Journal of Physics: Conference Series **411** (2013) 012021 doi:10.1088/1742-6596/411/1/012021

Bilinear equations, Bell polynomials and linear superposition principle

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Abstract. A class of bilinear differential operators is introduced through assigning appropriate signs and used to create bilinear differential equations which generalize Hirota bilinear equations. The resulting bilinear differential equations are characterized by a special kind of Bell polynomials and the linear superposition principle is applied to the construction of their linear subspaces of solutions. Illustrative examples are made by an algorithm using weights of dependent variables.

1. Introduction

Nonlinear differential equations play a significant role in exploring physical phenomena in depth. Hirota presented a direct method to solve a kind of specific bilinear differential equations [1]; and soliton solutions are, despite their diversity, a universal phenomenon that Hirota bilinear equations describe [2].

It is known that under $u = 2(\ln f)_{xx}$, the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1.1}$$

can be transformed into

$$(D_xD_t + D_x^4)f \cdot f = 0, \tag{1.2}$$

which reads

$$f_{xt}f - f_xf_t + f_{xxxx}f - 4f_{xxx}f_x + 3f_{xx}^2 = 0. \tag{1.3}$$

Through this bilinear form, general Wronskian solutions, including solitons and complexitons, are presented for the KdV equation [3, 4]. The Hirota D-operators [5] are defined to be

$$D_x^n f \cdot g = (\partial_x - \partial_{x'})^n f(x)g(x')|_{x'=x} = \partial_{x'}^n f(x+x')g(x-x')|_{x'=0}. \tag{1.4}$$

For example, we have

$$\begin{cases} D_xf \cdot g = f_xg - fg_x, \\ D_x^2 f \cdot g = f_{xx}g - 2f_xg_x + fg_{xx}, \\ D_x^3 f \cdot g = f_{xxx}g - 3f_{xx}g_x + 3f_xg_{xx} - fg_{xxx}. \end{cases} \tag{1.5}$$

It is very interesting that most integrable equations possess the Hirota bilinear form. Soliton solutions, particularly three-soliton solutions and Wronskian, Grammian and Pfaffian solutions, to Hirota bilinear equations can be generated by the Hirota perturbation and Pfaffian techniques [4, 6, 7, 8].

However, Hirota bilinear equations are special and there are many other bilinear differential equations which are not written in the Hirota bilinear form. This report will introduce a kind of generalized bilinear differential operators and their corresponding bilinear equations, which still possess nice mathematical properties. More importantly, we will talk about links of the presented bilinear equations with multivariate Bell exponential polynomials and their linear subspaces of solutions by the linear superposition principle.

2. Bilinear differential operators and bilinear equations

2.1. Bilinear D_p-operators

Let p be a given natural number. We introduce bilinear differential operators as follows:

$$(D_{p,x}^n f\cdot g)(x)=(\partial_x+\alpha\partial_{x'})^n f(x)g(x')\big|_{x'=x}=\sum_{i=0}^{n}\binom{n}{i}\alpha^i(\partial_x^{n-i}f)(x)(\partial_x^i g)(x),\ n\ge 1, \tag{2.1}$$

where the powers of α are determined by

$$\alpha^i=(-1)^{r(i)},\ \text{where } i=r(i) \bmod p \text{ with } 0\le r(i)<p,\ i\ge 0. \tag{2.2}$$

Obviously, the case of $p=1$ gives the normal derivatives, and the cases of $p=2k,\ k\in\mathbb{N}$, reduce to Hirota bilinear operators.

We can observe that the powers α^i read

$$p=3:\ +,\ -,\ +,\ +,\ -,\ +,\ \cdots\ \text{for } i=0,1,2,\cdots; \tag{2.3}$$

$$p=5:\ +,\ -,\ +,\ -,\ +,\ +,\ -,\ +,\ -,\ +,\ \cdots\ \text{for } i=0,1,2,\cdots; \tag{2.4}$$

$$p=7:\ +,\ -,\ +,\ -,\ +,\ -,\ +,\ +,\ -,\ +,\ -,\ +,\ -,\ +,\ \cdots\ \text{for } i=0,1,2,\cdots; \tag{2.5}$$

and thus

$$D_{3,x}^3 f\cdot g=f_{3x}g-3f_{2x}g_x+3f_xg_{2x}+fg_{3x},$$

$$D_{5,x}^5 f\cdot g=f_{5x}g-5f_{4x}g_x+10f_{3x}g_{2x}-10f_{2x}g_{3x}+5f_xg_{4x}+fg_{5x},$$

$$D_{7,x}^7 f\cdot g=f_{7x}g-7f_{6x}g_x+21f_{5x}g_{2x}-35f_{4x}g_{3x}+35f_{3x}g_{4x}-21f_{2x}g_{5x}+7f_xg_{6x}+fg_{7x},$$

which are different from the Hirota bilinear differential expressions [1].

