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Rapidity-only evolution of TMDs

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Abstract. The most known scheme to regulates the rapidity/UV divergences of the Transverse Momentum Distribution operators due to the infinite light-like gauge links are the Collis Soper Sterman (CSS) formalism or the Soft Collinear Effective Theory (SCET) formalism. An alternative procedure is provided by the scheme used in the small-x_B physics. The corresponding evolution equations differ already in leading order. Because of the future Electron-Ion Collider experiment, which will probe the TMDs at values of the Bjorken x in the region between small-x_B to $x_B \sim 1$, the different formalisms need to be reconciled. I will discuss the conformal properties of TMD operators and present the result of the conformal rapidity evolution of TMD operators in the Sudakov region. In particular, I will present the calculation of the scale of the coupling constant obtained using the BLM procedure.

1 Introduction

Recent developments in the study of transverse-momentum dependent parton distributions (TMDs) [1–3] have greatly expanded their use in processes such as semi-inclusive deep inelastic scattering (SIDIS) and particle production in hadron-hadron collisions. TMDs, which are defined as quark or gluon matrix elements accompanied by light-like gauge links, are subject to rapidity divergences due to the nature of these links. To deal with this, various regularization schemes have been proposed. The most common approaches involve the CSS or SCET formalisms, but alternative methods from small-x physics have also gained attention. However, discrepancies between these methods, even at leading order, pose theoretical challenges that need to be resolved, especially in view of the upcoming Electron-Ion Collider (EIC) experiments, which will probe a wide range of Bjorken-x values, from small-x_B to values closer to 1.

A key strategy to tackle these discrepancies is to derive leading-order evolution equations from conformal considerations. Since perturbative QCD (pQCD) is conformally invariant at leading order, one might be able to derive evolution equations without an explicit running coupling at this order. Given that TMD operators include light-like Wilson lines, their behavior should, in principle, follow the transformations of the conformal subgroup that preserves this light-like direction. However, regularizing rapidity divergences remains an obstacle, as no regularization method currently preserves full conformal invariance. A conformal regulator for leading-order calculations could still be a valuable tool, with higher-order calculations

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accounting for the breaking of conformal symmetry due to the QCD running coupling. Insights from $\mathcal{N} = 4$ Super-Yang-Mills (SYM) theory, which retains conformal symmetry, could provide a useful framework, as results in this theory often match the most complex parts of pQCD calculations. This approach has been applied to the rapidity evolution of color dipoles, leading to conformal invariance in the leading-order evolution of the Balitsky-Kovchegov (BK) equation, and offers a promising direction for similar studies in TMD evolution.

Th focus is the extension of the TMD formalism to small-x_B processes, a region that is becoming increasingly important for future experimental studies like those at the EIC. Traditionally, TMDs have been used at larger Bjorken-x, but expanding the formalism to the small-x_B regime has led to the development of new evolution equations based on rapidity cutoffs inspired by small-x_B physics [5, 6, 10, 11]. These new equations can smoothly interpolate between linear evolution at moderate x_B and the nonlinear dynamics at small x_B, capturing important transitions between different evolution regimes such as DGLAP, BFKL, and Sudakov forms. Despite this success, the evolution equations in the intermediate region are complex and present practical difficulties, such as the indeterminate argument of the coupling constant at leading order, which can only be determined through next-to-leading order calculations. This issue has been addressed through the Brodsky-Lepage-Mackenzie (BLM) [9] approach, which has been successfully applied to similar small-x evolution studies in color dipoles.

In this context, the results presented here aim to explore these theoretical issues and propose pathways to construct a unified framework for TMD evolution that is applicable across a broad range of kinematic regimes, from small to moderate x_B, in preparation for the future data from the EIC and other experiments.

