Wilson loops and their gravity duals in AdS_4/CFT_3

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To my wife and my parents

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Statement of originality

This thesis presents results from the papers [1–3] of the author in collaboration with James Sparks [1–3] as well as Dario Martelli and Jakob Lorenzen [2]. Not all the content of those papers have been included in this document due to space constraints. On the other hand, for clarity and self-consistence, some of the details have been expanded upon. The second chapter of this thesis is based on [1] whereas the third and fourth chapters are based on [2] and [3] respectively.

Abstract

In the first part of this thesis, we study the duality of Wilson loops and M2-branes in $\mathrm{AdS}_4/\mathrm{CFT}_3$. We focus on supersymmetric M-theory solutions on $\mathrm{AdS}_4 \times Y_7$ that have a superconformal dual description on $S^3 = \partial \mathrm{AdS}_4$. We will find that the Hamiltonian function h_M for the M-theory circle plays an important role in the duality. We show that an M2-brane wrapping the M-theory circle is supersymmetric precisely at the critical points of h_M, and moreover the value of this function at those points determines the M2-brane actions. Such a configuration determines the holographic dual of a Wilson loop for a Hopf circle in S^3. We find agreement in large classes of examples between the Wilson loop and its dual M2-brane and also that the image $h_M(Y_7)$ determines the range of support of the eigenvalues in the dual large N matrix model, with the critical points of h_M mapping to points where the derivative of the eigenvalue density is discontinuous. We will then move away from the three-sphere and construct gravity duals to a broad class of $\mathcal{N} = 2$ supersymmetric gauge theories defined on a general class of three-manifold geometries. The gravity backgrounds are based on Euclidean self-dual solutions to four-dimensional gauged supergravity. As well as constructing new examples, we prove in general that for solutions defined on the four-ball the gravitational free energy depends only on the supersymmetric Killing vector. Our result agrees with the large N limit of the free energy of the dual gauge theory, computed using localisation. This constitutes an exact check of the gauge/gravity correspondence for a very broad class of gauge theories defined on a general class of background three-manifold geometries. To further verify that our gravitational backgrounds are indeed dual to field theories on their boundaries, we compute Wilson loops and their M2-brane duals in this general setting. We find that the Wilson loop is given by a simple closed formula which depends on the background geometry only through the supersymmetric Killing vector field. The supergravity dual M2-brane precisely reproduces this large N field theory result. This constitutes a further check of $\mathrm{AdS}_4/\mathrm{CFT}_3$ for a very broad class of examples.

Contents

Chapter 1

Introduction

The AdS/CFT correspondence, also sometimes referred to as gauge/gravity duality, is a duality in string theory that relates gravitational theories to conformal gauge theories. It is an instance of the holographic principle which states that in a quantum theory of gravity there are the same number of gravitational degrees of freedom that live in a region of space as non-gravitational degrees of freedom on the boundary of that space. The famous Bekenstein-Hawking formula [4] showing that the entropy of a black-hole is proportional to its area is an early example of holography.

String theory was originally studied in order to describe strong interactions but was quickly replaced by QCD. It is not until the discovery of a massless spin-two particle in the spectrum of string theories that physicists became interested in string theory again and realised that it can be seen as a quantum theory of gravity. All string theories have a massless spin-two particle that can be interpreted as a graviton and gauge theories are naturally present. String theory is thus the first successful attempt at combining general relativity and gauge theories.

The AdS/CFT correspondence relates a gravitational theory coming from string theory to a conformal field theory on the boundary of the background geometry. In principle, any observable on one side of the correspondence can be matched to an observable on the other side of the correspondence and their value must agree. The beauty of AdS/CFT lies in the fact that when the gravitational theory is weakly coupled, the dual field theory is strongly coupled and vice-versa. This allows us to study strongly coupled systems and check field theory results against classical gravitational calculations. One might even hope that one day we will be able to use AdS/CFT to gain insight into M-theory at strong coupling. Studying the gauge/gravity duality is thus of importance for the advancement of string theory and quantum gravity but also for other fields, like strongly-coupled condensed matter systems, where the correspondence can be applied.

Testing the conjecture is not easy in general, precisely because of its strong/weak nature. One of the useful practical applications of AdS/CFT is to gain insight into condensed matter systems by assuming that the duality holds and use classical gravity to compute field theory results. However, the technique of localisation, that we will review in chapter 2, allows to do gauge theory calculations exactly and independently of the coupling regime. Using this method, it is possible to check exact field theory results against classical gravity calculations and see that they indeed agree as predicted by AdS/CFT. This is currently the best we can do to probe the correspondence as strongly coupled quantum gravity is not understood at all.

The first example of AdS/CFT was given by Maldacena [5]. The duality relates type IIB string theory on $AdS_5 \times S^5$ to $\mathcal{N} = 4$ super-Yang-Mills in flat four-dimensional Minkowski space-time. The $AdS_5 \times S^5$ geometry is obtained as the near-horizon limit of a stack of N D3-branes in supergravity while the geometry remains flat far away from the branes. On the other hand, the world volume theory of those N coincident D3-branes created by strings stretching between the D-branes is $\mathcal{N} = 4$ super-Yang-Mills. Hence, those two theories were conjectured to be equivalent to each other. It is then surprising, but nonetheless true, that the gravitational degrees of freedom in ten dimensions are encrypted into a field theory on the boundary of AdS_5; the gravitational theory can be seen as a holographic projection of the boundary field theory. A crucial point for the duality to work is that the global symmetries of the two dual theories are the same. In the case at hand, the isometry group of $AdS_5 \times S^5$ corresponds to the conformal group multiplied by the internal R-symmetry group of supersymmetry in $\mathcal{N} = 4$ super-Yang-Mills.

Known AdS/CFT pairs are not limited to AdS_5 and four-dimensional conformal field theories. In this thesis we will focus on $\mathrm{AdS}_4/\mathrm{CFT}_3$. The first example of such a pair appeared in the seminal paper [6] by Aharony, Bergman, Jafferis, and Maldacena (ABJM). Starting from ABJM we now have large classes of supersymmetric $\mathrm{AdS}_4 \times Y_7$ gravity backgrounds of M-theory that are associated with particular three-dimensional supersymmetric gauge theories, typically Chern-Simons theories coupled to matter. The construction of the gauge theory usually relies on a dual description in terms of type IIA string theory, which in turn involves a choice of M-theory circle $U(1)_M$ acting on Y_7. In [6] the highly supersymmetric case where $Y_7 = S^7/\mathbb{Z}_k$, equipped with its round Einstein metric and with N units of flux through this internal space, was related to a large N dual description as an $\mathcal{N} = 6$ superconformal $U(N) \times U(N)$ Chern-Simons-matter theory (the ABJM theory), with $k \in \mathbb{Z}$ being the Chern-Simons coupling. Here $\mathbb{Z}_k \subset U(1)_M$, with the M-theory circle action $U(1)_M$ being the Hopf action on S^7, so that $S^7/U(1)_M = \mathbb{CP}^3$. Hence, one can equivalently chose to work with M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$, which is the near-horizon geometry of a stack of N M2-branes, or type IIA string theory on $\mathrm{AdS}_4 \times \mathbb{CP}^3$. There are now many families of examples of a similar type [7–23], generally with $\mathcal{N} \geq 2$ supersymmetry, in which Y_7 is a Sasaki-Einstein seven-manifold and the dual description typically involves supersymmetric Chern-Simons-matter theories whose gauge groups are products of unitary groups, and with matter in various representations.

In chapter 2 of this thesis, we will be interested in looking at Wilson loops in conformal field theories that have a dual M-theory description on $\mathrm{AdS}_4 \times Y_7$. Wilson loops are natural observables in gauge theories and are defined for a closed loop and a representation of the gauge group. Roughly speaking, a Wilson loop is (classically) the holonomy of the gauge field. A supersymmetric version of the loop is also available for supersymmetric gauge theories. In QCD, Wilson loops measure the free energy of a pair of quark-antiquark propagating along the loop. More abstractly, Wilson loops are interesting to study because they are non-local gauge invariant observables and provide additional non-trivial information about the gauge theory. In the AdS/CFT context, one expects a Wilson loop to be dual to a fundamental string when viewed from a type IIA perspective [24], with the string worldsheet having boundary on the loop. Equivalently, this fundamental string can be viewed as an M2-brane in M-theory and this is the point of view that will be taken in this thesis. As we will explore in detail in chapter 2, the computations for the Wilson loops and the M2-branes agree in large classes of examples as predicted by the AdS/CFT correspondence.

As noted before, there has been many new pairs of dual theories discovered since ABJM. One can try to look for new pairs by changing the internal space $S^7/\mathbb{Z}_k$ of ABJM by another manifold Y_7. Because the Calabi-Yau cone over the internal manifold Y_7 is the moduli space of the CFT for $N = 1$, the brane constructions of those theories can be significantly different from ABJM and give various field theories, from Chern-Simons matter theory to super-Yang-Mills theories. In all those cases, the field theory lives on a three-sphere S^3 at the boundary of AdS_4. One can then wonder if it is possible to deform the three-sphere with AdS/CFT still applying. The answer to that question must be positive because it is the symmetries and asymptotic form of AdS that matter for the correspondence to hold. Some examples have be constructed for some particular deformations of the sphere in [25–28] and it was shown that the free energy of the CFT indeed matches the action of the supergravity dual [26, 29–31]. More recently, the partition function of a large class of three-dimensional Chern-Simons theories defined on a general manifold with three-sphere topology was computed explicitly in [32]. This has provided a unified understanding of all previous computations on deformed three-spheres. On the gravity side this yields a universal prediction for the action of the corresponding supergravity solutions. In chapter 3, we will study the gravity duals on a four-manifold M_4 of those supersymmetric theories defined on a general class of three-manifold $M_3 = \partial M_4$. It will be shown that the gravity action precisely matches the field theories partition function in the large N limit. This constitutes an exact check of the gauge/gravity correspondence for a broad class of gauge theories defined on a general class of background three-manifold geometries. We are now a long way from the original ABJM whose duality related $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ to a conformal field theory on S^3 as we now have large classes of supergravities on $M_4 \times Y_7$ dual to field theories on M_3.

At this point, it is natural to ask if it is possible to construct Wilson loops on M_3 and find their M2-brane gravity duals in $M_4 \times Y_7$. Relying on the results of chapters 2 and 3 we will construct those objects and compute them in chapter 4. As one can anticipate, those observables match, thus verifying the duality of supergravity on $M_4 \times Y_7$ and field theories on M_3 beyond the matching of the supergravity action and the free energy.

Chapter 2

Wilson loops, matrix models and Hamiltonian geometry

2.1 Introduction

Our understanding of the AdS_4/CFT_3 correspondence has improved considerably over the last few years. Broadly speaking, this has involved developments on two fronts. Firstly, as mentioned in the introduction, we now have large classes of very explicit examples of dual pairs; that is, gravity backgrounds for which we have some precise description of the dual superconformal field theories. Secondly, there are new quantitative tests of these conjectured dualities, based on supersymmetric localisation in the field theories. The aim of this chapter is to extend this quantitative analysis further, by examining the computation of certain BPS Wilson loops on both sides of the correspondence. In the process we will also understand how other structures are related via the duality.

Quantitative tests of these conjectured dualities arise by putting the Euclidean field theories on a compact three-manifold. The simplest case, in which this three-manifold is taken to be S^3 equipped with its round metric, was studied in [33–35]. This can be done for a completely general $\mathcal{N} = 2$ supersymmetric gauge theory, in such a way to preserve supersymmetry. Moreover, using a standard argument [36] one can show that the path integral, with any BPS operator inserted, reduces exactly to a finite-dimensional matrix integral. This implies that the VEVs of BPS operators may be computed exactly using a matrix model description, with the large N limit of this then expected to reproduce certain supergravity results. In practice this has been used to compute the free energy $\mathcal{F}$ (minus the logarithm of the partition function) on both sides of the correspondence [37–44], where on the supergravity side this is proportional to $N^{3/2}$ with a coefficient depending only on the volume of Y_7.[1]

It is natural to try to extend these results further, by inserting non-trivial BPS operators into the path integral, computing the corresponding large N behaviour in the matrix model, and comparing to an appropriate dual semi-classical supergravity computation. In the original papers on the ABJM theory [33,37,45–48] the supersymmetric Wilson loop for the gauge field around a Hopf circle $S^1 \subset S^3$ was studied. This is 1/2 BPS, and is readily computed in the large N matrix model [33, 37]. Generally speaking, one expects such a Wilson loop to be dual to a fundamental string when viewed from a type IIA perspective [24], with the Euclidean string worldsheet having boundary on the Hopf S^1 at conformal infinity. More precisely, this will be semi-classically a supersymmetric minimal surface Σ_2 in Euclidean AdS_4, with the VEV calculated via the regularised area of the string worldsheet. Such a string must then be pointlike in the internal space, and for the ABJM theory this is $\mathbb{CP}^3 = S^7/U(1)_M$. Equivalently, this IIA string lifts to an M2-brane wrapping the M-theory circle. Notice that since $\mathbb{CP}^3$ is a homogeneous space all positions for the IIA string are equivalent. The two computations (large N matrix model and area) of course agree.[2]

This Wilson loop is 1/2 BPS in a general $\mathcal{N} = 2$ supersymmetric gauge theory on S^3, as we review in section 2.3, and can be computed using the large N matrix model

[1]For a general $\mathrm{AdS}_4 \times Y_7$ solution this is the *contact volume* of Y_7, rather than the Riemannian volume, as we shall review in section 2.3.

[2]Similar Wilson loops have been considered in five-dimensional superconformal field theories on S^5 [49], which may also be computed using localisation techniques. The gravity duals are described by warped $\mathrm{AdS}_6 \times S^4/\mathbb{Z}_n$ solutions of massive IIA supergravity, and thus the geometry of the internal spaces here is fixed and in fact unique [50].

description. The supergravity dual computation will naturally involve an M2-brane wrapping the M-theory circle, leading to the same fundamental string configuration in Euclidean AdS_4 (see Figure 2.1). The only issue is which copy of the M-theory circle is relevant? When the internal space is $Y_7 = S^7/\mathbb{Z}_k$ all choices are equivalent by symmetry, but on a general Sasaki-Einstein manifold Y_7, this is clearly not the case. Equivalently we may ask which IIA fundamental strings in $\mathrm{AdS}_4 \times M_6$, that are pointlike in $M_6 = Y_7/U(1)_M$, preserve any supersymmetry.

2.2 Summary of results

Given the technical nature of the computation of the the action of the M2-brane in supergravity and in particular the use of various differential geometric tools to achieve it, we will start by summarising the results of this chapter in order for the reader to get a better view of what will be done. The mathematical details will be explained and expanded upon in the subsequent sections. The starting point is to consider BPS M2-branes in general $\mathcal{N} = 2$ supersymmetric $\mathrm{AdS}_4 \times Y_7$ solutions of eleven-dimensional supergravity. These backgrounds were studied in detail in [51,52], where it was shown that provided the quantised M2-brane charge N of the background (measured by a certain flux integral) is non-zero, then there is always a canonical *contact one-form* η defined on Y_7. Concretely, η is constructed as a bilinear in the Killing spinors on Y_7, and it was shown in the latter reference that this contact structure entirely captures both the gravitational free energy of the background, and also the scaling dimensions of BPS operators arising from supersymmetric M5-branes wrapped on five-manifolds $\Sigma_5 \subset Y_7$.

In this chapter we will show that the same contact form η captures the Wilson loop VEV $\langle W \rangle$ of interest, computed semi-classically from the action of a BPS M2-brane.

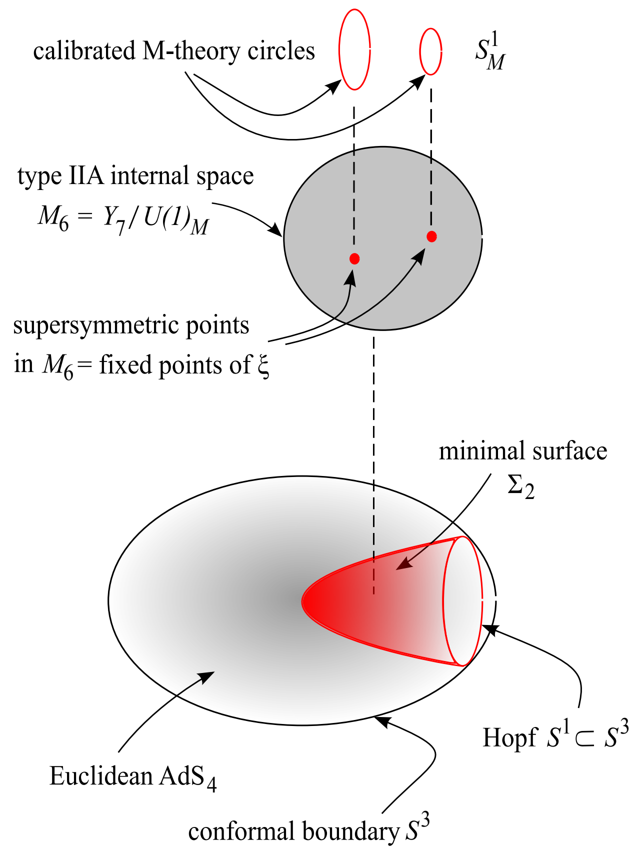

Figure 2.1: A depiction of the total spacetime $\mathrm{AdS}_4 \times Y_7$, with a choice of M-theory circle $U(1)_M$, together with the supersymmetric M2-branes of interest which are shown in red. These M2-branes are pointlike in the type IIA internal space $M_6 = Y_7/U(1)_M$, wrapping copies of the M-theory circle over these points, and are calibrated by the contact form η. The supersymmetric points in M_6 are precisely the points where the projection of the R-symmetry/Reeb vector field ξ is zero (giving fixed points on M_6), and in general the calibrated circles over such points have different lengths. The remaining worldvolume of the M2-brane wraps a minimal supersymmetric surface Σ_2 in Euclidean AdS_4. The latter may be viewed as a hyperbolic 4-ball, with conformal boundary S^3, and Σ_2 then has the topology of a 2-ball, with boundary a Hopf $S^1 \subset S^3$.

Concretely, we derive the general formula

$$\log \langle W \rangle_{\text{gravity}} = \frac{(2\pi)^2 \int_{S^1_M} \eta}{\sqrt{96\, \mathrm{Vol}_\eta(Y_7)}} N^{1/2} \ , \qquad (2.2.1)$$

where we have defined the *contact volume* of Y_7 as

$$\mathrm{Vol}_\eta(Y_7) \equiv \frac{1}{48}\int_{Y_7} \eta\wedge(\mathrm{d}\eta)^3 \ . \tag{2.2.2}$$

In particular, a supersymmetric M2-brane is *calibrated* with respect to η, which is why the integral of η along the M-theory circle S^1_M appears in the formula (2.2.1). A contact form η always has an associated unique *Reeb vector field* ξ, defined via the equations $\xi\lrcorner\eta = 1$, $\xi\lrcorner\mathrm{d}\eta = 0$, and in [51, 52] it was shown that ξ is also the *R-symmetry* Killing vector field, that is expected since an $\mathcal{N} = 2$ superconformal theory in three dimensions has a $\mathtt{u}(1)_R$ symmetry in the superconformal algebra. We will show that an M2-brane wrapping a copy of the M-theory circle S^1_M is supersymmetric precisely when the generating vector field ζ_M of $U(1)_M$ is proportional to ξ. Geometrically, this means that the corresponding fundamental string at a point $p \in M_6$ is supersymmetric precisely when p is a fixed point of ξ, considered as a vector field on M_6 (on Y_7, on the other hand, ξ is always nowhere zero).

There is another way to describe which wrapped M2-branes are supersymmetric which involves the *Hamiltonian function* for the M-theory circle, defined as

$$h_M \equiv \eta(\zeta_M) \ . \tag{2.2.3}$$

This is a real function $h_M : Y_7 \to \mathbb{R}$, invariant under ζ_M, and we show that the supersymmetric M-theory circles $S^1_M \subset Y_7$ lie precisely on the critical set $\mathrm{d}h_M = 0$. The action of a supersymmetric M2-brane corresponding to a point $p \in M_6$ may then also be written as

$$-S_{\mathrm{M2}} = \frac{(2\pi)^3 h_M(\hat{p})}{\sqrt{96\,\mathrm{Vol}_\eta(Y_7)}} N^{1/2} \ , \tag{2.2.4}$$

where $\hat{p} \in Y_7$ is any point that projects to $p \in M_6 = Y_7/U(1)_M$. Since (2.2.4) depends only on η we may compute this expression in examples using the same methods employed in [51, 52], [53–57]. For example, for toric solutions (2.2.4) may be computed entirely using toric geometry methods. In general there are multiple supersymmetric S^1_M circles, which can have different lengths with respect to η and thus leading to different actions (2.2.4). In the semi-classical computation one should *sum* over all such configurations, which in the large N limit then implies that in (2.2.1) it is the *longest* S^1_M that gives the leading contribution to the Wilson loop.

In the families of examples that we shall study, the dual field theory computation of the Wilson loop VEV reduces to a computation in a large N matrix model. As we shall review in section 2.3, in this matrix model the eigenvalues at large N take the general form $\lambda^I = xN^{1/2} + \mathrm{i}y^I(x)$, where the index I runs over the number of factors of $U(N)$ in the gauge group $G = \prod_I U(N)$, and are described by an eigenvalue density function $\rho(x)$ which is supported on some interval $[x_{\min}, x_{\max}] \subset \mathbb{R}$. To leading order at large N it is straightforward to compute

$$\log \langle W \rangle_{\mathrm{QFT}} = x_{\max} N^{1/2} \ , \tag{2.2.5}$$

which should be compared to the dual supergravity result (2.2.1).

Remarkably, in all examples that we study we find that the interval $[x_{\min}, x_{\max}]$ in the matrix model coincides, in a precise way, with the image of the Hamiltonian function $h_M(Y_7)$. Since Y_7 is compact and connected, the latter image is also necessarily a closed interval, and more precisely we find $h_M(Y_7) = [c_{\min}, c_{\max}]$, where the field theory quantity x is proportional to the geometrical quantity c:

$$x \ = \ \frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}_\eta(Y_7)}}\, c \ . \tag{2.2.6}$$

The Hamiltonian h_M is a Morse-Bott function on the symplectic cone over Y_7, and on general grounds we know that the image interval $[c_{\min}, c_{\max}]$ is divided into P subintervals $c_{\min} = c_1 < c_2 < \cdots < c_{P+1} = c_{\max}$, where the critical set maps as $h_M(\{\mathrm{d}h_M = 0\}) = \{c_i \mid i = 1, \ldots, P+1\}$. For all $c \in (c_i, c_{i+1})$ the level surfaces $h_M^{-1}(c) \subset Y_7$ are diffeomorphic to a fixed six-manifold, with the topology changing precisely as one passes a critical point c_i. Even more remarkable is that we find that the corresponding points x_i, related to c_i via (2.2.6), are precisely the points where $\rho'(x)$ has a jump discontinuity in the matrix model. These points are then also related to the fixed points of the Reeb vector ξ on M_6. Hence, every point c_i where $\rho'(x)$ is discontinuous corresponds to a BPS M2-brane whose action $-S_{M2} = x_i N^{1/2}$ can be computed from c_i using (2.2.6), or equivalently computed via (2.2.4), and the largest action of those M2-branes equals de Wilson loop VEV.

The outline of the rest of this chapter is as follows. In section 2.3 we review the definition of the BPS Wilson loop in $\mathcal{N} = 2$ Chern-Simons-matter theories, and how it may be computed in the large N matrix model. Section 2.4 analyses supersymmetric M2-branes in a general class of $\mathrm{AdS}_4 \times Y_7$ backgrounds in M-theory, and we derive the general formula for the action (2.2.4), leading to the holographic Wilson loop result (2.2.1). Finally, in section 2.5 we compute the Wilson loop on both sides of the correspondence in a variety of examples.

2.3 Wilson loops in $\mathcal{N} = 2$ gauge theories on S^3

The dual superconformal field theories of interest are $\mathcal{N} = 2$ Chern-Simons gauge theories with matter on S^3. We begin in this section by defining the BPS Wilson loop in such a theory, summarise how it localises in the matrix model, and explain how it can be efficiently calculated. This section is mainly a review of material already in the literature.

2.3.1 The Wilson loop

In $\mathcal{N} = 2$ supersymmetric gauge theories the gauge field $\mathcal{A}_i$ is part of a vector multiplet that also contains two real scalars σ and D, that are auxiliary fields, and a two-component spinor λ, all of which are in the adjoint representation of the gauge group G. The BPS Wilson loop in a representation $\mathfrak{R}$ of G is given by

$$W = \frac{1}{\dim \mathfrak{R}} \mathrm{Tr}_{\mathfrak{R}} \left[\mathcal{P} \exp \left(\oint_{\upsilon} \mathrm{d}s (\mathrm{i}\mathcal{A}_i \dot{x}^i + \sigma |\dot{x}|) \right) \right] , \tag{2.3.1}$$

where $x^i(s)$ parametrises the worldline $\upsilon \subset S^3$ of the Wilson line and the path ordering operator has been denoted by $\mathcal{P}$. For a Chern-Simons theory the gauge multiplet has a kinetic term described by the supersymmetric Chern-Simons action

$$S_{\mathrm{Chern-Simons}} = \frac{\mathrm{i}k}{4\pi} \int \mathrm{Tr} \left(\mathcal{A} \wedge \mathrm{d}\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} + (2D\sigma - \lambda^\dagger \lambda) \mathrm{vol}_3 \right) , \tag{2.3.2}$$

where vol_3 is the volume form of the round metric on S^3, and k denotes the Chern-Simons coupling. When G is a product of unitary groups, $G = \prod_I U(N_I)$, one can in general take different $k_I \in \mathbb{Z}$ for each factor. In this case we will denote $k = \gcd\{k_I\}$ [8].

There are four Killing spinors on S^3, two satisfying each choice of sign in the equation $\nabla_i \varepsilon = \pm \frac{\mathrm{i}}{2} \tau_i \varepsilon$, where the gamma matrices τ_i in an orthonormal frame generate the Clifford algebra Cliff$(3,0)$, and may thus be taken to be the Pauli matrices. A natural orthonormal frame $\{e^m_{S^3}\}_{m=1,2,3}$ on S^3 is provided by the left (or right) invariant one-forms under the isometry group $SU(2)_{\mathrm{left}} \times SU(2)_{\mathrm{right}}$.

The full supersymmetry transformations for a vector multiplet and matter multiplet may be found in [33–35]. For our purposes we need note only that localisation of the path integral, discussed in the next section, requires one to choose a Killing spinor ε, which without loss of generality we assume solves $\nabla_i \varepsilon = \frac{\mathrm{i}}{2} \tau_i \varepsilon$. Supersymmetry generators must be Killing in order to be able to construct invariant supersymmetric actions. This choice of Killing spinor then has the two associated supersymmetry transformations

$$\begin{aligned} \delta \mathcal{A}_i &= -\frac{\mathrm{i}}{2} \lambda^\dagger \tau_i \varepsilon , \\ \delta \sigma &= -\frac{1}{2} \lambda^\dagger \varepsilon . \end{aligned} \tag{2.3.3}$$

If one varies the Wilson loop (2.3.1) under the latter supersymmetry transformation one obtains

$$\delta W \propto \frac{1}{2}\lambda^\dagger(\tau_i \dot{x}^i - |\dot{x}|)\varepsilon \ . \tag{2.3.4}$$

The Wilson loop is then invariant under supersymmetry provided

$$(\tau_i \dot{x}^i - |\dot{x}|)\varepsilon = 0 \ . \tag{2.3.5}$$

Choosing s to parametrise arclength, so that $|\dot{x}| = 1$ along the loop, we see that $\tau_i \dot{x}^i$ must be constant. In the left-invariant orthonormal frame $e^m_{S^3}$ one may then align $\dot{x}^i$ along one direction, say $e^3_{S^3}$. The integral curve of this vector field is a Hopf $S^1 \subset S^3$ (or equivalently a great circle). The supersymmetry condition then becomes

$$(\tau_3 - 1)\varepsilon = 0 \ . \tag{2.3.6}$$

The two possible choices of ε satisfying $\nabla_i \varepsilon = \frac{i}{2}\tau_i \varepsilon$ have opposite chirality and only one of them survives the projection condition above. This implies that the Wilson loop (2.3.1) is indeed a 1/2 BPS operator provided one takes υ to be a Hopf circle. We will see later on that the condition (2.3.6), plus the fact that the supersymmetry generators are Killing spinors, also arises as the condition for supersymmetry of a probe M2-brane.

2.3.2 Localisation in the matrix model

The VEV of the BPS Wilson loop (2.3.1) is, by definition, obtained by inserting W into the path integral for the theory on S^3. The computation of this is greatly simplified by the fact that this path integral *localises* onto supersymmetric configurations of fields. We summarise the main steps and results in this section, following in particular [33, 34, 37, 38], and refer the reader to the original papers for further details.

The central idea is that the path integral, with W inserted, is invariant under the supersymmetry variation δ corresponding to the Killing spinor ε satisfying (2.3.6). We have written two of the supersymmetry variations in (2.3.3), and the variations of other fields (including fields in the chiral matter multiplets) may be found in the above references. Crucially, $\delta^2 = 0$ is nilpotent. There is then a form of *fixed point theorem* that implies that the only net contributions to this path integral come from field configurations that are invariant under δ [58].

Alternatively, and more practically for computation, one may add a conveniently chosen δ-exact positive definite term to the action, which a standard argument shows does not affect the expectation value of any supersymmetric (δ-invariant) operator. For the vector multiplet one can add the term $t\mathrm{Tr}[(\delta\lambda)^\dagger\delta\lambda]$ to the action (a similar term exists for a matter multiplet), without affecting the path integral. Sending $t \to \infty$ one notes that, due to the form of this term added to the Lagrangian, only configurations with $\delta\lambda = 0$ contribute to the path integral in a saddle point approximation. This saddle point then gives the same value as if the path integral had been calculated with $t = 0$, which is the quantity we are interested in. The saddle point approximation requires one to compute a one-loop determinant around the δ-invariant field configurations, which in the terminology of fixed point theorems is the contribution from the normal bundle to the fixed point set in field space.

For the $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theories of interest, one finds that the δ-invariant configurations on S^3 are particularly simple:

$$\mathcal{A}_i = 0 \ , \quad \text{and} \quad D \ = \ -\sigma \ = \ \text{constant} \ , \tag{2.3.7}$$

with all fields in the matter multiplet set identically to zero. Here we may diagonalise σ by a gauge transformation. For a $U(N)$ gauge group we may thus write $\sigma = \mathrm{diag}(\frac{\lambda_1}{2\pi}, \ldots \frac{\lambda_N}{2\pi})$, thus parametrising $2\pi\sigma$ by its eigenvalues λ_i. The theories of interest will have a product gauge group of the form $G = \prod_{I=1}^{g} U(N)$, and for $t = \infty$ the partition function then takes the saddle point form

$$Z = \frac{1}{(N!)^g} \int \left(\prod_{I=1}^{g} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i^I}{2\pi} \right) \exp \left[\mathrm{i} \sum_{I=1}^{g} \frac{k_I}{4\pi} \sum_{i=1}^{N} (\lambda_i^I)^2 \right] \mathrm{e}^{-\mathcal{F}_{\text{one-loop}}} , \tag{2.3.8}$$

where the one-loop determinant is given by

$$\mathrm{e}^{-\mathcal{F}_{\text{one-loop}}} = \prod_{I=1}^{g} \prod_{i \neq j} 2 \sinh \frac{\lambda_i^I - \lambda_j^I}{2} \cdot \prod_{\text{matter } \alpha} \det_{\mathfrak{R}_\alpha} \exp \left[\ell(1 - \Delta_\alpha + \mathrm{i}\sigma) \right] . \tag{2.3.9}$$

Here the first exponential term in (2.3.8) is simply the classical Chern-Simons action in (2.3.2), evaluated on the localised constant field configuration (2.3.7). The one-loop determinant factorises, and the first term in (2.3.9) is the one-loop determinant for the vector multiplet. The second term in (2.3.9) involves a product over chiral matter multiplets, labelled by α. We have taken the α^{th} multiplet to be in representation $\mathfrak{R}_\alpha$, and with R-charge Δ_α. The determinant in the representation $\mathfrak{R}_\alpha$ is understood to be a product over weights ϱ in the weight-space decomposition of this representation, and σ is then understood to mean $\varrho(\sigma)$ in (2.3.9). Finally,

$$\ell(z) = -z \log \left(1 - \mathrm{e}^{2\pi \mathrm{i} z} \right) + \frac{\mathrm{i}}{2} \left[\pi z^2 + \frac{1}{\pi} \mathrm{Li}_2 \left(\mathrm{e}^{2\pi \mathrm{i} z} \right) \right] - \frac{\mathrm{i}\pi}{12} . \tag{2.3.10}$$

In this set-up, the VEV of the BPS Wilson loop (2.3.1) reduces to

$$\langle W \rangle = \frac{1}{Z (N!)^g \dim \mathfrak{R}} \int \left(\prod_{I=1}^{g} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i^I}{2\pi} \right) \mathrm{e}^{\mathrm{i} \sum_{I=1}^{g} \frac{k_I}{4\pi} \sum_{i=1}^{N} (\lambda_i^I)^2} \, \mathrm{Tr}_{\mathfrak{R}} \left(\mathrm{e}^{2\pi\sigma} \right) \mathrm{e}^{-\mathcal{F}_{\text{one-loop}}} . \tag{2.3.11}$$

Notice the integrand is the same as that for the partition function (2.3.8), with an additional insertion of $\mathrm{Tr}_{\mathfrak{R}}(\mathrm{e}^{2\pi\sigma})$ arising from the Wilson loop operator. Note also that we have normalised the VEV relative to the partition function Z, so that $\langle 1 \rangle = 1$, as is usual in quantum field theory.

Localisation has reduced the partition function Z and the Wilson loop VEV to finite-dimensional integrals (2.3.8), (2.3.11) over the eigenvalues λ_i^I of σ, but in practice these are difficult to evaluate explicitly due to the complicated one-loop effective potential (2.3.9). For comparison to the dual supergravity results we must take the $N \to \infty$ limit, where the number of eigenvalues, and hence integrals, tends to infinity. One can then attempt to compute this limit using a saddle point approximation of the integral (this is then our second application of the saddle point method). With the exception of the $\mathcal{N} = 6$ supersymmetric ABJM theory, where this matrix model is well-understood [59], for general $\mathcal{N} = 2$ theories the large N limit of the matrix integrals is not understood rigorously. However, in [37] a simple *ansatz* for the large N limit of the saddle point eigenvalue distribution was introduced. This ansatz is based on a partial analytic analysis of the matrix model, and also on a numerical approach to computing the saddle point. One seeks saddle points with eigenvalues of the form

$$\lambda_i^I = x_i N^{\beta} + \mathrm{i} y_i^I \ , \tag{2.3.12}$$

with x_i and y_i^I real and assumed to be $\mathcal{O}(1)$ in a large N expansion, and $\beta > 0$. In the large N limit the real part is assumed to become dense. Ordering the eigenvalues so that the x_i are strictly increasing, the real part becomes a continuous variable x, with density $\rho(x)$, while y_i^I becomes a continuous function of x, $y^I(x)$.

Substituting this ansatz into the partition function expression (2.3.8), the sums over eigenvalues become Riemann integrals over x, and one finds that the *double sums* appearing in the one-loop expression (2.3.9) effectively have a delta function contribution which reduces them to single integrals over x. Writing $Z = \mathrm{e}^{-\mathcal{F}}$ one then obtains a functional $\mathcal{F}[\rho(x), y^I(x)]$, with x supported on some interval $[x_{\mathrm{min}}, x_{\mathrm{max}}]$, and to apply the saddle point method one then extremises $\mathcal{F}$ with respect to $\rho(x)$, $y^I(x)$, subject to the constraint that $\rho(x)$ is a density

$$\int_{x_{\mathrm{min}}}^{x_{\mathrm{max}}} \rho(x) \mathrm{d}x = 1 \ . \tag{2.3.13}$$

The existence of such a saddle point fixes the exponent $\beta = \frac{1}{2}$ in (2.3.12). One then finally also extremises over the choice of interval, by varying with respect to $x_{\rm min}$, $x_{\rm max}$, to obtain the saddle point eigenvalue distribution $\rho(x)$, $y^I(x)$.

We shall be interested in evaluating the Wilson loop VEV (2.3.11) in the fundamental representation because it is the one dual to a fundamental string. In this case, the Wilson loop is proportional to $\sum_{I=1}^{g}\sum_{i=1}^{N}\mathrm{e}^{\lambda_i^I}$. In the large N limit, described by the saddle point density $\rho(x)$ and imaginary parts $y^I(x)$ of the eigenvalues, the VEV reduces simply to

$$\begin{aligned} \langle W\rangle_{\rm QFT} &= \mathrm{Tr}_{\mathfrak{R}}\left(e^{2\pi\sigma}\right)\big|_{\text{at the saddle point}} \\ &= \sum_{I=1}^{g}\int_{x_{\rm min}}^{x_{\rm max}} e^{\lambda^I(x)}\rho(x)\mathrm{d}x \\ &= \sum_{I=1}^{g}\int_{x_{\rm min}}^{x_{\rm max}} \mathrm{e}^{xN^{1/2}+\mathrm{i}y^I(x)}\rho(x)\mathrm{d}x\ . \end{aligned} \tag{2.3.14}$$

Because of the form of $\mathcal{F}[\rho(x),y^I(x)]$ for $\mathcal{N}=2$ Chern-Simons-matter theories, the saddle point eigenvalue density $\rho(x)$ is always a continuous, piecewise linear function on $(x_{\rm min},x_{\rm max})$, see for example section 2.5. A simple computation then shows that, to leading order in the large N limit, the matrix model VEV (2.3.14) reduces to

$$\log\langle W\rangle_{\rm QFT} = x_{\rm max}\,N^{1/2}\ . \tag{2.3.15}$$

This is our final formula for the large N limit of the Wilson loop VEV. We see that it computes the *maximum* value of the real part of the saddle point eigenvalues.

In our summary above we have suppressed the dependence on the R-charges Δ_α of the matter multiplets, labelled by α, appearing in (2.3.9). If these are left arbitrary, one obtains a free energy $\mathcal{F}$ that is a function of Δ_α, and according to [34] the superconformal R-symmetry of an $\mathcal{N}=2$ superconformal field theory further extremises $\mathcal{F}$ as a function of Δ_α (in fact maximising $\mathcal{F}$ [60]). For theories with

M-theory duals of the form $\mathrm{AdS}_4 \times Y_7$ one finds the expected supergravity result

$$\mathcal{F} = \sqrt{\frac{2\pi^6}{27\,\mathrm{Vol}_\eta(Y_7)}} N^{3/2} , \tag{2.3.16}$$

but as a *function* of R-charges Δ_α [38], where on the right hand side it is in general the *contact volume* (2.2.2) of Y_7 that appears. This has by now been demonstrated in many classes of examples in the literature [37–44, 52].