A common feature that the D_p-operators share is the Taylor expansion

$$f(x+\delta)g(x+\alpha\delta)=\sum_{i=0}^{\infty}\frac{1}{i!}(D_{p,x}^i f\cdot g)\delta^i, \tag{2.6}$$

if we define

$$g(x+\alpha\delta)=\sum_{i=0}^{\infty}\frac{(\partial_x^i g)(x)}{i!}\alpha^i\delta^i. \tag{2.7}$$

The case of bilinear operators with more than one dependent variables can be similarly defined as follows:

$$\begin{aligned}&(D_{p,x_1}^{n_1}\cdots D_{p,x_l}^{n_l} f\cdot g)(x_1,\cdots,x_l)\\&=(\partial_{x_1}+\alpha\partial_{x_1'})^{n_1}\cdots(\partial_{x_l}+\alpha\partial_{x_l'})^{n_l}f(x_1,\cdots,x_l)g(x_1',\cdots,x_l')|_{x_i'=x_i},\ n_1,\cdots,n_l\ge 1.\end{aligned} \tag{2.8}$$

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2.2. Bilinear equations

A bilinear differential equation associated with a multivariate polynomial $\mathcal{F} = \mathcal{F}(x_1, \cdots, x_l)$ is defined by

$$\mathcal{F}(D_{p,x_1}, \cdots, D_{p,x_l}) f \cdot f = 0, \tag{2.9}$$

which reduces a Hirota bilinear equation if $p = 2k,\ k \in \mathbb{N}$. When $p = 5$, we particularly have the generalized bilinear KdV equation

$$(D_{5,x} D_{5,t} + D_{5,x}^4) f \cdot f = 2\, f_{xt} f - 2\, f_x f_t + 2 f_{4x} f - 8 f_{3x} f_x + 6\, f_{2x}^2 = 0, \tag{2.10}$$

the generalized bilinear Boussinesq equation

$$(D_{5,t}^2 + D_{5,x}^4) f \cdot f = 2\, f_{2t} f - 2\, f_t^2 + 2 f_{4x} f - 8 f_{3x} f_x + 6\, f_{2x}^2 = 0, \tag{2.11}$$

and the generalized bilinear KP equation

$$(D_{5,t} D_{5,x} + D_{5,x}^4 + D_{5,y}^2) f \cdot f = 2\, f_{xt} f - 2\, f_x f_t + 2 f_{4x} f - 8 f_{3x} f_x + 6\, f_{2x}^2 + 2\, f_{2y} f - 2\, f_y^2 = 0. \tag{2.12}$$

Such generalized bilinear equations have two common characteristics:

- Bilinear: The nearest neighbors to linear equations.
- Using the D_p-operators: Nice mathematical operators.

Two basic questions in the mathematical theory of bilinear equations are:

- How can one characterize bilinear equations defined by (2.9)?
- What kind of exact solutions are there to bilinear equations defined by (2.9)?

In this report, we would like to discuss those two questions, and provide solutions to both questions through the Bell exponential polynomials and the linear superposition principle, respectively.

3. Relations with Bell exponential polynomials

3.1. Bell polynomials

To begin with, let y be a C^∞ function of x and introduce

$$y_r = \partial_x^r y,\ r \geq 1. \tag{3.1}$$

The Bell polynomials are defined by

$$Y_{nx}(y) = Y_n(y_1, \cdots, y_n) = e^{-y} \partial_x^n e^y,\ n \geq 1, \tag{3.2}$$

in combinatorial mathematics [9]. A direct computation tells

$$\begin{cases} Y_1 = y_1,\ Y_2 = y_1^2 + y_2,\ Y_3 = y_1^3 + 3y_1y_2 + y_3, \\ Y_4 = y_1^4 + 6y_1^2y_2 + 4y_1y_3 + 3y_2^2 + y_4, \\ Y_5 = y_1^5 + 10y_1^3y_2 + 10y_1^2y_3 + 15y_1y_2^2 + 5y_1y_4 + 10y_2y_3 + y_5. \end{cases} \tag{3.3}$$

