

GEOMETRY OF DISCRIMINANTS AND COHOMOLOGY OF MODULI SPACES

EEN WETENSCHAPPELIJKE PROEVE OP HET GEBIED
VAN DE NATUURWETENSCHAPPEN, WISKUNDE EN INFORMATICA

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR
AAN DE RADBOUD UNIVERSITEIT NIJMEGEN
OP GEZAG VAN DE RECTOR MAGNIFICUS PROF. DR. C.W.P.M. BLOM,
VOLGENS BESLUIT VAN HET COLLEGE VAN DECANEN
IN HET OPENBAAR TE VERDEDIGEN OP MAANDAG 28 NOVEMBER 2005
DES NAMIDDAGS OM 1.30 UUR PRECIES
DOOR

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GEBOREN OP 29 NOVEMBER 1974
TE TRIËST, ITALIË.

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ISBN-10: 90-9019997-7

ISBN-13: 978-90-9019997-9

Acknowledgments

It is a pleasure to me to thank everybody who directly or indirectly contributed to the development of this thesis. First of all, I would like to thank my advisor Joseph Steenbrink for proposing the subject of this thesis as my Ph.D. project. Indeed, I have enjoyed my ‘discriminants’ very much. Furthermore, I am indebted to him for many useful discussions and suggestions.

I would like to thank Carel Faber, Ezra Getzler and Gert Heckman for the interest shown in my work. I am particularly grateful to Carel Faber for the invitation to visit KTH in the Fall of 2004. He also suggested the project that led to what became chapter II in this thesis.

I would also like to take this opportunity to thank the people with whom I worked during my Ph.D. studies: Jonas Bergström, Remke Kloosterman and Emilia Mezzetti. They all gave their time to develop our joint projects, and I appreciate very much their kindness. I am grateful to Jonas for letting me take a look at the wonderful world of the equivariant count of points of moduli spaces, and for his patience with my endless lists of questions and observations. I am very grateful to Emilia for her encouragement and for helping me to remain in contact with the Italian mathematical world.

Finally, I would like to thank all people who helped me feel at home during these four years in the Netherlands. In particular, thanks are due to Martijn and Marc, who were my roommates during my four years at the Radboud University Nijmegen, for their company and their kindness; to Trees van den Eem and Willy van de Sluis for their help and secretarial support; to Erik and Stan for many friendly conversations; and in general to all my fellow Ph.D. students. Outside my mathematical life, I wish to thank Hessie and Bindert for their sympathy and their daring to be among the first ones to speak Dutch with me.

I would like to thank Remke for being there.

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Introduction

1. Moduli spaces of curves

The object of this thesis is to investigate the topology of certain moduli spaces of curves. We are interested in a basic topological invariant of moduli spaces of complex curves, their *cohomology with rational coefficients* (also called rational cohomology). In particular, we are interested in studying the cases in which it is possible to relate the rational cohomology of the moduli space with that of the complement of a discriminant in a complex vector space.

When dealing with moduli spaces of curves, there are basically two types of curves to be considered, namely, non-singular curves and stable curves.

By a non-singular algebraic curve defined over a field k of we mean a non-singular projective irreducible scheme defined over k of dimension one. The basic invariant used to classify curves is the *genus*, which can be defined in different equivalent ways. We consider the *geometric genus* of a non-singular complete curve C as the dimension of the space of regular differentials over C . Regular differentials are differential forms which can be locally written as linear combinations of differentials of regular functions. This definition makes sense for complete non-singular curves defined over any field. If we assume that the ground field is the field \mathbf{C} of complex numbers, complete smooth algebraic curves can be viewed as real manifolds of dimension 2. The surfaces we get in this way are compact and complete, and are called *Riemann surfaces*. Indeed, the two concepts of complete connected complex non-singular curves and compact Riemann surfaces are equivalent. The topological genus of a compact Riemann surface equals the geometric genus of the algebraic curve corresponding to the surface. This provides a very easy way to visualize what genus is, since the topological genus is just the number of “holes” the surface has.

Let us look at how the correspondence between algebraic curves and Riemann surfaces works for low genus. The projective line \mathbf{P}^1 is an example of a non-singular curve of genus 0. Viewing $\mathbf{P}^1(\mathbf{C})$ as a Riemann surface yields a two-dimensional sphere, which has no holes. All non-singular curves of genus one are homeomorphic to complex tori in the analytic topology. This means that the associated Riemann surface looks very much like a doughnut. A Riemann surface of genus 2 is a surface with two holes. We can obtain it by deleting a small disc from two tori and gluing them together along the borders of the discs.

An n -pointed non-singular curve of genus g is simply an $(n+1)$ -tuple (C, p_1, \dots, p_n) where C is a non-singular curve and p_1, \dots, p_n are n distinct points on C . An isomorphism $\varphi : (C, p_1, \dots, p_n) \rightarrow (C', q_1, \dots, q_n)$ of n -pointed curves is defined to be an isomorphism from C to C' mapping p_i to q_i for every $i = 1, \dots, n$. An isomorphism from an n -pointed curve to itself is called an automorphism.

When the Euler characteristic $2 - 2g - n$ of $C \setminus \{p_1, \dots, p_n\}$ is negative, the group of automorphisms of (C, p_1, \dots, p_n) is always finite. In these cases, by the results of geometric invariant theory (see [MFK94, Theorem 5.11]), there exists a quasi-projective irreducible scheme $\mathcal{M}_{g,n}$ of dimension $3g + n - 3$ that parametrizes isomorphism classes of n -pointed curves. The variety $\mathcal{M}_{g,n}$ is what is called a *coarse moduli space*. Roughly speaking, this means that, though $\mathcal{M}_{g,n}$ does parametrize isomorphism classes of n -pointed curves quite well, it is not possible to use $\mathcal{M}_{g,n}$ to study families of curves. This problem is solved by giving $\mathcal{M}_{g,n}$ a more sophisticated structure, that of a stack. Then the moduli spaces $\mathcal{M}_{g,n}$ are smooth irreducible Deligne-Mumford stacks defined over $\text{Spec } \mathbf{Z}$. Note that when $n < 2g + 2$, the coarse moduli spaces $\mathcal{M}_{g,n}$ are singular, with few exceptions when g and n are both very small. The singular points of the coarse moduli space correspond to isomorphism classes of curves with automorphism group bigger than the general one. Note that the converse is not true, i.e., not all curves with bigger automorphism group give singular points of the coarse moduli space. The singularities of $\mathcal{M}_{g,n}$ are locally quotient singularities over the field of complex numbers.

Though moduli spaces of non-singular curves have many nice properties, they are not complete (with the only exception of $\mathcal{M}_{0,3}$, which is a point). The reason is that a family whose general elements are non-singular curves can nevertheless contain exceptional fibres that are singular curves. Geometric invariant theory provides a natural way to complete the moduli spaces $\mathcal{M}_{g,n}$ by providing the concept of *stable* curves. A stable n -pointed curve of genus g is an $(n + 1)$ -tuple (C, p_1, \dots, p_n) , where C is a connected nodal curve of genus g , and p_1, \dots, p_n are distinct non-singular points on C . Here the genus of C is defined as the geometric genus of its resolution of singularities; when the latter is reducible, the sum of the genera of the components is taken. The moduli stacks $\overline{\mathcal{M}}_{g,n}$ are irreducible non-singular proper Deligne-Mumford stacks defined over $\text{Spec } \mathbf{Z}$, and the underlying coarse moduli spaces are irreducible projective varieties, with locally quotient singularities over \mathbf{C} (for instance, see [DM69] and [Knu83]).

Note that in general we will abuse notation and do not distinguish between moduli stacks and underlying coarse moduli spaces. The reason is that the properties we consider are the same for both structures. Indeed, the rational cohomology of a stack (over \mathbf{C}), which is the main object of this thesis, coincide with that of the underlying moduli space, and an analogous statement holds for the number of points defined over finite fields taken into account in chapter II.

2. Discriminants

Let us consider a projective variety $\tilde{\Sigma}$ in a projective space \mathbf{P}^M . In many cases, the elements of \mathbf{P}^M can be seen as parametrizing subvarieties of another projective variety Z . The most natural example is that of hypersurfaces in projective space. Hypersurfaces are the subvarieties of $Z := \mathbf{P}^m$ defined by one equation. Once coordinates x_0, x_1, \dots, x_m on \mathbf{P}^m are chosen, every hypersurface of degree d is given by the vanishing of a homogeneous polynomial of degree d in x_0, \dots, x_m , and this polynomial is unique up to multiplication by a constant. For this reason, there is a bijection between the set of hypersurfaces in \mathbf{P}^m and the projectivization of the vector space $\mathbf{C}[x_0, \dots, x_m]_d$, which is a projective space \mathbf{P}^M of dimension $M = \binom{d+m}{d} - 1$.

Inside the \mathbf{P}^M of hypersurfaces, there is a locus of hypersurface with degenerate properties, the *singular* hypersurfaces. A hypersurface is singular at a point $p \in \mathbf{P}^M$ when

it is defined by a polynomial f such that $\frac{\partial f}{\partial x_0}(p) = \dots = \frac{\partial f}{\partial x_m}(p) = 0$. Singular hypersurfaces are hypersurfaces that have at least one singular point. The locus of singular hypersurfaces of degree d is an irreducible hypersurface in \mathbf{P}^M , called the *discriminant*.

It is straightforward to generalize this definition of discriminant to the following situation. Let Z be a complex projective variety, and L a vector bundle on Z . We define V to be a linear subspace of the space of sections of L . Note that the 0-locus of a section of a vector bundle remains the same when the section is multiplied by a non-zero constant, hence it is more appropriate to work with the projectivization $\mathbf{P}(V)$ of V . Inside $\mathbf{P}(V)$ there is a closed subvariety $\tilde{\Sigma}$ of sections whose zero scheme is not smooth of the expected dimension. We call $\tilde{\Sigma}$ the *discriminant*. Note that we expect the discriminant to be a hypersurface, but that this is not always the case.

Note that this characterization of the discriminant is compatible with the classical concept of discriminants as hypersurfaces in \mathbf{P}^M which are the dual of a lower-dimensional variety. Recall that the hyperplanes in \mathbf{P}^M can be seen as points of the dual projective space $\check{\mathbf{P}}^M$, which can be identified with \mathbf{P}^M once coordinates are chosen. For every variety Z in \mathbf{P}^M , it makes sense to consider the set of hyperplanes tangent to Z . If Z is singular, we consider the set of hyperplanes tangent to Z at the non-singular points of Z , and then take the closure of this in $\check{\mathbf{P}}^M$ in the Zariski topology, that is, the smallest projective variety containing all these points. The variety so obtained is called the *dual variety* of Z , and is expected to be a hypersurface in $\check{\mathbf{P}}^M \cong \mathbf{P}^M$.

Indeed, it is not difficult to interpret the dual variety as a discriminant in the sense explained above. Every hyperplane Λ in $\check{\mathbf{P}}^M \cong \mathbf{P}^M$ defines a subvariety of Z , namely, the intersection $Z \cap \Lambda$. If $Z \cap \Lambda$ is singular, then Λ must be tangent to Z , and also the converse holds (with the only exception of the case where Λ is a component of Z).

More information on the general theory of discriminants can be found in [GKZ94]. Note that, in this thesis, we will consider discriminants not in projective spaces, but in vector spaces. This simply means that we consider not the variety $\tilde{\Sigma} \subset \mathbf{P}^M$, but the variety Σ in \mathbf{C}^{M+1} defined by the same homogeneous equation as $\tilde{\Sigma}$, i.e., the affine cone over Σ .

Now that the concept of discriminant has been introduced, we explain the relationship between moduli spaces of non-singular curves over \mathbf{C} and discriminants in complex vector spaces. We start with an example: the moduli space \mathcal{M}_2 of non-singular curves of genus 2. It is known that every curve of genus two has a canonical structure as the double cover of \mathbf{P}^1 ramified at six points. This allows us to identify \mathcal{M}_2 with the moduli space of unordered configurations of six distinct points on the projective line, or, equivalently, with the moduli space of homogeneous binary polynomials of degree six without multiple roots. Denote by V the vector space $\mathbf{C}[x, y]_6$ of homogeneous polynomials of degree six. The action of the general linear group $\mathrm{GL}(2)$ on \mathbf{P}^1

$$\begin{array}{ccc} \mathrm{GL}(2) \times \mathbf{P}^1 & \longrightarrow & \mathbf{P}^1 \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [x, y] \right) & \longmapsto & [ax + by, cx + dy] \end{array}$$

induces an action of $\mathrm{GL}(2)$ on V . Denote by Σ the locus of singular polynomials in V , i.e., the discriminant. As we explained at the beginning of this section, Σ is the cone over an irreducible hypersurface in $\mathbf{P}(V)$. By the considerations above, the moduli space \mathcal{M}_2 is the quotient of $X = V \setminus \Sigma$ by the action of $\mathrm{GL}(2)$.

For curves of higher genus, the whole moduli space cannot be described as the geometric quotient of the complement of a discriminant. Nevertheless, for a fixed genus g , it is possible to find subloci $\mathcal{N} \subset \mathcal{M}_g$ that are isomorphic to the geometric quotient of a quasi-projective variety $X_{\mathcal{N}}$ by a linear action of a reductive group $G_{\mathcal{N}}$. We are only interested in the case where the variety $X_{\mathcal{N}}$ is the complement of a (not necessarily irreducible) divisor Σ in a complex projective space V . This is the situation in which the methods in this thesis apply. In general, it is possible to associate with every vector in V a curve of genus g (if the vector is in $X_{\mathcal{N}}$) or the degeneration of such a curve, which is expected to be a singular curve. Hence, the divisor Σ can be thought of as the closure of the locus of singular curves. For this reason, we can consider Σ as a discriminant in the complex vector space V .

It is important to note that, in the case of geometric quotients of complements of discriminants, it is possible to relate the cohomology of the complement $X_{\mathcal{N}}$ of the discriminant to that of the quotient $\mathcal{N} = X_{\mathcal{N}}/G_{\mathcal{N}}$ by means of the Leray spectral sequence associated to the quotient map $X_{\mathcal{N}} \rightarrow \mathcal{N}$. Furthermore, Peters and Steenbrink gave in [PS03] necessary conditions ensuring that this spectral sequence degenerates at E_2 , when the cohomology is taken with rational coefficients. Their theorem (see Theorem I.1.1, Theorem II.5.1) is a generalization of the classical Leray-Hirsch theorem, and implies that, in all cases in which we are interested, the rational cohomology of $X_{\mathcal{N}}$ is isomorphic as graded vector space to the tensor product of the cohomology of \mathcal{N} and that of the group $G_{\mathcal{N}}$. Hence, since the cohomology of reductive groups is well known, the problem of determining the rational cohomology of the moduli space \mathcal{N} is equivalent to that of determining the rational cohomology of the complement of the discriminant, $X_{\mathcal{N}} = V \setminus \Sigma$.

In this thesis, all computations of the cohomology of complements of discriminants in complex vector spaces are performed using Vassiliev-Gorinov's method ([Vas99], [Gor05], [Tom05]). In chapter I we present the version we use of this method. Our exposition is written in the language of cubical schemes. In particular, this ensures that Gorinov-Vassiliev's method respects the mixed Hodge structure of the space X . Hence we can use it to find the mixed Hodge structure of the cohomology of the complement of the discriminant.

Vassiliev-Gorinov's method consists of two steps. First, the computation of the cohomology of X is reduced to that of the (Alexander dual) Borel-Moore homology of its complement Σ . Secondly, the study of possible singular loci of the elements of Σ allows to construct a cubical space whose geometrical realization $\mathbf{D}_{\Sigma} \rightarrow \Sigma$ is a simplicial resolution of Σ , which is a homotopy equivalence that induces an isomorphism of Borel-Moore homology groups. Hence, the computation of the Borel-Moore homology of Σ is reduced to that of the domain of the geometrical realization. The latter is achieved by studying a filtration on \mathbf{D}_{Σ} based on the classification of the singular loci of the elements of Σ .

We can get an intuition of how the method works, by desingularizing Σ . Recall that we assumed that every element of Σ has a non-empty singular locus contained in a projective variety Z . Suppose that the map

$$\begin{aligned} \mathcal{J}_1 := \{(v, p) \in \Sigma \times Z : v \text{ is singular at } p\} \\ \downarrow \\ \Sigma \end{aligned}$$

is a desingularization of Σ (this is certainly true in the cases in which \mathcal{J}_1 is the total space of a vector bundle over Z , which, as we will see below, is what we expect). This desingularization is injective over the open subvariety $\Sigma_1 \subset \Sigma$ of sections with exactly one singular point. Again, the locus $\Sigma \setminus \Sigma_1$ can (in general) be desingularized by taking

$$\begin{aligned} \mathcal{J}_2 := \{(v, \{p, q\}) \in \Sigma \times \text{Sym}^2 Z : p \neq q, v \text{ is singular at } p, q, \} \\ \downarrow \\ \Sigma \setminus \Sigma_1, \end{aligned}$$

which injects to the open subvariety $\Sigma_2 \subset \Sigma \setminus \Sigma_1$ of sections with exactly two singular points. Clearly, we can go on like this, by defining inductively $\Sigma_0 = \Sigma$, constructing for every $M \geq 0$ a desingularization

$$\psi_M : \mathcal{J}_M \rightarrow \Sigma \setminus \bigcup_{1 \leq j \leq M-1} \Sigma_j$$

and setting Σ_M to be the locus of points with exactly one preimage in \mathcal{J}_M . Moreover, computing the Borel-Moore homology of the desingularizations \mathcal{J}_M reduces to the computation of the Borel-Moore homology of the spaces of singular configurations of the elements of Σ_M . For instance, there is a natural projection $pr_1 : \mathcal{J}_1 \rightarrow Z$, and if for every point p of Z the vectors in V that are singular at p form a linear subspace of a fixed dimension d_1 , the map pr_1 gives \mathcal{J}_1 the structure of a complex vector bundle of rank d_1 over Z . Analogously, it is natural to expect that the projection

$$pr_2 : \mathcal{J}_2 \rightarrow \{\{p, q\} \in \text{Sym}^2 Z : p \neq q\}$$

is a vector bundle over the space of unordered configurations of two distinct points on Z . For all discriminants we consider, for every M there are maps pr_M from \mathcal{J}_M to the space of singular configurations of the elements of Σ_M that give to \mathcal{J}_M the structure of a complex vector bundle (even if it may be necessary to refine the stratification of Σ in order to obtain this). Unfortunately, there is no clear way to relate the Borel-Moore homology of Σ to that of the spaces \mathcal{J}_M . This is why the more complicated construction, involving simplicial spaces (in the language of Vassiliev and Gorinov) or cubical spaces (in the language of chapter I, Section 2.1) is needed.

3. Outline of the results

In this thesis, we apply the constructions explained above to investigate the rational cohomology of moduli spaces of non-singular curves of genus 3 and 4. In particular, we simplify the computation of the rational cohomology by producing a stratification of \mathcal{M}_g ($g \in \{3, 4\}$) by geometric quotients of the form described in Section 2. Indeed, the construction of such a stratification on \mathcal{M}_3 and \mathcal{M}_4 follows in a very straightforward way from the study of the canonical map of the curves.

In the case of curves of genus 4, the stratification we get consists of three strata. The closed stratum is the hyperelliptic locus, i.e., the locus of curves that are double coverings of the projective line branched at 10 points. A locally closed stratum is given by the locus of curves whose canonical model lies on a quadric cone in \mathbf{P}^3 . The open stratum is the locus of curves whose canonical model lies on a non-singular quadric surface in \mathbf{P}^3 . It turns out that in these cases it is possible to compute the cohomology of all the complements of discriminants involved in the stratification. This allows us to compute in chapter I the rational cohomology of \mathcal{M}_4 .

In the case of \mathcal{M}_3 , the stratification consists of just two pieces: the hyperelliptic locus \mathcal{H}_3 and its complement \mathcal{Q} . In particular, every curve in \mathcal{Q} has a plane non-singular curve of degree 4 as its canonical model, hence \mathcal{Q} is the moduli space of non-singular quartic curves in the projective plane. Note that the rational cohomology of \mathcal{M}_3 and \mathcal{Q} was computed by Looijenga in [Loo93]. Given the simplicity of the construction of the stratification, we exploited our constructions to get results on the cohomology of $\mathcal{M}_{3,1}$ and $\mathcal{M}_{3,2}$ as well. Again, we stratify these moduli spaces $\mathcal{M}_{3,n}$ in such a way that the strata are locally trivial fibrations with as fibre the complement of a discriminant in some vector space. In the case of $\mathcal{M}_{3,1}$, we study the cohomology of its strata and obtain its rational cohomology (see Corollary II.5.5). Note that the Hodge Euler characteristic of $\mathcal{M}_{3,1}$ was already known (see [GL], [Loo93]), but its rational cohomology was not completely determined.

In the case of $\mathcal{M}_{3,2}$, it seems that these methods are not enough to recover a complete result. In particular, the interplay between the cohomology of the two strata $\mathcal{H}_{3,2}$ and \mathcal{Q}_2 inside $\mathcal{M}_{3,2}$ cannot be obtained from the construction considered in this thesis. For this reason, we chose to limit ourselves to the computation of the Euler characteristic of the rational cohomology of $\mathcal{M}_{3,2}$ in the Grothendieck group of mixed Hodge structures, or, equivalently, the Hodge Euler characteristic of $\mathcal{M}_{3,2}$. We recover in this way the result obtained by Jonas Bergström by a different method (see [Berb] and chapter II, Section 4.3). Furthermore, we studied the rational cohomology of the moduli space $\mathcal{H}_{g,2}$ of hyperelliptic curves of a given genus $g \geq 2$ with two marked points, and determined its rational cohomology groups with their mixed Hodge structures and their structure as representations of the symmetric group \mathfrak{S}_2 (see Theorem III.2.2).

Moreover, as a logical application of the above, we compute the rational cohomology of the moduli space $\overline{\mathcal{M}}_4$ of complex stable curves of genus 4. This is done in chapter II, which is joint work with Jonas Bergström. The computation of the cohomology of $\overline{\mathcal{M}}_4$ is obtained by studying the cohomology with compact support of its stratification by topological type of the curves. Roughly speaking, this stratification is constructed as follows. The moduli space \mathcal{M}_4 is open and dense in $\overline{\mathcal{M}}_4$. The complement $\partial\mathcal{M}_4$ of \mathcal{M}_4 has three divisors as components. We can consider all possible intersections and self-intersections of these components, and use them to define a stratification in $\overline{\mathcal{M}}_4$ by disjoint locally closed subloci. It is a known fact in the geometry of moduli spaces of stable curves that all strata so obtained are products of moduli spaces of stable curves of a smaller genus, or quotients of such products by the action of a finite group. Therefore, it is intuitive to expect that many results on the cohomology of $\overline{\mathcal{M}}_4$ can be obtained by studying \mathcal{M}_4 and moduli spaces of curves of genus ≤ 3 . Indeed, there is a formula by Getzler and Kapranov ([GK98]) that gives the relationship between the generating functions of the \mathfrak{S}_n -equivariant Hodge Euler characteristic of $\mathcal{M}_{g,n}$ and of $\overline{\mathcal{M}}_{g,n}$. Recall that the Hodge Euler characteristic is the Euler characteristic of the rational cohomology with compact supports of a space in the Grothendieck group of rational mixed Hodge structures, and that the \mathfrak{S}_n -equivariant Hodge Euler characteristic is an adaptation of the Hodge Euler characteristic that also keeps trace of the structure of the cohomology groups as \mathfrak{S}_n -representations.

In chapter II, we use Getzler-Kapranov's formula to compute the Hodge Euler characteristic of $\overline{\mathcal{M}}_4$. Since $\overline{\mathcal{M}}_4$ is complete and satisfies Poincaré duality, its Hodge Euler characteristic is enough to recover the whole structure of the rational cohomology as

graded vector space with mixed Hodge structures. To get the result, we need the knowledge of the \mathfrak{S}_n -equivariant Hodge Euler characteristic of all moduli spaces $\mathcal{M}_{g,n}$ with $g \leq 4$, $n \leq 2g - 8$. Many of these Euler characteristics were already known (see [Get95], [Get99], [Get98b], [Loo93] and [GL]), the others follow from the results in this thesis or from Jonas Bergström's equivariant counts of the number of points of $\mathcal{M}_{g,n}$ defined over finite fields. The relationship between the number of points of $\mathcal{M}_{g,n}$ defined over finite fields and Hodge Euler characteristics follows from a theorem by Van den Bogaart-Edixhoven ([BE05]), and is explained in Section 3 of chapter II.

Notations and conventions

Symbol	Description
\mathbf{C}^n	the Euclidean space $\mathbf{C}^n \cong \mathbf{R}^{2n}$
\mathbf{P}^n	complex projective space of dimension n over \mathbf{C}
$G(m, V)$	Grassmannian of linear subspaces of dimension m in the vector space V
\mathfrak{S}_n	symmetric group in n elements
$\mathrm{GL}(n)$	general linear group of linear automorphisms of \mathbf{C}^n
$\mathrm{PGL}(n)$	projective linear group of automorphisms of \mathbf{P}^{n-1}
$H^\bullet(Z, S)$	cohomology of the space Z in the local system of coefficients S
$\tilde{H}^\bullet(Z, S)$	reduced cohomology of the space Z in the local system of coefficients S
$H_c^\bullet(Z, S)$	cohomology with compact support of the space Z in the local system of coefficients S
$\bar{H}_\bullet(Z, S)$	Borel-Moore homology of the space Z in the local system of coefficients S
Δ_N	N -dimensional standard simplex in \mathbf{R}^{N+1}
$\mathring{\Delta}_N$	the interior of Δ_N
$\mathbf{Q}(k)$	Tate Hodge structure on \mathbf{Q} of weight $-2k$ (for the definition, see [Del71])
$H(k)$	Tate twist $H \otimes_{\mathbf{Q}} \mathbf{Q}(k)$ of the rational Hodge structure H
$\mathrm{K}_0(\mathbf{C})$	Grothendieck group of the abelian category \mathbf{C}
\mathbf{L}	class of $\mathbf{Q}(-1)$ in the Grothendieck group of rational Hodge structures.

In this paper we make an extensive use of Borel-Moore homology, i.e., homology with locally finite support. A reference for its definition and for the properties we use can be for instance [Ful98, Chapter 19].

Following [Loo93], when H^\bullet is a rational mixed Hodge structure, we will consider the Poincaré-Serre polynomial of H^\bullet , defined as the polynomial in $\mathbf{Z}[t, u, u^{-1}]$ such that the coefficient of $t^i u^j$ is the dimension of the weight j subquotient of H^i . The Poincaré-Serre polynomial of a complex variety Z is the Poincaré-Serre polynomial of its rational cohomology. Note that in all cases we will consider, we will work with mixed Hodge structures which are sums of (rational) Tate Hodge structures. In this special case, giving the Poincaré-Serre polynomial is equivalent to giving the whole rational mixed Hodge structure.

All complex varieties will be considered with the analytic topology. If not otherwise specified, all cohomology and Borel-Moore homology groups are considered with rational coefficients.

CHAPTER I

Rational cohomology of \mathcal{M}_4

This chapter is based on the paper **[Tom05]**.

1. Introduction and results

In this chapter, we compute the rational cohomology of the moduli space \mathcal{M}_4 of non-singular complex genus 4 curves. This is achieved by considering a natural stratification of \mathcal{M}_4 , and determining the cohomology of each stratum. Non-singular genus 4 curves can be divided into 3 classes (see **[Har77, IV.5.2.2 and IV.5.5.2]**):

- (1) curves whose canonical model is the complete intersection of a cubic surface and a non-singular quadric surface in \mathbf{P}^3 ;
- (2) curves whose canonical model is the complete intersection of a cubic surface and a quadric cone in \mathbf{P}^3 ;
- (3) hyperelliptic curves.

Denote by C_0 the locus in M_4 of curves of type (1), by C_1 the locus of curves of type (2) and by C_2 the hyperelliptic locus. We have a three-steps filtration

$$(1) \quad C_2 \subset \overline{C_1} \subset \overline{C_0} = \mathcal{M}_4.$$

The space C_2 can be studied from the theory of binary forms, and it is easy to show that it has the rational cohomology of a point. The spaces C_0 and C_1 are moduli spaces of smooth complete intersections, and their rational cohomology was not known. We choose to consider the elements of C_0 as representing isomorphism classes of non-singular curves of type (3, 3) on $\mathbf{P}^1 \times \mathbf{P}^1$. Analogously, elements of C_1 can be regarded as isomorphism classes of non-singular curves of degree 6 in the weighted projective space $\mathbf{P}(1, 1, 2)$. The space C_0 is then the quotient of an open subset of the space of bihomogeneous polynomials of bidegree (3, 3) in the two sets of indeterminates x_0, x_1 and y_0, y_1 , by the action of the automorphism group of $\mathbf{P}^1 \times \mathbf{P}^1$. Analogously, C_1 is the quotient of an open subset of $\Gamma(\mathcal{O}_{\mathbf{P}(1,1,2)}(6))$ by the action of the automorphism group of $\mathbf{P}(1, 1, 2)$.

This means that in both cases we are in the situation of the paper **[PS03]**. As a standard choice of notation, we denote by V the vector space, its dimension by N , and the open subset which we are interested in quotienting by X . The complement of X in V is a hypersurface Σ , called the *discriminant hypersurface*. We denote the group acting on X by G , the geometric quotient by $\varphi : X \rightarrow X/G$, and the orbit map by ρ :

$$\begin{array}{ccc} G & \longrightarrow & X \\ g & \longmapsto & g(x) \end{array}$$

where x is a fixed point of X . Recall (see **[Che52]**) that $H^\bullet(G)$ is an exterior algebra freely generated by classes $\eta_i \in H^{2r_i-1}(G)$. We show that in both cases in which we are interested, the following generalization of Leray-Hirsch theorem applies:

THEOREM 1.1 ([PS03]). *Suppose there are subschemes $Y_i \subset \Sigma$ of pure codimension r_i in V whose fundamental classes map to a non-zero multiple of η_i under the composition*

$$\bar{H}_{2(N-r_i)}(Y_i) \rightarrow \bar{H}_{2(N-r_i)}(\Sigma) \xrightarrow{\sim} H^{2r_i-1}(X) \xrightarrow{\rho^*} H^{2r_i-1}(G),$$

where \bar{H}_\bullet indicates the Borel-Moore homology of the space. Denote the image of $[Y_i]$ in $H^\bullet(X; \mathbf{Q})$ by y_i ; then the map $a \otimes \eta_i \mapsto \varphi^*a \cup y_i$, $a \in H^\bullet(X/G; \mathbf{Q})$ extends to an isomorphism of graded \mathbf{Q} -vector spaces

$$H^\bullet(X/G; \mathbf{Q}) \otimes H^\bullet(G; \mathbf{Q}) \xrightarrow{\sim} H^\bullet(X; \mathbf{Q}).$$

Note that Theorem 1.1 gives no information about the ring structure of the cohomology of X, G and X/G .

We use Theorem 1.1 to interpret $H^\bullet(X; \mathbf{Q})$ as a tensor product of the rational cohomology of X/G and G . This allows us to recover $H^\bullet(C_j; \mathbf{Q})$ from the other data. The problem is then essentially reduced to the computation of the rational cohomology of X . Vassiliev invented a general topological method for calculating the cohomology of complements of discriminants, or, more generally, of spaces of non-singular functions. First, the computation of the cohomology of X is reduced to that of the (Alexander dual) Borel-Moore homology of its complement Σ . Secondly, Vassiliev constructed a simplicial resolution of Σ , and a stratification of this resolution, based on the study of possible configurations of the singularities of the elements of Σ . The spectral sequence associated to this filtration is proved to converge to the Borel-Moore homology of Σ . Vassiliev used this method in [Vas99] to determine, for instance, the real cohomology of the space of non-singular quadric curves in \mathbf{P}^2 , and of non-singular cubic surfaces in \mathbf{P}^3 . Gorinov ([Gor05]) modified Vassiliev's method so that it applies to a wider range of situations. In this way he could calculate the real cohomology of the space of quintic curves in \mathbf{P}^2 . We compute the cohomology of the complements of the discriminants we are interested in, by a modification of Gorinov-Vassiliev's method. We include in this paper an exposition of the method we use. The functoriality of the whole construction is emphasized by describing it in the language of cubical schemes. Note that Gorinov-Vassiliev's method respects the mixed Hodge structure of the space X . Hence we can use it to find the mixed Hodge structure of the cohomology of the complement of the discriminant.

By applying the techniques above, we will show the following results. They are stated by means of Poincaré-Serre polynomials. Recall that the Poincaré-Serre polynomial of a variety Z is the polynomial in two indeterminates t, u such that the coefficient of $t^i u^j$ is the dimension of the weight j subquotient of the cohomology of Z in degree i .

THEOREM 1.2. *The space C_0 of non-singular genus 4 curves whose canonical model lies on a non-singular quadric in \mathbf{P}^3 has Poincaré-Serre polynomial $1 + t^5 u^6$.*

THEOREM 1.3. *The space C_1 of smooth genus 4 curves whose canonical model lies on a quadric cone in \mathbf{P}^3 has the rational cohomology of a point.*

Theorems 1.2 and 1.3 allow us to compute the spectral sequence associated to the filtration (1). This establishes the main result:

THEOREM 1.4. *The Poincaré-Serre polynomial of \mathcal{M}_4 is $1 + t^2 u^2 + t^4 u^4 + t^5 u^6$.*

In particular, Theorem 1.4 yields the Euler characteristic of the cohomology with compact support of \mathcal{M}_4 in the Grothendieck group of rational mixed Hodge structures as $\mathbf{L}^9 + \mathbf{L}^8 + \mathbf{L}^7 - \mathbf{L}^6$.

Theorem 1.4 agrees with what was previously known about the cohomology of \mathcal{M}_4 . In particular, its Euler characteristic had been computed by Harer and Zagier in [HZ86], and found to be 2. It also agrees with the known algebraic classes of \mathcal{M}_4 , as computed in [Fab90].

The plan of the paper is as follows. In Section 2 we explain our version of Gorinov-Vassiliev's method for computing cohomology of complements of discriminants. We formulate it by using the language of cubical schemes, which is also introduced. Moreover, we show or recall some useful homological results. In Section 3 and Section 4 we calculate explicitly the rational cohomology of respectively C_0 and C_1 , establishing Theorems 1.2 and 1.3. In Section 5 we give a short proof, in the style of the paper, of the fact that the moduli space of hyperelliptic curves of genus $g \geq 2$ has the rational cohomology of a point.

2. The cohomology of complements of discriminants

In this section we introduce the method we use for the computation of the cohomology of complements of discriminants. This method was originally developed by Vassiliev, see for instance [Vas99]. Gorinov ([Gor05]) modified it in a way that allows to apply it to a wider range of situations. We present Gorinov-Vassiliev's method in the slightly modified version of [Tom05]. This implies, in particular, that we do not need to assume that the variety on which we consider the configurations is smooth. Moreover, we present Gorinov-Vassiliev's construction by using the language of cubical spaces and geometrical realizations. This allows us to obtain information also on the mixed Hodge structures of the cohomology groups.

2.1. Gorinov-Vassiliev's method. Gorinov-Vassiliev's method is based on the study of the geometry of the singular configurations of the elements of the discriminant.

DEFINITION 2.1. Let Z be a projective variety. A subset $S \subset Z$ is called a *configuration* in Z if it is compact and non-empty. The space of all configurations in Z is denoted by $\text{Conf}(Z)$.

We can choose a metric on $Z \subset \mathbf{P}^N$, by considering the restriction μ of the Fubini-Study metric on \mathbf{P}^N . We have that (Z, μ) is a compact complete metric space. The metric μ induces a function $\tilde{\mu} : \text{Conf}(Z) \times \text{Conf}(Z) \rightarrow \mathbf{R}$, by posing

$$\tilde{\mu}(S_1, S_2) = \max\{\mu(x, S_2) : x \in S_1\} + \max\{\mu(S_1, y) : y \in S_2\}.$$

PROPOSITION 2.2 ([Gor05]). *The pair $(\text{Conf}(Z), \tilde{\mu})$ is a compact complete metric space.*

Let us fix Z , and consider a vector space V such that there is a map

$$\begin{aligned} V &\longrightarrow \text{Conf}(Z) \cup \{\emptyset\} \\ v &\longmapsto K_v, \end{aligned}$$

such that $K_0 = Z$, and $L(K) := \{v \in V : K \subset K_v\}$ is a linear space for all $K \in \text{Conf}(Z)$.

The most natural case is that in which the elements of V are sections of a line bundle on Z and K_v is the set of singular points of v .

We define the discriminant as

$$\Sigma := \{v \in V : K_v \neq \emptyset\}.$$

The method is based on the fact that there is a direct relation between the cohomology of the complement of Σ in V and the Borel-Moore homology of Σ . The cap product with the fundamental class $[\Sigma]$ of the discriminant induces for all indices i an isomorphism

$$\tilde{H}^i(V \setminus \Sigma; \mathbf{Q}) \cong H^{i+1}(V, V \setminus \Sigma; \mathbf{Q}) \xrightarrow{\cap [\Sigma]} \bar{H}_{2M-1-i}(\Sigma; \mathbf{Q})(-M),$$

where we have denoted by M the complex dimension of V , and by \bar{H} the reduced cohomology of the space.

Our aim is to compute the Borel-Moore homology of Σ . Gorinov constructed a simplicial resolution for Σ , starting from a collection X_1, \dots, X_N of families of configurations in Z . For his construction to work, the X_i 's have to satisfy some axioms. We list below the properties which we will require for X_1, \dots, X_N .

LIST 2.3.