2.4 BPS M2-branes

This section will analyse the dual objects to the Wilson loops that have been just built. From a type IIA perspective, the dual object is a fundamental string and it can be equivalently viewed as an M2-brane from a M-theory viewpoint. The supersymmetric, or BPS, probe M2-branes must therefor reproduce the holographic dual of the Wilson loop VEV (2.3.15). First, we review the form of the supergravity backgrounds. We then define the M2-brane of interest and recast its condition of supersymmetry into a geometric condition. After that, we derive the formula (2.2.4) for the action of the M2-brane, and finally describe how this may be computed in practice using different geometric methods.

2.4.1 Supergravity backgrounds

We will study the general class of $\mathcal{N} = 2$ supersymmetric $\mathrm{AdS}_4 \times Y_7$ backgrounds of M-theory described in [51, 52]. We begin by recalling some relevant results and formulae.

The bosonic field content of $D = 11$ supergravity consists of [61] a metric g_{11} and a three form C with four form field strength $G = \mathrm{d}C$. The signature of the metric is taken to be $(-,+,\ldots,+)$ and the action is

$$S_{11} \;=\; \frac{1}{(2\pi)^8\ell_p^9}\int \mathcal{R}*_{11}\mathbf{1} - \frac{1}{2}\mathrm{d}C\wedge *_{11}\mathrm{d}C - \frac{1}{6}C\wedge \mathrm{d}C\wedge \mathrm{d}C\ . \tag{2.4.1}$$

The metric g_{11} has Ricci scalar $\mathcal{R}$, C is the three-form potential and ℓ_p denotes the eleven-dimensional Planck length. The equations of motion for the metric and C-field follow immediately:

$$\begin{aligned} \mathcal{R}_{AB} - \frac{1}{12}(G_{AC_1C_2C_3}G_B{}^{C_1C_2C_3} - \frac{1}{12}g_{AB}G^2) &= 0\ , \\ \mathrm{d}*_{11}G + \frac{1}{2}G\wedge G &= 0\ , \end{aligned} \tag{2.4.2}$$

where we have defined $G\equiv \mathrm{d}C$, $\mathcal{R}_{AB}$ is the Ricci tensor and $A,B,C=1,\ldots,11$. We consider AdS_4 solutions of the warp product form:

$$\begin{aligned} g_{11} &= \mathrm{e}^{2\Delta}\left(\frac{1}{4}g_{\mathrm{AdS}_4} + g_{Y_7}\right)\ , \\ G &= \frac{m}{16}\mathrm{vol}_4 + F_4\ , \end{aligned} \tag{2.4.3}$$

where the metric on AdS_4 has unit AdS radius, with volume form vol_4. The warp factor Δ is taken to be a function on Y_7, m is a constant, and F_4 is a four-form on Y_7. The Bianchi identity $\mathrm{d}G=0$ requires that the four-form F_4 be closed. This is the most general ansatz compatible with the symmetries of AdS_4. The eleven-dimensional Majorana spinor takes the form

$$\epsilon_{11} = \mathrm{e}^{\Delta/2}\psi_+\otimes\chi_+ + \mathrm{e}^{\Delta/2}\psi_-\otimes\chi_- + \text{charge conjugate}\ , \tag{2.4.4}$$

where $\chi_\pm$ are complex spinors on Y_7, $\psi_\pm$ are the usual Killing spinors on AdS_4 (the $\pm$ signs are related to the charge under the R-symmetry, discussed below), and the factors of $\mathrm{e}^{\Delta/2}$ have been introduced for convenience.

In general, the vanishing of the variation of the gravitino, giving the general

Killing spinor equation in eleven dimensions, implies that the spinors $\chi_\pm$ solve quite a complicated system of coupled first order equations on Y_7, that may be found in [51, 52]. These equations are then necessary and sufficient for supersymmetry of the $\mathrm{AdS}_4 \times Y_7$ background. For our purposes we need note only a few key formulae. We first define the real one-forms

$$\xi \equiv \mathrm{i}\bar{\chi}^c_+ \gamma_{(1)} \chi_- \ , \qquad \eta \ \equiv \ -\frac{6}{m}\mathrm{e}^{3\Delta}\bar{\chi}_+\gamma_{(1)}\chi_+ \ , \tag{2.4.5}$$

where in general we denote $\gamma_{(n)} \equiv \frac{1}{n!}\gamma_{m_1\cdots m_n}\mathrm{d}y^{m_1}\wedge\cdots\wedge\mathrm{d}y^{m_n}$, with $y^1,\ldots,y^7$ local coordinates on Y_7, and the superscript c on the spinors denotes charge conjugation. By an abuse of notation, we will more generally regard ξ as the dual vector field defined by the metric g_{Y_7}. We then note that the differential equations for $\chi_\pm$ imply the equations

$$\begin{aligned}
\bar{\chi}_+\chi_+ \ = \ \bar{\chi}_-\chi_- \ &= \ 1 \ , \quad \frac{m}{6}\mathrm{e}^{-3\Delta} \ = \ -\mathrm{Im}\left[\bar{\chi}^c_+\chi_-\right] \ , \quad \mathrm{Re}\left[\bar{\chi}^c_+\chi_-\right] \ = \ 0 \ , \\
\mathrm{Re}\left[\bar{\chi}^c_+\gamma_{(1)}\chi_-\right] \ &= \ 0 \ , \quad \bar{\chi}_+\gamma_{(1)}\chi_+ \ = \ -\bar{\chi}_-\gamma_{(1)}\chi_- \ , \\
\mathrm{d}\eta \ &= \ -\frac{12}{m}\mathrm{e}^{3\Delta}\mathrm{Re}\left[\bar{\chi}^c_+\gamma_{(2)}\chi_-\right] \ .
\end{aligned} \tag{2.4.6}$$

These equations may all be found in reference [52].

The one-form η is a *contact form* on Y_7, meaning that the top form $\eta\wedge(\mathrm{d}\eta)^3$ is nowhere zero, see section 2.4.4 for more detail on contact geometry. Indeed, one finds [52] that

$$\eta\wedge(\mathrm{d}\eta)^3 = \frac{2^7 3^4}{m^3}\mathrm{e}^{9\Delta}\mathrm{vol}_7 \ , \tag{2.4.7}$$

where vol_7 is the Riemannian volume form defined by g_{Y_7}. It is a general fact that a contact form η has associated to it a unique *Reeb vector field*, defined by the relations

$$\xi \lrcorner \eta = 1 \ , \qquad \xi \lrcorner \mathrm{d}\eta \ = \ 0 \ , \tag{2.4.8}$$

and remarkably one finds that ξ and η defined by (2.4.5) indeed satisfy these equations. Moreover, ξ is a Killing vector field under which $\chi_\pm$ carry charges ± 2, i.e. $\mathcal{L}_\xi \chi_\pm = \pm 2i\chi_\pm$, and as such is the expected R-symmetry vector field.

Dirac quantisation in this background implies that

$$N = -\frac{1}{(2\pi\ell_p)^6}\int_{Y_7} *_{11}G + \frac{1}{2}C\wedge G \tag{2.4.9}$$

should be an integer. This may be identified with the M2-brane charge of the background, and a computation [51, 52] gives

$$N = \frac{1}{(2\pi\ell_p)^6}\frac{m^2}{2^5 3^2}\int_{Y_7}\eta\wedge(\mathrm{d}\eta)^3\ , \tag{2.4.10}$$

relating the quantised M2-brane charge to the contact volume (2.2.2) of Y_7 and m. Since this is proportional to m^2, in fact the contact form in (2.4.5) may be defined only when this charge is non-zero, so that $m \neq 0$. We assume this henceforth.

The above supergravity solution of M-theory is valid only in the large N limit, even for solutions with non-trivial warp factor Δ and internal four-form flux F_4. To see this [52], note that the scaling symmetry of eleven-dimensional supergravity in which the metric g_{11} and four-form G have weights two and three, respectively, leads to a symmetry in which one shifts $\Delta \to \Delta + \kappa$ and simultaneously scales $m \to \mathrm{e}^{3\kappa}m$, $F_4 \to \mathrm{e}^{3\kappa}F_4$, where κ is any real constant. We may then take the metric g_{Y_7} on Y_7 to be of order $\mathcal{O}(1)$ in N, and conclude from the quantisation condition (2.4.10), which has weight 6 on the right hand side, and the expression for $m\mathrm{e}^{-3\Delta}$ in (2.4.6) that $\mathrm{e}^\Delta = \mathcal{O}(N^{1/6})$. It follows that the AdS_4 radius, while dependent on Y_7 in general, is $R_{\mathrm{AdS}_4} = \mathrm{e}^\Delta = \mathcal{O}(N^{1/6})$, and that the supergravity approximation we have been using is valid only in the $N \to \infty$ limit.

2.4.2 Choice of M-theory circle

In addition to the supergravity background we must also pick a choice of M-theory circle. Geometrically, this means we also choose a $U(1) = U(1)_M$ action on Y_7. In terms of the supergravity solution described in the previous section, a choice of $U(1)_M$ implies the choice of a (non-$U(1)_R$) Killing vector field ζ_M on (Y_7, g_{Y_7}), whose flow generates the M-theory circle action. In particular ζ_M should preserve the Killing spinors $\chi_\pm$ on Y_7, and hence also the contact one-form η. The type IIA spacetime is then a warped product $\mathrm{AdS}_4 \times M_6$, where $M_6 \equiv Y_7/U(1)_M$ is the quotient space.

Of course globally we must be careful when writing $M_6 = Y_7/U(1)_M$. Although in principle one might choose any $U(1)_M$ action on Y_7, in practice the gauge theories we study arise from 'nice' actions of $U(1)_M$. In particular, if the action is *free*, i.e. only the identity fixes any point on Y_7, then M_6 inherits the structure of a smooth manifold from Y_7, the simplest example being that of the ABJM theory with $M_6 = \mathbb{CP}^3 = S^7/U(1)_{\mathrm{Hopf}}$. If one embeds $S^7 \subset \mathbb{C}^4$ as a unit sphere in the obvious way, then recall that $U(1)_{\mathrm{Hopf}}$ may be taken to have weights $(1, 1, -1, -1)$ on the four complex coordinates (z_1, z_2, z_3, z_4) on $\mathbb{C}^4$. In this case the dual field theory is the $\mathcal{N} = 6$ ABJM theory, which in $\mathcal{N} = 2$ language is a $U(N) \times U(N)$ Chern-Simons gauge theory with two chiral matter fields A_1, A_2 in the bifundamental $(\mathbf{N}, \overline{\mathbf{N}})$ representation of this gauge group, two chiral matter fields B_1, B_2 in the conjugate $(\overline{\mathbf{N}}, \mathbf{N})$ representation, and a quartic superpotential.

2.4.3 BPS M2-brane probes

The supersymmetric M2-brane which is conjectured to be holographically dual to the Wilson loop on S^3 must necessarily have as boundary a Hopf circle in S^3. A convenient explicit form for the Euclidean AdS_4 metric can be taken to be

$$g_{\mathrm{AdS}_4} = \frac{\mathrm{d}q^2}{1+q^2} + q^2 \mathrm{d}\Omega_3 \ , \tag{2.4.11}$$

with $\mathrm{d}\Omega_3$ the round metric on the unit sphere S^3, and $q \in [0, \infty)$ a radial coordinate. The M2-branes of interest then wrap $\Sigma_2 \times S^1_M$, where the surface $\Sigma_2 \subset \mathrm{AdS}_4$ has boundary $\partial\Sigma_2 = S^1_{\mathrm{Hopf}} \subset S^3$, and $S^1_M \subset Y_7$ is the M-theory circle. The submanifold Σ_2 is then parametrised by the radial direction q in AdS_4, and a geodesic Hopf circle

$S^1_{\rm Hopf}$ in S^3, whilst $S^1_M \subset Y_7$ is *a priori* arbitrary (imposing supersymmetry will later give restrictions on S^1_M). The area of the surface Σ_2 in AdS$_4$ is divergent, but can be regularised by subtracting the length of its boundary, *i.e.* the length of the $S^1_{\rm Hopf}$ geodesic in S^3 at $q \to \infty$. Notice this is then a local boundary counterterm. Including also the warp factor one finds the regularised area to be

$$\mathrm{Vol}(\Sigma_2) = -\frac{\pi}{2}\mathrm{e}^{2\Delta} \ . \tag{2.4.12}$$

The action of the M2-brane then reads

$$\begin{aligned} S_{\rm M2} &= \frac{1}{(2\pi)^2\ell_p^3}\left[\mathrm{Vol}(\Sigma_2\times S^1_M) + \int_{\Sigma_2\times S^1_M} C\right] \\ &= -\frac{1}{(2\pi)^2\ell_p^3}\frac{\pi}{2}\int_{S^1_M}\mathrm{e}^{3\Delta}\mathrm{vol}_{S^1_M} \ , \end{aligned} \tag{2.4.13}$$

where $\mathrm{vol}_{S^1_M}$ is the volume form on S^1_M induced from the metric g_{Y_7} and it is easily seen from the form of $G = \mathrm{d}C$ in (2.4.3) that the C field does not contribute to the action.

As mentioned above, imposing that the M2-brane $\Sigma_2 \times S^1_M$ is supersymmetric gives restrictions on the possible circles S^1_M. To see this, we need to split the Clifford algebra Cliff$(11,0)$ generated by gamma matrices $\tilde{\Gamma}_A$ satisfying $\{\tilde{\Gamma}_A, \tilde{\Gamma}_B\} = 2\delta_{AB}$ into Cliff$(4,0) \otimes$ Cliff$(7,0)$ via

$$\tilde{\Gamma}_\mu \ = \ \Gamma_\mu \otimes 1 \ , \qquad \tilde{\Gamma}_{a+3} \ = \ \Gamma_5 \otimes \gamma_a \ , \tag{2.4.14}$$

where $\mu, \nu = 0,1,2,3$ and $a, b = 1,\ldots,7$ are orthonormal frame indices for Euclidean AdS$_4$ and Y_7 respectively, $\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$ and we have defined $\Gamma_5 \equiv \Gamma_0\Gamma_1\Gamma_2\Gamma_3$. If we denote by X^M the embedding coordinates of the worldvolume of the M2-brane into the target geometry, the amount of preserved supersymmetry is equal to the number of spinors ϵ_{11}, as in (2.4.4), satisfying the projection condition [62]

$$\mathbb{P}\epsilon_{11} \;=\; 0\;, \quad \text{where} \quad \mathbb{P} \;\equiv\; \frac{1}{2}\left(1-\frac{\mathrm{i}}{3!}\varepsilon^{\alpha\beta\gamma}\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P\tilde{\Gamma}_{MNP}\right)\;, \qquad (2.4.15)$$

with α, β, γ indices on the worldvolume. We now choose an orthonormal frame in eleven-dimensions as (*c.f.* (2.4.3))

$$E^0 = \frac{1}{2}\mathrm{e}^{\Delta}\frac{\mathrm{d}q}{\sqrt{1+q^2}}\;, \quad E^m \;=\; \frac{1}{2}\mathrm{e}^{\Delta}q e^m_{S^3}\;, \quad E^{3+a} \;=\; \mathrm{e}^{\Delta}e^a_{Y_7}\;, \qquad (2.4.16)$$

where $\{e^m_{S^3}\}_{m=1,2,3}$ is an orthonormal frame on S^3 and $\{e^a_{Y_7}\}_{a=1,\ldots 7}$ is an orthonormal frame on (Y_7, g_7), with $e^1_{Y_7}$ (or rather its dual vector field) aligned along the M-theory circle vector field ζ_M. Taking $e^3_{S^3}$ to be aligned along the Hopf circle, as in section 2.3.1, the projector $\mathbb{P}$ then takes the simple form

$$\mathbb{P} = \frac{1}{2}\left(1 - \mathrm{i}\Gamma_5\Gamma_{03}\otimes\gamma_1\right)\;, \qquad (2.4.17)$$

and the constraints that follow on the spinors $\psi_\pm$, $\chi_\pm$ on Euclidean AdS_4 and Y_7, respectively, are

$$(1-\mathrm{i}\Gamma_5\Gamma_{03})\psi_\pm \;=\; 0\;, \quad \text{and} \quad (1-\gamma_1)\chi_\pm \;=\; 0\;. \qquad (2.4.18)$$

In order to determine how much supersymmetry is preserved by the brane in AdS_4, we must count the number of Killing spinors $\psi_\pm$ that satisfy the last projection equation. We may decompose the four-dimensional gamma matrices into $\Gamma_0 = 1\otimes\tau_3$ and $\Gamma_i = \tau_i\otimes\tau_1$, with the Pauli matrices τ_i, $i=1,2,3$. These matrices act on spinors of the form $\psi = (\psi_1, \psi_2)^T$, with $\psi_{1,2}$ 2-component spinors. The Killing spinors on AdS_4 may then be constructed from Killing spinors on the S^3 at fixed radial coordinate q. Explicitly, if ε solves the Killing spinor equation

$$\nabla_i\varepsilon = \frac{\mathrm{i}}{2}\tau_i\varepsilon\;, \qquad (2.4.19)$$

on S^3, then

$$\psi \;=\; \begin{pmatrix} (q+\sqrt{1+q^2})^{1/2}\varepsilon \\ (q+\sqrt{1+q^2})^{-1/2}\varepsilon \end{pmatrix} \;, \qquad (2.4.20)$$

is a Killing spinor on Euclidean AdS_4. Equation (2.4.19) has two solutions, one being chiral and one anti-chiral, *i.e.* $\tau_3\varepsilon = \pm\varepsilon$. One then easily shows that the first projection equation in (2.4.18) is satisfied if we restrict to chiral ε in the last solution, which singles out one of these two spinors on AdS_4[3]. Hence the M2-brane preserves half of the supersymmetry in AdS_4. Note that the same positive chirality condition also appeared in the supersymmetry condition derived in the field theory context, *c.f.* (2.3.6).

The second projection equation in (2.4.18) tells us which circles S^1_M give rise to supersymmetry-preserving M2-branes. Following a standard argument one notices that

$$\bar{\chi}_+\left(\frac{1-\gamma_1}{2}\right)\chi_+ \;=\; \bar{\chi}_+\left(\frac{1-\gamma_1}{2}\right)^\dagger\left(\frac{1-\gamma_1}{2}\right)\chi_+ \;=\; \left|\left(\frac{1-\gamma_1}{2}\right)\chi_+\right|^2 \;\geq\; 0\;, \qquad (2.4.21)$$

using $\gamma_1 = \gamma_1^\dagger$ and $\gamma_1^2 = 1$. This immediately gives $\mathrm{vol}_{S^1_M} \geq \bar{\chi}_+\gamma_{(1)}\chi_+$ (with a pull-back understood), with equality if and only if some supersymmetry is preserved by S^1_M. The action (2.4.13) for a supersymmetric brane is then

$$S_{\mathrm{M2}} \;=\; \frac{\mathrm{Vol}(\Sigma_2\times S^1_M)}{(2\pi)^2\ell_p^3} \;=\; -\frac{1}{(2\pi)^2\ell_p^3}\frac{\pi}{2}\int_{S^1_M}\mathrm{e}^{3\Delta}\bar{\chi}_+\gamma_{(1)}\chi_+\;. \qquad (2.4.22)$$

With the help of equations (2.4.5) and (2.4.10) the action of a supersymmetric M2-brane can be rewritten in terms of the contact form η as (taking a convention in which $m<0$)

$$S_{\mathrm{M2}} = -\frac{(2\pi)^2\int_{S^1_M}\eta}{\sqrt{2\int_{Y_7}\eta\wedge(\mathrm{d}\eta)^3}}N^{1/2}\;. \qquad (2.4.23)$$

[3]The other two Killing spinors on AdS_4 are constructed from spinors on S^3 satisfying $\nabla_i\varepsilon = -\frac{\mathrm{i}}{2}\tau_i\varepsilon$. We set the corresponding spinors to zero in section 2.3, as they are not used in the supersymmetric localisation. Again, one chirality is broken by the M2-brane.

2.4.4 Contact geometry, Hamiltonian functions and Sasaki-Einstein manifolds

Because the next two sections are more mathematical and require the use of some results coming from differential geometry, this section will present some definitions and results that will be of importance.

As we mentioned in section 2.4.1, the supergravity solution naturally yields a contact form η on Y_7. More generally, a manifold Y of dimension $2n-1$ is called *contact* if there exists a one-form η, called the *contact form*, such that the top form $\eta \wedge (\mathrm{d}\eta)^{n-1}$ is nowhere zero. It is a general fact that a contact form η has associated to it a unique *Reeb vector field*, defined by the relations

$$\xi \lrcorner \eta = 1 \, , \qquad \xi \lrcorner \mathrm{d}\eta \ = \ 0 \, . \tag{2.4.24}$$

Moreover, $\mathrm{d}\eta$ is a symplectic form on $\ker \eta$, the rank $(2n-2)$ subbundle of the tangent bundle TY of Y defined as vectors having zero contraction with η. We recall that a symplectic form is a closed non-degenerate differential two-form. Hence, since this means that $\mathrm{d}\eta$ is non-degenerate on this rank $(2n-2)$ bundle, the tangent bundle can be decomposed as $TY = \ker \eta \oplus \langle \xi \rangle$, where $\langle \xi \rangle$ is the real line bundle spanned by vectors proportional to ξ.

Every contact manifold is equivalent to a *symplectic cone* $(X = C(Y) = \mathbb{R}^+ \times Y, \omega = \frac{1}{2}\mathrm{d}(r^2\eta))$ with metric

$$\mathrm{d}s_X^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s_Y^2 \tag{2.4.25}$$

and symplectic form ω. On a symplectic manifold, if there exists a vector field V generating a $U(1)$ action on X such that

$$V \lrcorner \omega = -\mathrm{d}H \tag{2.4.26}$$

then $H : X \to \mathbb{R}$ is called a *moment map* or *Hamiltonian function* for V. It is a general fact that any component of the moment map for a compact group action on a symplectic manifold is a *Morse-Bott function.* A Morse-Bott function on a compact and connected symplectic manifold X is such that the image interval $H(X) = [h_{\min}, h_{\max}]$ is divided into P subintervals $h_{\min} = h_1 < h_2 < \cdots < h_{P+1} = h_{\max}$, where the *critical set* maps as $H(\{\mathrm{d}H = 0\}) = \{h_i \mid i = 1, \ldots, P+1\}$. For all $h \in (h_i, h_{i+1})$ the level surfaces $H^{-1}(h) \subset X$ are diffeomorphic to a fixed $(2n-2)$-manifold, with the topology changing precisely as one passes a critical point h_i.

If a $2n$-dimensional manifold X is equipped with a complex structure J, i.e. a smooth vector bundle isomorphism $J : TX \to TX$ which squares to $J^2 = -1$ and whose Nijenhuis tensor vanishes, then X is said to be a *complex manifold* and holomorphic coordinates can be defined on X. When (X, ω, J, g) is symplectic and complex with Riemannian metric g and if the triplet (ω, J, g) is compatible, i.e. $\omega(u, v) = g(Ju, v)$ for all $u, v \in TX$ then (X, J, g) is called a *Kähler manifold* with *Kähler form* ω. Kähler manifolds have very interesting properties. For example, their holonomy group is contained in $U(n)$ and there always exists a canonical spin$^{\mathrm{c}}$ spinor on such manifolds. This fact will turn out to be very useful in the next chapter.

If the symplectic cone $X = C(Y)$ defined above is also Kähler then the manifold Y is by definition called *Sasakian.* In this case, the contact volume defined in equation (2.2.2) coincides with the Riemannian volume. Furthermore, if X is Kähler and Ricci-flat, which is our definition of *Calabi-Yau*, then the base manifold Y is necessarily an *Einstein manifold*, i.e. its Ricci tensor is proportional to its metric, and we will call it a Sasaki-Einstein manifold. Interestingly, if the manifold Y is Einstein but not necessarily Sasakian, its contact volume is also equal to its Riemannian volume. In all examples of Wilson loops and M2-branes calculations that we will look at, Y_7 is a Sasaki-Einstein manifold with an extra *toric* structure. Those complex differential geometric structures will allow us to develop a rather general and simple method to compute the M2-brane actions, see section 2.4.6.

2.4.5 M-theory Hamiltonian function

In this section we further elucidate the geometry associated to the supersymmetric M2-branes. This geometric structure will both be of practical use, when we come to compute the M2-brane actions (2.4.23) in examples, and also, as we will see, is realised rather directly in the large N dual matrix model.

We begin by introducing the *M-theory Hamiltonian function*

$$h_M \equiv \eta(\zeta_M) \ = \ \zeta_M \lrcorner \eta \ , \tag{2.4.27}$$

where ζ_M generates the M-theory circle action. This is a real function on Y_7, and since ζ_M is assumed to preserve the Killing spinors and metric on Y_7, it follows that ζ_M preserves h_M and commutes with the Reeb vector field ξ. We then have that the contact length of an M-theory circle S^1_M over a point $p \in M_6 = Y_7/U(1)_M$ is given by $\int_{S^1_M} \eta = 2\pi h_M(\hat{p})$, where $\hat{p} \in Y_7$ is any lift of the point p. This directly leads to the form of the M2-brane action (2.2.4).

One way to characterise the *supersymmetric* M-theory circles S^1_M is to note that on $TY_7 \mid_{S^1_M}$ the vector ζ_M is necessarily proportional to the Reeb vector. Indeed, using (2.4.6) one can show that at these supersymmetric points

$$\zeta_M \lrcorner \mathrm{d}\eta = 0 \ . \tag{2.4.28}$$

To see this one takes the projection condition (2.4.18) with χ_-, applies $\bar{\chi}^c_+ \gamma_a$ on the left, and then takes the real part of the resulting equation. Using $\mathrm{Re}\,[\bar{\chi}^c_+ \chi_-] = \mathrm{Re}\,[\bar{\chi}^c_+ \gamma_a \chi_-] = 0$ and the relation between $\mathrm{d}\eta$ and $\mathrm{Re}\,[\bar{\chi}^c_+ \gamma_{(2)} \chi_-]$ in (2.4.6) then leads to (2.4.28). That this then implies $\zeta_M \propto \xi$ follows from the fact that $TY_7 = \ker \eta \oplus \langle \xi \rangle$ and that $\mathrm{d}\eta$ is non-degenerate on $\ker \eta$ because Y_7 is a contact manifold. Equation (2.4.28) implies that the projection of ζ_M onto $\ker \eta$ is zero, *i.e.* that $\zeta_M \propto \xi$.

The condition (2.4.28) is then also the condition that we are at a critical point of the Hamiltonian h_M. To see this, ζ_M preserving η is written $\mathcal{L}_{\zeta_M} \eta = 0$ and, using the Cartan formula, (2.4.28) is equivalent to

$$\mathrm{d}(\zeta_M \lrcorner \eta) = 0 \qquad \Leftrightarrow \qquad \mathrm{d}h_M \ = \ 0 \ . \tag{2.4.29}$$

Thus the supersymmetric M2-branes lie precisely on the critical set $\{\mathrm{d}h_M = 0\}$, and their action (2.2.4) is determined by h_M evaluated at the critical point. Recall that the cone $C(Y_7)$ is symplectic, with symplectic form $\omega = \frac{1}{2}\mathrm{d}\,(r^2\eta)$. The M-theory circle action then gives a $U(1)_M$ action on this cone, with moment map $\mu = \frac{1}{2}r^2 h_M$ because $\zeta_M \lrcorner \omega = -\mathrm{d}\mu$. Thus μ is Morse-Bott, and the restriction of μ to Y_7 at $r = 1$ is our Hamiltonian function $h_M/2$. We thus know that the image $h_M(Y_7) = [c_{\min}, c_{\max}]$ is a closed interval, and this is further subdivided into P intervals via $c_{\min} = c_1 < c_2 < \cdots < c_{P+1} = c_{\max}$, where the c_i are images under h_M of the critical set $\{\mathrm{d}h_M = 0\}$. On each open interval $c \in (c_i, c_{i+1})$ the level surfaces $h_M^{-1}(c)$ are all diffeomorphic to the same fixed six-manifold, with the topology changing as one crosses a critical point c_i.

Finally, since at a supersymmetric S^1_M we have $\zeta_M \propto \xi$, it follows that the corresponding point $p \in M_6 = Y_7/U(1)_M$ is a *fixed point* under the induced Reeb vector action on $M_6 = Y_7/U(1)_M$. That is, over every fixed point $p \in M_6$ of ξ, there exists a calibrated and supersymmetric M-theory circle $S^1_{M,p}$ whose corresponding supersymmetric M2-brane action is given by (2.2.4).

In the holographic computation of the Wilson loop VEV via the M2-brane action, one should *sum* $\mathrm{e}^{-S_{\mathrm{M2},p}}$ over all contributions. In some cases we shall find that the supersymmetric points $p \in M_6$ form *submanifolds* which are fixed by ξ, and this sum in fact becomes an integral over the different connected submanifolds. Notice that h_M is constant on each connected component of the fixed point set. In any case, in the large N limit only the *longest* circle S^1_M survives, the others being exponentially suppressed relative to it in the sum/integral, hence proving formula (2.2.1).

The calculation of the action of a supersymmetric M2-brane can be completely carried out once the Reeb vector field ξ and the M-theory circle generator ζ_M are known. Indeed, the contact volume $\mathrm{Vol}_\eta(Y_7)$ is a function only of the Reeb vector [53], and the length of a calibrated circle $\int_{S^1_{M,p}} \eta = 2\pi h_M(\hat{p})$ depends only on ξ, ζ_M and the point p. Even though this could appear to be involved, the computation of these two quantities is relatively straightforward for appropriate classes of Y_7. In particular, if we focus on toric Sasaki-Einstein manifolds, some geometrical techniques can be exploited to straightforwardly find *all* calibrated circles, *i.e.* the connected components of the critical set $\{\mathrm{d}h_M = 0\} \subset Y_7$, as well as the contact volume [57]. This is the subject of the next section.

2.4.6 Toric Sasaki-Einstein manifolds and BPS M2-brane actions

For appropriate classes of examples, namely toric Sasaki-Einstein manifolds Y_7, various quantities we have been discussing can be efficiently computed. When Y_7 is *toric* there exists a $U(1)^4$ action that acts on Y_7 and preserves the contact form η. In this case there are some pretty geometric methods, first developed in [56, 57], that may be utilised to calculate the length of the calibrated M-theory circles, as well as the volumes of the internal spaces. We will thus focus on this class of solutions.

Let us begin with the symplectic cone $C(Y)$ of section 2.4.4 in general dimension $2n$. Equivalently, (Y, η) is contact with $\dim Y = 2n - 1$. The toric condition means that $U(1)^n$ acts on the symplectic cone $C(Y)$ preserving the symplectic form ω, and we may parametrise the generating vector fields as ∂_{ϕ_i}, with $\phi_i \in [0, 2\pi)$ and $i = 1, \ldots, n$. This allows one to introduce symplectic coordinates (y_i, ϕ_i) in which the symplectic form on $C(Y)$ has the simple expression

$$\omega \;=\; \sum_{i=1}^{n} \mathrm{d}y_i \wedge \mathrm{d}\phi_i \; . \tag{2.4.30}$$

The coordinates y_i are moment maps for the $U(1)^n$ generated by the ∂_{ϕ_i} as

$$\partial_{\phi_i} \lrcorner \omega = -\mathrm{d}y_i \ . \tag{2.4.31}$$

Moreover, when the toric cone is such that ξ lies in the Lie algebra of $U(1)^n$, which will be assumed here, the coordinates y_i take values in a convex polyhedral cone $\mathcal{C}^* \subset \mathbb{R}^n$ [63]. If this cone has d facets, we have corresponding outward primitive normal vectors to these facets, $v_a \in \mathbb{Z}^n$, $a = 1, \ldots, d$, with the facets corresponding to the fixed point sets of $U(1) \subset U(1)^n$ with weights v_a. In particular this set-up applies to toric Sasakian Y [56], in which the symplectic cone $C(Y)$ is also Kähler. In this case, if one adds the condition that $C(Y)$ is Calabi-Yau, it is equivalent to the existence of an $SL(n, \mathbb{Z})$ transformation such that the normal vectors take the form $v_a = (1, w_a)$, for all a, with $w_a \in \mathbb{Z}^{n-1}$. In this basis, the first component of the Reeb vector is necessarily $\xi_1 = n$ [56].

In general the components of $\xi = \sum_{i=1}^n \xi_i \partial_{\phi_i}$ form a vector $\vec{\xi} = (\xi_1, \ldots, \xi_n)$ that defines the *characteristic hyperplane* in $\mathbb{R}^n$: $\{\vec{y} \in \mathbb{R}^n \mid \vec{\xi} \cdot \vec{y} = \frac{1}{2}\}$. This hyperplane interesects $\mathcal{C}^*$ to form a finite polytope Δ_ξ, and the contact volume of the base Y is related to the volume of this polytope by[4]

$$\mathrm{Vol}_\eta(Y) \ = \ 2n(2\pi)^n \mathrm{Vol}(\Delta_\xi) \ . \tag{2.4.32}$$

Moreover, each of the d facets $\mathfrak{F}_a$, intersected with the characteristic hyperplane, are images under the moment map of $(2n-3)$-dimensional subspaces Σ_a of Y. The volumes of these submanifolds may be calculated once the volumes of the facets are known, for

$$\mathrm{Vol}_\eta(\Sigma_a) \ = \ (2n-2)(2\pi)^{n-1} \frac{1}{|v_a|} \mathrm{Vol}(\mathfrak{F}_a) \ . \tag{2.4.33}$$

[4]Remember that in the Sasakian case the Riemannian volume and contact volumes coincide.

In addition, the volume of the base manifold Y is simply given by

$$\mathrm{Vol}_\eta(Y) \;=\; \frac{(2\pi)^n}{\xi_1}\sum_{a=1}^{d}\frac{1}{|v_a|}\mathrm{Vol}(\mathfrak{F}_a)\ . \qquad (2.4.34)$$

In [57] the idea is to study the space of Kähler cone metrics on $C(Y)$, and thus Sasakian structures on Y. One then considers the Einstein-Hilbert action (with a fixed positive cosmological constant) restricted to this space of Sasakian metrics on Y, so that a Sasaki-Einstein metric on Y is a critical point. In fact the action is minimised and proportional to the volume of the base $\mathrm{Vol}(Y)$ when the metric on Y is Sasaki-Einstein. In this case there is *unique* Reeb vector of the form $\vec{\xi} = (n, \xi_2, \ldots, \xi_n)$ such that the Einstein-Hilbert action, or equivalently $\mathrm{Vol}(Y)$, is minimised as a function of ξ. Thus, for any given toric diagram one calculates $\mathrm{Vol}(Y)$ with formula (2.4.34) as a function of the Reeb vector, and determines $\vec{\xi}$ for the Sasaki-Einstein metric on Y by minimising this function[5].

In this thesis we need only apply this method for $n = 4$. A way to compute $\mathrm{Vol}(\mathfrak{F}_a)$ as a function of the Reeb vector for $n = 4$ has been described in [13]. If the facet $\mathfrak{F}_a$ is a tetrahedron, its vertex is at the origin in $\mathcal{C}^*$ and its base is a triangle lying in the characteristic hyperplane. This is generated by three edges passing from the characteristic hyperplane to the origin, and bounded by four hyperplanes creating the polyhedron. In addition to v_a, three other facets are then involved in the construction of the tetrahedron, and we denote their normal vectors as $v_{a,1}, v_{a,2}, v_{a,3}$. The volume of the tetrahedron may be expressed as

$$\frac{1}{|v_a|}\mathrm{Vol}(\mathfrak{F}_a) \;=\; \frac{1}{48}\frac{(v_a, v_{a,1}, v_{a,2}, v_{a,3})^2}{|(\xi, v_a, v_{a,1}, v_{a,2})(\xi, v_a, v_{a,1}, v_{a,3})(\xi, v_a, v_{a,2}, v_{a,3})|}\ , \qquad (2.4.35)$$

with $(\cdot,\cdot,\cdot,\cdot)$ the determinant of a 4×4 matrix. If the facet $\mathfrak{F}_a$ is not a tetrahedron, *i.e.* there are more than 3 edges that meet at a vertex in the toric diagram (*c.f.* below), the volume can be computed with the same formula by breaking up the facet

[5]That the Sasaki-Einstein metric indeed always exists was proven in [64].

into tetrahedrons.

The *toric diagram* for the toric Calabi-Yau cone is by definition the convex hull of the lattice vectors w_a in $n-1=3$ dimensions. To each vertex in this diagram corresponds a facet $\mathfrak{F}_a$. If the vertex is located at the intersection of three planes, or equivalently three edges of the toric diagram meet at the vertex, then it corresponds to a tetrahedron. If instead four edges meet at the vertex, the facet is a pyramid that can be split into two tetrahedrons, and so on. A given facet $\mathfrak{F}_a$ then corresponds to a vector $v_a=(1,w_a)$, with w_a a vertex in the toric diagram; the other three vectors $v_{a,1},v_{a,2},v_{a,3}$ are the outward-pointing primitive vectors corresponding in the toric diagram to the three edges that meet at the vertex v_a. Let us also note that the base Y_7 of the cone is a *smooth manifold* only if each face of the toric diagram is a triangle, and there are no lattice points internal to any edge or face.

It should now be clear that once a toric diagram is given for a toric Calabi-Yau cone $C(Y)$, one can calculate the volume of the base $\mathrm{Vol}_\eta(Y)$ as a function of the toric data and the Reeb vector that is parametrised by $\vec{\xi}=(4,\xi_2,\xi_3,\xi_4)$. After minimising the volume with respect to ξ_2,ξ_3,ξ_4, one obtains the Reeb vector and $\mathrm{Vol}_\eta(Y)$ as a function of the toric data only.

Next we turn to the M-theory Hamiltonian function h_M, and the computation of the calibrated circles in Y_7 and their lengths. This involves, by definition, the choice of an M-theory circle ζ_M, as described in section 2.4.2. As we proved in this section, supersymmetric calibrated S^1_M exist where ζ_M is parallel to ξ. This is equivalent to

$$\zeta_M \;=\; \eta(\zeta_M)\xi \;=\; h_M\xi \;, \tag{2.4.36}$$

as follows by taking the contraction of each side with η. We can conclude that if we know the proportionality constant between ζ_M and ξ, the length of the corresponding calibrated M-theory circle, located over a fixed point p under ξ in M_6, is then simply $2\pi h_M(\hat{p})$ with $\hat{p}\in Y_7$ any point projecting to p. In terms of the toric geometry above, notice that with $\zeta_M = \sum_{i=1}^n \zeta_M^i \partial_{\phi_i}$ we have

$$\zeta_M \lrcorner \omega = -\mathrm{d}\left(\sum_{i=1}^n y_i \zeta_M^i\right) = -\mathrm{d}\left(\frac{1}{2} r^2 h_M\right) \ , \tag{2.4.37}$$

and it follows that on Y_7, i.e. at $r = 1$,

$$h_M = 2\sum_{i=1}^n y_i \zeta_M^i \ , \tag{2.4.38}$$

This may be regarded as a function on the polytope Δ_ξ, that is the image of Y_7 under the moment map.