A special case of the Faà di Bruno formula (see, e.g., [10]) presents the Bell polynomials precisely:

$$Y_{nx}(y) = \sum \frac{n!}{m_1! \cdots m_n! (1!)^{m_1} \cdots (n!)^{m_n}} y_1^{m_1} \cdots y_n^{m_n}, \tag{3.4}$$

where the sum is over all n-tuples of nonnegative integers $(m_1, \cdots, m_n)$ satisfying the constraint $m_1 + 2m_2 + \cdots + nm_n = n$. The Bell polynomials can also be computed from

$$\exp(\sum_{r=1}^{\infty} \frac{y_r}{r!} t^r) = 1 + \sum_{n=1}^{\infty} \frac{Y_n(y_1, \cdots, y_n)}{n!} t^n. \tag{3.5}$$

The general formula (3.4) immediately tells the homogeneous property

$$Y_n(\alpha y_1, \alpha^2 y_2, \cdots, \alpha^n y_n) = \alpha^n Y_n(y_1, \cdots, y_n), \tag{3.6}$$

whose left-hand side is evaluated through first substituting all $\alpha y_1, \alpha^2 y_2, \cdots, \alpha^n y_n$ into Y_n and then collecting powers of α and computing them by the rule (2.2). On the other hand, the general Leibniz rule

$$(fg)^{-1}\partial_x^n (fg) = \sum_{i=0}^{n} \binom{n}{i} (f^{-1}\partial_x^{n-i} f)(g^{-1}\partial_x^i g) \tag{3.7}$$

shows the addition formula for the Bell polynomials:

$$Y_{nx}(y + y') = \sum_{i=0}^{n} \binom{n}{i} Y_{(n-i)x}(y) Y_{ix}(y'). \tag{3.8}$$

Those two properties will be used to link bilinear equations to a special kind of Bell polynomials.

3.2. Binary Bell polynomials

We first explore a relation of the Bell polynomials to the D_p-operators. For the sake of computational convenience, we assume that

$$f = e^{\xi(x)},\ g = e^{\eta(x)}. \tag{3.9}$$

Then using (3.6) and (3.8), we have

$$\begin{aligned} (fg)^{-1} D_{p,x}^n f \cdot g &= \sum_{i=0}^{n} \alpha^i \binom{n}{i} (f^{-1}\partial_x^{n-i} f)(g^{-1}\partial_x^i g) \\ &= \sum_{i=0}^{n} \alpha^i \binom{n}{i} Y_{(n-i)x}(\xi) Y_{ix}(\eta) \\ &= Y_n(y_1, \cdots, y_n)|_{y_r = \xi_{rx} + \alpha^r \eta_{rx}}, \end{aligned} \tag{3.10}$$

where $\xi_{rx} = \partial_x^r \xi$ and $\eta_{rx} = \partial_x^r \eta,\ r \geq 1$.

Similarly to the case of the Hirota D-operators [11], we introduce binary Bell polynomials

$$\mathcal{Y}_{p;nx}(v, w) = Y_n(y_1, \cdots, y_n)|_{y_r = \frac{1}{2}(w_{rx} + v_{rx}) + \frac{1}{2}\alpha^r(w_{rx} - v_{rx})},\ n \geq 1, \tag{3.11}$$

where $v_{rx} = \partial_x^r v$ and $w_{rx} = \partial_x^r w,\ r \geq 1$. For example, we have

$$\begin{cases} \mathcal{Y}_{3;x}(v, w) = v_x,\ \mathcal{Y}_{3;2x}(v, w) = v_x^2 + w_{2x},\ \mathcal{Y}_{3;3x}(v, w) = v_x^3 + 3v_x w_{2x} + w_{3x}, \\ \mathcal{Y}_{3;4x}(v, w) = v_x^4 + 6v_x^2 w_{2x} + 3w_{2x}^2 + 4v_x w_{3x} + v_{4x}, \\ \mathcal{Y}_{3;5x}(v, w) = {v_x}^5 + 10v_x^3 w_{2x} + 15v_x w_{2x}^2 + 10v_x^2 w_{3x} + 10w_{2x} w_{3x} + 5v_x v_{4x} + w_{5x}. \end{cases} \tag{3.12}$$

This way, upon setting that

$$w = \xi + \eta,\ v = \xi - \eta, \tag{3.13}$$

from (3.10), we have a combinatorial formula for the D_p-operators:

$$(fg)^{-1}D_{p,x}^n f\cdot g = \mathcal{Y}_{p;nx}(v=\ln\frac{f}{g}, w=\ln fg). \tag{3.14}$$

To characterize bilinear equations, we further introduce $\mathcal{P}$-polynomials:

$$\mathcal{P}_{p;nx}(q)=\mathcal{Y}_{p;nx}(0,q), \tag{3.15}$$

the first few of which in the case of $p=3$ read

$$\begin{cases} \mathcal{P}_{3;x}(q)=0,\ \mathcal{P}_{3;2x}(q)=q_{2x},\ \mathcal{P}_{3;3x}(q)=q_{3x},\ \mathcal{P}_{3;4x}(q)=3q_{2x}^2, \\ \mathcal{P}_{3;5x}=q_{5x}+10q_{2x}q_{3x},\ P_{3;6x}=q_{6x}+15q_{2x}^2+10q_{3x}^2. \end{cases} \tag{3.16}$$