- (1) For every element $v \in \Sigma$, K_v must belong to some X_i .
- (2) If $x \in X_i$, $y \in X_j$, $x \subsetneq y$, then $i < j$.
- (3) $X_i \cap X_j = \emptyset$ for $i \neq j$.
- (4) Any $x \in \bar{X}_i \setminus X_i$ belongs to some X_j with $j < i$.
- (5) For every index $j = 1, \dots, N$ the space $L(x) \subset \Sigma$ has the same dimension d_j for every configuration $x \in X_j$. Moreover, for all indices j the map:

$$\begin{aligned} \varphi_j : X_j &\longrightarrow G(d_j, V) \\ v &\longmapsto L(K_v) \end{aligned}$$

from the configuration space X_j to the Grassmannian of linear subspaces of dimension d_j of V , is continuous.

- (6) For every i the space

$$\mathcal{T}_i = \{(p, x) \in Z \times X_i : p \in x\},$$

with the evident projection, is the total space of a locally trivial bundle over X_i .

- (7) Suppose X_i consists of finite configurations. Then for all y, x such that $x \in X_i$, $y \subsetneq x$, the configuration y belongs to X_j for some index $j < i$.

Note that the maps φ_j of condition 5 are always continuous, if $L(\{z\})$ has the same dimension d for all $z \in Z$ and the map to the Grassmannian of linear subspaces of dimension d of V ,

$$\begin{aligned} L : Z &\longrightarrow G(d, V) \\ z &\longmapsto L(\{z\}) \end{aligned}$$

is continuous.

DEFINITION 2.4. A *cubical space* over an index set $\underline{N} := \{1, 2, \dots, N\}$ (briefly, an \underline{N} -cubical space) is a collection of topological spaces $\{Y(I)\}_{I \subset \underline{N}}$ such that for each inclusion $I \subset J$ we have a natural continuous map $f_{IJ} : Y(J) \rightarrow Y(I)$ such that $f_{IK} = f_{IJ} \circ f_{JK}$ whenever $I \subset J \subset K$.

We will always consider cubical spaces such that all spaces $Y(I)$ are complex algebraic varieties and all maps f_{IJ} are regular morphisms.

A standard example of cubical space is given by the cube $\square(\bullet)$. For every $I \subset \underline{N}$, we define $\square(I) := \{g : I \longrightarrow [0, 1]\}$. For every inclusion $I \subset J \subset \underline{N}$, the map $f_{IJ} : \square(J) \longrightarrow$

$\square(I)$ is the restriction map $g \mapsto g|_I$. We can interpret $\square(\underline{N})$ as an N -dimensional cube; note that its vertices are in one-to-one correspondence with the subsets $I \subset \underline{N}$.

A sort of dual construction of that of the cube is given by the simplex Δ_\bullet . It is the collection of spaces

$$\Delta_I := \left\{ (g : \underline{N} \rightarrow [0, 1]) : \sum_{a \in \underline{N}} g(a) = 1, g|_{\underline{N} \setminus I} = 0 \right\}$$

for $I \subset \underline{N}$. Also in this case, whenever we have an inclusion $I \subset J \subset \underline{N}$, we can associate with it a natural map $e_{IJ} : \Delta_I \rightarrow \Delta_J$, given by the inclusion of Δ_I in Δ_J . We are ready to define the cubical spaces we work with.

$$\Lambda(I) := \{A \in \prod_{i \in I} X_i : i < j \Rightarrow A_i \subset A_j\}.$$

$$\begin{aligned} \mathcal{X}(I) &:= \{(F, A) \in \Sigma \times \Lambda(I) : K_F \supset A_{\max(I)}\} \text{ if } I \neq \emptyset, \\ \mathcal{X}(\emptyset) &:= \Sigma. \end{aligned}$$

Moreover, we will use the following (auxiliary) cubical spaces:

$$\tilde{\Lambda}(I) := \{A \in \prod_{i \in I} \overline{X}_i : i < j \Rightarrow A_i \subset A_j\},$$

$$\begin{aligned} \tilde{\mathcal{X}}(I) &:= \{(F, A) \in \Sigma \times \tilde{\Lambda}(I) : K_F \supset A_{\max(I)}\} \text{ if } I \neq \emptyset, \\ \tilde{\mathcal{X}}(\emptyset) &:= \Sigma. \end{aligned}$$

DEFINITION 2.5. Let $Y(\bullet)$ be a cubical space over an index set \underline{N} . Note that the cubical space $Y(\bullet)$ has a natural augmentation towards $Y(\emptyset)$, induced by the inclusion of the empty set in every subset of \underline{N} . The *geometric realization* of $Y(\bullet)$ is defined as the map

$$|\epsilon| : |Y(\bullet)| \longrightarrow Y(\emptyset)$$

induced from the natural augmentation on the space

$$|Y(\bullet)| = \coprod_{I \subset \underline{N}} (\Delta_I \times Y(I)) / R,$$

where R is the equivalence relation given by

$$(f, y) R (f', y') \Leftrightarrow f' = e_{IJ}(f), y = f_{IJ}(y').$$

We construct the geometric realization of all the cubical spaces defined above. We can define a surjective map φ from $|\tilde{\Lambda}(\bullet)|$ to $|\Lambda(\bullet)|$ as follows. Let $(t, A) \in \Delta_I \times \tilde{\Lambda}(I)$, and $[t, A]$ the corresponding class in $|\tilde{\Lambda}(\bullet)|$. Note that by conditions 3 and 4 in List 2.3, for each A_i ($i \in I$) there exists a unique index $k(i) \in \underline{N}$ such that $A_i \in X_{k(i)}$.

We define $\varphi([t, A])$ as the class in $|\Lambda(\bullet)|$ of the element $(s, B) \in \Delta_J \times \Lambda(J)$, where

$$J := \{k \in \underline{N} : A_i \in X_k \text{ for some } i \in I\} = \{k(i) : i \in I\};$$

$$B := \prod_{k \in J} B_k, \quad B_k := A_i \text{ for any index } i : k(i) = k;$$

$$s : J \longrightarrow [0, 1], \quad s(k) := \sum_{i \in I : k(i) = k} t(i).$$

Observe that φ acts by contracting all closed simplices in $|\tilde{\Lambda}(\bullet)|$ corresponding to inclusions of configurations of the form $x = x = x = \dots = x$.

We define analogously the map

$$\begin{aligned} \psi : \quad |\tilde{\mathcal{X}}(\bullet)| &\longrightarrow |\mathcal{X}(\bullet)| \\ [(F, A), t] &\longmapsto (F, \varphi(A, t)). \end{aligned}$$

In the rest of the paper, we consider the spaces $|\tilde{\Lambda}(\bullet)|$ and $|\tilde{\mathcal{X}}(\bullet)|$ with the quotient topology under the equivalence relation R of the direct product topology of the $\tilde{\Lambda}(I)$'s (respectively, $\mathcal{X}(I)$'s). The topology on $|\Lambda(\bullet)|$ and $|\mathcal{X}(\bullet)|$ is the topology induced by φ (respectively, ψ).

PROPOSITION 2.6 ([Gor05]). *The geometric realization of $\mathcal{X}(\bullet)$,*

$$|\epsilon| : |\mathcal{X}(\bullet)| \longrightarrow \mathcal{X}(\emptyset) = \Sigma,$$

is a homotopy equivalence and induces an isomorphism on Borel-Moore homology groups.

PROOF. It is enough to prove that $|\epsilon|$ is a proper map, and that its fibers are contractible. Let us show first that $|\epsilon|$ is also a proper map. Fix a compact subset $W \subset \Sigma$. We define a cubical space as follows:

$$\begin{aligned} \tilde{\mathcal{X}}_W(I) &:= \{(F, A) \in W \times \tilde{\Lambda}(I) : K_F \supset A_{\max(I)}\} \text{ if } I \neq \emptyset, \\ \tilde{\mathcal{X}}_W(\emptyset) &:= W. \end{aligned}$$

Note that $\tilde{\mathcal{X}}_W(I)$ is compact for all $I \subset \underline{N}$, and so is the space $|\tilde{\mathcal{X}}_W(\bullet)|$.

Then $|\epsilon|^{-1}(W)$ is compact, because it coincides with the image of the continuous map

$$\begin{aligned} \psi_W : \quad |\tilde{\mathcal{X}}_W(\bullet)| &\longrightarrow |\mathcal{X}(\bullet)| \\ [(F, A), t] &\longmapsto (F, \varphi(A, t)). \end{aligned}$$

Hence the map $|\epsilon|$ is proper.

We show next that the fibres of $|\epsilon|$ are contractible. Consider the fiber over $v \in \Sigma$. By conditions 1 and 3 of List 2.3, there is a unique index j such that $K_v \in X_j$. By definition, $|\epsilon|^{-1}(v)$ is a cone with vertex $[(K_v, v)] \in |\mathcal{X}|_{\{j\}}(\bullet) \hookrightarrow |\mathcal{X}(\bullet)|$, so it is clearly contractible. \square

COROLLARY 2.7. *The Borel-Moore homology of the geometric realization of $\mathcal{X}(\bullet)$ has a natural mixed Hodge structure, induced by the mixed Hodge structure on the Borel-Moore homology of the spaces $\mathcal{X}(I)$. From the theory of cubical spaces, the isomorphism $\bar{H}_\bullet(|\mathcal{X}(\bullet)|; \mathbf{Q}) \cong \bar{H}_\bullet(\Sigma; \mathbf{Q})$ of Proposition 2.6 is also an isomorphism of mixed Hodge structures.*

For every index set $I \subset \underline{N}$, we can restrict $\Lambda(\bullet)$ and $\mathcal{X}(\bullet)$ to the index set I , getting the two I -cubical spaces $\Lambda|_I(\bullet)$ and $\mathcal{X}|_I(\bullet)$.

If we consider the geometric realizations of the $\Lambda|_I(\bullet)$'s and $\mathcal{X}|_I(\bullet)$'s, we have that for every $I \subset J \subset \underline{N}$, there are natural embeddings $|\Lambda|_I(\bullet) \hookrightarrow |\Lambda|_J(\bullet)$ and $|\mathcal{X}|_I(\bullet) \hookrightarrow |\mathcal{X}|_J(\bullet)$. In this way we can define an increasing filtration on $|\Lambda(\bullet)|$ by posing

$$\text{Fil}_j |\Lambda(\bullet)| := \text{Im} \left(|\Lambda|_{\underline{j}}(\bullet) \hookrightarrow |\Lambda(\bullet)| \right)$$

for $j = 1, \dots, N$. We define analogously the filtration $\text{Fil}_j |\mathcal{X}(\bullet)|$ on $|\mathcal{X}(\bullet)|$. We use the notation $F_j := \text{Fil}_j |\mathcal{X}(\bullet)| \setminus \text{Fil}_{j-1} |\mathcal{X}(\bullet)|$, $\Phi_j := \text{Fil}_j |\Lambda(\bullet)| \setminus \text{Fil}_{j-1} |\Lambda(\bullet)|$.

Note that $\text{Fil}_j |\Lambda(\bullet)|$ is closed in $\text{Fil}_{j+1} |\Lambda(\bullet)|$ for every j , $1 \leq j \leq N-1$. The same holds for the filtration on $|\mathcal{X}(\bullet)|$.

These filtrations correspond to the chains of restriction maps

$$\Lambda|_{\{1\}}(\bullet) \leftarrow \Lambda|_{\{1,2\}}(\bullet) \leftarrow \dots \leftarrow \Lambda|_{\underline{N-1}}(\bullet) \leftarrow \Lambda|_{\underline{N}}(\bullet)$$

and

$$\mathcal{X}|_{\{1\}}(\bullet) \leftarrow \mathcal{X}|_{\{1,2\}}(\bullet) \leftarrow \dots \leftarrow \mathcal{X}|_{\underline{N-1}}(\bullet) \leftarrow \mathcal{X}|_{\underline{N}}(\bullet).$$

These considerations, together with Proposition 2.6, imply the following:

PROPOSITION 2.8. *The filtration $\text{Fil}_j |\mathcal{X}(\bullet)|$ defines a spectral sequence that converges to the Borel-Moore homology of Σ . Its term $E_{p,q}^1$ is isomorphic to $\bar{H}_{p+q}(F_p; \mathbf{Q})$. Note that the spaces F_p have mixed Hodge structures induced by those on $\mathcal{X}(\bullet)$.*

PROPOSITION 2.9 ([Gor05]).

1. *For every $j = 1, \dots, N$, the stratum F_j is a complex vector bundle of rank d_j over Φ_j . The space Φ_j admits a natural surjective map to X_j which is a locally trivial fibration.*
2. *If X_j consists of configurations of m points, the fiber of Φ_j over any $x \in X_j$ is an $(m-1)$ -dimensional open simplex, which changes its orientation under the homotopy class of a loop in X_j interchanging a pair of points in x_j .*
3. *If $X_N = \{Z\}$, F_N is the open cone with vertex a point (corresponding to the configuration Z), over $\text{Fil}_{N-1} |\Lambda(\bullet)|$.*

We recall below the topological definition of an open cone.

DEFINITION 2.10. Let B be a topological space. Then a space is said to be an *open cone* over B with vertex a point if it is homeomorphic to the space

$$B \times [0, 1)/R,$$

where the equivalence relation is $R = (B \times \{0\})^2$.

PROOF OF PROPOSITION 2.9. The second and the third point are clear by construction. The first point is trivial for F_j . For the map $\Phi_j \rightarrow X_j$, the fiber over a configuration x is given by a union of simplices with vertices determined by the points of $x \subset Z$. Thus the fibration is locally trivial as a consequence of condition 6 in List 2.3. \square

2.2. Homological lemmas. The fiber bundle $\Phi_j \rightarrow X_j$ of Proposition 2.9 is in general non-orientable. As a consequence, we have to consider the homology of X_j with coefficients not in \mathbf{Q} , but in some local system of rank one. Therefore we recall here some results and constructions about Borel-Moore homology of configuration spaces, with twisted coefficients.

DEFINITION 2.11. Let Z be a topological space. Then for every $k \geq 1$ we have the space of ordered configurations of k points in Z ,

$$F(Z, k) = Z^k \setminus \bigcup_{1 \leq i < j \leq k} \{(z_1, \dots, z_k) \in Z^k : z_i = z_j\}.$$

There is a natural action of the symmetric group S_k on $F(k, Z)$. The quotient is called the space of unordered configurations of k points in Z ,

$$B(Z, k) = F(Z, k)/S_k.$$

The *sign representation* $\pi_1(B(Z, k)) \rightarrow \text{Aut}(\mathbf{Z})$ maps the paths in $B(Z, k)$ defining odd (respectively, even) permutations of k points to multiplication by -1 (respectively, 1). The local system $\pm\mathbf{Q}$ over $B(Z, k)$ is the one locally isomorphic to \mathbf{Q} , but with monodromy representation equal to the sign representation of $\pi_1(B(Z, k))$. We will often call $\bar{H}_\bullet(B(Z, k), \pm\mathbf{Q})$ the *Borel-Moore homology of $B(Z, k)$ with twisted coefficients*, or, simply, the *twisted Borel-Moore homology of $B(Z, k)$* .

LEMMA 2.12. $\bar{H}_\bullet(B(\mathbf{C}^N, k); \pm\mathbf{Q}) = 0$ if $k \geq 2$.

PROOF. In [Vas92, Theorem 4.3, Corollary 2], it is proved that the Borel-Moore homology of $B(\mathbf{R}^{2N}, k)$ with coefficients in the system $\pm\mathbf{Z}$ is a finite group. Since $\pm\mathbf{Q} = \pm\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}$, the claim follows. \square

LEMMA 2.13 ([Vas99]). $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q}) = H_{\bullet-k(k-1)}(G(k, \mathbf{C}^{N+1}); \mathbf{Q})$.

In particular, $\bar{H}_\bullet(B(\mathbf{P}^N, k); \pm\mathbf{Q})$ is trivial if $k > N + 1$.

LEMMA 2.14. (1) The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{C}^*, 2); \mathbf{Q})$ is $t^2(1 + tu^{-2})^2$.

(2) The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{C}^*, k); \pm\mathbf{Q})$ is $t^k + t^{k+1}u^{-2}$ for every $k \geq 1$. If we consider the action of \mathfrak{S}_2 on \mathbf{C}^* induced by $\mathbf{C}^* \ni \tau \mapsto \frac{1}{\tau}$, we have that the Borel-Moore homology classes of even degree are invariant and those of odd degree are anti-invariant.

PROOF. We recover the Borel-Moore homology of $B(\mathbf{C}^*, 2)$ from the known situation for $B(\mathbf{C}, 2)$. Since $\bar{H}_\bullet(B(\mathbf{C}, 2); \pm\mathbf{Q})$ is trivial, we have that $\bar{H}_j(B(\mathbf{C}, 2); \mathbf{Q})$ and $\bar{H}_j(\mathbf{C}^2 \setminus \{(x, y) \in \mathbf{C}^2 : x = y\}; \mathbf{Q})$ are isomorphic, and in particular they both have Poincaré-Serre polynomial $t^3u^{-2} + t^4u^{-4}$.

The configuration space $B(\mathbf{C}^*, 2)$ can be considered as an open subset of $B(\mathbf{C}, 2)$, with complement isomorphic to \mathbf{C}^* . Then (1) follows from the long exact sequence

$$\cdots \rightarrow \bar{H}_j(\mathbf{C}^*; \mathbf{Q}) \rightarrow \bar{H}_j(B(\mathbf{C}, 2); \mathbf{Q}) \rightarrow \bar{H}_j(B(\mathbf{C}^*, 2); \mathbf{Q}) \rightarrow \bar{H}_{j-1}(\mathbf{C}^*; \mathbf{Q}) \rightarrow \cdots.$$

We will prove (2) by induction. For $k = 1$, the claim clearly holds. Suppose we have proved the claim for $B(\mathbf{C}^*, k)$. We can identify $B(\mathbf{C}^*, k+1)$ with the subset of $B(\mathbf{C}, k)$ of configurations containing the point 0. We have the long exact sequence

$$\cdots \rightarrow \bar{H}_j(B(\mathbf{C}, k+1); \pm\mathbf{Q}) \rightarrow \bar{H}_j(B(\mathbf{C}^*, k+1); \pm\mathbf{Q}) \rightarrow \bar{H}_{j-1}(B(\mathbf{C}^*, k); \pm\mathbf{Q}) \rightarrow \cdots.$$

By Lemma 2.12, this implies $\bar{H}_\bullet(B(\mathbf{C}^*, k)) \cong \bar{H}_{\bullet+1}(B(\mathbf{C}^*, k+1))$, which proves the claim as far as Poincaré-Serre polynomials are concerned. We determine the action of \mathfrak{S}_2 by looking at spaces of configurations on \mathbf{P}^1 . Let us consider $B(\mathbf{P}^1, k+2)$ and its subsets

$$C_{k+2} := \{S \in B(\mathbf{P}^1, k+2) : \{0, \infty\} \cap S \neq \emptyset\},$$

$$D_{k+2} := \{S \in B(\mathbf{P}^1, k+2) : \{0, \infty\} \subset S\}.$$

The space D_{k+2} is isomorphic to $B(\mathbf{C}^*, k)$, hence we have

$$\bar{H}_\bullet(D_{k+2}, \pm\mathbf{Q}) \cong \bar{H}_\bullet(B(\mathbf{C}^*, k); \pm\mathbf{Q}).$$

Note that the \mathfrak{S}_2 -actions on the Borel-Moore homology groups do not coincide, but invariant classes in the Borel-Moore homology of $B(\mathbf{C}^*, k)$ correspond to anti-invariant classes in that of D_{k+2} , and vice versa. The space $C_{k+2} \setminus D_{k+2}$ is isomorphic to the disjoint union of two copies of $B(\mathbf{C}^*, k+1)$ interchanged by the \mathfrak{S}_2 -action. Finally, we have $B(\mathbf{P}^1, k+2) \setminus B(\mathbf{C}^*, k+2) = C_{k+2}$.

The filtration $D_{k+2} \subset C_{k+2} \subset B(\mathbf{P}^1, k+2)$ induces the following spectral sequence converging to the Borel-Moore homology of $B(\mathbf{P}^1, k+2)$:

$$\begin{array}{c|ccc} k-1 & \mathbf{Q}(1) & \mathbf{Q}(1)^2 & \mathbf{Q}(1) \\ \hline k-2 & \mathbf{Q} & \mathbf{Q}^2 & \mathbf{Q} \end{array}$$

This spectral sequence makes sense also when $k = 0$. In that case, the $\mathbf{Q}(1)$ in the first column does not appear, and the \mathbf{Q} in the first column is a \mathfrak{S}_2 -anti-invariant class. This implies the wished \mathfrak{S}_2 -behaviour of the twisted Borel-Moore homology of $B(\mathbf{C}^*, 2)$. When $k \geq 1$, all rows of the spectral sequence must be exact by Lemma 2.13. This implies that the isomorphism $\bar{H}_\bullet(B(\mathbf{C}^*, k)) \cong \bar{H}_{\bullet+2}(B(\mathbf{C}^*, k+2))$ respects the action of \mathfrak{S}_2 . From this the claim follows. \square

LEMMA 2.15. *The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{P}^1 \times \mathbf{P}^1, 1); \pm \mathbf{Q})$ is $(t^2 u^{-2} + 1)^2$.*

The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{P}^1 \times \mathbf{P}^1, 2); \pm \mathbf{Q})$ is $2t^2 u^{-2}(t^4 u^{-4} + t^2 u^{-2} + 1)$.

The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{P}^1 \times \mathbf{P}^1, 3); \pm \mathbf{Q})$ is $t^4 u^{-4}(t^2 u^{-2} + 1)^2$.

The Poincaré-Serre polynomial of $\bar{H}_\bullet(B(\mathbf{P}^1 \times \mathbf{P}^1, 4); \pm \mathbf{Q})$ is $t^8 u^{-8}$.

The twisted Borel-Moore homology of $B(\mathbf{P}^1 \times \mathbf{P}^1, k)$ is trivial for $k \geq 5$.

PROOF. We modify here Vassiliev's arguments in the proof of Lemma 2.13 in [Vas99]. The technique we use is that of decomposing $B(\mathbf{P}^1 \times \mathbf{P}^1, k)$ into spaces of which the twisted Borel-Moore homology is known. In particular, it is possible to decompose $\mathbf{P}^1 \times \mathbf{P}^1$ by fixing two lines l, m in different rulings and considering the filtration $S_1 \subset S_2 \subset S_3 \subset S_4$, where

$$S_1 := l \cap m, S_2 := l, S_3 := l \cup m, S_4 := \mathbf{P}^1 \times \mathbf{P}^1.$$

This means $\mathbf{P}^1 \times \mathbf{P}^1$ is the disjoint union of spaces isomorphic to $\{\ast\}$, \mathbf{C} , \mathbf{C} , \mathbf{C}^2 respectively.

Let us fix $k \geq 1$. To any configuration of points in $B(\mathbf{P}^1 \times \mathbf{P}^1, k)$ we can associate an ordered partition (a_1, a_2, a_3, a_4) , where a_i is the number of points contained in $S_i \setminus S_{i-1}$. We can consider each possible partition of k as defining a stratum in $B(\mathbf{P}^1 \times \mathbf{P}^1, k)$, and order such strata by lexicographic order of the index of the partition. Note that all strata with $a_i \geq 2$ for some i have no twisted Borel-Moore homology by Lemma 2.12, so we need not consider them. This is the case for all strata, when $k \geq 5$.

As an example, we consider explicitly the case $k = 2$. The situation is analogous for the other values. There are 6 admissible partitions for $k = 2$:

- (1,1,0,0), (1,0,1,0): Both these strata are isomorphic to \mathbf{C} , hence they have homology $\mathbf{Q}(1)$ in degree 2, and trivial homology in all other degrees.
- (1,0,0,1), (0,1,1,0): Both these strata are isomorphic to \mathbf{C}^2 , hence they have homology $\mathbf{Q}(2)$ in degree 4, and trivial homology in all other degrees.
- (0,1,0,1), (0,0,1,1): These strata are both isomorphic to \mathbf{C}^3 , hence they have homology $\mathbf{Q}(3)$ in degree 6, and trivial homology in all other degrees.

This gives precisely that $\bar{H}_j(B(\mathbf{P}^1 \times \mathbf{P}^1, 2); \pm \mathbf{Q})$ is $\mathbf{Q}(j/2)^2$ for $j = 2, 4, 6$, and is trivial otherwise. \square

LEMMA 2.16. *Let us consider a quadric cone $U = \{[u_0, u_1, u_2, u_3] \in \mathbf{P}^3 : u_1^2 - u_2 u_3 = 0\}$. Define U_0 as U minus its vertex $[1, 0, 0, 0]$. Then $\bar{H}_\bullet(B(U_0, 2); \pm \mathbf{Q})$ is $\mathbf{Q}(3)$ in degree 6 and is trivial in all other degrees. The twisted Borel-Moore homology $\bar{H}_\bullet(B(U_0, k); \pm \mathbf{Q})$ is always trivial for $k \geq 3$.*

PROOF. Analogous as that of the previous lemma, by using the fact that U_0 can be decomposed as the disjoint union of \mathbf{C} (a line of the ruling) and a space homeomorphic to \mathbf{C}^2 . \square

LEMMA 2.17. *Let us consider the local system T on \mathbf{C}^* locally isomorphic to \mathbf{Q} and changing its sign when the point moves along a loop in \mathbf{C}^* whose homotopy class is an odd multiple of the generator of $\pi_1(\mathbf{C}^*) \cong \mathbf{Z}$. Then $\bar{H}_\bullet(\mathbf{C}^*; T) = 0$.*

PROOF. Let us consider the map $q : \mathbf{C}^* \longrightarrow \mathbf{C}^*$, $q(t) = t^2$. Clearly $q_* \mathbf{Q} = \mathbf{Q} \oplus T$. Since $\bar{H}_\bullet(\mathbf{C}^*; q_* \mathbf{Q}) \cong \bar{H}_\bullet(\mathbf{C}^*; \mathbf{Q})$, we have $\bar{H}_\bullet(\mathbf{C}^*; T) = 0$. \square

2.3. Some cases where the Borel-Moore homology is trivial. In Sections 3 and 4 we use the fact that most strata of the filtration we consider over the domain of the geometric realization of discriminants give no contribution to its Borel-Moore homology. We give here some ideas about the reason why it happens. In general they are either situations where the space is a locally trivial fiber bundle whose fibers have trivial Borel-Moore homology in the induced system of coefficients, or situations where non-discrete configurations are involved.

LEMMA 2.18. *Let C be an open cone with vertex a point over a compact connected space B . Then C admits a compactification \bar{C} with border homeomorphic to B . There are the following isomorphisms:*

$$\bar{H}_\bullet(C; \mathbf{Q}) \cong H_\bullet(\bar{C}, B; \mathbf{Q}) \cong H_{\bullet-1}(B, \text{point}; \mathbf{Q}).$$

PROOF. The first isomorphism comes from the characterization of Borel-Moore homology as relative homology of the one point compactification of the space, modulo the added point. The second is the border isomorphism of the exact sequence associated to the triple $(\text{point}, B, \bar{C})$. \square

LEMMA 2.19. *Suppose we have a variety Z and the following families of configurations in Z :*

$$\begin{aligned} X_1 &= B(Z, 1); \\ X_2 &= \{\{p, q\} \in B(Z, 2) : p \text{ and } q \text{ lie on a line } l \subset Z\}; \\ X_3 &= \{\{p, q, r\} \in B(Z, 3) : p, q \text{ and } r \text{ lie on a line } l \subset Z\}; \\ X_4 &= \{\text{lines on } Z\}. \end{aligned}$$

Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in Section 2.1. Then the space Φ_4 has trivial Borel-Moore homology.

PROOF. The space Φ_4 is a fiber bundle over X_4 . Its fibre Ψ over a line $l \subset Z$ is the union of all simplices with 4 vertices p, q, r, l for all $\{p, q, r\} \in B(Z, 3)$. Note that we have to take *closed* simplices minus their face with vertices p, q, r , so they are indeed open cones with vertex l over the closed simplices with vertices corresponding to the points p, q, r . The system of coefficients induced on the fibre changes its sign if we interchange

two of the points p, q, r . The union of all *open* simplices with vertices p, q, r, l is a non-oriented simplices bundle over $B(l, 3)$. Hence the Borel-Moore homology of the union of open simplices with three vertices on l is trivial, because $\bar{H}_\bullet(B(l, 3); \pm \mathbf{Q})$ is trivial (see Lemma 2.13). This means that we need to consider only the Borel-Moore homology of the union of the external faces of the simplices considered before. This space admits a filtration of the form: $A_0 = \{l\}$, A_1 is the union of the open segments joining l and a point of l , A_2 is the union of the open simplices with vertices l and two distinct points on l . This gives a spectral sequence converging to the Borel-Moore homology of Ψ , with the following E^1 -term:

$$\begin{array}{c|cc} 1 & \mathbf{Q} & \mathbf{Q} \\ 0 & & \\ -1 & \mathbf{Q} & \mathbf{Q} \\ \hline & 1 & 2 & 3 \end{array}$$

Our space is an open cone, and it is a consequence of Lemma 2.18 that open cones have trivial Borel-Moore homology in degree 0. This implies that the row $q = -1$ of the spectral sequence is exact. As for the row $q = 1$, it follows from the shape of a generator of $E_{2,1}^1$ that it maps to a generator of $E_{1,1}^1$ under the differential d^1 . Thus the Borel-Moore homology of Ψ is trivial, which proves the claim. \square

REMARK 1. A slight modification of Lemma 2.19 allows us to conclude that also strata with singular configurations which are union of a line and a fixed finite number k of points have trivial Borel-Moore homology.

As before, we consider the projection to the configuration space and look at the fiber Ψ over a fixed configuration $\{a_1, a_2, \dots, a_k\} \cup l$. Let $h \geq 3$ be the maximal number of isolated points lying on the same line which appear in the previous strata of $|\Lambda(\bullet)|$. For any choice of distinct points b_1, \dots, b_h on l , the union of the $(k+h)$ -dimensional open simplices with vertices identified with $a_1, \dots, a_k, b_1, \dots, b_h, l$ is contained in Ψ . Indeed, Ψ is the union of the open simplex D with vertices a_1, \dots, a_k, l and the union of such simplices for every choice of b_1, \dots, b_h . This means that Ψ has a natural projection π to D . Each point p of $\Psi \setminus D$ is contained in exactly one open simplex with vertices b_1, \dots, b_h, d for some $d \in D$; we pose $\pi(p) = d$. On $D \subset \Psi$ the projection π coincides with the identity. If we look at the fibres of π , we see that they are homeomorphic to the fibres of the map $\Phi_4 \longrightarrow X_4$ studied in the proof of Lemma 2.19. Then we can apply the result found there, and conclude that the fibres of π have trivial Borel-Moore homology. Then also Ψ has trivial Borel-Moore homology, which implies the claim.

We consider next the case of the union of two rational curves, intersecting in one point. For simplicity, we state the result in the case of lines.

LEMMA 2.20. *Suppose we have a variety Z and the following families of configurations in Z :*

$$\begin{aligned} X_1 &= B(Z, 1); \\ X_2 &= B(Z, 2); \\ X_3 &= B(Z, 3); \\ X_4 &= B(Z, 4); \\ X_5 &= \{\text{lines on } Z\}; \end{aligned}$$

$$\begin{aligned}
X_6 &= \{\{p, q, r, s, t\} \in B(Z, 5) : p, q \text{ lie on a line } l \subset Z, \\
&\quad r, s \text{ lie on a line } m \subset Z, l \cap m = \{t\}\}; \\
X_7 &= \{l \cup \{p\} : l \subset Z \text{ line}, p \notin l\}; \\
X_8 &= \{l \cup \{p, q\} : l \subset Z \text{ line}, p, q \notin l, p \neq q\}; \\
X_9 &= \{l \cup m : l, m \subset Z \text{ line}, \#(l \cap m) = 1\}.
\end{aligned}$$

Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in Section 2.1. Then the space Φ_9 has trivial Borel-Moore homology.

PROOF. Consider the fiber Ψ of the projection $\Phi_9 \rightarrow X_9$ over a configuration $l \cup m$, such that $l \cap m = \{t\}$. The maximal chains of inclusions of configurations we can construct are of the following forms:

- (a) $\{p_1, p_2, p_3, p_4\} \subset l \subset l \cup \{q_1, q_2\} \subset l \cup m$, where $p_i \in l, q_j \in m \setminus \{t\}$;
- (b) $\{q_1, q_2, q_3, q_4\} \subset m \subset m \cup \{p_1, p_2\} \subset l \cup m$, where $q_i \in m, p_j \in l \setminus \{t\}$;
- (c) $\{p_1, p_2, t, q_1, q_2\} \subset l \cup \{q_1, q_2\} \subset l \cup m$, where $p_i \in l \setminus \{t\}, q_j \in m \setminus \{t\}$;
- (d) $\{p_1, p_2, t, q_1, q_2\} \subset m \cup \{p_1, p_2\} \subset l \cup m$, where $p_i \in l \setminus \{t\}, q_j \in m \setminus \{t\}$.

The simplices constructed from chains of inclusions of type (c) and (d) are contained in the simplices arising from chains of type (a) and (b), so that we have to consider just these. Denote by U_1 the union of closed simplices with vertices given by the configurations $p_1, p_2, p_3, p_4, q_1, q_2, l$ with $p_i \in l$ and $q_j \in m \setminus \{t\}$. Analogously, denote by U_2 the union of closed simplices with vertices given by the configurations $q_1, q_2, q_3, q_4, p_1, p_2, m$. The fiber Ψ is then the open cone (with vertex corresponding to the configuration $l \cup m$) over $U_1 \cup U_2$. Note that the intersection of U_1 and U_2 is given by the union of closed simplices with vertices p_1, p_2, q_1, q_2, t . It follows from the results of Remark 1 that the Borel-Moore homology of the open cone over $U_1 \cap U_2$ is trivial. The same can be said for the open cone over the union of simplices with 4 points of l and two points of m as vertices. The remaining part of U_1 is an open cone with vertex l (over simplices with 4 points of l and two points of m as vertices). This space is clearly contractible, so that the open cone over it has trivial Borel-Moore homology by Lemma 2.18. We can conclude exactly the same for U_2 . Then the claim holds. \square

LEMMA 2.21. Suppose Z is the product of \mathbf{C} and a variety M . Consider the following families of configurations in Z :

$$\begin{aligned}
X_1 &= B(Z, 1); \\
X_2 &= B(Z, 2); \\
X_3 &= \{\{a, b, c\} \in B(Z, 3) : a, b, c \in \mathbf{C} \times \{p\}, p \in M\}; \\
X_4 &= \{\{a, b, c\} \in B(Z, 3) : b, c \in \mathbf{C} \times \{p\}, p \in M, a \notin \mathbf{C} \times \{p\}\}; \\
X_5 &= \{\{a, b, c, d\} \in B(Z, 4) : a, b, c, d \in \mathbf{C} \times \{p\}\}; \\
X_6 &= \{\{a, b, c, d\} \in B(Z, 4) : a, b \in \mathbf{C} \times \{p\}, c, d \in \mathbf{C} \times \{q\}, p \neq q\}.
\end{aligned}$$

Construct the cubical space $\Lambda(\bullet)$, its geometric realization and the filtration as in Section 2.1. Then the space Φ_6 has trivial Borel-Moore homology.

PROOF. The space Φ_6 is an open non-orientable simplicial bundle over X_6 . We study the Borel-Moore homology of X_6 in the system of coefficients locally isomorphic to \mathbf{Q} , with orientation induced by the orientation of the simplices. We look at the ordered situation. Let $Y = F(M, 2) \times F(\mathbf{C}, 2) \times F(\mathbf{C}, 2)$. Every point $(p, q, a, b, c, d) \in Y$ gives

an ordered configuration of points $((a, p), (b, p), (c, q), (d, q))$, and the twisted Borel-Moore homology of X_6 can be identified with the part of $\bar{H}_\bullet(Y; \mathbf{Q})$ which is

- anti-invariant under the action of loops interchanging a and b ;
- anti-invariant under the action of loops loops interchanging c and d ;
- invariant under the action of loops interchanging p and q , a and c , b and d .

It is clear that such homology classes cannot exist, because

$$\bar{H}_\bullet(Y; \mathbf{Q}) = \bar{H}_\bullet(F(M, 2); \mathbf{Q}) \otimes \bar{H}_\bullet(F(\mathbf{C}, 2); \mathbf{Q}) \otimes \bar{H}_\bullet(F(\mathbf{C}, 2); \mathbf{Q})$$

and $\bar{H}_\bullet(F(\mathbf{C}, 2); \mathbf{Q})$ contains no classes that are anti-invariant with respect to the interchange of points (see Lemma 2.12). \square

A variation of the situation of the above lemma is given, for instance, by the case of configurations of two triplets of collinear points in $\mathbf{P}^1 \times \mathbf{P}^1$. In that case we have to use the fact that there are no anti-invariant homological classes in $\bar{H}_\bullet(F(\mathbf{P}^1, 3); \mathbf{Q})$.

3. Curves on a non-singular quadric

Any non-singular quadric surface is isomorphic to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in \mathbf{P}^3 . Such a surface is covered by two families of lines: The family of lines of the form $\mathbf{P}^1 \times \{q\}$ (which we call first ruling of the quadric) and that of lines of the form $\{p\} \times \mathbf{P}^1$ (second ruling). A curve on the Segre quadric is always given by the vanishing of a bihomogeneous polynomial in the two sets of variables x_0, x_1 and y_0, y_1 . The bidegree (n, m) of the polynomial has a geometrical interpretation as giving the number of points (counted with multiplicity) in the intersection of the curve with a general line of respectively the second and the first ruling. A curve on $\mathbf{P}^1 \times \mathbf{P}^1$ is said to be *of type* (n, m) if it is defined by the vanishing of a polynomial of bidegree (n, m) .

