quad (\text{A8a})$$

$$\delta S_2 := \epsilon \sum_j \left\langle \psi_j \left| \frac{i}{2\epsilon} (\delta\psi_{j+1,p} - \delta\psi_{j-1,p}) - \mathcal{H} \delta\psi_j \right. \right\rangle. \quad (\text{A8b})$$

If we choose $\delta\psi = 0$, then $\delta S = \delta S_1$, and requiring $\delta S = 0$ for any $\delta\psi^\dagger$ trivially delivers the correct EOMs. To see that this also works if $\delta\psi^\dagger = 0$, so that $\delta S = \delta S_2$, we need to do some work on δS_2 . First of all, if j runs from an initial j_i to a final j_f and p from p_i to p_f , then because of the symmetric lattice derivatives the action $S_{\text{DQW}}^{\text{asym.}}$ in equation (A2a) must be defined with a $\sum_{j=j_i+1}^{j_f-1}$ and a $\sum_{p=p_i+1}^{p_f-1}$, i.e. omitting the first and last indices. Then, from (A8b) we have that

$$\delta S_2 = \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle \psi_j \left| \frac{i}{2\epsilon} \delta\psi_{j+1} \right. \right\rangle + \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle \psi_j \left| -\frac{i}{2\epsilon} \delta\psi_{j-1} \right. \right\rangle + \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle \psi_j \left| -\mathcal{H} \delta\psi_j \right. \right\rangle. \quad (\text{A9})$$

After (a) performing a discrete integration by parts (i.e. a shift) in j in the two first terms, and (b) using the definition of the adjoint of \mathcal{H} in the third one, we obtain

$$\delta S_2 = \epsilon \sum_{j=j_i+2}^{j_f} \left\langle -\frac{i}{2\epsilon} \psi_{j-1} \left| \delta \psi_j \right. \right\rangle + \epsilon \sum_{j=j_i}^{j_f-2} \left\langle \frac{i}{2\epsilon} \psi_{j+1} \left| \delta \psi_j \right. \right\rangle + \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle -\mathcal{H}^\dagger \psi_j \left| \delta \psi_j \right. \right\rangle. \quad (\text{A10})$$

At this point of the derivation of such a variational problem, it is frequent to say that we keep the ends of ψ_j fixed, i.e. choose the lattice Neumann boundary conditions

$$\delta \psi_{j_i} = \delta \psi_{j_i+1} = \delta \psi_{j_f-1} = \delta \psi_{j_f} = 0, \quad (\text{A11})$$

so as not to have to deal with them. Note that this is not necessary for now, but it will be in the final reasoning where it ensures that the solution ψ is determined, i.e. unique. With these boundary conditions the first term of the right-hand side of equation (A10) becomes

$$\epsilon \sum_{j=j_i+2}^{j_f} \left\langle -\frac{i}{2\epsilon} \psi_{j-1} \left| \delta \psi_j \right. \right\rangle = \epsilon \sum_{j=j_i+2}^{j_f-1} \left\langle -\frac{i}{2\epsilon} \psi_{j-1} \left| \delta \psi_j \right. \right\rangle + \left\langle -\frac{i}{2\epsilon} \psi_{N-1} \left| \underbrace{\delta \psi_N}_{=0} \right. \right\rangle + \left\langle -\frac{i}{2\epsilon} \psi_0 \left| \underbrace{\delta \psi_1}_{=0} \right. \right\rangle \quad (\text{A12a})$$

$$= \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle -\frac{i}{2\epsilon} \psi_{j-1} \left| \delta \psi_j \right. \right\rangle. \quad (\text{A12b})$$

Similarly, the second term of the right-hand side of equation (A10) becomes

$$\epsilon \sum_{j=j_i}^{j_f-2} \left\langle \frac{i}{2\epsilon} \psi_{j+1} \left| \delta \psi_j \right. \right\rangle = \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle \frac{i}{2\epsilon} \psi_{j+1} \left| \delta \psi_j \right. \right\rangle. \quad (\text{A13})$$

Inserting equations (A12b) and (A13) into equation (A10) and using the Hermiticity of \mathcal{H} , we finally obtain that

$$\delta S_2 = \epsilon \sum_{j=j_i+1}^{j_f-1} \left\langle \frac{i}{2\epsilon} (\psi_{j+1,p} - \psi_{j-1,p}) - \mathcal{H} \psi_j \left| \delta \psi_j \right. \right\rangle. \quad (\text{A14})$$

If we choose $\delta \psi^\dagger = 0$, then $\delta S = \delta S_2$, and requiring $\delta S = 0$ for any $\delta \psi$ now trivially delivers the correct EOM of equation (16). This completes the proof that the action of equation (34) is a valid one for unitary fermions up to not being real-valued, an issue which is immediately solved by considering the real-valued action of equation (36) instead.

If we had not fixed the ends of ψ_j we would have obtained (A14) with additional boundary terms. Then, to arrive at the EOM in the ‘bulk’ one has to successively choose particular variation functions $\delta \psi_j$ as usual for this kind of proofs, with in the present case the (additional) requirement that all these variations functions have vanishing ends. Fixing the ends in the beginning like we did is thus not necessary but merely convenient, and we could have and would have obtained the same EOM if we had done the derivation with loose ends.