2 Rapidity factorization for particle production in hadron-hadron collisions

As a working example of the of the rapidity-only formalism, let us consider the Drell-Yan (DY) hadronic tensor $W_{\mu\nu}(q)$, which in the TMD-factorization scheme [7] can be written as

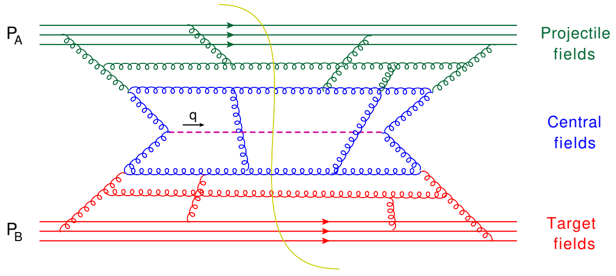

Figure 1. Diagrammatic representation of rapidity factorization in particle production in hadron-hadron collisions.

$$W_{\mu\nu}(q) = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}^{(i)}_{f/A}(x_A, k_\perp)\mathcal{D}^{(i)}_{f/B}(x_B, q_\perp - k_\perp)C_{\mu\nu}(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ Y} - \text{terms} \tag{1}$$

Here $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a quark f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$ is a similar quantity for hadron B, and coefficient functions $C_i(q, k)$ are determined by the cross section $\sigma(ff \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two quarks.

The TMD densities $\mathcal{D}_{f/A}(x_A, k_\perp)$ and $\mathcal{D}_{f/B}(x_B, k_\perp)$ are defined by quark-antiquark operators with gauge links going to $-\infty$. For example, the TMD f_1 responsible for the total DY cross section for unpolarized hadrons is defined by

$$f_1^f(x_B, k_\perp) \;=\; \frac{1}{16\pi^3}\int dz_+ d^2z_\perp\, e^{-ix_B z^+\sqrt{\frac{s}{2}}+i(k,z)_\perp}\langle p_N|\bar\psi_f(z_+, z_\perp)[z, z-\infty n]\not{n}\psi_f(0)|p_N\rangle \tag{2}$$

where $|p_N\rangle$ is an unpolarized nucleon with momentum $p_N \simeq p_N^-$ and $n = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})$ is a lightlike vector in the "+" direction (almost) collinear to vector p_A. For a straight-line gauge link, known as Wilson line, connecting points x and y we use the notation

$$[x, y] \;\equiv\; \mathrm{P}e^{ig\int du\;(x-y)^\mu A_\mu(ux+(1-u)y)} \tag{3}$$

Here, we will consider the rapidity-only evolution of the quark operator

$$\bar\psi(x^+, x_\perp)[x, x\pm\infty n]\;[\pm\infty n + x_\perp, \pm\infty n + y_\perp]\Gamma[\pm\infty n + y, y]\psi(y^+, y_\perp) \tag{4}$$

with Γ is one of the matrices $\gamma^-, \gamma^-\gamma_5, \gamma^-\gamma_\perp$ so we single out "good" projections in the light-cone language.

A diagrammatic representation of the factorization in rapidity is given in Fig. 1. Here the fields are cut with respect to the light-cone component of their momenta. The gluon fields, which are the dominant degree of freedom at this regime, are factorized in target fields (in red) with light-cone component $|k^+| < \sigma_b$, projectile fields (in green) with light-cone component $|k^-| < \sigma_a$, and central fields (in blue) for the coefficient functions. Here, σ_a and σ_b are two parameters which cut the gluon longitudinal momenta to distinguish them in target, projectile and central fields.

In section 4 we will present the evolution of the TMDs operator arising either from the target fields or the projectile ones with respect to the rapidity cut-off σ.

3 Conformal invariance of TMD Operators

Before delving into the calculation of the evolution equation, let us present the TMDs conformal group. The algebra of the full conformal group $SO(4,2)$ consists of 16 generators: four operators P^μ, six $M^{\mu\nu}$, four special conformal generators K^μ, and dilatation operator D. One can easily check that in the leading order the following 11 operators act on TMDs covariantly

$$P^i, P^-, M^{12}, M^{-i}, D, K^i, K^-, M^{-+} \tag{5}$$

while the action of operators P^+, M^{+i}, and K^+ do not preserve the form of the TMD operator. The action of the generators (5) on the TMD operator is the same as the action on the field F^{-i} without gauge link attachments. The corresponding group consists of transformations which leave the hyperplane $z^- = 0$ and vector n invariant. Those include shifts in transverse and "+″ directions, rotations in the transverse plane, Lorentz rotations/boosts created by M^{-i}, dilatations, and special conformal transformations