The curves which are the intersection of the quadric with a cubic surface are the curves of type $(3, 3)$. This suggests that the space C_0 can be obtained as a quotient of the space of polynomials of bidegree $(3, 3)$ by the action of the automorphism group of $\mathbf{P}^1 \times \mathbf{P}^1$. We denote that vector space by

$$V := \mathbf{C}[x_0, x_1, y_0, y_1]_{3,3} \cong \mathbf{C}^{16}.$$

In V we can consider the discriminant locus Σ of polynomials defining singular curves in $\mathbf{P}^1 \times \mathbf{P}^1$. The discriminant Σ is closed in the Zariski topology, and is a 15-dimensional cone with the origin as vertex. It is also irreducible, because it is the cone over the dual variety of the Segre embedding of the product of two rational normal curves of degree 3. In other words, the projectivization of Σ is the dual of the image of the map

$$\begin{array}{ccccc} \mathbf{P}^1 \times \mathbf{P}^1 & \xrightarrow{v_3 \times v_3} & \mathbf{P}^3 \times \mathbf{P}^3 & \xrightarrow{\sigma} & \mathbf{P}(V) \\ ([x_0, x_1], [y_0, y_1]) & \longmapsto & [x_0^3 y_0^3, x_0^2 x_1 y_0^3, \dots, x_1^3 y_1^3]. \end{array}$$

The locus of polynomials giving non-singular curves will be denoted by $X = V \setminus \Sigma$.

The automorphism group of $\mathbf{P}^1 \times \mathbf{P}^1$ is of the form $G \rtimes \mathfrak{S}_2$. The factor \mathfrak{S}_2 is generated by the involution v interchanging the two rulings:

$$\begin{array}{ccc} v : & \mathbf{P}^1 \times \mathbf{P}^1 & \longrightarrow \mathbf{P}^1 \times \mathbf{P}^1 \\ & ([x_0, x_1], [y_0, y_1]) & \longmapsto ([y_0, y_1], [x_0, x_1]). \end{array}$$

The connected component containing the identity of the automorphism group is the group

$$G := \mathrm{GL}(2) \times \mathrm{GL}(2) / \{\lambda I, \lambda^{-1} I\}_{\lambda \in \mathbb{C}^*},$$

and its action on $\mathbf{P}^1 \times \mathbf{P}^1$ is induced by the action of $\mathrm{GL}(2)$ on \mathbf{P}^1 .

Being the quotient of a reductive group, G is itself reductive. The action of G on binary polynomials induces an action on V . The space X is clearly invariant under this action of G , and all its points are G -stable. Thus there exists a geometric invariant theory quotient X/G , which is a double cover of C_0 , the involution of the double cover being induced by v .

3.1. Generalized Leray-Hirsch theorem. The aim of this section is to prove that the action of G on X satisfies the hypotheses of Theorem 1.1.

We need first to compute the cohomology of G . Consider the map:

$$\begin{aligned} \iota : \mathbb{C}^* \times \mathrm{SL}(2) \times \mathrm{SL}(2) &\longrightarrow G \\ (\lambda, A, B) &\longmapsto [(\lambda A, \lambda B)]. \end{aligned}$$

The map ι is an isogeny of connected algebraic groups. Its kernel is finite, hence ι induces an isomorphism of rational cohomology groups. As a consequence, the cohomology of G is an exterior algebra on three independent generators: a generator ξ of degree 1 and two generators η_1, η_2 of degree 3.

What we need, is to show the surjectivity of the map (induced by the orbit map $\rho : G \rightarrow X$)

$$\rho^* : H^i(X; \mathbb{Q}) \longrightarrow H^i(G; \mathbb{Q}).$$

We will do it by studying the map

$$\begin{array}{ccccccc} \psi : & H^i(X; \mathbb{Q}) & \longrightarrow & H^i(G; \mathbb{Q}) & \longrightarrow & H^i(\mathrm{GL}(2) \times \mathrm{GL}(2); \mathbb{Q}) \\ & \downarrow & & & & \downarrow \\ & \bar{H}_{31-i}(\Sigma; \mathbb{Q}) & \longrightarrow & \bar{H}_{15-i}((D \times M) \cup (M \times D); \mathbb{Q}), & & & \end{array}$$

where we have considered the embedding of $\mathrm{GL}(2)$ in the space M of 2×2 matrices, and written $D = M \setminus \mathrm{GL}(2)$ for the hypersurface defined by the vanishing of the determinant. Note that $H^3(G; \mathbb{Q})$ is isomorphic to $H^3(\mathrm{GL}(2) \times \mathrm{GL}(2); \mathbb{Q})$ and that $H^1(G; \mathbb{Q})$ can be identified with the part of $H^1(\mathrm{GL}(2) \times \mathrm{GL}(2); \mathbb{Q})$ that is invariant with respect to the interchange of the two factors of the product.

We need only to show that the generators of $H^\bullet(G; \mathbb{Q})$ are contained in the image of ψ . We will see that in this case we can write the generators of the cohomology groups of X quite explicitly, by means of fundamental classes.

The cohomology of $\mathrm{GL}(2) \times \mathrm{GL}(2)$ is determined by the Borel-Moore homology of the discriminant $D \subset M$. We can compute it by looking at the desingularization

$$\begin{aligned} \tilde{D} &= \{(p, A) \in \mathbf{P}^1 \times M : Ap = 0\}, \\ \mathbf{P}^1 &\xleftarrow{\tau} \tilde{D} \longrightarrow D. \end{aligned}$$

As a consequence, $\bar{H}_\bullet(\tilde{D}; \mathbb{Q}) \cong \bar{H}_{\bullet-4}(\mathbf{P}^1; \mathbb{Q})$. Hence the only non-trivial groups are $\bar{H}_6(\tilde{D}; \mathbb{Q})$, which is generated by the fundamental class of \tilde{D} , and $\bar{H}_4(\tilde{D}; \mathbb{Q})$, which is generated by the fundamental class of the preimage \tilde{R} of a point in \mathbf{P}^1 . As Borel-Moore homology is covariant for proper morphisms, there is a natural map $\bar{H}_\bullet(\tilde{D}; \mathbb{Q}) \rightarrow \bar{H}_\bullet(D; \mathbb{Q})$, which must be an isomorphism in degrees 4,6 because in those cases the two groups have

the same dimension. Thus we know generators for $\bar{H}_\bullet(D; \mathbf{Q})$. The fundamental class of D is a generator of degree 6, and the fundamental class of the image R of \tilde{R} is a generator of degree 4. In particular, we can choose R to be the subvariety of matrices with only zeroes on the first column.

We have natural projections

$$D \times M \xrightarrow{a_1} D \xleftarrow{a_2} M \times D$$

and natural immersions

$$\begin{array}{ccccc} D \times M & \xleftarrow{i_1} & D & \xrightarrow{i_2} & M \times D \\ (A, I) & \longleftarrow & A & \longrightarrow & (I, A). \end{array}$$

Then we have

$$\bar{H}_{14}((D \times M) \cup (M \times D); \mathbf{Q}) \cong \bar{H}_{14}(D \times M; \mathbf{Q}) \oplus \bar{H}_{14}(M \times D; \mathbf{Q}) \cong \mathbf{Q}\langle a_1^*([D]), a_2^*([D]) \rangle,$$

and the part which comes from the cohomology of G is generated by $a_1^*([D]) + a_2^*([D])$. Analogously, in degree 12 we have again

$$\bar{H}_{12}((D \times M) \cup (M \times D); \mathbf{Q}) \cong \bar{H}_{12}(D \times M; \mathbf{Q}) \oplus \bar{H}_{12}(M \times D; \mathbf{Q}) \cong \mathbf{Q}\langle a_1^*([R]), a_2^*([R]) \rangle,$$

where the first isomorphism is a consequence of the fact that the two space have the same dimension.

We can associate to Σ the variety

$$\tilde{\Sigma} := \{(p, v) \in \mathbf{P}^1 \times \mathbf{P}^1 \times V : \text{the curve defined by } v = 0 \text{ is singular at } p\},$$

$$\mathbf{P}^1 \times \mathbf{P}^1 \xleftarrow{\nu} \tilde{\Sigma} \xrightarrow{\pi} \Sigma.$$

The map π gives $\tilde{\Sigma}$ the structure of a \mathbf{C}^{13} -bundle over $\mathbf{P}^1 \times \mathbf{P}^1$. Hence $\tilde{\Sigma}$ is a desingularization of Σ , and is homotopy equivalent to $\mathbf{P}^1 \times \mathbf{P}^1$. This ensures that the Borel-Moore homology group of degree 28 of $\tilde{\Sigma}$ is generated by the fundamental classes $[\tilde{D}_1], [\tilde{D}_2]$, where $\tilde{D}_1 = \nu^{-1}(\{p_0\} \times \mathbf{P}^1)$, $\tilde{D}_2 = \nu^{-1}(\mathbf{P}^1 \times \{q_0\})$.

If we choose a point $q_1 \in \mathbf{P}^1 \setminus \{q_0\}$, the orbit map defines a map

$$\begin{array}{ccc} \rho_1 : & \tilde{D} & \longrightarrow \tilde{\Sigma} \\ & (p, A) & \longmapsto ((p, q_1), \rho(A, I)). \end{array}$$

The map ρ_1 is well defined, because $\rho(A, I)$ is the union of the line $\{p\} \times \mathbf{P}^1$, with multiplicity 3, and three lines of the other ruling. Hence it is always singular at (p, q_1) .

Since $\rho_1^{-1}(\tilde{D}_1) = \tau^{-1}(\{p_0\})$, $\rho_1^{-1}(\tilde{D}_2) = \emptyset$, we have

$$\rho_1^*([\tilde{D}_1]) = [\tau^{-1}(\{p_0\})], \quad \rho_1^*([\tilde{D}_2]) = 0.$$

Analogously, if we fix $p_1 \neq p_0$, the map

$$\begin{array}{ccc} \rho_2 : & \tilde{D} & \longrightarrow \tilde{\Sigma} \\ & (p, A) & \longmapsto ((p_1, q), \rho(I, A)) \end{array}$$

satisfies $\rho_2^*([\tilde{D}_2]) = [\tau^{-1}(\{q_0\})]$, $\rho_2^*([\tilde{D}_1]) = 0$.

We define D_1 and D_2 as the images of \tilde{D}_1 and \tilde{D}_2 respectively under the map $\tilde{\Sigma} \rightarrow \Sigma$. The following diagrams are commutative:

$$\begin{array}{ccc} D & \xrightarrow{i_1} & D \times M \xrightarrow{\rho} \Sigma \\ \uparrow & & \uparrow \\ \tilde{D} & \xrightarrow{\rho_1} & \tilde{\Sigma}, \end{array} \quad \begin{array}{ccc} D & \xrightarrow{i_2} & M \times D \xrightarrow{\rho} \Sigma \\ \uparrow & & \uparrow \\ \tilde{D} & \xrightarrow{\rho_2} & \tilde{\Sigma}. \end{array}$$

This implies that we can use our knowledge about ρ_1 and ρ_2 to study the map induced by ρ on Borel-Moore homology. Thus we find

$$\begin{aligned} \rho^*([\Sigma]) &= a_1^*([D]) + a_2^*([D]), \\ \rho^*([D_1]) &= a_1^*([R]), \\ \rho^*([D_2]) &= a_2^*([R]), \end{aligned}$$

which is exactly what we wanted to prove.

3.2. Application of Gorinov-Vassiliev's method. We can associate to each $v \in V$ its zero locus on the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$, which is a non-singular quadric $Q \subset \mathbf{P}^3$. For $v \neq 0$ we always get a curve of type $(3,3)$ on Q . Recall that Q has two rulings, and that curves on Q are classified by considering the number of points of intersection with a general line of, respectively, the second and the first ruling. We want to know what the singular locus of such a curve can be. The singular points of a curve are the union of the singular points of each irreducible component of it and the points that are pairwise intersection of components. Note that for each component, the number of singular points allowed is bounded by the arithmetic genus of the component. By writing all possible ways a $(3,3)$ -curve can decompose into components, we get all possibilities in Table 2.

Note that for all configurations of singularities in the same class, the elements of V which are singular at least at the chosen configurations always form a vector space, which has always the same codimension c . We write this codimension in the second column of the table.

Each item in Table 2 can be used to define a family of configurations on Q . For every $j = 1, \dots, 26$, we can define X_j as the space of all configurations of type (j) . In this way we get a sequence of subsets X_1, X_2, \dots, X_{26} , which satisfies conditions 1-3 and 6 in List 2.3. Conditions 4 and 7 do not hold. This problem can be solved by enlarging the list.

In order for condition 7 to be satisfied, it suffices to include all subconfigurations of finite configurations of Table 2. Condition 4 is more delicate. We have to consider all possible limit positions of configurations of singular points. For instance, points in general position can become collinear, and conics of maximal rank can degenerate to the union of two lines.

In this way, we can construct a list of configurations that verifies all conditions in List 2.3. This new list is really long, so we do not report it here. Luckily, most configurations give no contribution to the Borel-Moore homology of Σ . In particular, by Lemma 2.13, configurations with more than two points on a rational curve give no contribution, and, by Lemma 2.12, the same holds for configurations with at least two points on a rational curve minus a point (which is $\cong \mathbf{C}$). Also all configurations containing curves give no

TABLE 2. Singular sets of (3,3)-curves.

type	c	description.
1	3	One point.
2	6	Two points.
3	7	Three collinear points.
4	8	A line.
5	9	Three non-collinear points.
6	10	Three collinear points plus a point not lying on the same line.
7	11	Four points lying on a non-singular conic.
8	11	5 points: two pairs of points lying each on a line of a different ruling, and the intersection of the two lines.
9	11	A line plus another point.
10	12	4 points, not lying on the same plane in \mathbf{P}^3 .
11	12	A conic (possibly degenerating into two lines).
12	13	5 points: three points on a line of some ruling and two more points not coplanar with them.
13	14	5 points: a point p and 4 more points lying on a non-singular conic not passing through p . These points have to be distinct from the intersection of the conic with the two lines of the rulings of Q that pass through p .
14	14	6 points: two points on a line of the first ruling, two on a line of the second ruling, the intersection of the two lines and an additional point, not lying on the same line of a ruling of Q as any of the others.
15	14	6 points: union of two triplets of collinear points (on lines of the same ruling). No two points of the configuration can lie on the same line of the other ruling.
16	14	A line and two non-collinear points.
17	15	5 points in general position (no two of them on a line of any ruling, no 4 on a conic).
18	15	6 points: two points lying on a line, three points lying on a non-singular conic and the intersection of the line and the conic. In this configuration only the first two points lie on a line contained in Q .
19	15	6 points: intersection of Q with three concurrent lines in \mathbf{P}^3 .
20	15	7 points, intersection of components of the union of two lines of a ruling, one of the other and an irreducible curve of type (2, 1) or (1, 2).
21	15	7 points, intersection of components of two lines of different rulings and two non-singular conics on Q .
22	15	8 points, intersection of components of the union of two pairs of lines of different rulings and an irreducible conic.
23	15	9 points lying on 6 lines (three in every ruling).
24	15	A line and three points on a line of the same ruling.
25	15	A conic (possibly reducible) plus an extra point.
26	16	The whole Q .

contribution, because we can apply Lemma 2.19 and Lemma 2.20, or a slight modification of them.

We give below a list limited to the interesting configurations:

LIST 3.1.

- (A) One point.
- (B) Two points.
- (C) Three points in general position.
- (D) Four points in general position.
- (E) Six points, intersection of Q with three concurrent lines.
- (F) Eight points, intersection of components of two pairs of lines for every ruling and a conic.
- (G) The whole quadric.

A configuration is said to be *general* if no three points of it lie on the same line, and no four points of it lie on the same plane in \mathbf{P}^3 .

3.3. Columns (A)-(D). The space F_A is a \mathbf{C}^3 -bundle over $X_A \cong \mathbf{P}^1 \times \mathbf{P}^1$.

The space F_B is a $\mathbf{C}^6 \times \mathring{\Delta}_1$ -bundle over $X_B \cong B(\mathbf{P}^1 \times \mathbf{P}^1, 2)$.

The space F_C is a $\mathbf{C}^9 \times \mathring{\Delta}_2$ -bundle over X_C , which has the same twisted Borel-Moore homology as $B(\mathbf{P}^1 \times \mathbf{P}^1, 3)$ (the non-general configurations form a space with trivial twisted Borel-Moore homology).

Analogously, F_D is a $\mathbf{C}^{12} \times \mathring{\Delta}_3$ -bundle over X_D , which has the same twisted Borel-Moore homology as $B(\mathbf{P}^1 \times \mathbf{P}^1, 4)$.

Recall that the simplicial bundles are non-orientable, so that we have to consider the Borel-Moore homology with coefficients in the local system $\pm \mathbf{Q}$. This means that we can compute all the terms in these columns by Lemma 2.15.

3.4. Column (E). Configurations of type (E) are the singular loci of (3,3)-curves which are the union of three conics lying on Q . This gives three pairs of points on the Segre quadric, which are the intersection of it with three concurrent lines in \mathbf{P}^3 . Then it is natural to consider this configuration space as a fiber bundle X_E over $\mathbf{P}^3 \setminus Q$. The projection is given by mapping each configuration to the common point of the three lines. The fiber over a point $p \in \mathbf{P}^3 \setminus Q$ is the space of configurations of three lines through p , all three not tangent to Q . It is isomorphic to the space $\tilde{B}(\mathbf{P}^2 \setminus C, 3)$ of configurations of three non-collinear points on $\mathbf{P}^2 \cong \mathbf{P}(T_p \mathbf{P}^3)$ minus an irreducible conic $C \subset \mathbf{P}^2$. Note that the orientation of the simplicial bundle over X_E changes under the action of a non-trivial element of the fundamental group of $\mathbf{P}^2 \setminus C$, which is \mathfrak{S}_2 . This means that we have to compute the Borel-Moore homology of $\tilde{B}(\mathbf{P}^2 \setminus C, 3)$ with a system of coefficients locally isomorphic to \mathbf{Q} , which changes its orientation when a point of the configuration moves along a loop in $\mathbf{P}^2 \setminus C$ with non-trivial homotopy class. We denote this local system by W .

We consider $\tilde{B}(\mathbf{P}^2 \setminus C, 3)$ as a quotient of the corresponding space of ordered configurations, $\tilde{F} = (\mathbf{P}^2 \setminus C)^3 \setminus \Delta$, where

$$\Delta = \{(x, y, z) \in (\mathbf{P}^2 \setminus C)^3 : \dim \langle x, y, z \rangle \leq 1\}.$$

By abuse of notation, we denote also the local system on $(\mathbf{P}^2 \setminus C)^3$ by W .

The closed immersion $\Delta \hookrightarrow (\mathbf{P}^2 \setminus C)^3$ induces the exact sequence in Borel-Moore homology

$$(2) \quad \cdots \rightarrow \bar{H}_{i+1}(\tilde{F}; W) \rightarrow \bar{H}_i(\Delta; W) \rightarrow \bar{H}_i((\mathbf{P}^2 \setminus C)^3; W) \rightarrow \bar{H}_i(\tilde{F}; W) \rightarrow \cdots$$

The map associating to a point $p \in (\mathbf{P}^2 \setminus C)$ the intersection with C of the line polar to p with respect to C , induces an isomorphism $(\mathbf{P}^2 \setminus C) \cong B(C, 2) \cong B(\mathbf{P}^1, 2)$. In particular, we have

$$\bar{H}_\bullet((\mathbf{P}^2 \setminus C)^3; W) \cong (\bar{H}_\bullet(B(\mathbf{P}^1, 2); \pm \mathbf{Q}))^{\otimes 3},$$

which implies that $\bar{H}_\bullet((\mathbf{P}^2 \setminus C)^3; W)$ is $\mathbf{Q}(3)$ in degree 6, and trivial in all other degrees. Note that the generator of $\bar{H}_6((\mathbf{P}^2 \setminus C)^3; W)$ is invariant under the action of \mathfrak{S}_3 .

Next, we compute the Borel-Moore homology of Δ by considering the following filtration:

$$\begin{aligned} \Delta &= \Delta_3 \supset \Delta_2 \supset \Delta_1, \\ \Delta_1 &:= \{(x, y, z) \in \Delta : x = y = z\} \\ \Delta_2 &:= \{(x, y, z) \in \Delta : \exists l \in C^\circ (x, y, z \in l)\}. \end{aligned}$$

The first term Δ_1 of the filtration is isomorphic to $\mathbf{P}^2 \setminus C$, and hence to $B(\mathbf{P}^1, 2)$, so that its Borel-Moore homology with coefficients in W is $\mathbf{Q}(1)$ in degree 2.

Let us come back to the original construction, and consider ordered configurations of three lines with a common point p . Then the configurations in $\Delta_2 \setminus \Delta_1$ correspond to triples of lines lying on a plane tangent to Q . This means that the space is fibered over the family of such planes, which is parametrized by $Q^\circ \cap p^\circ$, a non-singular conic. In particular, $Q^\circ \cap p^\circ$ is simply connected, hence the only system of coefficients we can have there is the constant one.

The fibre over a plane Π is given by all ordered triples of lines passing through p , lying in Π and not tangent to Q , such that not all lines coincide. This is the space $(\mathbf{C}^3 \setminus \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 = z_2 = z_3\})$, which has Borel-Moore homology $\mathbf{Q}(1)$ in degree 3 and $\mathbf{Q}(3)$ in degree 6. All elements of $\bar{H}_\bullet(\{z_1 = z_2 = z_3\}; \mathbf{Q})$ are invariant with respect to the \mathfrak{S}_3 -action. We can conclude that the Borel-Moore homology of $\Delta_2 \setminus \Delta_1$ has Poincaré-Serre polynomial $t^3u^{-2} + t^5u^{-4} + t^6u^{-6} + t^8u^{-8}$.

For later use, we consider the action induced by the involution v interchanging the two rulings of $\mathbf{P}^1 \times \mathbf{P}^1$ on $\bar{H}_8(\Delta_2 \setminus \Delta_1; \mathbf{Q})$. This Borel-Moore homology group is obtained as the tensor product of $\bar{H}_4(Q^\circ \cap p^\circ; \mathbf{Q})$ and the Borel-Moore homology group of degree 6 of the fiber. Both factors are invariant under the action induced by v , hence the whole group is invariant under it. Note that the action of v on the fiber can be seen as interchanging the two lines of intersection of the plane Π and Q . This means that, if we consider the configuration of points of intersections of the three lines and Q , the action of v interchanges three pairs of points.

The space $\Delta_3 \setminus \Delta_2$ is a fiber bundle over $\tilde{\mathbf{P}}^2 \setminus C^\circ$. The fiber is isomorphic to $(\mathbf{C}^*)^3 \setminus \delta$, $\delta := \{(x, y, z) \in (\mathbf{C}^*)^3 : x = y = z\}$. The local system W' induced by W coincides with that induced by the local system T on \mathbf{C}^* locally isomorphic to \mathbf{Q} and changing its sign if the point moves along a loop in \mathbf{C}^* whose homotopy class is an odd multiple of the generator of $\pi_1(\mathbf{C}^*) \cong \mathbf{Z}$.

We have again an exact sequence

$$\cdots \rightarrow \bar{H}_{i+1}((\mathbf{C}^*)^3 \setminus \delta; W') \rightarrow \bar{H}_i(\delta, T) \rightarrow \bar{H}_i((\mathbf{C}^*)^3, T^{\otimes 3}) \rightarrow \bar{H}_i((\mathbf{C}^*)^3 \setminus \delta; W') \rightarrow \cdots$$

Since $\bar{H}_\bullet(\mathbf{C}^*, T)$ is trivial by Lemma 2.17, $\Delta_3 \setminus \Delta_2$ gives no contribution to the homology of Δ .

Let us consider the spectral sequence associated to the filtration Δ_i . The only possibly non-trivial differential is that from $E_{1,2}$ to $E_{1,1}$, and this must be an isomorphism for dimensional reasons ($V \setminus \Sigma$ is affine of complex dimension 16). As a consequence, $\bar{H}_k(\Delta; W)^{\mathfrak{S}_3}$ is isomorphic to $\bar{H}_k(\Delta; W)$, and has Poincaré-Serre polynomial $t^5u^{-4} + t^6u^{-6} + t^8u^{-8}$.

From (2) we get that the Poincaré-Serre polynomial of $\bar{H}_k(\tilde{B}(\mathbf{P}^2 \setminus C, 3); W)$ is $t^6u^{-4} + t^9u^{-8}$.

The Borel-Moore homology of X_E is then by Leray's theorem the tensor product of the above Borel-Moore homology and that of $\mathbf{P}^3 \setminus Q$, which is $\mathbf{Q}(1)$ in degree 3 and $\mathbf{Q}(3)$ in degree 6. The local system induced on $\mathbf{P}^3 \setminus Q$ is indeed the constant one.

We consider the action induced by the involution v on the Borel-Moore homology of X_E and F_E . We are only interested in determining this action on the graded piece of highest degree of these homology groups. It is easy to find that $\bar{H}_3(\mathbf{P}^3 \setminus Q; \mathbf{Q})$ is anti-invariant with respect to the involution v , and $\bar{H}_6(\mathbf{P}^3 \setminus Q; \mathbf{Q})$ is invariant. The group $\bar{H}_{15}(X_E; \pm \mathbf{Q})$ is the tensor product of $\bar{H}_6(\mathbf{P}^3 \setminus Q; \mathbf{Q})$ and $\bar{H}_9(\tilde{B}(\mathbf{P}^2 \setminus C, 3); W)$. The latter group is isomorphic to $\bar{H}_8(\Delta_2 \setminus \Delta_1; \mathbf{Q})$, which we already showed to be invariant. We also saw that the action of v on $\Delta \setminus \Delta_1$ produces the interchange of an *odd* number of pairs of points, so that the action of v on the generator of the Borel-Moore homology of the fiber of the bundle $\Phi_E \rightarrow X_E$ changes its orientation. Hence the invariance of $\bar{H}_{15}(X_E; \pm \mathbf{Q})$ under v implies the anti-invariance of both $\bar{H}_{20}(\Phi_E; \mathbf{Q})$ and $\bar{H}_{22}(F_E; \mathbf{Q})$.

3.5. Column (F). The configurations of type (F) are the configurations of eight points which are intersection of components of two pairs of lines for every ruling (say, l_1, l_2, m_1, m_2) and a conic C . The situation has to be sufficiently general to give exactly 8 points of intersection.

First let us fix the conic. If it is singular, each pair of lines must be an element of $B(\mathbf{C}, 2)$, which has trivial Borel-Moore homology with twisted coefficients. This means that we have to consider only non-singular conics. On each such conic, the configuration is univocally determined by the intersection points of the four lines with it. Denote by Ψ the space of two pairs of points in \mathbf{P}^1 , i.e., it is the quotient of $F(\mathbf{P}^1, 4)$ by the relation $(z_1, z_2, z_3, z_4) \sim (z_2, z_1, z_3, z_4)$, $(z_1, z_2, z_3, z_4) \sim (z_1, z_2, z_4, z_3)$. The configuration space we have to consider is isomorphic to the product $(\check{\mathbf{P}}^3 \setminus Q^\sim) \times \Psi$.

The local system of coefficients is the tensor product of \mathbf{Q} on $\check{\mathbf{P}}^3 \setminus Q^\sim$ and the rank one local system R on Ψ that changes its sign along the loops exchanging respectively only the first pair of points, or only the second.

The following two lemmas can be also deduced from the results of Getzler on the cohomology of $\mathcal{M}_{0,n}$ ([Get95]).

LEMMA 3.2. *The Poincaré-Serre polynomial of the Borel-Moore homology of $F(\mathbf{P}^1, 4)$ is $2t^4u^{-2} + t^5u^{-4} + 2t^7u^{-6} + t^8u^{-8}$.*

PROOF. The action of $\mathrm{PGL}(2)$ induces a quotient map $F(\mathbf{P}^1, 4) \rightarrow \mathcal{M}_{0,4} \cong \mathbf{P}^1 \setminus \{0, 1, \infty\}$. Notice that there is a natural action of \mathfrak{S}_4 on both $F(\mathbf{P}^1, 4)$ and $\mathcal{M}_{0,4}$, and that the quotient map is equivariant with respect to this action. The \mathfrak{S}_4 -quotient of $F(\mathbf{P}^1, 4)$ is the space $\mathbf{P}U_{4,1}$ of non-zero square-free homogeneous quartic polynomials in two variables, up to scalar multiples. By [PS03] the map $H^\bullet(\mathbf{P}U_{4,1}; \mathbf{Q}) \rightarrow H^\bullet(\mathrm{PGL}(2); \mathbf{Q})$

is an isomorphism (it is induced by any orbit map). Hence also the pull-back of the orbit map $H^\bullet(F(\mathbf{P}^1, 4); \mathbf{Q}) \rightarrow H^\bullet(\mathrm{PGL}(2); \mathbf{Q})$ is surjective.

We can apply the generalized Leray-Hirsch Theorem 1.1, and get that the rational cohomology of $F(\mathbf{P}^1, 4)$ is isomorphic to $H^\bullet(\mathcal{M}_{0,4}; \mathbf{Q}) \otimes_{\mathbf{Q}} H^\bullet(\mathrm{PGL}(2); \mathbf{Q})$. The cohomology of $\mathrm{PGL}(2)$ is \mathbf{Q} in degree 0 and $\mathbf{Q}(-2)$ in degree 3. Then the claim follows from the fact that $F(\mathbf{P}^1, 4)$ is smooth of complex dimension 4, so that the cap product with the fundamental class $[F(\mathbf{P}^1, 4)]$ induces an isomorphism $\bar{H}_\bullet(F(\mathbf{P}^1, 4); \mathbf{Q}) \cong H^{8-\bullet}(F(\mathbf{P}^1, 4); \mathbf{Q})(4)$. \square

LEMMA 3.3. *The Poincaré-Serre polynomial of $\bar{H}_\bullet(\Psi; R)$ is $t^4u^{-2} + t^7u^{-6}$. Moreover, $\bar{H}_\bullet(\Psi; R)$ is invariant with respect to the \mathfrak{S}_2 -action induced by the interchange of (z_1, z_2, z_3, z_4) and (z_3, z_4, z_1, z_2) .*

PROOF. We refer to the preceding Lemma.

$\bar{H}_\bullet(\Psi; R)$ can be identified with the part of $\bar{H}_\bullet(F(\mathbf{P}^1, 4); \mathbf{Q})$ which is anti-invariant with respect to the action of the transpositions $(1, 2)$ and $(3, 4)$. By the equivariance of the quotient map, it is sufficient to determine the action on $\mathcal{M}_{0,4}$. The action of \mathfrak{S}_4 on $H^\bullet(\mathcal{M}_{0,4}; \mathbf{Q})$ factorizes via \mathfrak{S}_3 , hence the action of any pair of commuting transpositions must coincide. The anti-invariant part of $H^\bullet(\mathcal{M}_{0,4}; \mathbf{Q})$ with respect to $(1, 2) \in \mathfrak{S}_4$ is $\mathbf{Q}(-1)$ in degree 1. The actions of the two transpositions $(1, 3)$ and $(2, 4)$ on $\mathcal{M}_{0,4}$ coincide, and their product is the identity. If we pass from cohomology to Borel-Moore homology, this implies the claim. \square

Lemma 3.3 implies that the Poincaré-Serre polynomial of Borel-Moore homology of X_F with coefficients in $\pm \mathbf{Q}$ is $t^7u^{-4} + 2t^{10}u^{-8} + t^{13}u^{-12}$.

Moreover, we can compute the action on $\bar{H}_\bullet(X_F; \pm \mathbf{Q})$ induced by the involution v interchanging the two rulings of the quadric Q . Fix a configuration in X_F . Up to a choice of coordinates, we can assume that the action of v permutes the two lines of the first ruling with the two of the second ruling in the configuration. This means that the action of v on Ψ can be identified with that interchanging $(z_1, z_2, z_3, z_4) \in F(\mathbf{P}^1, 4)$ with (z_3, z_4, z_1, z_2) . As a consequence, $\bar{H}_{13}(X_F; \pm \mathbf{Q})$ is invariant with respect to v . If we look at the configuration itself, three pairs of points are exchanged. Namely, the four points of intersection of the conic and the lines are exchanged in pairs, and also exactly two of the points of intersection between the lines are exchanged. Hence the Borel-Moore homology of the fiber of the bundle $\Phi_F \rightarrow X_F$ is anti-invariant for v . We can conclude that the highest class in the Borel-Moore homology of F_F (i.e., that of degree 22) is anti-invariant for the action of v .

3.6. Column (G). By Proposition 2.9, the space F_G is an open cone. The Borel-Moore homology of its base space can be computed by the spectral sequence in Table 3. Its columns coincide with those of the main spectral sequence, but are shifted by twice the dimension of the fibre of the complex vector bundle we considered for each of them.

For dimensional reasons, the rows of indices 1, 3, 5 and 7 of this spectral sequence are exact. The reason is that if it were not so, there would be non-trivial elements in the general spectral sequence, in a position such that they could not disappear. Hence they would give non-trivial elements in $H^j(V \setminus \Sigma) \cong \bar{H}_{31-j}(\Sigma)$ for $j > 16$, which is impossible because $V \setminus \Sigma$ is affine of dimension 16.

As a consequence, the spectral sequence here degenerates at E^2 . Then Lemma 2.18 allows us to obtain the Borel-Moore homology of the open cone.

TABLE 3. Spectral sequence converging to the base of F_G .

15				$\mathbf{Q}(7)$	
14					$\mathbf{Q}(6)$
13					
12				$\mathbf{Q}(5)^2$	
11					$\mathbf{Q}(4)^2$
10					
9				$\mathbf{Q}(3)$	
8					$\mathbf{Q}(2)$
7		$\mathbf{Q}(4)$	$\mathbf{Q}(4)$		
6					
5		$\mathbf{Q}(3)^2$	$\mathbf{Q}(3)^2$		
4					
3	$\mathbf{Q}(2)$	$\mathbf{Q}(2)^2$	$\mathbf{Q}(2)$		
2					
1	$\mathbf{Q}(1)^2$	$\mathbf{Q}(1)^2$			
0					
-1	\mathbf{Q}				
	A	B	C	D	E
					F

3.7. Spectral sequence. We know now all columns of the spectral sequence associated to the filtration

$$\text{Fil}_A(|\mathcal{X}(\bullet)|) \subset \cdots \subset \text{Fil}_G(|\mathcal{X}(\bullet)|).$$

Its E^1 term is represented in Table 4.

This spectral sequence degenerates at E^1 . Indeed, the only possible non-trivial differentials are between the first 4 columns. We know a priori that $H^0(C_0; \mathbf{Q})$ has dimension 1. Then Theorem 1.1 implies that the cohomology of $X = V \setminus \Sigma$ must contain a copy of the cohomology of the group G . This is impossible if any of the differentials in columns (A)-(D) is non-zero.

We can compute the whole cohomology of X from the Borel-Moore homology of Σ , using the isomorphism induced by the cap product with the fundamental class of the discriminant

$$\tilde{H}^\bullet(X; \mathbf{Q}) \cong \bar{H}_{31-\bullet}(\Sigma; \mathbf{Q})(-d).$$

In view of the results in Section 3.1 on the rational cohomology of G , Theorem 1.1 implies that the Poincaré-Serre polynomial of X/G is $1 + t^5u^6 + t^9(u^{16} + u^{18})$.

The quotient X/G is a double cover of C_0 , the \mathfrak{S}_2 -action being generated by the involution v interchanging the two rulings of the Segre quadric Q . The cohomology of X is invariant with respect to this involution in the degrees 0, 5. The cohomology in degree 9 comes from the terms $E_{E,17}^1, E_{F,16}^1$ in the spectral sequence. During the computation of columns (E) and (F) we observed that these terms are anti-invariant with respect to the action of \mathfrak{S}_2 on X . Then C_0 has no cohomology in degree 9.

We can conclude that the rational cohomology of C_0 is \mathbf{Q} in degree 0 and $\mathbf{Q}(-3)$ in degree 5, as we claimed in Theorem 1.2.

TABLE 4. Spectral sequence converging to $\bar{H}_k(\Sigma; \mathbf{Q})$.

29	$\mathbf{Q}(15)$					
28						
27	$\mathbf{Q}(14)^2$					
26						
25	$\mathbf{Q}(13)$	$\mathbf{Q}(13)^2$				
24						
23		$\mathbf{Q}(12)^2$				
22						
21		$\mathbf{Q}(11)^2$	$\mathbf{Q}(11)$			
20						
19			$\mathbf{Q}(10)^2$			
18						
17			$\mathbf{Q}(9)$	$\mathbf{Q}(8)$		
16					$\mathbf{Q}(7)$	
15				$\mathbf{Q}(8)$		
14				$\mathbf{Q}(6)^2$		$\mathbf{Q}(7) + \mathbf{Q}(6)$
13					$\mathbf{Q}(5)^2$	
12						
11				$\mathbf{Q}(4)$		$\mathbf{Q}(5)^2 + \mathbf{Q}(4)^2$
10					$\mathbf{Q}(3)$	
9						
8						$\mathbf{Q}(3) + \mathbf{Q}(2)$
	A	B	C	D	E	F
						G

4. Curves on a quadric cone

4.1. The space C_1 as geometric quotient. The aim of this section is to realize C_1 as a geometric quotient, satisfying the hypotheses of Theorem 1.1. We perform it by regarding the elements of C_1 as curves of degree 6 on a quadric cone.

After a choice of coordinates, we can identify the quadric cone with $\mathbf{P}(1, 1, 2)$. Then we consider the vector space $\mathbf{C}[x, y, z]_6$, where $\deg x = \deg y = 1$, $\deg z = 2$.