Appendix B. Equivalence of the asymmetric and symmetric actions of unitary fermions

Let us show that the asymmetric action of unitary fermions, equation (34), is equal to the symmetric action of equation (36) up to boundary terms. We start from the asymmetric action:

$$S_{\text{DQW}}^{\text{asym.}} = \epsilon^2 \sum_n \psi_n^\dagger (\tilde{\gamma}^0)^{-1} \left[\left(i \tilde{\gamma}^\mu \frac{1}{2\epsilon} (\mathcal{T}_\mu^{-1} - \mathcal{T}_\mu) - m \right) \psi \right]_n \quad (\text{B1a})$$

$$= \epsilon \frac{i}{2} \sum_n \psi_n^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^\mu \psi_{n+\hat{\mu}} + \epsilon \left(\frac{-i}{2} \right) \sum_n \psi_n^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^\mu \psi_{n-\hat{\mu}} - \epsilon \sum_n \epsilon m \psi_n^\dagger (\tilde{\gamma}^0)^{-1} \psi_n \quad (\text{B1b})$$

We consider the second term, and perform discrete integrations by parts on this term, i.e. a shift of one lattice site in μ for $\mu = 0, 1$. To avoid cumbersome notations, we write explicitly $n = (j, p)$. Then, the second term of $S_{\text{DQW}}^{\text{asym.}}$ becomes

$$T^{(2)} := \epsilon \left(\frac{-i}{2} \right) \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \psi_{j,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^\mu \psi_{(j,p)-\hat{\mu}} \quad (\text{B2a})$$

$$= \epsilon \left(\frac{-i}{2} \right) \left[\sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \psi_{j,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j-1,p} + \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \psi_{j,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^1 \psi_{j,p-1} \right] \quad (\text{B2b})$$

$$= \epsilon \left(\frac{-i}{2} \right) \left[\underbrace{\sum_{j=j_i}^{j_f-2} \sum_{p=p_i+1}^{p_f-1} \psi_{j+1,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j,p}}_{A^{(0)}} + \underbrace{\sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i}^{p_f-2} \psi_{j,p+1}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^1 \psi_{j,p}}_{A^{(1)}} \right]. \quad (\text{B2c})$$

Extracting the boundary term $j = j_i$ out of $A^{(0)}$ and adding and effective zero yields

$$A^{(0)} = \sum_{j=j_i+1}^{j_f-2} \sum_{p=p_i+1}^{p_f-1} \psi_{j+1,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j,p} + \sum_{p=p_i+1}^{p_f-1} \psi_{j_i+1,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j_i,p} + \underbrace{\sum_{p=p_i+1}^{p_f-1} \psi_{j_f,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j_f-1,p} - \sum_{p=p_i+1}^{p_f-1} \psi_{j_f,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j_f-1,p}}_{=0} \quad (\text{B3a})$$

$$= \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \psi_{j+1,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j,p} + \underbrace{\sum_{p=p_i+1}^{p_f-1} \left(\psi_{j_i+1,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j_i,p} - \psi_{j_f,p}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^0 \psi_{j_f-1,p} \right)}_{B^{(0)}}. \quad (\text{B3b})$$

Similarly, we obtain

$$A^{(1)} = \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \psi_{j,p+1}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^1 \psi_{j,p} + \underbrace{\sum_{j=j_i+1}^{j_f-1} \left(\psi_{j,p_i+1}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^1 \psi_{j,p_i} - \psi_{j,p_f}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^1 \psi_{j,p_f-1} \right)}_{B^{(1)}}. \quad (\text{B4})$$

Inserting these expressions, equations (B3b) and (B4), back into (B2c), we obtain

$$T^{(2)} = \epsilon \left(\frac{-i}{2} \right) \sum_n \psi_{n+\hat{\mu}}^\dagger (\tilde{\gamma}^0)^{-1} \tilde{\gamma}^\mu \psi_{j,p} + B^{(0)} + B^{(1)}. \quad (\text{B5})$$

Now, equating

$$(\tilde{\gamma}^0)^{-1} \tilde{\gamma}^\mu = \frac{\gamma^0}{\mu_\epsilon} \tilde{\gamma}^\mu \gamma^0 \gamma^0 = \gamma^0 \tilde{\gamma}^\mu \gamma^0 \frac{\gamma^0}{\mu_\epsilon} = (\tilde{\gamma}^\mu)^\dagger ((\tilde{\gamma}^0)^{-1})^\dagger, \quad (\text{B6})$$

and inserting the last equality into equation (B5) delivers

$$T^{(2)} = \epsilon \left(\frac{-i}{2} \right) \sum_n \psi_{n+\hat{\mu}}^\dagger (\tilde{\gamma}^\mu)^\dagger ((\tilde{\gamma}^0)^{-1})^\dagger \psi_{j,p} + B^{(0)} + B^{(1)}. \quad (\text{B7})$$

Inserting this back into equation (B1b), we obtain

$$S_{\text{DQW}}^{\text{asym.}} = S_{\text{DQW}} + B^{(0)} + B^{(1)}, \quad (\text{B8})$$

where S_{DQW} has been defined by equation (36).