$$z'_\mu \;=\; \frac{z_\mu - a_\mu z^2}{1 - 2a\cdot z + a^2z^2} \tag{6}$$

with $a = (a^+, 0, a_\perp)$. In terms of "embedding formalism" [8] defined in 6-dim space, this subgroup is isomorphic to "Poincare + dilatations" group of the 4-dim subspace orthogonal to our physical light-like "+" and "-" directions. In ref. [10], we have shown that the LO evolution equation in the Sudakov approximation is invariant under the TMDs group represented by the generators (5) provided that the cut-off is chosen in the appropriate way.

4 Rapidity-only evolution equation with running coupling

The evolution equation of the quark TMD (4) is obtained calculating the diagrams in Fig.2 with rapidity cut-off defined as

$$A_\mu^\sigma(x) = \int \frac{d^4k}{16\pi^4}\, \theta(\sigma\varrho - |k^+|)e^{-ik\cdot x}A_\mu(k) \tag{7}$$

However, instead of a rigid one imposed by θ-function, it turned out that it is more convenient

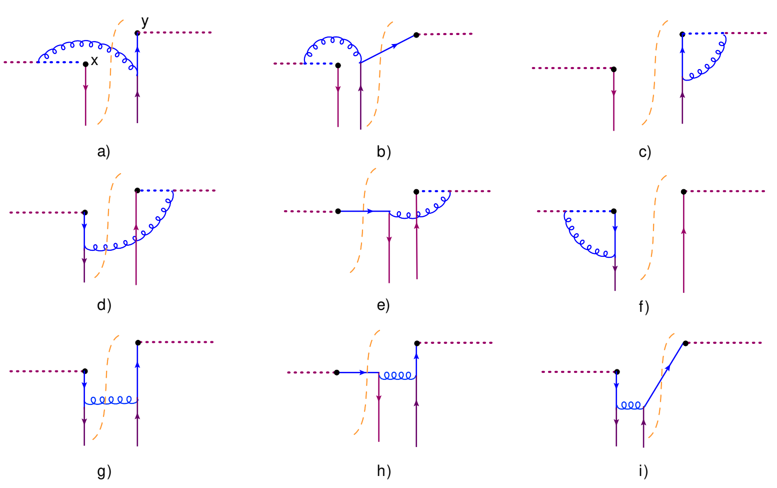

Figure 2. One-loop diagrams for TMD operator (4) in the background quark field. The dashed lines denote gauge links.

to use smooth cutoff in $|k^+|$. This can be encoded in coordinate space by defining the rapidity-regularized operators as

$$\bar\psi^\sigma(x^+, x_\perp) \equiv \bar\psi(x^+, x_\perp, -\frac{1}{\rho\sigma})[x^+, -\infty]_x, \quad \psi^\sigma(y^+, y_\perp) \equiv [-\infty, y^+]_y\psi(y^+, y_\perp, -\frac{1}{\rho\sigma}) \tag{8}$$

We will consider the evolution equation in the Sudakov region $\sigma x_B s \gg k_\perp^2 \sim q_\perp^2$, where s is the Mandelstam variable, $q_\perp^2$ is the transverse momenta of the produced lepton-pair or Higgs boson in DY process (or of the produced hadron in SIDIS), while $k_\perp^2$ is the typical transverse momenta of the parton participating in the interaction.