This space has complex dimension 16. The polynomials defining singular curves form the *discriminant* Σ , which has in this case two irreducible components of dimension 15. One component, which we denote by H , is the locus of curves passing through the vertex of the cone. Such a curve is always singular, and in general this singularity can be resolved by considering the proper transform in the blowing up of the vertex of the cone. The other component, which we denote by S , is the closure in \mathbf{C}^{16} of the locus of curves which are singular in a point different from the vertex of the cone. More specifically, S is the locus of curves such that the proper transform in the blowing up of the vertex of the cone is singular, or tangent to the exceptional locus. Note that H is the hyperplane defined by the condition that the coefficient of z^3 is zero. The other component S is the affine cone over the dual variety of the Veronese embedding of $\mathbf{P}(1, 1, 2)$ in \mathbf{P}^{15} .

Next, we find generators of the cohomology of X in degree 0 and 2 which have an interpretation as fundamental classes of subvarieties of $S \cup H$. The whole cohomology of $X = \mathbf{C}^{16} \setminus (S \cup H)$ will be calculated in Section 4.2.

We can compute the Borel-Moore homology of $S \cup H$ by considering the incidence correspondence

$$\mathcal{T} = \{(f, p) \in (S \cup H) \times \mathbf{P}(1, 1, 2) : f \text{ is singular at } p\}.$$

The incidence correspondence has a natural projection π to the cone $\mathbf{P}(1, 1, 2)$. The fiber over the vertex $[0, 0, 1]$ is simply $\{[0, 0, 1]\} \times H$. If we restrict to the preimage of $\mathbf{P}(1, 1, 2) \setminus \{[0, 0, 1]\}$, π is a complex vector bundle of rank 13. This allows us to compute the Borel-Moore homology of \mathcal{T} , and obtain $2t^{30}u^{-30} + t^{28}u^{-28}$ as its Poincaré-Serre polynomial.

Although π is no desingularization of $S \cup H$, it is a proper map, hence it induces a homomorphism of Borel-Moore homology groups. By the results in Section 4.2 the map induced by π in degrees 28 and 30 must be an isomorphism. In particular, this implies that $\bar{H}_{28}(S \cup H; \mathbf{Q})$ is generated by the fundamental class of the locus S' of curves with a singularity on a chosen line. Obviously $\bar{H}_{30}(S \cup H; \mathbf{Q})$ is generated by $[S]$ and $[H]$.

Automorphisms of the graded ring $\mathbf{C}[x, y, z]$ are of the form

$$\begin{cases} x \mapsto \alpha x + \beta y \\ y \mapsto \gamma x + \delta y \\ z \mapsto \epsilon z + q(x, y), \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ are complex numbers such that $\epsilon(\alpha\delta - \beta\gamma) \neq 0$ and $q \in \mathbf{C}[x, y]_2$.

They form a group G of dimension 8. By contracting the vector space $\mathbf{C}[x, y]_2 \cong \mathbf{C}^3$ to a point, we get that G is homotopy equivalent to $\mathrm{GL}(2) \times \mathbf{C}^*$. This means that we can apply Theorem 1.1 to the action of $\mathrm{GL}(2) \times \mathbf{C}^*$ instead of the whole G . In order to be able to apply the Theorem in Section 4.2, we check that its hypotheses are satisfied. Namely, for each generator η of degree $2r - 1$ of the cohomology of $\mathrm{GL}(2) \times \mathbf{C}^*$, we want to define a subscheme Y of the discriminant, of pure codimension r , whose fundamental class maps to a non-zero multiple of η under the composition

$$\bar{H}_{2(16-r)}(Y) \rightarrow \bar{H}_{2(16-r)}(\Sigma) \xrightarrow{\sim} H^{2r-1}(X) \xrightarrow{\rho^*} H^{2r-1}(G),$$

where ρ denotes any orbit map from G to \mathbf{C}^{16} .

The cohomology of the product $\mathrm{GL}(2) \times \mathbf{C}^*$ decomposes naturally into that of its subgroups $\{I\} \times \mathbf{C}^*$ and $\mathrm{GL}(2) \times \{1\}$. We can reduce the study of the orbit map to that of the two maps

$$\rho_1 : \mathbf{C}^* \longrightarrow X, \quad \rho_1(t) = \rho(I, t)$$

and

$$\rho_2 : \mathrm{GL}(2) \longrightarrow X, \quad \rho_2(A) = \rho(A, 1).$$

The map induced by ρ_1 on cohomology is

$$\bar{H}_{31-\bullet}(H \cup S) \cong H^\bullet(X; \mathbf{Q}) \xrightarrow{\rho_1^*} H^\bullet(\mathbf{C}^*; \mathbf{Q}).$$

The cohomology of \mathbf{C}^* is generated by the fundamental class of $\{0\} \subset \mathbf{C}$. If we extend ρ_1 to a map $\mathbf{C} \longrightarrow \mathbf{C}^{16}$, we find that 0 is mapped to a curve which is the union of 6 distinct lines through the vertex of the cone. This means that the preimage of H coincides with

0, while the preimage of S is empty. Hence $\rho_1^*([H])$ is a non-zero multiple of $[0]$ (in fact, by direct computation, it is $3[0]$).

We consider next ρ_2 . It induces the map

$$\bar{H}_{31-\bullet}(H \cup S) \cong H^\bullet(X; \mathbf{Q}) \xrightarrow{\rho_2^*} H^\bullet(\mathrm{GL}(2); \mathbf{Q}).$$

Recall from Section 3.1 that the cohomology of $\mathrm{GL}(2)$ is generated by the fundamental class of the complement D of $\mathrm{GL}(2)$ in the space M of 2×2 matrices and that of the subspace R of matrices with only zeroes on the first column. The Borel-Moore homology class $[D] \in \bar{H}_6(D; \mathbf{Q})$ corresponds to a class of degree 1 in the cohomology of $\mathrm{GL}(2)$, and the Borel-Moore homology class $[R]$ corresponds to a class in $H^3(\mathrm{GL}(2); \mathbf{Q})$.

We look at the extension of ρ_2 to $M \rightarrow \mathbf{C}^{16}$. The elements in D are mapped to curves which are the union of three non-singular conics, having the same tangent line in a common point. These are always elements of $S \setminus H$. If we choose S' as the locus of curves singular at some point of the line $\{y = 0\} \subset \mathbf{P}(1, 1, 2)$, then we have that the preimage of S' is exactly R . These considerations imply the surjectivity of the orbit map on cohomology, hence the hypotheses of Theorem 1.1 are established.

4.2. Application of Vassiliev-Gorinov's method. As a starting point for the application of Gorinov-Vassiliev's method, we study the possible singular loci of elements of Σ . This is achieved by considering all possible decompositions in irreducible components of a curve of degree 6 on $\mathbf{P}(1, 1, 2)$. Then we consider how many singular points can lie on each component, and where the pairwise intersection of components can lie. As a result, we get a list similar to that given in Table 2.

For two configurations of the same type, the linear subspace of V of curves that are singular in the one and the linear subspace of curves that are singular in the other have always the same codimension. Hence this codimension depends only on the type of the configuration.

The families of configurations defined by the possible singular loci do not satisfy the conditions in List 2.3. We have to refine it in order to have all finite subconfigurations of finite configurations included. In this way we get a new list as follows:

LIST 4.1. Sequence of families of configurations which satisfies the conditions on List 2.3. The number under square brackets is the codimension of the space of polynomials singular at a chosen configuration of that type.

In this list, a configuration of points is said to be *general* if it does not contain the vertex, no two points of the configuration lie on the same line of the ruling of the cone and at most 3 points lie on the same conic contained in the cone.

- (A) The vertex. [1]
- (B) A general point. [3]
- (C) The vertex and a general point. [4]
 - Two points distinct from the vertex, lying on the same line of the ruling. [6]
 - The vertex and two points on the same line of the ruling. [6]
- (D) Two general points. [6]
 - Three collinear points, distinct from the vertex. [7]
 - The vertex and three collinear points. [7]
 - Four collinear points, distinct from the vertex. [7]
 - The vertex and four collinear points. [7]

- Five collinear points, distinct from the vertex . [7]
- The vertex and five collinear points. [7]
- A singular line. [7]

(E) The vertex and 2 general points. [7]

- Two points on the same line of the ruling and a general point. [9]
- Three general points. [9]
- The vertex, two points on a line of the ruling and a general point. [9]
- The vertex and 3 general points. [10]
- Three collinear points (distinct from the vertex) and a general point. [10]
- The vertex, three collinear points and a general point. [10]
- Four collinear points (distinct from the vertex) and a general point. [10]
- The vertex, four collinear points and a general point. [10]
- A singular line and a general point. [10]
- Two points on a line of the ruling, and two points on another. [11]
- The vertex, two points on a line of the ruling, and two points on another. [11]
- Four points on a non-singular conic. [11]
- Two points on a line of the ruling and three points on another. All points are different from the vertex. [12]
- The vertex, two points on a line and four points on another. [12]
- Two points on a line of the ruling and three points on another. All points are different from the vertex. [12]
- The union of a line and 2 points on a line of the ruling. [12]
- Two points on a line of the ruling and 2 general points. [12]
- Four general points. [12]
- The vertex, two collinear points and 2 general points. [12]
- The vertex and 4 points on a non-singular conic. [12]
- Two lines. [12]
- A non-singular conic. [12]
- The vertex and a non-singular conic. [13]
- The union of a line and two general points. [13]
- Four points on a conic and a point lying on the line joining one of the 4 points with the vertex. [14]
- The vertex, 4 points on a conic and a point lying on the line joining one of the 4 points with the vertex. [14]
- Two points on a line of the ruling, two points on another and a general point. [14]
- The vertex, two points on a line of the ruling, two points on another and a general point. [14]
- Four points on a non-singular conic and a general point. [14]
- A rational normal cubic. [15]

(F) The intersection of components of the union of two lines and two conics, excluding the vertex. [15]

(G) The intersection of components of the union of two lines and two conics. [15]

(H) 6 points, intersection of the cone with three concurrent lines. [15]

- Three lines. [15]
- Union of a line and a non-singular conic. [15]

(I) The whole cone. [16]

By the results in Sections 2.2 and 2.3, or an adaptation of them, the only configurations giving non-trivial contribution to the Borel-Moore homology are those indicated with (A)-(I). We will consider only them.

Case A: Obviously, $F_A \cong \mathbf{C}^{15}$. The only Borel-Moore homology is in degree 30.

Case B: The space F_B is a \mathbf{C}^{13} -bundle over the cone minus its vertex.

Case C: The space F_C is a $\mathbf{C}^{10} \times \mathring{\Delta}_1$ -bundle over the cone minus its vertex.

Case D: The space Φ_D is a non-orientable bundle of open simplices of dimension 1 over the subspace of $B(\mathbf{C} \times \mathbf{P}^1, 2)$ consisting of points not on the same line. This has the same Borel-Moore homology with twisted coefficients of $B(\mathbf{C} \times \mathbf{P}^1, 2)$, which is non-trivial only in degree 6. The space F_D is a \mathbf{C}^{10} -bundle over Φ_D .

Case E: The space Φ_E is a non-orientable bundle of open simplices of dimension 2 over the subspace of $B(\mathbf{C} \times \mathbf{P}^1, 2)$ consisting of points not on the same line. The space F_E is a \mathbf{C}^9 -bundle over Φ_E .

Cases F and G: We can consider configurations of types (F) and (G) together.

We have then that $\Phi_F \cup \Phi_G$ is a complex vector bundle of rank one over $F_F \cup F_G$.

We claim that Borel-Moore homology of $F_F \cup F_G$ is trivial. Let us consider the fiber Ψ of its projection to X_F (which is canonically isomorphic to X_G). Then Ψ is a simplex with vertices $t, a_1, a_2, a_3, a_4, a_5, a_6$, where $\{a_i\} \in X_F$ and t is the top of the cone. We have to consider the closed simplex S , minus all external faces containing t . Denote by B the union of such external faces. Observe that both B and S can be contracted to the vertex t . Then $\bar{H}_\bullet(\Psi; \mathbf{Q}) = H_\bullet(S, B; \mathbf{Q}) = 0$, which yields the claim.

Case H: The space F_H is a $\mathbf{C} \times \mathring{\Delta}_5$ -bundle over X_H . We claim that X_H has no Borel-Moore homology in the system of coefficients induced by F_H .

Each configuration in X_H is determined by three concurrent lines in \mathbf{P}^3 . This implies that X_H is fibred over the complement of the cone in \mathbf{P}^3 . The fiber over a point p is the space of configurations of three distinct lines through p , subject to certain conditions. Namely, the lines cannot be tangent to the cone, any such triple must span \mathbf{P}^3 , and no two of the lines can span a plane passing through the vertex of the cone. The condition of being tangent to the cone defines the union of two lines l, m inside $\mathbf{P}^2 \cong \mathbf{P}(T_p \mathbf{P}^3)$. Then what we need is to compute the Borel-Moore homology of the subset $B(\mathbf{C} \times \mathbf{C}^*, 3) \setminus S$, where S is the subset of the configurations in which exactly two points of the configuration lie on a line through $l \cap m$, or the three points are collinear (in this case, we can distinguish whether the line on which they lie passes through $l \cap m$ or not).

The system of coefficients we have to consider changes its sign every time that a point moves along a loop around one of the lines l or m . This system of coefficients is induced by a system T' on $\mathbf{C} \times \mathbf{C}^*$, and Lemma 2.17 implies that $\bar{H}_\bullet(\mathbf{C} \times \mathbf{C}^*; T')$ is trivial. Hence, $B(\mathbf{C} \times \mathbf{C}^*, 3)$ has trivial Borel-Moore homology in the system of coefficients we are interested in. However, S also has trivial Borel-Moore homology in the chosen system of coefficients, because each of its three disjoint substrata has trivial Borel-Moore homology in the system of coefficients induced by T' . This is again a consequence of Lemma 2.17. We can

TABLE 5. Spectral sequence converging to $\bar{H}_k(\Sigma; \mathbf{Q})$.

30					
29	$\mathbf{Q}(15)$				
28		$\mathbf{Q}(15)$			
27					
26		$\mathbf{Q}(14)$	$\mathbf{Q}(14)$		
25					
24			$\mathbf{Q}(13)$		
23				$\mathbf{Q}(13)$	
22					
21					$\mathbf{Q}(12)$
	A	B	C	D	E

conclude that $B(\mathbf{C} \times \mathbf{C}^*, 3) \setminus S$ has trivial Borel-Moore homology in the system of coefficients we are interested in.

Case I: By Proposition 2.9, point (3), the space F_I is an open cone over $\text{Fil}_G |\Lambda(\bullet)|$. The differentials between columns B and C, and between columns D and E of the spectral sequence converging to the Borel-Moore homology of the base of the cone must be isomorphisms for dimensional reasons. This implies that F_I has trivial Borel-Moore homology.

Putting information over all columns together, we have that the spectral sequence converging to Borel-Moore homology of $\Sigma = S \cup H$ has E^1 term as in Table 5. Columns from (F) to (I) have been omitted because they give trivial contributions.

The differential $E_{C,26}^1 \rightarrow E_{B,26}^1$ is trivial because $H^\bullet(X; \mathbf{Q})$ must contain a copy of the cohomology of G . This means that this spectral sequence degenerates at E^1 . As a consequence, $H^\bullet(X; \mathbf{Q}) = H^\bullet(G; \mathbf{Q})$. Theorem 1.1 gives that $C_1 = X/G$ has the rational cohomology of a point. Theorem 1.3 is now established.

5. Hyperelliptic locus

The moduli space \mathcal{H}_g of smooth hyperelliptic curves of genus $g \geq 2$ has always the cohomology of a point. We have that \mathcal{H}_g coincides with the moduli space of configurations of $2g + 2$ distinct points on \mathbf{P}^1 , which is in turn the quotient of the space of binary polynomials of degree $2g + 2$ without double roots, for the action of $\text{GL}(2)$. This is a special case of a moduli space of hypersurfaces. It is shown in [PS03] that in the hypersurface case the hypotheses of Theorem 1.1 are always satisfied. Thus all we need to prove is that the cohomology of the space of polynomials without double roots is generated by the elements mapped to the generators of the cohomology of $\text{GL}(2)$.

LEMMA 5.1. *Let $V = \mathbf{C}[x, y]_d$ be the vector space of homogeneous binary polynomials of degree $d \geq 4$, and $\Delta \subset V$ the discriminant, i.e., the locus of polynomials with multiple roots. Then $V \setminus \Delta$ has Poincaré-Serre polynomial $1 + tu^2 + t^3u^4 + t^4u^6$.*

PROOF. We apply Gorinov-Vassiliev's method and compute the Borel-Moore homology of Δ . For every $v \in \Delta$, we denote by K_v the locus in \mathbf{P}^1 which is the projectivization of the multiple roots of v . Denote by X_k the family of all configurations of k distinct

points in \mathbf{P}^1 . Then $X_1, \dots, X_{[d/2]}, \{\mathbf{P}^1\}$ satisfy the conditions in List 2.3. The dimension d_i of $L(x)$ for $x \in X_i$ is always $d + 1 - 2i$. This means that we can construct a geometrical realization $|\mathcal{X}(\bullet)| \rightarrow \Delta$. Note that in the filtration only the first two terms have non-trivial Borel-Moore homology, because $X_k \cong B(\mathbf{P}^1, k)$ has trivial twisted Borel-Moore homology for $k \geq 3$.

Clearly, the spectral sequence for the Borel-Moore homology of Δ degenerates at E_1 . This implies the claim. \square

CHAPTER II

Rational cohomology of $\overline{\mathcal{M}}_4$

This chapter is joint work with Jonas Bergström (KTH, Stockholm).

1. Introduction and results

In this chapter, we compute the rational cohomology of the moduli space $\overline{\mathcal{M}}_4$ of stable complex curves of genus 4, together with its Hodge structures.

The moduli space $\overline{\mathcal{M}}_4$ has a natural stratification based on the topological type of the curves, or equivalently, on the dual graph of a curve. The locally closed substrata obtained in this way are finite quotients of products of moduli spaces of pointed non-singular curves. More precisely, if we denote by $\mathcal{M}_{g,n}$ the moduli space of non-singular curves of genus g with n marked points, we find that only the spaces $\mathcal{M}_{g,n}$ with $g \leq 4$, $n \leq 8 - 2g$ are involved in the construction of the strata of $\overline{\mathcal{M}}_4$.

The moduli space $\overline{\mathcal{M}}_4$ is a proper smooth stack and satisfies therefore Poincaré duality. Hence, there are isomorphisms in all degrees between cohomology and cohomology with compact support, and all cohomology groups are pure with Hodge weight equal to the degree. For this reason, the rational cohomology of $\overline{\mathcal{M}}_4$ can be read off from the Hodge Euler characteristic of $\overline{\mathcal{M}}_4$, i.e., the Euler characteristic

$$e(\overline{\mathcal{M}}_4) := \sum_{j \in \mathbf{Z}} (-1)^j [H_c^j(\overline{\mathcal{M}}_4; \mathbf{Q})]$$

of $H_c^\bullet(\overline{\mathcal{M}}_4; \mathbf{Q})$ in the Grothendieck group of rational mixed Hodge structures.

Since the Hodge Euler characteristic is additive, it is possible a priori to determine the Hodge Euler characteristic of $\overline{\mathcal{M}}_4$ from that of its strata. As was said above, each stratum is the quotient of a product of spaces of the form $\mathcal{M}_{g,n}$ by the action of a finite group. The Hodge Euler characteristic of all $\mathcal{M}_{g,n}$'s occurring in the strata does not give enough information to compute the Hodge Euler characteristic of the strata, since it does not keep track of the action of the symmetric group. What we need instead is the \mathfrak{S}_n -equivariant Hodge characteristic of the $\mathcal{M}_{g,n}$'s, which also encodes the structure of the cohomology groups as representations of the symmetric group \mathfrak{S}_n , which acts naturally on $\mathcal{M}_{g,n}$ by permuting the marked points.

Once all \mathfrak{S}_n -equivariant Hodge Euler characteristics of the spaces $\mathcal{M}_{g,n}$ are known for $g \leq 4$, $n \leq 8 - 2g$, determining the Hodge Euler characteristic of $\overline{\mathcal{M}}_4$ becomes a problem in the combinatorics of graphs. Indeed, this is the approach of Getzler and Kapranov in [GK98], where they develop the theory of *modular operads* and apply it to the homology of the moduli spaces of curves. In particular, this enables them to give a formula ([GK98, 8.13]), that expresses the relationship of Euler characteristics between $\overline{\mathcal{M}}_{g,n}$ and the $\mathcal{M}_{\tilde{g},\tilde{n}}$ appearing in its stratification. As remarked in [Get98a], the formula applies to Hodge Euler characteristics, and in general to all Euler characteristics taking

values in the Grothendieck group of a symmetric monoidal category that is additive over a field of characteristic 0 and has finite colimits.

The computation of the cohomology of $\overline{\mathcal{M}}_4$ is thus essentially reduced to the computation of the equivariant Hodge Euler characteristic of all spaces $\mathcal{M}_{g,n}$ with $g \leq 4$, $n \leq 8 - 2g$. The equivariant Hodge Euler characteristics of all moduli spaces $\mathcal{M}_{0,n}$ and $\mathcal{M}_{1,n}$, and of $\mathcal{M}_{2,n}$ ($n \leq 3$), $\mathcal{M}_{3,n}$ ($n \leq 1$) are already known (see [Get95], [Get99], [Get98b], [Loo93], [GL]). In this chapter, we present two methods that allow us to calculate the Hodge Euler characteristic in the remaining cases. As a result, we establish the following.

THEOREM 1.1. *The Hodge Euler characteristic of $\overline{\mathcal{M}}_4$ is equal to*

$$\mathbf{L}^9 + 4\mathbf{L}^8 + 13\mathbf{L}^7 + 32\mathbf{L}^6 + 50\mathbf{L}^5 + 50\mathbf{L}^4 + 32\mathbf{L}^3 + 13\mathbf{L}^2 + 4\mathbf{L} + 1$$

where \mathbf{L} denotes the class of the Tate Hodge structure of weight 2 in the Grothendieck group of rational Hodge structures.

We conclude that $\overline{\mathcal{M}}_4$ has no odd cohomology. This is in agreement with the result in the article [AC98] by Arbarello-Cornalba, that the first, third and fifth cohomology group of $\overline{\mathcal{M}}_{g,n}$ is zero for all g and n .

The first method we present is based on the equivariant count of the number of points of $\mathcal{M}_{g,n}$ defined over finite fields. It has previously been used to get cohomological information on the moduli space of curves of genus two by Faber-Van der Geer in [FG04a], [FG04b] and on the moduli spaces of pointed curves of genus 0 by Kisin-Lehrer in [KL02].

Here we use it to get the wanted information for curves of genus two and three. The key remark is that if we consider the number of points of $\mathcal{M}_{2,n}$ or $\mathcal{M}_{3,n}$ defined over a finite field k as a function of the number q of elements of k , the function we get is a polynomial in q if n is sufficiently small. This is not merely true for the number of points of $\mathcal{M}_{g,n}$, i.e., the trace of Frobenius on the étale cohomology with compact support of $(\mathcal{M}_{g,n})_{\overline{k}}$. Also the trace of the composition of Frobenius with any automorphism of étale cohomology induced by the action of an element of \mathfrak{S}_n on $\mathcal{M}_{g,n}$ is a polynomial in q . We say that in these cases, the *equivariant count* of the number of points of $\mathcal{M}_{g,n}$ gives a polynomial. This is true for all moduli spaces occurring in the boundary of $\overline{\mathcal{M}}_4$. The equivariant count of points of $\mathcal{M}_{g,n}$ for $g \in \{2, 3\}$ is performed for a wider range of indexes n and explained in greater detail by the first author in [Berb] and [Bera].

The equivariant count of the number of points of $\mathcal{M}_{2,n}$ is based on the fact that all curves of genus two are hyperelliptic. In general, if the characteristic of k is odd, the moduli space \mathcal{H}_g of hyperelliptic curves of a fixed genus $g \geq 2$ is isomorphic to the moduli space of binary forms of degree $2g + 2$ without multiple roots. This allows us to reduce the equivariant count of the number of points defined over a finite field k of the moduli space of genus g hyperelliptic curves with n marked points, to a computation involving only the quadratic characters of finite extensions of k of the values of square-free monic polynomials of the appropriate degrees, at points on the projective line.

The equivariant count of the number of points of $\mathcal{Q}_n := \mathcal{M}_{3,n} \setminus \mathcal{H}_{3,n}$ is based on the fact that this space coincides with the moduli space of non-singular n -pointed quartic curves in \mathbf{P}^2 . Hence, in order to know the number of points defined over a finite field k of \mathcal{Q}_n it is enough to count the number of $(n+1)$ -tuples $(C, p_1, \dots, p_n) \in \mathbf{P}^{14}(k) \times (\mathbf{P}^2(k))^n$, where $\mathbf{P}^{14}(k)$ is identified with the space of quartic curves in \mathbf{P}^2 with coefficients in k ,

C is a non-singular curve and p_1, \dots, p_n are distinct points on C . It turns out that it is easier to count the number of singular curves than the number of non-singular ones. Hence, it is possible to obtain the equivariant count of points in \mathcal{Q}_n by counting, for each choice of an n -tuple defined over k of distinct points in \mathbf{P}^2 , the number of singular curves passing through these points. The latter count is performed by a modified version of the sieve principle.

From the fact that the equivariant count of points of $\mathcal{M}_{g,n}$ gives polynomials in the cases we investigate, it follows that we can use these polynomials to obtain the Hodge Euler characteristic of the moduli spaces. A theorem by Van den Bogaart and Edixhoven ([BE05]) ensures that the cohomology of $\overline{\mathcal{M}}_{g,n}$ is pure and of Tate Hodge type whenever the number of points of $\overline{\mathcal{M}}_{g,n}$ is a polynomial in q . Moreover, the Betti numbers and the Hodge weights of the cohomology classes are given by the coefficients of the polynomial. It is not difficult to extend this result so to keep track of the action of \mathfrak{S}_n on the cohomology of $\overline{\mathcal{M}}_{g,n}$. Note that if the equivariant count of points gives polynomials for all $\mathcal{M}_{\tilde{g},\tilde{n}}$ occurring in the stratification of $\overline{\mathcal{M}}_{g,n}$, also the count of the number of points of $\overline{\mathcal{M}}_{g,n}$ will give a polynomial. This implies that in this situation we can determine the \mathfrak{S}_n -equivariant Hodge characteristic of $\mathcal{M}_{g,n}$ from the \mathfrak{S}_n -equivariant count of the number of its points. Specifically, this allows us to establish the following.

THEOREM 1.2. *The equivariant Hodge Euler characteristic (written in terms of the Schur polynomials, see Section 2) of $\mathcal{M}_{2,4}$ is equal to*

$$(\mathbf{L}^7 + \mathbf{L}^6 - \mathbf{L}^5 - \mathbf{L}^4 - \mathbf{L}^2 + \mathbf{L} + 1)s_4 + (\mathbf{L}^6 - \mathbf{L}^5 - \mathbf{L}^2)s_{31} - \mathbf{L}^2 s_{22} + (\mathbf{L}^2 - 1)s_{21^2}.$$

THEOREM 1.3. *The equivariant Hodge Euler characteristic of $\mathcal{M}_{3,2}$ is equal to*

$$(\mathbf{L}^8 + 2\mathbf{L}^7 + \mathbf{L}^6 - \mathbf{L}^5 + \mathbf{L}^2 + \mathbf{L})s_2 + (\mathbf{L}^7 + \mathbf{L}^6 + \mathbf{L})s_{1^2}.$$

The second method we present is based on the fact that \mathcal{M}_4 and \mathcal{M}_3 have a stratification such that all strata can be interpreted as quotients of complements of discriminants in a complex vector space by the action of a reductive group. On the one hand, the rational cohomology of the complement of such a discriminant can be computed by Vassiliev-Gorinov's method (see [Vas99], [Gor05] and Section 2.1 of chapter I). On the other hand, by a generalization by Peters and Steenbrink of the Leray-Hirsch theorem ([PS03]), the rational cohomology of these complements of discriminants is the tensor product of the rational cohomology of the quotient and that of the group acting. As a consequence, if the cohomology of the complement of the discriminant is known, it is straightforward to obtain the cohomology of the quotient. Moreover, we show that it is easy to apply Peters-Steenbrink's generalized Leray-Hirsch theorem to incidence correspondences. This implies that we can use the same methods to compute the rational cohomology of moduli spaces of non-singular pointed curves. As a first easy example, we present here the calculation of the rational cohomology of $\mathcal{M}_{3,1}$. The rational cohomology of $\mathcal{M}_{3,1}$ is given in Corollary 5.5, which refines the result given in [GL], which corrects an error in [Loo93, 4.10].

The outline of the chapter is as follows. In Section 2, we present the formula of Getzler and Kapranov on the generating function of the Euler characteristic of modular operads, and use it to establish Theorem 1.1 from the Hodge Euler characteristics of the spaces $\mathcal{M}_{g,n}$ with $g \leq 4$, $n \leq 8g - 2$. In Section 3, following [BE05], we explain why the equivariant count of points of $\mathcal{M}_{g,n}$ for $g \leq 3$, $n \leq 8g - 2$ gives the equivariant Hodge Euler

characteristic of these spaces. This requires us to use an equivariant version of [BE05, Theorem 2.1]. In Section 4, we introduce the methods used for the equivariant counts of the number of points of $\mathcal{M}_{g,n}$ in the cases we are interested in. In Section 5 we present the relation between the study of the cohomology of moduli spaces of curves of genus 3 and of genus 4 and the cohomology of the complement of certain discriminants. Moreover we introduce the generalized Leray-Hirsch theorem and explain why it is applicable also in the case of moduli spaces of curves with marked points. We recall the result in [Tom05] on the rational cohomology of \mathcal{M}_4 , and compute the rational cohomology of $\mathcal{M}_{3,1}$. These computations are achieved using Vassiliev-Gorinov's method, as presented in I.2.1.

Acknowledgments. The cooperation between the two authors started with a visit of the second author at KTH in Fall 2004. The second author would like to thank Carel Faber for the invitation and KTH for the hospitality. Moreover, we would like to thank Theo van den Bogaart and Bas Edixhoven for allowing us to read early versions of their article [BE05].

Notation. Let $\mathcal{M}_{g,n}$ be the moduli space of irreducible, non-singular, projective curves of genus g with n distinct marked points. Furthermore, let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of Deligne-Mumford stable curves of genus g with n distinct marked points. Both these moduli spaces are smooth Deligne-Mumford stacks defined over \mathbf{Z} . The symmetric group \mathfrak{S}_n of permutations of n elements acts on both $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ by permuting the n marked points on the curves.

For every pair of non-negative integers g, n such that $2g + n - 2 > 0$, we let

$$\mathcal{D}_{g,n} := \{(\hat{g}, \hat{n}) : 0 \leq \hat{g} \leq g, \max\{0, 3 - 2\hat{g}\} \leq \hat{n} \leq 2(g - \hat{g}) + n\}.$$

To state the results in this chapter, we need to consider the Grothendieck groups of several categories. We will denote by $\mathbf{MHS}_{\mathbf{Q}}$ the category of rational mixed Hodge structures, and by \mathbf{Gal} the category of $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ -representations. For any abelian category \mathbf{C} , we will denote by $K_0(\mathbf{C})$ the Grothendieck group of \mathbf{C} . Note that $K_0(\mathbf{MHS}_{\mathbf{Q}})$ coincides with $K_0(\mathbf{HS}_{\mathbf{Q}})$, the Grothendieck group of (pure) Hodge structures.

We denote by $\mathbf{Q}(-k)$ the one-dimensional rational Tate Hodge structure of weight $2k$. The class of $\mathbf{Q}(-1)$ in $K_0(\mathbf{HS}_{\mathbf{Q}})$ is denoted by \mathbf{L} . For any rational mixed Hodge structure H , we denote by $H(-k)$ the Tate twist $H \otimes \mathbf{Q}(-k)$. Similarly, we denote by $\mathbf{Q}_l(k)$ the k -th Tate twist of the trivial $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ -representation.

2. The stratification of $\overline{\mathcal{M}}_{G,N}$

The strategy followed in this chapter is to compute the rational cohomology of the moduli space $\overline{\mathcal{M}}_4$ of stable curves (over the complex numbers), from what we know on the cohomology of some moduli spaces of smooth n -pointed curves of genus $g \leq 4$.

The stratification based on the dual graph gives a way to divide every moduli space $\overline{\mathcal{M}}_{G,N}$ into strata which are explicitly related to the moduli spaces $\mathcal{M}_{g,n}$ with $(g, n) \in \mathcal{D}_{G,N}$. This gives the intuition that all information on $\overline{\mathcal{M}}_{G,N}$ can be deduced from the appropriate information on moduli spaces of smooth pointed curves.

The aim of this section is to present in Theorem 2.6 a formula by Getzler and Kapranov ([GK98]) which expresses the relationship between the \mathfrak{S}_N -equivariant Euler characteristic of $\overline{\mathcal{M}}_{G,N}$ and the \mathfrak{S}_n -equivariant Euler characteristics of the $\mathcal{M}_{g,n}$'s with $(g, n) \in \mathcal{D}_{G,N}$.

Then we will apply this theorem to $\overline{\mathcal{M}}_4$, in the case of Hodge Euler characteristics, to give a proof of Theorem 1.1.

First, we explain what the dual graph of a stable curve is. Let (C, p_1, \dots, p_N) be a stable curve of genus G with N marked points. Its dual graph is the labelled graph Γ such that:

- Every vertex v_i of Γ corresponds to an irreducible component C_i of C , and its label g_i is the geometric genus of C_i .
- The edges of Γ correspond to the nodes of C (e.g., two distinct vertices are joined by an edge if and only if the corresponding components intersect).
- There are N half-edges, labelled from 1 to N , corresponding to the N marked points.

For instance, a non-singular N -pointed curve of genus G corresponds to a tree with one vertex of genus G and N half-edges. Finally, an automorphism of a labelled graph is defined to be an automorphism of the underlying non-labelled graph that preserves the labelling of the vertices and that fixes each half-edge.

For any connected labelled graph Γ with N half-edges and such that

$$\sum_i (g_i - 1) + \#\{\text{Edges}\} + 1 = G,$$

we can consider the moduli space $\mathcal{M}(\Gamma)$ of curves with dual graph Γ . Each $\mathcal{M}(\Gamma)$ is locally closed in $\overline{\mathcal{M}}_{G,N}$. Note that the spaces $\mathcal{M}(\Gamma)$ can be expressed in terms of moduli spaces of non-singular curves. Indeed, $\mathcal{M}(\Gamma) \cong \prod_i \mathcal{M}_{g_i, n_i} / \text{Aut}(\Gamma)$, where g_i denotes, as usual, the label of the i -th vertex v_i of Γ , and n_i is the sum of the number of edges and half-edges incident to the vertex v_i .

Since $\overline{\mathcal{M}}_{G,N}$ is complete and satisfies Poincaré duality, the knowledge of the cohomology of $\overline{\mathcal{M}}_{G,N}$ as a graded vector space with mixed Hodge structure is equivalent to that of the *Hodge Euler characteristic* of $\overline{\mathcal{M}}_{G,N}$. That is, the Euler characteristic of $H_c^\bullet(\overline{\mathcal{M}}_{G,N}; \mathbf{Q})$ in the Grothendieck group of rational mixed Hodge structures,

$$\mathbf{e}(\overline{\mathcal{M}}_{G,N}) := \sum_{j \in \mathbf{Z}} (-1)^j [H_c^j(\overline{\mathcal{M}}_{G,N}; \mathbf{Q})] \in K_0(\mathbf{MHS}_{\mathbf{Q}}).$$

This Euler characteristic does not contain all the information we need. Namely, it does not give information on the action of the symmetric group \mathfrak{S}_n , which acts on $\mathcal{M}_{g,n}$ (and $\overline{\mathcal{M}}_{g,n}$) by permuting the marked points. In order to express the induced action on cohomology, we will make use of symmetric functions in our notation, given the well-known correspondence between them and the characters of the symmetric group.

We denote by Λ_n the ring of symmetric functions in n variables. The ring Λ_n is generated by the complete symmetric functions ${}_{(n)}h_1, \dots, {}_{(n)}h_n$, where

$${}_{(n)}h_j := \sum_{1 \leq i_1 \leq \dots \leq i_j \leq n} x_{i_1} x_{i_2} \dots x_{i_j}.$$

There is a natural projection $\Lambda_{n+1} \rightarrow \Lambda_n$ that maps ${}_{(n+1)}h_j$ to ${}_{(n)}h_j$ if $j \leq n$ and ${}_{(n+1)}h_{n+1}$ to 0. This allows us to define

$$\Lambda := \varprojlim \Lambda_n.$$

Note that the elements of Λ are infinite series in the variables $\{x_j\}_{j \geq 1}$, and can be written as (possibly infinite) sums of *complete symmetric functions* $h_j := \sum_{1 \leq i_1 \leq \dots \leq i_j} x_{i_1} \dots x_{i_j}$. For every n , the ring Λ_n can be identified with the subring $\mathbf{Z}[h_1, \dots, h_n]$ of Λ .

The correspondence between elements of Λ_n and representations of \mathfrak{S}_n (for some fixed n) is given by the Schur polynomials. The Schur polynomials of degree n are in one to one correspondence with partitions of n and we denote by s_λ the Schur polynomial corresponding to the partition $\lambda = (\lambda_1, \dots, \lambda_s)$ (with $\lambda_1 \geq \dots \geq \lambda_s \geq 1$). As is well known, the set of Schur polynomials freely generates Λ as an abelian group.

We will also consider another set of symmetric functions, namely the *power sums* $p_n := \sum_i x_i^n$, which constitute a \mathbf{Q} -basis of $\Lambda \otimes \mathbf{Q}$.