In terms of

$$q=w-v=2\ln g,\ v=\ln\frac{f}{g}, \tag{3.17}$$

the combinatorial formula (3.14) becomes

$$(fg)^{-1}D_{p,x}^n f\cdot g=\mathcal{Y}_{p;nx}(v,v+q). \tag{3.18}$$

Letting $f=g$, this tells a relation between bilinear expressions and the $\mathcal{P}$-polynomials:

$$f^{-2}D_{p,x}^n f\cdot f=\mathcal{P}_{p;nx}(q=2\ln f). \tag{3.19}$$

Therefore, a bilinear equation

$$\mathcal{F}(D_{p,x})f\cdot f=0 \ \text{ with }\ \mathcal{F}(x)=\sum_{i=0}^{n} c_i x^i \tag{3.20}$$

is equivalent to an equation on a linear combination of $\mathcal{P}$-polynomials in $q=2\ln f$:

$$\sum_{i=0}^{n} c_i\mathcal{P}_{p;ix}(q=2\ln f)=0. \tag{3.21}$$

This is a characterization for our generalized bilinear equations in one dimensional case.

3.3. Multivariate binary Bell polynomials

For a C^∞ function $y=y(x_1,\cdots,x_l)$, define the variables [12]:

$$y_{r_1,\cdots,r_l}=y_{r_1x_1,\cdots,r_lx_l}=\partial_{x_1}^{r_1}\cdots\partial_{x_l}^{r_l}y(x_1,\cdots,x_l),\ r_1,\cdots,r_l\ge 0,\ \sum_{i=1}^{l} r_i\ge 1, \tag{3.22}$$

and the multivariate Bell polynomials

$$Y_{n_1x_1,\cdots,n_lx_l}(y)=Y_{n_1,\cdots,n_l}(y_{r_1,\cdots,r_l})=e^{-y}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l}e^y,\ n_1,\cdots,n_l\ge 0,\ \sum_{i=1}^{l} n_i\ge 1, \tag{3.23}$$

which can be computed through

$$\exp\Big(\sum_{\substack{r_1+\cdots+r_l\ge 1\\ r_1,\cdots,r_l\ge 0}}\frac{y_{r_1,\cdots,r_l}}{r_1!\cdots r_l!}t_1^{r_1}\cdots t_l^{r_l}\Big)=1+\sum_{\substack{n_1+\cdots+n_l\ge 1\\ n_1,\cdots,n_l\ge 0}}\frac{Y_{n_1,\cdots,n_l}}{n_1!\cdots n_l!}t_1^{n_1}\cdots t_l^{n_l}. \tag{3.24}$$

Three examples in differential polynomial form are

$$\begin{cases} Y_{x,t} = y_{xt} + y_x y_t,\ Y_{2x,t} = y_{2x,t} + y_{2x} y_t + 2y_{xt} y_x + y_x^2 y_t, \\ Y_{3x,t} = y_{3x,t} + y_{3x} y_t + 3y_{2x,t} y_x + 3y_{2x} y_{xt} + 3y_{2x} y_x y_t + 3y_x^2 y_{xt} + y_x^3 y_t. \end{cases} \tag{3.25}$$

Based on (3.24), we can show the homogeneous property:

$$Y_{n_1,\cdots,n_l}(\alpha^{r_1+\cdots r_l} y_{r_1,\cdots,r_l}) = \alpha^{n_1+\cdots+n_l} Y_{n_1,\cdots,n_l}(y_{r_1,\cdots,r_l}), \tag{3.26}$$

and the general Lebnitz rule

$$(fg)^{-1}\partial_{x_1}^{n_1}\cdots\partial_{x_l}^{n_l} fg = \sum_{i_1=0}^{n_1}\cdots\sum_{i_l=0}^{n_l}\prod_{j=1}^{l}\binom{n_j}{i_j}(f^{-1}\partial_{x_1}^{n_1-i_1}\cdots\partial_{x_l}^{n_l-i_l} f)(g^{-1}\partial_{x_1}^{i_1}\cdots\partial_{x_l}^{i_l} g) \tag{3.27}$$

implies the addition formula for the multivariate Bell polynomials:

$$Y_{n_1x_1,\cdots,n_lx_l}(y+y') = \sum_{i_1=0}^{n_1}\cdots\sum_{i_l=0}^{n_l}\prod_{j=1}^{l}\binom{n_j}{i_j} Y_{(n_1-i_1)x_1,\cdots,(n_l-i_l)x_l}(y) Y_{i_1x_1,\cdots,i_lx_l}(y'). \tag{3.28}$$