The only remaining question is how to find where the two vectors ζ_M, ξ are proportional to each other, or equivalently what the critical points of h_M are, and also what the value of h_M at those points is. With the formalism at hand, this is straightforward to answer. Once a toric diagram and ζ_M are given, the Reeb vector and the volume can be found with the method described above. We may then find the solutions to the equation

$$\zeta_M \ = \ \beta\xi + \sum_{a\in I} \alpha_a v_a \ , \tag{2.4.39}$$

with β, α_a real numbers, and $I \subset \{1, \dots, d\}$ a subset of three facets which *intersect*. Geometrically, the intersection of three facets defines an edge of $\mathcal{C}^*$, which corresponds to a *circle* $S^1 \subset Y_7$. This circle is a fixed point set of $U(1)^3 \subset U(1)^4$ defined by the three vectors v_a, $a \in I$, meaning that the generating $U(1)$ vector fields corresponding to v_a are zero over this circle, and hence ζ_M is parallel to ξ. Thus this S^1 is precisely a calibrated circle. The proportionality constant is then $h_M = \eta(\zeta_M) = \beta$, and its length is $2\pi h_M$. Thus our problem boils down to linear algebra on the polyhedral cone.

We make a few further geometrical observations. First, if (2.4.39) holds with $\beta = 0$ then ζ_M actually fixes the S^1. The M-theory circle then has zero length on such loci, formally leading to M2-branes with zero action; if ζ_M acts freely on Y_7 this cannot happen. Next we note that (2.4.39) cannot hold with $\alpha_a = 0$ for *all* $a \in I$, since then ζ_M would be *everywhere* parallel to ξ, and this cannot happen since ζ_M is a non-R symmetry. However, it may happen that (2.4.39) holds with one or two (but not all three) of the coefficients $\alpha_a = 0$. Geometrically, this means that in this case ζ_M is parallel to ξ over the intersection of the corresponding two or one facets with non-zero α_a coefficients, leading to three-dimensional or five-dimensional subspaces of Y_7 which are fibred by calibrated S^1_M circles. These then descend to two-dimensional or four-dimensional fixed point sets of ξ on $M_6 = Y_7/U(1)_M$, respectively. We shall see examples of this in section 2.5. Finally, if the toric diagram contains faces which have more than three sides, then (2.4.39) may hold for I being the corresponding set of 4 or more vectors v_a. In this case the manifold has a locus of singularities along the corresponding S^1 in Y_7, and our theory above does not directly apply to these singular circles.

Even though the above theoretical background may appear cumbersome, it is effectively not difficult to find the volume of Y_7, its Reeb vector ξ and all the calibrated circles and their lengths. Thanks to equation (2.2.4), the action for each corresponding M2-brane follows straightforwardly, and can be compared to the data extracted from the matrix model of the dual field theory. We examine these computations in a variety of examples in section 2.5.

2.4.7 Hamiltonian function and density

In [65, 66] a relation was also found between $\rho(x)$, and other matrix model variables, and certain geometric invariants. In particular, $\rho(x)$ is related to the derivative of a function that counts operators in the chiral ring of the gauge theory according to their R-charge and monopole charges. In the language of the current paper, the monopole charge is the charge under $U(1)_M$. With our notations and conventions, using [66] one can rewrite their conjecture for $\rho(x)$ in the following form:

$$\begin{aligned} \rho(x) &= \frac{4}{\pi^2}\frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}_\eta(Y_7)}}\frac{\partial_r \mathrm{vol}(P_{rc})}{|\xi\wedge\zeta_M|}\bigg|_{r=1} , \\ \text{where} \quad P_{rc} &\equiv \left\{ y\in\mathcal{C}^* \middle| \; \vec{y}\cdot\vec{\xi} = \frac{r}{2} \; , \; \vec{y}\cdot\vec{\zeta}_M = \frac{c}{2} \right\} , \end{aligned} \tag{2.4.40}$$

where the variable c is related to x by (2.2.6). Using equation (2.4.38), we know that for the toric case $\vec{y}\cdot\vec{\zeta}_M = \frac{1}{2}h_M$. If we introduce

$$P_c \equiv \{y\in\mathcal{C}^*| \; h_M = c\} , \tag{2.4.41}$$

we see that P_{rc} is nothing but the intersection of P_c with the characteristic hyperplane. But since the pre-image under the moment map of P_c in Y_7 is the same as $h_M^{-1}(c)$, which changes topology every time we pass through a critical point of h_M, we know that the topology of the pre-image of P_{rc} in Y_7 also changes every time a critical point is crossed. Thus we expect a change of behaviour of $\mathrm{vol}(P_{rc})$ and hence $\rho(x)$ at the critical points x_i that are related to the c_i by (2.2.6). In other words, the eigenvalue density has a different behaviour in each subset (c_i, c_{i+1}), as we will see in the examples in the next section, because there are supersymmetric M2 branes located at the c_i, which are critical points of a Hamiltonian function. That explains why the function $\rho(x)$ has a jump in its derivative precisely at the critical points.

2.5 Examples

In this section we illustrate the duality between geometries and matrix models in a wide variety of examples. In particular we will compute the image of the M-theory Hamiltonian $h_M(Y_7) = [c_{\min}, c_{\max}]$, and show that it coincides with the support of the matrix model eigenvalues $[x_{\min}, x_{\max}]$ via (2.2.6). The critical points of h_M will be shown to map to the points $x = x_i$ where $\rho'(x)$ has a jump discontinuity, with the matching of Wilson loops being a corollary of this result for $x = x_{\max}$.

2.5.1 Duals to the round S^7

We begin by studying two superconformal duals to $\mathrm{AdS}_4 \times S^7$, where S^7 is equipped with its standard round Einstein metric. These differ in the choice of M-theory circle $U(1)_M$ acting on S^7. In this case the geometry is particularly simple, allowing us to illustrate the geometric structures we have described very explicitly.

ABJM theory

The ABJM theory [6] is an $\mathcal{N}=6$ superconformal $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory. In $\mathcal{N}=2$ language, there are two chiral matter fields A_1, A_2 in the bifundamental $(\mathbf{N}, \overline{\mathbf{N}})$ representation of this gauge group, two chiral matter fields B_1, B_2 in the conjugate $(\overline{\mathbf{N}}, \mathbf{N})$ representation, and a quartic superpotential. Here the subscript $k \in \mathbb{Z}$ in $U(N)_k$ denotes the Chern-Simons level for the particular copy of $U(N)$, as in (2.3.2). This theory is dual to $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ with N units of flux (2.4.10), where $\mathbb{Z}_k \subset U(1)_{\mathrm{Hopf}} = U(1)_M$.

We may realise S^7 as the unit sphere $S^7 \subset \mathbb{R}^8 \cong \mathbb{C}^4$ and take $U(1)_{\mathrm{Hopf}}$ to have weights $(1,1,-1,-1)$ on the four complex coordinates (z_1, z_2, z_3, z_4) on $\mathbb{C}^4$. In this description the $U(1)_R$ symmetry of the $\mathcal{N}=2$ subalgebra of the $\mathcal{N}=6$ manifest superconformal symmetry of the theory has weights $(1,1,1,1)$ on $\mathbb{C}^4$, which gives a different Hopf action on $\mathbb{C}^4$.

In these coordinates $S^7 = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1\}$, while the M-theory Hamiltonian function on $S^7/\mathbb{Z}_k$ is

$$h_M = \frac{1}{k}\left(|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2\right) . \tag{2.5.1}$$

In the toric geometry language of section 2.4.6, we have the symplectic coordinates $y_i = \frac{1}{2}|z_i|^2$. The level sets $h_M^{-1}(c)$ are diffeomorphic to $S^3 \times S^3/\mathbb{Z}_k$ for $c \in (-\frac{1}{k}, \frac{1}{k})$. Indeed, notice that these level sets are also described by

$$|z_1|^2 + |z_2|^2 = \frac{1}{2}(1+ck) , \qquad |z_3|^2 + |z_4|^2 \;=\; \frac{1}{2}(1-ck) . \tag{2.5.2}$$

When $c \to \pm\frac{1}{k}$ the $S^3 \times S^3/\mathbb{Z}_k$ level sets thus collapse to two copies of $S^3/\mathbb{Z}_k$ at $\{z_3 = z_4 = 0\}$ and $\{z_1 = z_2 = 0\}$, respectively. Thus the image $h_M(S^7) = \left[-\frac{1}{k}, \frac{1}{k}\right]$, with the endpoints $c_{\max} = -c_{\min} = \frac{1}{k}$ being the only two critical points of the Morse-Bott function h_M.

The contact form in these coordinates is

$$\eta = \frac{\mathrm{i}}{2r^2} \sum_{i=1}^{4} (z_i \mathrm{d}\bar{z}_i - \bar{z}_i \mathrm{d}z_i) \ , \qquad r^2 \ \equiv \ \sum_{i=1}^{4} |z_i|^2 \ . \tag{2.5.3}$$

Being Einstein, the contact volume of $S^7/\mathbb{Z}_k$ is equal to the Riemannian volume, with

$$\mathrm{Vol}(S^7/\mathbb{Z}_k) = \frac{\pi^4}{3k} \ . \tag{2.5.4}$$

Our general formula (2.2.6) thus implies that the matrix model variable x should be related to the geometric quantity c above via

$$x \ = \ \frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}(S^7/\mathbb{Z}_k)}}\, c \ = \ \pi\sqrt{2k}\, c \ . \tag{2.5.5}$$

The large N saddle point eigenvalue distribution for the ABJM theory was given in [37]. The eigenvalues for the two gauge groups are related by

$$\lambda^1(x) = \bar{\lambda}^2(x) \ = \ xN^{1/2} + \mathrm{i}y(x) \ , \tag{2.5.6}$$

where

$$\rho(x) = \frac{\sqrt{k}}{2\pi\sqrt{2}} \ , \qquad y(x) \ = \ \frac{\sqrt{k}}{2\sqrt{2}} x \ , \tag{2.5.7}$$

and the eigenvalues are supported on $[x_{\min}, x_{\max}]$, where $x_{\max} = -x_{\min} = \pi\sqrt{2/k}$. This of course agrees with the geometric formula (2.5.5), and since the density $\rho(x)$ is constant on $(x_{\min}, x_{\max})$ (see Figure 2.2) its derivative is in particular continuous on this region. It is then automatic that the gravity formula (2.2.1) agrees with the

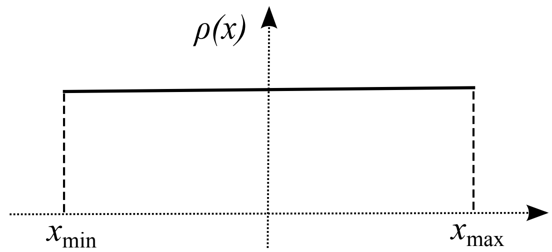

Figure 2.2: Eigenvalue density $\rho(x)$ for the ABJM theory.

field theory formula (2.2.5) for the Wilson loop, giving in both cases

$$\log \langle W \rangle = \pi \sqrt{\frac{2}{k}} N^{1/2} \ . \tag{2.5.8}$$

Mirror theory

The mirror to the ABJM theory (with $k = 1$) arises by choosing a different M-theory circle action on S^7. We will not go into the details of why this theory is called mirror to ABJM but rather see it as another example of an AdS/CFT pair where Wilson loops can be studied. The field theory [18] is $\mathcal{N} = 8$ $U(N)$ super-Yang-Mills theory coupled to two additional fields q, $\tilde{q}$ in the fundamental and anti-fundamental representation of $U(N)$, respectively. The superpotential is

$$\mathcal{W} = \text{Tr} \left(q X_1 \tilde{q} + X_3 [X_1, X_2] \right) \ , \tag{2.5.9}$$

where X_1, X_2, X_3 are the adjoint chiral fields of the $\mathcal{N} = 8$ theory, in $\mathcal{N} = 2$ language. In this case the M-theory circle $U(1)_M$ has weights $(1, -1, 0, 0)$ on $S^7 \subset \mathbb{C}^4$, which has a codimension four fixed point set $\mathscr{F} = S^3 = \{z_1 = z_2 = 0\} \subset S^7$. It follows that the type IIA internal space is $M_6 = S^6$.

Although the background geometry is the same as in the previous subsection, i.e.

$\mathrm{AdS}_4 \times S^7$, the M-theory Hamiltonian is now[6]

$$h_M = |z_1|^2 - |z_2|^2 \ . \tag{2.5.10}$$

The level surfaces $h_M^{-1}(c)$ are described by

$$2|z_1|^2 + |z_3|^2 + |z_4|^2 = 1 + c \ , \qquad 2|z_2|^2 + |z_3|^2 + |z_4|^2 \ = \ 1 - c \ , \tag{2.5.11}$$

so that $c \in [-1, 1]$. However, the critical point set of h_M is quite different to that for the ABJM model. The endpoints $c = +1$, $c = -1$ are now the copies of $S^1 \subset S^7$ at $\{z_2 = z_3 = z_4 = 0\}$ and $\{z_1 = z_3 = z_4 = 0\}$, respectively. (Compare to the ABJM model, where for $k = 1$ also $c \in [-1, 1]$, but with the endpoints being images of copies of S^3, rather than S^1.) Moreover, there is an *additional* critical point at $c = 0$. Indeed, on S^7 we have

$$\begin{aligned} \mathrm{d}h_M &= (z_1 \mathrm{d}\bar{z}_1 + \bar{z}_1 \mathrm{d}z_1) - (z_2 \mathrm{d}\bar{z}_2 + \bar{z}_2 \mathrm{d}z_2) \ , \\ 0 &= \sum_{i=1}^{4} (z_i \mathrm{d}\bar{z}_i + \bar{z}_i \mathrm{d}z_i) \qquad \Leftrightarrow \qquad 0 \ = \ \mathrm{d}r \ . \end{aligned} \tag{2.5.12}$$

Thus in addition to the endpoints $\{z_2 = z_3 = z_4 = 0\}$ and $\{z_1 = z_3 = z_4 = 0\}$, we also have $\mathrm{d}h_M = 0$ at $\{z_1 = z_2 = 0\} = S^3$, which is the fixed point set of $U(1)_M$ where $h_M = 0$. Thus we have the three critical points $c_1 = c_{\min} = -1$, $c_2 = 0$, $c_3 = c_{\max} = 1$.

The topology of the level sets $h_M^{-1}(c)$ is the same for $c \in (-1, 0)$ and $c \in (0, 1)$, but with different circles collapsing on each side. For $c \in (0, 1)$ we may 'solve' $h_M = c$ as $|z_1|^2 = |z_2|^2 + c > 0$, and note that consequently $z_1 \neq 0$ on this locus. From (2.5.11) it follows that $h_M^{-1}(c) \cong S^1_1 \times S^5$, where S^1_1 is parametrised by the phase of $z_1 = |z_1|\mathrm{e}^{\mathrm{i}\phi_1}$. On the other hand, for $c \in (-1, 0)$ instead we solve $h_M = c$ as $|z_2|^2 = |z_1|^2 - c > 0$,

[6] We could similarly choose to quotient by $\mathbb{Z}_k \subset U(1)_M$. However, here we restricted to $k = 1$ in order to compare to the $k = 1$ ABJM theory, which is also dual to $\mathrm{AdS}_4 \times S^7$ (the point being that the $\mathbb{Z}_k$ quotients in each case are different). In fact the general k case is $a = k$, $b = 0$ of section 2.5.3.

so that $h_M^{-1}(c) \cong S_2^1 \times S^5$, where S_2^1 is parametrised by the phase of $z_2 = |z_2|\mathrm{e}^{\mathrm{i}\phi_2}$.

The general formula (2.2.6) implies that the matrix model variable x should be related to the geometric quantity c again via

$$x \;=\; \frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}(S^7)}}\,c \;=\; \pi\sqrt{2}\,c\ , \qquad (2.5.13)$$

which is the same formula as for the ABJM model with $k=1$. The large N saddle point eigenvalue distribution is in fact a special case of the models in section 2.5.3, with $a=1$, $b=0$ in the notation of that section, and appears in [40]. In this case there is only a single gauge group, and one finds the eigenvalue density

$$\rho(x) \;=\; \begin{cases} \frac{1}{2\pi^2}(x - x_{\mathrm{min}})\ , & x_{\mathrm{min}} < x < 0 \\ \frac{1}{2\pi^2}(x_{\mathrm{max}} - x)\ , & 0 < x < x_{\mathrm{max}} \end{cases}\ , \qquad (2.5.14)$$

where $x_{\mathrm{max}} = -x_{\mathrm{min}} = \pi\sqrt{2}$, thus agreeing with (2.5.13). Moreover, the derivative of

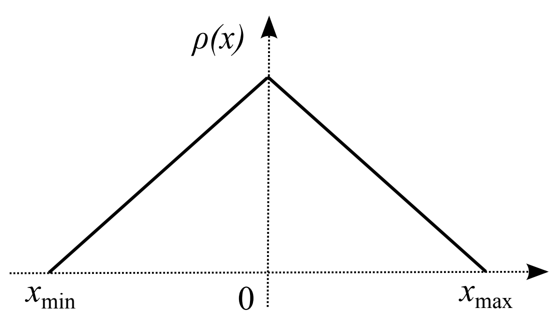

Figure 2.3: Eigenvalue density as a function of x. There are three points where $\rho'(x)$ is discontinuous, corresponding to critical points of h_M.

ρ is discontinuous at the endpoints *and* at the point $x=0$. The Wilson loop is again given by (2.5.8), with $k=1$.

2.5.2 Dual to $Q^{1,1,1}/\mathbb{Z}_k$

Our next example is that of the homogeneous and toric Sasaki-Einstein manifold $Q^{1,1,1}/\mathbb{Z}_k$. The manifold $Q^{1,1,1}$ is the total space of an S^1 fibration over the product of three copies of S^2, *i.e.* $S^1 \hookrightarrow Q^{1,1,1} \to S^2 \times S^2 \times S^2$, which describes its structure as a regular Sasaki-Einstein manifold. Even though this manifold is toric, and the geometrical techniques described in section 2.4.6 can be applied, we will instead take advantage of the fact that the metric is known explicitly on this space.

The Sasaki-Einstein metric on $Q^{1,1,1}$ can be written as

$$g_{Y_7} \;=\; \frac{1}{16}\left(\mathrm{d}\psi + \sum_{i=1}^{3}\cos\theta_i \mathrm{d}\varphi_i\right)^2 + \frac{1}{8}\sum_{i=1}^{3}(\mathrm{d}\theta_i^2 + \sin^2\theta_i \mathrm{d}\varphi_i^2)\ , \tag{2.5.15}$$

where the coordinates $\theta_i \in [0,\pi]$ and $\varphi_i \in [0,2\pi)$ are the usual S^2 coordinates, and the coordinate $\psi \in [0,4\pi)$ parametrises the S^1 fibre. The contact form is simply

$$\eta \;=\; \frac{1}{4}\left(\mathrm{d}\psi + \sum_{i=1}^{3}\cos\theta_i \mathrm{d}\varphi_i\right)\ , \tag{2.5.16}$$

and for the field theory model below the M-theory circle is generated by $\zeta_M = \frac{1}{k}(\partial_{\varphi_1} + \partial_{\varphi_2})$. The M-theory Hamiltonian follows straightforwardly and reads

$$h_M \;=\; \eta(\zeta_M) \;=\; \frac{1}{4k}(\cos\theta_1 + \cos\theta_2)\ . \tag{2.5.17}$$

The length of a supersymmetric M-theory circle is always given by $2\pi h_M(\hat{p})$, where $\hat{p} \in Y_7$ covers a fixed point p of ξ, with $p \in M_6 = Y_7/U(1)_M$. However, when the Sasaki-Einstein manifold is *regular*, as in the case at hand, we may also describe the supersymmetric M-theory circles in terms of the base Kähler-Einstein manifold $B_6 = Y_7/U(1)_R$, where $U(1)_R$ is generated by the Reeb vector ξ. In this point of view, the supersymmetric M-theory circles cover fixed points of ζ_M on B_6, which in the case at hand is $B_6 = S^2 \times S^2 \times S^2$ because $\xi = 4\partial_\psi$. These points are located at $\{(\theta_1,\theta_2) \mid \theta_1 \in \{0,\pi\},\ \theta_2 \in \{0,\pi\}\}$. Thus one obtains three critical values $c_1 = c_{\min} = -\frac{1}{2k}$, $c_2 = 0$, $c_3 = c_{\max} = \frac{1}{2k}$. Notice these are S^2 loci of critical points, parametrised by (θ_3,φ_3).

Being Einstein, the contact volume of $Q^{1,1,1}/\mathbb{Z}_k$ is equal to the Riemannian volume, with

$$\mathrm{Vol}(Q^{1,1,1}/\mathbb{Z}_k) \;=\; \frac{\pi^4}{8k} \;, \tag{2.5.18}$$

and as usual the $\mathbb{Z}_k$ quotient is along $U(1)_M$ generated by ζ_M. The general formula (2.2.6) tells us that the matrix model variable $x_{\max} = -x_{\min}$ predicted from the gravity calculation is

$$x_{\max} \;=\; \frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}(Q^{1,1,1}/\mathbb{Z}_k)}}\, c_{\max} \;=\; \frac{2\pi}{\sqrt{3k}} \;. \tag{2.5.19}$$

A dual field theory to $Q^{1,1,1}/\mathbb{Z}_k$ has been proposed in [16, 18]. This theory is closely related to the ABJM theory. In addition to the bifundamental fields A_i, B_i, a pair of field in the (anti-) fundamental representation is added to each gauge group node, and one adds a cubic term to the superpotential

$$\mathcal{W}_{\text{cubic}} \;=\; \mathrm{Tr}\left(q_1 A_1 \tilde{q}_1 + q_2 A_2 \tilde{q}_2\right) \;. \tag{2.5.20}$$

The corresponding matrix model has been worked out in [39], where it was found that the density of the real part of the eigenvalues is

$$\rho(x) \;=\; \frac{k}{4\pi^2}(2x_{\max} - |x|) \quad \text{for} \quad x_{\min} < x < x_{\max} \;, \tag{2.5.21}$$

with $x_{\max} = \frac{2\pi}{\sqrt{3k}}$, thus agreeing with (2.5.19). Moreover, the derivative of ρ is dis-

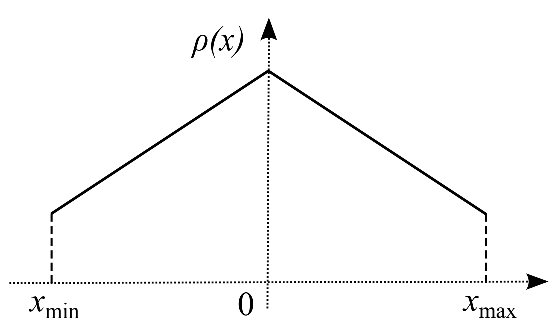

Figure 2.4: Eigenvalue density $\rho(x)$. There are three points where $\rho'(x)$ is discontinuous, associated with supersymmetric M-theory circles.

continuous at the endpoints and at the point $x = 0$, as predicted by c_1, c_2 and c_3 above. The Wilson loop calculated from the field theory then agrees with the gravity computation, and reads

$$\log \langle W \rangle = \frac{2\pi}{\sqrt{3k}} N^{1/2} . \qquad (2.5.22)$$

2.5.3 $\mathcal{N} = 8$ super-Yang-Mills with flavour

In this section we consider a family of theories that generalise the mirror to the ABJM theory discussed in section 2.5.1. These were discussed in [40], having been first introduced in [18].

One begins with $\mathcal{N} = 8$ super-Yang-Mills with gauge group $U(N)$, which is the theory on N D2-branes in flat space. In $\mathcal{N} = 2$ language we have three adjoint chiral matter fields X_1, X_2, X_3, together with the cubic superpotential $\mathrm{Tr}\, X_3[X_1, X_2]$. To this we add matter fields in the fundamental and anti-fundamental representations, which breaks the supersymmetry generically to $\mathcal{N} = 2$. More precisely, we add n_1 fields $(q_j^{(1)}, \tilde{q}_j^{(1)})$, n_2 fields $(q_j^{(2)}, \tilde{q}_j^{(2)})$ and n_3 fields $(q_j^{(3)}, \tilde{q}_j^{(3)})$, together with the cubic superpotential

$$\mathcal{W} = \mathrm{Tr}\left[\sum_{j=1}^{n_1} q_j^{(1)} X_1 \tilde{q}_j^{(1)} + \sum_{j=1}^{n_2} q_j^{(2)} X_2 \tilde{q}_j^{(2)} + \sum_{j=1}^{n_3} q_j^{(3)} X_3 \tilde{q}_j^{(3)} + X_3[X_1, X_2]\right] , \qquad (2.5.23)$$

so that the mirror theory of section 2.5.1 is simply $n_1 = 1$, $n_2 = n_3 = 0$.

In [18] it was shown that the quantum corrected moduli space of vacua of these theories, for $\mathcal{N} = 1$, may be parametrised by the three coordinates X_1, X_2, X_3, together with the monopole operators T, $\tilde{T}$, which satisfy the constraint

$$T\tilde{T} = X_1^{n_1} X_2^{n_2} X_3^{n_3} . \qquad (2.5.24)$$

This defines a Calabi-Yau cone $C(Y_7)$ as a hypersurface singularity in $\mathbb{C}^5$. The M-theory circle is straightforward to identify in this case, since by definition the monopole operators T, $\tilde{T}$ have charges ± 1, respectively, under $U(1)_M$, while the X_i

are uncharged.

The matrix model for this gauge theory can be analysed as described in section 2.3.2 and carried out in [40]. The eigenvalue density is given by

$$\rho(x) = \begin{cases} \frac{(\sum_{i=1}^{3} n_i\Delta_i - 2\Delta_m)}{8\pi^2\Delta_1\Delta_2\Delta_3}(x - x_{\text{min}}) \ , & x_{\text{min}} < x < 0 \\ \frac{(\sum_{i=1}^{3} n_i\Delta_i + 2\Delta_m)}{8\pi^2\Delta_1\Delta_2\Delta_3}(x_{\text{max}} - x) \ , & 0 < x < x_{\text{max}} \end{cases} \ , \tag{2.5.25}$$

and the endpoints are

$$x_{\text{max/min}} = \pm\sqrt{\frac{8\pi^2\Delta_1\Delta_2\Delta_3(\sum_{i=1}^{3} n_i\Delta_i \mp 2\Delta_m)}{(\sum_{i=1}^{3} n_i\Delta_i)(\sum_{i=1}^{3} n_i\Delta_i \pm 2\Delta_m)}} \ . \tag{2.5.26}$$

Here $\Delta_i = \Delta(X_i)$, $i = 1, 2, 3$, are the R-charges of the fields X_i, while $\Delta_m = \Delta(T) = \Delta(\tilde{T})$ is the R-charge of the monopole operators. As described in section 2.3.2, these may be left *a priori* arbitrary at this point, the only restriction being that the superpotential $\mathcal{W}$ has R-charge $\Delta(\mathcal{W}) = 2$. This leads to the constraint $\sum_{i=1}^{3} \Delta_i = 2$. The shape of ρ as a function of x is shown in Figure 2.5.

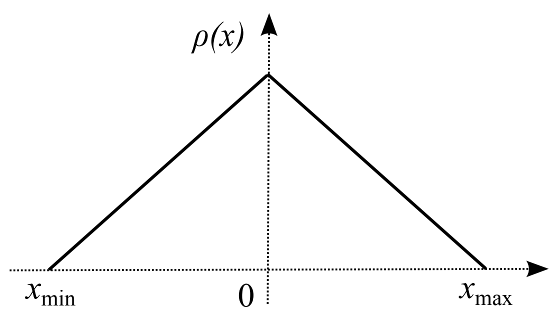

Figure 2.5: Eigenvalue density as a function of x. There are three points where $\rho'(x)$ is discontinuous, and we correspondingly expect to find three critical points of h_M, with associated supersymmetric circles.

The superconformal R-charges are determined by maximising the free energy $\mathcal{F}$ as a function of the R-charges. This immediately leads to $\Delta_m = 0$, and then

$$\mathcal{F} = \frac{2\sqrt{2}\pi\sqrt{\Delta_1\Delta_2\Delta_3\left(\sum_{i=1}^{3} n_i\Delta_i\right)}}{3} N^{3/2} \ , \tag{2.5.27}$$

which must be further maximised subject to the constraint $\sum_{i=1}^{3}\Delta_i = 2$. In practice the formulae are rather too unwieldy for general n_i, so following [40] we restrict to the case $n_1 = a, n_2 = b, n_3 = 0$. In this case the free energy is maximised by

$$\Delta_1 = \frac{a-2b+\sqrt{a^2+b^2-ab}}{2(a-b)} \ , \ \Delta_2 = \frac{b-2a+\sqrt{a^2+b^2-ab}}{2(b-a)} \ , \ \Delta_3 = \frac{1}{2} \ , \tag{2.5.28}$$

and thus

$$x_{\text{max/min}} = \pm 2\pi\sqrt{\frac{\Delta_1\Delta_2}{a\Delta_1+b\Delta_2}} \ . \tag{2.5.29}$$

The moduli space equation (2.5.24) correspondingly reduces to $T\tilde{T} = X_1^a X_2^b$. The field X_3 is then unconstrained, and the Calabi-Yau cone takes the product form $C(Y_7) = \mathbb{C}\times C(Y_5)$, where X_3 is a coordinate on $\mathbb{C}$ and $C(Y_5)$ is precisely the $Y_5 = L^{a,b,a}$ toric singularity. The toric diagram has lattice vectors

$$\begin{aligned} w_1 &= (0,0,0) \ , \quad w_2 = (0,1,0) \ , \quad w_3 = (1,0,0) \ , \\ w_4 &= (0,0,a) \ , \quad w_5 = (0,1,b) \ , \end{aligned} \tag{2.5.30}$$

and is shown in Figure 2.6. Recall that we parametrise the Reeb vector by $\xi = (4,\xi_2,\xi_3,\xi_4)$, and that the four-dimensional outward-pointing vectors to the facets are $v_a = (1, w_a)$. With the method described earlier in section 2.4.6, the volume of the base Y_7 and the Reeb vector can be found and expressed in terms of Δ_1 and Δ_2, and one finds

$$\text{Vol}(Y_7) = \frac{\pi^4}{6}\frac{1}{\Delta_1\Delta_2(a\Delta_1+b\Delta_2)} \ , \tag{2.5.31}$$

and

$$\vec{\xi} = (4,1,2\Delta_2, a\Delta_1+b\Delta_2) \ . \tag{2.5.32}$$

The M-theory circle in this basis is given by $\zeta_M = (0,0,0,-1)$; one can derive this by writing the functions $T,\tilde{T},X_i$ in terms of the toric geometry formalism above (see, for example, section 4.3 of [53]). Recall also that in this formalism the M-theory Hamiltonian function is given by (2.4.38). Thus in this case we have simply

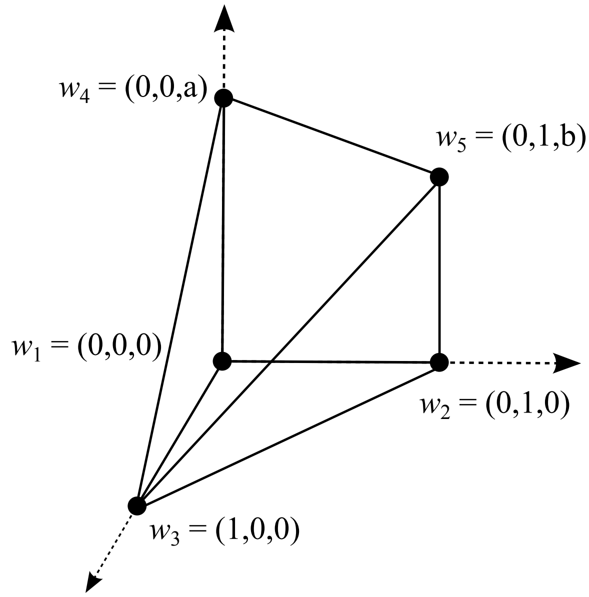

Figure 2.6: Toric diagram corresponding to $C(Y_7) = \{T\tilde{T} = X_1^a X_2^b\} \times \mathbb{C}$. The apex is not an isolated singularity, as one sees from the non-triangular face with vertices $(1, 2, 4, 5)$.

$h_M = -2y_4$. The critical points of h_M must always lie on the boundary of the polyhedral cone, which are coordinate singularities, and thus it is easiest to determine this critical set using the method described at the end of section 2.4.6. We denote the face of the toric diagram which has vertices $\{v_a,\ v_b,\ v_c, \ldots\}$ by $(a, b, c, \ldots)$. Equation (2.4.39) then has two types of solution:

$$\begin{aligned} h_M &= 0 && \text{on} \quad (2,3,5),\ (1,3,4),\ (1,2,4,5)\ , \\ |h_M| &= \frac{1}{a\Delta_1 + b\Delta_2} && \text{on} \quad (1,2,3),\ (3,4,5)\ , \end{aligned} \tag{2.5.33}$$

and correspondingly one has the critical values $h_M = c_i$ given by

$$c_3 = -c_1 = \frac{1}{a\Delta_1 + b\Delta_2}\ , \quad \text{and} \quad c_2 = 0\ . \tag{2.5.34}$$

Notice here that the face $(1, 2, 4, 5)$ (being non-triangular) corresponds to the S^1 locus of $L^{a,b,a}$ conical singularities in Y_7. Using the general formula (2.2.6) we then find that these values of c_i precisely match the corresponding positions x_1, x_2, x_3 at which the derivative of the eigenvalue density $\rho'(x)$ is discontinuous. Finally, using (2.2.4) the Wilson loop is

$$\log \langle W \rangle_{\text{gravity}} = 2\pi \sqrt{\frac{\Delta_1 \Delta_2}{a\Delta_1 + b\Delta_2}} N^{1/2} \;=\; x_{\max} N^{1/2} \;=\; \log \langle W \rangle_{\text{QFT}} \;, \qquad (2.5.35)$$

where we used (2.5.29).

2.5.4 $L^{a,2a,a}$ Chern-Simons-quivers

In this section and the next we study two families of examples whose matrix models were first analysed in [44].

The $\mathcal{N} = 2$ field theories begin life as low-energy theories on N D2-branes at an $L^{a,b,a}$ Calabi-Yau three-fold singularity. This may be simply described as the hypersurface $\{wz = u^a v^b\} \subset \mathbb{C}^4$, where (w, z, u, v) are the coordinates on $\mathbb{C}^4$. This geometry also appeared in the previous subsection of course, but there the M-theory Calabi-Yau four-fold was a product $\mathbb{C} \times C(L^{a,b,a})$, whereas here instead $C(L^{a,b,a})$ arises as the type IIA spacetime. The low-energy theory on the N D2-branes is known from [67–69], and is described by a $U(N)^{a+b}$ gauge theory, with a superpotential $\mathcal{W}$ consisting of both cubic and quartic terms in the bifundamental and adjoint chiral matter fields. Without loss of generality we may take $b \geq a$, in which case there are $b - a$ adjoint chiral superfields associated to $b - a$ of the $a + b$ $U(N)$ gauge group factors, and a total of $2(a + b)$ bifundamental fields. We refer the reader to the above references for further details of these gauge theories.

Following [17] and in particular the construction in [11], the D2-brane theories become M2-brane theories at a Calabi-Yau four-fold. Geometrically the M-theory circle is fibred over the base $C(L^{a,b,a})$, and Chern-Simons couplings for the gauge group are introduced in the field theory, described by a vector of Chern-Simons levels $\vec{k} = (k_1, \ldots, k_{a+b}) = (k_1, \ldots, k_{b-a} \| k_{b-a+1}, \ldots, k_{a+b})$, where the double bar separates the copies of $U(N)$ with adjoint fields from those without. This construction is described in more detail in [44].

Our first class of examples arise from $L^{a,2a,a}$ quiver theories, where the vector of Chern-Simons levels is $\vec{k} = (0, \ldots, 0, -2k \| k, k, -k, k, -k, \ldots, k, -k, k)$, with $k \in \mathbb{Z}$. These theories generalise the model first studied in [38]. The matrix model may be solved using the general large N saddle point method described in section 2.3.2, and one finds [44] the eigenvalue density

$$\rho(x) \quad = \quad \begin{cases} \frac{4ak\pi x(1-\Delta)+\mu}{16a\pi^3(1-\Delta)\Delta^2} \,, & -\frac{\mu}{4ak\pi(1-\Delta)} < x < -\frac{\mu}{2ak\pi(2-\Delta)} \\ \frac{\mu}{16a\pi^3\Delta(2-\Delta)(1-\Delta)} \,, & -\frac{\mu}{2ak\pi(2-\Delta)} < x < \frac{\mu}{2ak\pi(2-\Delta)} \\ -\frac{4ak\pi x(1-\Delta)-\mu}{16a\pi^3(1-\Delta)\Delta^2} \,, & \frac{\mu}{2ak\pi(2-\Delta)} < x < \frac{\mu}{4ak\pi(1-\Delta)} \end{cases} , \qquad (2.5.36)$$

where we have defined[7]

$$\mu = 8a\pi^2 \sqrt{\frac{k\Delta(2-3\Delta+\Delta^2)^2}{4-3\Delta}} \ . \qquad (2.5.37)$$

Here the single R-charge variable Δ parametrises the R-charges of all the chiral matter fields, as in [44]. The eigenvalue density $\rho(x)$ is shown in Figure 2.7.

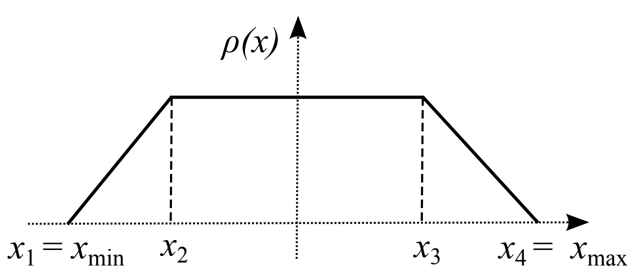

Figure 2.7: Eigenvalue density as a function of x. There are 4 points x_1, x_2, x_3, x_4 where $\rho'(x)$ is discontinuous, corresponding to critical points of h_M.

The free energy, as a function of Δ, is given by

$$\mathcal{F} = \frac{8a\pi}{3} \sqrt{\frac{k\Delta(1-\Delta)^2(2-\Delta)^2}{(4-3\Delta)}} N^{3/2} \ . \qquad (2.5.38)$$

[7]The variable μ arises as a Lagrange multiplier, enforcing that $\rho(x)$ is a density satisfying (2.3.13).

One may then maximise $\mathcal{F}$ to determine the superconformal Δ, finding the cubic irrational

$$\Delta = \frac{1}{18}\left[19 - \frac{37}{\left(431-18\sqrt{417}\right)^{1/3}} - \left(431-18\sqrt{417}\right)^{1/3}\right] \simeq 0.319 \ . \qquad (2.5.39)$$

This agrees with the value computed in [38], which was for the particular case $a = 1$.