In order to compute $\mathbf{e}(\overline{\mathcal{M}}_4)$, we use the formula by Getzler and Kapranov given in Theorem 2.6. Its formulation and proof are based on the theory of *modular operads*. Since there is an excellent exposition of it in [GK98], we will not explain this theory here, but refer to [GK98] for all definitions and properties we use. Following [Get99], we will formulate the theory for any symmetric monoidal category \mathbf{C} that is additive over a field of characteristic 0 and has finite colimits. Recall (see for instance [JS93]) that a monoidal category is a category \mathbf{C} with a functor

$$\begin{aligned} \mathbf{C} \times \mathbf{C} &\longrightarrow \mathbf{C} \\ (X, Y) &\longmapsto X \otimes Y \end{aligned}$$

such that there is a system of isomorphisms $(X \otimes Y) \otimes Z \xrightarrow[\sim]{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$ for every objects X, Y, Z of \mathbf{C} , satisfying some compatibility conditions which ensure that \otimes is associative (up to a composition of isomorphisms of the form α) also when more than three object are involved. A symmetric monoidal category possesses also a system of isomorphisms $X \otimes Y \xrightarrow{S_{X,Y}} Y \otimes X$ for every objects X, Y of \mathbf{C} . The isomorphisms $S_{X,Y}$ have to be compatible with the associativity conditions given by the isomorphisms α , and have to satisfy $S_{Y,X} = S_{X,Y}^{-1}$. We always assume that a monoidal category has a unit object $\mathbf{1}$, that is, an object $\mathbf{1}$ such that there are functorial isomorphisms $X \xrightarrow{\sim} X \otimes \mathbf{1}$ for every object X .

A category \mathbf{C} is additive over a field k if the sets of morphisms $\mathbf{C}(X, Y)$ are k -vector spaces for all objects, the composition of morphisms always is a linear map and every finite set of objects of \mathbf{C} has a direct sum.

To any such category \mathbf{C} we associate the category $\mathbf{C}^{\mathfrak{S}_n}$ whose objects are objects of \mathbf{C} equipped with an action of the symmetric group \mathfrak{S}_n by morphisms in \mathbf{C} .

THEOREM 2.1 ([Get96, Theorem 4.8]). *There is a canonical isomorphism*

$$K_0(\mathbf{C}^{\mathfrak{S}_n}) \cong K_0(\mathbf{C}) \otimes \Lambda_n.$$

DEFINITION 2.2 ([GK98]). A stable \mathfrak{S} -module \mathcal{V} in the category \mathbf{C} is a collection, for all $g, n \geq 0$, of chain complexes $\{\mathcal{V}((g, n))_i\}$ of objects of $\mathbf{C}^{\mathfrak{S}_n}$ such that $\mathcal{V}((g, n)) = 0$ if $2g + n - 2 \leq 0$.

DEFINITION 2.3. Let $R := \{R_i\}$ be a finite chain complex of objects of $\mathbf{C}^{\mathfrak{S}_n}$ for some $n \geq 0$. The characteristic of R is defined as

$$\text{ch}_n(R) := \sum_i (-1)^i [R_i] \in K_0(\mathbf{C}^{\mathfrak{S}_n}) \cong K_0(\mathbf{C}) \otimes \Lambda_n.$$

When \mathcal{X} is an algebraic stack over \mathbf{C} , with an action of the symmetric group \mathfrak{S}_n , the rational cohomology with compact support of \mathcal{X} has a natural structure of a chain complex $\{C_i\}$ of objects of $\mathbf{MHS}_{\mathbf{Q}}^{\mathfrak{S}_n}$ by setting $C_i := H_c^i(\mathcal{X}; \mathbf{Q})$ and defining all differentials to be zero. We define the \mathfrak{S}_n -equivariant Hodge Euler characteristic \mathcal{X} as

$$\mathbf{e}^{\mathfrak{S}_n}(\mathcal{X}) := \text{ch}_n(H_c^\bullet(\mathcal{X}; \mathbf{Q})) \in K_0(\mathbf{MHS}_{\mathbf{Q}}) \otimes \Lambda_n.$$

DEFINITION 2.4. Let \mathcal{V} be a stable \mathfrak{S} -module. Then the *characteristic* of \mathcal{V} is defined as

$$\mathbf{Ch}(\mathcal{V}) := \sum_{2g+n-2>0} \hbar^{g-1} \text{ch}_n(\mathcal{V}((g, n))) \in K_0(\mathbf{C}) \otimes \Lambda((\hbar)),$$

where $\Lambda((\hbar))$ is the ring of Laurent series with coefficients in Λ .

REMARK 2. On Λ there is a unique associative operation, called *plethysm*, satisfying the conditions

- (1) $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$;
- (2) $(f_1 f_2) \circ g = (f_1 \circ g)(f_2 \circ g)$;
- (3) If $f = f(p_1, p_2, \dots)$ then $p_n \circ f = f(p_n, p_{2n}, \dots)$.

Following [Get96], the plethysm \circ can be extended to a map $\Lambda \times (K_0(\mathbf{C}) \otimes \Lambda) \rightarrow K_0(\mathbf{C}) \otimes \Lambda$ by setting $h_n \circ [R]$ for every $n \geq 0$, $R \in \mathbf{C}$, to be the class in $K_0(\mathbf{C})$ of the n -th symmetric product of R . In particular, when \mathbf{C} is the category of rational mixed Hodge structures, we have $p_n \circ \mathbf{L} = \mathbf{L}^n$, where \mathbf{L} is the class of the Tate Hodge structure of weight 2. We also extend the plethysm to $\Lambda((\hbar))$ by letting $p_n \circ \hbar = \hbar^n$.

DEFINITION 2.5. (1) Suppose $f = \sum_{\alpha \in \mathbf{Z}} f_\alpha [R_\alpha] \hbar^\alpha \in K_0(\mathbf{C}) \otimes \Lambda((\hbar))$, where $f_\alpha \in \Lambda$ and $[R_\alpha] \in K_0(\mathbf{C})$, is such that for all $\alpha \in \mathbf{Z}$ all monomials occurring in f_α have degree at least $1 - 2\alpha$.

Then the plethystic exponential of f is defined by the expression:

$$\text{Exp}(f) = \sum_{n=0}^{\infty} h_n \circ f.$$

(2) We denote the inverse of the plethystic exponential by:

$$\text{Log}(f) = \sum_{n=1}^{\infty} \frac{\mu(n) \log(p_n) \circ f}{n}.$$

(3) On $\Lambda((\hbar))$, an analogue of the Laplacian is given by

$$\Delta = \sum_{n=1}^{\infty} \hbar^n \left(\frac{n}{2} \frac{\partial^2}{\partial p_n^2} + \frac{\partial}{\partial p_{2n}} \right).$$

Recall from [GK98] that it is possible to associate with every stable \mathfrak{S} -module the free modular operad $\mathbb{M}\mathcal{V}$ generated by \mathcal{V} . The following theorem gives the relationship between the characteristics of the stable \mathfrak{S} -modules \mathcal{V} and $\mathbb{M}\mathcal{V}$.

THEOREM 2.6 ([GK98, Thm 8.13]). *Let \mathcal{V} be an \mathfrak{S} -module in the category \mathbf{C} . Then*

$$\mathbf{Ch}(\mathbb{M}\mathcal{V}) = \text{Log}(\exp(\Delta) \text{Exp}(\mathbf{Ch}(\mathcal{V}))).$$

PROOF OF THEOREM 1.1. Let us consider $\mathcal{V}((g, n)) := H_c^\bullet(\mathcal{M}_{g,n}; \mathbf{Q})$. From the results in [GK98] it follows that \mathcal{V} is a modular operad in the category of stable \mathfrak{S} -modules with mixed Hodge structures. The free modular operad associated with \mathcal{V} is $H_c^\bullet(\overline{\mathcal{M}}_{g,n}; \mathbf{Q})$. In this situation, Theorem 2.6 can be used to compute the generating function

$$\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V}) = \sum_{2g+n-2>0} \hbar^{g-1} \mathbf{e}^{\mathfrak{S}_n}(H_c^\bullet(\overline{\mathcal{M}}_{g,n}; \mathbf{Q})),$$

from the generating function $\mathbb{C}\mathrm{h}(\mathcal{V})$.

The Hodge Euler characteristic of $\overline{\mathcal{M}}_4$ is exactly the part of the coefficient of \hbar^3 in $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V})$ with degree 0 in Λ . It is not necessary to compute the whole $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V})$ in order to obtain this. Since the Euler characteristic $2 - 2g - n$ of the dual graph behaves additively throughout the whole computation, only the $\mathcal{V}((g, n))$'s with Euler number not less than $-6 = 2 - 2 \cdot 4$ give a contribution. Therefore, all we need to know is the \mathfrak{S}_n -equivariant Euler Hodge characteristic of $\mathcal{M}_{g,n}$ for all (g, n) in $\mathcal{D}_{4,0}$.

For $g = 0$, $g = 1$, $g = 2$ and $n \leq 3$ and $g = 3$, $n \leq 1$, these results are already known (see [Get95], [Get99], [Get98b], [Loo93], [GL]). The equivariant Hodge Euler characteristic of $\mathcal{M}_{2,4}$ and $\mathcal{M}_{3,2}$ are given in Theorems 1.2 and 1.3 respectively. These theorems follow from the results in Sections 3 and 4 and rely on the equivariant count of points made by the first author in [Berb] and [Bera].

The rational cohomology of \mathcal{M}_4 has been given in Theorem I.1.4 (see also [Tom05]). In particular, the Hodge Euler characteristic of \mathcal{M}_4 was found to be $\mathbf{e}(\mathcal{M}_4) = \mathbf{L}^9 + \mathbf{L}^8 + \mathbf{L}^7 - \mathbf{L}^6$.

Note that for all $(g, n) \in \mathcal{D}_{4,0}$, the \mathfrak{S}_n -equivariant Hodge Euler characteristic of $\mathcal{M}_{g,n}$ lies in $\mathbf{Q}[\mathbf{L}] \otimes \Lambda$. Hence, our computation only involves polynomials in \mathbf{L} and certain symmetric functions.

To obtain the result, we used a computer program (by Del Baño), based on Theorem 2.6, which computes the part of $\mathbb{C}\mathrm{h}(\mathbb{M}\mathcal{V})$ of degrees $G - 1$ in \hbar and N in Λ , whenever $\mathrm{ch}_n(\mathcal{V}((g, n)))$ is given and is a polynomial in \mathbf{L} for every (g, n) in $\mathcal{D}_{G,N}$. This computer program makes extensive use of Stembridge's package **SF** ([Ste]) for computations with symmetric functions. \square

3. From counting to cohomology

For every integer q which is the power of a prime number, denote by \mathbf{F}_q the finite field with q elements and by F_q its Frobenius map. Fix two non-negative integers G and N such that $2G + N - 2 > 0$, and assume that the functions

$$f_{g,n,\sigma}(q) = |(\mathcal{M}_{g,n})_{\mathbf{F}_q}^{F_q \cdot \sigma}|$$

are polynomials in q for every (g, n) in $\mathcal{D}_{G,N}$ and for every $\sigma \in \mathfrak{S}_n$.

Then we will see that we can determine the étale cohomology of $(\overline{\mathcal{M}}_{G,N})_{\overline{\mathbf{Q}}}$ with its structure as $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ - and \mathfrak{S}_N -representation. Moreover, from this we can determine the action of the symmetric group and the Hodge structure on the Betti cohomology of the complex stack $(\overline{\mathcal{M}}_{G,N})_{\mathbf{C}}$.

3.1. Étale cohomology. In this section we will apply the theory of Getzler and Kapranov, presented in Section 2, to the stacks $\mathcal{M}_{g,n}$ in the setting of l -adic étale cohomology.

Let $k = \mathbf{F}_q$ denote the finite field with q elements. All varieties and stacks mentioned in the following will be assumed to be defined over \bar{k} if nothing else is specified.

We denote by $H_{\text{ét}}^{\bullet}(-, \mathbf{Q}_l)$ compactly supported l -adic étale cohomology. Observe that the c , standing for compactly supported, is omitted in this notation. Unless otherwise stated we assume that l does not divide $|k| = q$. Let $\text{Gal}_{\mathbf{Q}_l}$ be the category of \mathbf{Q}_l -vector spaces equipped with the l -adic topology that have a continuous action of the absolute Galois group $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$. This is a symmetric monoidal additive category over \mathbf{Q}_l , with finite colimits. Hence, Theorem 2.6 holds in this category.

DEFINITION 3.1. Suppose we have a space \mathcal{X} with an action of \mathfrak{S}_n commuting with the action of the absolute Galois group. Denote by $H_{\text{ét},\lambda}^j(\mathcal{X}, \mathbf{Q}_l)$ the Galois subrepresentation of $H_{\text{ét}}^j(\mathcal{X}, \mathbf{Q}_l)$ which is the sum of all copies of the irreducible representation of \mathfrak{S}_n indexed by the partition λ that appear in $H_{\text{ét}}^j(\mathcal{X}, \mathbf{Q}_l)$. Then for every partition λ of n we define $\mathbf{e}_{\text{ét},\lambda}(\mathcal{X})$ to be the Euler characteristic of $H_{\text{ét},\lambda}^j(\mathcal{X}, \mathbf{Q}_l)$ in the Grothendieck group of $\text{Gal}_{\mathbf{Q}_l}$.

Define a stable \mathfrak{S} -module \mathcal{V} in $\text{Gal}_{\mathbf{Q}_l}$ by

$$\mathcal{V}((g, n)) := H_{\text{ét}}^{\bullet}(\mathcal{M}_{g,n}, \mathbf{Q}_l).$$

Taking the characteristic of $\mathcal{V}((g, n))$ in the Grothendieck group of equivariant Galois representations gives

$$\text{ch}_n(\mathcal{V}((g, n))) = \sum_{\lambda \vdash n} (\chi_{\lambda}(id))^{-1} \mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n}) s_{\lambda} \in K_0(\text{Gal}_{\mathbf{Q}_l}) \otimes \Lambda_n.$$

The properties of Euler characteristics now ensure that

$$\text{ch}_N(\mathbb{M}\mathcal{V}((G, N))) = \sum_{\lambda \vdash N} (\chi_{\lambda}(id))^{-1} \mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{G,N}) s_{\lambda} \in K_0(\text{Gal}_{\mathbf{Q}_l}) \otimes \Lambda_N.$$

Applying 2.6 to \mathcal{V} , we can express $\mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{G,N})$ in terms of $\mathbf{e}_{\text{ét},\mu}(\mathcal{M}_{g,n})$ for partitions μ of n and for $(g, n) \in \mathcal{D}_{G,N}$. Note that the plethysm of $p_j \in \Lambda$ with a Galois representation of the form $\mathbf{Q}_l(i)$ for some $i \in \mathbf{Z}$ equals $[\mathbf{Q}_l(i \cdot j)] \in K_0(\text{Gal}_{\mathbf{Q}_l})$. Hence, if $\mathbf{e}_{\text{ét},\mu}(\mathcal{M}_{g,n})$ is a sum of Tate twists of the trivial Galois representation for all partitions μ of n and for all $(g, n) \in \mathcal{D}_{G,N}$ then this also holds for $\mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{G,N})$.

3.2. The Lefschetz trace formula. Let F_q be the geometric Frobenius map belonging to the finite field k .

Let \mathcal{X}_k be a smooth Deligne-Mumford stack of finite type over k that has an action of \mathfrak{S}_n and define $\mathcal{X} := \mathcal{X}_k \otimes_k \bar{k}$.

By an *equivariant count* of the number of points defined over k of \mathcal{X} we mean a count, for each $\sigma \in \mathfrak{S}_n$, of the number of fixed points of the automorphism $\sigma \cdot F_q$ on \mathcal{X} . Note in particular that the number of fixed points is the same for all σ with the same cycle type.

Since the fixed points of $\sigma \cdot F_q$ all have multiplicity one, the Lefschetz trace formula, generalized in [Beh93] to Deligne-Mumford stacks, gives the equality

$$|\mathcal{X}^{F_q \cdot \sigma}| = \text{Tr}(F_q \cdot \sigma, \mathbf{e}_{\text{ét}}(\mathcal{X})).$$

Let χ_{λ} be the character of the irreducible representation of \mathfrak{S}_n indexed by λ . Then the endomorphism

$$\pi_{\lambda} := \frac{1}{n!} \chi_{\lambda}(id) \sum_{\sigma \in \mathfrak{S}_n} \chi_{\lambda}(\sigma) \sigma$$

is the projection of $H_{\text{ét}}^i(\mathcal{X}, \mathbf{Q}_l)$ onto $H_{\text{ét},\lambda}^i(\mathcal{X}, \mathbf{Q}_l)$ (see for instance [FH91, 2.31]). Therefore,

$$(1) \quad \frac{1}{n!} \chi_\lambda(id) \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) |\mathcal{X}^{\sigma \cdot F_q}| = \text{Tr}(F_q \cdot \pi_\lambda, \mathbf{e}_{\text{ét}}(\mathcal{X})) = \text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\mathcal{X}))$$

which gives a formula expressing $\text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\mathcal{X}))$ as a function of the $|\mathcal{X}^{F_q \cdot \sigma}|$ with $\sigma \in \mathfrak{S}_n$.

3.3. The Galois action. In this section we will present an equivariant version of Theorem 2.1 in [BE05].

THEOREM 3.2. *Let \mathcal{X} be a Deligne-Mumford stack defined over \mathbf{Z} which is proper, smooth, of pure relative dimension d and that has an action of \mathfrak{S}_n .*

For every partition λ of n , denote by χ_λ the character of the irreducible representation of \mathfrak{S}_n indexed by λ . Furthermore let S be a set of primes of Dirichlet density 1.

$P_\lambda(t) \in \mathbf{Q}[t]$ such that Assume that for a partition λ of n there exists a polynomial $P_\lambda(t) \in \mathbf{Q}[t]$ such that

$$(2) \quad \frac{1}{n!} \chi_\lambda(id) \sum_{\sigma \in \mathfrak{S}_n} \chi_\lambda(\sigma) |\mathcal{X}_{\bar{\mathbf{F}}_p}^{\sigma \cdot F_{p^r}}| = P_\lambda(p^r)$$

for all $r \in \mathbf{Z}_{\geq 1}$ and $p \in S$.

Then $P_\lambda(t)$ has degree d and non-negative integer coefficients. Furthermore, if we let b_j be the coefficient of q^j in P_λ , then for all primes l and all $i \geq 0$ there is an isomorphism of $\text{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ -representations

$$H_{\text{ét},\lambda}^i(\mathcal{X}_{\bar{\mathbf{Q}}}, \mathbf{Q}_l) \simeq \begin{cases} 0 & \text{if } i \text{ is odd} \\ \mathbf{Q}_l(-i/2)^{b_{i/2}} & \text{if } i \text{ is even} \end{cases}$$

PROOF. The proof follows very closely that of Theo van den Bogaart and Bas Edixhoven, thus we will only include the vital steps.

Assume Condition (2) holds for a partition λ . Using equation (1) we get, for some $b_i \in \mathbf{Q}$,

$$(3) \quad \text{Tr}(F_{p^r}, \mathbf{e}_{\text{ét},\lambda}(\mathcal{X}_{\bar{\mathbf{F}}_p})) = \sum_i b_i p^{ri}$$

for all $r \in \mathbf{Z}_{\geq 1}$ and $p \in S$. Note here that $\text{Tr}(F_{p^r}, [\mathbf{Q}_l(-i)]) = p^{ri}$.

In the proof of Theorem 2.1 of [BE05] it is shown that (if $p \neq l$) the Galois representation $H_{\text{ét}}^i(\mathcal{X}_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$ is unramified and all eigenvalues of the Frobenius map F_{p^r} have complex absolute value $p^{ri/2}$. Since $H_{\text{ét},\lambda}^i(\mathcal{X}_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$ is a Galois subrepresentation of $H_{\text{ét}}^i(\mathcal{X}_{\bar{\mathbf{F}}_p}, \mathbf{Q}_l)$, it will inherit these properties. Thus we can use the arguments of [BE05] to conclude from equation (3) that the semisimplification of $H_{\text{ét},\lambda}^i(\mathcal{X}_{\bar{\mathbf{Q}}}, \mathbf{Q}_l)$ has dimension zero if i is odd and is isomorphic to $\mathbf{Q}_l(-i/2)^{P_{i/2}}$ if i is even. Then the claim follows, analogously as in [BE05], from the fact that $H_{\text{ét},\lambda}^i(\mathcal{X}_{\bar{\mathbf{Q}}}, \mathbf{Q}_l)$ is potentially semistable. This is, again, a consequence of the fact that it is a subrepresentation of $H_{\text{ét}}^i(\mathcal{X}_{\bar{\mathbf{Q}}}, \mathbf{Q}_l)$, which is shown to be potentially semistable in [BE05]. \square

Note that the moduli spaces $\overline{\mathcal{M}}_{g,n}$ are indeed proper and smooth Deligne-Mumford stacks defined over \mathbf{Z} , of pure relative dimension and with an action of \mathfrak{S}_n (see [DM69] and [Knu83, Theorem 2.7]).

3.4. The mixed Hodge structures. Using comparison theorems we get a theorem corresponding to 3.2, but for Betti cohomology. This new theorem is an equivariant version of Corollary 5.3 of [BE05].

Let \mathcal{X} be an algebraic stack (or simply a scheme) defined over \mathbf{C} together with an action of \mathfrak{S}_n . Analogously to the case of l -adic cohomology in Section 3.1, we denote by $H_{c,\lambda}^i(\mathcal{X}; \mathbf{Q})$ the sum of all copies of the irreducible representation of \mathfrak{S}_n indexed by the partition λ that appear in the Betti cohomology groups with compact support $H_c^i(\mathcal{X}; \mathbf{Q})$. We denote by $\mathbf{e}_\lambda(\mathcal{X})$ the Euler characteristic of $H_{c,\lambda}^\bullet(\mathcal{X}; \mathbf{Q})$ in $K_0(\mathbf{MHS}_{\mathbf{Q}})$.

Since all the isomorphisms used in the proof of Corollary 5.1 of [BE05] respect the action of the symmetric group we have a proof of the following theorem.

THEOREM 3.3. *In the situation of Theorem 3.2 suppose furthermore that the coarse moduli space of the stack $\mathcal{X}_{\mathbf{Q}}$ is the quotient of a smooth projective \mathbf{Q} -scheme by a finite group.*

Then for all partitions λ of n and for all $i \geq 0$, there is an isomorphism of pure \mathbf{Q} -Hodge structures

$$H_{c,\lambda}^i(\mathcal{X}(\mathbf{C}), \mathbf{Q}) \simeq \begin{cases} 0 & \text{if } i \text{ is odd} \\ \mathbf{Q}(-i/2)^{b_{i/2}} & \text{if } i \text{ is even} \end{cases}$$

where the left hand side is equipped with the canonical Hodge structure of [Del74].

Note that the moduli spaces $\overline{\mathcal{M}}_{g,n}$ fulfill the additional condition of Theorem 3.3 (see [BP00]).

3.5. The main theorem. The following theorem summarizes what cohomological information we can get from polynomial counts of the points of the spaces $\mathcal{M}_{g,n}$.

THEOREM 3.4. *Assume that, for all $\mathcal{M}_{g,n}$ with $(g, n) \in \mathcal{D}_{G,N}$ and for all partitions λ of n , equation (2) of Theorem 3.2 is fulfilled for some set S and polynomial $P_{\lambda,g,n}(t)$.*

Then the following holds for all $(g, n) \in \mathcal{D}_{G,N}$ and all partitions λ of n .

For $\overline{\mathcal{M}}_{g,n}$ equation (2) of Theorem 3.2 is fulfilled for some set S and polynomials $Q_\lambda(t)$. Hence, both Theorem 3.2 and Theorem 3.3 hold for $\overline{\mathcal{M}}_{g,n}$.

Moreover,

- (i) $\mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}) = P_{\lambda,g,n}([\mathbf{Q}_l(-1)]) \in K_0(\mathbf{Gal}_{\mathbf{Q}_l})$
- (ii) $\mathbf{e}_\lambda(\mathcal{M}_{g,n}(\mathbf{C})) = P_{\lambda,g,n}(\mathbf{L}) \in K_0(\mathbf{MHS}_{\mathbf{Q}})$.

PROOF. This is proved by induction on (g, n) . The starting case is $\mathcal{M}_{0,3}$ for which the result is clear since we can apply Theorems 3.2 and 3.3 to $\mathcal{M}_{0,3} = \overline{\mathcal{M}}_{0,3}$.

Say that the theorem holds for $\mathcal{D}_{g,n} \setminus \{(g, n)\}$. In the stratification of $\overline{\mathcal{M}}_{g,n}$, $\mathcal{M}_{g,n}$ is the open part and it does not contribute to the cohomology of the boundary $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$. From the additivity of Euler characteristics we get an equality

$$(4) \quad \mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}) = \mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}) + \mathbf{e}_{\text{ét},\lambda}(\partial\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}).$$

Thus, using Theorem 2.6 and Section 3.1 we can express $\mathbf{e}_{\text{ét},\lambda}(\partial\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$ in terms of $\mathbf{e}_{\text{ét},\mu}(\mathcal{M}_{\hat{g},\hat{n}} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$ for partitions μ of n and $(\hat{g}, \hat{n}) \in \mathcal{D}_{g,n} \setminus \{(g, n)\}$. Since, by induction, $\mathbf{e}_{\text{ét},\mu}(\mathcal{M}_{\hat{g},\hat{n}} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$ is known for these indices and is a sum of elements $[\mathbf{Q}_l(-j)]$, we can determine $\mathbf{e}_{\text{ét},\lambda}(\partial\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$ and it will also be a sum of elements $[\mathbf{Q}_l(-j)]$.

Equation (4) takes place in the category $K_0(\text{Gal}_{\mathbf{Q}_l})$ and thus we can take the trace of Frobenius on both sides. By the assumption together with Section 3.2, for almost all finite fields k we can compute $\text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{k}))$, which is equal to $\text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}))$, see [BE05]. Just as $\text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\partial \mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}))$, it will be polynomial in $|k| = q$.

Hence we can use equation (4) to compute $\text{Tr}(F_q, \mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}}))$ and we know that the answer will be polynomial in q . Theorems 3.2 and 3.3 are therefore applicable for $\overline{\mathcal{M}}_{g,n}$.

Thus we know both $\mathbf{e}_{\text{ét},\lambda}(\overline{\mathcal{M}}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$ and $\mathbf{e}_{\text{ét},\lambda}(\partial \mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{\mathbf{Q}})$, which together with equation (4) gives (i). But equation (4) also holds in the category of mixed Hodge structures and therefore we can use the same argument to conclude (ii). Here the induction ends. \square

REMARK 3. In [KL02], Kisin and Lehrer made an equivariant count of the number of points of $\mathcal{M}_{0,N}$ for $N \geq 3$ and these numbers fulfill the hypotheses of Theorem 3.4.

The first author has made an equivariant count of the number of points of $\mathcal{M}_{1,N}$ for $N \leq 6$, which also were found to be polynomial. It was achieved by applying a method similar to that described in Section 4.3, and based on the embedding of a genus one curve with a marked point P , via the divisor $3P$, in \mathbf{P}^2 . This computation agreed with Getzler's results on the Hodge Euler characteristic of $\mathcal{M}_{1,N}$ for $N \geq 1$ (see [Get99]).

4. Counting points over finite fields

For all finite fields k , with the possible exception of a finite number of characteristics, we wish to make an \mathfrak{S}_n -equivariant count (for the definition see Section 3.2) of the number of points defined over k of $\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{k}$ for all $(g, n) \in \mathcal{D}_{4,0} \setminus \{(4, 0)\}$.

Following Section 3.2, we will present results of equivariant counts of the number of points in the form of traces of Frobenius on the \mathfrak{S}_n -equivariant Euler characteristics in the category of Galois representations

$$\mathbf{e}_{\text{ét}}^{\mathfrak{S}_n}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{k}) := \sum_{\lambda \vdash n} \chi_{\lambda}(id)^{-1} \mathbf{e}_{\text{ét},\lambda}(\mathcal{M}_{g,n} \otimes_{\mathbf{Z}} \bar{k}) s_{\lambda} \in K_0(\text{Gal}) \otimes \Lambda_n.$$

4.1. Preparation. In the following we will always assume that $\mathcal{M}_{g,n}$ is defined over \bar{k} . Recall that F_q denotes the geometric Frobenius belonging to the finite field k .

DEFINITION 4.1. An n -tuple of distinct points (p_1, \dots, p_n) is called a *conjugate n -tuple* if $F(p_i) = p_{i+1}$ for $1 \leq i \leq n-1$ and $F(p_n) = p_1$. Moreover, if $\lambda = (\lambda_1, \dots, \lambda_{\nu})$ then a $|\lambda|$ -tuple of distinct points $(p_1, \dots, p_{|\lambda|})$ is called a λ -tuple if $(p_{\sum_{i=1}^{j-1} \lambda_i + 1}, p_{\sum_{i=1}^{j-1} \lambda_i + 2}, \dots, p_{\sum_{i=1}^j \lambda_i})$ is a conjugate λ_j -tuple for every $1 \leq j \leq \nu$.

From the definition of the \mathfrak{S}_n -action on $\mathcal{M}_{g,n}$ we see that if $\sigma \in \mathfrak{S}_n$ has cycle type λ then $|\mathcal{M}_{g,n}^{\sigma, F_q}|$ is equal to the number of curves $C \in \mathcal{M}_g(k)$ together with a λ -tuple of points (p_1, \dots, p_n) lying on the curve C .

The Lefschetz fixed point formula (see for instance [Mil80, Theorem V.2.5] shows that for all $C \in \mathcal{M}_g(k)$ and $m \geq 1$

$$|C(k_m)| = \sum_i (-1)^i \text{Tr}(F_q^m, H_{\text{ét}}^i(C_{\bar{k}}, \mathbf{Q}_l)) = 1 + q^m - a_m(C)$$

where $a_m(C) := \text{Tr}(F_q^m, H_{\text{ét}}^1(C_{\bar{k}}, \mathbf{Q}_l))$.

Note that the number of conjugate m -tuples on a curve C equals

$$\sum_{d|m} \mu(m/d) |C(k_m)|$$

where μ is the Möbius function. Hence, we see that

$$|\mathcal{M}_{g,n}^{\sigma, F_q}| = \sum_{C \in \mathcal{M}_g(k)} Q_\lambda(q, a_1(C), \dots, a_n(C))$$

where $Q_\lambda(x_0, x_1, \dots, x_n)$ is a polynomial with coefficients in \mathbf{Z} . If we assign degree i to the indeterminate x_i , then $Q_\lambda(x_0, x_1, \dots, x_n)$ has a single monomial of the highest degree $|\lambda|$, namely $(-1)^{|\lambda|} x_{\lambda_1} \cdots x_{\lambda_\nu}$.

Thus, if $|\mathcal{M}_{g,m}^{\tau, F_q}|$ is known for all $m < n$ and for all $\tau \in \mathfrak{S}_m$, the only part missing to compute $|\mathcal{M}_{g,n}^{\sigma, F_q}|$ is

$$\sum_{C \in \mathcal{M}_g(k)} a_{\lambda_1}(C) \cdots a_{\lambda_\nu}(C).$$

We can simplify these computations in the following way. Instead of summing over all elements in $\mathcal{M}_g(k)$, that is curves of genus g defined over k up to \bar{k} -isomorphism, one can sum over all curves defined over k up to k -isomorphism but with weight the reciprocal of the number of k -automorphisms of the curve (see [GV92]).

4.2. Hyperelliptic curves. In this section we assume that k is a finite field of odd characteristic.

Let $\mathcal{H}_{g,n}$ denote the subset of $\mathcal{M}_{g,n}$ of hyperelliptic curves. We wish to make an equivariant count of the number of points defined over k of the space $\mathcal{H}_{g,n}$ for all $g \in \{2, 3\}$ and $n \leq 8 - 2g$.

In view of the results in section 4.1, it is equivalent to compute

$$\sum_{C \in \mathcal{H}_g(k)} \prod_{i=1}^{\nu} a_{\lambda_i}(C)$$

for all partitions $\lambda = (\lambda_1, \dots, \lambda_\nu)$ of weight n . Instead of summing over elements in $\mathcal{H}_g(k)$, we can sum over k -isomorphism classes of hyperelliptic curves of genus g defined over k , but with weight the reciprocal of the number of automorphisms.

Viewing the hyperelliptic curves of genus $g \geq 2$ defined over k as double covers of \mathbf{P}^1 , we can write them in the form $y^2 = f(x)$, where x is a local coordinate on \mathbf{P}^1 and f is in P_g , the set of square-free polynomials of degree $2g + 1$ or $2g + 2$ with coefficients in k . It follows that we can sum over the set of curves corresponding to the elements of P_g and then divide by the number of elements of the group of isomorphisms defined over k of these curves, that is, by $|\mathrm{GL}_2(k)|$.

If we denote by C_f the curve corresponding to the polynomial $f \in P_g$, then by Lefschetz fixed point theorem we have

$$a_m(C_f) = - \sum_{\alpha \in \mathbf{P}^1(k_m)} \chi_{2,m}(f(\alpha)),$$

where $\chi_{2,m}$ is the quadratic character of the field k_m . Recall that the quadratic character of a field is the function mapping 0 to 0, all non-zero squares of the field to 1, and all other elements to -1 .

We conclude that

$$\sum_{C \in \mathcal{H}_g(k)} \prod_{i=1}^{\nu} a_{\lambda_i}(C) = \frac{(-1)^{|\lambda|}}{|\mathrm{GL}_2(k)|} \sum_{f \in P_g} \prod_{i=1}^{\nu} \left(\sum_{\alpha \in \mathbf{P}^1(k_{\lambda_i})} \chi_{2,\lambda_i}(f(\alpha)) \right).$$

This sum splits into sums of the form

$$u_g(\alpha_1, \dots, \alpha_{\nu}) := \sum_{f \in P_g} \prod_{i=1}^{\nu} \chi_{2,\lambda_i}(f(\alpha_i)),$$

for every choice of points $\alpha_i \in \mathbf{P}^1(k_{\lambda_i})$. Note that if n is odd, all these sums are equal to zero.

EXAMPLE 1. We consider the case $\lambda = (1, 1)$ and $(\alpha_1, \alpha_2) = (0, \infty)$. We want to compute $u_g := u_g(0, \infty)$, for all $g \geq 0$. If f is a monic polynomial, representing a hyperelliptic curve, then $f(\infty) = 0$ if it is of odd degree and $f(\infty) = 1$ if it is of even degree. Thus u_g is equal to the sum of $\chi_{2,1}(f(\alpha_1))$ multiplied with $q - 1$, over all monic square-free polynomials f of degree $2g + 2$. Since there are as many non-zero squares as non-squares in k , the sum of $\chi_{2,1}(h(\alpha_1))$ over all monic polynomials h is zero. Hence we get the following recursion formula for $g \geq 0$,

$$0 = u_g + u_{g-1}(q - 1) + u_{g-2}(q - 1)q + \dots + u_0(q - 1)q^{g-1} + (q - 1)^2 q^g$$

from which we conclude that $u_g(0, \infty) = -(q - 1)^2$ for all $g \geq 0$.

The example above can be generalized in the following sense. Fix a λ -tuple $(\alpha_1, \dots, \alpha_{\nu})$ of points in \mathbf{P}^1 and set $u_g := u_g(\alpha_1, \dots, \alpha_{\nu})$. Then we can determine u_g recursively, once we know the integers

$$U_g = \sum_h \prod_{i=1}^{\nu} \chi_{2,\lambda_i}(f(\alpha_i)),$$

where the sum is over all monic polynomials h of degree $2g + 1$ or $2g + 2$.

Define b_j to be the number of monic polynomials l of degree j such that $l(\alpha_i)$ is non-zero for all i .

If n is even then $u_g = (q - 1) \sum_{f \in \hat{P}_g} \prod_{i=1}^{\nu} \chi_{2,\lambda_i}(f(\alpha_i))$ where \hat{P}_g is the set of monic polynomials in P_g . Any monic polynomial h has a unique decomposition of the form $h = f \cdot l^2$ where f is a square-free monic polynomial. Thus for any point $\alpha \in \mathbf{P}^1(k_m)$, $\chi_{2,m}(h(\alpha)) = \chi_{2,m}(f(\alpha))$ as long as $l(\alpha)$ is non-zero. From this we can conclude that

$$(5) \quad U_g = b_g + \sum_{i=1}^g u_i b_{g-i}.$$

If h is a monic polynomial of degree greater than n , there is a one to one correspondence between the set of possible values of the first n coefficients of h and the set of possible values of $h(\alpha_i)$ for $1 \leq i \leq \nu$. Using this, and the fact that half of the non-zero elements of any finite field of odd characteristic are squares and half are non-squares, we can determine U_g for all g satisfying $2g + 1 \geq n$.

This shows that, if we know the values of u_g for all g such that $2g + 1 < n$, we can use (5) to determine their value for all possible g .