The boundary term $B^{(0)} + B^{(1)}$ is irrelevant in the variational problem for determining the EOMs. Notice that it is possible to write the boundary term $B^{(0)} + B^{(1)}$ as a (lattice) $(1 + 1)$ -divergence as it is usually done in the continuum, although for the current defined in that way to be real we have to consider the Hermitian conjugate part of the action: this is done in a more general framework in section 3 and in appendix D.

Appendix C. Euler–Lagrange equations (from a real action)

Consider the real action of equation (37), and a field ψ that extremalizes it. Varying the field arbitrarily by $\delta\psi$ results in the action

$$S'_F := \sum_n \mathcal{L} \left(\psi_n + \delta\psi_n, \psi_{n+\hat{\mu}} + \delta\psi_{n+\hat{\mu}}, \psi_n^\dagger + \delta\psi_n^\dagger, \psi_{n+\hat{\mu}}^\dagger + \delta\psi_{n+\hat{\mu}}^\dagger \right) \tag{C1a}$$

$$= S_F + \sum_n \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n \delta\psi_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_n \delta\psi_{n+\hat{\mu}} + \delta\psi_n^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \delta\psi_{n+\hat{\mu}}^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_n \right), \tag{C1b}$$

where in the second equation we have only kept terms up to first order since the action is linear in its arguments, and with implicit sums over μ in the second and fourth terms inside the brackets.

Equation (C1b) can be rewritten as

$$\delta S := S'_F - S_F = \delta S_1 + \delta S_2, \tag{C2}$$

where δS_1 is the variation of the fields and δS_2 that of the conjugate fields, i.e.

$$\delta S_1 := \sum_n \left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n \delta\psi_n + \delta S_{11} \tag{C3a}$$

$$\delta S_2 := \sum_n \left(\delta\psi_n^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \delta\psi_{n+\hat{\mu}}^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_n \right), \tag{C3b}$$

where

$$\delta S_{11} := \sum_n \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_n \delta\psi_{n+\hat{\mu}}. \tag{C4}$$

‘Splitting time and space’ gives

$$\delta S_{11} = \delta S_{11}^{(0)} + S_{11}^{(1)}, \tag{C5}$$

where

$$\delta S_{11}^{(0)} := \sum_{j=j_i}^{j_f-1} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_j \delta\psi_{j+1} \tag{C6a}$$

$$\delta S_{11}^{(1)} := \sum_{j=j_i}^{j_f-1} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \right|_p \delta\psi_{p+1}, \tag{C6b}$$

where we have written down the index only when it is the ‘working’ index, i.e. the index on which we are going to perform modifications. Using the boundary conditions of equation (A11), we have that

$$\delta S_{11}^{(0)} = \sum_{j=j_i+1}^{j_f} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j-1} \delta\psi_j \tag{C7a}$$

$$= \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j-1} \delta\psi_j + \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j_i-1} \underbrace{\delta\psi_{j_f}}_{=0} + \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j_i-1} \underbrace{\delta\psi_{j_i}}_{=0} - \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j_i-1} \underbrace{\delta\psi_{j_i}}_{=0} \tag{C7b}$$

$$= \sum_{j=j_i}^{j_f-1} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \right|_{j-1} \delta\psi_j. \tag{C7c}$$

Similarly, we can show that

$$\delta S_{11}^{(1)} = \sum_{j=j_i}^{j_f-1} \sum_{p=p_i}^{p_f-1} \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \right|_{p-1} \delta\psi_p, \tag{C8}$$

so that in total we obtain

$$\delta S_{11} = \sum_n \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_{n-\hat{\mu}} \delta \psi_n, \tag{C9}$$

with an implicit sum over μ . Inserting this into equation (C3a) yields

$$\delta S_1 = \sum_n \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_{n-\hat{\mu}} \right) \delta \psi_n. \tag{C10}$$

Similarly, we can show that

$$\delta S_2 = \sum_n \delta \psi_n^\dagger \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_{n-\hat{\mu}} \right), \tag{C11}$$

and so

$$\begin{aligned} \delta S &= \sum_n \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_{n-\hat{\mu}} \right) \delta \psi_n \\ &\quad + \sum_n \delta \psi_n^\dagger \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_{n-\hat{\mu}} \right). \end{aligned} \tag{C12}$$

Recall that ψ extremalizes the action, i.e. $\delta S = 0$. By choosing $\delta \psi_n = 0$, we must have $\delta S_2 = 0$ for any variation $\delta \psi_n^\dagger$. Similarly, by choosing $\delta \psi_n^\dagger = 0$, implies that $\delta S_1 = 0$ for any variation $\delta \psi_n$. Together, these conditions give the Euler–Lagrange equations

$$\left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_{n-\hat{\mu}} = 0 \tag{C13a}$$

$$\left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_{n-\hat{\mu}} = 0. \tag{C13b}$$

It might be possible to prove that for a real action these equations imply each other (more precisely, by proving that they are the Hermitian conjugates of each other), we leave it as an open problem.