Turning to the dual geometry, the Calabi-Yau four-fold that arises as the Abelian $\mathcal{N} = 1$ moduli space of these theories has toric data (for $k = 1$)

$$\begin{aligned} w_1 &= (0, 2a, 0) \ , \quad w_2 = (-1, a, 0) \ , \quad w_3 = (-1, 0, 0) \ , \\ w_4 &= (0, a, a) \ , \quad w_5 = (0, a, -a) \ , \quad w_6 = (0, 0, 0) \ , \end{aligned} \qquad (2.5.40)$$

and with toric diagram shown in Figure 2.8. The volume of Y_7 may be computed as

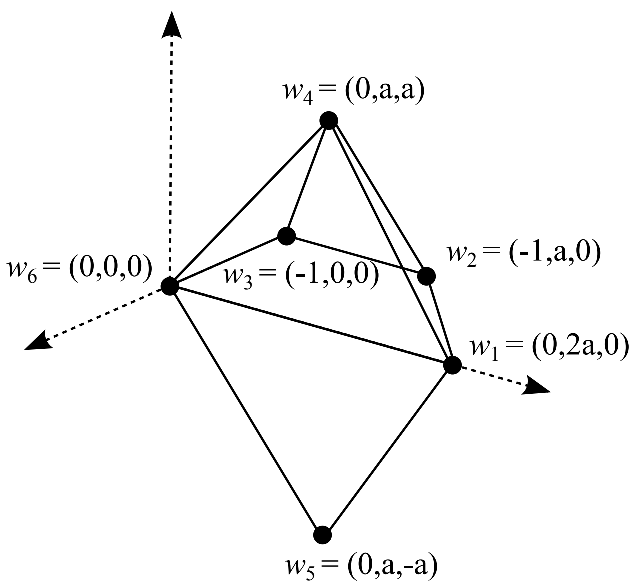

Figure 2.8: Toric diagram of the $L^{a,2a,a}$ Chern-Simons-quiver theories with $k = 1$.

described in section 2.4.6, and one obtains

$$\mathrm{Vol}(Y_7) = \frac{\pi^4(4-3\Delta)}{96a^2k\Delta(\Delta-1)^2(\Delta-2)^2} \ , \qquad (2.5.41)$$

with corresponding Reeb vector field

$$\vec{\xi} = (4, -4\Delta, 2a(2-\Delta), 0) \ . \tag{2.5.42}$$

The M-theory circle for this field theory is given in this basis by $\zeta_M = (0,0,0,1)$, so that Y_7 is given by a $\mathbb{Z}_k$ quotient of the geometry appearing in Figure 2.8, with $\mathbb{Z}_k \subset U(1)_M$. We may again compute the critical points of the M-theory Hamiltonian $h_M = 2y_4$ using the method at the end of section 2.4.6. Equation (2.4.39) has solutions associated to the following faces of the toric diagram:

$$\begin{aligned}
\Big(h_M &= 0 && \text{on} \quad (1,4,5,6)\Big) \ , \\
|h_M| &= \frac{1}{4a(1-\Delta)} && \text{on} \quad (2,3,4) \ , \ (2,3,5) \ , \\
|h_M| &= \frac{1}{2a(2-\Delta)} && \text{on} \quad (1,2,4) \ , \ (1,2,5) \ , \ (3,4,6) \ , \ (3,5,6) \ ,
\end{aligned} \tag{2.5.43}$$

and correspondingly one has the critical values $h_M = c_i$ given by

$$c_4 = -c_1 = \frac{1}{4ak(1-\Delta)} \ , \quad \text{and} \quad c_3 = -c_2 = \frac{1}{2ka(2-\Delta)} \ . \tag{2.5.44}$$

Note here that the face $(1,4,5,6)$ describes a *singular* S^1 locus in Y_7, and thus although $h_M = 0$ here, formally leading to zero-action M2-branes, the tangent space is singular. Using the general formula (2.2.6) we then find that these values of c_i precisely match the corresponding positions x_1, x_2, x_3, x_4 at which the derivative of the eigenvalue density $\rho'(x)$ is discontinuous. Explicitly, the actions of M2-branes wrapped on the corresponding calibrated $S^1 \subset Y_7$ are then

$$\begin{aligned}
-S_{\mathrm{M2}}(c_2) &= 4\pi(1-\Delta)\sqrt{\tfrac{\Delta}{k(4-3\Delta)}}N^{1/2} \ , \\
\log\langle W\rangle = -S_{\mathrm{M2}}(c_4) &= 2\pi(2-\Delta)\sqrt{\tfrac{\Delta}{k(4-3\Delta)}}N^{1/2} \ ,
\end{aligned} \tag{2.5.45}$$

with the latter determining the Wilson loop VEV, and showing that the field theory and gravity computations of it agree.

2.5.5 $L^{a,b,a}$ Chern-Simons-quivers

Our second family within this class are the $L^{a,b,a}$ Chern-Simons theories, with the vector of Chern-Simons levels now given by $\vec{k} = (0, \ldots, k, -2k \| k, 0, \ldots, 0)$. One finds the eigenvalue density [44]

$$\rho(x) \;=\; \begin{cases} \frac{4k\pi x(1-\Delta)+\mu}{16\pi^3(1-\Delta)\Delta((b-2)(1-\Delta)+a\Delta)} \;, & -\frac{\mu}{4k\pi(1-\Delta)} < x < -\frac{\mu}{2k\pi(b(1-\Delta)+a\Delta)} \\ \frac{\mu}{16\pi^3(1-\Delta)\Delta(b(1-\Delta)+a\Delta)} \;, & -\frac{\mu}{2k\pi(b(1-\Delta)+a\Delta)} < x < \frac{\mu}{2k\pi(b(1-\Delta)+a\Delta)} \\ -\frac{4k\pi x(1-\Delta)-\mu}{16\pi^3(1-\Delta)\Delta((b-2)(1-\Delta)+a\Delta)} \;, & \frac{\mu}{2k\pi(b(1-\Delta)+a\Delta)} < x < \frac{\mu}{4k\pi(1-\Delta)} \end{cases} \tag{2.5.46}$$

where we have defined

$$\mu = 8\pi^2 \sqrt{\frac{k\Delta(1-\Delta)^2(b(1-\Delta)+a\Delta)^2}{(b-2)(1-\Delta)+a\Delta}} \;. \tag{2.5.47}$$

Again, the R-charge variable Δ parametrises the R-charges of all the chiral matter fields, as detailed in [44]. The eigenvalue density $\rho(x)$ is shown in Figure 2.9.

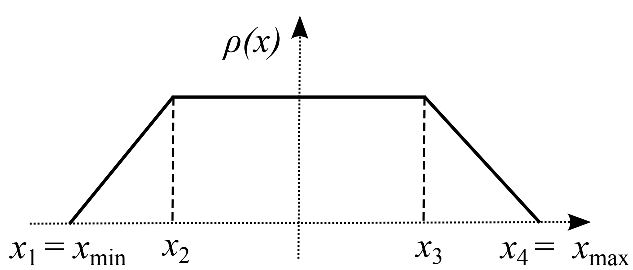

Figure 2.9: Eigenvalue density as a function of x. There are again 4 points x_1, x_2, x_3, x_4 where $\rho'(x)$ is discontinuous, corresponding to critical points of h_M.

The free energy, as a function of Δ, is given by

$$\mathcal{F} = \frac{8\pi}{3} \sqrt{\frac{k(1-\Delta)^2\Delta(b(1-\Delta)+a\Delta)^2}{(b+2)(1-\Delta)+a\Delta}} N^{3/2} \;. \tag{2.5.48}$$

One may then maximise $\mathcal{F}$ to find an expression (not presented) for the superconformal Δ that depends on a and b.

The corresponding Calabi-Yau four-fold that arises as the Abelian $\mathcal{N} = 1$ moduli space of these theories has toric data (for $k = 1$)

$$\begin{aligned} w_1 &= (0,0,0) \ , \ w_2 = (1,-1,0) \ , \ w_3 = (1,1,0) \ , \ w_4 = (b-1,-1,0) \ , \\ w_5 &= (b-1,1,0) \ , \ w_6 = (b,0,0) \ , \ w_7 = (0,0,1) \ , \ w_8 = (a,0,1) \ , \end{aligned} \tag{2.5.49}$$

and with toric diagram shown in Figure 2.10. The volume of Y_7 may be computed

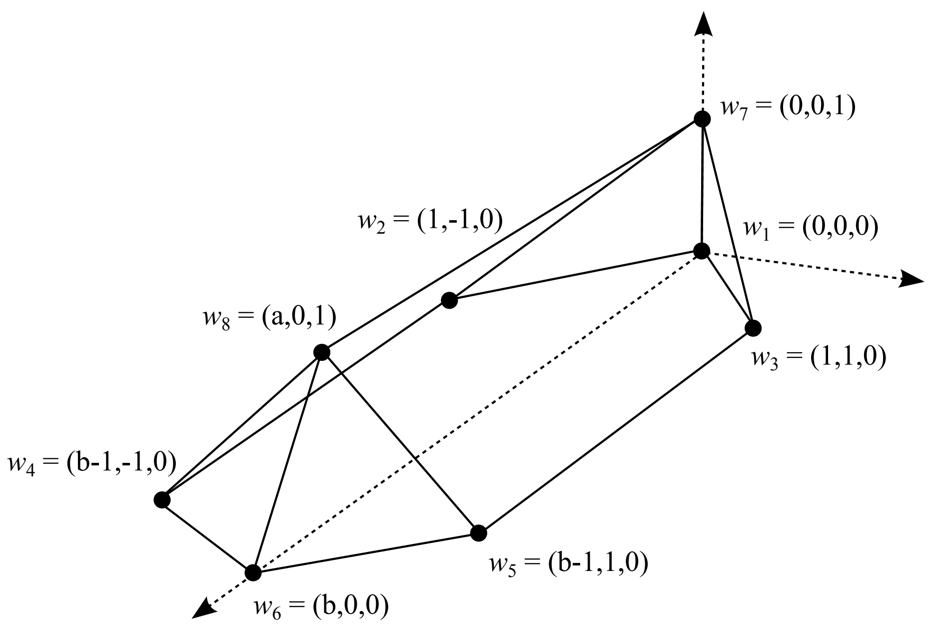

Figure 2.10: Toric diagram of the $L^{a,b,a}$ Chern-Simons-quiver theories with $k = 1$.

as described in section 2.4.6, and one obtains

$$\mathrm{Vol}(Y_7) = \frac{\pi^4((b+2)(1-\Delta)+a\Delta)}{96k\Delta(1-\Delta)^2(b(1-\Delta)+a\Delta)^2} \ , \tag{2.5.50}$$

with corresponding Reeb vector

$$\vec{\xi} = (4, 2(b(1-\Delta)+a\Delta), 0, 4\Delta) \ . \tag{2.5.51}$$

The M-theory circle for this field theory is given in this basis by $\zeta_M = (0,0,1,0)$, so that again Y_7 is given by a $\mathbb{Z}_k$ quotient of the geometry appearing in Figure 2.10, with $\mathbb{Z}_k \subset U(1)_M$. The M-theory Hamiltonian is $h_M = 2y_3$, and its critical points may be computed from equation (2.4.39), which has solutions on the following faces of the toric diagram:

$$\begin{aligned}
\Big(h_M &= 0 && \text{on} \quad (1,2,3,4,5,6)\ ,\Big) \\
|h_M| &= \frac{1}{4(1-\Delta)} && \text{on} \quad (2,4,7,8)\ ,\ (3,5,7,8)\ , \\
|h_M| &= \frac{1}{2(b(1-\Delta)+a\Delta)} && \text{on} \quad (1,2,7)\ ,\ (1,3,7)\ ,\ (4,6,8)\ ,\ (5,6,8)\ ,
\end{aligned} \tag{2.5.52}$$

and correspondingly one has critical values $h_M = c_i$ given by

$$c_4 = -c_1 = \frac{1}{4k(1-\Delta)}\ , \quad \text{and} \quad c_3 = -c_2 = \frac{1}{2k(b(1-\Delta)+a\Delta)}\ . \tag{2.5.53}$$

Using the general formula (2.2.6) we then find that these values of c_i precisely match the corresponding positions x_1, x_2, x_3, x_4 at which the derivative of the eigenvalue density $\rho'(x)$ is discontinuous. Explicitly, the actions of M2-branes wrapped on the corresponding calibrated $S^1 \subset Y_7$ are

$$\begin{aligned}
-S_{\rm M2}(c_2) &= 4\pi(1-\Delta)\sqrt{\tfrac{\Delta}{k((2+b)(1-\Delta)+a\Delta)}}N^{1/2}\ , \\
\log\langle W\rangle = -S_{\rm M2}(c_4) &= 2\pi(b(1-\Delta+a\Delta))\sqrt{\tfrac{\Delta}{k((2+b)(1-\Delta)+a\Delta)}}N^{1/2}\ ,
\end{aligned} \tag{2.5.54}$$

with the latter determining the Wilson loop VEV, and showing that the field theory and gravity computations of it agree.

Chapter 3

Gravity duals of field theories on three-manifolds

3.1 Introduction

Non-perturbative computations can be performed in certain supersymmetric field theories defined on curved Euclidean manifolds, using the technique of localisation described in the last chapter. This has motivated the systematic study of rigid supersymmetry in curved space [70], and it has also prompted the exploration of the gauge/gravity duality in situations when the boundary supersymmetric field theories are defined on non-trivial curved manifolds. This programme has been initiated in [26], where a simple Euclidean supersymmetric solution of four-dimensional minimal gauged supergravity was proposed as the dual to three-dimensional supersymmetric Chern-Simons theories defined on a squashed three-sphere (ellipsoid), for which the exact partition function had been computed previously in [25]. Generalisations have been discussed in [29–31].

Using localisation, the partition function Z of a large class of $\mathcal{N} = 2$ three-dimensional Chern-Simons theories defined on a general manifold with three-sphere topology was computed *explicitly* in [32]. This has provided a unified understanding of all previous localisation computations on deformed three-spheres [25–28], and has shown that the partition function on these manifolds depends only on a single parameter b_1/b_2, related to a choice of almost contact structure. Specifically, for a general toric metric on the three-sphere, the real numbers b_1, b_2 specify a choice of Killing vector K in the torus of isometries. For a broad class of Chern-Simons-quiver theories, the large N limit of the free energy $\mathcal{F} = -\log|Z|$ can be computed using saddle points methods [26], giving the general result

$$\lim_{N\to\infty} \mathcal{F}_{\frac{b_1}{b_2}} = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot \mathcal{F}_{\text{round}} \ , \tag{3.1.1}$$

where $\mathcal{F}_{\text{round}}$ is the large N limit of the free energy on the round three-sphere scaling with $N^{3/2}$, see equation (2.3.16). All computations in chapter 2 were done for a round three-sphere boundary and we did not emphasise it by adding the index 'round' to the free energy and the Wilson loops. However, we will now denote the free energy and Wilson loops by $\mathcal{F}$ and W for the more generic backgrounds developed in this chapter and use the notation $\mathcal{F}_{\text{round}}$ and W_{round} for the free energy and Wilson loops of chapter 2 on the round sphere.

On the gravity side, (3.1.1) yields a universal prediction for the holographically renormalised on-shell action of the corresponding supergravity solutions. Indeed, the on-shell action of the solutions of [26], [29], [30], and [31] reproduced this formula, for certain choices of metrics *and* background gauge fields. More precisely, these are all supersymmetric solutions of minimal four-dimensional gauged supergravity in Euclidean signature, and comprise a negatively curved Einstein anti-self-dual metric on the four-ball[1], with a specific choice of gauge field with anti-self-dual curvature, that we refer to as an instanton. The result of [32] raises two questions:

- Given an arbitrary toric metric on the three-sphere, with a background gauge field satisfying the rigid Killing spinor equations [71, 72], can one construct a

[1]References [29] and [30] also discuss several solutions with topology different from the four-ball; however, at present the precise field theory constructions dual to these remain unknown. In this thesis we will not discuss topologies different from the four-ball.

dual supergravity solution?

- Assuming such a supergravity solution exists, can one compute the corresponding holographic free energy and show that it matches (3.1.1)?

The purpose of this chapter is to address these two questions. Working in the context of four-dimensional[2] minimal gauged supergravity, and assuming an ansatz that the solutions are *anti-self-dual* and have the topology of the *ball*, we will be able to provide rather general answers to both these questions.

Regarding the first question, we will show that given an anti-self-dual metric on the ball with $U(1)^2$ isometry, and a choice of an arbitrary Killing vector therein, we can construct an instanton configuration, such that together these give a smooth supersymmetric solution of minimal gauged supergravity. Moreover, assuming this metric is asymptotically locally AdS, we will show that on the conformal boundary the four-dimensional solution reduces to a three-dimensional geometry solving the rigid Killing spinor equations of [71, 72], in the form presented in [32]. We will illustrate this construction through several examples, including previously known as well as new solutions.

We will be able to answer the second question, regarding the computation of the holographic free energy, *independently* of the details of a specific solution. Namely, assuming only that a smooth solution with given boundary conditions exists, we will show that the holographically renormalised on-shell action takes the form

$$I = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot I_{\text{round}} \ , \tag{3.1.2}$$

precisely matching the large N field theory prediction from localisation (3.1.1). Here, we made use of the fact that $\mathcal{F}_{\text{round}} = I_{\text{round}}$ has been checked in many classes of examples, see end of section 2.3.2 for a reminder. We emphasise that (3.1.2) will

[2] We will show in section 3.2 how the eleven-dimensional supergravity can be reduced to four-dimensional supergravity for the the purpose of finding a gravity dual to field theories on generic three-manifolds.

be derived without reference to a specific solution, and that it receives non-zero contributions from the boundary, as well as from the bulk, specifically from the 'centre' of the ball.

The rest of this chapter is organised as follows. In section 3.2 we explain how finding a supergravity dual to a superconformal field theory on a generic three-manifold can be reduced to minimal four-dimensional gauged supergravity computations. In section 3.3 we discuss the local geometry of Euclidean supersymmetric solutions of minimal four-dimensional gauged supergravity. In section 3.4 we turn to global and smooth asymptotically locally Euclidean AdS solutions, with the topology of the four-ball. Section 3.5 contains the derivation of the general formula (3.1.2) for the holographic free energy. Finally, in section 3.6 we present specific examples. Some details about the geometry can be found in appendices A and B.

3.2 Reduction to four-dimensional supergravity

Our starting point on the gravity side is eleven-dimensional supergravity in Euclidean signature. An Euclidean signature will be used for the rest of this thesis. We are interested in a class of $\mathcal{N}=2$ supersymmetric $M_4 \times Y_7$ backgrounds of M-theory. In Euclidean signature there are certain factors of i that appear relative to the eleven-dimensional supergravity solution in Lorentzian signature of [73]. Those factors will be very important for correctly computing M2-brane actions in chapter 4.

The action of $D = 11$ supergravity in Euclidean signature is

$$S_{11} \;=\; -\frac{1}{(2\pi)^8 \ell_p^9}\left(\int \mathrm{d}^{11}x\sqrt{g_{11}}\,\mathcal{R} - \int \frac{1}{2}\mathrm{d}C\wedge *_{11}\mathrm{d}C + \frac{\mathrm{i}}{6}C\wedge \mathrm{d}C\wedge \mathrm{d}C\right)\;. \quad (3.2.1)$$

Here we have denoted by g_{11} the eleven-dimensional metric, with associated Ricci scalar $\mathcal{R}$, C is the three-form potential and ℓ_p denotes the eleven-dimensional Planck length. The equations of motion for the metric and C-field follow immediately:

$$\begin{aligned} \mathcal{R}_{AB} - \frac{1}{12}(G_{AC_1C_2C_3}G_B{}^{C_1C_2C_3} - \frac{1}{12}g_{AB}G^2) &= 0 \ , \\ \mathrm{d} *_{11} G + \frac{\mathrm{i}}{2} G \wedge G &= 0 \ , \end{aligned} \tag{3.2.2}$$

where we have defined $G \equiv \mathrm{d}C$, $\mathcal{R}_{AB}$ is the Ricci tensor and $A, B, C = 1, \ldots, 11$. It is also useful to define $G_7 = \mathrm{i}(*_{11}G + \frac{\mathrm{i}}{2}C \wedge G)$ so that the equation of motion for G is simply $\mathrm{d}G_7 = 0$.

An ansatz to the last system of equations in Lorentz signature was given in [73]. There is an internal space Y_7 taken to be any Sasaki-Einstein seven-manifold Y_7 with contact one-form η, transverse Kähler-Einstein six-metric $\mathrm{d}s^2_T$ with Kähler form $\omega_T = \mathrm{d}\eta/2$, and with the seven-dimensional metric normalised so that $\mathrm{Ric} = 6g_{Y_7}$. The ansatz in Euclidean signature is then

$$\begin{aligned} \mathrm{d}s^2_{11} &= R^2 \left[\frac{1}{4}\mathrm{d}s^2_{M_4} + \left(\eta + \frac{1}{2}A\right)^2 + \mathrm{d}s^2_T \right] \ , \\ G &= -\mathrm{i}R^3 \left(\frac{3}{8}\mathrm{vol}_4 - \frac{1}{4} *_4 F \wedge \mathrm{d}\eta \right) \ , \end{aligned} \tag{3.2.3}$$

and is compatible with (2.4.3). The warp factor Δ of the last chapter is now constant and encoded into R and the metric on Y_7 is more explicit. It is the more specific[3] form of solutions (3.2.3) that will be used for the rest of this thesis. Here, $\mathrm{d}s^2_{M_4}$ is a four-dimensional metric on a manifold M_4 with abelian gauge field A, field-strength $F = \mathrm{d}A$ and volume form vol_4. The radius R is

$$R^6 = \frac{(2\pi\ell_p)^6 N}{6\mathrm{Vol}(Y_7)} \ , \tag{3.2.4}$$

[3]This solution is more specific on the internal part as the metric on Y_7 couples to A and the G form has a precise dependence on η but it allows us to generalise the AdS_4 part to a more generic four-manifold M_4.

where N is the number of units of flux

$$N = \frac{1}{(2\pi\ell_p)^6} \int_{Y_7} G_7 \ . \tag{3.2.5}$$

Substituting the ansatz (3.2.3) into the equations of motion (3.2.2), we find the latter are equivalent to the metric $g_{\mu\nu}$ corresponding to $\mathrm{d}s^2_{M_4}$ and F satisfying

$$\begin{aligned} R_{\mu\nu} + 3g_{\mu\nu} &= 2\left(F_\mu{}^\rho F_{\nu\rho} - \frac{1}{4}F^2 g_{\mu\nu}\right) \ , \\ \mathrm{d} *_4 F &= 0 \ . \end{aligned} \tag{3.2.6}$$

The ansatz (3.2.3) then solves the eleven-dimensional Euclidean equations of motion if and only if the four-dimensional metric $g_{\mu\nu}$ and gauge field A are a solution of minimal four-dimensional Euclidean gauged supergravity. One can show that if the ansatz (3.2.3) is plugged into the action (3.2.1), we get the four-dimensional supergravity action [74]

$$I^{\mathrm{SUGRA}} = -\frac{1}{16\pi G_4} \int \left(R + 6 - F^2\right) \sqrt{\det g}\, \mathrm{d}^4 x \ , \tag{3.2.7}$$

where R denotes the Ricci scalar of the four-dimensional metric $g_{\mu\nu}$, we have defined $F^2 \equiv F_{\mu\nu}F^{\mu\nu}$, and we have a cosmological constant $\Lambda = -3$. The four-dimensional Newton constant G_4 is defined by

$$\frac{1}{16\pi G_4} = N^{3/2} \sqrt{\frac{\pi^2}{2^5 3^3 \mathrm{Vol}_\eta(Y_7)}} \ . \tag{3.2.8}$$

Interestingly, if we vary the action (3.2.7) for the four-dimensional metric $g_{\mu\nu}$ and gauge field A we recover the equations of motion (3.2.6) showing that the ansatz (3.2.3) is consistent.

A truncation to a lower dimensional theory is said to be consistent if the solution to the lower dimensional theory necessarily solves the original equations of motion. More generally, one has to carry out a Kaluza-Klein reduction on the internal manifold (Y_7 here) which leads to a lower dimensional theory with an infinite tower of fields. The reduction is consistent if it is in fact consistent to set all the non-lowest energy fields in the Kaluza-Klein tower to zero and obtain the equations of motion for the lower dimensional theory. As shown in [73], the supergravity solution (3.2.3) consistently reduces to minimal four-dimensional gauged supergravity and we can safely focus on the four-dimensional geometry in what follows.

At this point, it is interesting to note that when $M_4 = \mathrm{AdS}_4$, the *regularised* supergravity action together with the appropriate counterterms, see examples in section 3.6, becomes

$$I_{\text{round}} = \frac{\pi}{2G_4} = \sqrt{\frac{2\pi^6}{27\ \mathrm{Vol}_\eta(Y_7)}} N^{3/2} = \mathcal{F}_{\text{round}} \ . \tag{3.2.9}$$

where we have used (3.2.8) and the expression for $\mathcal{F}_{\text{round}}$ in (2.3.16).

In what follows, we assume that the eleven-dimensional supergravity solution has the form given in equation (3.2.3) and we will work with four-dimensional supergravity to construct gravity duals to supersymmetric field theories on generic three-manifolds.

3.3 Local geometry of self-dual solutions

The action for the bosonic sector of four-dimensional $\mathcal{N} = 2$ gauged supergravity is given in equation (3.2.7). In our context, this action is seen as a truncation to four dimensions of the full eleven-dimensional supergravity as explained above. The graviphoton is an Abelian gauge field A with field strength $F = \mathrm{d}A$. The equations of motion are (3.2.6). They are simply Einstein-Maxwell theory with a cosmological constant $\Lambda = -3$. Notice that when F is anti-self-dual, i.e. $*_4 F = -F$, the right hand side of the Einstein equation in (3.2.6) is zero, so that the metric $g_{\mu\nu}$ is necessarily Einstein as $R_{\mu\nu} = -3g_{\mu\nu}$, and that $\mathrm{d} *_4 F = 0$.

A solution is supersymmetric provided it admits a Dirac spinor ϵ satisfying the

Killing spinor equation

$$\left(\nabla_\mu - \mathrm{i}A_\mu + \frac{1}{2}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\epsilon = 0 \ . \tag{3.3.1}$$

This takes the same form as in Lorentzian signature, except that here the gamma matrices generate the Clifford algebra Cliff$(4,0)$ in an orthonormal frame, so $\{\Gamma_\mu, \Gamma_\nu\} = 2g_{\mu\nu}$. Notice that we may define the charge conjugate of the spinor ϵ as $\epsilon^c \equiv B\epsilon^*$, where B is the charge conjugation matrix satisfying $B^{-1}\Gamma_\mu B = \Gamma_\mu^*$, $BB^* = -1$ and may be chosen to be antisymmetric $B^T = -B$ [26]. Then provided the gauge field A is real (as it will be here) ϵ^c satisfies (3.3.1) with $A \to -A$.

In [75, 76] the authors studied the local geometry of Euclidean supersymmetric solutions to the above theory for which F is anti-self-dual, $*_4F = -F$. It follows that the metric $g_{\mu\nu}$ then has anti-self-dual Weyl tensor, and adopting a standard abuse of terminology we shall refer to such solutions as 'self-dual'. Supersymmetry also equips this background geometry with a Killing vector field K. Self-dual Einstein metrics with a Killing vector have a rich geometric structure that has been well-studied (see for example [77]), and are well-known to be related by a conformal rescaling to a local Kähler metric with zero Ricci scalar. Such metrics are described by a solution to a single PDE, known as the Toda equation, and this solution also specifies uniquely the background gauge field A. In fact we will show that $F = \mathrm{d}A$ is $\frac{1}{2}$ the Ricci-form of the conformally related Kähler metric. Moreover, we will reverse the direction of implication in [75, 76] and show that any self-dual Einstein metric with a choice of Killing vector field admits locally a solution to the Killing spinor equation (3.3.1). This may be constructed from the canonically defined spin^c spinor that exists on any Kähler manifold.

3.3.1 Local form of the solution

In this section we briefly review the local geometry determined in [75, 76]. The existence of a non-trivial solution to the Killing spinor equation (3.3.1), together with the ansatz that F is anti-self-dual and real, implies that the metric $g_{\mu\nu}$ is Einstein with anti-self-dual Weyl tensor. Because F is real for this solution we assume that A is real throughout this thesis. There is then a canonically defined local coordinate system in which the metric takes the form[4]

$$\mathrm{d}s^2_{\mathrm{SDE}} = \frac{1}{y^2}\left[V^{-1}(\mathrm{d}\psi+\phi)^2 + V\left(\mathrm{d}y^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z}\right)\right] \ , \tag{3.3.2}$$

where

$$V = 1 - \frac{1}{2}y\partial_y w \ , \tag{3.3.3}$$

$$\mathrm{d}\phi = \mathrm{i}\partial_z V \mathrm{d}y\wedge\mathrm{d}z - \mathrm{i}\partial_{\bar{z}}V\mathrm{d}y\wedge\mathrm{d}\bar{z} + 2\mathrm{i}\partial_y(V\mathrm{e}^w)\mathrm{d}z\wedge\mathrm{d}\bar{z} \ , \tag{3.3.4}$$

and $w = w(y, z, \bar{z})$ satisfies the Toda equation

$$\partial_z\partial_{\bar{z}}w + \partial_y^2\mathrm{e}^w = 0 \ . \tag{3.3.5}$$

Notice that the function w determines entirely the metric. The two-form $\mathrm{d}\phi$ is easily verified to be closed provided the Toda equation (3.3.5) is satisfied, implying the existence of a local one-form ϕ.

The vector $K = \partial_\psi$ is a Killing vector field, and arises canonically from supersymmetry as a bilinear $K^\mu \equiv \mathrm{i}\epsilon^\dagger\Gamma^\mu\Gamma_5\epsilon$, where ϵ is the Killing spinor solving (3.3.1) and $\Gamma_5 \equiv \Gamma_{0123}$. Using the Killing spinor equation one can verify that K^μ is indeed Killing as $\nabla_{(\mu}K_{\mu)} = 0$. Notice that the corresponding bilinear in the charge conjugate spinor ϵ^c is $\mathrm{i}(\epsilon^c)^\dagger\Gamma^\mu\Gamma_5\epsilon^c = -K^\mu$. Thus as in the discussion after equation (3.3.1) we may change variables to $\tilde{\epsilon} = \epsilon^c$, $\tilde{A} = -A$. In the tilded variables the equations of motion (3.2.6) and Killing spinor equation (3.3.1) are identical to the untilded equations, but now $\tilde{A} = -A$ and $\tilde{K} = -K$. Thus the sign of the instanton is correlated with a choice of sign for the supersymmetric Killing vector, with charge conjugation of the

[4]SDE stands for self-dual Einstein. As explained before, we call the metric self-dual for convenience even though it has anti-self-dual Weyl tensor.

spinor changing the signs of both A and K.

As we shall see in the next section, the coordinate y determines the conformal factor for the conformally related Kähler metric, and is also the Hamiltonian function for the vector field $K = \partial_\psi$ with respect to the associated symplectic form. The graviphoton field is given by

$$A = -\frac{1}{4}V^{-1}\partial_y w(\mathrm{d}\psi + \phi) + \frac{\mathrm{i}}{4}\partial_z w \mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w \mathrm{d}\bar{z} \ . \tag{3.3.6}$$

We are of course free to make gauge transformations of A, and we stress that (3.3.6) is in general valid only locally.

Having summarised the results of [75, 76], in the next two sections we study this local geometry further. In particular we show that any self-dual Einstein metric with Killing vector $K \equiv \partial_\psi$, which then takes the form (3.3.2), admits a Killing spinor ϵ solving (3.3.1), where A is given by (3.3.6).

3.3.2 Conformal Kähler metric

As already mentioned, every self-dual Einstein four-metric with a Killing vector is conformally related to a scalar-flat Kähler metric. This is given by

$$\begin{aligned} \mathrm{d}s^2_{\mathrm{Kahler}} &\equiv \mathrm{d}\hat{s}^2 = y^2 \mathrm{d}s^2_{\mathrm{SDE}} \\ &= V^{-1}(\mathrm{d}\psi + \phi)^2 + V\left(\mathrm{d}y^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z}\right) \ . \end{aligned} \tag{3.3.7}$$

Introducing an associated local orthonormal frame of one-forms

$$\hat{e}^0 = V^{1/2}\mathrm{d}y \ , \quad \hat{e}^1 = V^{-1/2}(\mathrm{d}\psi + \phi) \ , \quad \hat{e}^2 + \mathrm{i}\hat{e}^3 = 2(V\mathrm{e}^w)^{1/2}\mathrm{d}z \ , \tag{3.3.8}$$

the Kähler form is

$$\omega = \hat{e}^{01} + \hat{e}^{23} \ , \tag{3.3.9}$$

where we have denoted $\hat{e}^0 \wedge \hat{e}^1 = \hat{e}^{01}$, *etc.* That (3.3.9) is indeed closed follows immediately from the expression for $\mathrm{d}\phi$ in (3.3.4). The Kähler form is self-dual with respect to the natural orientation on a Kähler manifold, namely $\hat{e}^{0123}$ above, and it is with respect to this orientation that the curvature F and Weyl tensor are anti-self-dual. We denote the corresponding orthonormal frame for the self-dual Einstein metric (3.3.2) as $e^a = y^{-1}\hat{e}^a$, $a = 0, 1, 2, 3$.

Next we introduce the Hodge type $(2, 0)$-form

$$\Omega \equiv (\hat{e}^0 + \mathrm{i}\hat{e}^1) \wedge (\hat{e}^2 + \mathrm{i}\hat{e}^3) \ , \tag{3.3.10}$$

and recall that the metric (3.3.7) is Kähler if and only if

$$\mathrm{d}\Omega = \mathrm{i}\mathcal{Q} \wedge \Omega \ , \tag{3.3.11}$$

where $\mathcal{Q}$ is then the Ricci one-form, with Ricci two-form $\mathcal{R} = \mathrm{d}\mathcal{Q}$. It is straightforward to compute $\mathrm{d}\Omega$ for the metric (3.3.7), and one finds that

$$\mathcal{Q} = 2A \ , \tag{3.3.12}$$

where A is given by (3.3.6). The curvature is correspondingly $F = \mathrm{d}A = \frac{1}{2}\mathcal{R}$, where recall that $\mathcal{R}_{\mu\nu} = \frac{1}{2}\hat{R}_{\mu\nu\rho\sigma}\omega^{\rho\sigma}$ where $\hat{R}_{\mu\nu\rho\sigma}$ denotes the Riemann tensor for the Kähler metric. A computation gives

$$-2\mathcal{R} \wedge \omega = \frac{1}{V\mathrm{e}^w}\left[\partial_z\partial_{\bar{z}}w + \partial_y^2\mathrm{e}^w\right]\hat{e}^{0123} \ , \tag{3.3.13}$$

so that the Kähler metric is indeed scalar flat if the Toda equation holds. An explicit computation shows that with respect to the frame (3.3.8) $F = \mathrm{d}A$ is

$$\begin{aligned} F \ = \ & -\frac{1}{4}\partial_y\left[V^{-1}\partial_y w\right]\left(\hat{e}^{01} - \hat{e}^{23}\right) + \frac{1}{8\mathrm{e}^{w/2}}\Big[\mathrm{i}(\partial_z - \partial_{\bar{z}})[V^{-1}\partial_y w]\left(\hat{e}^{02} + \hat{e}^{13}\right) \\ & -(\partial_z + \partial_{\bar{z}})[V^{-1}\partial_y w]\left(\hat{e}^{03} - \hat{e}^{12}\right)\Big] \ , \end{aligned} \tag{3.3.14}$$

which is then manifestly anti-self-dual. One can also derive the formula

$$F \;=\; -\left(\tfrac{1}{2}y\mathrm{d}K^\flat + y^2 K^\flat \wedge JK^\flat\right)^- \ , \tag{3.3.15}$$

where $K^\flat$ denotes the one-form dual to the Killing vector K (in the self-dual Einstein metric), and J is the complex structure tensor for the Kähler metric (3.3.7), and a further short computation leads to

$$F = \left(\frac{1}{y}\mathrm{i}\partial\bar{\partial}y\right)^- \;=\; \frac{1}{y}\mathrm{i}\partial\bar{\partial}y + \frac{1}{4y}(\hat{\Delta}y)\omega \ , \tag{3.3.16}$$

where $\bar{\partial}$ denotes the standard operator on a Kähler manifold, the superscript "$-$" in (3.3.16) denotes anti-self-dual part, and $\hat{\Delta}$ denotes the scalar Laplacian for the Kähler metric.

Let us note that the Kähler form is explicitly

$$\omega = \mathrm{d}y \wedge (\mathrm{d}\psi + \phi) + 2\mathrm{i}V\mathrm{e}^w \mathrm{d}z \wedge \mathrm{d}\bar{z} \ . \tag{3.3.17}$$

Thus $\mathrm{d}y = -\partial_\psi \lrcorner \omega$, which identifies the coordinate y as the Hamiltonian function for the Killing vector $K = \partial_\psi$. Of course, y^2 is also the conformal factor relating the self-dual Einstein metric to the Kähler metric in (3.3.7).

3.3.3 Killing spinor: sufficiency

In this section we show that a self-dual Einstein metric with Killing vector $K = \partial_\psi$, which necessarily takes the form (3.3.2), admits a solution to the Killing spinor equation (3.3.1) with gauge field given by (3.3.6). The key to this construction is to begin with the canonically defined spin^c spinor that exists on any Kähler manifold.

On any Kähler manifold there is always a complex spinor ζ satisfying the spin^c

Killing spinor equation[5]

$$\left(\hat{\nabla}_\mu - \tfrac{\mathrm{i}}{2}\mathcal{Q}_\mu\right)\zeta = 0 \ . \tag{3.3.18}$$

Here the hat denotes that we will apply this to the conformal Kähler metric (3.3.7) in the case at hand, and $\mathcal{Q}$ is the Ricci one-form potential we encountered above. Using the result earlier that $\mathcal{Q} = 2A$ the equation (3.3.18) may be rewritten as

$$\left(\hat{\nabla}_\mu - \mathrm{i}A_\mu\right)\zeta = 0 \ , \tag{3.3.19}$$

which may already be compared with the Killing spinor equation (3.3.1).

More concretely, the solution to (3.3.18), or equivalently (3.3.19), is simply given by a constant spinor ζ, so that $\partial_\mu\zeta = 0$. This equation makes sense globally as ζ may be identified with a complex-valued function. To see this it is useful to take the following projection conditions

$$\hat{\Gamma}_1\zeta = \mathrm{i}\hat{\Gamma}_0\zeta \ , \qquad \hat{\Gamma}_3\zeta \ = \ \mathrm{i}\hat{\Gamma}_2\zeta \ , \tag{3.3.20}$$

following *e.g.* reference [78]. Here $\hat{\Gamma}_a$, $a = 0, 1, 2, 3$, denote the gamma matrices in the orthonormal frame (3.3.8)[6]. The covariant derivative of ζ is then computed to be

$$\hat{\nabla}_\mu\zeta = \left(\partial_\mu + \frac{1}{4}\hat{\omega}_\mu{}^{\nu\rho}\hat{\Gamma}_{\nu\rho}\right)\zeta \ = \ \partial_\mu\zeta + \tfrac{\mathrm{i}}{2}\left(\hat{\omega}_\mu{}^{01} + \hat{\omega}_\mu{}^{23}\right)\zeta \ = \ \partial_\mu\zeta + \mathrm{i}A_\mu\zeta \ , \tag{3.3.21}$$

where $\hat{\omega}_\mu{}^{\nu\rho}$ is the spin connection of the conformal Kähler metric, and we have used the explicit form of this in appendix A together with the formula (3.3.6) for A. It follows that simply taking ζ to be constant, $\partial_\mu\zeta = 0$, solves (3.3.18). This is a general phenomenon on any Kähler manifold.