In [Berb] the method described above, which follows a suggestion by Nicholas M. Katz, together with the fact that if C has genus zero then $a_m(C) = 0$ for all m , is used to

TABLE 1. Equivariant counts of some $\mathcal{H}_{g,n}$.

n	$\text{Tr}(F_q, \mathbf{e}_{\text{ét}}^{\mathfrak{S}_n}(\mathcal{H}_{2,n} \otimes_{\mathbf{Z}} \bar{k}))$
0	$q^3 s_0$
1	$(q^4 + q^3)s_1$
2	$(q^5 + q^4 - 1)s_2 + q^4 s_{1^2}$
3	$(q^6 + q^5 - q^4 - q)s_3 + (q^5 - q)s_{21}$
4	$(q^7 + q^6 - q^5 - q^4 - q^2 + q + 1)s_4 + (q^6 - q^5 - q^2)s_{31} - q^2 s_{2^2} + (q^2 - 1)s_{21^2}$
n	$\text{Tr}(F_q, \mathbf{e}_{\text{ét}}^{\mathfrak{S}_n}(\mathcal{H}_{3,n} \otimes_{\mathbf{Z}} \bar{k}))$
0	$q^5 s_0$
1	$(q^6 + q^5)s_1$
2	$(q^7 + q^6)s_2 + (q^6 - 1)s_{1^2}$

make an equivariant count of the number of points defined over k of $\mathcal{H}_{g,n}$ for any value of g and all $n \leq 5$.

The results, for the cases relevant here, are given in Table 1. In particular, in view of Theorem 3.4, the equivariant count of points of $\mathcal{H}_{2,4}$ implies Theorem 1.2.

4.3. Quartic curves. Let \mathcal{Q}_n be the complement of $\mathcal{H}_{3,n}$ in $\mathcal{M}_{3,n}$. Using the canonical embedding we can identify \mathcal{Q}_n with the moduli space of n -pointed plane non-singular quartic curves. By the results in Section 4.1 we find that an equivariant count of the number of points of \mathcal{Q}_n is the same as an equivariant count of the number of plane n -pointed non-singular quartic curves defined over k divided by the number of elements, defined over k , of the group of isomorphisms acting on these curves, which is equal to $\text{PGL}_3(k)$.

From now on all curves mentioned in this section will be assumed to be plane curves.

Fix a partition λ of weight n . We want to compute $|\mathcal{Q}_n^{\sigma \cdot F_q}|$, where σ is any permutation with cycle type λ . In other words, we want to compute the sum, over all λ -tuples P of points in the plane, of the number of non-singular quartic curves that contain P .

DEFINITION 4.2. Let us identify the space of quartic curves defined over k with $\mathbf{P}^{14}(k)$. For every λ -tuple P of points, we denote by L_P the linear subspace of $\mathbf{P}^{14}(k)$ of curves that contain P .

Moreover, a set S of m distinct points in the plane is called an *unordered μ -tuple* if there is a μ -tuple (p_1, \dots, p_m) such that $S = \{p_1, \dots, p_m\}$. For any unordered μ -tuple S of points in the plane, we define $L_{P,S}$ to be the linear subspace of L_P of curves that have singularities at the points of S .

The locus of singular curves in L_P is the union of all linear spaces $L_{P,S}$ for every $m \geq 1$ and unordered m -tuple S . We want to use the sieve principle to compute the number of elements of this union. That is, sum the numbers $(-1)^{i+1} |L_{P,S_1} \cap \dots \cap L_{P,S_i}|$, for each $i \geq 1$ and for each unordered choice of distinct sets S_1, \dots, S_i where S_j is an unordered m_j -tuple. If this procedure terminates, every singular curve in L_P will have been counted exactly once. Hence, taking $|L_P|$ minus the resulting number gives the number of non-singular curves in L_P .

Note that, if $S = S_1 \cup S_2$ for an unordered μ_1 -tuple S_1 and an unordered μ_2 -tuple S_2 , then clearly $L_{P,S_1} \cap L_{P,S_2} = L_{P,S}$. Thus, to be able to use the sieve principle in this way we need only find the dimensions of all linear spaces of the form $L_{P,S}$.

Unfortunately, determining the dimension of all linear subspaces of the form $L_{P,S}$ is not always easy. Moreover, as we said, in order to apply the sieve principle, we need to know that there is a number M such that $L_{P,S}$ is empty as soon as S consists of more than M points. Since there are curves with infinitely many singularities, namely the non-reduced ones, such a number M does not exist.

Instead, we choose a number M and define a modified sieve principle as follows.

DEFINITION 4.3. Let P be a λ -tuple of points in the plane, and M a positive integer. We define the *modified sieve principle* as the computation of the sum of $(-1)^{i+1}|L_{P,S_1} \cap \dots \cap L_{P,S_i}|$ for each $i \geq 1$ and for each unordered choice of distinct sets S_1, \dots, S_i where S_j is an unordered m_j -tuple such that $\sum_{j=1}^i m_j \leq M$. Denote the resulting number by $s_{M,P}$.

If we subtract $s_{M,P}$ from $|L_P|$, all curves with at most M singularities will have been removed from L_P exactly once. Thus having computed the sum $\sum_P (|L_P| - s_{M,P})$ where P runs over all λ -tuples, we need to amend for the curves with more than M singularities to obtain the sum over all λ -tuples P of the number of non-singular curves in L_P .

For any partition μ of weight $m > M$, define t_μ to be the sum over all choices of λ -tuples P and unordered μ -tuples S of the number of curves that contain P and that have singularities at the points of S , and nowhere else. Note that t_μ will be non-zero only for a finite number of μ .

For every curve with more than M singularities we can easily find how many times it would be removed or added when applying the modified sieve principle, as in the above, to compute $s_{M,P}$. Note that the multiplicity with which a curve with more than M singularities is counted in the computation of $s_{M,P}$ is the same for all curves which are singular exactly at a μ -tuple. Therefore, after adding a suitable multiple of t_μ to the number $\sum_P (|L_P| - s_{M,P})$ for each μ of weight greater than M , every singular curve will have been removed exactly once from each space L_P and hence we will have computed the number of points of $\mathcal{Q}_n^{\sigma, F_q}$.

This method was used in [Bera], where M was taken to be two, and gave the equivariant answers for the number of points defined over k of \mathcal{Q}_n for all n less than or equal to six. In the following, we will review the method in a little more detail, in the case when M equals two.

The first part of the method consists in fixing a partition λ of weight n and then computing $|L_P| - s_{2,P}$ for all λ -tuples P . To do this, we need to determine, for every P , the dimension of the linear spaces of curves passing through P and that have singularities at zero, one or two points.

EXAMPLE 2. Suppose that P consists of the point p_1 . In this case L_P has dimension 13. If p_1 is not contained in S , the subspace $L_{P,S}$ will have the expected dimension; if this is not the case, the dimension of $L_{P,S}$ will be one more than expected. Hence, we have to distinguish the following five types of singular curves passing through P : curves singular in p_1 , singular in a point different from p_1 , singular in a conjugate 2-tuple, singular in p_1 and another k -point and finally those singular in two k -points both different from p_1 .

Note that these cases are not mutually exclusive. By choosing the appropriate signs we get,

$$\begin{aligned} |L_P| - s_{2,P} = & |\mathbf{P}^{13}(k)| - |\mathbf{P}^{11}(k)| - (q^2 + q)|\mathbf{P}^{10}(k)| - \frac{1}{2}(q^4 - q)|\mathbf{P}^7(k)| + \\ & + (q^2 + q)|\mathbf{P}^8(k)| + \frac{1}{2}(q^2 + q)(q^2 + q - 1)|\mathbf{P}^7(k)|. \end{aligned}$$

The second part consists in computing t_μ for all partitions μ of weight $m > 2$. This requires to determine the number of points and singularities (with their fields of definition) of each quartic curve with at least three singularities.

To do this we distinguish types of quartic curves defined over k according to the following information,

- degrees and multiplicities of the irreducible components;
- over which fields the components are defined;
- the number of singularities of each irreducible component;
- over which fields the singularities are defined;
- in how many points every pair of irreducible components intersect;
- over which fields the intersection points are defined.

Hence if we know the type of a curve we only need to find its number of points to deduce its contribution to t_μ .

All curves of a given type with all irreducible components of degree one or two have the same number of points, which is easy to compute. This follows from the fact that irreducible curves of degree one or two are non-singular rational curves.

EXAMPLE 3. Let us consider quartic curves defined over k consisting of a conic together with one tangent line and one transversal line intersecting the conic in two points defined over k distinct from the tangency point of the other line. In this case all components and intersection points will be defined over k . Furthermore all components are rational curves, so the number of points defined over k_i of such a curve equals $3|\mathbf{P}^1(k_i)| - 4$. The number of such curves is easily computed to be $|\mathrm{PGL}_3(k)|/2$.

In case the normalization of a singular component has genus zero we only need to find how many points there are in the inverse image of the singularities under the normalization map and over which fields they are defined, to be able to compute its number of points.

EXAMPLE 4. There are $(q - 2)|\mathrm{PGL}_3(k)|/6$ quartic curves defined over k consisting of a cuspidal cubic together with a transversal line intersecting the cubic in three points defined over k . The fact that the singularity of the cubic is a cusp means that the inverse image of the singularity under the normalization map is only one point which is defined over the same field as the singularity. This shows that curves of this type have $2|\mathbf{P}^1(k_i)| - 3$ points defined over k_i .

All quartic curves with at least three singularities are such that all components (or their normalizations) have genus zero, with the exception of those consisting of a non-singular cubic together with a transversal line.

Non-singular cubic curves have genus one, which makes it harder to use the same approach as above. But non-singular cubic curves can be counted in the same fashion as

TABLE 2. Equivariant counts of points of \mathcal{Q}_n for $n \leq 2$.

n	$\text{Tr}(F_q, \mathbf{e}_{\text{ét}}^{\mathfrak{S}_n}(\mathcal{Q}_n \otimes_k \bar{k}))$
0	$(q^6 + 1)s_0$
1	$(q^7 + q^6 + q + 1)s_1$
2	$(q^8 + q^7 - q^5 + q^2 + q)s_2 + (q^7 + q + 1)s_{12}$

we are counting the non-singular quartic curves, see also Remark 3. Hence after adding the transversal line we can find the contribution of these curves to t_μ .

The results for the cases relevant here can be found in Table 2. In particular, the results for \mathcal{Q}_2 , together with the results on $\mathcal{H}_{3,2}$ given in Table 1, imply Theorem 1.3.

5. Rational cohomology of geometric quotients

If g and n are small enough, it is possible to determine exactly what the rational cohomology of $\mathcal{M}_{g,n}$ is. This applies at least for the moduli space of curves of genus 3 and 4. The reason is that in these cases the canonical map provides a very simple description of each such curve. This allows to divide the coarse moduli space into locally closed subschemes which are geometric quotients of the complement of a discriminant in a complex vector space, by the action of a reductive group. In this section, we explain how this description applies to moduli spaces of (n -pointed) non-singular curves of genus 3 or 4. Moreover, following [PS03], we explain which relation holds between the cohomology of the complement of the discriminant and the cohomology of the geometric quotient.

Note that, throughout this section, we will always consider not the moduli stacks $\mathcal{M}_{g,n}$, but the underlying coarse moduli spaces. Since the rational cohomology of the two coincides, we will abuse notation and not distinguish between the two concepts.

Throughout this section and the following one, all results on the cohomology (or the Borel-Moore homology) will be given by means of *Poincaré-Serre polynomials* (following [Loo93]). If H^\bullet is a rational mixed Hodge structure, the Poincaré-Serre polynomial of H^\bullet is defined as the polynomial in $\mathbf{Z}[t, u, u^{-1}]$ such that the coefficient of $t^i u^j$ is the dimension of the weight j subquotient of H^i . The Poincaré-Serre polynomial of a complex variety Z is the Poincaré-Serre polynomial of its rational cohomology. Note that in all cases we will consider, the mixed Hodge structures will be sums of (rational) Tate Hodge structures. In this special case, giving the Poincaré-Serre polynomial is equivalent to giving the whole rational mixed Hodge structure.

5.1. Moduli spaces of smooth subvarieties. Let $Z \subset \mathbf{P}^N$ be a complex projective variety, and L a vector bundle on Z . We define V to be a linear subspace of the space of sections of L . Inside V there is a closed subvariety Σ of sections whose zero scheme is not smooth of the expected dimension. We call Σ the *discriminant*. In the following, we will always assume that Σ is of pure complex codimension 1 in V .

Suppose further that there is a complex affine algebraic group G acting on Z and L , and that this action induces an action on V .

We are interested in studying the quotient of $X := V \setminus \Sigma$ by the action of G (if such a quotient exists), and specifically in determining its rational cohomology. In many

examples the geometric quotient X/G can be interpreted as the moduli space of the smooth varieties defined by the vanishing of elements in V .

Moreover, for every $n \geq 0$ we can consider the incidence correspondence

$$\mathcal{I}_n := \{(v, p_1, \dots, p_n) \in X \times F(Z, n) : v(p_i) = 0 \ \forall i = 1, \dots, n\},$$

where $F(Z, n)$ denotes the space of ordered configurations of n distinct points on Z . The action of G on Z clearly induces an action on \mathcal{I}_n and if the geometric quotient X/G exists, so does \mathcal{I}_n/G .

We recall Peters-Steenbrink's generalization of the Leray-Hirsch theorem:

THEOREM 5.1 ([PS03]). *Let $\varphi : X \rightarrow Y$ be a geometric quotient for the action of a connected group G , such that for all $x \in X$ the connected component of the stabilizer of x is contractible. Consider the orbit map*

$$\begin{aligned} \rho : \quad G &\longrightarrow \quad X \\ g &\longmapsto \quad gx_0, \end{aligned}$$

where $x_0 \in X$ is a fixed point. Suppose that for all $k > 0$ there exist classes $e_1^{(k)}, \dots, e_{n(k)}^{(k)} \in H^k(X; \mathbf{Q})$ that restrict to a basis for $H^k(G; \mathbf{Q})$ under the map induced by ρ on cohomology. Then the map

$$a \otimes \rho^*(e_i^{(k)}) \longmapsto \varphi^*a \cup e_i^{(k)}$$

extends linearly to an isomorphism of graded linear spaces

$$H^\bullet(Y; \mathbf{Q}) \otimes H^\bullet(G; \mathbf{Q}) \cong H^\bullet(X; \mathbf{Q}).$$

REMARK 4. The hypotheses of Theorem 5.1 are apparently very natural in all cases where G is a reductive group and $X = V \setminus \Sigma$ is the complement of the discriminant. In particular, they are known to hold in the following cases:

- (1) *Moduli spaces of smooth hypersurfaces* [PS03, Theorem 1]
 $Z = \mathbf{P}^N$, V is the vector space of sections of $L = \mathcal{O}_{\mathbf{P}^N}(d)$ ($d \geq 3$) and $G = \mathrm{GL}(N, \mathbf{C})$.
- (2) *Moduli space of smooth curves on a quadric cone* ([Tom05, 4.1], I.4.1)
 $Z = \mathbf{P}(1, 1, 2)$, V is the vector space of sections of $L = \mathcal{O}_Z(6)$ and G is the automorphism group of the graded ring $\mathbf{C}[x, y, z]$ where $\deg x = \deg y = 1$, $\deg z = 2$. This can also be generalized to curves of even degree ≥ 4 on a quadric cone.
- (3) *Moduli space of smooth $(3, 3)$ -curves on $\mathbf{P}^1 \times \mathbf{P}^1$* ([Tom05, 3.1], I.3.1)
 $Z = \mathbf{P}^1 \times \mathbf{P}^1$, V is the vector space of sections of $L = \mathcal{O}_{\mathbf{P}^1}(3) \otimes \mathcal{O}_{\mathbf{P}^1}(3)$ and G is the connected component of the identity of the automorphism group of the $\mathbf{Z} \times \mathbf{Z}$ -graded ring $\mathbf{C}[x_0, x_1; y_0, y_1]$. Note that this result can be easily generalized to smooth (m, n) -curves on $\mathbf{P}^1 \times \mathbf{P}^1$, with $m, n \geq 2$.

THEOREM 5.2. *Suppose X satisfies the hypotheses of Theorem 5.1 for the action of an affine algebraic group G . Then the action of G on \mathcal{I}_n satisfies the hypotheses of Theorem 5.1, for every $n \geq 0$.*

PROOF. Consider the natural projection $\pi_n : \mathcal{I}_n \rightarrow X$. The map π_n is equivariant with respect to the action of G . Then the claim follows from the fact that, for an appropriate choice of the base points, the orbit map from G to X is the composition of π_n with the orbit map from G to \mathcal{I}_n . \square

5.2. Moduli spaces of curves of genus three and four. The considerations above can be applied directly to the study of the rational cohomology of moduli spaces of curves of genus 3 and 4, with or without marked points. Indeed, every non-hyperelliptic curve of genus three has a plane quartic curve as its canonical model ([Har77, IV.5.2.1]). This means that we are in the situation of Remark 4, case 1, with $N = 2$ and $d = 4$. In the notation of that case, we have that

$$H^\bullet(\mathcal{Q}_n; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(3); \mathbf{Q}) \cong H^\bullet(\mathcal{I}_n; \mathbf{Q}),$$

where \mathcal{Q}_n is the complement in $\mathcal{M}_{3,n}$ of the hyperelliptic locus $\mathcal{H}_{3,n}$. Hence, determining the cohomology of \mathcal{Q}_n is equivalent to determining that of \mathcal{I}_n .

We are interested in determining the cohomology of \mathcal{I}_n for the first values of n . For $n = 0$, we have $\mathcal{I}_0 = X = V \setminus \Sigma$. The cohomology of this space has been computed by Vassiliev in [Vas99]. Vassiliev's results imply that the Poincaré-Serre polynomial of \mathcal{I}_0 is equal to

$$(1 + u^2t)(1 + u^4t^3)(1 + u^6t^5)(1 + u^{12}t^6).$$

By Theorem 5.2, this implies that the Poincaré-Serre polynomial of \mathcal{Q}_0 is $1 + u^{12}t^6$. Since the hyperelliptic locus has the rational cohomology of a point, this yields an alternative proof of the following result of Looijenga.

THEOREM 5.3 ([Loo93, 4.7]). *The Poincaré-Serre polynomial of \mathcal{M}_3 is $1 + u^2t^2 + u^{12}t^6$.*

Consider next the case $n = 1$. The incidence correspondence \mathcal{I}_1 has a natural forgetful morphism $\pi_1 : \mathcal{I}_1 \longrightarrow \mathbf{P}^2$. The map π_1 is a locally trivial fibration, and the fibre F_1 is isomorphic to the space of non-singular homogeneous polynomials of degree 4, vanishing at a fixed point $p \in \mathbf{P}^2$. The rational cohomology of this space is computed with Vassiliev-Gorinov's method in Section 5.3 (see (6) at page 64). In particular, its Poincaré-Serre polynomial is

$$(1 + u^2t)(1 + 2u^4t^3 + u^8t^6 + u^{12}t^6 + 2u^{16}t^9 + u^{20}t^{12}).$$

The Leray spectral sequence associated to π_1 is given in Table 3.

By Theorem 5.2, we know that the rational cohomology of \mathcal{I}_1 is the tensor product of the cohomology of $\mathrm{GL}(3)$ and that of \mathcal{Q}_1 . This is only possible if all non-trivial differentials between column 0 and column 4 in the spectral sequence have rank 1. This yields the following.

PROPOSITION 5.4. *The Poincaré-Serre polynomial of \mathcal{I}_1 is $(1 + u^2t)(1 + u^4t^3)(1 + u^6t^5)(1 + u^2t^2)(1 + u^{12}t^6)$.*

COROLLARY 5.5. *The Poincaré-Serre polynomials of \mathcal{Q}_1 and of $\mathcal{M}_{3,1}$ are, respectively, $(1 + u^2t^2)(1 + u^{12}t^6)$ and $(1 + u^2t^2)(1 + u^2t^2 + u^{12}t^6)$.*

PROOF. This follows from the above Proposition, together with Theorem 5.2, and the fact that $H^\bullet(\mathcal{H}_{3,1}; \mathbf{Q}) \cong H^\bullet(\mathbf{P}^1; \mathbf{Q})$. \square

Note that Corollary 5.5 refines the result on the Poincaré-Serre polynomial of $\mathcal{M}_{3,1}$ given in [GL], which corrects an error in [Loo93, 4.10].

The situation with curves of genus 4 is similar. Therefore, we review here the results of [Tom05] and chapter I. The canonical model of a non-hyperelliptic curve of genus 4 is the complete intersection of a quadric and a cubic surface in \mathbf{P}^3 ([Har77, IV.5.5.2]). As a consequence, we can stratify \mathcal{M}_4 by taking the space \mathcal{M}_4^0 of curves which are the complete intersection of a cubic surface and a quadric of maximal rank, the space \mathcal{M}_4^1 of

TABLE 3. Leray spectral sequence converging to $H^\bullet(\mathcal{I}_1; \mathbf{Q})$.

13	$\mathbf{Q}(-11)$	$\mathbf{Q}(-12)$	$\mathbf{Q}(-13)$	
12	$\mathbf{Q}(-10)$	$\mathbf{Q}(-11)$	$\mathbf{Q}(-12)$	
11				
10	$\mathbf{Q}(-9)^2$	$\mathbf{Q}(-10)^2$	$\mathbf{Q}(-11)^2$	
9	$\mathbf{Q}(-8)^2$	$\mathbf{Q}(-9)^2$	$\mathbf{Q}(-10)^2$	
8				
7	$\mathbf{Q}(-5) + \mathbf{Q}(-7)$	$\mathbf{Q}(-6) + \mathbf{Q}(-8)$	$\mathbf{Q}(-7) + \mathbf{Q}(-9)$	
6	$\mathbf{Q}(-4) + \mathbf{Q}(-6)$	$\mathbf{Q}(-5) + \mathbf{Q}(-7)$	$\mathbf{Q}(-6) + \mathbf{Q}(-8)$	
5				
4	$\mathbf{Q}(-3)^2$	$\mathbf{Q}(-4)^2$	$\mathbf{Q}(-5)^2$	
3	$\mathbf{Q}(-2)^2$	$\mathbf{Q}(-3)^2$	$\mathbf{Q}(-4)^2$	
2				
1	$\mathbf{Q}(-1)$	$\mathbf{Q}(-2)$	$\mathbf{Q}(-3)$	
0	\mathbf{Q}	$\mathbf{Q}(-1)$	$\mathbf{Q}(-2)$	
	0	1 2	3	4

curves which are the complete intersection of a cubic surface and a quadric cone, and the hyperelliptic locus \mathcal{H}_4 . Then, by Remark 4, case 2 and 3, we can determine the cohomology of both \mathcal{M}_4^1 and \mathcal{M}_4^0 by computing that of the complements of the discriminants involved in each case.

This construction allows us to compute the rational cohomology of \mathcal{M}_4 . Let us consider \mathcal{M}_4^0 first. Every element of \mathcal{M}_4^0 has a canonical embedding to a curve of type $(3, 3)$ on $\mathbf{P}^1 \times \mathbf{P}^1$. The cohomology of $V \setminus \Sigma$ can be obtained by applying Vassiliev-Gorinov's method (as described in Section I.2.1), starting from a classification of all possible singular loci of a $(3, 3)$ -curve. This yields the following result:

PROPOSITION 5.6 ([Tom05, 1.2], I.1.2). *The cohomology of the moduli space of non-singular $(3, 3)$ -curves on $\mathbf{P}^1 \times \mathbf{P}^1$ has Poincaré-Serre polynomial equal to $1 + u^6t^5$.*

Analogously, it is possible to apply Vassiliev-Gorinov's method to the space of non-singular curves of degree 6 on a quadric cone in \mathbf{P}^3 and prove that its cohomology is isomorphic to that of the automorphism group of the cone. As a consequence, we have the following.

THEOREM 5.7 ([Tom05, 1.4], I.1.3). *The Poincaré-Serre polynomial of the cohomology of \mathcal{M}_4 is $1 + u^2t^2 + u^4t^4 + u^6t^5$.*

5.3. Quartic curves through a point. In this section, we will apply Vassiliev-Gorinov's method to a concrete case, namely, to the case of quartic curves through a fixed point. This is an adaptation of the computation of Vassiliev in [Vas99] for the space of quartic curves.

Let us fix a point $p \in \mathbf{P}^2$, and choose coordinates x_0, x_1, x_2 on \mathbf{P}^2 . Denote by V the vector subspace of $\mathbf{C}[x_0, x_1, x_2]_4$ of homogeneous polynomials of degree four vanishing at p . The singular polynomials inside V form the discriminant locus $\Sigma \subset V$. We want to

apply Vassiliev-Gorinov's method in order to determine the Borel-Moore homology of Σ and the cohomology of its complement in V .

We start by listing in Table 4 all possible singular loci of elements of Σ . Note that the list of configurations in Table 4 defines a collection of families of configurations that already satisfies Gorinov's axioms. Therefore, we can use this collection to construct the cubical spaces $\mathcal{X}(\bullet)$ and $\Lambda(\bullet)$ as in Section I.2.1.

The strata defined by configurations of types 3a, 3b, 4b, 5, 6, 7, 8, 9, 11 and 12 do not contribute to the Borel-Moore homology of the domain of the geometric resolution of $\mathcal{X}(\bullet)$. This follows either from the fact that the twisted Borel-Moore homology of the corresponding configuration spaces is trivial, or from the fact that the configurations are curves. It remains to calculate the Borel-Moore homology of the other strata.

- 1a The space F_{1a} is isomorphic to \mathbf{C}^{12} . We have $\Phi_{1a} \cong \{p\}$.
- 1b The space F_{1b} is a complex vector bundle of rank 11 over $\Phi_{1b} \cong (\mathbf{P}^2 \setminus \{p\})$.
- 2a The space F_{2a} is a complex vector bundle of rank 9 over Φ_{2a} , which in turn is a non-orientable Δ_1 -bundle over $\mathbf{P}^2 \setminus \{p\}$.
- 2b The space F_{2b} is a complex vector bundle of rank 8 over Φ_{2b} , which in turn is a non-orientable $\mathring{\Delta}_1$ -bundle over $B(\mathbf{P}^2 \setminus \{p\}, 2)$. After the choice of a line through p , the complement of p in \mathbf{P}^2 can be seen as the disjoint union of an affine line and an affine two-space, hence we have $B(\mathbf{P}^2 \setminus \{p\}, 2) = B(\mathbf{C}, 2) \sqcup \mathbf{C} \times \mathbf{C}^2 \sqcup B(\mathbf{C}^2, 2)$. The induced local system is $\pm \mathbf{Q}$ on the first and the third component. It is a standard fact in the theory of complements of unions of hyperplanes that $\bar{H}_\bullet(B(\mathbf{C}^N, j); \mathbf{Q}) = 0$ when $j \geq 2$. Hence, $\bar{H}_\bullet(B(\mathbf{P}^2 \setminus \{p\}, 2); \pm \mathbf{Q}) \cong \bar{H}_\bullet(\mathbf{C}^3; \mathbf{Q})$.
- 4a The space F_{4a} is a complex vector bundle of rank 6 over Φ_{4a} , which in turn is a non-orientable $\mathring{\Delta}_2$ -bundle over $B(\mathbf{P}^2 \setminus \{p\}, 2)$.
- 10 The space F_{10} is a complex vector bundle of rank 1 over Φ_{10} , which in turn is a non-orientable $\mathring{\Delta}_5$ -bundle over the configuration space X_{10} . It is easier to consider X_{10} as a subspace of $B(\mathbf{P}^{2^\sim}, 4)$. Then X_{10} is the space of all configurations of four points in general position such that at least one point lies on the line p^\sim . The local system induced by the orientation of the simplicial bundle is the constant one, because interchanging two lines interchanges two pairs of intersection points.

The Borel-Moore homology with constant coefficients of $\tilde{B}(\mathbf{P}^{2^\sim}, 4)$ is easy to compute, because it coincides with that of $\mathrm{PGL}(3)$. Its Poincaré-Serre polynomial is $u^{-16}t^{16} + u^{-12}t^{13} + u^{-10}t^{11} + u^{-6}t^8$.

Consider the following variety:

$$\mathcal{S} := \{(l_1, l_2, l_3, l_4, p) \in \tilde{F}(\mathbf{P}^2, 4) \times \mathbf{P}^2 : p \in \bigcup_i l_i\},$$

where $\tilde{F}(\mathbf{P}^{2^\sim}, 4)$ denotes the space of four lines in \mathbf{P}^2 in general position. If we consider the projection of \mathcal{S} onto $\tilde{F}(\mathbf{P}^{2^\sim}, 4)$, it is easy to see that the \mathfrak{S}_4 -invariant part of the Borel-Moore homology of \mathcal{S} coincides with the tensor product of the Borel-Moore homology of $\tilde{F}(\mathbf{P}^{2^\sim}, 4) \cong \mathrm{PGL}(3)$ and that of \mathbf{P}^1 . If we consider the projection $p_2 : \mathcal{S} \rightarrow \mathbf{P}^2$, we obtain that the fibre of p_2 is a \mathfrak{S}_4 -covering of X_{10} . This implies that the cohomology of X_{10} coincides with the \mathfrak{S}_4 -invariant part of the cohomology of this fibre. A direct computation based on the Leray spectral sequence associated to p_2 yields that the Poincaré-Serre polynomial of the Borel-Moore homology of X_{10} must be $u^{-14}t^{14} + 2u^{-10}t^{11} + u^{-6}t^8$.

TABLE 4. Singular sets of quartic polynomials vanishing at p .

The number in the last column is the dimension of the linear subspace of V of polynomials singular in a configuration of the given type. The position of n points is called *general* when no three of them lie on the same line.

Type	Description	Dim.
1a	The point p .	12
1b	Any other point.	11
2a	The point p and another point.	9
2b	Any other pair of points.	8
3a	Three points lying on a line through p .	7
3b	Three points on a line not passing through p .	6
4a	The point p and two other points in general position with respect to p .	6
4b	Three points in general position, different from p .	5
5a	A line through p .	6
5b	A line not passing through p .	5
6a,b,c	Three points on the same line and one point outside it. There are 3 subcases, according to the position of p .	4,4,3
7a,b	Four points in general position. There are 2 subcases, depending on the position of p .	3,2
8a,b,c	A line and a point outside it. There are 3 subcases, depending on the position of p .	3,3,2
9a,b,c	Two pairs of points, lying on different lines, and the intersection point of the two lines they span. There are 3 subcases: p may be the central point, or lie on one of the lines, or the conic is uniquely determined as the conic passing through p and the other 4 points.	2,2,1
10	Six points, pairwise intersection of a general configuration of four lines in \mathbf{P}^2 . At least one line passes through p .	1
11	A non-singular conic through p .	1
12	A conic of rank 2, passing through p .	1
13	The whole \mathbf{P}^2 .	0

13 The space F_{13} is an open cone with base $\text{Fil}_{13} |\Lambda(\bullet)| = \bigcup_{1 \leq i \leq 13} \Phi_i$. The spectral sequence associated to the filtration $\text{Fil}_i (|\Lambda(\bullet)|)$ is such that the differential d_1 is an isomorphism between the columns corresponding to the contributions of Φ_{2a} and Φ_{1b} . The same holds for Φ_{4a} and Φ_{2b} . From this it follows that the

TABLE 5. Spectral sequence converging to $\bar{H}_\bullet(\Sigma; \mathbf{Q})$.

24		$\mathbf{Q}(13)$					
23	$\mathbf{Q}(12)$						
22		$\mathbf{Q}(12)$					
21							
20			$\mathbf{Q}(11)$				
19				$\mathbf{Q}(11)$			
18			$\mathbf{Q}(10)$				
17							
16							
15				$\mathbf{Q}(9)$	$\mathbf{Q}(8)$		
14							
13						$\mathbf{Q}(7)$	
12					$\mathbf{Q}(6)^2$		
11							
10						$\mathbf{Q}(5)^2$	
9					$\mathbf{Q}(4)$		
8							
7						$\mathbf{Q}(3)$	
6							
	1a	1b	2a	2b	4a	10	13

only contribution comes from configurations of type 10, and that $\bar{H}_\bullet(F_{13}; \mathbf{Q}) \cong \bar{H}_{\bullet-1}(\Phi_{10}; \mathbf{Q})$.

The spectral sequence converging to the Borel-Moore homology of Σ has E^1 term as in Table 5.

From the above, and considering the fact that the relationship between Borel-Moore homology of Σ and reduced cohomology of $X = V \setminus \Sigma$ is given by

$$\tilde{H}^\bullet(X; \mathbf{Q}) \cong \bar{H}_{27-\bullet}(\Sigma; \mathbf{Q})(-14),$$

we get the Poincaré-Serre polynomial of the rational cohomology of $V \setminus \Sigma$

$$(6) \quad (1 + u^2t)(1 + 2u^4t^3 + u^8t^6 + u^{12}t^6 + 2u^{16}t^9 + u^{20}t^{12}).$$

This result is used in Section 5.2 to compute the cohomology of $\mathcal{M}_{3,1}$.

CHAPTER III

Moduli spaces of curves with two marked points

1. Introduction

In this chapter, we apply the methods explained in Section 5 of chapter II to investigate the rational cohomology of moduli spaces of non-singular curves with two marked points. We study moduli spaces of curves that are defined by the vanishing of polynomials in a vector space V . As we will see below, in the case of hyperelliptic curves it is more practical to take an affine space as V . We consider only cases in which the locus of singular polynomials is a hypersurface $\Sigma \subset V$. We consider then the incidence correspondence

$$\mathcal{I}_2 := \{(f, \alpha, \beta) : f \in V \setminus \Sigma, \alpha \neq \beta, f(\alpha) = f(\beta) = 0\}.$$

We deal with moduli spaces of curves with two marked points that can be viewed as the quotient of \mathcal{I}_2 (for an appropriate choice of V and Σ) by the action of an affine algebraic group G . We will make use of the generalized Leray-Hirsch theorem for incidence correspondences (Theorem 5.2 of chapter II) to study their rational cohomology. Theorem II.5.2 ensures that the rational cohomology of \mathcal{I}_2 , as graded vector space, is isomorphic to the tensor product of the rational cohomology of G and that of the moduli space. This reduces the computation of the rational cohomology of the chosen moduli space of 2-pointed non-singular curves to that of \mathcal{I}_2 . The latter computation will be performed by considering the Leray spectral sequence associated to the projection $\pi_2 : (f, \alpha, \beta) \longmapsto (\alpha, \beta)$. The fibres of this projection are linear sections of $V \setminus \Sigma$. Therefore, they can be viewed as discriminant loci in a linear subspace of V . We use this description of the fibres of π_2 to compute their rational cohomology with Vassiliev-Gorinov's method (as explained in Section 2.1 of chapter I). Then, the Leray spectral sequence associated to π_2 and the generalized Leray-Hirsch theorem will enable us to draw conclusions on the rational cohomology of our moduli space.

We will deal with two different moduli spaces of curves. We start in Section 2 with the moduli space $\mathcal{H}_{g,2}$ of hyperelliptic curves of a fixed genus $g \geq 2$ with two marked points. We give in Corollary 2.2 the rational cohomology of $\mathcal{H}_{g,2}$, with its Hodge structures and its structure as \mathfrak{S}_2 -representation. The result we obtain is compatible with the results on the Hodge Euler characteristics of Section II.4.2, and has been obtained independently from the computations of [Bera]. Our construction is based on the fact that every hyperelliptic curve of genus g can be viewed as a curve of degree $2g + 2$ on a weighted projective space $\mathbf{P}(1, 1, g + 1)$. Hence, every hyperelliptic curve is defined by the vanishing of a homogeneous polynomial of degree $2g + 2$ in three indeterminates x_0, x_1, y of degree respectively 1, 1 and $g + 1$. Since the coefficient of y^2 is always different from 0 for polynomials defining hyperelliptic curves, we will study the space V of homogeneous polynomials of degree $2g + 2$ such that the coefficient of y^2 equals 1. If we denote by Σ the locus inside V of singular polynomials, we have that the moduli space of hyperelliptic curves of genus g is the quotient of $V \setminus \Sigma$ by the action of a group G

which is homotopy equivalent to $GL(2)$. It is easy to prove that the action of G on $V \setminus \Sigma$ satisfies the hypothesis of the generalized Leray-Hirsch Theorem I.1.1. This allows us to apply the construction explained above and view $\mathcal{H}_{g,2}$ as the quotient of the incidence correspondence $\mathcal{I}_2 \subset (V \setminus \Sigma) \times F(\mathbf{P}(1, 1, g+1), 2)$. Indeed, the projection π_2 from \mathcal{I}_2 to $F(\mathbf{P}(1, 1, g+1), 2)$ is not a locally trivial fibration. The reason is that the fibres of π_2 at configurations of two points lying on the same line in $\mathbf{P}(1, 1, g+1)$ are not homeomorphic to the general fibres of π_2 . We solve this problem by restricting to an open set $\mathcal{H}_{g,2}^2$ of $\mathcal{H}_{g,2}$, that is the quotient by G of the total space of a locally trivial fibration over the space of ordered configurations of two points in $\mathbf{P}(1, 1, g+1)$ not lying on a line of the ruling of $\mathbf{P}(1, 1, g+1)$. The rational cohomology of the fibre is computed with Vassiliev-Gorinov's method in Section 2.1. In order to do that, we have to change slightly the construction of the method in Section 2.1 of chapter I, in order to allow the spaces of polynomials with prescribed singularities to be complex affine spaces instead of complex vector spaces. This is a very straightforward adaptation, which does not really affect the constructions.

The cohomology of the complement of $\mathcal{H}_{g,2}^2$ is easy to determine, because $\mathcal{H}_{g,2}^1 = \mathcal{H}_{g,2} \setminus \mathcal{H}_{g,2}^2$ is a moduli space of configurations of points on the projective line. Though the cohomology of moduli spaces of configurations of points on \mathbf{P}^1 is well known (see [Get95]), it seemed interesting to compute $H^\bullet(\mathcal{H}_{g,2}^1; \mathbf{Q})$ by applying successively Vassiliev-Gorinov's method and the generalized Leray-Hirsch Theorem. This is done in Section 2.2.