Appendix D. Proof of the lattice Noether’s theorem for internal symmetries

To the title of this appendix, we should actually add ‘that depend only on a single real parameter α ’, but as mentioned in section 3 the generalization to an arbitrary number of real parameters exhibits no major difficulty.

Under the transformation $\psi_n \rightarrow \psi'_n = \psi_n + C_n \delta \alpha$ of equation (41), the action defined in equation (37) becomes

$$S'_F := \sum_n \mathcal{L}(\psi'_n, \psi'_{n+\hat{\mu}}, (\psi'_n)^\dagger, (\psi'_{n+\hat{\mu}})^\dagger) \tag{D1a}$$

$$= \sum_n \mathcal{L}(\psi_n + C_n \delta \alpha, \psi_{n+\hat{\mu}} + C_{n+\hat{\mu}} \delta \alpha, \psi_n^\dagger + C_n^\dagger \delta \alpha, \psi_{n+\hat{\mu}}^\dagger + C_{n+\hat{\mu}}^\dagger \delta \alpha) \tag{D1b}$$

$$\begin{aligned} &= \sum_n \mathcal{L}(\psi_n, \psi_{n+\hat{\mu}}, \psi_n^\dagger, \psi_{n+\hat{\mu}}^\dagger) + \left(\left. \frac{\partial \mathcal{L}}{\partial \psi_n} \right|_n C_n + \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \right|_n C_{n+\hat{\mu}} + C_n^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \right|_n + C_{n+\hat{\mu}}^\dagger \left. \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \right|_n \right) \\ &\quad \times \delta \alpha + O(\delta \alpha^2), \end{aligned} \tag{D1c}$$

where we implicitly sum over μ in the second and fourth terms of the big bracket. Hence, we obtain

$$\delta S := S'_F - S_F = (Q_1 + Q_2) \delta \alpha, \tag{D2}$$

where

$$Q_1 := \sum_n \left(\frac{\partial \mathcal{L}}{\partial \psi_n} \Big|_n C_n + C_n^\dagger \frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \Big|_n \right) \tag{D3a}$$

$$Q_2 := \sum_n \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \Big|_n C_{n+\hat{\mu}} + C_{n+\hat{\mu}}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \Big|_n \right). \tag{D3b}$$

In going from the first to the second equality in equations (D2), we only kept, as in previous appendices, the first order in $\delta\alpha$, because the action is again assumed to be linear in its arguments.

We are going to perform a discrete integration by parts on Q_2 . To this end, let us ‘split time and space’:

$$Q_2 = Q_2^{(0)} + Q_2^{(1)}, \tag{D4}$$

with temporal and the spatial terms

$$Q_2^{(0)} := \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \Big|_j C_{j+1} + C_{j+1}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}^\dagger} \Big|_j \right) \tag{D5a}$$

$$Q_2^{(1)} := \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \Big|_p C_{p+1} + C_{p+1}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}^\dagger} \Big|_p \right). \tag{D5b}$$

Unless otherwise mentioned, for brevity and clarity we only write explicitly the coordinate at which we consider an expression whenever any of its factors is shifted, since this is the only coordinate on which ‘we are going to work’. Let us now perform a temporal discrete integration by parts on $Q_2^{(0)}$:

$$Q_2^{(0)} = \sum_{j=j_i+2}^{j_f} \sum_{p=p_i+1}^{p_f-1} \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \Big|_{j-1} C_j + C_j^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}^\dagger} \Big|_{j-1} \right) \tag{D6a}$$

$$= \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \Big|_{j-1} C_j + C_j^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}^\dagger} \Big|_{j-1} \right) + b^{(0)}, \tag{D6b}$$

where we have introduced the following boundary term,

$$b^{(0)} := \sum_{p=p_i+1}^{p_f-1} \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \Big|_{j_f-1} C_{j_f} + C_{j_f}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}^\dagger} \Big|_{j_f-1} \right) - \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}} \Big|_{j_i} C_{j_i+1} + C_{j_i+1}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{0}}^\dagger} \Big|_{j_i} \right) \right]. \tag{D7}$$

This boundary term can be rewritten as the telescoping sum

$$b^{(0)} = \sum_{p=p_i+1}^{p_f-1} \sum_{j=j_i+1}^{j_f-1} (J_j^0 - J_{j-1}^0) = \sum_n \epsilon (d_0^T J^0)_n, \tag{D8}$$

where J^0 has been defined in equation (43). Similarly, we perform a spatial discrete integration by parts on $Q_2^{(1)}$, which gives

$$Q_2^{(1)} = \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \Big|_{p-1} C_p + C_p^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}^\dagger} \Big|_{p-1} \right) + b^{(1)}, \tag{D9}$$

with boundary term

$$b^{(1)} := \sum_{j=j_i+1}^{j_f-1} \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \Big|_{p_f-1} C_{p_f} + C_{p_f}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}^\dagger} \Big|_{p_f-1} \right) - \left(\frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}} \Big|_{p_i} C_{p_i+1} + C_{p_i+1}^\dagger \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{1}}^\dagger} \Big|_{p_i} \right) \right]. \tag{D10}$$