Similarly for the sake of computational convenience, we assume that

$$f = e^{\xi(x_1,\cdots,x_l)},\ g = e^{\eta(x_1,\cdots,x_l)}. \tag{3.29}$$

Then by (3.26) and (3.28), we can compute that

$$\begin{aligned} &(fg)^{-1} D_{p,x_1}^{n_1}\cdots D_{p,x_l}^{n_l} f\cdot g \\ &= \sum_{i_1=0}^{n_1}\cdots\sum_{i_l=0}^{n_l}\prod_{j=1}^{l}\alpha^{i_j}\binom{n_j}{i_j}(e^{-\xi}\partial_{x_1}^{n_1-i_1}\cdots\partial_{x_l}^{n_l-i_l} e^{\xi})(e^{-\eta}\partial_{x_1}^{i_1}\cdots\partial_{x_l}^{i_l} e^{\eta}) \\ &= \sum_{i_1=0}^{n_1}\cdots\sum_{i_l=0}^{n_l}\prod_{j=1}^{l}\alpha^{i_j}\binom{n_j}{i_j} Y_{(n_1-i_1)x_1,\cdots,(n_l-i_l)x_l}(\xi) Y_{i_1x_1,\cdots,i_lx_l}(\eta) \\ &= \sum_{i_1=0}^{n_1}\cdots\sum_{i_l=0}^{n_l}\prod_{j=1}^{l}\binom{n_j}{i_j}\ Y_{(n_1-i_1)x_1,\cdots,(n_l-i_l)x_l}(\xi_{r_1,\cdots,r_l}) Y_{i_1x_1,\cdots,i_lx_l}(\alpha^{r_1+\cdots+r_l}\eta_{r_1,\cdots,r_l}) \\ &= Y_{n_1,\cdots,n_l}(y_{r_1,\cdots,r_l} = \xi_{r_1,\cdots,r_l} + \alpha^{r_1+\cdots+r_l}\eta_{r_1,\cdots,r_l}). \end{aligned} \tag{3.30}$$

Let us now introduce binary multivariate Bell polynomials in differential polynomial form:

$$\mathcal{Y}_{p;n_1x_1,\cdots,n_lx_l}(v,w) = Y_{n_1,\cdots,n_l}\left(y_{r_1,\cdots,r_l} = \xi_{r_1x_1,\cdots,r_lx_l} + \alpha^{r_1+\cdots+r_l}\eta_{r_1x_1,\cdots,r_lx_l}\right), \tag{3.31}$$

where

$$w = \xi + \eta,\ v = \xi - \eta. \tag{3.32}$$

Then from (3.30), a combinatorial formula follows

$$(fg)^{-1} D_{p,x_1}^{n_1}\cdots D_{p,x_l}^{n_l} f\cdot g = \mathcal{Y}_{p;n_1x_1,\cdots,n_lx_l}(v = \ln\frac{f}{g}, w = \ln fg). \tag{3.33}$$

Further setting the following multivariate $\mathcal{P}$-polynomials:

$$\mathcal{P}_{p;n_1x_1,\cdots,n_lx_l}(q) = \mathcal{Y}_{p;n_1x_1,\cdots,n_lx_l}(v=0, w=q). \tag{3.34}$$

For example, we have

$$\mathcal{P}_{3;x,t}(q) = 0,\ \mathcal{P}_{3;2x,t} = \frac{1}{4}q_x^2 q_t,\ \mathcal{P}_{3;3x,t} = \frac{3}{4}q_x^2 q_{xt} + \frac{3}{8}q_x^3 q_t + \frac{3}{4}q_x q_{2x} q_t. \tag{3.35}$$

It now follows that

$$f^{-2} D_{p,x_1}^{n_1} \cdots D_{p,x_l}^{n_l} f \cdot f = \mathcal{P}_{p;n_1 x_1,\cdots,n_l x_l}(q = 2\ln f). \tag{3.36}$$

Thus, a bilinear equation

$$\mathcal{F}(D_{p,x_1}, \cdots D_{p,x_l}) f \cdot f = 0 \quad \text{with} \quad \mathcal{F}(x_1, \cdots, x_l) = \sum_{i_1,\cdots,i_l=0}^{n} c_{i_1,\cdots,i_l} x_1^{i_1} \cdots x_l^{i_l} \tag{3.37}$$

is equivalent to an equation on a linear combination of multivariate $\mathcal{P}$-polynomials in $q = 2\ln f$:

$$\sum_{i_1,\cdots,i_l=0}^{n} c_{i_1,\cdots,i_l} \mathcal{P}_{p;i_1 x_1,\cdots,i_l x_l}(q = 2\ln f) = 0, \tag{3.38}$$

where the coefficients $c_{i_1,\cdots,i_l}$'s are constants. This is a characterization for generalized bilinear equations defined through the D_p-operators.