Finally, in Section 3 we compute the Hodge Euler characteristic of the moduli space \mathcal{Q}_2 of non-singular quartic curves in the projective plane with two marked points. In Theorem 3.1, we give the Hodge Euler characteristic of \mathcal{Q}_2 (which of course agrees with Jonas Bergström's equivariant count of the number of points of \mathcal{Q}_2 , see Section II.4.3), and recover the result on the Hodge Euler characteristic of $\mathcal{M}_{3,2}$ already established in Theorem II.1.3. The methods are the same as in the computation of the rational cohomology of $\mathcal{M}_{3,1}$ in Section 5.2 of chapter II. As an application of Theorem 3.1, we compute the rational cohomology of the moduli space $\overline{\mathcal{M}}_{3,2}$ with the same methods used for $\overline{\mathcal{M}}_4$ in chapter II. This result is given in Theorem 3.2.

2. Moduli of hyperelliptic curves with two marked points

We are interested in determining the rational cohomology of the moduli space $\mathcal{H}_{g,2}$ of hyperelliptic curves of genus $g \geq 2$ with two marked points. In order to do this, we divide hyperelliptic curves with two marked points into two classes:

- (1) curves such that the two marked points map to the same point in \mathbf{P}^1 ;
- (2) curves such that the two marked points map to distinct points in \mathbf{P}^1 .

In this way we get two subloci $\mathcal{H}_{g,2}^1$ and $\mathcal{H}_{g,2}^2$ of the moduli space $\mathcal{H}_{g,2}$. The space $\mathcal{H}_{g,2}^1$ can be identified with the moduli space of collections of points in \mathbf{P}^1 of the form $(p, \{q_1, \dots, q_{2g+2}\})$. It will be shown in Section 2.2 that this moduli space has the rational cohomology of a point. Of course, this is a standard result on the cohomology of configuration spaces and can also be deduced from Getzler's results on the action of \mathfrak{S}_n on the cohomology of $\mathcal{M}_{0,n}$ (see [Get95]).

Denote by \mathbf{P} the weighted projective space $\mathbf{P}(1, 1, g+1)$. Note that \mathbf{P} can be embedded in \mathbf{P}^{g+2} as the cone over a rational normal curve of degree $g+1$ with a point as vertex. In particular, \mathbf{P} is covered by the lines through the singular point $[0, 0, 1] \in \mathbf{P}$. Denote by $\tilde{F}(\mathbf{P}, 2)$ the configuration space of ordered pairs of points in \mathbf{P} such that the two points

do not lie on the same line through the singular point $[0, 0, 1]$ of \mathbf{P} . Every hyperelliptic curve of genus g is defined by the vanishing of a polynomial of the form

$$y^2 + h(x_0, x_1)y + k(x_0, x_1) = 0,$$

where h and k are homogeneous polynomials of degree $g + 1$ and $2g + 2$ respectively, in the coordinates x_0, x_1, y of \mathbf{P} with $\deg x_i = 1$ and $\deg y = g + 1$. The space $V \cong \mathbf{C}[x_0, x_1]_{g+1} \times \mathbf{C}[x_0, x_1]_{2g+2}$ of such polynomials is an affine space of dimension $3g + 5$. As we are interested in 2-pointed hyperelliptic curves, we are going to study the incidence correspondence

$$\mathcal{J}_{g,2} = \{(p, q, f) \in \tilde{F}(\mathbf{P}, 2) \times V : f \text{ non-singular, } f(p) = f(q) = 0\}.$$

If we consider the action on $\mathcal{J}_{g,2}$ induced by the action of $\mathrm{GL}(2)$ on V , Theorem 5.2 of chapter II implies that the rational cohomology of $\mathcal{J}_{g,2}$ is the tensor product of that of $\mathcal{H}_{g,2}^2$ and that of $\mathrm{GL}(2)$. Moreover, the \mathfrak{S}_2 -invariant part of the cohomology of $\mathcal{J}_{g,2}$ is the tensor product of that of $\mathcal{H}_{g,2}^2/\mathfrak{S}_2$ and $\mathrm{GL}(2)$.

The natural projection from $\mathcal{J}_{g,2}$ to $\tilde{F}(\mathbf{P}(1, 1, g + 1), 2)$ is locally trivial with fibre isomorphic to an open set $Z_g \subset \mathbf{C}^{3g+3}$, where \mathbf{C}^{3g+3} is the affine subspace of V of polynomials vanishing at two chosen points p and q . The fibre Z_g can be viewed as the complement of a discriminant in \mathbf{C}^{3g+3} . Therefore, we can use Gorinov-Vassiliev's method to compute the rational cohomology of Z_g (see Proposition 2.4 in the next section). Note that we are working with complex affine spaces instead of vector spaces as in the definition of the method in chapter I. Since the study of the cohomology of affine bundles is not more complicated than that of the cohomology of vector bundles, it is easy to extend the constructions of chapter I, Section 2.1 to this slightly different situation.

The space $\tilde{F}(\mathbf{P}, 2)$ is a \mathbf{C}^2 -bundle over $F(\mathbf{P}^1, 2)$, which in turn is homotopy equivalent to \mathbf{P}^1 . Its cohomology is \mathfrak{S}_2 -invariant in degree 0 and anti-invariant in degree 2. The considerations above, together with the result on the rational cohomology of Z_g given in Proposition 2.4, allow us to compute the E_2 term of the Leray spectral sequence in rational cohomology associated to the projection $\mathcal{J}_{g,2} \longrightarrow \tilde{F}(\mathbf{P}, 2)$. We give the E_2 term of this spectral sequence in Table 1.

In view of Theorem II.5.2, we can conclude the following.

THEOREM 2.1. *The Poincaré-Serre polynomial of $\mathcal{H}_{g,2}^2$ is $1 + u^2t^2 + u^{4g+2}t^{2g+1}$. The Poincaré-Serre polynomial of $\mathcal{H}_{g,2}^2/\mathfrak{S}_2$ is $1 + u^{4g+2}t^{2g+1}$ if g is even and 1 if g is odd.*

Since $\mathcal{H}_{g,2}^1$ has the rational cohomology of a point, the theorem above implies the following.

COROLLARY 2.2. *The Poincaré-Serre polynomial of $\mathcal{H}_{g,2}$ is $1 + 2u^2t^2 + u^{4g+2}t^{2g+1}$. The Poincaré-Serre polynomial of $\mathcal{H}_{g,2}/\mathfrak{S}_2$ is $1 + u^2t^2 + u^{4g+2}t^{2g+1}$ if g is even and $1 + u^2t^2$ if g is odd.*

2.1. Hyperelliptic curves through two fixed points. In this section, we study non-singular curves of degree $2g + 2$ on $\mathbf{P} = \mathbf{P}(1, 1, g + 1)$ passing through two fixed points p, q lying on different lines of the ruling of \mathbf{P} . For this reason, we work with an open subset $Z_g \subset \mathbf{C}^{3g+3}$, where \mathbf{C}^{3g+3} is identified with the affine subspace of $V \cong \mathbf{C}[x_0, x_1]_{g+1} \times \mathbf{C}[x_0, x_1]_{2g+2}$ of polynomials vanishing at p and q . We will always consider \mathbf{P} as embedded in \mathbf{P}^{g+2} as the cone with vertex a point over a rational normal curve of degree $g + 1$ in $\mathbf{P}^{g+1} \subset \mathbf{P}^{g+2}$.

TABLE 1. Spectral sequence converging to $H^\bullet(\mathcal{H}_{g,2}^2; \mathbf{Q}) \otimes H^\bullet(\mathrm{GL}(2); \mathbf{Q})$.

$2g + 3$	$\mathbf{Q}(-2g - 3)$	$\mathbf{Q}(-2g - 4)$
$2g + 2$	$\mathbf{Q}(-2g - 2)^2$	$\mathbf{Q}(-2g - 3)^2$
$2g + 1$	$\mathbf{Q}(-2g - 1)$	$\mathbf{Q}(-2g - 2)$
$2g$		
\vdots		
5		
4	$\mathbf{Q}(-3)$	$\mathbf{Q}(-4)$
3	$\mathbf{Q}(-2)$	$\mathbf{Q}(-3)$
2		
1	$\mathbf{Q}(-1)$	$\mathbf{Q}(-2)$
0	\mathbf{Q}	$\mathbf{Q}(-1)$
	0	1
		2

TABLE 2. List of singular configurations.

We always assume that the points in a configuration lie on different lines of the ruling of the cone \mathbf{P} . The number in the last column is the dimension of the complex affine space singular at a configuration of the given type.

1.b	p or q .	$3g + 1$
1.c	any point not on $l_p \cup l_q$.	$3g$
2.a	p and q .	$3g - 1$
2.b	p (or q) and another point not on $l_p \cup l_q$.	$3g - 2$
2.c	two points not on $l_p \cup l_q$.	$3g - 3$

For $k = 3, \dots, g$

k.a	p, q and $k - 2$ points not on $l_p \cup l_q$.	$3g - 3k + 5$
k.b	p (or q) and $k - 1$ other points not on $l_p \cup l_q$.	$3g - 3k + 4$
k.c	k points not on $l_p \cup l_q$.	$3g - 3k + 3$
$g + 1.a$	p, q and $g - 1$ points not on $l_p \cup l_q$.	2
$g + 1.b$	p (or q) and g other points not on $l_p \cup l_q$.	1
$g + 1.c$	$g + 1$ points not on $l_p \cup l_q$, generating a hyperplane with p, q .	1
$g + 2$	$g + 1$ points in general position, not on $l_p \cup l_q$.	0
$g + 3$	irreducible hyperplane sections of \mathbf{P} passing through p and q .	0

Note that the cap product with the fundamental class of the discriminant $\mathbf{C}^{3g+3} \setminus Z_g$ gives an isomorphism (called Alexander's duality)

$$\tilde{H}^\bullet(Z_g; \mathbf{Q}) \cong \bar{H}_{6g+5-\bullet}(\mathbf{C}^{3g+3} \setminus Z_g; \mathbf{Q})(-3g - 3).$$

Again, we start by listing in Table 2 all possible singular sets of the elements of the discriminant $\mathbf{C}^{3g+3} \setminus Z_g$.

We study next the spectral sequence converging to the Borel-Moore homology of $\mathbf{C}^{3g+3} \setminus Z_g$. Observe that, for every $1 \leq k \leq g + 1$, all configurations of types $k.c, k.b$ and

(for $k > 1$) $k.a$ consist of the same number of points. This means that $\Phi_{k.a}$, $\Phi_{k.b}$ and $\Phi_{k.c}$ are simplices bundles with the same fibre $\mathring{\Delta}_{k-1}$. Therefore, it makes sense to study the contributions of strata $k.a$, $k.b$ and $k.c$ together. Define $F_k \subset |\mathcal{X}(\bullet)|$ to be the union of $F_{k.a}$, $F_{k.b}$ and $F_{k.c}$.

PROPOSITION 2.3. *For every k , $1 \leq k \leq g$, the space F_k is a non-orientable simplicial bundle of rank $k-1$ over a space that has the same twisted Borel-Moore homology as an affine bundle of rank $3g-2k+3$ over $B(\mathbf{P}^1, k)$.*

PROOF. For every k , $1 \leq k \leq g$, the configuration spaces $X_{k.a}$, $X_{k.b}$ and $X_{k.c}$ are contained in the space of configurations of k points on the cone \mathbf{P} , such that all k points lie on different lines of the ruling of \mathbf{P} . If we identify the ruling of \mathbf{P} with \mathbf{P}^1 , we have that all configuration spaces $X_{k.i}$ where $i \in \{a, b, c\}$ map to $B(\mathbf{P}^1, k)$. This allows us to define a natural map $m_k : F_k \longrightarrow B(\mathbf{P}^1, k)$. The fibre of this map over an element $K \in B(\mathbf{P}^1, k)$ can be interpreted as the product of an open simplex (whose vertices correspond to the elements of K) and a second space parametrizing the choice of the singular points on each line in K , and the choice of the singular polynomial. When K does not contain l_p and l_q , the fibre is simply the product of $\mathring{\Delta}_{k-1}$ with an affine bundle of rank $3g-3k+4$ over the \mathbf{C}^k parametrizing the choice of the k points on the lines in K .

Blow up \mathbf{P} at the points p and q and contract the proper transforms of the lines l_p and l_q to a point A . Denote by $\tilde{\mathbf{P}}$ the complement of A in the space so obtained. In particular, $\tilde{\mathbf{P}} \cup \{A\}$ contains the exceptional divisors E_p and E_q of the two blow-ups. Note that $\tilde{\mathbf{P}}$ is a \mathbf{C} -bundle over \mathbf{P}^1 , and that the points on $E_p \setminus \{A\}$ (respectively, $E_q \setminus \{A\}$) represent the directions tangent to \mathbf{P} at p (respectively, q) and different from the direction of the line l_p (respectively, l_q). The fact that we exclude A from E_p (respectively, E_q) means that we are not considering the direction tangent to the line l_p (respectively, l_q).

We will use the construction of $\tilde{\mathbf{P}}$ to construct desingularizations of the spaces \tilde{F}_k . Observe that the space $F_{k.c}$ can be extended to a $\mathring{\Delta}_{k-1} \times \mathbf{C}^{3g-3k+3}$ -bundle \tilde{F}_k over the space of configurations of k points in $\tilde{\mathbf{P}}$ such that the k points lie above k different points in \mathbf{P}^1 . If the configuration K of points in $\tilde{\mathbf{P}}$ contains a point α in E_p , then the polynomials in the fibre above K are requested to define a curve singular at p that contains the tangent direction α in its tangent cone. An analogous condition is posed for configurations containing a point of E_q . Since \tilde{F}_k is clearly non-singular, the natural map from \tilde{F}_k to F_k is a desingularization of F_k . To establish the claim it is enough to prove that the map $\tilde{F}_k \rightarrow F_k$ induces an isomorphism in Borel-Moore homology.

Let us start with considering the case $k = 1$ in some detail. The space \tilde{F}_1 parametrizes pairs (α, f) in $\tilde{\mathbf{P}} \times \mathbf{C}^{3g+3}$ such that:

- if $\alpha \notin E_p \cap E_q$, then f is singular at the point of \mathbf{P} corresponding to α ;
- if $\alpha \in E_p$, then f is singular at p and the direction α lies in the tangent cone of $\{f = 0\}$ at p ;
- if $\alpha \in E_q$, then f is singular at q and the direction α lies in the tangent cone of $\{f = 0\}$ at q .

Denote by ρ the natural map $\tilde{F}_1 \longrightarrow F_1$. The discriminant of the resolution, i.e., the closure of the locus where the map ρ is not an isomorphism, is $D = m_1^{-1}(\{p, q\})$. It has two affine spaces of dimension \mathbf{C}^{3g+1} as disjoint irreducible components. Its preimage \tilde{D} in ρ also has two irreducible components, which are affine bundles of rank \mathbf{C}^{3g} over \mathbf{C} . In

particular, both \tilde{D} and D are smooth. The restriction of ρ to \tilde{D} is a finite map of degree two. Indeed, curves on \mathbf{P} defined by an equation of the form $y^2 - h(x_0, x_1)y + k(x_0, x_1) = 0$ can only have double points as singularities. Furthermore, the directions in the tangent cone are always different from the vertical direction (i.e., the direction of the line of the ruling through the singular point). Thus, the map $\rho|_{\tilde{D}}$ is a double cover of D , and it induces an isomorphism in Borel-Moore homology because both components of \tilde{D} are simply connected. Concluding, what we have got is a diagram

$$(1) \quad \begin{array}{ccc} \tilde{D} & \xrightarrow{\rho|_{\tilde{D}}} & D \\ \downarrow & & \downarrow \\ \tilde{F}_1 & \xrightarrow{\rho} & F_1, \end{array}$$

where all vertical arrows denote inclusions, and all horizontal arrows denote proper maps. Since ρ is an isomorphism when restricted to $\tilde{F}_1 \setminus \tilde{D}$, and induces an isomorphism on Borel-Moore homology when restricted to \tilde{D} , we can conclude that ρ induces an isomorphism between the Borel-Moore homology of \tilde{F}_1 and F_1 . This establishes the claim for $k = 1$.

When $k \geq 2$, the reasoning above can be adapted to prove that the space

$$S_k := \{(p_1, \dots, p_k, f) \in F(\mathbf{P}, k) \times \mathbf{C}^{3g+3} : \{p_1, \dots, p_k\} \in \bigcup_{i \in \{a, b, c\}} X_{k,i}, f \text{ is singular at } p_1, \dots, p_k\}$$

has the same Borel-Moore homology as an affine bundle \tilde{S}_k of rank $3g - 3k + 3$ over the locus in $F(\tilde{\mathbf{P}}, k)$ of configurations of points lying on distinct lines of the ruling. This ensures that both S_k and \tilde{S}_k have natural projections to $F(\mathbf{P}^1, k)$. Then the claim follows from the fact that the commutative diagram

$$\begin{array}{ccc} \tilde{S}_k & \longrightarrow & F(\mathbf{P}^1, k) \\ \downarrow & & \parallel \\ S_k & \longrightarrow & F(\mathbf{P}^1, k) \end{array}$$

is compatible with the action of the symmetric group \mathfrak{S}_k permuting the singular points p_1, \dots, p_k . \square

Proposition 2.3 implies that we can use the results on the twisted Borel-Moore homology of $B(\mathbf{P}^1, x)$ given in Lemma 2.13 of chapter I to compute the E^1 term of the spectral sequence converging to the Borel-Moore homology of the domain of the geometric realization of $\mathcal{X}(\bullet)$. This spectral sequence is given in Table 3. In particular, Lemma 2.13 of chapter I implies that all spaces F_k with $3 \leq k \leq g$ give a trivial contribution to the spectral sequence. For this reason, in Table 3 we omit them. Moreover, also the contributions of $F_{g+1,a}$, $F_{g+1,b}$ and $F_{g+1,c}$ kill each other in the spectral sequence. Indeed, if it were not so, the extra terms we would get in the spectral sequence of Table 1 would create a contradiction with Theorem II.5.2. This means that the contribution of F_{g+1} to the spectral sequence of Table 3 is trivial, and hence we can omit it. Also the last stratum F_{g+3} gives no contribution. This follows from the fact that the configurations in X_{g+3} are rational curves, together with Lemma I.2.19.

If we consider Table 3 from the point of view of the behaviour of the Borel-Moore homology groups with respect to the involution interchanging p and q , we have that all terms in the first two columns are invariant. This follows from the fact that the whole

TABLE 3. Spectral sequence converging to the Borel-Moore homology of $\mathbf{C}^{3g+4} \setminus Z_g$.

$6g + 3$	$\mathbf{Q}(3g + 2)$		
$6g + 2$			
$6g + 1$	$\mathbf{Q}(3g + 1)$		
$6g$			
$6g - 1$		$\mathbf{Q}(3g)$	
\vdots			
$4g + 1$			$\mathbf{Q}(g + 2)$
$4g$			$\mathbf{Q}(g + 1)^2$
$4g - 1$			$\mathbf{Q}(g)$
	1	2	$g + 2$

Borel-Moore homology of \mathbf{P}^1 is invariant with respect to any involution of \mathbf{P}^1 , and the same can be said for the twisted Borel-Moore homology of $B(\mathbf{P}^1, 2)$.

The space F_{g+2} is a non-orientable $\mathring{\Delta}_g$ -bundle over the configuration space X_{g+2} of $g + 1$ distinct points on $\mathbf{P} \setminus (l_p \cup l_q)$, lying above $g + 1$ distinct lines in \mathbf{P}^{g+2} and not lying on the same hyperplane in \mathbf{P} . If we embed X_{g+2} in the space X'_{g+2} of all configurations of $g + 1$ points on $g + 1$ distinct lines in $\mathbf{P} \setminus (l_p \cup l_q)$, we get that the complement of X_{g+2} is the configurations space $X_{g+1,c}$. It is easy to see that the spaces $X_{g+1,c}$ and X'_{g+2} are the total spaces of complex vector bundles of rank g and $g + 1$ respectively, over the configuration space $B(\mathbf{C}^*, g + 1)$. In particular, we can use Lemma I.2.14 to determine the twisted Borel-Moore homology of $X_{g+1,c}$ and X'_{g+2} . Next, we compute the twisted Borel-Moore homology of X_{g+2} from the long exact sequence associated to the closed inclusion $X_{g+1,c} \hookrightarrow X'_{g+2}$. This gives the Borel-Moore homology of F_{g+2} as in the last column of Table 3.

Lemma I.2.14 also implies that the behaviour of the classes in column $g + 2$ depends on the parity of g . Specifically, if g is odd, then the term in row $4g - 1$ is \mathfrak{S}_2 -invariant and the term in row $4g$ has an invariant one-dimensional subspace; all the rest is anti-invariant. If g is even, the behaviour is reversed: The term in row $4g + 1$ is \mathfrak{S}_2 -invariant and the term in row $4g$ has a one-dimensional \mathfrak{S}_2 -invariant subspace; all the rest is anti-invariant.

By Alexander's duality, this establishes the following:

PROPOSITION 2.4. *The Poincaré-Serre polynomial of Z_g is*

$$1 + u^2t + u^4t^3 + u^6t^4 + u^{4g+2}t^{2g+1} + 2u^{4g+4}t^{2g+2} + u^{4g+6}t^{2g+3}.$$

The Poincaré-Serre polynomial of the \mathfrak{S}_2 -invariant part of the cohomology of Z_g is

$$1 + u^2t + u^4t^3 + u^6t^4 + u^{4g+4}t^{2g+2} + u^{4g+6}t^{2g+3}$$

if g is odd, and

$$1 + u^2t + u^4t^3 + u^6t^4 + u^{4g+2}t^{2g+1} + u^{4g+4}t^{2g+2}$$

if g is even.

TABLE 4. List of singular configurations.

For $k = 1, \dots, \left[\frac{d}{2} \right]$		
$k.a$	The point p and $k - 1$ other points	$d - 2k$
$k.b$	$k + 1$ points different from p	$d - 2k - 1$
If d is even:		
$(\frac{d}{2} + 1).a$	\mathbf{P}^1	0
If d is odd:		
$\frac{d+1}{2}.a$	p and other $\frac{d-1}{2}$ points	1
$\frac{d+1}{2}.b$	\mathbf{P}^1	0

2.2. Collections of points the form $(p, \{q_1, \dots, q_d\})$. In the following, we consider the vector space of homogeneous binary polynomials of degree $d + 1$, vanishing at a fixed point $p \in \mathbf{P}^1$. We identify this vector space with \mathbf{C}^{d+1} . We are interested in computing the cohomology of the locus $F_{1,d} \subset \mathbf{C}^{d+1}$ of non-singular polynomials.

We apply Vassiliev-Gorinov's method. We start by listing in Table 4 all possible singular configurations of elements in $\mathbf{C}^{d+1} \setminus F_{1,d}$.

It is easy to check that only the configurations of types (1.a), (1.b) and (2.a) contribute to the Borel-Moore homology of the discriminant. The reason is that $\bar{H}_\bullet(B(\mathbf{C}, k); \pm \mathbf{Q})$ is trivial for $k > 1$ (see Lemma 2.12 in chapter I).

The spectral sequence converging to the Borel-Moore homology of the discriminant has E^1 term as follows.

$2d - 1$	$\mathbf{Q}(d)$		
$2d - 2$		$\mathbf{Q}(d)$	
$2d - 3$			
$2d - 4$			$\mathbf{Q}(d - 1)$
	1.a	1.b	2.a

This implies that the Poincaré-Serre polynomial of $F_{1,d}$ is $(1 + u^2 t)^2$.

Denote by $\mathcal{N}_{1,d}$ the moduli space of collections of $d + 1$ distinct points of the form $(p, \{q_1, \dots, q_d\}) \in \mathbf{P}^1 \times B(\mathbf{P}^1, d)$. We consider $\mathcal{N}_{1,d}$ as the quotient of the incidence correspondence $\mathcal{I}_{1,d}$ by the action of $\mathrm{GL}(2)$, where

$$\mathcal{I}_{1,d} = \{(p, f) \in \mathbf{P}^1 \times (V_{d+1} \setminus \Sigma_{d+1}) : f(p) = 0\},$$

V_{d+1} is the space of homogeneous polynomials of degree $d + 1$ in two indeterminates x_0, x_1 , and Σ_{d+1} denotes the discriminant locus in V_{d+1} .

PROPOSITION 2.5. *The moduli space $\mathcal{N}_{1,d}$ has the rational cohomology of a point.*

PROOF. We can compute the cohomology of $\mathcal{I}_{1,d}$ by considering its natural projection to \mathbf{P}^1 . This is a locally trivial fibration with fibre isomorphic to $F_{1,d}$, whose Poincaré-Serre polynomial is $(1 + u^2 t)^2$. The Leray spectral sequence associated to the projection of $\mathcal{I}_{1,d}$ to \mathbf{P}^1 is as in Table 5.

By Theorem 5.2, we have $H^\bullet(\mathcal{I}_{1,d}; \mathbf{Q}) \cong H^\bullet(\mathrm{GL}(2); \mathbf{Q}) \otimes H^\bullet(\mathcal{N}_{1,d}; \mathbf{Q})$. This implies that all differentials between the first and the third column of the spectral sequence have maximal possible rank. This proves that $\mathcal{N}_{1,d}$ has the rational cohomology of a point. \square

TABLE 5. Leray spectral sequence converging to $H^\bullet(\mathcal{I}_{1,d}; \mathbf{Q})$.

2	$\mathbf{Q}(-2)$		$\mathbf{Q}(-3)$
1	$\mathbf{Q}(-1)^2$		$\mathbf{Q}(-2)^2$
0	\mathbf{Q}		$\mathbf{Q}(-1)$

3. Hodge Euler characteristic of \mathcal{Q}_2

Recall that we are considering the vector space V of homogeneous polynomials of degree 4 in three indeterminates, and its discriminant locus Σ . Let us fix once and for all two distinct points $p, q \in \mathbf{P}^2$. We denote by V_1 (respectively, V_2) the hyperplane in V of polynomials vanishing at p (respectively, q). We are interested in studying the incidence correspondence

$$\mathcal{I}_2 = \{(f, \alpha, \beta) \in (V \setminus \Sigma \times F(\mathbf{P}^2, 2) : f(\alpha) = f(\beta) = 0)\},$$

with its natural projection π_2 onto $F(\mathbf{P}^2, 2)$. It is a locally trivial fibration with fibre isomorphic to $V_1 \cap V_2 \setminus \Sigma$.

In this section we determine the \mathfrak{S}_2 -equivariant Hodge Euler characteristic $\mathbf{e}^{\mathfrak{S}_2}(\mathcal{Q}_2)$ of \mathcal{Q}_2 , that is, the Euler characteristic of the rational cohomology with compact support of \mathcal{Q}_2 in the Grothendieck group of rational mixed Hodge structures with an action of \mathfrak{S}_2 (for the definition, see Section 2 of chapter II). Since the coarse moduli space \mathcal{Q}_2 has locally quotient singularities, the Hodge Euler characteristic of \mathcal{Q}_2 can be found from the \mathfrak{S}_2 -equivariant Euler characteristic of $H^\bullet(\mathcal{Q}_2; \mathbf{Q})$ in the Grothendieck group of rational mixed Hodge structures. Following the notation of Section 2 of chapter II, we indicate the latter Euler characteristic by $\text{ch}_2(H^\bullet(\mathcal{Q}_2; \mathbf{Q}))$.

We start by computing the \mathfrak{S}_2 -equivariant Euler characteristic of the cohomology of $(V_1 \cap V_2) \setminus \Sigma$ with Vassiliev-Gorinov's method (see Section 2.1 of chapter I). Next, we consider the Leray spectral sequence for the natural projection $\pi_2 : \mathcal{I}_2 \longrightarrow F(\mathbf{P}^2, 2)$. Finally, we use the fact that the rational cohomology of \mathcal{I}_2 is the tensor product of the rational cohomology of \mathcal{Q}_2 and that of $\text{GL}(3)$ (by Theorem II.5.2) to deduce from the previous results what the \mathfrak{S}_2 -equivariant Euler characteristic of the cohomology of \mathcal{Q}_2 in the Grothendieck group of $\text{MHS}_{\mathbf{Q}}$ is.

3.1. Quartic curves through two fixed points. By Alexander's duality, we have

$$\tilde{H}^\bullet((V_1 \cap V_2) \setminus \Sigma; \mathbf{Q}) \cong \bar{H}_{25-\bullet}(V_1 \cap V_2 \cap \Sigma; \mathbf{Q})(-13).$$

The Borel-Moore homology of $V_1 \cap V_2 \cap \Sigma$ coincides with that of the domain of the geometric realization

$$|\epsilon| : |\mathcal{X}(\bullet)| \longrightarrow \mathcal{X}(\emptyset) = V_1 \cap V_2 \cap \Sigma,$$

where the cubical space $\mathcal{X}(\bullet)$ is constructed as in Chapter I, Section 2.1, starting from an ordered list of all possible singular sets of the elements in $V_1 \cap V_2 \cap \Sigma$. We can easily obtain such a list by an adaptation of the list of possible singular configurations of quartic curves passing through a fixed point given in Section 5.3 of chapter II. In the new situation, we will have to distinguish further whether the singular points are in general position with

respect to p and q , or not (the easiest example of the latter situation is when p or q are contained in the singular configuration).

Analogously as in Section 5.3 of chapter II, we will have that most types of configurations correspond to strata with trivial (twisted) Borel-Moore homology. This is the case for types of configurations defining a configurations space with trivial twisted Borel-Moore homology. This is the case, for all spaces of configurations of ≥ 3 points on a projective space, for spaces of configurations of ≥ 2 points on an affine complex space, and for spaces directly related to these. In such cases, the fact that the contribution of the stratum is trivial follows from Lemma I.2.13 or from Lemma I.2.12. Other types of configurations giving trivial contribution are those containing a rational curve (see Lemma I.2.19).

We list below in Table 6 all remaining configurations, i.e., all singular configurations indexing strata that give a non-trivial contribution to the Borel-Moore homology of the domain of the geometric realization. In the same table, we also give a description of the strata of $|\Lambda(\bullet)|$ and $|\mathcal{X}(\bullet)|$ corresponding to each configuration. The most difficult strata (corresponding to configurations of type 8, 9 and 10) are studied separately in Section 3.2. From the description of the strata and the results of Section 3.2, it is straightforward to compute the \mathfrak{S}_2 -equivariant Euler characteristics of the strata Φ_j of the domain of the geometric realization of the cubical space $\Lambda(\bullet)$. The same can be said for the strata F_j of the domain of the geometrical realization of $\mathcal{X}(\bullet)$. These Euler characteristics are listed in Table 7.

The \mathfrak{S}_2 -equivariant Euler characteristic of the Borel-Moore homology of the discriminant in the Grothendieck group of mixed Hodge structures can be obtained by taking the sum of the Euler characteristics of all strata F_j . This yields

$$\begin{aligned} \text{ch}_2(\bar{H}_\bullet(\Sigma, \mathbf{Q})) = & (\mathbf{L}^{-12} + 2\mathbf{L}^{-11} - 2\mathbf{L}^{-10} - \mathbf{L}^{-9} + \mathbf{L}^{-8} - \mathbf{L}^{-7} + \mathbf{L}^{-6} + 2\mathbf{L}^{-5} \\ & - 2\mathbf{L}^{-4} - \mathbf{L}^{-3} + \mathbf{L}^{-2})s_2 + (\mathbf{L}^{-11} - 2\mathbf{L}^{-10} + 2\mathbf{L}^{-8} - \mathbf{L}^{-7})s_{1,1}. \end{aligned}$$

Alexander's duality now implies that the \mathfrak{S}_2 -equivariant Euler characteristic of the rational cohomology of $(V_1 \cap V_2) \setminus \Sigma$ in $K_0(\text{MHS}_{\mathbf{Q}})$ is

$$\begin{aligned} (2) \quad \text{ch}_2(H^\bullet((V_1 \cap V_2) \setminus \Sigma; \mathbf{Q})) \\ = (1 - \mathbf{L})(1 - 2\mathbf{L}^2 + \mathbf{L}^4 + \mathbf{L}^6 - 2\mathbf{L}^8 + \mathbf{L}^{10})s_2 + (1 - \mathbf{L})(-\mathbf{L}^2 + \mathbf{L}^3 + \mathbf{L}^4 - \mathbf{L}^5)s_{1,1}. \end{aligned}$$

Next, we compute $\text{ch}_2(H^\bullet(\mathcal{I}_2; \mathbf{Q}))$ by studying the fibration $\pi_2 : \mathcal{I}_2 \longrightarrow F(\mathbf{P}^2, 2)$. The space $F(\mathbf{P}^2, 2)$ is simply connected and if we consider the action of \mathfrak{S}_2 on $F(\mathbf{P}^2, 2)$ generated by the involution interchanging $(f, \alpha, \beta) \in \mathcal{I}_2$ and (f, β, α) , we have

$$\text{ch}_2(H^\bullet(\mathcal{I}_2; \mathbf{Q})) = (1 + \mathbf{L} + \mathbf{L}^2)(s_2 + \mathbf{L}s_{1,1}).$$

Considering the Leray spectral sequence in rational cohomology associated with π_2 yields the equality

$$\begin{aligned} \text{ch}_2(H^\bullet(\mathcal{I}_2; \mathbf{Q})) &= \text{ch}_2(H^\bullet((V_1 \cap V_2) \setminus \Sigma; \mathbf{Q})) \otimes \text{ch}_2(H^\bullet(F(\mathbf{P}^2, 2); \mathbf{Q})) \\ &= (1 - \mathbf{L})(1 - \mathbf{L}^2)(1 - \mathbf{L}^3) ((1 + \mathbf{L} - \mathbf{L}^3 + \mathbf{L}^6 + \mathbf{L}^7)s_2 + (\mathbf{L} + \mathbf{L}^7 + \mathbf{L}^8)s_{1,1}). \end{aligned}$$

Since the Euler characteristic of the rational cohomology of $\text{GL}(3)$ is precisely $(1 - \mathbf{L})(1 - \mathbf{L}^2)(1 - \mathbf{L}^3)$, the result above gives the \mathfrak{S}_2 -equivariant Euler characteristic of \mathcal{Q}_2 in the Grothendieck group of mixed Hodge structures. Passing from cohomology

TABLE 6. Singular configurations and corresponding strata.

	configuration	corresponding stratum
1	The point p or the point q .	F_1 is a \mathbf{C}^{11} -bundle over $\Phi_1 = \{p, q\}$
2	Any point different from p, q .	F_2 is a \mathbf{C}^{10} -bundle over $\Phi_2 \cong \mathbf{P}^2 \setminus \{p, q\}$.
3	The pair $\{p, q\}$.	F_3 is a \mathbf{C}^9 -bundle over $\Phi_3 \cong \mathring{\Delta}_1$.
4a	Pairs of points on the line pq , different from $\{p, q\}$.	F_{4a} is a \mathbf{C}^8 -bundle over Φ_{4a} , which is a non-orientable $\mathring{\Delta}_1$ -bundle over a space which can be decomposed as the disjoint union of \mathbf{C}^* and $B(\mathbf{C}, 2)$.
4b	Pairs of points $\{a, b\}$ with $a \in \{p, q\}$, $b \notin pq$.	F_{4b} is a \mathbf{C}^8 -bundle over Φ_{4b} , which is a non-orientable $\mathring{\Delta}_1$ -bundle over the disjoint union of two copies of \mathbf{C}^2 .
5	Pairs of points $\{a, b\}$ with $a \in (pq \setminus \{p, q\})$, $b \notin pq$.	F_5 is a \mathbf{C}^7 -bundle over Φ_5 , which is a non-orientable $\mathring{\Delta}_1$ -bundle over $\mathbf{C}^* \times \mathbf{C}^2$.
6	Triplets consisting of p, q and another point outside pq .	F_6 is a \mathbf{C}^6 -bundle over Φ_6 , which is a $\mathring{\Delta}_2$ -bundle over \mathbf{C}^2 .
7	Triplets with two points on pq (not both in $\{p, q\}$) and another point outside pq .	F_7 is a \mathbf{C}^5 -bundle over Φ_7 , which is a non-orientable bundle over a space that can be decomposed as the union of $\mathbf{C}^* \times \mathbf{C}^2$ and $B(\mathbf{C}, 2) \times \mathbf{C}^2$.
8	Five points $a, b, c, d, e \in \mathbf{P}^2$, such that a, b, d, e, p, q lie on a conic different from $ab \cup de$, $\{c\} = ab \cap de \not\subset \{p, q\}$ and $\{p, q\} \not\subset \{a, b, d, e\}$.	F_8 is a \mathbf{C} -bundle over Φ_8 , which is a $\mathring{\Delta}_4$ -bundle over the configuration space X_8 of Section 3.2.
9	Six points that are the pairwise intersection of pq and three more lines in general position.	F_9 and F_{10} are \mathbf{C} -bundles over Φ_9 and Φ_{10} respectively, which are $\mathring{\Delta}_5$ -bundles over the configuration spaces X_9 and X_{10} of Section 3.2.
10	Six points that are the pairwise intersection of a set of 4 lines in general position, all different from pq .	In both cases, the simplices bundle does not change its orientation when two lines in the configuration are interchanged.
11	The entire \mathbf{P}^2 .	F_{11} is an open cone over the space $ \Lambda(\bullet) $, which is the union of all strata Φ_j with $j \leq 10$.

to cohomology with compact supports, we get the following, where we have used the computation of the rational cohomology of $\mathcal{H}_{3,2}$ in Corollary 2.2.