Again, this boundary term can be rewritten in terms of J^1 defined in equation (43) as

$$b^{(1)} = \sum_{j=j_i+1}^{j_f-1} \sum_{p=p_i+1}^{p_f-1} (J_p^1 - J_{p-1}^1) = \sum_n \epsilon (d_1^T J^1)_n. \tag{D11}$$

Inserting equations (D6b) and (D9) into equation (D4), and then the resulting expression into equation (D2), we obtain, after rearranging the first two sums and using the final expressions of the boundary terms, equations (D8) and (D11), the following expression,

$$\delta S = \left(\sum_n \left[\left(\frac{\partial \mathcal{L}}{\partial \psi_n} \Big|_n + \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}} \Big|_{n-\hat{\mu}} \right) C_n + C_n^\dagger \left(\frac{\partial \mathcal{L}}{\partial \psi_n^\dagger} \Big|_n + \frac{\partial \mathcal{L}}{\partial \psi_{n+\hat{\mu}}^\dagger} \Big|_{n-\hat{\mu}} \right) \right] + \sum_n \epsilon (d_\mu^\dagger J^\mu)_n \right) \delta \alpha. \quad (\text{D12})$$

As in appendix C one can check that the first two summands are Euler–Lagrange expressions which vanish on shell¹³. Thus, finally, we obtain that, on shell,

$$\delta S = \sum_n \epsilon (d_\mu^\dagger J^\mu)_n \delta \alpha. \quad (\text{D13})$$

If the considered transformation is a symmetry, i.e. if $\delta S = 0$, then this implies that J is conserved on the lattice, i.e. equation (44), which ends the proof of our lattice Noether’s theorem.

Appendix E. Current conservation from the EOM

In the main text, we obtained the U(1)-charge Noether-current conservation equation from our lattice Noether’s theorem. Here, we show that this conservation equation can be derived from the EOMs (as in the continuum), either the one-step or the two-step one. As above, we write only the indices that are shifted from (j, p) .

E.1. From the one-step EOM

From the (generic) one-step EOM, equation (18) with \mathcal{W} as in equation (20), we immediately obtain the following four equalities

$$\psi^\dagger \psi_{j+1} = \psi^\dagger (W_{-1} \psi_{p+1} + W_1 \psi_{p-1} + W_0 \psi) \quad (\text{E1a})$$

$$\psi_{j+1}^\dagger \psi = \left(\psi_{p+1}^\dagger W_{-1}^\dagger + \psi_{p-1}^\dagger W_1^\dagger + \psi^\dagger W_0^\dagger \right) \psi \quad (\text{E1b})$$

$$\psi_{j-1}^\dagger \psi = \left(\psi_{p-1}^\dagger W_{-1} + \psi_{p+1}^\dagger W_1 + \psi^\dagger W_0 \right) \psi \quad (\text{E1c})$$

$$\psi^\dagger \psi_{j-1} = \psi^\dagger \left(W_{-1}^\dagger \psi_{p-1} + W_1^\dagger \psi_{p+1} + W_0^\dagger \psi \right). \quad (\text{E1d})$$

Summing the first two equations, equations (E1b) and (E1a), and subtracting the last two, equations (E1d) and (E1c), we obtain after some simplifications and rearrangements the following current-conservation equation,

$$d_\mu^\dagger Q^\mu = 0, \quad (\text{E2})$$

where the ‘temporal’ and the ‘spatial’ components of the current Q are

$$Q^0 := \frac{\epsilon}{2} \left(\psi^\dagger \psi_{j+1} + \psi_{j+1}^\dagger \psi \right) \quad (\text{E3a})$$

$$Q^1 := \frac{\epsilon}{2} \left(\psi^\dagger \left(W_1^\dagger - W_{-1} \right) \psi_{p+1} + \psi_{p+1}^\dagger \left(W_1 - W_{-1}^\dagger \right) \psi_p \right). \quad (\text{E3b})$$

The temporal component of the current, Q^0 , is actually equal to $J_{\text{U}(1)}^0$ defined in equation (47). Moreover, with the choices of equations (27b) and (30), we have that

$$\mu_\epsilon \alpha^1 = B_- := W_1 - W_{-1} \quad (\text{E4a})$$

$$\mu_\epsilon = B_+ := W_1 + W_{-1}. \quad (\text{E4b})$$

¹³ We leave it as an open problem whether for a real action these two Euler–Lagrange expressions are Hermitian conjugates of each other.

This implies

$$W_1 = \frac{\mu_\epsilon}{2}(1 + \alpha^1) = W_1^\dagger \quad (\text{E5a})$$

$$W_{-1} = \frac{\mu_\epsilon}{2}(1 - \alpha^1) = W_{-1}^\dagger, \quad (\text{E5b})$$

so that

$$W_1 - W_{-1}^\dagger = \mu_\epsilon \alpha^1 \quad (\text{E6a})$$

$$W_1^\dagger - W_{-1} = \mu_\epsilon \alpha^1, \quad (\text{E6b})$$

which, inserted into equation (E3b), yields $Q^1 = J_{U(1)}^1$, which ends our proof.