4. Linear superposition principle

4.1. Subspaces of solutions

Let $\mathcal{F}(x_1, \cdots, x_l)$ be a multivariate polynomial. Consider a bilinear equation

$$\mathcal{F}(D_{p,x_1}, \cdots, D_{p,x_l}) f \cdot f = 0. \tag{4.1}$$

Define a set of N wave variables

$$\theta_i = k_{1,i} x_1 + \cdots + k_{l,i} x_l,\ 1 \le i \le N, \tag{4.2}$$

where the $k_{j,i}$'s are constants, and form a linear combination of N exponential waves

$$f = \sum_{i=1}^{N} \varepsilon_i\, e^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i\, e^{k_{1,i} x_1 + \cdots + k_{l,i} x_l}, \tag{4.3}$$

where all ε_i's are arbitrary constants.

Note that we have the bilinear identities:

$$\mathcal{F}(D_{p,x_1}, \cdots, D_{p,x_l})\, e^{\theta_i} \cdot e^{\theta_j} = \mathcal{F}(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j})\, e^{\theta_i + \theta_j},\ 1 \le i, j \le N, \tag{4.4}$$

where the powers of α obey the rule (2.2). Therefore, we can have the following criterion for obtaining the linear subspaces of solutions defined by (4.3) (see also [13, 14]).

Theorem: *Let $N \ge 1$. An arbitrary linear combination of N exponential waves defined by* (4.3) *solves the generalized bilinear equation* (4.1) *iff the constants $k_{j,i}$'s satisfy*

$$\mathcal{F}(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + \mathcal{F}(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0,\ 1 \le i \le j \le N. \tag{4.5}$$

Let $\mathcal{F}$ be a multivariate polynomial defined as in (3.37). Obviously, the formula (3.33) yields

$$D_{p,x_1}^{i_1} \cdots D_{p,x_l}^{i_l} e^{\theta_i} \cdot e^{\theta_j} = \mathcal{Y}_{p;i_1x_1,\cdots,i_lx_l}(\theta_i - \theta_j, \theta_i + \theta_j) e^{\theta_i+\theta_j}, \ 1 \le i, j \le N. \quad (4.6)$$

Thus, we obtain an equivalent theorem on the linear subspaces of exponential N-wave solutions.

Theorem': *Let $\mathcal{F}$ be defined by* (3.37) *and $N \ge 1$. An arbitrary linear combination of N exponential waves defined by* (4.3) *presents a solution to the generalized bilinear equation* (4.1) *iff the wave variables θ_i's satisfy*

$$\sum_{i_1,\cdots,i_l=0}^{n} c_{i_1,\cdots,i_l}[\mathcal{Y}_{p;i_1x_1,\cdots,i_lx_l}(\theta_i - \theta_j, \theta_i + \theta_j) + \mathcal{Y}_{p;i_1x_1,\cdots,i_lx_l}(\theta_j - \theta_i, \theta_j + \theta_i)] = 0, \ 1 \le i \le j \le N. \quad (4.7)$$

This theorem has an advantage that the wave variables θ_i's can be nonlinear functions of dependent variables, but the first theorem only works for linear wave variables θ_i's.

Given a multivariate polynomial $\mathcal{F}$, below is one way of solving the system (4.5) or (4.7) for $k_{j,i}$ and $c_{i_1,\cdots,i_l}$, in order to obtain bilinear equations and their linear subspaces of solutions (see, e.g., [13, 15]). We adopt a kind of parameterization for wave numbers and frequencies and list the sequential solution procedure as follows:

- Introduce weights for the independent variables:

$$(w(x_1), \cdots, w(x_l)) = (w_1, \cdots, w_l), \quad (4.8)$$

  where the weights w_i's can be both positive and negative.
- Form a homogeneous polynomial $\mathcal{F}(x_1, \cdots, x_l)$, defined by (3.37), in some weight.
- Parameterize $k_{1,i}, \cdots, k_{l,i}$ using a parameter k_i:

$$k_{j,i} = b_j k_i^{w_j}, \ 1 \le j \le l, \quad (4.9)$$

  and then determine the proportional constants b_j's and the coefficients $c_{i_1,\cdots,i_l}$'s by solving the system (4.5) or (4.7).