Using the canonical spinor ζ we may construct a spinor ϵ that is a solution to the

[5] We sometimes refer to this spinor as the canonical Killing spinor of the Kähler manifold, or the canonical spinor for simplicity.

[6] Strictly speaking the hats are redundant, but we keep them as a reminder that in this section the orthonormal frame is for the Kähler metric.

Killing spinor equation (3.3.1). Specifically, we find

$$\epsilon = \frac{1}{\sqrt{2y}} \left(1 + V^{-1/2} \hat{\Gamma}_0 \right) \zeta \ . \tag{3.3.22}$$

To verify this one first notes that the spin connections of the Kähler metric and the self-dual Einstein metric are related by

$$\hat{\nabla}_\mu \zeta = \nabla_\mu \zeta + \frac{1}{2} \hat{\Gamma}_\mu{}^\nu (\partial_\nu \log y) \zeta \ , \tag{3.3.23}$$

where $\hat{\Gamma}_\mu = y \Gamma_\mu$ in a coordinate basis. The Killing spinor equation (3.3.1) then takes the form

$$\left[\partial_\mu + \frac{1}{4} \hat{\omega}_\mu{}^{\nu\rho} \hat{\Gamma}_{\nu\rho} - \frac{1}{2} \hat{\Gamma}_\mu{}^\nu (\partial_\nu \log y) \ - \mathrm{i} A_\mu + \frac{1}{2y} \hat{\Gamma}_\mu + \frac{\mathrm{i}}{4} y F_{\nu\rho} \hat{\Gamma}^{\nu\rho} \hat{\Gamma}_\mu \right] \epsilon \ = \ 0 \ . \tag{3.3.24}$$

To verify this is solved by (3.3.22) one simply substitutes (3.3.22) directly into the left-hand-side of (3.3.24). Using the explicit expressions for the spin connection, the gauge field, the field strength, as well as the projection conditions on the canonical spinor ζ and (3.3.19), one sees that (3.3.24) indeed holds.

From this analysis we can conclude that the self-dual Einstein metric (3.3.2) and the gauge field (3.3.6), which are solutions to Einstein-Maxwell theory in four dimensions, yield a Dirac spinor ϵ that is solution to the Killing spinor equation (3.3.1). This implies that these self-dual Einstein backgrounds are always locally supersymmetric solutions of Euclidean $\mathcal{N} = 2$ gauged supergravity. Using those backgrounds, one can lift them to supersymmetric solutions of eleven-dimensional supergravity with the help of (3.2.3). We turn to global issues in the next section.

3.4 Asymptotically locally AdS solutions

In this section and the next we will assume that we are given a complete self-dual Einstein metric with a Killing vector, which then necessarily takes the local form

(3.3.2). Moreover, we shall assume this metric is asymptotically locally Euclidean AdS[7], and in later subsections also that the four-manifold M_4 on which the metric is defined is topologically a ball. A two-parameter family of such self-dual solutions on the four-ball, generalising all previously known solutions of this type, was constructed in [31]. In section 3.6 we shall review these solutions, and also introduce a number of further generalisations. In particular, the results of the current section allow us to deform the choice of Killing vector (which was essentially fixed in previous results), and we will also explain how to generalise to an *infinite-dimensional* family of solutions satisfying the above properties, starting with the local metrics in [79].

With the above assumptions in place, we begin in this section by showing that if the Killing vector $K = \partial_\psi$ is nowhere zero in a neighbourhood of the conformal boundary three-manifold M_3 then it is a Reeb vector field for an almost contact structure on M_3. We then reproduce the same geometric structure on M_3 studied from a purely three-dimensional viewpoint in [72]. In particular the asymptotic expansion of the Killing spinor ϵ leads to the same Killing spinor equation as [72]. This is important, as it shows that the dual field theory is defined on a supersymmetric background of the form studied in [72], for which the exact partition function of a general $\mathcal{N} = 2$ supersymmetric gauge theory was computed in [32] using localisation. Having studied the conformal boundary geometry, we then turn to the bulk in 3.4.4. In particular we show that, with an appropriate restriction on the Killing vector K, the conformal Kähler structure of section 3.3.2 is everywhere non-singular. This allows us to prove in turn that the instanton and Killing spinor defined by the Kähler structure are everywhere non-singular.

In particular this means that each of the self-dual Einstein metrics in section 3.6 leads to a one-parameter family (depending on the choice of Killing vector K) of smooth supersymmetric solutions. In other words, if the self-dual Einstein metric depends on n parameters, the complete solution will depend on $n + 1$ parameters.

[7]Since the metric has Euclidean signature one might more accurately describe this boundary condition as *asymptotically locally hyperbolic*, which is often used in the mathematics literature.

We emphasise that in the previously known solutions the only example of this phenomenon is the solution of [26]. There the Einstein metric was simply AdS_4, which does not have any parameters.

3.4.1 Conformal boundary at $y = 0$

We are interested in self-dual Einstein metrics of the form (3.3.2) which are asymptotically locally Euclidean AdS (hyperbolic), in order to apply to the gauge/gravity correspondence. From the assumptions described above there is a single asymptotic region where the metric approaches $\frac{\mathrm{d}r^2}{r^2} + r^2 \mathrm{d}s^2_{M_3}$ as $r \to \infty$, where M_3 is a smooth compact three-manifold. In fact the metrics (3.3.2) naturally have such a conformal boundary at $y = 0$. More precisely, we impose boundary conditions such that $w(y, z, \bar{z})$ is analytic around $y = 0$, so

$$w(y, z, \bar{z}) = w_{(0)}(z, \bar{z}) + y w_{(1)}(z, \bar{z}) + \tfrac{1}{2} y^2 w_{(2)}(z, \bar{z}) + \mathcal{O}(y^3) \ . \tag{3.4.1}$$

It follows that

$$V(y, z, \bar{z}) = 1 - \tfrac{1}{2} y w_{(1)}(z, \bar{z}) - \tfrac{1}{2} y^2 w_{(2)}(z, \bar{z}) + \mathcal{O}(y^3) \ , \tag{3.4.2}$$

and that the metric (3.3.2) is

$$\mathrm{d}s^2_{\mathrm{SDE}} = [1 + \mathcal{O}(y)] \frac{\mathrm{d}y^2}{y^2} + \frac{1}{y^2} \left[(\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}} \mathrm{d}z \mathrm{d}\bar{z} + \mathcal{O}(y) \right] \ . \tag{3.4.3}$$

Setting $r = 1/y$ this is to leading order

$$\mathrm{d}s^2_{\mathrm{SDE}} \simeq \frac{\mathrm{d}r^2}{r^2} + r^2 \left[(\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}} \mathrm{d}z \mathrm{d}\bar{z} \right] \ , \tag{3.4.4}$$

as $r \to \infty$, so that the metric is indeed asymptotically locally Euclidean AdS around $y = 0$. Here we have also expanded the one-form tangent to M_3

$$\phi(y, z, \bar{z}) \mid_{M_3} = \phi_{(0)}(z, \bar{z}) + y\phi_{(1)}(z, \bar{z}) + \mathcal{O}(y^2). \tag{3.4.5}$$

In fact by expanding (3.3.4) one can show that $\phi_{(1)} = 0$. Of course, as usual one is free to redefine $r \to r\Omega(\psi, z, \bar{z})$, where Ω is any smooth, nowhere zero function on M_3, resulting in a conformal transformation of the boundary metric $\mathrm{d}s^2_{M_3} \to \Omega^2 \mathrm{d}s^2_{M_3}$. However, in the present context notice that $r = 1/y$ is a *natural* choice of radial coordinate.

With the analytic boundary condition (3.4.1) for w it follows automatically that $K = \partial_\psi$ is nowhere zero in a neighbourhood of the conformal boundary $y = 0$ because $||K||^2 = 1/(y^2 V) \neq 0$ near the conformal boundary. The ansatz (3.4.1) is certainly a restriction on the class of possible globally regular solutions, although all examples in section 3.6 have choices of Killing vector for which this expansion holds.

Returning to the case at hand, the conformal boundary is a compact three-manifold M_3 (by assumption), and from the above discussion a natural choice of representative for the metric is

$$\mathrm{d}s^2_{M_3} = (\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}} \mathrm{d}z \mathrm{d}\bar{z} \ . \tag{3.4.6}$$

Notice that the form of the metric (3.4.6) is precisely of the form studied in [32] where localisation of $\mathcal{N} = 2$ supersymmetric field theories on generic three-manifolds is carried out. In that reference an important role is played by the one-form

$$\sigma \ \equiv \ \mathrm{d}\psi + \phi_{(0)} \ , \tag{3.4.7}$$

which has exterior derivative

$$\mathrm{d}\sigma \ = \ \mathrm{d}\phi_{(0)} = 2\mathrm{i}\partial_y(V\mathrm{e}^w) \mid_{y=0} \mathrm{d}z \wedge \mathrm{d}\bar{z} \ = \ \mathrm{i}w_{(1)}\mathrm{e}^{w_{(0)}} \mathrm{d}z \wedge \mathrm{d}\bar{z} \ . \tag{3.4.8}$$

The form σ is a global *almost contact one-form* on M_3. The most straightforward way to derive this is to note the form of the boundary Killing spinor equation in section 3.4.2 and appeal to the results of [72].

The Killing vector $K = \partial_\psi$ is the Reeb vector for the almost contact form σ, as follows from the equations

$$K \lrcorner \sigma = 1 \; , \qquad K \lrcorner \mathrm{d}\sigma \;=\; 0 \; . \tag{3.4.9}$$

The orbits of K thus foliate M_3, and moreover this foliation is transversely holomorphic with local complex coordinate z. When the orbits of K all close it generates a $U(1)$ symmetry of the boundary structure, and the orbit space $M_3/U(1)$ is in general a complex surface, on which z may be regarded as a local complex coordinate. On the other hand, if K has at least one non-closed orbit then since the isometry group of a compact manifold is compact, we deduce that M_3 admits at least a $U(1)\times U(1)$ symmetry, and the structure defined by σ is a *toric* almost contact structure. In this case we may introduce standard 2π-period coordinates φ_1, φ_2 on the torus $U(1) \times U(1)$ and write

$$K = \partial_\psi \;=\; b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \; . \tag{3.4.10}$$

From (3.4.8) we deduce that the Taylor coefficient $w_{(1)}$ is a globally defined *basic* function on M_3 – that is, it is invariant under $K = \partial_\psi$. Moreover, the almost contact form σ is a *contact form* precisely when the function $w_{(1)}$ is everywhere positive. We shall see later that there are examples for which σ is contact and not contact. On the other hand, the coefficient $w_{(0)}$ is in general only a locally defined function of $z, \bar{z}$, as one sees by noting that the transverse metric $g_T = \mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}$ is a global two-tensor[8]. It will be useful in what follows to define a corresponding transverse volume form

$$\mathrm{vol}_T \equiv 2\mathrm{i}\mathrm{e}^{w_{(0)}}\mathrm{d}z \wedge \mathrm{d}\bar{z} \; . \tag{3.4.11}$$

[8]If it was not singular, the transverse metric would be defined everywhere making the space M_3 non-compact, which we assume is not true.

This tensor is related to the contact form σ by

$$\mathrm{d}\sigma = \mathrm{d}\phi_{(0)} \;=\; \frac{w_{(1)}}{2}\mathrm{vol}_T \; . \tag{3.4.12}$$

3.4.2 Boundary Killing spinor

In this section we show that the Killing spinor ϵ induces a Killing spinor χ on the conformal boundary M_3 that solves the Killing spinor equation in [72].

We begin by recalling the orthonormal frame of one-forms

$$e^0 = \frac{1}{y}V^{1/2}\mathrm{d}y \; , \quad e^1 \;=\; \frac{1}{y}V^{-1/2}(\mathrm{d}\psi + \phi) \; , \quad e^2 + \mathrm{i}e^3 \;=\; \frac{2}{y}(V\mathrm{e}^w)^{1/2}\mathrm{d}z \; , \tag{3.4.13}$$

for the self-dual Einstein metric (3.3.2). We introduce a corresponding frame for the three-metric $\mathrm{d}s^2_{M_3}$ on the conformal boundary:

$$e^1_{(3)} \;=\; \mathrm{d}\psi + \phi_{(0)} \; , \qquad e^2_{(3)} + \mathrm{i}e^3_{(3)} \;=\; 2\mathrm{e}^{w_{(0)}/2}\mathrm{d}z \; , \tag{3.4.14}$$

and will use indices $i, j, k = 1, 2, 3$ for this orthonormal frame.

We next expand the four-dimensional Killing spinor equation (3.3.1) as a Taylor series in y. One starts by noting that $\Gamma^\mu = e^\mu{}_a\Gamma^a = \mathcal{O}(y)$. But as $\Gamma_\mu = e^a{}_\mu\Gamma_a = \mathcal{O}(1/y)$ and the field strength expands as $F = F_{(0)} + yF_{(1)} + \mathcal{O}(y^2)$ we see that

$$\frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu = \mathcal{O}(y) \; . \tag{3.4.15}$$

After a computation we then obtain

$$\left[\nabla^{(3)}_\mu - \mathrm{i}A_{(0)\mu} + \frac{1}{2y}\left(1 + \frac{1}{4}yw_{(1)}\right)e^i_{(3)\mu}(\Gamma_i - \Gamma_{i0}) + \mathcal{O}(y)\right]\epsilon \;=\; 0 \; , \tag{3.4.16}$$

where $\mu = \psi, z, \bar{z}$, and where

$$A_{(0)} \;=\; -\frac{1}{4}w_{(1)}e^1_{(3)} + \frac{\mathrm{i}}{8}\mathrm{e}^{-w_{(0)}/2}(\partial_z - \partial_{\bar{z}})w_{(0)}e^2_{(3)} - \frac{1}{8}\mathrm{e}^{-w_{(0)}/2}(\partial_z + \partial_{\bar{z}})w_{(0)}e^3_{(3)} \; , \tag{3.4.17}$$

is the lowest order term in the expansion of A given by (3.3.6). The Killing spinor ϵ then expands as

$$\epsilon = \frac{1}{\sqrt{2y}}\Big[1+\Gamma_0+\frac{1}{4}yw_{(1)}\Gamma_0+\mathcal{O}(y^2)\Big]\zeta_0\ , \tag{3.4.18}$$

where ζ_0 is the lowest order (y-independent) part of the Kähler spinor ζ. Substituting this into (3.4.16) gives a leading order term that is identically zero. The subleading term then reads

$$\Big[\Big(\nabla_i^{(3)}-\mathrm{i}A_{(0)i}\Big)(1+\Gamma_0)+\frac{1}{8}w_{(1)}(\Gamma_{i0}-\Gamma_i)\Big]\zeta_0=0\ . \tag{3.4.19}$$

The projections (3.3.20), in the current context, read

$$\Gamma_1\zeta_0=\mathrm{i}\Gamma_0\zeta_0\ , \qquad \Gamma_3\zeta_0 = \mathrm{i}\Gamma_2\zeta_0\ . \tag{3.4.20}$$

We may choose the following representation of the gamma matrices:

$$\Gamma_i = \begin{pmatrix} 0 & \tau_i \\ \tau_i & 0 \end{pmatrix}\ , \qquad \Gamma_0 = \begin{pmatrix} 0 & \mathrm{i}\mathbb{I}_2 \\ -\mathrm{i}\mathbb{I}_2 & 0 \end{pmatrix}\ , \tag{3.4.21}$$

with τ_i the Pauli matrices[9]. The projection conditions then force ζ_0 to take the form[10]

$$\zeta_0 = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{where} \quad \chi = \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix}\ . \tag{3.4.22}$$

Here χ is a two-component spinor and χ_0 is simply a constant. The three-dimensional

[9]In this basis the charge conjugation matrix B, appearing in $\epsilon^c \equiv B\epsilon^*$, is $B = \begin{pmatrix} \varepsilon & 0 \\ 0 & -\varepsilon \end{pmatrix}$ where $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

[10]Notice that although our frame coincides with that of [72], our three-dimensional gamma matrices are a permutation of those in the latter reference, which is why the spinor solution takes a slightly different form.

Killing spinor equation then becomes

$$\left(\nabla_i^{(3)} - \mathrm{i}A_{(0)i} - \frac{\mathrm{i}}{8}w_{(1)}\tau_i\right)\chi = 0 \ . \qquad (3.4.23)$$

This three-dimensional Killing spinor equation is precisely of the form found in [72], and studied in [32]. More precisely, this is the form of the Killing spinor equation in the case where the background geometry has real-valued fields, with the metric given by (3.4.6), and the Killing spinor χ and its charge conjugate χ^c give rise to an $\mathcal{N} = 2$ supersymmetric background. In the notation of these references we have that the three-dimensional gauge field $V = 0$ (or rather there exists a gauge in which this is true – see appendix B), while $A = A_{(0)}$ and the function $H = -\frac{\mathrm{i}}{4}w_{(1)}$. This result shows that there indeed exists a spinor χ with the required properties to construct supersymmetric field theories on M_3. Thus our four-dimensional self-dual Einstein manifolds with background gauge field A are the supergravity duals to supersymmetric field theories on M_3 with metric (3.4.6) and Killing spinors χ and χ^c.

We close this subsection by remarking that supersymmetry singled out a natural representative (3.4.6) of the conformal class of the boundary metric. However, one is free to make the change in radial coordinate $r \to r\Omega$, with Ω any smooth, nowhere zero function on M_3, resulting in a conformal transformation of (3.4.6) by $\mathrm{d}s^2_{M_3} \to \Omega^2 \mathrm{d}s^2_{M_3}$. In particular, in the metric (3.4.6) the Killing vector $K = \partial_\psi$ has length 1, while the latter conformal rescaling gives $\|K\|_{M_3} = \Omega$. In this case one instead finds that the vector V in [32, 72] is non-zero, with gauge-invariant and generically non-zero components $V_2 = \partial_3 \log\Omega$ and $V_3 = -\partial_2 \log\Omega$. This is then in agreement with the three-dimensional results of [32]. For further details of this conformal rescaling we refer the reader to appendix B.

3.4.3 Non-singular gauge

In a neighbourhood of the conformal boundary the Kähler metric is defined on $[0,\epsilon)\times M_3$, for some $\epsilon > 0$. This follows since via the conformal rescaling (3.3.7) the Kähler metric asymptotes to

$$\mathrm{d}s^2_{\text{Kahler}} \simeq \mathrm{d}y^2 + \mathrm{d}s^2_{M_3} \ , \tag{3.4.24}$$

near the conformal boundary $y = 0$. In particular the Kähler structure is smooth and globally defined in a neighbourhood of this boundary. In this section we analyse the case where $M_3 \cong S^3$. The gauge field A restricts to a one-form $A_{(0)}$ on the conformal boundary, but as we shall see the explicit representative (3.4.17) is in a singular gauge. Correspondingly, since the boundary Killing spinor χ is constructed with the help of A, equation (3.3.19), the solution (3.4.22) to (3.4.23) is similarly in a singular gauge. In this section we correct this by writing $A_{(0)}$ as a global one-form on $M_3 \cong S^3$.

The expression (3.4.17) for the restriction of A to the conformal boundary is of course only well-defined up to gauge transformations. We may rewrite the expression in (3.4.17) as

$$A^{\text{local}}_{(0)} = -\frac{1}{4}w_{(1)}(\mathrm{d}\psi + \phi_{(0)}) + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar z} w_{(0)}\mathrm{d}\bar z \ , \tag{3.4.25}$$

adding the superscript label 'local' to emphasise that in general this is only a local one-form. The first term is $-\frac{1}{4}w_{(1)}\sigma$, which is always a global one-form on M_3, independently of the topology of M_3. However, the last two terms are not globally defined in general. We may remedy this in the case where $M_3 \cong S^3$ by making a gauge transformation, adding an appropriate multiple of $\mathrm{d}\psi$:

$$A_{(0)} = -\frac{1}{4}w_{(1)}\sigma + \gamma\left[\mathrm{d}\psi + \frac{\mathrm{i}}{4\gamma}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4\gamma}\partial_{\bar z} w_{(0)}\mathrm{d}\bar z\right] \ . \tag{3.4.26}$$

This is then a global one-form on $M_3 \cong S^3$ if and only if the curvature two-form of the connection in square brackets lies in the same basic cohomology class as $\mathrm{d}\sigma = \mathrm{d}\phi_{(0)}$.

Concretely, we write

$$\gamma \mathrm{d}\psi + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z} \equiv \gamma\mathrm{d}\psi + B \ \equiv \ \gamma\sigma + \alpha \ , \tag{3.4.27}$$

and compute

$$\begin{aligned} \mathrm{d}B &= -\frac{\mathrm{i}}{2}\partial_z\partial_{\bar{z}} w_{(0)}\mathrm{d}z\wedge\mathrm{d}\bar{z} \ = \ \left(w_{(1)}^2 + w_{(2)}\right)\mathrm{e}^{w_{(0)}}\frac{\mathrm{i}}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z} \\ &= \frac{1}{4}\left(w_{(1)}^2 + w_{(2)}\right)\mathrm{vol}_T \ , \end{aligned} \tag{3.4.28}$$

where we used the Toda equation (3.3.5) and Taylor expanded. Since σ is a global one-form on $M_3 \cong S^3$, it follows that (3.4.26) is a global one-form precisely if α defined via (3.4.27) is a global *basic* one-form, *i.e.* α is invariant under $\mathcal{L}_{\partial_\psi}$ and satisfies $\partial_\psi \lrcorner \alpha = 0$. In this case we have

$$\int_{M_3} \sigma \wedge \frac{1}{\gamma}\mathrm{d}B = \int_{M_3} \sigma\wedge\mathrm{d}\sigma \ , \tag{3.4.29}$$

which may be interpreted as saying that $[\frac{1}{\gamma}\mathrm{d}B] = [\mathrm{d}\sigma] \in H^2_{\mathrm{basic}}(M_3) \cong \mathbb{R}$ lie in the same basic cohomology class. Indeed, this is the case if and only if $\frac{1}{\gamma}\mathrm{d}B$ and $\mathrm{d}\sigma$ differ by the exterior derivative of a global basic one-form.

The integral on the right hand side of (3.4.29) is the *almost contact volume* of M_3:

$$\mathrm{Vol}_\sigma \equiv \int_{M_3}\sigma\wedge\mathrm{d}\sigma \ = \ \int_{M_3}\frac{w_{(1)}}{2}\sigma\wedge\mathrm{vol}_T \ = \ \int_{M_3}\frac{w_{(1)}}{2}\sqrt{\det g_{M_3}}\,\mathrm{d}^3x \ . \tag{3.4.30}$$

This played an important role in computing the classical localised Chern-Simons action in [32], which contributes to the field theory partition function on M_3. Using (3.4.28), (3.4.29) and (3.4.30) we see that $A_{(0)}$ in (3.4.26) is a global one-form if we choose the constant γ via

$$\frac{1}{4\gamma}\int_{M_3}\left(w_{(1)}^2 + w_{(2)}\right)\ \sqrt{\det g_{M_3}}\,\mathrm{d}^3x = \mathrm{Vol}_\sigma \ . \tag{3.4.31}$$

We shall return to this formula in section 3.4.5

3.4.4 Global conformal Kähler structure

Recall that at the beginning of this section we assumed we were given a complete self-dual Einstein metric with Killing vector $K = \partial_\psi$, of the local form (3.3.2). We would like to understand when the conformal Kähler structure, studied locally in section 3.3.2, is globally non-singular. As we shall see, this is not automatically the case. Focusing on the case of toric metrics on a four-ball (all examples in section 3.6 are of this type), with an appropriate restriction on K we will see that the conformal Kähler structure is indeed everywhere regular. It follows in this case that the Kähler spinor and instanton $F = \frac{1}{2}\mathcal{R}$ are globally non-singular, and thus that the Killing spinor ϵ given by (3.3.22) is also globally defined and non-singular. Before embarking on this section, we warn the reader that the discussion is a little involved, and this section is probably better read in conjunction with the explicit examples in section 3.6. In fact the Euclidean AdS_4 metric in section 3.6.1 displays almost all of the generic features we shall encounter.

The self-dual Einstein metrics of section 3.6 are all toric, and we may thus parameterise a choice of toric Killing vector K as

$$K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \ , \tag{3.4.32}$$

where we have introduced standard 2π-period coordinates φ_1, φ_2 on the torus $U(1) \times U(1)$. It will be important to fix carefully the orientations here. Since the metrics are defined on a ball, diffeomorphic to $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ with $U(1) \times U(1)$ acting in the obvious way, we choose ∂_{φ_i} so that the orientations on $\mathbb{R}^2$ induce the given orientation on $\mathbb{R}^4$ (with respect to which the metric has anti-self-dual Weyl tensor). This fixes the relative sign of b_1 and b_2. Given that we have also assumed that K has no fixed points near the conformal boundary, we must also have b_1 and b_2 non-zero. Thus $b_1/b_2 \in \mathbb{R} \setminus \{0\}$, and its sign will be important in what follows.

Since the self-dual Einstein metric is assumed regular, the one-form $K^\flat$ and its exterior derivative $\mathrm{d}K^\flat$ are both globally defined and regular. We introduce the self-dual two-form

$$\Psi \equiv \left(\mathrm{d}K^\flat\right)^+ \equiv \frac{1}{2}(\mathrm{d}K^\flat + *\mathrm{d}K^\flat) , \tag{3.4.33}$$

and the invariant definition of the function/coordinate y in section 3.3 is given in terms of its norm by

$$\frac{2}{y^2} = \|\Psi\|^2 \equiv \frac{1}{2!}\Psi_{\mu\nu}\Psi^{\mu\nu} . \tag{3.4.34}$$

The complex structure tensor for the conformal Kähler structure is correspondingly

$$J^\mu{}_\nu = -y\Psi^\mu{}_\nu , \tag{3.4.35}$$

where indices are raised and lowered using the self-dual Einstein metric. It is then an algebraic fact that $J^2 = -1$. The conformal Kähler structure will thus be everywhere regular, provided the functions y and $1/y$ are not zero. Of course $y = 0$ is the conformal boundary (which is at infinity, and is not part of the self-dual Einstein space). We are free to choose the sign when taking a square root of (3.4.34), and without loss of generality we take $y > 0$ in a neighbourhood of the conformal boundary at $y = 0$. Everything is regular, and in particular the norm of Ψ cannot diverge anywhere (except at infinity), and thus $y \neq 0$ in the interior of the bulk M_4. It follows that y is everywhere positive on M_4.

The Killing vector K is zero only at the 'NUT', namely the fixed origin of $\mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$. At this point the two-form $\mathrm{d}K^\flat$, in an orthonormal frame, is a skew-symmetric 4×4 matrix whose weights are precisely the coefficients b_1, b_2 in (3.4.32).[11] It follows from the definitions (3.4.33) and (3.4.34), together with a little

[11] This is perhaps easiest to see by noting that to leading order the metric is flat at the NUT, so one can compute $\mathrm{d}K^\flat$ in an orthonormal frame at the NUT using the flat Euclidean metric on $\mathbb{R}^2 \oplus \mathbb{R}^2$.

linear algebra in such an orthonormal frame, that

$$y_{\rm NUT} = \frac{1}{|b_1+b_2|} \ . \tag{3.4.36}$$

The conformal Kähler structure will thus be regular everywhere, except potentially where $1/y = 0$. Suppose that $1/y = 0$ at a point $p \in M_4 \setminus \{\rm NUT\}$. Then $K = \partial_\psi \mid_p \neq 0$, and thus from the metric (3.3.2) we see that $||K||^2 = 1/(Vy^2) \mid_p \neq 0$. It follows that the function V must tend to zero as $1/y^2$ as one approaches p. We may thus write $V = \frac{c}{y^2} + O(1/y^2)$, where $c = c(z,\bar{z})$ is non-zero at p. Using the definition of V in terms of w in (3.3.3) we thus see that $\partial_y w = \frac{2}{y} - \frac{2c}{y^3} + o(1/y^3)$. There are then various ways to see that the corresponding supersymmetric supergravity solution is *singular*. Perhaps the easiest is to note from the Killing spinor formula (3.3.22), together with the fact that we may normalise $\zeta^\dagger\zeta = 1$, we have

$$\epsilon^\dagger\epsilon = \frac{1}{2y}\left(1+V^{-1}\right) \ , \tag{3.4.37}$$

which from the above behaviour of V then diverges as we approach the point p. It follows that the Killing spinor ϵ is divergent at p, and the solution is singular.

The solutions are thus singular on $M_4 \setminus \{\rm NUT\}$ if and only if $\{1/y = 0\} \setminus \{\rm NUT\}$ is non-empty. Since $y_{\rm NUT} = 1/|b_1+b_2|$, the analysis will be a little different for the cases $b_1/b_2 = -1$ and $b_1/b_2 \neq -1$. We thus assume the latter (generic) case for the time being. As in the last paragraph, let us suppose $1/y \mid_p = 0$. Due to the behaviour of V and w near p, it follows from the form of the metric (3.3.2) that p must lie on one of the axes, *i.e.* at $\rho_1 = 0$ or at $\rho_2 = 0$, where (ρ_i, φ_i) are standard polar coordinates on each copy of $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4 \cong M_4$, $i = 1, 2$.[12] This must be so because $\mathrm{d}s^2_{M_4}|_p \sim \mathrm{d}\psi^2$ but $\mathrm{d}\psi \sim \mathrm{d}\varphi_1 + \mathrm{d}\varphi_2$ is two-dimensional unless $\rho_1 = 0$ or $\rho_2 = 0$. In either case there is then an $S^1 \ni p$ locus of points where $1/y = 0$, as follows by following the orbits of the Killing vector ∂_{φ_2} or ∂_{φ_1}, respectively.

[12]Notice that when $b_1/b_2 = -1$ in fact $1/y = 0$ at the NUT itself, $\rho_1 = \rho_2 = 0$.

To see when this happens, our analysis will be based on the fact that, since the Killing vector has finite norm in the interior of M_4, one can straightforwardly show that y diverges if and only if $||\mathrm{d}y|| = 0$. This is because $||\mathrm{d}y||^2 = V/y^2$ and $V = \frac{c}{y^2} + O(1/y^2)$ when y diverges, as explained above. It is then convenient to consider the function y restricted to the relevant axis, *i.e.* $y\mid_{\{\rho_1=0\}} \equiv y_2(\rho_2)$ or $y\mid_{\{\rho_2=0\}} \equiv y_1(\rho_1)$. We have $y_1(0) = y_2(0) = y_{\mathrm{NUT}} > 0$. Suppose that $y_i(\rho)$ (for either $i = 1, 2$) starts out decreasing along the axis as we move away from the NUT. Then in fact it must remain monotonically decreasing along the whole axis, until it reaches $y = 0$ at conformal infinity where $\rho = \infty$. The reason for this is simply that if $y_i(\rho)$ has a turning point then $\mathrm{d}y = 0$, which we have already seen can happen only where y diverges: but this contradicts the fact that $y_i(\rho)$ is decreasing from a positive value at $\rho = 0$ (and is bounded below by 0). On the other hand, suppose that $y_i(\rho)$ starts out increasing at the NUT. Then since at conformal infinity $y_i(\infty) = 0$, it follows that $y_i(\rho)$ must have a turning point at some finite $\rho > 0$. At such a point y will diverge, and from our above discussion the solution is singular.

This shows that the key is to examine $\mathrm{d}y$ at the NUT itself. Recall that the coordinate y is a Hamiltonian function for the Killing vector K, *i.e.* $\mathrm{d}y = -K\lrcorner\omega$. From (3.4.35), we also know that ω is related to the two-form $\Psi = \left(\mathrm{d}K^\flat\right)^+$ by $\omega = -y^3\Psi$, yielding $\mathrm{d}y = y^3 K\lrcorner\left(\mathrm{d}K^\flat\right)^+$. At the NUT we may again use the polar coordinates (ρ_i, φ_i) for the two copies of $\mathbb{R}^2$, where the metric is to leading order the metric on flat space. In the usual orthonormal frame for these polar coordinates, using the above formulae we then compute to leading order

$$(\mathrm{d}y)|_{\mathrm{NUT}} \simeq \begin{pmatrix} -\frac{b_1}{(b_1+b_2)^2}\ \mathrm{sign}(b_1+b_2)\rho_1 \\ 0 \\ -\frac{b_2}{(b_1+b_2)^2}\ \mathrm{sign}(b_1+b_2)\rho_2 \\ 0 \end{pmatrix} . \qquad (3.4.38)$$

Thus when $b_1/b_2 > 0$ we see that $y_i(\rho)$ starts out decreasing at the NUT, for *both*

$i = 1, 2$, and from the previous paragraph it follows that the solution is then globally non-singular. On the other hand, the case $b_1/b_2 < 0$ splits further into two subcases. For simplicity let us describe the case where $b_2 > 0$ (with the case $b_2 < 0$ being similar). Then when $b_1/b_2 < -1$ we have $y_2(\rho)$ starts out increasing at the NUT, which then leads to a singularity along the axis $\rho_1 = 0$ at some finite value of ρ_2; on the other hand, when $-1 < b_1/b_2 < 0$ we have that $y_1(\rho)$ starts out increasing at the NUT, which then leads to a singularity along the axis $\rho_2 = 0$ at some finite value of ρ_1. Notice these two subcases meet where $b_1/b_2 = -1$, when we know that $1/y = 0$ at the NUT itself, $\rho_1 = \rho_2 = 0$.

This leads to the simple picture that all solutions with $b_1/b_2 > 0$ are globally regular, while all solutions with $b_1/b_2 < 0$ are singular, *except* when $b_1/b_2 = -1$. In this latter case y is infinity at the NUT. As one moves out along either axis y is then necessarily monotonically decreasing to zero, by similar arguments to those above. Thus the $b_1/b_2 = -1$ solution is in fact also non-singular, although qualitatively different from the solutions with $b_1/b_2 > 0$. One can show that, regardless of the values of b_1 and b_2, the complex structure (3.4.35) is always the standard complex structure on flat space at the NUT, meaning that when $b_1/b_2 > 0$ the induced complex structure at the NUT is $\mathbb{C}^2$, while when $b_1/b_2 = -1$ the NUT becomes a point at infinity in the conformal Kähler metric, with the Kähler metric being asymptotically Euclidean. In particular the instanton is zero at the NUT in this case, and so is regular there.

Notice that, for the regular solutions, since K is nowhere zero away from the NUT we may deduce that also $\mathrm{d}y = -K \lrcorner \omega$ is nowhere zero (as ω is a global symplectic form on $M_4 \setminus \{\mathrm{NUT}\}$). In particular y is a global Hamiltonian function for K, and in particular it is a Morse-Bott function on M_4. This implies that y has no critical points on $M_4 \setminus \{\mathrm{NUT}\}$, and thus that y_{NUT} is the *maximum* value of y on M_4. Moreover, the Morse-Bott theory tells us that constant y surfaces on $M_4 \setminus \{\mathrm{NUT}\}$ are all diffeomorphic to $M_3 \cong S^3$.

We shall see all of the above behaviour very explicitly in section 3.6 for the case when the self-dual Einstein metric is simply Euclidean AdS_4. The more complicated Einstein metrics in that section of course also display these features, although the corresponding formulae become more difficult to make completely explicit as the examples become more complicated.

3.4.5 Toric formulae

In this subsection we shall obtain some further formulae, valid for any toric self-dual Einstein metric on the four-ball. These will be useful for computing the holographic free energy in the next section.

We first note that for $M_3 \cong S^3$ with Reeb vector (3.4.10) the almost contact volume in (3.4.30) may be computed using equivariant localisation to give

$$\mathrm{Vol}_\sigma \ = \ \int_{M_3} \sigma \wedge \mathrm{d}\sigma = -\frac{(2\pi)^2}{b_1 b_2} \ . \tag{3.4.39}$$

This formula also appeared in [32], although in the present paper we have been more careful with sign conventions. One proves (3.4.39) by using *equivariant localisation*, explained below, but we first need to rewrite the integral in (3.4.39) as an integral on the manifold M_4. We define a two-form

$$\tilde{\omega} \equiv \tfrac{1}{2}\mathrm{d}(\varrho^2\sigma) \ , \tag{3.4.40}$$

on M_4, where ϱ is a choice of radial coordinate with the NUT at $\varrho = 0$ and the conformal boundary at $\varrho = \infty$. Note how this form is similar to the symplectic structure on a symplectic cone defined in section 2.4.4 of the last chapter. A straightforward computation shows that the almost contact volume can be written

$$\mathrm{Vol}_\sigma = -\int_{M_4} \mathrm{e}^{-\varrho^2/2}\frac{1}{2}\tilde{\omega} \wedge \tilde{\omega} \ . \tag{3.4.41}$$

The minus sign arises here because the natural orientation on M_3 defined in our set-up is opposite to that on the right hand side of (3.4.41). Specifically, y is decreasing towards the boundary of M_4, so that $\mathrm{d}y$ points inwards from $M_3 = \partial M_4$, while ϱ is increasing towards the boundary, with $\mathrm{d}\varrho$ pointing outwards[13]. One then evaluates the right hand side of (3.4.41) using equivariant localisation.

Equivariant localisation

Before getting into the detail of the computation of (3.4.41), let us explain how equivariant localisation works on a general even-dimensional manifold M. Let M be a manifold of dimension $2n$ that admits a Riemannian metric g with a Killing vector V for the Levi-Civita connection. The space of k-form over M is denoted $\Lambda^k M$ and we define $\Lambda M \equiv \oplus_{k=0}^{n} \Lambda^{2k} M$. The *equivariant derivative* is defined by

$$\mathrm{d}_V \equiv \mathrm{d} + V \lrcorner \qquad (3.4.42)$$

and is an operator that acts on ΛM. We say that $\alpha \in \Lambda M$ is d_V-closed if and only if

$$\mathrm{d}_V \alpha = 0. \qquad (3.4.43)$$

Note that any d_V-closed form automatically satisfies $\mathcal{L}_V \alpha = \mathrm{d}_V^2 \alpha = 0$, where we used the Cartan formula. The integral of any form $\alpha \in \Lambda M$ is defined by the integral of its top form component. In other words, any $\alpha \in \Lambda M$ is given by $\alpha = \alpha^{(2n)} + \alpha^{(2n-2)} + \ldots + \alpha^{(2)} + \alpha^{(0)}$ with $\alpha^{(k)} \in \Lambda^k M$ and its integral over M is defined by

$$\int_M \alpha \equiv \int_M \alpha^{(2n)} \, . \qquad (3.4.44)$$

[13]Notice that we could have avoided this by choosing y to be strictly negative on the interior of M_4, rather than strictly positive.

We can now state the *Berline-Vergne theorem*. If the Killing vector V has a discrete zero set, then for any d_V-closed form α we have the following localisation formula

$$\int_M \alpha \;=\; (2\pi)^n \sum_{p\in\{\mathrm{zero}(V)\}} \frac{\alpha^{(0)}}{\lambda(p)} \tag{3.4.45}$$

where $\lambda(p)$ is the product of the weights of V at p.