E.2. Directly from the two-step EOM

Multiplying the two-step EOM, equation (16), from the left by $2i\psi^\dagger$ we obtain

$$\psi^\dagger \psi_{j+1} = \psi^\dagger \psi_{j-1} - \psi^\dagger \tilde{\alpha}^1 [\psi_{p+1} - \psi_{p-1}] - 2im\psi^\dagger \tilde{\alpha}^0 \psi. \quad (\text{E7})$$

Similarly, taking the adjoint of equation (16) and multiplying it from the right by $2i\psi$ yields

$$\psi_{j+1}^\dagger \psi = \psi_{j-1}^\dagger \psi - [\psi_{p+1}^\dagger - \psi_{p-1}^\dagger] \tilde{\alpha}^1 \psi + 2im\psi^\dagger \tilde{\alpha}^0 \psi. \quad (\text{E8})$$

The sum of these equations (E7) and (E8) gives the same current-conservation equation as that obtained from Noether's theorem, namely, equation (49). Note that the above derivation has a direct parallel with the case of naive fermions.

Appendix F. Continuum limit of the gauge-field action

It is easy to show that

$$F_{\mu\nu} F^{\mu\nu} = \sum_l (-F_{0l}^2 - F_{l0}^2) + \sum_{k,l} F_{kl}^2. \quad (\text{F1})$$

Moreover, we have after a few lines of computation that, at lowest order in ϵ ,

$$S_G^{\text{time}} = -\frac{1}{2} \sum_n \sum_l \epsilon^4 (-F_{0l}_n^2), \quad (\text{F2})$$

and, since $(F_{0l})^2 = (F_{l0})^2$, equation (F2) can be rewritten as

$$S_G^{\text{time}} = -\frac{1}{4} \sum_n \sum_l \epsilon^4 (-F_{0l}_n^2 - F_{l0}_n^2). \quad (\text{F3})$$

We also have after a few lines of computation that, at lowest order ϵ ,

$$S_G^{\text{space}} = -\frac{1}{2} \sum_n \sum_{\substack{k,l \\ k<l}} \epsilon^4 (F_{kl}_n)^2 \quad (\text{F4a})$$

$$= -\frac{1}{4} \sum_n \sum_{\substack{k,l \\ k<l}} \epsilon^4 [(F_{kl}_n)^2 + (F_{lk}_n)^2] \quad (\text{F4b})$$

$$= -\frac{1}{4} \sum_n \epsilon^4 \left(\sum_{\substack{k,l \\ k<l}} (F_{kl}_n)^2 + \sum_{\substack{k,l \\ k<l}} (F_{lk}_n)^2 \right) \quad (\text{F4c})$$

$$= -\frac{1}{4} \sum_n \epsilon^4 \left(\sum_{\substack{k,l \\ k < l}} (F_{kl})_n^2 + \sum_{\substack{k,l \\ l < k}} (F_{kl})_n^2 \right) \quad (\text{F4d})$$

$$= -\frac{1}{4} \sum_n \sum_{k,l} \epsilon^4 (F_{kl})_n^2. \quad (\text{F4e})$$

Summing equations (F3) and (F4e) and taking into account equation (F1), we obtain the desired continuum limit, equation (76).