4.2. Illustrative examples

To present illustrative examples, we consider the $(3 + 1)$-dimensional case with

$$(w(x), w(y), w(z), w(t)) = (w_x, w_y, w_z, w_t), \quad (4.10)$$

and

$$\theta_i = k_i x + l_i y + m_i z - \omega_i t, \ l_i = b_1 k_i^{w_y}, \ m_i = b_2 k_i^{w_z}, \ \omega_i = -b_3 k_i^{w_t}, \ 1 \le i \le N. \quad (4.11)$$

Then, upon forming a homogeneous multivariate polynomial in some weight

$$\mathcal{F} = \sum_{i_1,i_2,i_3,i_4=1}^{n} c_{i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} t^{i_4}, \quad (4.12)$$

we solve the system (4.5) or (4.7) for the proportional constants b_1, b_2, b_3 and the coefficients c_{i_1,i_2,i_3,i_4}'s so that we can determine the corresponding bilinear equations and their associated linear subspaces of solutions consisting of linear combinations of exponential waves. We are going to present two concrete illustrative examples by applying this general idea below.

Example 1 - Example with positive weights

Let us set the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, 2, 4, 5), \tag{4.13}$$

and consider a polynomial being homogeneous in weight 6:

$$\mathcal{F} = c_1x^6 + c_2x^4y + c_3x^2z + c_4xt + c_5yz. \tag{4.14}$$

Following the parameterization of wave numbers and frequencies in (4.11), we set the wave variables

$$\theta_i = k_ix + b_1k_i^2y + b_2k_i^4z + b_3k_i^5t,\ 1 \le i \le N, \tag{4.15}$$

where k_i, $1 \le i \le N$, are arbitrary constants but the proportional constants b_1, b_2 and b_3 are to be determined by (4.5).

Now, a direct computation shows that the corresponding bilinear equation reads

$$\begin{aligned}\mathcal{F}(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f\cdot f &= 2c_1f_{6x}f + 20c_1f_{3x}^2 + 2c_2f_{4x,y}f - 2c_2f_{4x}f_y - 8c_2f_{3x,y}f_x \\ &+8c_2f_{3x}f_{xy} + 12c_2f_{2x,y}f_{2x} + 2c_3f_{2x,z}f + 2c_4f_{xt}f - 2c_4f_xf_t + 2c_5f_{yz}f - 2c_5f_yf_z = 0.\end{aligned} \tag{4.16}$$

The corresponding linear subspace of N-wave solutions is given by

$$f = \sum_{i=1}^{N} \varepsilon_i\, e^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i\, e^{k_ix + b_1k_i^2y + b_2k_i^3z + b_3k_i^4t}, \tag{4.17}$$

where $\varepsilon_i,\ 1 \le i \le N$, are arbitrary constants but the proportional constants b_1, b_2 and b_3 are defined by

$$b_1 = -\frac{25c_3}{8c_5},\ b_2 = \frac{5c_2}{c_5},\ b_3 = \frac{10c_1}{c_4}, \tag{4.18}$$

when the coefficients of the polynomial $\mathcal{F}$ satisfy

$$4c_1c_5 = 5c_2c_3. \tag{4.19}$$

Example 2 - Example with positive and negative weights

Let us set the weights of independent variables:

$$(w(x), w(y), w(z), w(t)) = (1, -1, -2, 3), \tag{4.20}$$

and consider a polynomial being homogeneous in weight 3:

$$\mathcal{F} = c_1x^3 + c_2x^5z + c_3xyt. \tag{4.21}$$

Following the parameterization of wave numbers and frequencies in (4.11), we set the wave variables

$$\theta_i = k_ix + b_1k_i^{-1}y + b_2k_i^{-2}z + b_3k_i^3t,\ 1 \le i \le N, \tag{4.22}$$

where k_i, $1 \le i \le N$, are arbitrary constants but the proportional constants b_1, b_2 and b_3 are to be determined by (4.5).

Similarly, a similar direct computation shows that the corresponding bilinear equation reads

$$\mathcal{F}(D_{3,x}, D_{3,y}, D_{3,z}, D_{3,t})f\cdot f = 2c_1f_{3x}f + 2c_2f_{5x,z}f + 20c_2f_{3x}f_{2x,z} + 2c_3f_{xyt}f = 0, \tag{4.23}$$

and it possesses the linear subspace of N-wave solutions determined by

$$f = \sum_{i=1}^{N} \varepsilon_i\, e^{\theta_i} = \sum_{i=1}^{N} \varepsilon_i\, e^{k_ix + b_1k_i^{-1}y + b_2k_i^{-2}z + b_3k_i^3t}, \tag{4.24}$$

where the ε_i's and k_i's are arbitrary but b_1, b_2 and b_3 need to satisfy

$$c_1 + 11c_2b_2 + c_3b_1b_3 = 0. \tag{4.25}$$

5. Conclusion and further questions

We created a kind of generalized bilinear differential operators $D_{p,x}$, discussed their links with the Bell polynomials, and applied the linear superposition principle to the corresponding bilinear equations. Two illustrative examples in the case of $p = 3$ were made to shed light on the general framework.