Let us now apply the localisation formula to the integral in (3.4.41). We define the form α by

$$\alpha \equiv -\exp\left[-\frac{\varrho^2}{2}+\tilde{\omega}\right] = -e^{-\rho^2/2}\left(1+\tilde{\omega}+\frac{1}{2}\tilde{\omega}\wedge\tilde{\omega}\right)\ , \tag{3.4.46}$$

Using (3.4.9) and the definition of $\tilde{\omega}$, we have $K\lrcorner\tilde{\omega} = -\mathrm{d}(\frac{\varrho^2}{2})$ and it is easy to show that α is equivariantly closed under d_K. The NUT is the only point where K has a vanishing action and the corresponding weights are b_1 and b_2. The Berline-Vergne theorem then gives

$$\begin{aligned}\int_{M_4}\alpha \;&=\; -\int_{M_4} \mathrm{e}^{-\varrho^2/2}\frac{1}{2}\tilde{\omega}\wedge\tilde{\omega} = \mathrm{Vol}_\sigma \\ &=\; -(2\pi)^2\,\frac{e^{-\rho^2/2}}{b_1b_2}\bigg|_{\mathrm{NUT}} = -\frac{(2\pi)^2}{b_1b_2}\ ,\end{aligned} \tag{3.4.47}$$

where we have used (3.4.41) in the first line and the fact that the NUT is located at $\rho=0$ in the second line. This then proves (3.4.39) for the almost contact volume[14].

Finally, let us return to the equation (3.4.31). In fact there is another interpretation of the constant γ, in terms of the charge of the Killing spinor under K. To see this, recall that the solution (3.4.22) to the three-dimensional Killing spinor equation (3.4.23) is simply constant in our frame, but that was for the case where the gauge field $A_{(0)}$ is given by (3.4.25), which as we saw in section 3.4.3 is always in a singular gauge on $M_3\cong S^3$. The gauge transformation $A_{(0)}\to A_{(0)}+\gamma\mathrm{d}\psi$ that we made in

[14]This formula is known as the Duistermaat-Heckman formula when $\tilde{\omega}$ is a symplectic form, *i.e.* when σ is a contact form.

(3.4.26) to obtain a non-singular gauge implies that the correct global spinor χ has a phase dependence

$$\chi^{\text{global}} = \mathrm{e}^{\mathrm{i}\gamma\psi} \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} , \tag{3.4.48}$$

where χ_0 is a constant complex number. Since the frame is invariant under $K = \partial_\psi$, we thus deduce that γ is precisely the charge of the Killing spinor ϵ under K. On the other hand, the total four-dimensional spinor is constructed from the canonical spinor ζ on the conformal Kähler manifold, via (3.3.22). Thus γ is also the charge of ζ under K.

Let us now explicitly show how γ is related to b_1 and b_2 when the metric is regular, i.e. $b_1/b_2 > 0$ or $b_1/b_2 = -1$. When $b_1/b_2 > 0$ the associated complex structure identifies $M_4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{C}^2$. The orientation in which the Weyl tensor is anti-self-dual is the same as the canonical orientation on $\mathbb{C}^2$. One can then introduce standard complex coordinates $z_i = \rho_i \mathrm{e}^{\mathrm{i}\psi_i}$, $i = 1, 2$, on $\mathbb{C}^2$. The spinor ζ being the canonical spinor that exists on any Kähler manifold we have

$$\mathcal{L}_{\partial_{\psi_i}} \epsilon = \frac{\mathrm{i}}{2} \epsilon \, , \qquad i \ = \ 1, 2 \, . \tag{3.4.49}$$

Denoting the complex structure tensor by J we also have that $J(V^{-1}\partial_y) = \partial_\psi = K$. Since y is *decreasing* as we move away from the origin of $\mathbb{C}^2$, where recall that the origin is at $y = y_{\text{NUT}} > 0$, this means that for $b_1 > 0$ and $b_2 > 0$ we must then identify $\varphi_i = -\psi_i$, where φ_i are the coordinates on $U(1) \times U(1)$ in (3.4.32). This is because for r any radial coordinate on $\mathbb{C}^2$ we have $J(r\partial_r) = a_1 \partial_{\psi_1} + a_2 \partial_{\psi_2}$ where necessarily $a_1, a_2 > 0$ (that is, the Reeb cone is the positive quadrant in $\mathbb{R}^2$ – see, for example, [80]). On the other hand for $b_1 < 0$ and $b_2 < 0$ we instead have $\varphi_i = +\psi_i$, $i = 1, 2$.

The other non-singular case is $b_1/b_2 = -1$. This is qualitatively different from the case $b_1/b_2 > 0$ in the last paragraph, as here $y_{\text{NUT}} = \infty$ (3.4.36). Moreover, the origin $y = y_{\text{NUT}}$ of $M_4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ is now identified with the point at infinity in $\mathbb{C}^2$, rather than the origin, and the origin is at $y = 0$. One can see this from the conformal Kähler metric $\mathrm{d}s^2_{\text{Kahler}} = y^2 \mathrm{d}s^2_{\text{SDE}}$, which is asymptotically Euclidean around $y = y_{\text{NUT}}$. Thus now y *increases* as we move away from the origin and $V^{-1}\partial_y$ has the correct orientation for a radial vector on $\mathbb{C}^2$. We deduce that for $b_1 < 0$ and $b_2 > 0$ we have $\varphi_1 = -\psi_1$, $\varphi_2 = +\psi_2$, while for $b_1 > 0$ and $b_2 < 0$ we instead have $\varphi_1 = +\psi_1$, $\varphi_2 = -\psi_2$.

Putting all of the above together, we may compute the charge of the Killing spinor ϵ under the supersymmetric Killing vector $K = \partial_\psi$:

$$\mathcal{L}_K \epsilon = \mathrm{i}\gamma\epsilon \ , \tag{3.4.50}$$

where

$$\gamma \equiv -\mathrm{sign}\left(\frac{b_1}{b_2}\right) \cdot \frac{|b_1| + |b_2|}{2} \ . \tag{3.4.51}$$

This immediately allows us to write down that

$$|\gamma| = \frac{|b_1| + |b_2|}{2} \ . \tag{3.4.52}$$

Now that we have an expression for A in a global gauge and that the value of γ is known, we can turn to the computation of the free energy.

3.5 Holographic free energy

In this section we compute the regularised holographic free energy for a supersymmetric self-dual asymptotically locally Euclidean AdS solution defined on the four-ball, deriving the remarkably simple formula (3.1.2) quoted in the introduction.

The computation of the holographic free energy follows by now standard holographic renormalisation methods [81, 82]. The total on-shell action is

$$I = I^{\text{grav}}_{\text{bulk}} + I^F + I^{\text{grav}}_{\text{bdry}} + I^{\text{grav}}_{\text{ct}} \ . \tag{3.5.1}$$

Here the first two terms are the bulk Euclidean supergravity action (3.2.7)

$$I^{\text{SUGRA}} = I^{\text{grav}}_{\text{bulk}} + I^{F} \equiv -\frac{1}{16\pi G_4}\int_{M_4}\left(R+6-F^2\right)\sqrt{\det g}\,\mathrm{d}^4x\ , \qquad (3.5.2)$$

evaluated on a particular solution with topology M_4. The boundary term $I^{\text{grav}}_{\text{bdry}}$ in (3.5.1) is the Gibbons-Hawking-York term, required so that the equations of motion (3.2.6) follow from the bulk action (3.5.2) for a manifold M_4 with boundary. This action is divergent, but we may regularise it using holographic renormalisation. Introducing a cut-off at a sufficiently small value of $y = \delta > 0$, with corresponding hypersurface $\mathcal{S}_\delta = \{y = \delta\} \cong M_3$, we have the following total boundary terms

$$I^{\text{grav}}_{\text{bdry}} + I^{\text{grav}}_{\text{ct}} = \frac{1}{8\pi G_4}\int_{\mathcal{S}_\delta}\left(-K + 2 + \tfrac{1}{2}R(h)\right)\sqrt{\det h}\,\mathrm{d}^3x\ . \qquad (3.5.3)$$

Here $R(h)$ is the Ricci scalar of the induced metric h_{ij} on $\mathcal{S}_\delta$, and K is the trace of the second fundamental form of $\mathcal{S}_\delta$, the latter being the Gibbons-Hawking-York boundary term. It is convenient to rewrite it using

$$\int_{\mathcal{S}_\delta} K\sqrt{\det h}\,\mathrm{d}^3x = \mathcal{L}_n\int_{\mathcal{S}_\delta}\sqrt{\det h}\,\mathrm{d}^3x\ , \qquad (3.5.4)$$

where n is the outward pointing normal vector to the boundary $\mathcal{S}_\delta$. In the rest of this section we evaluate the total free energy (3.5.1) in the case of a supersymmetric self-dual solution on the four-ball $M_4 \cong B^4 \cong \mathbb{R}^4$.

We deal with each term in (3.5.1) in turn, beginning with the gauge field contribution

$$I^F = \frac{1}{16\pi G_4}\int_{M_4} F^2\sqrt{\det g}\,\mathrm{d}^4x = -\frac{1}{8\pi G_4}\int_{M_4} F\wedge F = \int_{M_3} A_{(0)}\wedge F_{(0)}\ . \qquad (3.5.5)$$

Here in the second equality we have used the fact that $*_4F = -F$ is anti-self-dual, while in the last equality we used the fact that on the four-ball $M_4 = B^4 \cong \mathbb{R}^4$ the curvature $F = \mathrm{d}A$ is globally exact. Thus we may apply Stokes' theorem with

$M_3 = \partial M_4$, recalling that the natural orientation on M_3 is induced from an inward-pointing normal vector[15]. Notice also that here the gauge field action is already finite, so there is no need to realise the conformal boundary M_3 as the limit $\lim_{\delta\to 0} \mathcal{S}_\delta$. Next we compute the integrand in (3.5.5) using the global form of $A_{(0)}$ (3.4.26). Recall that this reads

$$A_{(0)} = -\frac{1}{4}w_{(1)}\sigma + \gamma \mathrm{d}\psi + B \ = \ -\frac{1}{4}w_{(1)}\sigma + \gamma\sigma + \alpha \ , \tag{3.5.6}$$

where in particular α is a global basic one-form. We then compute

$$\begin{aligned} A_{(0)} \wedge F_{(0)} \ &= \ \frac{w_{(1)}^3}{32}\sigma \wedge \mathrm{vol}_T - \frac{1}{4}w_{(1)}\sigma \wedge \mathrm{d}B - \frac{\gamma}{8}w_{(1)}^2\sigma \wedge \mathrm{vol}_T \\ &\quad + \gamma\sigma \wedge \mathrm{d}B - \frac{1}{4}\alpha \wedge \mathrm{d}w_{(1)} \wedge \sigma \ . \end{aligned} \tag{3.5.7}$$

When we integrate this over M_3, the last term may be integrated by parts, giving an integral that is equal to the integral of $-\frac{1}{4}w_{(1)}\sigma \wedge \mathrm{d}\alpha$, which then combines with the first line of (3.5.7). On the other hand, the first term on the second line of (3.5.7) may be evaluated in the $U(1) \times U(1)$ toric case using (3.4.28), the integral (3.4.31) and the formula (3.4.52) for $|\gamma|$. This leads to

$$\begin{aligned} I^F \ &= \ -\frac{\pi}{2G_4} \cdot \frac{(|b_1| + |b_2|)^2}{4b_1 b_2} + \frac{1}{8\pi G_4}\int_{M_3} \frac{w_{(1)}^3}{32}\sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \\ &\quad - \frac{1}{8\pi G_4}\int_{M_3} \frac{1}{8}(w_{(1)}^3 + w_{(1)}w_{(2)})\sqrt{\det g_{M_3}}\, \mathrm{d}^3 x \ . \end{aligned} \tag{3.5.8}$$

Notice that the first term closely resembles the free energy appearing in (3.1.2) – we shall see momentarily that this combines with a term coming from the gravitational contribution.

We turn next to the bulk gravity part of the action, which when evaluated on-shell

[15]Concretely, the integral over y is $\int_{y_{\mathrm{NUT}}}^{0} \mathrm{d}y$, where we chose the convention that $y_{\mathrm{NUT}} > 0$.

is

$$I^{\text{grav}}_{\text{bulk}} = \frac{1}{16\pi G_4} \int_{M_4^\delta} 6\text{vol}_4 \ . \tag{3.5.9}$$

Here M_4^δ is cut off along the boundary $\mathcal{S}_\delta = \{y = \delta\} \cong M_3$, which is necessary as the volume is of course divergent. The volume form of interest is

$$\text{vol}_4 = \frac{1}{y^4}\text{d}y \wedge (\text{d}\psi + \phi) \wedge V\text{e}^w 2\text{id}z \wedge \text{d}\bar{z} \ . \tag{3.5.10}$$

A computation reveals that this may be written as the exact form

$$-3\text{vol}_4 = \text{d}\Upsilon \ , \tag{3.5.11}$$

where we have defined the three-form

$$\Upsilon \equiv \frac{1}{2y^2}(\text{d}\psi + \phi) \wedge \text{d}\phi + \frac{1}{y^3}(\text{d}\psi + \phi) \wedge V\text{e}^w 2\text{id}z \wedge \text{d}\bar{z} \ . \tag{3.5.12}$$

We may then integrate over M_4^δ using Stokes' theorem. To do this let us define ϱ to be the geodesic distance from the NUT. We then more precisely cut off the space also at small $\varrho > 0$ and let $\varrho \to 0$, so that we are integrating over $M_4^{\delta,\varrho}$. The form Υ may be written

$$\Upsilon = \frac{1}{2y^2}(\text{d}\psi + \phi) \wedge \text{d}\phi + \frac{1}{y^3}(\text{d}\psi + \phi) \wedge \omega \ , \tag{3.5.13}$$

where ω is the conformal Kähler form. As argued in section 3.4.4, when y_{NUT} is finite ω is everywhere a smooth two-form, and thus in particular in polar coordinates near the NUT at $\varrho = 0$ it takes the form $\omega \simeq \varrho\text{d}\varrho \wedge \beta_1 + \varrho^2\beta_2$ to leading order, where β_1 and β_2 are pull-backs of smooth forms on the $S^3 = S^3_{\text{NUT}}$ at constant $\varrho > 0$. Because of this, the second term in (3.5.13) does not contribute to the integral around the NUT (if $y_{\text{NUT}} = \infty$ this term also clearly does not contribute) but does contribute around $y = \delta \to 0$. Notice that Stokes' theorem allows to write

$$0 = \int_{M_4} \underbrace{\mathrm{d}\left[(\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi\right]}_{=0} = \int_{M_3^{y=0}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi - \int_{S^3_{\mathrm{NUT}}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi \ , \quad (3.5.14)$$

which then gives

$$\int_{S^3_{\mathrm{NUT}}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi = \int_{M_3^{y=0}} (\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi \ = \mathrm{Vol}_\sigma \ = \ -\frac{(2\pi)^2}{b_1 b_2} \ , \quad (3.5.15)$$

where we have used the almost contact volume (3.4.39). Using the fact that $y_{\mathrm{NUT}} = 1/|b_1 + b_2|$ one thus obtains

$$\int_{M_4^\delta} \mathrm{vol}_4 = \frac{(2\pi)^2 |b_1 + b_2|^2}{6 b_1 b_2} + \int_{M_3^{y=0}} \left[\frac{1}{3\delta^3} + \frac{w_{(1)}}{4\delta^2}\right] \sqrt{\det g_{M_3}} \, \mathrm{d}^3 x \ , \quad (3.5.16)$$

so that

$$\begin{aligned} I^{\mathrm{grav}}_{\mathrm{bulk}} \ &= \ \frac{\pi}{2G_4} \cdot \frac{|b_1 + b_2|^2}{2 b_1 b_2} + \frac{1}{8\pi G_4} \cdot \frac{1}{\delta^3} \int_{M_3^{y=0}} \sqrt{\det g_{M_3}} \, \mathrm{d}^3 x \\ &\quad + \frac{3}{32\pi G_4} \cdot \frac{1}{\delta^2} \int_{M_3^{y=0}} w_{(1)} \sqrt{\det g_{M_3}} \, \mathrm{d}^3 x \ . \end{aligned} \quad (3.5.17)$$

In particular notice that the $\mathcal{O}(0)$ term at the conformal boundary is zero. This follows from the identity

$$\int_{M_3} \left(w_{(1)}^3 + 3 w_{(1)} w_{(2)} + w_{(3)}\right) \sqrt{\det g_{M_3}} \, \mathrm{d}^3 x = 0 \ , \quad (3.5.18)$$

which arises from Taylor expanding the Toda equation (3.3.5) as

$$\begin{aligned} 0 \ &= \ \partial_z \partial_{\bar{z}} w_{(0)} + \mathrm{e}^{w_{(0)}} \left(w_{(1)}^2 + w_{(2)}\right) \\ &\quad + y \left[\partial_z \partial_{\bar{z}} w_{(1)} + \mathrm{e}^{w_{(0)}} \left(w_{(1)}^3 + 3 w_{(1)} w_{(2)} + w_{(3)}\right)\right] + \mathcal{O}(y^2) \ . \end{aligned} \quad (3.5.19)$$

In particular, because $w_{(1)}$ is a smooth global function on M_3, the second line implies (3.5.18).

It remains to evaluate the boundary terms $I^{\rm grav}_{\rm bdry}+I^{\rm grav}_{\rm ct}$. After a computation, and again using (3.5.18), one obtains

$$\begin{aligned} I^{\rm grav}_{\rm bdry}+I^{\rm grav}_{\rm ct} &= -\frac{1}{8\pi G_4\delta^3}\int_{M_3^{y=0}}\sqrt{\det g_{M_3}}\,{\rm d}^3x-\frac{3}{32\pi G_4\delta^2}\int_{M_3^{y=0}}w_{(1)}\sqrt{\det g_{M_3}}\,{\rm d}^3x \\ &\quad +\frac{1}{256\pi G_4}\int_{M_3}\left(3w_{(1)}^3+4w_{(1)}w_{(2)}\right)\sqrt{\det g_{M_3}}\,{\rm d}^3x\ . \end{aligned} \tag{3.5.20}$$

Adding (3.5.20) to the bulk gravity term (3.5.17) we see that the divergent terms do indeed precisely cancel, and further combining with (3.5.8) we see that the terms involving the integrals of $w_{(i)}$ also all cancel.

The computations we have done are valid only for globally regular solutions, and recall these divide into the two cases $b_1/b_2>0$, and $b_1/b_2=-1$. In the first case the first term in (3.5.8) combines with the first term in (3.5.17) to give

$$I=\frac{\pi}{2G_4}\cdot\frac{(|b_1|+|b_2|)^2}{4|b_1b_2|}\ , \tag{3.5.21}$$

where notice $|b_1+b_2|=|b_1|+|b_2|$. On the other hand the isolated case with $b_1/b_2=-1$ has $b_1+b_2=0$, so that the free energy comes entirely from the first term in (3.5.8), which remarkably is then also given by the formula (3.5.21). Thus for all regular supersymmetric solutions we have shown that (3.1.2) holds and is indeed equal to the free energy (3.1.1) computed using localisation of a supersymmetric field theory on M_3.

3.6 Examples

In this section we illustrate our general results by discussing three explicit families of solutions. These consist of three sets of self-dual Einstein metrics on the four-ball, studied previously in [26, 29–31]. We begin with AdS_4 in section 3.6.1. Although the metric is trivial, the one-parameter family of instantons given by our general results is non-trivial, and it turns out that this family is identical to that in [26].

The solutions in sections 3.6.2 and 3.6.3 each add a deformation parameter, meaning that the metrics in each subsequent section generalise that in the previous section. *Particular* supersymmetric instantons on these backgrounds were found in [29–31], but our general results allow us to study the most general choice of instanton, leading to new solutions. Furthermore, in section 3.6.4 we indicate how to generalise these metrics further by adding an *arbitrary* number of parameters.

3.6.1 AdS_4

The metric on Euclidean AdS_4 can be written as

$$\mathrm{d}s^2_{\mathrm{EAdS}_4} = \frac{\mathrm{d}q^2}{1+q^2} + q^2\left(\mathrm{d}\vartheta^2 + \cos^2\vartheta\mathrm{d}\varphi_1^2 + \sin^2\vartheta\mathrm{d}\varphi_2^2\right) \ . \tag{3.6.1}$$

Here q is a radial variable with $q \in [0,\infty)$, so that the NUT is at $q=0$ while the conformal boundary is at $q=\infty$. The coordinate $\vartheta \in [0,\frac{\pi}{2}]$, with the endpoints being the two axes of $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4$. The AdS_4 metric is of course both self-dual and anti-self-dual.

Writing a general choice of Reeb vector field as $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$, as in our general discussion (3.4.32), the function y is then defined in terms of K via (3.4.33) and (3.4.34). Using these formulae one easily computes

$$y(q,\vartheta) \ = \ \frac{1}{\sqrt{(b_2+b_1\sqrt{q^2+1})^2\cos^2\vartheta + (b_1+b_2\sqrt{q^2+1})^2\sin^2\vartheta}} \ . \tag{3.6.2}$$

Notice that indeed $y_{\mathrm{NUT}} = 1/|b_1+b_2|$, in agreement with (3.4.36). Using (3.6.2) one can also verify the general behaviour of section 3.4.4 very explicitly. In particular we see the very different global behaviour, depending on the sign of b_1/b_2. If $b_1/b_2 > 0$ then $1/y$ is nowhere zero, while if $b_1/b_2 < 0$ instead $1/y$ has a zero on M_4. More precisely, if $-1 < b_1/b_2 < 0$ then $1/y = 0$ at $\{\vartheta = 0, q = \sqrt{b_2^2 - b_1^2}/|b_1|\}$, while if $b_1/b_2 < -1$ then $1/y = 0$ at $\{\vartheta = \frac{\pi}{2}, q = \sqrt{b_1^2 - b_2^2}/|b_2|\}$. These are each a copy of S^1 at one of the 'axes' of $\mathbb{R}^2 \oplus \mathbb{R}^2$, at the corresponding radius given by q. In the special case that $b_1 = -b_2$ we have $1/y = 0$ at the NUT itself, where the axes meet. These comments of course all agree with the general analysis in section 3.4.4, except here all formulae can be made completely explicit.

We thus indeed obtain smooth solutions for all $b_1/b_2 > 0$, as well as the isolated non-singular solution with $b_1/b_2 = -1$. In fact it is not difficult to check that the former are precisely the solutions first found in [26], where the parameter $b^2 = b_2/b_1$. To see this we may compute the instanton using the formulae in section 3.3, finding

$$A = \frac{\left(b_1 + b_2\sqrt{q^2+1}\right)\mathrm{d}\varphi_1 + \left(b_2 + b_1\sqrt{q^2+1}\right)\mathrm{d}\varphi_2}{2\sqrt{(b_2 + b_1\sqrt{q^2+1})^2\cos^2\vartheta + (b_1 + b_2\sqrt{q^2+1})^2\sin^2\vartheta}}\ , \tag{3.6.3}$$

which agrees with the corresponding formula in [26]. In particular one can check that this gives a regular instanton when $b_1/b_2 > 0$, with the particular cases that $b_1/b_2 = \pm 1$ giving a *trivial* instanton, and correspondingly the conformal Kähler structure is flat. We shall comment further on this below. Moreover, one can also check that the singular instantons with $b_1/b_2 < 0$ are singular at precisely the locus that $1/y = 0$, again in agreement with our general discussion.

In this case we may also compute all other functions appearing in sections 3.3, 3.4 and 3.5 very explicitly. For example, we find

$$V(q,\vartheta) \ = \ \frac{(b_2 + b_1\sqrt{q^2+1})^2\cos^2\vartheta + (b_1 + b_2\sqrt{q^2+1})^2\sin^2\vartheta}{q^2(b_1^2\cos^2\vartheta + b_2^2\sin^2\vartheta)}\ , \tag{3.6.4}$$

while the functions $w_{(1)}$ and $w_{(2)}$ on $\partial M_4 = M_3 \cong S^3$ appearing in the free energy computations are given by

$$w_{(1)} = \frac{-4b_1b_2}{\sqrt{b_1^2\cos^2\vartheta + b_2^2\sin^2\vartheta}}\ , \quad w_{(2)} \ = \ \frac{-2\left(3b_1^2b_2^2 + b_1^4\cos^2\vartheta + b_2^4\sin^2\vartheta\right)}{b_1^2\cos^2\vartheta + b_2^2\sin^2\vartheta}\ . \tag{3.6.5}$$

Using these expressions one can verify all of the key formulae in our general analysis very explicitly. For example, the integrals in (3.4.39), (3.5.8), (3.5.17) and (3.5.20) are all easily computed in closed form.

Finally, let us return to discuss the special cases $b_1/b_2 = \pm 1$, where recall that the instanton is trivial and the conformal Kähler structure is flat. The latter is thus locally the flat Kähler metric on $\mathbb{C}^2$, but in fact in the two cases $b_1/b_2 = \pm 1$ the Euclidean AdS$_4$ metric is conformally embedded into *different* regions of $\mathbb{C}^2$. Notice this has to be the case, because the conformal factor y of the $b_1/b_2 = +1$ solution has $y_{\rm NUT} = 1/(2|b_1|)$, while for the $b_1/b_2 = -1$ solution instead $y_{\rm NUT} = \infty$. We may see this very concretely by writing the flat Kähler metric on $\mathbb{C}^2$ as

$$\mathrm{d}s^2_{\rm flat} = \mathrm{d}R^2 + R^2\left(\mathrm{d}\vartheta^2 + \cos^2\vartheta \mathrm{d}\varphi_1^2 + \sin^2\vartheta\mathrm{d}\varphi_2^2\right) \ . \tag{3.6.6}$$

In both cases the change of radial coordinate to (3.6.1) is

$$q(R) \ = \ \frac{2R}{|R^2-1|} \ . \tag{3.6.7}$$

However, for the $b_1/b_2 = +1$ case the range of R is $0 \le R < 1$, with the NUT being at $R = 0$ and the conformal boundary being at $R = 1$; while for the $b_1/b_2 = -1$ case the range of R is instead $1 < R \le \infty$, with the NUT being at $R = \infty$ (and the conformal boundary again being at $R = 1$). In particular the two conformal factors are

$$y(R) = \tfrac{1}{2|b_1|}|R^2-1| \ . \tag{3.6.8}$$

The two solutions $b_1/b_2 = \pm 1$ thus effectively fill opposite sides of the unit sphere in $\mathbb{C}^2$, and because of this they induce opposite orientations on S^3. Again, this may be seen rather explicitly in various formulae. For example, $w_{(1)} = \mp 4|b_1|$ in the two cases, so that the boundary Killing spinor equation (3.4.23) on the round S^3 becomes[16] respectively $\nabla^{(3)}_i\chi = \mp\frac{\mathrm{i}}{2}|b_1|\gamma_i\chi$, where one can take the gamma matrices to be the Pauli matrices $\gamma_i = \tau_i$ in an orthonormal frame.

[16]The gauge field $A_{(0)} = \frac{1}{2}(\mathrm{d}\varphi_1 + \mathrm{d}\varphi_2)$ can be gauged away and that is why it does not appear in the Killing spinor equation.

3.6.2 Taub-NUT-AdS$_4$

The Taub-NUT-AdS$_4$ metrics are a one-parameter family of self-dual Einstein metrics on the four-ball, and have been studied in detail in [29,30]. The metric may be written as

$$\mathrm{d}s_4^2 = \frac{r^2-s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2-s^2)(u_1^2+u_2^2) + \frac{4s^2\Omega(r)}{r^2-s^2}u_3^2 \ , \tag{3.6.9}$$

where

$$\Omega(r) = (r\mp s)^2[1+(r\mp s)(r\pm 3s)] \ , \tag{3.6.10}$$

and u_1, u_2, u_3 are left-invariant one-forms on $SU(2)\simeq S^3$. The latter may be written in terms of Euler angular variables as

$$u_1 + \mathrm{i}u_2 = \mathrm{e}^{-\mathrm{i}\varsigma}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi) \ , \qquad u_3 \ = \ \mathrm{d}\varsigma + \cos\theta\mathrm{d}\varphi \ . \tag{3.6.11}$$

Here ς has period 4π, while $\theta\in[0,\pi]$ with φ having period 2π. The radial coordinate r lies in the range $r\in[s,\infty)$, with the NUT (origin of the ball $\cong\mathbb{R}^4$) being at $r=s$. The parameter $s>0$ is referred to as the *squashing parameter*, with $s=\frac{1}{2}$ being the Euclidean AdS$_4$ metric studied in the previous section. Indeed, the metric is asymptotically locally Euclidean AdS as $r\to\infty$, with

$$\mathrm{d}s_4^2 \approx \frac{\mathrm{d}r^2}{r^2} + r^2(u_1^2+u_2^2+4s^2u_3^2) \ , \tag{3.6.12}$$

so that the conformal boundary at $r=\infty$ is a biaxially squashed S^3.

Using the results of this chapter we may write a general choice of Reeb vector field as $K=(b_1+b_2)\partial_\varphi+(b_1-b_2)\partial_\varsigma$, as in our general discussion (3.4.32), and the function y is then defined in terms of K via (3.4.33) and (3.4.34). Using these one computes

$$\begin{aligned}\frac{1}{y(r,\theta)^2} \ &= \ [2(b_1-b_2)(r-s)s+(b_1+b_2)(1+2(r-s)s)\cos\theta]^2 \\ &\quad +(b_1+b_2)^2\,[1+(r-s)(r+3s)]\sin^2\theta \ .\end{aligned} \tag{3.6.13}$$

Notice that indeed $y_{\rm NUT} = \lim_{r\to s} y(r,\theta) = 1/|b_1+b_2|$. We see that if $b_1/b_2>0$ or $b_1/b_2=-1$ then $1/y$ is indeed never zero (except at the NUT in the latter case), as expected. In this way we obtain a *two-parameter* family of regular supersymmetric solutions, parametrised by the squashing parameter s and b_1/b_2. One can also compute explicitly the corresponding instanton F for a general choice of s and b_1/b_2. This was done in [2] and the expression for F is not reported here. In the remainder of this subsection we shall instead discuss further some special cases, making contact with the previous results [29, 30].

While the Taub-NUT-AdS metric (3.6.9) has $SU(2)\times U(1)$ isometry, a generic choice of the Killing vector $K=(b_1+b_2)\partial_\varphi+(b_1-b_2)\partial_\varsigma$ breaks the symmetry of the full solution to $U(1)\times U(1)$. In particular, this symmetry is also broken by the corresponding instanton A. On the other hand, in [29,30] the $SU(2)\times U(1)$ symmetry of the metric was *also* imposed on the gauge field, which results in two one-parameter subfamilies of the above two-parameter family of solutions, which are 1/4 BPS and 1/2 BPS, respectively. In each case this effectively fixes the Killing vector K (or rather the parameter b_1/b_2) as a function of the squashing parameter s.

1/4 BPS solution: This solution is simple enough that it can be presented in complete detail. The coordinate transformation to the (3.3.2) form for the 1/4 BPS solution reads

$$r - s \;=\; 1/y\ , \qquad -2su_3 \;=\; \mathrm{d}\psi+\phi\ , \tag{3.6.14}$$

and

$$y^2(r^2-s^2) \;=\; \mathrm{e}^w V(1+|z|^2)^2\ , \qquad \frac{r^2-s^2}{\Omega(r)} \;=\; y^2 V\ . \tag{3.6.15}$$

Notice immediately that at the NUT $r=s$ we have $1/y=0$, so that this solution must have $b_1=-b_2$ – we shall find this explicitly below. The metric $(u_1^2+u_2^2)$ is diffeomorphic to the Fubini-Study metric on $\mathbb{CP}^1\cong S^2$:

$$u_1^2+u_2^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}\ . \tag{3.6.16}$$

The metric functions then simplify to

$$V(y) = \frac{1+2sy}{1+4sy+y^2} \ , \qquad w(y,z,\bar{z}) \ = \ \log \frac{1+4sy+y^2}{(1+|z|^2)^2} \ , \tag{3.6.17}$$

and it is straightforward to check these satisfy the defining equation (3.3.3) and Toda equation (3.3.5). The conformally related scalar-flat Kähler metric is

$$\mathrm{d}s^2_{\text{Kahler}} = \frac{1+2sy}{1+4sy+y^2}\mathrm{d}y^2 + (1+2sy)(u_1^2+u_2^2) + \frac{4s^2(1+4sy+y^2)}{1+2sy}u_3^2 \ , \tag{3.6.18}$$

with Kähler form

$$\omega = -\mathrm{d}y \wedge 2su_3 + (1+2sy)u_1 \wedge u_2 \ = \ -\mathrm{d}\left[(1+2sy)u_3\right] \ . \tag{3.6.19}$$

Using the formula (3.3.6) for the gauge field A, we compute

$$A = \frac{1}{2}(4s^2-1)\frac{r-s}{r+s}u_3 + \text{pure gauge} \ , \tag{3.6.20}$$

which we see reproduces the 1/4 BPS choice of instanton of [30][17]. The supersymmetric Killing vector is $K = \partial_\psi = -\frac{1}{2s}\partial_\varsigma$ and so generates the Hopf fibration of S^3. Since $\varsigma = \varphi_1 - \varphi_2$, $\varphi = \varphi_1 + \varphi_2$ we hence find

$$b_1 = -b_2 \ = \ -\frac{1}{4s} \ , \tag{3.6.21}$$

which using (3.1.2) yields

$$I_{1/4\,\text{BPS}} = \frac{\pi}{2G_4}. \tag{3.6.22}$$

This formula matches the result of [30].

[17] Notice that in [30] the opposite orientation convention was chosen, so that the instanton in [30] is self-dual, rather than anti-self-dual. Recall also from the discussion above equation (3.3.6) that the overall sign of the instanton is correlated with the sign of the supersymmetric Killing vector K. Here $K = -\frac{1}{2s}\partial_\varsigma$, which is minus the expression in [30], hence leading to the opposite sign for the instanton gauge field A.

1/2 BPS solution: The Taub-NUT-AdS metric (3.6.9) also admits a 1/2 BPS solution [29, 30]. We hence have two linearly independent Killing spinors, which may be parametrised by an arbitrary choice of constant two-component spinor $\chi_{(0)} = \begin{pmatrix} \mathsf{p} \\ \mathsf{q} \end{pmatrix} \in \mathbb{C}^2 \setminus \{0\}$. The corresponding Killing vector is given by the unlikely expression

$$\begin{aligned} K &= (2s+\sqrt{4s^2-1})\Big[2\mathrm{Im}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}]\partial_\theta + \big(|\mathsf{p}|^2 - |\mathsf{q}|^2 + 2\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}]\cot\theta\big)\,\partial_\varphi\Big] \quad (3.6.23) \\ &+ \Big[(|\mathsf{p}|^2+|\mathsf{q}|^2)\Big(\tfrac{1}{2s} - 2s - \sqrt{4s^2-1}\Big) - 2\mathrm{Re}\,[\mathrm{e}^{\mathrm{i}\varphi}\mathsf{p}\bar{\mathsf{q}}](2s+\sqrt{4s^2-1})\csc\theta\Big]\,\partial_\varsigma\ . \end{aligned}$$

Since multiplying $\chi_{(0)}$ by a non-zero complex number $\lambda \in \mathbb{C}^*$ simply rescales K by $|\lambda|^2$, this leads to a $\mathbb{CP}^1$ family of choices of Killing vector K in this case. Of course, the vector (3.6.23) is not toric for generic choice of $\chi_{(0)}$. Nevertheless, one can still compute the various geometric quantities in section 3.3. In particular one can check that the formula (3.3.15) for the instanton gives

$$A = s\sqrt{4s^2-1}\frac{r-s}{r+s}u_3 + \text{pure gauge}\ , \qquad (3.6.24)$$

for any choice of K in (3.6.23), which agrees with the expression in [29, 30]. Notice that the instanton is invariant under the $SU(2)\times U(1)$ symmetry of the metric, even though a choice of Killing vector K breaks this symmetry. Indeed, in this case the conformal factor $y = y(r,\theta)$ for toric solutions given by (3.6.13) depends non-trivially on both r and θ, thus also breaking the $SU(2)$ symmetry of the underlying Taub-NUT-AdS metric. This is to be contrasted with the 1/4 BPS solution, where instead (3.6.13) reduces simply to $y = y(r) = 1/(r-s)$ (see (3.6.14)).

The toric choices of K for these 1/2 BPS solutions correspond to the poles of the $\mathbb{CP}^1$ parameter space. For example, choosing $\mathsf{p} = 1$, $\mathsf{q} = 0$ above gives

$$K = \Big(2s+\sqrt{4s^2-1}\Big)\,\partial_\varphi + \Big(\tfrac{1}{2s} - 2s - \sqrt{4s^2-1}\Big)\,\partial_\varsigma\ , \qquad (3.6.25)$$

so that

$$b_1 = \frac{1}{4s} \ , \qquad b_2 \ = \ -\frac{1}{4s} + 2s + \sqrt{4s^2 - 1} \ . \tag{3.6.26}$$

The free energy (3.1.2) is thus

$$I = \frac{2\pi s^2}{G_4} \ , \tag{3.6.27}$$

which of course matches the result obtained in [30].

3.6.3 Plebanski-Demianski

The Taub-NUT-AdS metric has been extended to a two-parameter family of smooth self-dual Einstein metrics on the four-ball in [31], which lie in the Plebanski-Demianski class of local solutions [83] to Einstein-Maxwell theory. We will henceforth refer to the solution of [31] as 'Plebanski-Demianski'. The metric may be written as

$$\mathrm{d}s^2_{\mathrm{PD}} \ = \ \frac{\mathcal{P}(q)}{q^2 - p^2}(\mathrm{d}\tau + p^2\mathrm{d}\nu)^2 - \frac{\mathcal{P}(p)}{q^2 - p^2}(\mathrm{d}\tau + q^2\mathrm{d}\nu)^2 + \frac{q^2 - p^2}{\mathcal{P}(q)}\mathrm{d}q^2 - \frac{q^2 - p^2}{\mathcal{P}(p)}\mathrm{d}p^2, \tag{3.6.28}$$

where

$$\mathcal{P}(x) = (x - p_1)(x - p_2)(x - p_3)(x - p_4) \ . \tag{3.6.29}$$

The roots of the quartic $\mathcal{P}(x)$ can be expressed in terms of the two parameters of the solution, a and v, as

$$\begin{aligned} p_1 \ &= \ -\frac{1}{2} - \sqrt{1 + a^2 - v^2} \ , \quad p_3 = \frac{1}{2} - a \ , \\ p_2 \ &= \ -\frac{1}{2} + \sqrt{1 + a^2 - v^2} \ , \quad p_4 = \frac{1}{2} + a \ . \end{aligned} \tag{3.6.30}$$

The coordinate $p \in [p_3, p_4]$ is essentially a polar angle variable, while $q \in [p_4, \infty)$ plays the role of a radial coordinate, with the conformal boundary being at $q = \infty$. The NUT/origin of $\mathbb{R}^4$ is located at $p = p_3$, $q = p_4$. The Killing vectors ∂_τ, ∂_ν generate the $U(1)^2$ torus symmetry of the solution, with the coordinates related to our standard

2π-period coordinates φ_1, φ_2 on $U(1)^2$ via

$$\begin{aligned}\tau &= \frac{2p_3^2}{\mathcal{P}'(p_3)}\varphi_1 - \frac{2p_4^2}{\mathcal{P}'(p_4)}\varphi_2 \ , \\ \nu &= -\frac{2}{\mathcal{P}'(p_3)}\varphi_1 + \frac{2}{\mathcal{P}'(p_4)}\varphi_2 \ .\end{aligned} \tag{3.6.31}$$

In order that the metric is smooth on the four-ball the parameters must obey $v^2 > 2|a|$, with the $a = 0$ limit being the Taub-NUT-AdS metric of the previous section, and further setting $v = 1$ one recovers Euclidean AdS_4 (we refer the reader to [31] for further details).