THEOREM 3.1. *The \mathfrak{S}_2 -equivariant Hodge Euler characteristic of \mathcal{Q}_2 is $(\mathbf{L}^8 + \mathbf{L}^7 - \mathbf{L}^5 + \mathbf{L}^2 + \mathbf{L})s_2 + (\mathbf{L}^7 + \mathbf{L} + 1)s_{1,1}$.*

TABLE 7. Euler characteristics in $K_0(\mathbf{MHS}_Q)$ of the strata.

j	$\text{ch}_2(H_\bullet(F_j, \mathbf{Q}))$	$\text{ch}_2(H_\bullet(\Phi_j, \mathbf{Q}))$
1	$\mathbf{L}^{-11}s_2 + \mathbf{L}^{-11}s_{1,1}$	$s_2 + s_{1,1}$
2	$(\mathbf{L}^{-12} + \mathbf{L}^{-11})s_2 - \mathbf{L}^{-10}s_{1,1}$	$(\mathbf{L}^{-2} + \mathbf{L}^{-1})s_2 - s_{1,1}$
3	$-\mathbf{L}^{-9}s_{1,1}$	$-s_{1,1}$
4	$(-\mathbf{L}^{-10} - \mathbf{L}^{-9})s_2 + (-\mathbf{L}^{-10} + \mathbf{L}^{-8})s_{1,1}$	$(-\mathbf{L}^{-2} - \mathbf{L}^{-1})s_2 + (-\mathbf{L}^{-2} + 1)s_{1,1}$
5	$-\mathbf{L}^{-10}s_2 + \mathbf{L}^{-9}s_{1,1}$	$-\mathbf{L}^{-3}s_2 + \mathbf{L}^{-2}s_{1,1}$
6	$\mathbf{L}^{-8}s_{1,1}$	$\mathbf{L}^{-2}s_{1,1}$
7	$\mathbf{L}^{-8}s_2 - \mathbf{L}^{-7}s_{1,1}$	$\mathbf{L}^{-3}s_2 - \mathbf{L}^{-2}s_{1,1}$
8	$(\mathbf{L}^{-7} - 2\mathbf{L}^{-6} + \mathbf{L}^{-5})s_2 + (-\mathbf{L}^{-6} + 2\mathbf{L}^{-5} - \mathbf{L}^{-4})s_{1,1}$	$(\mathbf{L}^{-6} - 2\mathbf{L}^{-5} + \mathbf{L}^{-4})s_2 + (-\mathbf{L}^{-5} + 2\mathbf{L}^{-4} - \mathbf{L}^{-3})s_{1,1}$
9	$(-\mathbf{L}^{-7} + \mathbf{L}^{-6} + \mathbf{L}^{-5} - \mathbf{L}^{-4})s_2$	$(-\mathbf{L}^{-6} + \mathbf{L}^{-5} + \mathbf{L}^{-4} - \mathbf{L}^{-3})s_2$
10	$(-\mathbf{L}^{-7} + \mathbf{L}^{-6} + \mathbf{L}^{-4} - \mathbf{L}^{-3})s_2 + (\mathbf{L}^{-6} - 2\mathbf{L}^{-5} + \mathbf{L}^{-4})s_{1,1}$	$(-\mathbf{L}^{-6} + \mathbf{L}^{-5} + \mathbf{L}^{-3} - \mathbf{L}^{-2})s_2 + (\mathbf{L}^{-5} - 2\mathbf{L}^{-4} + \mathbf{L}^{-3})s_{1,1}$
Total 1–10: $(1 - \mathbf{L}^{-6} + 2\mathbf{L}^{-4} - \mathbf{L}^{-2})s_2$		
11	$(\mathbf{L}^{-6} - 2\mathbf{L}^{-4} + \mathbf{L}^{-2})s_2$	$(\mathbf{L}^{-6} - 2\mathbf{L}^{-4} + \mathbf{L}^{-2})s_2$

The \mathfrak{S}_2 -equivariant Hodge Euler characteristic of $\mathcal{M}_{3,2}$ is $(\mathbf{L}^8 + 2\mathbf{L}^7 + \mathbf{L}^6 - \mathbf{L}^5 + \mathbf{L}^2 + \mathbf{L})s_2 + (\mathbf{L}^7 + \mathbf{L}^6 + \mathbf{L})s_{12}$.

Note that we recovered Theorem II.1.3, obtained from Jonas Bergström's equivariant count of the number of points of \mathcal{Q}_2 and $\mathcal{M}_{3,2}$. In particular, with the same techniques of Theorem II.1.1, Theorem 3.1 implies the following.

COROLLARY 3.2. *The \mathfrak{S}_2 -equivariant Hodge Euler characteristic of $\overline{\mathcal{M}}_{3,2}$ is*

$$(\mathbf{L}^8 + 7\mathbf{L}^7 + 31\mathbf{L}^6 + 74\mathbf{L}^5 + 100\mathbf{L}^4 + 74\mathbf{L}^3 + 31\mathbf{L}^2 + 7\mathbf{L} + 1)s_2 + (2\mathbf{L}^7 + 11\mathbf{L}^6 + 30\mathbf{L}^5 + 42\mathbf{L}^4 + 30\mathbf{L}^3 + 11\mathbf{L}^2 + 2\mathbf{L})s_{1,1}.$$

3.2. Configurations of five and six points. The aim of this section is to compute the contribution of the strata Φ_8 , Φ_9 and Φ_{10} to the Borel-Moore homology of $|\Lambda(\bullet)|$. We start by determining the twisted Borel-Moore homology of the underlying families of configurations X_8 and X_9 .

Recall that X_8 is defined as the space of configurations of five points $a, b, c, d, e \in \mathbf{P}^2$, such that a, b, d, e, p, q lie on a conic different from $ab \cup de$, $c = ab \cap de \not\subset \{p, q\}$ and $\{p, q\} \not\subset \{a, b, d, e\}$. In this case, the conic is uniquely determined, so that X_8 is isomorphic to the space of pairs $(C, \{l_1, l_2\})$ such that

- (1) C is a conic through p, q , l_1 and l_2 are lines not tangent to C ;
- (2) there are exactly five distinct points that are pairwise intersection of C , l_1 and l_2 ;
- (3) $\{p, q\} \not\subset C \cap (l_1 \cup l_2)$.

Let us define W to be the space of pairs $(C, \{l_1, l_2\})$ that satisfy only conditions 1 and 2.

LEMMA 3.3. *For the rank one local system τ on W and X_8 changing its sign when an odd permutation of the points of the set $C \cap (l_1 \cup l_2)$ occurs, and the action of \mathfrak{S}_2 interchanging the points p and q , we have*

- (1) $\text{ch}_2(\bar{H}_\bullet(W; \tau)) = (\mathbf{L}^{-6} - \mathbf{L}^{-5})s_2 - (\mathbf{L}^{-5} - \mathbf{L}^{-4})s_{1,1}.$
- (2) $\text{ch}_2(\bar{H}_\bullet(X_8; \tau)) = (\mathbf{L}^{-6} - 2\mathbf{L}^{-5} + \mathbf{L}^{-4})s_2 - (\mathbf{L}^{-5} - 2\mathbf{L}^{-4} + \mathbf{L}^{-3})s_{1,1}.$

To prove Lemma 3.3, we will reduce to the following situation. Let us fix four points in general position in \mathbf{P}^2 , and denote them by E_1, E_2, E_3 and E_4 . Denote by L the space of conics passing through the E_i 's and distinct from the reducible conic $\overline{E_1 E_2} \cup \overline{E_3 E_4}$. Note that L is isomorphic to an affine line. We define the following incidence correspondence:

$$R = \{(C, \alpha, \beta) \in L \times F(\mathbf{P}^2, 2) : p, q \in L\}.$$

Let us consider the subgroup G of \mathfrak{S}_4 generated by the permutations $(1, 2), (3, 4)$ and $(1, 3)(2, 4)$. We can identify each element σ of G with an automorphism of \mathbf{P}^2 mapping E_i to $E_{\sigma(i)}$. An appropriate choice of these identifications allows us to consider G as a subgroup of $\text{Aut}(\mathbf{P}^2)$. Since G is a subgroup of \mathfrak{S}_4 , it makes sense to restrict the sign representation to it.

LEMMA 3.4. *Consider the quotient of R by the action of G . Then the Borel-Moore homology of R/G in the local system of coefficients S induced by the sign representation on G is invariant with respect to the \mathfrak{S}_2 -action interchanging α and β , and has Poincaré-Serre polynomial $u^{-4}t^4$.*

PROOF. The space $Q = R/G$ can be decomposed as the union of a closed locus K containing all equivalence classes of triples (C, α, β) such that C is a singular conic, and an open part U , where the conic C is always non-singular. The locus K has two components, according to the position of the two points α, β . We denote by M the component of K such that α, β lie on the same irreducible component of C , and N the component in which α, β lie on two distinct components of C . The intersection $M \cap N$ corresponds to configurations in which either α or β is the singular point of C . We consider the following increasing filtration on Q :

$$Q_1 := M, \quad Q_2 := M \cup N, \quad Q_3 := Q.$$

Up to the G -action, the space M can be identified with the space of ordered configurations of two points on the projective line $\overline{E_1 E_3}$, hence the \mathfrak{S}_2 -equivariant Euler characteristic of $\bar{H}_\bullet(M; S)$ in $K_0(\mathbf{MHS}_\mathbf{Q})$ is $\mathbf{L}^{-2}s_2 + \mathbf{L}^{-1}s_{1,1}$.

Next, we identify the space $N \setminus M$ with a \mathfrak{S}_2 -quotient of the space of pairs (α, β) where α lies on $\overline{E_1 E_3}$, the point β lie on $\overline{E_2 E_4}$ and both points are distinct from the intersection points of these lines. The \mathfrak{S}_2 -action interchanges E_1 and E_3 , and E_2 and E_4 , and we have to take invariant classes with respect to it. This implies that the only Borel-Moore homology of $N \setminus M$ with S -coefficients is $\mathbf{Q}(2)$ in degree 4, and is invariant with respect to the involution interchanging the points α and β in the configurations.

Finally, we compute the Borel-Moore homology of U by lifting U to a G -invariant subset $U' \subset R$, and looking for the part of the Borel-Moore homology of U' that has the desired behaviour with respect to the G -action. We have that U' projects to the locus of non-singular conics in L , which is isomorphic to $\mathbf{C} \setminus \{\pm 1\}$. Note that the action of

TABLE 8.

3	$\mathbf{Q}(2)$			
2		$\mathbf{Q}(2)$	$\mathbf{Q}(2)$	
1	$\mathbf{Q}(1)$			
0			$\mathbf{Q}(1)$	
	1	2	3	

$(1, 2) \in G$ on $L \cong \mathbf{C}$ interchanges the two singular conics. The projection $U' \rightarrow \mathbf{C} \setminus \{\pm 1\}$ is a locally trivial fibration, with fibre $F(\mathbf{P}^1, 2)$. As we already said, the whole Borel-Moore homology of $F(\mathbf{P}^1, 2)$ is G -invariant, hence we have to consider the part of the Borel-Moore homology of $\mathbf{C} \setminus \{\pm 1\}$ which is anti-invariant for the involution $t \leftrightarrow -t$, which is \mathbf{Q} in degree 1. Note that this Borel-Moore homology group is invariant for the involution interchanging (α, β) and (β, α) .

Concluding, the spectral sequence in Borel-Moore homology with S -coefficients associated to the filtration $\{Q_i\}$ has E^1 term as in Table 8. From this the claim follows. \square

PROOF OF LEMMA 3.3. If we consider the action of the stabilizer of the points p and q in \mathbf{P}^2 , we have that there is a surjective map from W to the space R/G of Lemma 3.4. The geometric interpretation of this map is as follows. The elements of W are determined by a conic and two lines. If we choose a point (C, α, β) of R , every automorphism f of \mathbf{P}^2 mapping α to p and β to q gives a point of W , where the conic is $f(C)$ and the two lines are $f(\overline{E_1 E_2})$ and $f(\overline{E_3 E_4})$. Moreover, every point of W can be obtained in this way, and two points of R have the same image in W if and only if they are in the same class modulo G .

This implies that there is a natural map $W \rightarrow R/G$, which is a locally trivial fibration. Its fibre is the space of automorphisms of \mathbf{P}^2 mapping two fixed points to p, q , which is homeomorphic to the stabilizer $\text{Stab}(p, q)$ of p, q in $\text{PGL}(3)$. Considering the Leray spectral sequence in Borel-Moore homology associated to this fibration, together with Lemma 3.4, yields (1) in Lemma 3.3.

Next, the Borel-Moore homology of the space $W \setminus X_8$ can be described as follows. First of all, it is easy to prove that the subset of $W \setminus X_8$ such that p, q lie both on l_1 (or on l_2) has trivial Borel-Moore homology in the system of coefficients we are considering, which changes its orientation every time two points of intersection are interchanged. We remain with the locus of pairs $(C, \{l_1, l_2\})$ such that the line l_1 passes through p , the line l_2 passes through q , and both lines are different from pq . Once l_1 and l_2 are chosen (and their choice is parametrized by a \mathbf{C}^2), the choice of C is parametrized by a \mathbf{C} -bundle over $(\mathbf{C}^*)^2$. The action of \mathfrak{S}_2 interchanging p and q translates into the interchange of the two factors \mathbf{C}^* . As a consequence, the Borel-Moore homology of $W \setminus X_8$ has Poincaré-Serre polynomial $u^{-6}t^8 + 2u^{-8}t^9 + u^{-10}t^{10}$. The part of the Borel-Moore homology which is invariant for the action of \mathfrak{S}_2 has Poincaré-Serre polynomial $u^{-8}t^9 + u^{-10}t^{10}$.

Then (2) follows from part (1) and from the long exact sequence associated to the closed inclusion $X_8 \subset W$. \square

Recall from Section 3.1 that the space X_9 is the space of configurations of six points which are the pairwise intersection of four lines in general position, such that p and q are contained in the union of the lines.

LEMMA 3.5. *The Euler characteristic of the Borel-Moore homology of X_9 with values in $K_0(\text{MHS}_{\mathbf{Q}}^{\mathfrak{S}_2})$ is $(\mathbf{L}^{-6} - \mathbf{L}^{-5} - \mathbf{L}^{-4} + \mathbf{L}^{-3})s_2$.*

PROOF. If we consider a configuration in X_9 , we have that the three lines different from pq are determined by the position of their three intersection points. These three points may be any configuration of three points in $\mathbf{P}^2 \setminus pq$ not lying on the same line. We have

$$\bar{H}_\bullet(\tilde{B}(\mathbf{C}^2, 3); \mathbf{Q}) \cong \bar{H}_\bullet(\tilde{F}(\mathbf{C}^2, 3); \mathbf{Q})$$

i.e., in this case it is equivalent to work with ordered triples of points. The configuration space $F(\mathbf{C}^2, 3)$ of ordered triple of points in general position is isomorphic to the space of affine transformations of \mathbf{C}^2 , which is $\mathbf{C}^2 \times \text{GL}(2)$. This implies that the Borel-Moore homology of X_9 has Poincaré-Serre polynomial $u^{-6}t^8 + u^{-8}t^9 + u^{-10}t^{11} + u^{-12}t^{12}$, and is all invariant for the involution interchanging p and q . \square

LEMMA 3.6. *If we consider the action of \mathfrak{S}_2 induced by the interchange of the points p and q , the Euler characteristic in $K_0(\text{MHS}_{\mathbf{Q}}^{\mathfrak{S}_2})$ of the Borel-Moore homology of X_{10} is*

$$(\mathbf{L}^{-6} - \mathbf{L}^{-5} - \mathbf{L}^{-3} + \mathbf{L}^{-2})s_2 + (-\mathbf{L}^{-5} + 2\mathbf{L}^{-4} - \mathbf{L}^{-3})s_{1,1}.$$

PROOF. When one is working with ordered configurations of lines, choosing four lines in general position in \mathbf{P}^2 determines a reference system in \mathbf{P}^2 . This means that choosing four lines through p and q is the same as choosing two points a, b on $L := \{x_0x_1x_2(x_0 + x_1 + x_2) = 0\}$ and a change of coordinates mapping the point a to p and b to q . As we saw in the proof of Lemma 3.3, the space of these changes of coordinates is always isomorphic to $\text{Stab}(p, q) \subset \text{PGL}(3)$. Hence, the space of ordered quadruples of lines passing through p and q can be described as the total space Y of a locally trivial fibration over $F(L, 2)$, with fibre F isomorphic to the configuration space

$$F = \{(u_1, u_2) \in F(\mathbf{P}^2, 2) : p, q, u_1, u_2 \text{ are four points in general position}\}.$$

As the projection to $F(L, 2)$ is the restriction of the analogous map $\tilde{F}(\mathbf{P}^{2\vee}, 4) \rightarrow F(\mathbf{P}^2, 2)$, and $F(\mathbf{P}^2, 2)$ is simply connected, all Borel-Moore homology groups of F must induce the standard local system on $F(L, 2)$.

The configuration space $X_9 \cup X_{10}$ is the quotient of Y by the action of \mathfrak{S}_4 obtained by identifying every permutation $\sigma \in \mathfrak{S}_4$ with an automorphism of \mathbf{P}^2 sending the line l_i to l_{σ_i} for all i , $1 \leq i \leq 4$, where the lines l_i are the components of L .

Let us see what these considerations tell us about the Borel-Moore homology of X_{10} . The space X_{10} corresponds to the \mathfrak{S}_4 -quotient of the preimage of the locus of configurations of two points (a, b) not lying on the same component of L , in the map $Y \rightarrow F(L, 2)$. This locus can be decomposed according to whether a and b are or are not singular points of L into the loci

$$S_1 := \{(a, b) \in F(L, 2) : a \text{ and } b \text{ are both singular points}\},$$

$$S_2 := \{(a, b) \in F(L, 2) : \text{only one of the points } a \text{ and } b \text{ is a singular point of } L\},$$

$$S_3 := \{(a, b) \in F(L, 2) : a \text{ and } b \text{ are non-singular points of } L\}.$$

The quotient S_1/\mathfrak{S}_4 consists of only one point, the class of the pair $([1, 0, 0], [0, 1, -1])$. The quotient S_2/\mathfrak{S}_4 has two isomorphic components, according to which point (a or b) is a singular point of L . Consider the case in which a is singular. Up to the action of \mathfrak{S}_4 , we can assume that a is the point $[1, 0, 0]$ and b lies on $x_0 = 0$. By the definition of S_2 we know that b is different from the points $[0, 1, -1]$, $[0, 0, 1]$ and $[0, 1, 0]$. Note that, since we are working modulo \mathfrak{S}_4 , the coordinates of b are defined up to the involution interchanging x_1 and x_2 . This proves that both components of S_2/\mathfrak{S}_4 are isomorphic to \mathbf{C}^* .

Finally, we determine the Borel-Moore homology of the quotient of S_3 by the action of \mathfrak{S}_4 . Up to the action of the group, we can assume that a lies on the line l_3 and b on l_4 . The position of both points is determined up to the involution interchanging the lines l_1 and l_2 . If we identify l_3 and l_4 with \mathbf{P}^1 , and $l_3 \cap l_4$ with the point at infinity of these projective lines, we have that S_2/\mathfrak{S}_4 can be embedded into the quotient of $(\mathbf{C} \setminus \{\pm 1\})^2$ by the relation $(t, s) \sim (-t, -s)$. The complement of S_2/\mathfrak{S}_4 in this quotient is the locus such that either t or s are equal to ± 1 . We can study $(\mathbf{C} \setminus \{\pm 1\})^2 / \sim$ as follows:

$$\begin{array}{ccccccc} \mathbf{C}^2 & \xrightarrow{\text{mod } \sim} & \{(x, y, z) \in \mathbf{C}^3 : y^2 = xz\} & \xrightarrow{\text{mod } \mathfrak{S}_2} & \mathbf{C}^2 \\ (t, s) & \mapsto & (t^2, ts, s^2) & \mapsto & (t^2 + s^2, ts) \\ (1, s) & \mapsto & (1, s, s^2) & \mapsto & (s^2 + 1, s) \\ (t, 1) & \mapsto & (t^2, t, 1) & \mapsto & (t^2 + 1, t), \end{array}$$

where the second map denotes the quotient by the action of \mathfrak{S}_2 interchanging t and s . Hence we have $\text{ch}_2(\bar{H}_\bullet(S_3/\mathfrak{S}_4; \mathbf{Q})) = (\mathbf{L}^{-2} - \mathbf{L}^{-1})s_2 + (-\mathbf{L}^{-1} + 2)s_{1,1}$.

Recall from the proof of Lemma 3.3 that the Euler characteristic of the Borel-Moore homology of the fibre of $Y \rightarrow F(L, 2)$ in $\text{MHS}_\mathbf{Q}^{\mathfrak{S}_2}$ is $(\mathbf{L}^{-4} - \mathbf{L}^{-3})s_2 - (\mathbf{L}^{-3} - \mathbf{L}^{-2})s_{1,1}$. Taking the tensor product of this Euler characteristic with the sum of the Euler characteristics of the strata S_i yields the claim. \square

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Samenvatting

Dit proefschrift gaat over de cohomologie van moduliruimtes van krommen, in gevallen waar er een verband bestaat tussen de cohomologie van de moduliruimte en die van het complement van een zekere discriminant in een vectorruimte over het lichaam der complexe getallen. De vectorruimtes V die we gebruiken zijn altijd ruimtes van sneden van een vectorbundel over een algebraïsche variëteit Z . Dat betekent dat iedere vector in V een deelvariëteit van Z definiëert, gegeven door de punten waar de snede verdwijnt. De *discriminant* is dan de verzameling van de vectoren in V die een geassocieerde deelvariëteit hebben die singulier is, of die een te grote dimensie heeft. De discriminant is een gesloten algebraïsche deelverzameling van V , en heeft codimensie 1 in V in alle gevallen waar we in geïnteresseerd zijn.

De moduliruimtes waar we het over hebben zijn moduliruimtes van gladde krommen van een bepaald geslacht g met een aantal n gemarkeerde punten. Voor ieder paar gehele positieve getallen g, n zodanig dat $2g + n - 2 > 0$ bestaat er een moduliruimte $\mathcal{M}_{g,n}$, dat wil zeggen, een quasi-projectieve variëteit $\mathcal{M}_{g,n}$ die de klassen van isomorfisme van irreducibele gladde krommen van geslacht g met n gemarkeerde punten parametriseert. We zijn ook geïnteresseerd in de moduliruimtes van stabiele krommen. De moduliruimte $\overline{\mathcal{M}}_{g,n}$ van stabiele krommen van geslacht g met n gemarkeerde punten is een compactificatie van $\mathcal{M}_{g,n}$. Het begrip van stabiele krommen bevat, behalve gladde krommen, ook alle ontaardingen van gladde krommen met knopen als singulariteiten en zodanig dat de gemarkeerde punten gladde punten van de krommen zijn.

Onze resultaten betreffen de cohomologie met coëfficiënten in het lichaam \mathbf{Q} der rationale getallen (deze cohomologie wordt in het vervolg “rationale cohomologie” genoemd). Om de cohomologie van een ruimte van de vorm $\mathcal{M}_{g,n}$ te bestuderen, beginnen we om $\mathcal{M}_{g,n}$ te schrijven als de vereniging van locaal-afgesloten strata die het quotient zijn van het complement van de discriminant voor de actie van een bepaalde algebraïsche groep. Dat betekent dat we eigenlijk met de studie van moduliruimtes bezig zijn die kleiner zijn dan de hele $\mathcal{M}_{g,n}$. Op zo'n kleinere moduliruimte geldt de eigenschap dat iedere kromme gedefinieerd is door het verdwijnen van een polynoom, dat tot een vectorruimte V behoort. Aangezien we werken met moduliruimtes van gladde krommen, moeten we alle elementen van V die singuliëre krommen geven, weglaten. In andere woorden werken we niet met V , maar met het complement van de discriminant in V .

Het verband tussen de cohomologie van de moduliruimtes en die van het complement van de discriminant is gegeven door een stelling van Peters en Steenbrink ([PS03]) die garandeert dat onder bepaalde hypotheses de rationale cohomologie van het complement van de discriminant met het tensorproduct van de rationale cohomologie van de moduliruimte en die van de groep overeenkomt. Dit isomorfisme bewaart niet de ringstructuur van de cohomologiegroepen, maar wel hun gemengde Hodge structuren. Om de rationale cohomologie van de moduliruimte te weten te komen, is het dus voldoende

om de cohomologie van het complement van de discriminant uit te rekenen. Voor dit soort berekening gebruiken we altijd een methode van de topoloog Vassiliev (zie [Vas92], [Vas99]), in de versie ontworpen door Gorinov ([Gor05]).

In Hoofdstuk I worden de algemene technieken die we gebruiken, uitgelegd. Ze worden ook meteen toegepast om de rationale cohomologie van de moduliruimte \mathcal{M}_4 van gladde krommen van geslacht 4 te berekenen (Stelling I.1.4). Dit resultaat wordt bereikt door \mathcal{M}_4 in drie strata te verdelen, afhankelijk van het canonieke beeld van de krommen.

We kunnen de methodes van Hoofdstuk I ook gebruiken om resultaten over de cohomologie van moduliruimtes van krommen met gemarkeerde punten te bewijzen. Hoe de constructie aangepast kan worden, is in sectie 5.1 van Hoofdstuk II uitgelegd. Op deze manier kunnen we de rationale cohomologie van $\mathcal{M}_{3,1}$ bepalen (Gevolg II.5.5), en ook die van de moduliruimte van hyperelliptische krommen van geslacht $g \geq 2$ met twee gemarkeerde punten (Stelling III.2.2). In het geval van $\mathcal{M}_{3,2}$ zijn deze methodes niet sterk genoeg om een volledig resultaat over de rationale cohomologie op te leveren. Ze zijn wel voldoende om de Eulerkarakteristiek te krijgen van de rationale cohomologie van $\mathcal{M}_{3,2}$ met waarden in de Grothendieckgroep van gemengde Hodge structuren over \mathbf{Q} (Stelling III.3.1).

In Hoofdstuk II, dat samen met Jonas Bergström (KTH, Stockholm) geschreven is, berekenen we de rationale cohomologie van de moduliruimte $\overline{\mathcal{M}}_4$ van stabiele krommen van geslacht 4. Dit resultaat is gebaseerd op de stratificatie van $\overline{\mathcal{M}}_4$ volgens het topologische type van de krommen. Deze stratificatie kunnen we als volgt beschrijven. Ten eerste definiëren we $U_k \subset \overline{\mathcal{M}}_4$ voor alle $k \geq 0$ als de verzameling van alle klassen van isomorfisme van stabiele krommen met precies k singuliere punten. De stratificatie volgens topologisch type heeft als strata de irreducibele componenten van de U_k . Bijvoorbeeld, $U_0 = \mathcal{M}_4$ is irreducibel en is dus een van de strata. Uit U_1 krijgen we drie componenten. De eerste bevat de klassen van isomorfisme van irreducibele krommen met een singuliere punt, de tweede de klassen van isomorfisme van krommen met een component van geslacht 3 en een component van geslacht 1 die elkaar in een punt snijden, en de derde component van U_1 bevat de klassen van isomorfisme van krommen met twee componenten van geslacht 2, die elkaar in een punt snijden.

Het is bekend dat alle strata van de stratificatie volgens topologisch type het product zijn van moduliruimtes van gladde krommen van geslacht $g \leq 4$, of het quotient van zo'n product door de actie van een eindige groep. Dat geeft het idee dat er een verband is tussen de cohomologie van $\overline{\mathcal{M}}_4$ en die van moduliruimtes van gladde krommen van geslacht ≤ 4 . De ruimte $\overline{\mathcal{M}}_4$ is compleet en voldoet aan de dualiteit van Poincaré. Daarom is het mogelijk de rationale cohomologie van \mathcal{M}_4 af te lezen uit de Eulerkarakteristiek van de cohomologie met compacte dragers van $\overline{\mathcal{M}}_4$ met waarden in de Grothendieckgroep $K_0(\mathbf{MHS}_{\mathbf{Q}})$ van de categorie van rationale Hodge structuren. Voor Eulerkarakteristieken is er wel een formule door Getzler en Kapranov ([GK98, 8.13]) die het verband tussen de Eulerkarakteristiek van de cohomologie met compacte dragers van $\overline{\mathcal{M}}_4$ en die van de ruimtes $\mathcal{M}_{g,n}$ met $g \leq 4$ en $n \leq 8 - 2g$ belicht.

Vele van de Eulerkarakteristieken van deze $\mathcal{M}_{g,n}$ zijn al bekend (zie [Get95], [Get99], [Get98b], [Loo93] en [GL]). De ontbrekende Eulerkarakteristieken volgen of uit de resultaten van dit proefschrift, of uit de resultaten van Jonas Bergström ([Berb] en [Bera]) over het aantal punten van $\mathcal{M}_{g,n}$ over eindige lichamen. Het verband tussen het aantal punten van $\mathcal{M}_{g,n}$ over eindige lichamen en de Eulerkarakteristiek in $K_0(\mathbf{MHS}_{\mathbf{Q}})$ van de

cohomologie met compacte dragers van $\mathcal{M}_{g,n}$ wordt uitgelegd in sectie 3 van Hoofdstuk II, dat gebaseerd is op een stelling van Van den Bogaart-Edixhoven ([BE05]).

Riassunto della tesi

L'oggetto di questa tesi è lo studio della coomologia con coefficienti razionali (chiamata anche coomologia razionale) di spazi di moduli di curve, in casi in cui c'è un collegamento tra la coomologia dello spazio di moduli e quella del complementare di un discriminante in uno spazio vettoriale complesso. Lo spazio vettoriale V utilizzato sarà sempre lo spazio delle sezioni di un fibrato vettoriale su una varietà algebrica Z . In questi casi, a ogni vettore di V è possibile associare una sottovarietà di Z data dal luogo dei punti in cui la sezione si annulla. Questo permette di definire il *discriminante* come l'insieme dei vettori in V tali che la sottovarietà associata è singolare, oppure non ha la dimensione aspettata. Il discriminante è un chiuso in V e ha (in tutti i casi qui considerati) codimENSIONE 1 in tale spazio.

Si ricordi che, per ogni coppia di interi non negativi g ed n che soddisfano la condizione $2g + n - 2 > 0$, esiste uno spazio di moduli $\mathcal{M}_{g,n}$, ovvero una varietà quasi-proiettiva che parametrizza le classi di isomorfismo di curve lisce irriducibili di genere g con n punti marcati. Lo spazio $\mathcal{M}_{g,n}$ può essere compattificato considerando lo spazio di moduli $\overline{\mathcal{M}}_{g,n}$ delle curve stabili di genere g con n punti marcati. Il concetto di curva stabile comprende, oltre alle curve lisce, anche le loro degenerazioni con al massimo nodi come singolarità, e tali che i punti marcati sono punti semplici della curva.

Il metodo che usiamo per lo studio della coomologia razionale di spazi di moduli del tipo $\mathcal{M}_{g,n}$ prevede di dividere $\mathcal{M}_{g,n}$ in strati localmente chiusi che sono il quoziente del complementare di un discriminante per l'azione di un gruppo algebrico fissato. Si tratta, in sostanza, di restringersi allo studio di spazi di moduli più piccoli dell'intero $\mathcal{M}_{g,n}$, in modo da avere la proprietà che ogni curva sia definita dall'annullarsi di un polinomio che appartiene a un certo spazio vettoriale V . Siccome si stanno considerando moduli di curve lisce, è necessario escludere tutti gli elementi di V i cui zeri danno una curva singolare. Questo significa lavorare non con tutto V , ma solo con il complementare del discriminante di V . A questo punto, il legame tra la coomologia del complementare del discriminante di V e quella dello spazio di moduli è dato da un teorema di Peters e Steenbrink ([PS03]) che garantisce che sotto opportune ipotesi la coomologia razionale del complementare del discriminante altro non è che il prodotto tensoriale della coomologia razionale dello spazio di moduli e quella del gruppo. Si noti che tale isomorfismo non rispetta la struttura di anello dei gruppi di coomologia coinvolti, ma nondimeno rispetta le loro strutture di Hodge miste. L'esistenza di questo isomorfismo permette di ricavare la coomologia razionale dello spazio di moduli dalla coomologia razionale del complementare del discriminante. In questa tesi, il calcolo della coomologia del complementare di un discriminante viene sempre effettuato con un metodo dovuto al topologo Vassiliev (vedi [Vas92], [Vas99]), nella versione sviluppata da Gorinov ([Gor05]).

Queste tecniche vengono introdotte nel capitolo I, dove vengono utilizzate per il calcolo della coomologia razionale dello spazio di moduli \mathcal{M}_4 delle curve lisce complesse di genere 4

(teorema I.1.4). La costruzione prevede di dividere \mathcal{M}_4 in tre strati localmente chiusi, definiti in base al modello canonico delle curve.

I metodi del primo capitolo possono essere sfruttati anche per calcolare la coomologia razionale di spazi di moduli di curve con punti marcati, per lo meno nel caso in cui il numero di punti marcati è basso. In che modo le costruzioni debbano essere adattate è spiegato nella sezione 5.1 del capitolo II. L'applicazione dei metodi di tale sezione permette di calcolare la coomologia razionale di $\mathcal{M}_{3,1}$ (corollario II.5.5), e dello spazio di moduli delle curve iperellittiche di genere $g \geq 2$ con due punti marcati (teorema III.2.2). Nel caso di $\mathcal{M}_{3,2}$, invece, questi metodi non sembrano essere sufficientemente forti da fornire da soli un risultato completo sulla coomologia. Per questo motivo ci siamo limitati a calcolare la caratteristica di Eulero della coomologia razionale di $\mathcal{M}_{3,2}$ nel gruppo di Grothendieck delle strutture di Hodge (teorema III.3.1).

Il maggiore risultato del capitolo II, che è frutto della collaborazione con Jonas Bergström (KTH, Stoccolma), è la determinazione della coomologia razionale dello spazio di moduli $\overline{\mathcal{M}}_4$ delle curve stabili di genere 4. Tale calcolo viene effettuato considerando la stratificazione di $\overline{\mathcal{M}}_4$ per tipo topologico delle curve. Questa stratificazione si può ottenere nel modo seguente. Si comincia con definire $U_k \subset \overline{\mathcal{M}}_4$ per ogni $k \geq 0$ come il luogo delle classi di isomorfismo di curve stabili con esattamente k punti singolari. La stratificazione per tipo topologico ha come strati le componenti irriducibili dei luoghi U_k . Per esempio, $U_0 = \mathcal{M}_4$ è irriducibile ed è perciò uno degli strati. Il luogo U_1 ha esattamente tre componenti. Una di esse è formata dalle classi di isomorfismo di curve irriducibili con esattamente un punto singolare, un'altra contiene le classi di isomorfismo di curve con due componenti di genere 1 e 3 che si intersecano in un punto, e un'altra ancora ha come elementi le classi di isomorfismo di curve due componenti di genere 2 che si intersecano in un punto.

È un fatto noto che nella stratificazione per tipo topologico, ogni strato ha una descrizione come prodotto di spazi di moduli di curve lisce di genere ≤ 4 , oppure è il quoziente di un prodotto di questo tipo per l'azione di un gruppo finito. Pertanto, è naturale aspettarsi che molti risultati sulla coomologia di $\overline{\mathcal{M}}_4$ si possano dedurre dallo studio di spazi di moduli di curve lisce di genere ≤ 4 . Si noti che $\overline{\mathcal{M}}_4$ è uno spazio completo che soddisfa la dualità di Poincaré, e quindi la coomologia razionale di $\overline{\mathcal{M}}_4$ è univocamente determinata dalla conoscenza della caratteristica di Eulero della coomologia a supporto compatto di $\overline{\mathcal{M}}_4$ nel gruppo di Grothendieck della categoria delle strutture di Hodge razionali. Per questo tipo di caratteristiche di Eulero, esiste una formula di Getzler e Kapranov ([GK98, 8.13]), proveniente dalla teoria degli operad modulari, che permette di ottenere la caratteristica di Eulero di $\overline{\mathcal{M}}_4$ dalla caratteristica di Eulero della coomologia a supporto compatto di ognuno degli spazi $\mathcal{M}_{g,n}$ con $g \leq 4$ e $n \leq 8 - 2g$, nonché la scomposizione di questa caratteristica come somma di rappresentazioni del gruppo simmetrico \mathfrak{S}_n . Molte di queste caratteristiche di Eulero erano già note (vedi [Get95], [Get99], [Get98b], [Loo93] e [GL]), le altre si ricavano o dai risultati di questa tesi oppure dai risultati di Jonas Bergström ([Berb] e [Bera]) sul numero di punti di $\mathcal{M}_{g,n}$ definiti su campi finiti. La relazione tra il numero di punti di $\mathcal{M}_{g,n}$ definiti su campi finiti e la caratteristica di Eulero in $K_0(MHS_{\mathbb{Q}})$ della coomologia a supporto compatto di $\mathcal{M}_{g,n}$ segue da un teorema di Van den Bogaart-Edixhoven ([BE05]), ed è spiegata nella sezione 3 del capitolo II.

Curriculum vitae

Orsola Tommasi was born in Trieste (Italy) on November 29th, 1974. She received her education in Trieste, where she got her high school diploma at the Liceo Scientifico “G. Oberdan” in 1993. She graduated at the School of Viola of the Conservatorio di Musica “G. Tartini” in October 1998. She studied Mathematics at the University of Trieste (Italy), where she graduated cum laude in March 2001. Her Laurea thesis *Sulle varietà con applicazione di Gauss degeneri*, written under the supervision of prof. Emilia Mezzetti, won in 2002 the Marco Reni award for a Laurea thesis in Mathematics at the University of Trieste, 2000-2002.

She began her Ph.D. studies at Radboud University Nijmegen in September 2001, with prof. J.H.M. Steenbrink as supervisor. She was organizer of the conferences GAEI (Géometrie Algébrique en Liberté) X and XI (CIRM, Luminy (France), April 2003 and 2004). In 2004 she received a Frye Stipendium of Radboud University Nijmegen, which she used to visit prof. Getzler at Northwestern University (Evanston, IL), and to attend the AMS Summer Research Institute in Algebraic Geometry (Seattle, WA).