ORCID iD

Pablo Arnault  <https://orcid.org/0000-0003-2928-162X>

References

- [1] Rothe H J 2012 *Lattice Gauge Theories: An Introduction* 4th edn (Singapore: World Scientific)
- [2] Wilson K G 1974 Confinement of quarks *Phys. Rev. D* **10** 2445
- [3] Hernández M P 2011 Lattice field theory fundamentals *Modern Perspectives in Lattice QCD: Quantum Field Theory and High Performance Computing, Lecture Notes of the Les Houches Summer School: Volume 93, August 2009* (Oxford: Oxford University Press) pp 1–91
- [4] Wilson K and Kogut J 1974 The renormalization group and the ϵ expansion *Phys. Rep.* **12** 75
- [5] Kogut J and Susskind L 1975 Hamiltonian formulation of Wilson's lattice gauge theories *Phys. Rev. D* **11** 395
- [6] Weinberg S 1995 *The Quantum Theory of Fields I* (Cambridge: Cambridge University Press)
- [7] Bañuls M C, Cichy K, Cirac J I, Jansen K, Kühn S and Saito H 2017 Towards overcoming the Monte Carlo sign problem with tensor networks *EPJ Web Conf.* vol 137 p 04001
- [8] Lang C B 2008 The hadron spectrum from lattice QCD *Prog. Part. Nucl. Phys.* **61** 35
- [9] Aoki Y *et al* 2021 FLAG review 2021 (arXiv:2111.09849)
- [10] Brower R, Christ N, DeTar C, Edwards R and Mackenzie P 2018 Lattice QCD application development within the US DOE exascale computing project *EPJ Web Conf.* vol 175 p 09010
- [11] Troyer M and Wiese U-J 2005 Computational complexity and fundamental limitations to fermionic quantum Monte Carlo simulations *Phys. Rev. Lett.* **94** 170201
- [12] Preskill J 2018 Simulating quantum field theory with a quantum computer, PoS LATTICE 2018 024 (arXiv:1811.10085)
- [13] Feynman R P 1982 Simulating physics with computers *Int. J. Theor. Phys.* **21** 467
- [14] Nielsen M A and Chuang I L 2010 *Quantum Computation and Quantum Information: 10th Anniversary edn* (Cambridge: Cambridge University Press)
- [15] Zohar E and Reznik B 2011 Confinement and lattice quantum-electrodynamic electric flux tubes simulated with ultracold atoms *Phys. Rev. Lett.* **107** 275301
- [16] Banerjee D, Dalmonte M, Müller M, Rico E, Stebler P, Wiese U-J and Zoller P 2012 Atomic quantum simulation of dynamical gauge fields coupled to fermionic matter: from string breaking to evolution after a quench *Phys. Rev. Lett.* **109** 175302
- [17] Wiese U-J 2013 Ultracold quantum gases and lattice systems: quantum simulation of lattice gauge theories *Ann. Phys., Lpz.* **525** 777
- [18] Zohar E, Cirac J I and Reznik B 2015 Quantum simulations of lattice gauge theories using ultracold atoms in optical lattices *Rep. Prog. Phys.* **79** 014401
- [19] Weimer H, Müller M, Lesanovsky I, Zoller P and Büchler H P 2010 A Rydberg quantum simulator *Nat. Phys.* **6** 382
- [20] Tagliacozzo L, Celi A, Zamora A and Lewenstein M 2013 Optical Abelian lattice gauge theories *Ann. Phys., NY* **330** 160
- [21] Byrnes T and Yamamoto Y 2006 Simulating lattice gauge theories on a quantum computer *Phys. Rev. A* **73** 022328
- [22] Jordan S P, Lee K S M and Preskill J 2012 Quantum algorithms for quantum field theories *Science* **336** 1130
- [23] Jordan S P, Lee K S M and Preskill J 2014 Quantum computation of scattering in scalar quantum field theories *Quantum Inf. Comput.* **14** 1014–80
- [24] Jordan S P, Lee K S M and Preskill J 2014 Quantum algorithms for fermionic quantum field theories (arXiv:1404.7115)
- [25] Brennen G K, Rohde P, Sanders B C and Singh S 2015 Multiscale quantum simulation of quantum field theory using wavelets *Phys. Rev. A* **92** 032315
- [26] Marshall K, Pooser R, Siopsis G and Weedbrook C 2015 Quantum simulation of quantum field theory using continuous variables *Phys. Rev. A* **92** 063825
- [27] Jordan S P, Krovi H, Lee K S M and Preskill J 2018 BQP-completeness of scattering in scalar quantum field theory *Quantum* **2** 44
- [28] Alexandru A, Bedaque P F, Lamm H and Lawrence S 2019 (collaboration NuQS collaboration), σ models on quantum computers *Phys. Rev. Lett.* **123** 090501
- [29] Moosavian A H, Garrison J R and Jordan S P 2019 Site-by-site quantum state preparation algorithm for preparing vacua of fermionic lattice field theories (arXiv:1911.03505)
- [30] Kanwar G and Wagman M L 2021 Real-time lattice gauge theory actions: unitarity, convergence and path integral contour deformations *Phys. Rev. D* **104** 014513
- [31] Dalmonte M and Montangero S 2016 Lattice gauge theory simulations in the quantum information era *Contemp. Phys.* **57** 388
- [32] Bañuls M C *et al* 2020 Simulating lattice gauge theories within quantum technologies *Eur. Phys. J. D* **74** 165
- [33] Bañuls M C and Cichy K 2020 Review on novel methods for lattice gauge theories *Rep. Prog. Phys.* **83** 024401
- [34] Aidelsburger M *et al* 2021 Cold atoms meet lattice gauge theory *Phil. Trans. R. Soc. A* **380** 0064
- [35] Zohar E 2021 Quantum simulation of lattice gauge theories in more than one space dimension-requirements, challenges and methods *Phil. Trans. R. Soc. A* **380** 0069