It is straightforward, but tedious, to express the metric (3.6.28) in the form (3.3.2), with an arbitrary choice of toric Killing vector $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$. For the special case of the Killing vector/instanton in the solution of [31] the change of coordinates was worked out in [2].

In the (τ, ν) coordinates an arbitrary Killing vector may be written as

$$K = b_\tau\partial_\tau + b_\nu\partial_\nu \ , \tag{3.6.32}$$

where

$$b_\tau = \frac{2p_3^2}{\mathcal{P}'(p_3)}b_1 - \frac{2p_4^2}{\mathcal{P}'(p_4)}b_2 \ , \qquad b_\nu = -\frac{2}{\mathcal{P}'(p_3)}b_1 + \frac{2}{\mathcal{P}'(p_4)}b_2 \ . \tag{3.6.33}$$

Using (3.4.33) and (3.4.34) one can calculate

$$\begin{aligned}\frac{1}{y(p,q)^2} &= \frac{1}{4}\frac{1}{(q^2-p^2)^2}\left\{\left[\left(\frac{2\mathcal{P}(q)}{q-p} - \mathcal{P}'(q)\right)(b_\tau + b_\nu p^2)\right.\right. \\ &\quad \left.\left. - \left(\frac{2\mathcal{P}(p)}{q-p} + \mathcal{P}'(p)\right)(b_\tau + b_\nu q^2)\right]^2 - 4b_\nu^2\mathcal{P}(q)\mathcal{P}(p)(q+p)^2\right\} \ .\end{aligned} \tag{3.6.34}$$

Notice that this is a sum of two non-negative terms. Furthermore, these terms may vanish only when evaluated at the roots $p = p_3$, $p = p_4$ or $q = p_4$, which correspond to the axes of $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Let us calculate these limits:

$$\begin{aligned}
\lim_{p\to p_3} \frac{1}{y^2} &= \left(\frac{(b_1+b_2)v^2 + 2ab_1 + b_2(2q-1)}{v^2+2a}\right)^2 , \\
\lim_{p\to p_4} \frac{1}{y^2} &= \left(\frac{(b_1+b_2)v^2 - 2ab_2 + b_1(2q-1)}{v^2-2a}\right)^2 , \\
\lim_{q\to p_4} \frac{1}{y^2} &= \left(\frac{(b_1+b_2)v^2 - 2ab_2 + b_1(2p-1)}{v^2-2a}\right)^2 .
\end{aligned} \tag{3.6.35}$$

A careful analysis of the above limits shows that $1/y$ does not vanish, and hence the metric is regular, whenever $b_1/b_2 > 0$, while $1/y = 0$ only at the NUT when $b_1/b_2 = -1$. On the other hand, the solution is indeed singular if $b_1/b_2 < 0$ and $b_1/b_2 \neq -1$. Notice that we also easily recover the formula (3.4.36) for the conformal factor at the NUT: $\lim_{p\to p_3,\ q\to p_4} y = 1/|b_1+b_2|$.

In [31] particular supersymmetric instantons (particular choices of b_1/b_2 for fixed a and v) were studied for this two-parameter family of metrics, which by construction lie within the Plebanski-Demianski ansatz. The results of this subsection extend these results to a general choice of instanton on the same background, parametrised by b_1/b_2, leading to a *three-parameter* family of regular supersymmetric solutions. The general expression for this instanton is lengthy, but computable, and the interested reader may find the details in [2].

3.6.4 Infinite parameter generalisation

In each subsection we have generalised the metrics of the previous subsection by adding a parameter, and one might wonder whether one can find more general self-dual Einstein metrics on the four-ball. In fact from the gauge-gravity point of view it is more natural to ask the question of which conformal structures on S^3 may be filled by a self-dual Einstein metric. Of course one expects this problem to be overdetermined, and some general results in this direction appear in [84]. Roughly speaking, as long as the conformal class of the boundary metric $[g_{S^3}]$ is sufficiently close to the round metric $[g^0_{S^3}]$, then one can write $[g_{S^3}] = [g^0_{S^3}] + [g^+_{S^3}] + [g^-_{S^3}]$, where $[g^0_{S^3}] + [g^\pm_{S^3}]$ bound self-dual/anti-self-dual Einstein metrics on the four-ball B^4, respectively. Equivalently, viewed as self-dual fillings these induce opposite orientations on S^3. Another important general result is that these fillings are *unique*: that is, two self-dual Einstein four-manifolds $(M_4^{(1)}, g^{(1)})$, $(M_4^{(2)}, g^{(2)})$ inducing the same conformal structure on $M_3 = \partial M_4$ are isometric [85].

However, starting with a particular conformal three-metric and trying to construct a global filling explicitly is likely to be very difficult. In order to construct further explicit examples one might instead attempt to directly generalise the Plebanski-Demianski metrics of the previous subsection. In [79] the authors studied the general *local* geometry of toric self-dual Einstein metrics, which thus includes all the solutions (locally) above. In appropriate coordinates the metric takes the form

$$\begin{aligned} \mathrm{d}s^2_{\text{toric}} \;=\; & \frac{4\rho^2(\mathcal{F}_\rho^2 + \mathcal{F}_\upsilon^2) - \mathcal{F}^2}{4\mathcal{F}^2}\mathrm{d}s^2_{\mathcal{H}^2} + \frac{4}{\mathcal{F}^2(4\rho^2(\mathcal{F}_\rho^2 + \mathcal{F}_\upsilon^2) - \mathcal{F}^2)}\Big[\big(y^{\text{can}}_\rho \mathrm{d}\nu \\ & + (\upsilon y^{\text{can}}_\rho - \rho y^{\text{can}}_\upsilon)\mathrm{d}\varphi\big)^2 + \big(y^{\text{can}}_\upsilon \mathrm{d}\nu + (\rho y^{\text{can}}_\rho + \upsilon y^{\text{can}}_\upsilon - y^{\text{can}})\mathrm{d}\varphi\big)^2\Big] \, . \end{aligned} \tag{3.6.36}$$

where we have defined

$$y^{\text{can}}(\rho, \upsilon) \equiv \sqrt{\rho}\mathcal{F}(\rho, \upsilon) \ , \tag{3.6.37}$$

and

$$\mathrm{d}s^2_{\mathcal{H}^2} = \frac{\mathrm{d}\rho^2 + \mathrm{d}\upsilon^2}{\rho^2} \tag{3.6.38}$$

is the metric on hyperbolic two-space $\mathcal{H}^2$, regarded as the upper half plane with boundary at $\rho = 0$. The metric (3.6.36) is entirely determined by the choice of function $\mathcal{F} = \mathcal{F}(\rho, \upsilon)$, and the metric is self-dual Einstein if and only if this solves the eigenfunction equation

$$\Delta_{\mathcal{H}^2}\mathcal{F} = \frac{3}{4}\mathcal{F} \qquad \Longleftrightarrow \qquad \mathcal{F}_{\rho\rho} + \mathcal{F}_{\upsilon\upsilon} \;=\; \frac{3}{4\rho^2}\mathcal{F} \ , \tag{3.6.39}$$

where $\mathcal{F}_\rho \equiv \partial_\rho \mathcal{F}$, *etc.* Unlike the Toda equation (3.3.5) this is linear, and one may add solutions. In particular there is a basic solution

$$\mathcal{F}(\rho, \upsilon; \lambda) = \frac{\sqrt{\rho^2 + (\upsilon - \lambda)^2}}{\sqrt{\rho}} \ , \tag{3.6.40}$$

where λ is any constant. Via linearity

$$\mathcal{F}(\rho, \upsilon) = \sum_{i=1}^{m} \alpha_i \mathcal{F}(\rho, \upsilon; \lambda_i) \ , \tag{3.6.41}$$

also solves (3.6.39), for arbitrary constants α_i, λ_i, $i = 1, \ldots, m$. We refer to (3.6.41) as an *m-pole solution*. Of course, one could also replace the sum in (3.6.41) by an integral, smearing the monopoles in some chosen charge distribution.

Thus the *local* construction of toric self-dual Einstein metrics is very straightforward – the above gives an infinite-dimensional space. However, understanding when the above metrics extend to complete asymptotically locally hyperbolic metrics on a ball (or indeed any other topology for M_4) is more involved. In [2] some steps were taken in this direction by showing that the general 2-pole solution is simply Euclidean AdS_4, while the general 3-pole solution is precisely the Plebanski-Demianski solutions of section 3.6.3. This requires taking into account the symmetries of (3.6.36) (in particular the $PSL(2, \mathbb{R})$ symmetry of $\mathcal{H}^2$), and then making a number of rather non-trivial coordinate transformations. Some work has also been done on global properties of the metrics (3.6.36) in [86], although the focus in that paper is on constructing complete asymptotically locally Euclidean scalar-flat Kähler metrics, which are conformal to (3.6.36). It remains an interesting open problem to understand when the general m-pole metrics extend to complete metrics on the ball.

Chapter 4

M2-brane duals of Wilson loops on three manifolds

4.1 Introduction and summary

In the second chapter of this thesis, we showed how to compute Wilson loops on S^3 and find their M2-brane gravity duals in $\mathrm{AdS}_4 \times Y_7$ and compute their actions when Y_7 is any toric Sasaki-Einstein manifold. In the last chapter, we showed how the sphere S^3 can be replaced by any three-manifold M_3 with S^3 topology that allows for supersymmetric theories and how this yields a supergravity dual of the form $M_4 \times Y_7$ where M_4 is a self-dual Einstein four-manifold with metric (3.3.2) and background gauge field (3.3.6). At this point, a question naturally arises: is it possible to reproduce the Wilson loop/M2-brane computation of chapter 2 in the more general background of chapter 3 and match their respective actions, hence verifying the gauge/gravity duality beyond the matching of the free-energy for non-trivial $M_3 = \partial M_4 \cong S^3$ manifolds? The answer to that question is positive and it is what we will look at in this chapter.

The partition function Z of three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on M_3 depends on the background geometry only through the supersymmetric

Killing vector field K. As explained in the last chapter, when M_3 is diffeomorphic to S^3 with the standard action of $U(1) \times U(1)$ on $S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$, one finds the large N free energy $\mathcal{F} = -\log Z$ satisfies

$$\lim_{N\to\infty} \mathcal{F} = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot \mathcal{F}_{\text{round}} \ , \tag{4.1.1}$$

where $\mathcal{F}_{\text{round}}$ is the large N limit of the free energy on the *round* three-sphere, which scales as $N^{3/2}$. In the last chapter the field theory result (4.1.1) was reproduced in a dual computation in four-dimensional gauged supergravity. Here $M_3 \cong S^3$ arises as the conformal boundary of a self-dual Einstein four-manifold M_4, where the supersymmetric Killing vector K also extends over M_4. The asymptotically locally Euclidean AdS metric on M_4 is conformally Kähler, and supersymmetry requires one to turn on a graviphoton field A proportional to the Ricci one-form of this Kähler metric. A remarkable feature of the computation of the holographic free energy in section 3.5 is that one does not need to know the form of the Einstein metric on M_4 explicitly – rather (4.1.1) is proven for an *arbitrary* such metric.

In chapter 2 the vacuum expectation values of BPS Wilson loops on the round sphere were computed for a variety of gauge theories, and matched to regularised M2-brane actions in $\mathrm{AdS}_4 \times Y_7$. Here the choice of internal space Y_7 determines the gauge theory on M_3. In this chapter however, we extend these computations to general supersymmetric backgrounds $M_3 = \partial M_4$. A Wilson loop is BPS if it wraps an orbit of K, and we will find that the large N Wilson loop VEV satisfies

$$\lim_{N\to\infty} \log \langle W \rangle = \mathcal{S}_{b_1,b_2} \cdot \log \langle W_{\text{round}} \rangle \ , \tag{4.1.2}$$

where

$$\mathcal{S}_{b_1,b_2} \equiv \frac{|b_1| + |b_2|}{2} \ell \ . \tag{4.1.3}$$

Here $\langle W_{\text{round}} \rangle$ denotes the large N limit of the Wilson loop on the round sphere, given by (2.2.1) or equivalently (2.2.5), and $2\pi\ell$ denotes the length of the orbit of K.

Such orbits always close over the poles of S^3, *i.e.* at the origins of each copy of $\mathbb{R}^2$ in $S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$, where the lengths are then $\ell = 1/|b_1|$ and $\ell = 1/|b_2|$, respectively. For these Wilson loops (4.1.2) becomes a function of b_1/b_2, exactly as in (4.1.1). The supergravity dual configurations are given by M2-branes wrapping a supersymmetric copy of the M-theory circle in Y_7 and a complex curve $\Sigma_2 \subset M_4$, with boundary $\partial\Sigma_2 \subset M_3$ being the Wilson line. Identifying the logarithm of the VEV with minus the holographically renormalised M2-brane action, we prove that (4.1.2) holds in general, thus verifying the matching of this observable in AdS/CFT in a very broad class of backgrounds.

The outline of the rest of this chapter is as follows. In section 4.2 we review the geometry of M_3, the definition of the BPS Wilson loop and how it may be computed using localisation techniques in the large N limit to find (4.1.2). Section 4.3 analyses supersymmetric M2-branes in $M_4 \times Y_7$ backgrounds in M-theory and we also derive the formula (4.1.2) in supergravity. Since our arguments are for general backgrounds they are somewhat implicit; in section 4.4 we therefore look explicitly at AdS_4 and Taub-NUT-AdS_4, to exemplify our general formulae.

4.2 Wilson loops in $\mathcal{N} = 2$ gauge theories on M_3

In this section, we will review the geometry of M_3 and the computation of the BPS Wilson loops using localisation. Our discussion will be, of course, very similar to the one of section 2.3 as it is a generalisation of it. We will nonetheless expose all the steps of the computation in order to have the whole picture but we will be more concise than in section 2.3.

The field theories of interest have UV descriptions as $\mathcal{N} = 2$ Chern-Simons gauge theories coupled to matter on M_3, where M_3 is a supersymmetric three-manifold. After first reviewing the geometry of M_3, we explain how the Wilson loop VEVs localise in the matrix model and take the large N limit to derive (4.1.2).

4.2.1 Three-dimensional background geometry

The manifold M_3 belongs to a general class of 'real' supersymmetric backgrounds, with two supercharges related to one another by charge conjugation [72]. If χ denotes the Killing spinor on M_3 then there is an associated Killing vector field

$$K \equiv \chi^\dagger \gamma^\mu \chi \partial_\mu \ = \ \partial_\psi \ . \tag{4.2.1}$$

This Killing vector is nowhere zero and therefore defines a foliation of the three-manifold. This foliation is transversely holomorphic with local complex coordinate z. In terms of these coordinates the background metric may be written as[1]

$$\mathrm{d}s^2_{M_3} = (\mathrm{d}\psi + \phi_{(0)})^2 + 4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}\,, \tag{4.2.2}$$

where $\phi_{(0)} = \phi_{(0)}(z,\bar{z})\mathrm{d}z + \overline{\phi_{(0)}(z,\bar{z})}\mathrm{d}\bar{z}$ is a local one-form and $w_{(0)}(z,\bar{z})$ is a function. We use the orthonormal frame (3.4.14) for the three-metric $\mathrm{d}s^2_{M_3}$:

$$e^1_{(3)} \ = \ \mathrm{d}\psi + \phi_{(0)} \ , \qquad e^2_{(3)} + \mathrm{i}e^3_{(3)} \ = \ 2\mathrm{e}^{w_{(0)}/2}\mathrm{d}z \ , \tag{4.2.3}$$

with indices $i,j,k = 1,2,3$ for this frame.

It is important to stress here that *arbitrary* choices for $\phi_{(0)}$ and $w_{(0)}$ (subject to M_3 being smooth) lead to supersymmetric backgrounds. The corresponding Killing spinor equation for χ may be found in (3.4.23). Choosing the three-dimensional gamma matrices, in the frame (4.2.3), to be the Pauli matrices, one finds that the Killing spinor solution is

$$\chi = \mathrm{e}^{\mathrm{i}\alpha(\psi,z,\bar{z})} \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} \ , \tag{4.2.4}$$

where χ_0 is a constant and $\alpha(\psi,z,\bar{z})$ is a phase. From this three-dimensional point

[1] More generally there is a conformal factor for this metric [72]. However, as in the last chapter we are interested in conformal field theories with gravity duals, and we may hence set this conformal factor to 1.

of view, the phase $\alpha(\psi, z, \bar{z})$ is a priori an arbitrary function and does not play a significant role. As we already know, when viewed as a boundary supersymmetric field theory dual to supergravity this phase is chosen to be

$$\alpha(\psi, z, \bar{z}) = \gamma\psi, \quad \text{with} \quad \gamma = -\text{sign}\left(\frac{b_1}{b_2}\right) \cdot \frac{|b_1| + |b_2|}{2} \ , \tag{4.2.5}$$

in order to have a global expression for the gauge field A. Here, we have assumed that $M_3 \cong S^3$ with a toric structure, so that we have a $U(1) \times U(1)$ symmetry. If we realise $M_3 \cong S^3 \subset \mathbb{R}^2 \oplus \mathbb{R}^2$ then we may write

$$K = b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \ , \tag{4.2.6}$$

where φ_1, φ_2 are standard 2π-period coordinates on $U(1) \times U(1)$.

4.2.2 The Wilson loop

In $\mathcal{N} = 2$ supersymmetric gauge theories the gauge field $\mathcal{A}_i$ is part of a vector multiplet that also contains two real scalars σ and D and a two-component spinor λ, all of which are in the adjoint representation of the gauge group G. The BPS Wilson loop in a representation $\mathfrak{R}$ of G is given by

$$W = \frac{1}{\dim \mathfrak{R}} \text{Tr}_{\mathfrak{R}} \left[\mathcal{P} \exp\left(\oint_{\upsilon} \text{d}s (\text{i}\mathcal{A}_i \dot{x}^i + \sigma |\dot{x}|) \right) \right] \ , \tag{4.2.7}$$

where $x^i(s)$ parametrises the worldline $\upsilon \subset M_3$ of the Wilson loop and the path ordering operator has been denoted by $\mathcal{P}$. The supersymmetry transformations of the gauge field $\mathcal{A}_i$ and the scalar σ were given in equation (2.3.3) and we recall them here:

$$\delta\mathcal{A}_i = -\frac{\text{i}}{2}\lambda^\dagger \tau_i \chi \ , \qquad \delta\sigma \ = \ -\frac{1}{2}\lambda^\dagger \chi \ ,$$

where τ_i are the Pauli matrices. If one varies the Wilson loop (4.2.7) under the latter supersymmetry transformation one obtains

$$\delta W \propto \frac{1}{2}\lambda^\dagger(\tau_i \dot{x}^i - |\dot{x}|)\chi \ . \tag{4.2.8}$$

The Wilson loop is then invariant under supersymmetry provided

$$(\tau_i \dot{x}^i - |\dot{x}|)\chi = 0 \ . \tag{4.2.9}$$

Choosing s to parametrise arclength, so that $|\dot{x}| = 1$ along the loop, it is straightforward to show that (4.2.9) is satisfied if and only if the Wilson loop lies along the $e^1_{(3)}$ direction. From (4.2.3) we see that $e^1_{(3)}$ is the one-form dual to the supersymmetric Killing vector $K = \partial_\psi$. Thus the Wilson loop (4.2.7) is indeed a BPS operator provided one takes υ to be an orbit of K. Notice that the topology of M_3 has not been used in this subsection, and thus any Wilson loop wrapped along an orbit of K is BPS, regardless of the topology of M_3.

4.2.3 Localisation in the matrix model

The VEV of the BPS Wilson loop (4.2.7) is, by definition, obtained by inserting W into the path integral for the theory on M_3. The computation of this is greatly simplified by the fact that this path integral localises onto supersymmetric configurations of fields. The localisation of the Wilson loop was explained in detail in section 2.3 for the round S^3 case. This section generalises that discussion to a generic supersymmetric manifold $M_3 \cong S^3$.

The central idea is that the path integral, with W inserted, is invariant under the supersymmetry variation δ corresponding to the Killing spinor χ. Crucially, $\delta^2 = 0$ is nilpotent and the only net contributions to this path integral come from field configurations that are invariant under δ.

For the $\mathcal{N} = 2$ supersymmetric Chern-Simons-matter theories of interest, one finds that the δ-invariant configurations on $M_3 \cong S^3$ are particularly simple:

$$\mathcal{A}_i = 0 \ , \quad \sigma \ = \ \text{constant} \ , \quad D \ = \ -\sigma h \ , \tag{4.2.10}$$

where the function $h = \frac{1}{2} * (e^1_{(3)} \wedge \mathrm{d}e^1_{(3)})$, and with all fields in the matter multiplet set identically to zero [32]. Here we may diagonalise σ by a gauge transformation. The exact localised partition function then takes the saddle point form [32]

$$Z \ = \ \int \mathrm{d}\sigma \, \mathrm{e}^{-\frac{\mathrm{i}\pi k}{|b_1 b_2|} \mathrm{Tr}\, \sigma^2} \prod_{\alpha\in\Delta_+} 4 \sinh \frac{\pi\sigma\alpha}{|b_1|} \sinh \frac{\pi\sigma\alpha}{|b_2|} \prod_{\varrho} s_\beta \left[\frac{\mathrm{i}Q}{2}(1-r) - \frac{\varrho(\sigma)}{\sqrt{|b_1 b_2|}} \right] \ . \tag{4.2.11}$$

Note that b_1 and b_2 now appear in Z. Here k denotes the Chern-Simons level and the first product is over positive roots $\alpha \in \Delta_+$ of the gauge group, while the second product is over weights ϱ in the weight space decomposition for a chiral matter field in an arbitrary representation $\mathfrak{R}_{\rm matter}$ of the gauge group. We have also defined

$$\beta \equiv \sqrt{\left| \frac{b_1}{b_2} \right|} \ , \qquad Q \ \equiv \ \beta + \frac{1}{\beta} \ , \tag{4.2.12}$$

the R-charge of the matter field is denoted r, and $s_\beta(z)$ denotes the double sine function.

In this set-up, the VEV of the BPS Wilson loop (4.2.7) reduces to

$$\begin{aligned} \langle\, W \,\rangle \ = \ \frac{1}{Z \dim \mathfrak{R}} \int \ \mathrm{d}\sigma \ \ \mathrm{e}^{-\frac{\mathrm{i}\pi k}{|b_1 b_2|} \mathrm{Tr}\, \sigma^2} \prod_{\alpha\in\Delta_+} 4 \sinh \frac{\pi\sigma\alpha}{|b_1|} \sinh \frac{\pi\sigma\alpha}{|b_2|} \\ \times \prod_{\varrho} s_\beta \left[\frac{\mathrm{i}Q}{2}(1-r) - \frac{\varrho(\sigma)}{\sqrt{|b_1 b_2|}} \right] \mathrm{Tr}_{\mathfrak{R}} \left(\mathrm{e}^{2\pi\ell\sigma} \right) \ . \end{aligned} \tag{4.2.13}$$

Notice the integrand is the same as that for the partition function (4.2.11), with an additional insertion of $\mathrm{Tr}_{\mathfrak{R}}(\mathrm{e}^{2\pi\ell\sigma})$ arising from the Wilson loop operator. Note also that, as in chapter 2, we have normalised the VEV relative to the partition function

Z, so that $\langle 1 \rangle = 1$, as is usual in quantum field theory. We have also defined

$$\oint_{\upsilon} \mathrm{d}s \;=\; 2\pi\ell \qquad (4.2.14)$$

so that ℓ parametrises the length of the Wilson line. More precisely, the integral (4.2.14) is well-defined only for a closed orbit of the Killing vector K. A generic orbit is closed only when $b_1/b_2 \in \mathbb{Q}$ is rational, so that K generates a circle subgroup of $U(1) \times U(1)$. Writing $b_1/b_2 = m/n$ with $m, n \in \mathbb{Z}$ relatively prime integers, these define torus knots via $\upsilon \subset T^2 \subset S^3$, where the homology class $[\upsilon] = (m, n) \in H_1(T^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. These have been studied in the present context in [87]. If on the other hand b_1/b_2 is irrational, then the *only* closed orbits are at the two 'poles' of $M_3 \cong S^3$, where $\partial_{\varphi_1} = 0$ and $\partial_{\varphi_2} = 0$, respectively. Over these poles $\oint_{\upsilon} \mathrm{d}s = 2\pi/|b_2|$, $2\pi/|b_1|$, respectively. Wherever the loop is located, we denote its length $\oint_{\upsilon} \mathrm{d}s$ by $2\pi\ell$ as above.

For a $U(N)$ gauge group we may write $\sigma = \mathrm{diag}(\frac{\lambda_1}{2\pi}, \ldots \frac{\lambda_N}{2\pi})$, thus parametrising $2\pi\sigma$ by its eigenvalues λ_i. Localisation has then reduced the partition function Z and the Wilson loop VEV to finite-dimensional integrals (4.2.11), (4.2.13) over these eigenvalues, but in practice the formulae are difficult to evaluate explicitly. For comparison to the dual supergravity results we must take the $N \to \infty$ limit, where the number of eigenvalues, and hence integrals, tends to infinity. One can then attempt to compute this limit using a saddle point approximation of the integral. As in chapter 2, the large N limit of the saddle point eigenvalue distribution is assumed to take the form

$$\lambda_i = x_i N^{1/2} + \mathrm{i} y_i \ , \qquad (4.2.15)$$

with x_i and y_i real and assumed to be $\mathcal{O}(1)$ in a large N expansion. In the large N limit the real part is assumed to become dense. Ordering the eigenvalues so that the x_i are strictly increasing, the real part becomes a continuous variable x, with density $\rho(x)$, while y_i becomes a continuous function of x, $y(x)$.

Writing $Z = \mathrm{e}^{-\mathcal{F}}$ one then obtains a functional $\mathcal{F}[\rho(x), y(x)]$, with x supported on some interval $[x_{\rm min}, x_{\rm max}]$, and to apply the saddle point method one then extremises $\mathcal{F}$ with respect to $\rho(x)$, $y(x)$, subject to the constraint that $\rho(x)$ is a density

$$\int_{x_{\rm min}}^{x_{\rm max}} \rho(x){\rm d}x = 1 \ . \tag{4.2.16}$$

One then finally also extremises over the choice of interval, by varying with respect to $x_{\rm min}$, $x_{\rm max}$, to obtain the saddle point eigenvalue distribution $\rho(x)$, $y(x)$.

As it turns out, if one caries out the large N limit with the ansatz (4.2.15), one finds a very simple relation between the round sphere results $\mathcal{F}_{\rm round}$ and $\log \langle W_{\rm round} \rangle$ and their squashed counterparts (with arbitrary b_1 and b_2) $\mathcal{F}$ and $\log \langle W \rangle$. To obtain this result for $\mathcal{F}$, one may first relabel σ as $|b_2|\sigma$ in (4.2.11). The partition function then takes the same form as that in [26], where the large N limit was computed in detail. In particular in the latter reference it was shown that in the large N limit $\mathcal{F}[\rho(x), y(x)]$ is simply a rescaling of the round sphere result by a factor $(\beta Q)^3/2^3\beta^2$, *provided* one also rescales the Chern-Simons coupling k as $k \to (2/\beta Q)^2 \cdot k$. This then leads to the large N result (4.1.1).

The same logic may be applied to the calculation of the Wilson loop. For the class of $\mathcal{N} = 2$ supersymmetric Chern-Simons theories coupled to matter on the round three-sphere studied in chapter 2, $x_{\rm max}$ is always proportional to $1/\sqrt{k}$. According to the above prescription, the result for $x_{\rm max}$ on a general background M_3 is given by rescaling the round sphere result by $|b_2| \cdot (\beta Q/2) = (|b_1| + |b_2|)/2$. Here the factor of $|b_2|$ comes from the relabelling $\sigma \to |b_2|\sigma$, while the factor of $\beta Q/2$ comes from the rescaling of the Chern-Simons coupling. Thus

$$x_{\rm max} = \frac{|b_1| + |b_2|}{2} \, x_{\rm max}^{\rm round} \ , \tag{4.2.17}$$

where $x_{\rm max}^{\rm round}$ determines the supremum of the support of $\rho(x)$ for the field theory on the round three-sphere. For the field theories of interest, the eigenvalue density is always a continuous piecewise linear function supported on $[x_{\rm min}, x_{\rm max}]$. Using this fact, the large N limit of the Wilson loop (4.2.13) in the fundamental representation may be easily computed with a saddle point approximation, as explained in section 2.3, and we find

$$\log \langle W \rangle_{\mathrm{QFT}} \ = \ \ell \cdot x_{\mathrm{max}} \ N^{1/2} + o(N^{1/2}) \ . \tag{4.2.18}$$

Here recall that the length $\oint_v \mathrm{d}s$ is in general $2\pi\ell$. The round three-sphere Wilson loop in particular is obtained by setting $b_1 = b_2 = 1$ and $\ell = 1$ and is, equation (2.3.15),

$$\log \langle W_{\mathrm{round}} \rangle_{\mathrm{QFT}} \ = \ x_{\mathrm{max}}^{\mathrm{round}} \ N^{1/2} + o(N^{1/2}) \ . \tag{4.2.19}$$

We thus obtain

$$\lim_{N\to\infty} \frac{\log \langle W \rangle_{\mathrm{QFT}}}{\log \ \langle W_{\mathrm{round}} \rangle_{\mathrm{QFT}}} \ = \ \frac{|b_1| + |b_2|}{2}\ell \ . \tag{4.2.20}$$

This is the field theory result for the VEV of a supersymmetric Wilson loop on a general supersymmetric manifold $M_3 \cong S^3$. In the next section we will look at the M2-brane dual to this Wilson loop, and show quite generally that the holographic dual computation of the VEV agrees with (4.2.20).

4.3 Dual M2-branes

In this section we analyse the supersymmetric M2-brane probes that are relevant for computing the holographic dual of the Wilson loop VEV (4.2.20). The dual solution is constructed in four-dimensional gauged supergravity of the last chapter, and we begin by summarising the geometry of these solutions.

4.3.1 Supergravity dual

In chapter 3 it was shown that supersymmetric three-manifolds M_3 of the form described in section 4.2.1 arise as the conformal boundaries of Euclidean self-dual solutions to four-dimensional gauged supergravity. For $M_3 \cong S^3$ the four-dimensional supergravity solution is defined on a four-ball $M_4 \cong B^4$, and is asymptotically locally

Euclidean AdS with conformal boundary M_3. The Killing vector K defined by (4.2.1) extends as a Killing vector bilinear over M_4, and the four-metric is then Einstein, has anti-self-dual Weyl tensor, and is conformal to a Kähler metric. Supersymmetry also requires one to turn on a specific graviphoton field A.

The four-dimensional metric on the manifold M_4 takes the form (3.3.2)

$$\mathrm{d}s^2_{\mathrm{SDE}} = \frac{1}{y^2}\left[V^{-1}(\mathrm{d}\psi+\phi)^2 + V\left(\mathrm{d}y^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z}\right)\right] \ , \tag{4.3.1}$$

The metric (4.3.1) is equipped with the Killing vector $K = \partial_\psi$, which extends the vector (4.2.1) from the conformal boundary, which is at $y = 0$. On the boundary M_3, the graviphoton gauge field $A_{(0)}$ takes the *global* form, equation (3.4.26),

$$A_{(0)} = \gamma \mathrm{d}\psi - \frac{1}{4} w_{(1)}\sigma + \frac{\mathrm{i}}{4}\partial_z w_{(0)}\mathrm{d}z - \frac{\mathrm{i}}{4}\partial_{\bar{z}} w_{(0)}\mathrm{d}\bar{z} \ . \tag{4.3.2}$$

where γ is given in (4.2.5). We shall use the following orthonormal frame for the metric (4.3.1)

$$e^0 \ = \ \frac{1}{y}V^{1/2}\mathrm{d}y \ , \quad e^1 \ = \ \frac{1}{y}V^{-1/2}(\mathrm{d}\psi+\phi) \ , \quad e^2 + \mathrm{i}e^3 \ = \ \frac{2}{y}(Ve^w)^{1/2}\mathrm{d}z \ . \tag{4.3.3}$$

The four-dimensional geometry that we have just described, together with the gauge field A, form a supersymmetric solution to Euclidean gauged supergravity. There is correspondingly a Dirac spinor ϵ satisfying the Killing spinor equation of this theory. In the orthonormal frame (4.3.3) and using the gamma matrices

$$\Gamma_i = \begin{pmatrix} 0 & \tau_i \\ \tau_i & 0 \end{pmatrix} \ , \qquad \Gamma_0 \ = \ \begin{pmatrix} 0 & \mathrm{i}\mathbb{I}_2 \\ -\mathrm{i}\mathbb{I}_2 & 0 \end{pmatrix} \ , \tag{4.3.4}$$

with τ_i the Pauli matrices, the Killing spinor ϵ is given by

$$\epsilon = \frac{1}{\sqrt{2y}}\left(1 + V^{-1/2}\Gamma_0\right)\zeta \ , \tag{4.3.5}$$

with

$$\zeta = \begin{pmatrix} \chi \\ 0 \end{pmatrix} \quad \text{where} \quad \chi \ = \ \mathrm{e}^{\mathrm{i}\gamma\psi} \begin{pmatrix} \chi_0 \\ \chi_0 \end{pmatrix} \ . \tag{4.3.6}$$

We recall that we assume that the four-manifold M_4 is $M_4 \cong B^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ and that the torus $U(1)\times U(1)$ acts in the standard way on $\mathbb{R}^2 \oplus \mathbb{R}^2$. The Killing vector $K = \partial_\psi$ is then parametrised like in (4.2.6) on the conformal boundary. Remember that while the metric (4.3.1) is smooth by assumption, the instanton $F = \mathrm{d}A$ and Killing spinor ϵ are singular where the conformal Kähler metric is singular. Regularity is in fact equivalent to having either $b_1/b_2 > 0$ or $b_1/b_2 = -1$. Moreover, the origin $y = y_{\mathrm{NUT}}$ of M_4 is then at

$$y_{\mathrm{NUT}} = \frac{1}{|b_1 + b_2|} \ , \tag{4.3.7}$$

which is $y_{\mathrm{NUT}} = \infty$ when $b_1/b_2 = -1$.

In order to study the M2-branes dual to Wilson loops, we need to work with the full eleven-dimensional supergravity solution of section 3.2. We recall that the solution takes the form

$$\begin{aligned} \mathrm{d}s^2_{11} \ &= \ R^2 \left[\frac{1}{4}\mathrm{d}s^2_{\mathrm{SDE}} + \left(\eta + \frac{1}{2}A \right)^2 + \mathrm{d}s^2_T \right] \ , \\ G \ &= \ -\mathrm{i}R^3 \left(\frac{3}{8}\mathrm{vol}_4 - \frac{1}{4} *_4 F \wedge \mathrm{d}\eta \right) \ . \end{aligned} \tag{4.3.8}$$

The radius R is

$$R^6 = \frac{(2\pi\ell_p)^6 N}{6\mathrm{Vol}(Y_7)} \ , \tag{4.3.9}$$

where N is the number of units of flux defined in equation (3.2.5).

4.3.2 BPS M2-branes

We are interested in calculating the action of M2-branes that are dual to Wilson loops of gauge theories on M_3. These M2-branes wrap $\Sigma_2 \times S^1_M$, where the surface $\Sigma_2 \subset M_4$ has boundary given by the Wilson line $\partial\Sigma_2 = S^1 \subset M_3 = \partial M_4$, and $S^1_M \subset Y_7$ is a copy of the M-theory circle. In particular we will show that submanifolds $\Sigma_2 \subset M_4$ parametrised by the radial direction y in M_4 and an orbit of the Killing vector K are complex with respect to the complex structure J of the conformal Kähler metric to $\mathrm{d}s^2_{\mathrm{SDE}}$. The wrapped M2-brane is then supersymmetric provided S^1_M is calibrated by the contact one-form η. Over the poles $S^1 \subset M_3 \cong S^3$ the topology of Σ_2 is a disc, where $y \in (0, y_{\mathrm{NUT}}]$ serves as a radial coordinate with the origin of the disc at $y = y_{\mathrm{NUT}} > 0$.

The action of the M2-brane in Euclidean signature reads

$$S_{\mathrm{M2}} = \frac{1}{(2\pi)^2 \ell_p^3} \left[\mathrm{Vol}(\Sigma_2 \times S^1_M) + \mathrm{i} \int_{\Sigma_2 \times S^1_M} C \right] . \tag{4.3.10}$$

A supersymmetric M2-brane satisfies an appropriate projection condition, which may be written as

$$\mathbb{P}\epsilon_{11} \ = \ 0 \ , \quad \text{where} \quad \mathbb{P} \ \equiv \ \frac{1}{2}\left(1 - \frac{\mathrm{i}}{3!}\varepsilon^{\alpha\beta\gamma}\partial_\alpha X^M \partial_\beta X^N \partial_\gamma X^P \tilde{\Gamma}_{MNP}\right) \ , \tag{4.3.11}$$

with α, β, γ indices on the worldvolume. Here ϵ_{11} is the eleven-dimensional Killing spinor for the background (4.3.8), which is constructed as a tensor product of the four-dimensional spinor ϵ and the Killing spinor on the internal space Y_7. The $\tilde{\Gamma}_M$ are eleven-dimensional gamma matrices, with X^M describing the M2-brane embedding. One can analyse (4.3.11) precisely as we did in chapter 2. The upshot is that $S^1_M \subset Y_7$ must be a calibrated circle in Y_7, while taking $\Sigma_2 \subset M_4$ to be a surface at constant z, parametrised by y and ψ, one finds (4.3.11) is equivalent to the projection condition

$$(1 - \mathrm{i}\Gamma_5 \Gamma_{01})\epsilon \ = \ 0 \ . \tag{4.3.12}$$

Here we have used the orthonormal frame (4.3.3), and $\Gamma_5 \equiv \Gamma_0\Gamma_1\Gamma_2\Gamma_3$ with Γ_μ defined by (4.3.4) (in the orthonormal frame). Using the explicit form for ϵ in (4.3.5) it is trivial to see that (4.3.12) indeed holds. Moreover, Σ_2 is calibrated with respect to the Kähler form for the conformal Kähler metric, making it a complex curve. Equivalently, denoting the complex structure J we have $J(V^{-1}\partial_y) = \partial_\psi$ making Σ_2 a complex curve.