The diagrams contributing the LO evolution equation are given in Fig. 2. The calculation can be performed directly in coordinate space [10], or in momentum space and then perform

Fourier transform in coordinate space [11]. The result in coordinate space is

$$
\begin{aligned}
&\Big(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\Big)\bar\psi^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp)\\
&= \frac{\alpha_s}{4\pi^2}c_F\int dz^+\Big\{\Big[i\frac{\ln\varrho(-x^++z^++i\epsilon)-\ln\frac{\sigma b_\perp^2 s}{4}e^\gamma}{-x^++z^++i\epsilon}+\text{c.c.}\Big]\bar\psi^{\sigma'}(z^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp)\\
&+\Big[i\frac{\ln\varrho(-y^++w^++i\epsilon)-\ln\frac{\sigma' b_\perp^2 s}{4}e^\gamma}{-y^++w^++i\epsilon}+\text{c.c.}\Big]\bar\psi^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp)\Big\}
\end{aligned}
\tag{9}
$$

The solution of this equation has the form

$$
\begin{aligned}
\bar\psi^{\sigma'}(x^+,x_\perp)\Gamma\psi^{\sigma}(y^+,y_\perp) &= e^{-\frac{\alpha_s c_F}{4\pi}\left(\ln\frac{\sigma'}{\sigma'_0}\ln\sigma'\sigma'_0+\ln\frac{\sigma}{\sigma_0}\ln\sigma\sigma_0\right)}\\
\times\int dz^+\Big[\frac{i\Gamma(1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma'}{\sigma'_0})}{(z^+-x^++i\epsilon)^{1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma'}{\sigma'_0}}}+\text{c.c.}\Big]&\int dw^+\Big[\frac{i\Gamma(1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma}{\sigma_0})}{(w^+-y^++i\epsilon)^{1-\frac{\alpha_s c_F}{2\pi}\ln\frac{\sigma}{\sigma_0}}}+\text{c.c.}\Big]\\
\times\frac{1}{4\pi^2}(b_\perp^2e^\gamma\sqrt{s/8})^{-\frac{\alpha_s c_F}{2\pi}\left(\ln\frac{\sigma'}{\sigma'_0}+\ln\frac{\sigma}{\sigma_0}\right)}&\bar\psi^{\sigma'_0}(z^+,x_\perp)\Gamma\psi^{\sigma_0}(w^+,y_\perp)
\end{aligned}
\tag{10}
$$

The evolution equation (10) is invariant under the conformal sub-group represented by the generators (5) if we take $\sigma=\sigma'=\frac{\varsigma\sqrt{2}}{\varrho|\Delta_\perp|}$ where ς is an evolution parameter.

In the LO, the rapidity evolution equation has the argument of the coupling constant left undetermined. To include it, we adopted the BLM method: calculate the contribution of the first quark loop to our TMD evolution and promote $-\frac{1}{6\pi}n_f$ to full $b_0=\frac{11}{12\pi}N_c-\frac{1}{6\pi}n_f$. To this end, each gluon propagator in diagrams in Fig. 2 should be replaced by a one-loop correction, i.e.

$$
\begin{aligned}
\frac{1}{p^2+i\epsilon} &\rightarrow \frac{1}{p^2+i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde\mu^2}{-p^2-i\epsilon}\Big)\\
\frac{1}{p^2-i\epsilon} &\rightarrow \frac{1}{p^2-i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde\mu^2}{-p^2+i\epsilon}\Big)\\
2\pi\delta(p^2)\theta(p_0) &\rightarrow \frac{i\theta(p_0)}{p^2+i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde\mu^2}{-p^2-i\epsilon}\Big)-\frac{i\theta(p_0)}{p^2-i\epsilon}\Big(1+b_0\alpha_s(\mu)\ln\frac{\tilde\mu^2}{-p^2+i\epsilon}\Big)
\end{aligned}
\tag{11}
$$

where $\tilde\mu^2\equiv\bar\mu_{\rm MS}^2e^{5/3}$.