- [36] Klco N, Roggero A and Savage M J 2022 Standard model physics and the digital quantum revolution: thoughts about the interface *Rep. Prog. Phys.* **85** 064301
- [37] Martinez E A *et al* 2016 Real-time dynamics of lattice gauge theories with a few-qubit quantum computer *Nature* **534** 516
- [38] Arrighi P 2019 An overview of quantum cellular automata *Nat. Comput.* **18** 885
- [39] Farrelly T 2020 A review of quantum cellular automata *Quantum* **4** 368
- [40] Schumacher B and Werner R F 2004 Reversible quantum cellular automata (arXiv:quant-ph/0405174)
- [41] Destri C and De Vega H J 1987 Light-cone lattice approach to fermionic theories in 2D: The massive thirring model *Nucl. Phys. B* **290** 363
- [42] Bisio A, D'Ariano G M, Perinotti P and Tosini A 2018 Thirring quantum cellular automaton *Phys. Rev. A* **97** 032132
- [43] Arrighi P, Bény C and Farrelly T 2020 A quantum cellular automaton for one-dimensional QED *Quantum Inf. Process.* **19** 88
- [44] Sellapillay K, Arrighi P and Di Molfetta G 2022 A discrete relativistic spacetime formalism for 1+1-QED with continuum limits *Sci. Rep.* **12** 2198
- [45] Eon N, Di Molfetta G, Magnifico G and Arrighi P 2022 A relativistic discrete spacetime formulation of 3+1 QED (arXiv:2002.02643)
- [46] Farrelly T C, 2015 Insights from quantum information into fundamental physics, *PhD Thesis* School University of Cambridge
- [47] Farrelly T and Streich J 2020 Discretizing quantum field theories for quantum simulation (arXiv:2002.02643)
- [48] Yepez J 2016 Quantum lattice gas algorithmic representation of gauge field theory *Proc. SPIE* **9996** 99960N
- [49] Cedzich C, Joye A, Werner A H and Werner R F Exponential tail estimates for quantum lattice dynamics (In preparation)
- [50] Vogts H 2013 Discrete time quantum lattice systems *PhD Thesis* Technischen Universität zu Braunschweig
- [51] Bialynicki-Birula I 1994 Weyl, Dirac and Maxwell equations on a lattice as unitary cellular automata *Phys. Rev. D* **49** 6920
- [52] Farrelly T C and Short A J 2014 Discrete spacetime and relativistic quantum particles *Phys. Rev. A* **89** 062109
- [53] Bisio A, D'Ariano G M and Tosini A 2015 Quantum field as a quantum cellular automaton: the Dirac free evolution in one dimension *Ann. Phys., NY* **354** 244
- [54] Di Molfetta G, Debbasch F and Brachet M 2014 Quantum walks in artificial electric and gravitational fields *Physica A* **397** 157
- [55] Arnault P and Debbasch F 2016 Quantum walks and discrete gauge theories *Phys. Rev. A* **93** 052301
- [56] Márquez-Martín I, Arnault P, Molfetta G D and Pérez A 2018 Electromagnetic lattice gauge invariance in two-dimensional discrete-time quantum walks *Phys. Rev. A* **98** 032333
- [57] Cedzich C, Geib T, Werner A H and Werner R F 2019 Quantum walks in external gauge fields *J. Math. Phys.* **60** 012107
- [58] Arnault P, Di Molfetta G, Brachet M and Debbasch F 2016 Quantum walks and non-Abelian discrete gauge theory *Phys. Rev. A* **94** 012335
- [59] Di Molfetta G, Brachet M and Debbasch F 2013 Quantum walks as massless Dirac fermions in curved space *Phys. Rev. A* **88** 042301
- [60] Arnault P and Debbasch F 2017 Quantum walks and gravitational waves *Ann. Phys.* **383** 645
- [61] Arrighi P, Facchini S and Forets M 2016 Quantum walking in curved spacetime *Quantum Inf. Process.* **15** 3467
- [62] Arrighi P and Facchini S 2017 Quantum walking in curved spacetime: (3+1) dimensions and beyond *Quantum Info. Comput.* **17** 810
- [63] Arnault P, 2017 Discrete-time quantum walk and gauge theories *PhD Thesis* school Université Pierre et Marie Curie
- [64] Arrighi P, Di Molfetta G, Marquez-Martín I and Perez A 2019 From curved spacetime to spacetime-dependent local unitaries over the honeycomb and triangular quantum walks *Sci. Rep.* **9** 10904
- [65] Farrelly T C and Short A J 2014 Causal fermions in discrete space-time *Phys. Rev. A* **89** 012302
- [66] D'Ariano G M and Perinotti P 2016 Quantum cellular automata and free quantum field theory *Front. Phys.* **12** 120301
- [67] Debbasch F 2019 Action principles for quantum automata and Lorentz invariance of discrete time quantum walks *Ann. Phys.* **405** 340
- [68] Arnault P 2022 Clifford algebra from quantum automata and unitary Wilson fermions *Phys. Rev. A* **106** 012201
- [69] Susskind L 1977 Lattice fermions *Phys. Rev. D* **16** 3031
- [70] Arnault P, Pepper B and Pérez A 2020 Quantum walks in weak electric fields and Bloch oscillations *Phys. Rev. A* **101** 062324
- [71] Bachmann S 2007 Champs quantiques relativistes notes du cours du Professeur Mikhaïl Shaposhnikov.
- [72] Cedzich C, Rybár T, Werner A H, Alberti A, Genske M and Werner R F 2013 Propagation of quantum walks in electric fields *Phys. Rev. Lett.* **111** 160601
- [73] Cedzich C and Werner A H 2021 Anderson localization for electric quantum walks and skew-shift CMV matrices *Comm. Math. Phys.* **387** 1257
- [74] Christiansen S H and Halvorsen T G 2009 Convergence of lattice gauge theory for Maxwell's equations *BIT Numer. Math.* **49** 645
- [75] Arrighi P, Facchini S and Forets M 2014 Discrete Lorentz covariance for quantum walks and quantum cellular automata *New J. Phys.* **16** 093007
- [76] Debbasch F 2019 Discrete geometry from quantum walks *Condens. Matter* **4** 40