There are, however, many other questions which are worth further investigation. Below is just a few of them.

Question 1 - Mixed bilinear equations

We can mix D_p-operators with different natural numbers p to formulate a more general bilinear equation, for example,

$$\sum_{p_1,\cdots,p_l=1}^{m} \sum_{i_1,\cdots,i_l=0}^{n} c_{i_1,\cdots,i_l}^{p_1,\cdots,p_l} D_{p_1,x_1}^{i_1} \cdots D_{p_l,x_l}^{i_l} f \cdot f = 0,$$

where the coefficients $c_{i_1,\cdots,i_l}^{p_1,\cdots,p_l}$'s are constants. This kind of mixed combinations will bring diversity in establishing links with binary Bell polynomials and formulations for linear subspaces of solutions.

Question 2 - Geometries related to multivariate polynomials

What kind of geometries of a multivariate polynomial $\mathcal{F}$ does the equation

$$\mathcal{F}(k_1 + \alpha k_1', \cdots, k_l + \alpha k_l') + \mathcal{F}(k_1' + \alpha k_1, \cdots, k_l' + \alpha k_l) = 0$$

define? It determines an affine geometry of $\mathcal{F}$ when $p = 2k$.

Question 3 - Parameterizations achieved by multiple parameters

Parameterize $k_{1,i}, \cdots, k_{l,i}$ using multiple parameters, for example, two parameters k_i and l_i:

$$k_{j,i} = \sum_{r=0}^{w_i} b_{j,r} k_i^r l_i^{w_i - r}, \ 1 \le j \le l.$$

What kind of spaces can exist for the proportional constants $b_{j,r}$ which will solve the system

$$\mathcal{F}(k_{1,i} + \alpha k_{1,j}, \cdots, k_{l,i} + \alpha k_{l,j}) + \mathcal{F}(k_{1,j} + \alpha k_{1,i}, \cdots, k_{l,j} + \alpha k_{l,i}) = 0, \ 1 \le i \le j \le N?$$

Question 4 - Bilinear Bäcklund transformations

In the case of the Hirota D-operators, binary Bell polynomials are used to build bilinear Bäcklund transformations for soliton equations [16]. Is there any theory in the general case of D_p-operators? This case should be a little bit more difficult than the Hirota case, noticing a fact that the sign function $(-1)^{r(i)}$ in the definition (2.2) does not satisfy

$$(-1)^{r(i+j)} = (-1)^{r(i)+r(j)}, \ i, j \ge 0,$$

when $p > 1$ is odd, while it is true when p is one or even. It is obvious that the above property holds when p is one or even; but it does not hold because we have

$$(-1)^{r(p)} \ne (-1)^{r(p-1)+r(1)}, \text{ due to } (-1)^{r(p)} = 1, \ (-1)^{r(p-1)} = 1, \ (-1)^{r(1)} = -1,$$

when $p > 1$ is odd. The property is also crucial in deriving Lax pairs from bilinear Bäcklund transformations (see, e.g., [11, 12]).

Question 5 - Criterion for multivariate polynomials with one zero

While we used multivariate polynomials, which have one zero and only one zero, to determine bilinear equations with given linear subspaces of solutions, we came up with an interesting question [15]: How can one determine if a multivariate polynomial $\mathcal{F}(x_1, \cdots, x_l)$ with real coefficients has one and only one zero in $\mathbb{R}^l$? Two examples of such multivariate polynomials are as follows:

$$x^2 + y^2 - 2y + 1, \text{ zero } (x, y) = (0, 1);$$
$$5x^2 + 4xy + y^2 - 2x - 2y + 2, \text{ zero } (x, y) = (-1, 3).$$

This problem is more general than Hilbert's 17th problem, since all such multivariate polynomials satisfy all the conditions in Hilbert's 17th problem. It is hoped that there would be a definitive answer to the question.

Acknowledgments

The work was supported in part by the State Administration of Foreign Experts Affairs of China, the National Natural Science Foundation of China (Nos. 61072147 and 11071159), Chunhui Plan of the Ministry of Education of China, the Natural Science Foundation of Shanghai (No. 09ZR1410800) and the Shanghai Leading Academic Discipline Project (No. J50101).

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