The evolution equation for the TMD operator with running coupling takes the form

$$
\begin{aligned}
&\Big(\sigma\frac{d}{d\sigma}+\sigma'\frac{d}{d\sigma'}\Big)\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)=-\frac{c_F}{2\pi}\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp)\\
&\times\Big[\alpha_s(\mu_{\sigma'})\ln\Big(-\frac{i}{4}(\beta'_B+i\epsilon)\sigma' sb_\perp^2e^\gamma\Big)+\alpha_s(\mu_\sigma)\ln\Big(-\frac{i}{4}(\beta_B+i\epsilon)\sigma sb_\perp^2e^\gamma\Big)\Big]
\end{aligned}
\tag{12}
$$

where $b_\perp\equiv\Delta_\perp=(x-y)_\perp$ We see that in the Sudakov region we can define TMD operator (4) with two independent "left" and "right" cutoffs σ and σ' defined in Eqs. (8) and the evolutions with respect to those cutoffs are independent [except for $b_\perp=(x-y)_\perp$]. The solution of Eq. (12) has the form

$$
\begin{aligned}
\bar\psi^{\sigma'}(\beta'_B,x_\perp)\Gamma\psi^{\sigma}(\beta_B,y_\perp) &= e^{-\frac{2c_F}{\pi b_0^2}\left[\left(\frac{1}{\alpha_s(\tilde b_\perp^{-1})}-\frac{b_0}{2}\ln[-i(\tau'_B+i\epsilon)]\right)\ln\frac{\alpha_s(\mu_{\sigma'})}{\alpha_s(\mu_{\sigma'_0})}+\frac{1}{\alpha_s(\mu_{\sigma'})}-\frac{1}{\alpha_s(\mu_{\sigma'_0})}\right]}\\
\times e^{-\frac{2c_F}{\pi b_0^2}\left[\left(\frac{1}{\alpha_s(\tilde b_\perp^{-1})}-\frac{b_0}{2}\ln[-i(\tau_B+i\epsilon)]\right)\ln\frac{\alpha_s(\mu_{\sigma})}{\alpha_s(\mu_{\sigma_0})}+\frac{1}{\alpha_s(\mu_{\sigma})}-\frac{1}{\alpha_s(\mu_{\sigma_0})}\right]}&\bar\psi^{\sigma'_0}(\beta'_B,x_\perp)\Gamma\psi^{\sigma_0}(\beta_B,y_\perp)
\end{aligned}
\tag{13}
$$

where $\tilde b_\perp^2=\frac{b_\perp^2}{2}e^{\gamma/2}$ and $\tau_B=\frac{\beta_B}{|\beta_B|}$, $\tau'_B=\frac{\beta'_B}{|\beta'_B|}$, $\mu_\sigma^2\equiv b_\perp^{-1}\sqrt{\sigma|\beta_B|s}$, $\mu_{\sigma'}^2\equiv b_\perp^{-1}\sqrt{\sigma|\beta'_B|s}$. As at leading order, the structure of the Sudakov evolution (13) resembles two independent exponential factors which describe two independent evolutions of operators (8). However, this property should not be expected beyond the Sudakov region.

5 Conclusions

We described the rapidity evolution of quark (and gluon) TMDs using small-x_B methods. By employing a rapidity-only cutoff, we simplified the evolution to depend on a single parameter – rapidity. However, as highlighted in the introduction, the argument of the coupling constant in such evolution is not determined at leading order. To address this, we applied the BLM scale-setting procedure to both quark and gluon TMDs, leading to the result that the effective BLM scale for Sudakov evolution is halfway between the transverse momentum and the longitudinal energy of the TMD in logarithmic scale. This approach mirrors previous successful applications in NLO BK evolution [12].

Despite using small-x_B methods, our findings are valid for any value of x_B, provided certain kinematic conditions are met. The distinction between small and moderate x_B only emerges at the endpoint of evolution, where different evolution regimes, such as DGLAP for moderate x_B and BFKL/BK for small-x_B, should be applied.

We also observed the importance of choosing a smooth rapidity cutoff to preserve the analytic properties of Feynman diagrams and avoid infrared (IR) divergences, a consideration that distinguishes TMD evolution from dipole evolution. Finally, future work will focus on extending this framework to TMD factorization and calculating cross sections for processes like Higgs production and Drell-Yan at $q_\perp \sim$ a few GeV, as well as comparing our results with those from the CSS method at higher orders. This ongoing research aims to provide a more comprehensive understanding of TMD evolution across different kinematic regimes.

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