Let us now calculate the action (4.3.10) for our M2-brane. Using the supergravity solution that we briefly summarised in section 4.3.1, the C-field, where remember that $G = \mathrm{d}C$, is computed to be

$$C = -\mathrm{i}R^3 \left(-\frac{1}{8}\Upsilon + \frac{1}{4} F \wedge \eta \right) , \tag{4.3.13}$$

where

$$\Upsilon = \frac{1}{2y^2}(\mathrm{d}\psi + \phi) \wedge \mathrm{d}\phi + \frac{1}{y^3}(\mathrm{d}\psi + \phi) \wedge 2\mathrm{i}V\mathrm{e}^w \mathrm{d}z \wedge \mathrm{d}\bar{z} , \tag{4.3.14}$$

and $\mathrm{d}\Upsilon = -3\mathrm{vol}_4$. The area of the surface Σ_2 in M_4 is divergent, but can be regularised by subtracting the length of its boundary, *i.e.* the length of the S^1 in M_3^δ at $y = \delta \to 0$. Notice this is then a local boundary counterterm. If we denote by M_4^δ the manifold M_4 with boundary $M_3^\delta = \{y = \delta\}$ (with $0 < \delta < y_{\mathrm{NUT}}$), and similarly for Σ_2^δ *etc*, the action of the M2-brane is

$$S_{\mathrm{M2}} = \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \frac{R^3}{4} \mathrm{vol}_{S_M^1} \cdot \lim_{\delta \to 0} \left[\int_{\Sigma_2^\delta} \mathrm{vol}_{\Sigma_2} - \int_{\partial\Sigma_2^\delta} e_\mu^1 \mathrm{d}x^\mu + \int_{\Sigma_2^\delta} F \right] . \tag{4.3.15}$$

Here we have written $\mathrm{vol}_{S_M^1}$ for the volume form on S_M^1 induced from the metric g_{Y_7}, and similarly for vol_{Σ_2} and the metric g_{M_4}. Applying Stokes' theorem for the gauge field term $F = \mathrm{d}A$ we then compute[2]

$$\begin{aligned} S_{\mathrm{M2}} &= \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \mathrm{vol}_{S_M^1} \cdot \frac{\pi \ell R^3}{2} \lim_{\delta \to 0} \left[\left(\int_\delta^{y_{\mathrm{NUT}}} \frac{\mathrm{d}y}{y^2} - \frac{1}{\delta \sqrt{V(\delta, z, \bar{z})}} \right) - \frac{1}{2\pi\ell} \int_{\partial\Sigma_2^\delta} A \right] \\ &= \frac{1}{(2\pi)^2 \ell_p^3} \int_{S_M^1} \mathrm{vol}_{S_M^1} \cdot \frac{\pi \ell R^3}{2} \left[-\left(\frac{1}{y_{\mathrm{NUT}}} + \frac{1}{4} w_{(1)} \right) - \frac{1}{2\pi\ell} \int_{\partial\Sigma_2} A \right] . \end{aligned} \tag{4.3.16}$$

[2]The sign in front of the gauge field term arises because y is decreasing towards the boundary of M_4, and hence $\mathrm{d}y$ points inwards from M_3. Thus the natural orientation of the boundary we take is opposite to that in Stokes' theorem.

Recall here that $2\pi\ell$ denotes the length of the orbit of K, as in (4.2.14). The contribution of the M-theory circle S^1_M is exactly the same as for the $\mathrm{AdS}_4\times Y_7$ backgrounds studied in chapter 2, and is expressed in terms of the contact form η on Y_7 and the Dirac quantised number N. The gauge field integral is easily computed, thanks to (4.3.2)

$$\int_{\partial\Sigma_2} A \;=\; \int_{\partial\Sigma_2} A_{(0)} \;=\; 2\pi\ell\left(-\frac{1}{4}w_{(1)}+\gamma\right) . \qquad (4.3.17)$$

Putting everything together, and using the formula (4.3.7) for y_{NUT}, we have

$$\log\langle\, W\,\rangle_{\text{gravity}} \;=\; -S_{\mathrm{M2}} \;=\; \ell\,(|b_1+b_2|+\gamma)\cdot\frac{(2\pi)^2\int_{S^1_M}\eta}{\sqrt{2\int_{Y_7}\eta\wedge(\mathrm{d}\eta)^3}}\;N^{1/2} . \qquad (4.3.18)$$

Using the round sphere result of equation (2.4.23)

$$\log\langle\, W_{\text{round}}\,\rangle_{\text{gravity}} \;=\; \frac{(2\pi)^2\int_{S^1_M}\eta}{\sqrt{2\int_{Y_7}\eta\wedge(\mathrm{d}\eta)^3}}\;N^{1/2} , \qquad (4.3.19)$$

and the formula (4.2.5) for γ, in both cases $b_1/b_2>0$ and $b_1/b_2=-1$ we obtain

$$\log\langle\, W\,\rangle_{\text{gravity}} \;=\; \frac{|b_1|+|b_2|}{2}\ell\cdot\log\langle\, W_{\text{round}}\,\rangle_{\text{gravity}} . \qquad (4.3.20)$$

In chapter 2 it was shown in numerous families of examples that the large N limit of the Wilson loop on the round three-sphere and the M2-brane in AdS_4 have the same VEV, *i.e.* $\log\langle\, W_{\text{round}}\,\rangle_{\text{QFT}} \;=\; \log\langle\, W_{\text{round}}\,\rangle_{\text{gravity}}$ holds to leading order at large N. Assuming this to be the case, equations (4.2.20) and (4.3.20) mean that we have shown very generally that in the large N limit

$$\log\langle\, W\,\rangle_{\text{QFT}} \;=\; \log\langle\, W\,\rangle_{\text{gravity}} \qquad (4.3.21)$$

where now the field theory is defined on a general class of background three-manifolds M_3, with fillings M_4 in four-dimensional gauged supergravity.

We conclude this section with two further comments. Firstly, it is interesting to note that when the orbit of K is one of the poles of S^3, where correspondingly $\ell = 1/|b_1|$ or $\ell = 1/|b_2|$ respectively, the Wilson loops are then functions only of $|b_1/b_2|$, just as for the free energy (4.1.1). Secondly, in the case that $b_1/b_2 = m/n$ is rational and the Wilson line wraps a generic orbit $\upsilon \subset T^2 \subset S^3$ (*i.e.* not at either pole), then the curve $\Sigma_2 \subset M_4 \cong \mathbb{C}^2$ wrapped by the dual M2-brane is the Brieskorn-Pham curve $\{z_1^n = z_2^m\} \subset \mathbb{C}^2$. This follows since supersymmetry pairs the orbit of K with its complexification in $M_4 \cong \mathbb{C}^2$, meaning that Σ_2 is swept out as a generic $\mathbb{C}^*$ orbit of $(z_1, z_2) \to (\lambda^m z_1, \lambda^n z_2)$, with $\lambda \in \mathbb{C}^*$. The curve $\{z_1^n = z_2^m\}$ adds the origin in $\mathbb{C}^2$ at $y = y_{\rm NUT}$, which is a singular point when $m, n > 1$, although notice this does not affect our computation of the M2-brane action, which is finite. It is well-known that (m, n) torus knots in S^3 may be realised as links of the above Brieskorn-Pham curves, and it is interesting to see that this construction is realised as the holographic dual of the knot.

4.4 Examples

Our derivation of the formula (4.3.20) was necessarily somewhat indirect, as we have shown that it holds for a very general class of solutions. In particular we did not need to use the explicit form of the solution to the Toda equation (3.3.5). In this section we illustrate our general results by discussing two explicit families of solutions, where all quantities in the previous section may be written down in closed form. We will focus on the four-dimensional part of the M2-brane calculation, in particular showing how the factor $\ell(|b_1| + |b_2|)/2$ in (4.3.20) arises explicitly in these cases. In order to do so we will use the results of the previous section that allow us to write

$$\log \langle W \rangle_{\rm gravity} \; = \; \mathcal{S}_{b_1, b_2} \cdot \log \langle W_{\rm round} \rangle_{\rm gravity} \; , \qquad (4.4.1)$$

where

$$\mathcal{S}_{b_1,b_2} \equiv \frac{1}{2\pi}\left(-\int_{\Sigma_2} \mathrm{vol}_{\Sigma_2} + \int_{\partial\Sigma_2} \mathrm{vol}_{\partial\Sigma_2} + \int_{\partial\Sigma_2} A\right) . \tag{4.4.2}$$

Here we cut off Σ_2 at $y = \delta$, and (4.4.2) is then understood to be the limit $\delta \to 0$. We compute (4.4.2) directly in the examples, confirming that (4.3.20) indeed holds in these cases.

AdS$_4$

We begin with the metric on Euclidean AdS$_4$, which can be written

$$\mathrm{d}s^2_{\mathrm{EAdS}_4} = \frac{\mathrm{d}q^2}{1+q^2} + q^2\left(\mathrm{d}\vartheta^2 + \cos^2\vartheta \mathrm{d}\varphi_1^2 + \sin^2\vartheta \mathrm{d}\varphi_2^2\right) . \tag{4.4.3}$$

Here q is a radial variable with $q \in [0, \infty)$, so that the origin of $M_4 \cong \mathbb{R}^4$ is at $q = 0$ while the conformal boundary is at $q = \infty$. The coordinate $\vartheta \in [0, \frac{\pi}{2}]$, with the endpoints being the two axes of $\mathbb{R}^2 \oplus \mathbb{R}^2 \cong \mathbb{R}^4$.

Of course the metric (4.4.3) is conformally flat, which leads to a trivial graviphoton $A = 0$. However, we may instead pick a general supersymmetric Killing vector $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$. This leads to a family of conformal Kähler structures on $\mathbb{C}^2$, where the explicit formulae for the conformal factor y, the metric function $w(y, z, \bar{z})$ and the gauge field A were derived in section 3.6.1. Writing A as a global one-form and restricting to the conformal boundary at $q = \infty$ we obtain

$$A_{(0)} = \frac{b_2\mathrm{d}\varphi_1 + b_1\mathrm{d}\varphi_2}{2\sqrt{b_1^2\cos^2\vartheta + b_2^2\sin^2\vartheta}} - \frac{1}{2}(\mathrm{sign}(b_2)\mathrm{d}\varphi_1 + \mathrm{sign}(b_1)\mathrm{d}\varphi_2) . \tag{4.4.4}$$

In particular notice this is well-defined at both poles $\vartheta = 0$ and $\vartheta = \pi/2$. The submanifold Σ_2 is parametrised by the radial direction q in AdS$_4$ and the S^1 wrapping φ_1 or φ_2 when $\vartheta = 0$ or $\vartheta = \pi/2$, respectively.

We now turn to the computation of (4.4.2). Notice that the dependence on b_1 and

b_2 arises only via the gauge field A, and not from the metric. Indeed, we compute

$$\left[-\int_{\Sigma_2} \mathrm{vol}_{\Sigma_2} + \int_{\partial\Sigma_2} \mathrm{vol}_{\partial\Sigma_2}\right] = 2\pi \ , \tag{4.4.5}$$

and

$$\int_{\partial\Sigma_2} A_{(0)} \quad = \quad \begin{cases} \pi(\frac{b_2}{|b_1|} - \mathrm{sign}(b_2)) \cdot \mathrm{sign}(b_1) & \text{if } \vartheta = 0 \ , \\ \pi(\frac{b_1}{|b_2|} - \mathrm{sign}(b_1)) \cdot \mathrm{sign}(b_2) & \text{if } \vartheta = \pi/2 \ . \end{cases} \tag{4.4.6}$$

The overall factors of $\mathrm{sign}(b_1)$, $\mathrm{sign}(b_2)$ for $\vartheta = 0, \pi/2$ arise because the orientation of $\partial\Sigma_2$ is determined by K. Equation (4.4.2) immediately gives for all regular cases that

$$\mathcal{S}_{b_1,b_2} \quad = \quad \begin{cases} \dfrac{|b_1| + |b_2|}{2|b_1|} & \text{if } \vartheta = 0 \ , \\ \dfrac{|b_1| + |b_2|}{2|b_2|} & \text{if } \vartheta = \pi/2 \ . \end{cases} \tag{4.4.7}$$

In particular using the variable ℓ introduced previously, which is given by $\ell = 1/|b_1|$ and $1/|b_2|$ for the $\vartheta = 0$ pole and $\vartheta = \pi/2$ pole respectively, we obtain for both poles and all regular cases that

$$\mathcal{S}_{b_1,b_2} = \frac{|b_1| + |b_2|}{2}\ell \ , \tag{4.4.8}$$

as expected.

Taub-NUT-AdS$_4$

The Taub-NUT-AdS$_4$ metric may be written

$$\mathrm{d}s_4^2 = \frac{r^2 - s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2 - s^2)(u_1^2 + u_2^2) + \frac{4s^2\Omega(r)}{r^2 - s^2}u_3^2 \ , \tag{4.4.9}$$

where

$$\Omega(r) \equiv (r \mp s)^2[1 + (r \mp s)(r \pm 3s)] \ , \tag{4.4.10}$$

and u_1, u_2, u_3 are left-invariant one-forms on $SU(2) \simeq S^3$. The latter may be written in terms of Euler angle variables as

$$u_1 + \mathrm{i}u_2 = \mathrm{e}^{-\mathrm{i}\varsigma}(\mathrm{d}\theta + \mathrm{i}\sin\theta\mathrm{d}\varphi) \ , \qquad u_3 \ = \ \mathrm{d}\varsigma + \cos\theta\mathrm{d}\varphi \ . \tag{4.4.11}$$

Here ς has period 4π, while $\theta \in [0, \pi]$ with φ having period 2π. The radial coordinate r lies in the range $r \in [s, \infty)$, with the origin of the ball $B^4 \cong \mathbb{R}^4$ being at $r = s$. The parameter $s > 0$ is referred to as the squashing parameter, with $s = \frac{1}{2}$ being the Euclidean AdS_4 metric studied in the previous section.

Remember that the Taub-NUT-AdS metric (4.4.9) has $SU(2) \times U(1)$ isometry, but a generic choice of the Killing vector $K = (b_1 + b_2)\partial_\varphi + (b_1 - b_2)\partial_\varsigma$ breaks the symmetry of the full solution to $U(1) \times U(1)$. In particular, this symmetry is broken by the corresponding instanton A. If the $SU(2) \times U(1)$ symmetry of the metric is also imposed on the gauge field, it results in two subfamilies of the above solutions, which are 1/4 BPS and 1/2 BPS, respectively. In each case this effectively fixes the Killing vector K (or rather the parameter b_1/b_2) as a function of the squashing parameter s.

1/4 BPS solution: The supersymmetric Killing vector for this solution is $K = -\frac{1}{2s}\partial_\varsigma$ and we have

$$b_1 = -b_2 \ = \ -\frac{1}{4s} \ . \tag{4.4.12}$$

The boundary gauge field $A_{(0)}$ is, equation (3.6.20),

$$A_{(0)} = \frac{1}{2}(4s^2 - 1)u_3 \ , \tag{4.4.13}$$

which is a global one-form on $M_3 \cong S^3$. We may now take the surface Σ_2 wrapped by the M2-brane to be *any* S^1 orbit of the Hopf Killing vector ∂_ς (at any point on the base $S^2 = S^3/U(1)_\varsigma$), together with the radial direction r. This is supersymmetric, and the regularised volume of Σ_2 is

$$\left[-\int_{\Sigma_2}\mathrm{vol}_{\Sigma_2}+\int_{\partial\Sigma_2}\mathrm{vol}_{\partial\Sigma_2}\right]=8\pi s^2\ , \tag{4.4.14}$$

while the gauge field integral is

$$\int_{\partial\Sigma_2}A_{(0)}=-2\pi(4s^2-1)\ . \tag{4.4.15}$$

This leads to

$$\mathcal{S}_{b_1,b_2}=1\ =\ \frac{|b_1|+|b_2|}{2}\ell\ , \tag{4.4.16}$$

where $\ell=4s$ is the length of K divided by 2π.

1/2 BPS solution: The Taub-NUT-AdS metric (4.4.9) also admits a 1/2 BPS solution. There are thus two linearly independent Killing spinors, and an appropriate linear combination preserves $U(1)\times U(1)$ symmetry, leading to the Killing vector

$$K=\left(2s+\sqrt{4s^2-1}\right)\partial_\varphi+\left(\tfrac{1}{2s}-2s-\sqrt{4s^2-1}\right)\partial_\varsigma\ , \tag{4.4.17}$$

so that

$$b_1=\frac{1}{4s}\ ,\qquad b_2\ =\ -\frac{1}{4s}+2s+\sqrt{4s^2-1}\ . \tag{4.4.18}$$

The boundary gauge field is, equation (3.6.24),

$$A_{(0)}=s\sqrt{4s^2-1}u_3\ . \tag{4.4.19}$$

This time we take the Wilson loop to wrap one of the two poles $\theta=0$, $\theta=\pi$. These are both copies of S^1, and Σ_2 is again formed by adding the radial direction r. The boundary gauge field is

$$A_{(0)}\,|_{\text{pole}} \;=\; \begin{cases} 2s\sqrt{4s^2-1}\;\mathrm{d}\varphi_1 & \text{if } \theta = 0\ , \\ -2s\sqrt{4s^2-1}\;\mathrm{d}\varphi_2 & \text{if } \theta = \pi\ . \end{cases} \tag{4.4.20}$$

The regularised volume is again $8\pi s^2$, which then gives

$$\mathcal{S}_{b_1,b_2} \;=\; \begin{cases} 2s(2s+\sqrt{4s^2-1}) & \text{if } \theta = 0\ , \\ 2s(2s-\sqrt{4s^2-1}) & \text{if } \theta = \pi\ . \end{cases} \tag{4.4.21}$$

In both cases we indeed have

$$\mathcal{S}_{b_1,b_2} = \frac{|b_1|+|b_2|}{2}\ell\ , \tag{4.4.22}$$

where $\ell = 1/|b_1|$, $\ell = 1/|b_2|$ for the two poles.

Chapter 5

Conclusions

The AdS/CFT duality provides a way of better understanding field theories and their string theory counterparts. As we have seen in this thesis, the localisation technique has been very helpful to further explore the duality. Thanks to this method, we can make exact field theory computations at strong coupling and compare them to their supergravity duals. In this thesis, we have focused on the supergravity side of the duality, while summarising and further extending some field theory results, to check the duality for the Wilson loop and the free energy in a variety of examples.

In chapter 2, we have shown that the large N field theory and gravity computations of the BPS Wilson loop agree in a large class of three-dimensional $\mathcal{N} = 2$ superconformal field theories with $\mathrm{AdS}_4 \times Y_7$ gravity duals. In fact really this matching is a corollary of the fact that the image of the M-theory Hamiltonian $h_M(Y_7) = [c_{\min}, c_{\max}]$ is equal to the support $[x_{\min}, x_{\max}]$ of the real part of the saddle point eigenvalue distribution in the large N matrix model, with the proportionality factor between the variables x and c given by

$$x \;=\; \frac{(2\pi)^3}{\sqrt{96\,\mathrm{Vol}_\eta(Y_7)}}\, c\ . \tag{5.1}$$

Moreover, the critical points of h_M, which give the loci of supersymmetric M2-branes wrapping the M-theory circle, always map under h_M to the points at which $\rho'(x)$ is discontinuous in the matrix model. The fact that the eigenvalue density changes behaviour every time a critical point x_i is crossed is explained by (2.4.40) which relates $\rho(x)$ to the volume of a subspace of $h_M^{-1}(c)$ whose topology changes at the critical points c_i. All those relations show that field theory quantities, like $\rho(x)$ and x, seem to be captured by geometrical quantities on the gravity side. This is not so surprising after all, because supergravity computations are purely geometrical and are predicted to be dual to field theory computations.

Although shedding light on the relation between Wilson loops and M2 branes, the work presented in this thesis opens to way for future research. Even though we know that the image of the Hamiltonian function h_M is related to the range of the real part of the eigenvalues, it would be interesting to understand if this relation has a deeper meaning. In order to do so, one could consider representation of Wilson loops that differ from the fundamental representation and see how it would change the gravity dual object. It would also be interesting to study how a deformation of the Hopf $S^1 \subset S^3$ affects the supersymmetry of the brane and its relation to the M2-brane. Finally, we could also look at different field theory operators and try to seek their gravity duals. In [88, 89], it was shown how to compute the VEV of various vortex loops, a kind of defect operator, in supersymmetric field theories. Calculating the VEV of those operators in some specific field theories could help us find their dual objects and potentially uncover new geometrical relations.

We followed the Wilson loop/M2-brane computations by the construction of supergravity duals to generic supersymmetric field theories on three-manifolds in chapter 3. The main result of this chapter is the proof of the formula

$$I = \frac{(|b_1| + |b_2|)^2}{4|b_1 b_2|} \cdot I_{\text{round}} \ , \tag{5.2}$$

for the holographically renormalised on-shell action in minimal four-dimensional supergravity. Moreover, we have provided a general construction, that extends to eleven-dimensional supergravity, of regular supersymmetric solutions of this theory based on self-dual Einstein metrics on the four-ball equipped with a one-parameter family of instanton fields for the graviphoton. Specifically, if the self-dual Einstein metric admits n parameters, our construction produces an $(n+1)$-parameter family of solutions. We have shown that the renormalised on-shell action does *not* depend on the n metric parameters, but only on this last 'instanton parameter'. This matches beautifully the field theory results of [32].

We have also shown how all the previous examples in the literature, as well as some new examples that we have presented, can be understood in our general setting. In section 3.6.4 we have suggested that using a family of local metrics, it should be possible to construct global asymptotically locally Euclidean AdS self-dual Einstein metrics on the four-ball, thus obtaining an infinite family of completely explicit metrics. It would be interesting to analyse these m-pole solutions in more detail.

In this third chapter we have achieved a rather general understanding of the gauge/gravity duality for supersymmetric asymptotically locally Euclidean AdS_4 solutions. Nevertheless, there are a number of possible extensions of our work. One might further generalise our results by relaxing one or more of the assumptions we have made. For example, remaining in the context of minimal gauged supergravity, it would be very interesting to investigate the more general class of supersymmetric, but non-(anti-)self-dual solutions [75]. Several examples of such solutions were constructed in [29, 30], and these all turn out to have a bulk topology different from the four-ball. This suggests that self-duality and the topology of supersymmetric asymptotically AdS_4 solutions are two related issues, and it would be desirable to clarify this. On the other hand, at present it is unclear what the precise dual field theory implication of non-trivial two-cycles in the geometry is, and therefore this direction is both challenging and interesting. Perhaps related to this, one of our main results is that a smooth toric self-dual Einstein metric on the four-ball with supersymmetric Killing vector $K = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$ gives rise to a smooth supersymmetric solution only if $b_1/b_2 > 0$ or $b_1/b_2 = -1$. Specifically, for other choices of b_1/b_2 the conformal factor/Killing spinor are singular in the interior of the bulk. Nevertheless, the conformal boundary is smooth for all choices of b_1, b_2, and the question arises as to how to fill those boundaries smoothly within gauged supergravity. A natural conjecture is that these are filled with the non-self-dual solutions mentioned above.

Another assumption that should be straightforward to relax is in taking the gauge field A to be real. In general, if A is complex the existence of one Killing spinor does not imply that the metric possesses any isometry [75]. However, we expect that if one requires the existence of *two* spinors of opposite R-charge, then there will be canonically defined Killing vectors, and therefore it should be possible to analyse the solutions with the techniques of this thesis.

All the above extensions would be important conceptually, in order to address the issue of uniqueness of the filling of a given conformal boundary geometry. In fact, this could also motivate the study of this problem directly in eleven-dimensional supergravity.

Of course, in any of these more general set-ups a central issue will be to prove a generalised version of the formula (5.2) for the renormalised on-shell action. In this respect, some of the methods that were employed in [2] to derive this -not presented here- may be more amenable to generalisation than others. For example, an expression for I in terms of boundary conformal invariants and bulk topological invariants might extend to the class of non-self-dual metrics and/or non-ball topology.

In chapter 4, we looked at how the Wilson loop/M2-brane computations carried out in chapter 2 could be done in the more general geometry developed in chapter 3. We derived the formula

$$\lim_{N\to\infty} \log \langle W \rangle = \frac{|b_1| + |b_2|}{2} \ell \cdot \log \langle W_{\mathrm{round}} \rangle \,, \tag{5.3}$$

for the expectation values of large N BPS Wilson loops, in both gauge theory and in supergravity. A key feature of the gravity calculation is that we are able to evaluate the regularised M2-brane action, that is identified with the Wilson loop VEV, without using the explicit form of the metric and graviphoton field. This seems to be a general feature of such computations of BPS quantities in AdS/CFT, and allows us to verify the correspondence for these observables in a very broad class of solutions.

The results described in chapter 4 lead to a number of questions, and possible future directions to pursue. First, in supergravity we have restricted to self-dual solutions, while more generally there are also non-self-dual solutions to gauged supergravity as mentioned above. Presumably the methods we have used extend to this general class of solutions. In particular, the Wilson loop was computed for a charged topological black hole background in [90], and successfully compared to a field theory calculation. The non-self-dual solutions in [30] all have the feature that the bulk M_4 has non-trivial topology. It would be interesting to try to calculate Wilson loops in such examples, and compare to a dual field theory computation. Finally, it is now clear that similar results should also hold in higher dimensions. A very similar formula to (5.3) was found to hold for certain supersymmetric squashed five-sphere conformal boundaries and their gravity duals in [91, 92], and was conjectured to hold for general backgrounds in those references.

Appendix A

Spin connection of the Kähler metric

For the Kähler metric (3.3.7) in the frame (3.3.8) the spin connection reads

$$
\begin{aligned}
\hat{\omega}^{01} &= -\frac{\left(\partial_y w + y\partial_y^2 w\right)}{4V^{3/2}}\hat{e}^1 + \frac{\mathrm{i}y\partial_y\left(\partial_z - \partial_{\bar{z}}\right)w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^2 - \frac{y\partial_y\left(\partial_z + \partial_{\bar{z}}\right)w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^3 \,, \\
\hat{\omega}^{02} &= -\frac{y\partial_y(\partial_z + \partial_{\bar{z}})w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^0 + \frac{\mathrm{i}y\partial_y\left(\partial_z - \partial_{\bar{z}}\right)w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^1 + \frac{\left(\partial_y w + y\partial_y^2 w\right) - 2V\partial_y w}{4V^{3/2}}\hat{e}^2 \,, \\
\hat{\omega}^{03} &= -\frac{\mathrm{i}y\partial_y(\partial_z - \partial_{\bar{z}})w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^0 - \frac{y\partial_y\left(\partial_z + \partial_{\bar{z}}\right)w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^1 + \frac{\left(\partial_y w + y\partial_y^2 w\right) - 2V\partial_y w}{4V^{3/2}}\hat{e}^3 \,, \\
\hat{\omega}^{12} &= -\hat{\omega}^{03} \,, \\
\hat{\omega}^{13} &= \hat{\omega}^{02} \,, \\
\hat{\omega}^{23} &= -\frac{\left(\partial_y w - y(\partial_y w)^2 - y\partial_y^2 w\right)}{4V^{3/2}}\hat{e}^1 + \frac{\mathrm{i}\left[2V(\partial_z - \partial_{\bar{z}})w - y\partial_y(\partial_z - \partial_{\bar{z}})w\right]}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^2 \\
&\quad -\frac{2V(\partial_z + \partial_{\bar{z}})w - y\partial_y(\partial_z + \partial_{\bar{z}})w}{8V^{3/2}\mathrm{e}^{w/2}}\hat{e}^3 \,. \qquad \text{(A.1)}
\end{aligned}
$$

Here we have used both (3.3.3) and (3.3.4).

Appendix B

Weyl transformations of the boundary

In section 3.4 of the main text we studied the boundary geometry and Killing spinor equation using the radial coordinate $r = 1/y$ defined naturally by supersymmetry. This gives a preferred representative for the conformal class of the boundary metric on M_3. In this appendix we study the more general choice $r = 1/(\Omega y)$, where $\Omega = \Omega(z, \bar{z})$ is an arbitrary smooth, basic, nowhere zero function on M_3. This results in a Weyl transformation of the boundary geometry and corresponding Killing spinor equation. We will see that we precisely recover the boundary structure, derived from a purely three-dimensional perspective, in [32, 72].

For comparison with [32], we begin by rescaling the constant-norm Kähler spinor ζ as

$$\zeta \;\equiv\; \Omega^{-1/2}(z, \bar{z})\; \hat{\zeta}\;, \tag{B.1}$$

so that the norm of $\hat{\zeta}$ is $\Omega^{1/2}$ if we normalise ζ to have unit norm. We then also have a rescaling of the four-dimensional Killing spinor ϵ,

$$\hat{\epsilon} \;\equiv\; \Omega^{1/2}\epsilon \;=\; \frac{1}{\sqrt{2y}}\left(1 + V^{-1/2}\hat{\Gamma}_0\right)\hat{\zeta}\;. \tag{B.2}$$

Recall ϵ solves the Killing spinor equation (3.3.1), with the gauge field A_μ given by (3.3.6). Using instead $\hat{\epsilon}$ this Killing spinor equation reads

$$\left(\nabla_\mu - \mathrm{i}A_\mu - \frac{1}{2}\partial_\mu \log\Omega + \frac{1}{2}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\hat{\epsilon} = 0\ , \tag{B.3}$$

where the third term appears due to the rescaling[1]

With the new choice of radial coordinate the boundary metric is

$$\mathrm{d}s^2_{M_3} \ = \ \Omega^2(z,\bar{z})\left[(\mathrm{d}\psi+\phi_0)^2 + 4\mathrm{e}^{w_{(0)}}\mathrm{d}z\mathrm{d}\bar{z}\right]\ . \tag{B.4}$$

As always, we introduce an orthonormal frame for this metric:

$$e^1_{(3)} \ = \ \Omega(\mathrm{d}\psi+\phi_0)\ , \qquad e^2_{(3)} + \mathrm{i}e^3_{(3)} \ = \ 2\Omega\mathrm{e}^{w_{(0)}/2}\mathrm{d}z\ . \tag{B.5}$$

The four-dimensional geometry is the same as before, namely

$$\mathrm{d}s^2_{\mathrm{SDE}} \ = \ \frac{1}{y^2}\left[V^{-1}(\mathrm{d}\psi+\phi)^2 + V(\mathrm{d}y^2 + 4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z})\right]\ , \tag{B.6}$$

and we will use the frame

$$e^0 = \frac{1}{y}V^{1/2}\mathrm{d}y\ , \quad e^1 \ = \ \frac{1}{y}V^{-1/2}(\mathrm{d}\psi+\phi)\ , \quad e^2+\mathrm{i}e^3 \ = \ \frac{2}{y}(V\mathrm{e}^w)^{1/2}\mathrm{d}z\ . \tag{B.7}$$

Calculating the spin connection of (B.7), expanding in y and comparing to the spin

[1] As this term is a total derivative it can formally be absorbed into a *complex* gauge transformation of A_μ, although as we shall see all gauge fields will in the end be real.

connection of (B.5), we find

$$
\begin{aligned}
\omega^{12} &= \omega^{12}_{(3)} - \partial_2 \log\Omega\, e^1_{(3)} + \mathcal{O}(y)\ , \\
\omega^{13} &= \omega^{13}_{(3)} - \partial_3 \log\Omega\, e^1_{(3)} + \mathcal{O}(y)\ , \\
\omega^{23} &= \omega^{23}_{(3)} - \partial_3 \log\Omega\, e^2_{(3)} + \partial_2 \log\Omega\, e^3_{(3)} + \mathcal{O}(y)\ , \\
\omega^{0i} &= \frac{1}{y}\Omega^{-1}(1+\tfrac{1}{4}yw_{(1)})\, e^i_{(3)} + \mathcal{O}(y)\ ,
\end{aligned} \tag{B.8}
$$

with $i = 1, 2, 3$.

We next expand the Killing spinor equation with the rescaled spinor, $\hat{\epsilon}$. As in section 3.4 the term $\frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu = \mathcal{O}(y)$ does not contribute. One gets

$$
\begin{aligned}
&\Big[\nabla^{(3)}_\mu - \mathrm{i}A_{(0)\mu} - \frac{1}{2}\partial_\mu \log\Omega + \frac{1}{2y}\Omega^{-1}\left(1+\tfrac{1}{4}yw_{(1)}\right) e^i_{(3)\mu}(\Gamma_i - \Gamma_{i0}) \\
&- \frac{1}{2}\partial_2 \log\Omega e^i_{(3)\mu}\Gamma_{i2} - \frac{1}{2}\partial_3 \log\Omega e^i_{(3)\mu}\Gamma_{i3} + \mathcal{O}(y)\Big]\,\hat{\epsilon} \;=\; 0\ ,
\end{aligned} \tag{B.9}
$$

where $\mu = \psi, z, \bar{z}$, and $A_{(0)\mu}$ is the lowest order expansion of the gauge field (3.3.6), which in the frame (B.5) reads

$$
4\,A_{(0)} \;=\; -\Omega^{-1}w_{(1)}\, e^1_{(3)} + \partial_3 w_{(0)}\, e^2_{(3)} - \partial_2 w_{(0)}\, e^3_{(3)}\ . \tag{B.10}
$$

The Killing spinor $\hat{\epsilon}$ expands as

$$
\hat{\epsilon} = \frac{1}{\sqrt{2y}}\Big[1 + \Gamma_0 + \frac{1}{4}yw_{(1)}\Gamma_0 + \mathcal{O}(y^2)\Big]\hat{\zeta}_0\ , \tag{B.11}
$$

and when substituted into (B.9) gives a vanishing leading order term. The subleading term reads

$$
\begin{aligned}
&\Big[\Big(\nabla^{(3)}_i - \mathrm{i}A_{(0)i} - \frac{1}{2}\partial_i \log\Omega\Big)(1+\Gamma_0) - \frac{1}{8}w_{(1)}\Omega^{-1}(\Gamma_i - \Gamma_{i0}) \\
&-\frac{1}{2}\partial_2 \log\Omega\Gamma_{i2}(1+\Gamma_0) - \frac{1}{2}\partial_3 \log\Omega\Gamma_{i3}(1+\Gamma_0)\Big]\,\hat{\zeta}_0 \;=\; 0\ .
\end{aligned} \tag{B.12}
$$

The projection conditions (3.3.20) imply the following form for $\hat{\zeta}_0$,

$$\hat{\zeta}_0 = \begin{pmatrix} \hat{\chi} \\ 0 \end{pmatrix} \quad \text{where} \quad \hat{\chi} \ = \ \begin{pmatrix} \hat{\chi}_0 \\ \hat{\chi}_0 \end{pmatrix} \ . \tag{B.13}$$

The three-dimensional Killing spinor equation then becomes

$$\left[\nabla_i^{(3)} + \mathrm{i}(V_i - A_i^{(3)}) + \frac{1}{2}H\sigma_i + \frac{1}{2}\epsilon_{ijk}V_j\sigma_k\right]\hat{\chi} \ = \ 0 \ , \tag{B.14}$$

with

$$\begin{aligned} H \ &= \ -\frac{\mathrm{i}}{4}w_{(1)}\Omega^{-1} + \mathrm{i}V_1 \ , \qquad A_1^{(3)} \ = \ A_{(0)1} + \frac{3}{2}V_1 \ , \\ A_2^{(3)} \ &= \ A_{(0)2} - \frac{3}{2}\mathrm{i}V_3 - \frac{3}{2}\mathrm{i}\partial_2 \log\Omega + \frac{3}{2}\partial_3\log\Omega \ , \\ A_3^{(3)} \ &= \ A_{(0)3} + \frac{3}{2}V_3 \ , \\ V_2 + \mathrm{i}V_3 \ &= \ -\mathrm{i}\partial_2\log\Omega + \partial_3\log\Omega \ . \end{aligned} \tag{B.15}$$

The Killing spinor equation (B.14) is precisely of the form found in [72], which allows for the construction of supersymmetric field theories on M_3. The identifications of $A^{(3)}$, V and H are not unique because equation (B.14) has some symmetry properties, *c.f.* (4.2) of [72]. In particular this symmetry allows one to freely choose V_1, as shown in (2.10) of [32]. Recall that $A_{(0)}$ is real. If we demand also the boundary gauge field $A^{(3)}$ to be real, one finds from the equations in (B.15) that V is also real with

$$V_2 \ = \ \partial_3\log\Omega \ , \qquad V_3 \ = \ -\partial_2\log\Omega \ . \tag{B.16}$$

This is exactly the result obtained for V in [32] using the purely three-dimensional analysis of [72]. The remaining equations in (B.15) then further simplify to

$$H = -\frac{\mathrm{i}}{4} w_{(1)} \Omega^{-1} + \mathrm{i} V_1 \ , \tag{B.17}$$

$$A_i^{(3)} = A_{(0)i} + \frac{3}{2} V_i \ . \tag{B.18}$$

Again this is consistent with [32], where it was found (in our notation) that

$$A_\mu^{(3)} = -\frac{\mathrm{i}}{2} H e^1_{(3)\mu} + V_\mu + j_\mu \ , \tag{B.19}$$

where

$$j_\mu = \frac{\mathrm{i}}{4\Omega^2} \left(s \partial_\mu \bar{s} - \bar{s} \partial_\mu s \right) + \frac{1}{2} \omega_{\mu(3)}^{\ 23} \ , \tag{B.20}$$

and $|s| = \Omega$ is the square norm of the three-dimensional spinor,

$$\hat{\chi} = \sqrt{s(\psi, z, \bar{z})} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \ . \tag{B.21}$$

Hence we have $s = \Omega \mathrm{e}^{2\mathrm{i}\upsilon(\psi,z,\bar{z})}$. Equation (B.20) then reads

$$\begin{aligned} j_\mu &= \partial_\mu \upsilon + \frac{1}{2} \omega_{\mu(3)}^{\ 23} \qquad \text{(B.22)} \\ &= \partial_\mu \upsilon - \frac{1}{8} \Omega^{-1} w_{(1)} e^1_{(3)} + \frac{1}{4} \left(\partial_3 w_{(0)} + 2 \partial_3 \log \Omega \right) e^2_{(3)} - \frac{1}{4} \left(\partial_2 w_{(0)} + 2 \partial_2 \log \Omega \right) e^3_{(3)} \ , \end{aligned}$$

where we also used equation (3.3.4). Substituting equation (B.16), (B.17), and (B.22) into the right hand side of (B.19), this gives

$$\begin{aligned} A_\mu^{(3)} &= -\frac{1}{4} \Omega^{-1} w_{(1)} \, e^1_{(3)\mu} + \frac{1}{4} \partial_3 w_{(0)} \, e^2_{(3)\mu} - \frac{1}{4} \partial_2 w_{(0)} \, e^3_{(3)\mu} \ + \frac{3}{2} V_\mu + \partial_\mu \upsilon \\ &= A_{(0)\mu} + \frac{3}{2} V_\mu + \partial_\mu \upsilon \ , \qquad \text{(B.23)} \end{aligned}$$

where in the second line we used equation (B.10). As the last term in equation (B.23) is a total derivative, it can be absorbed into a gauge transformation of $A_{(0)}$. Thus we see that equation (B.19) reproduces (B.18) up to a gauge transformation. Indeed, such a gauge transformation with $\upsilon = \gamma\psi$ was shown in section 3.4.3 to be necessary in order for the gauge field to be globally well-defined on $M_3 \cong S^3$.

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