Proceedings

of the Second International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

Memorial Prof. W. FUSHCHYCH Conference

July 7–13, 1997, Kyiv, Ukraine

Organized by

Institute of Mathematics of the National Academy of Sciences of Ukraine
Ukrainian Pedagogical University

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Volume 1

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ISBN 966–02–0342–X

Symmetry in Nonlinear Mathematical Physics,
Editors: M. Shkil, A. Nikitin and V. Boyko.

ISBN 966–02–0343–8 (Volume 1) and **ISBN 966–02–0344–6** (Volume 2)

Printed in Ukraine, Kyiv, 1997.

Preface

The Second International Conference **"Symmetry in Nonlinear Mathematical Physics"** was organized mainly due to efforts of Professor Wilhelm Fushchych. It happened, however, that this outstanding scientist and kind person passed away. The Conference was dedicated to his memory. Among 80 participants of the Conference there were representatives of 14 countries from 5 continents. The main part of the lectures was devoted to investigation of symmetries and construction of exact solutions of nonlinear differential equations. Moreover, the latest trends in symmetry analysis, such as conditional symmetry, potential symmetries, discrete symmetries and differential geometry approach to symmetry analysis were represented efficiently. In addition, important branches of symmetry analysis such as representation theory, quantum groups, complete integrable systems and the corresponding infinite sets of conservation laws were widely discussed. The related papers as well as ones devoted to symmetries in physics and backgrounds of the nonlinear quantum mechanics, are included into the second volume of the Proceedings. We plan to continue the series of conferences dedicated to the memory of Professor Wilhelm Fushchych and hope that they will make an essential contribution to symmetry approach to nonlinear mathematical physics.

Anatoly NIKITIN
October, 1997

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 11–20.

Scientific Heritage of W. Fushchych

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Abstract

A very short discussion of the main scientific results obtained by Prof. W. Fushchych is presented.

Introduction

The scientific heritage of Prof. W. Fushchych is great indeed. All of us have obtained the list of his publications which includes more than 330 items. It is difficult to imagine that all this was produced by one man. But it is the case, and numerous students of Prof. Fushchych can confirm that he made a decisive contribution to the majority of his publications.

As one of the first students of Prof. W. Fushchych, I would like to say a few words about the style of his collaboration with us. He liked and appreciated any collaboration with him. He was a very optimistic person and usually believed in the final success of every complicated investigation, believed that his young collaborators are able to overcome all difficulties and to solve the formulated problem. In addition to his purely scientific contributions to research projects, such an emotional support was very important for all of us. He helped to find a way in science and life for great many of people including those of them who had never collaborated with him directly. His scientific school includes a lot of researchers, and all of them will remember this outstanding and kind person.

Speaking about scientific results obtained by my teacher, Prof. W. Fushchych, I have to restrict myself to the main ones only. In any case, our discussion will be fragmentary inasmuch it is absolutely impossible to go into details of such a large number of publications.

From the extremely rich spectrum of scientific interests of W. Fushchych, I selected the following directions:

1. Invariant wave equations.
2. Generalized Poincaré groups and their representations.
3. Non-Lie and hidden symmetries of PDE.
4. Symmetry analysis and exact solutions of nonlinear PDE.

I will try to tell you about contributions of Prof. W. Fushchych to any of the fields enumerated here. It is necessary to note that item 4 represents the most extended field of investigations of W. Fushchych, which generated the majority of his publications.

1. Invariant wave equations

1. Poincaré-invariant equations

W. Fushchych solved a fundamental problem of mathematical physics, which was formulated long ago and attacted much attention of such outstanding scientists as Wigner, Bargmann, Harish-Chandra, Gelfand and others. The essence of this problem is a description of multicomponent wave equations which are invariant with respect to the Poincaré group and satisfy some additional physical requirements.

In order to give you an idea about of this problem, I suggest to consider the Dirac equation

$$L\Psi = (\gamma^\mu p_\mu - m)\Psi = 0, \qquad p_\mu = -i\frac{\partial}{\partial x_\mu}, \qquad \mu = 0,1,2,3. \tag{1}$$

Here, γ_μ are 4×4 matrices satisfying the Clifford algebra:

$$\begin{aligned} &\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \\ &g_{00} = -g_{11} = -g_{22} = -g_{33} = 1, \qquad g_{\mu\nu} = 0, \quad \mu \neq \nu. \end{aligned} \tag{2}$$

Equation (1) is invariant with respect to the Poincaré group. Algebraic formulation of this statement is the following: there exist symmetry operators for (1)

$$P_\mu = i\frac{\partial}{\partial x_\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + S_{\mu\nu}, \tag{3}$$

where

$$S_{\mu\nu} = \frac{i}{4}[\gamma_\mu, \gamma_\nu], \qquad \mu,\nu = 0,1,2,3.$$

These operators commute with L of (1) and satisfy the Poincaré algebra $AP(1,3)$

$$[L, P_\mu] = [L, J_{\mu\nu}] = 0, \qquad [P_\mu, P_\nu] = 0, \tag{4}$$

$$[P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu). \tag{5}$$

It follows from (4) that generators P_μ, $J_{\mu\nu}$ transform solutions of (1) into solutions.

Of course, the Dirac equation is not the only one having this symmetry, and it is interesting to search for other equations invariant with respect to the Poincaré algebra. In papers of Bargmann, Harish-Chandra, Gelfand, Umezawa and many others, we can find a number of relativistic wave equations for particles of arbitrary spin. It happens, however, that all these equations are inconsistent inasmuch as they lead to violation of the causality principle for the case of a particle interacting with an external field. Technically speaking, these equations lose their hyperbolic nature if we take into account the interaction with an external field.

To overcome this difficulty, Fushchych proposed to search for Poincaré-invariant wave equations in the Schrödinger form

$$i\frac{\partial}{\partial x_0}\Psi = H\Psi, \qquad H = H(\vec{p}), \tag{6}$$

where H is a differential operator which has to be found starting with the requirement of Poincaré invariance of equation (6). In spite of the asymmetry between spatial and time variables, such an approach proved to be very fruitful and enables to find causal equations for arbitrary spin particles.

I will not enter into details but present a nice formulation of the Poincaré-invariance condition for equation (6):

$$\begin{cases} \Big[[H, x_a], [H, x_b]\Big] = -4iS_{ab}, \\ H^2 = \vec{p}^2 + m^2. \end{cases} \tag{7}$$

It is easy to verify that the Dirac Hamiltonian $H = \gamma_0\gamma_a\gamma_a + \gamma_0 m$ satisfies (7). The other solutions of these relations present new relativistic wave equations [3, 4].

It is necessary to note that relativistic wave equations found by W. Fushchych with collaborators present effective tools for solving physical problems related to the interaction of spinning particles with external fields. Here, I present the formula (obtained by using these equations) which describes the energy spectrum of a relativistic particle of spin s interesting with the Coulomb field:

$$E = n\left[1 + \frac{\alpha^2}{\left(n' + \frac{1}{2} + \left[\left(j + \frac{1}{2}\right)^2 - \alpha^2 - b_\lambda^{sj}\right]^{1/2}\right)^2}\right]^{1/2}. \tag{8}$$

Here $n' = 0, 1, 2, \ldots$, $j = \dfrac{1}{2}, \dfrac{3}{2}, \ldots, b_\lambda^{sj}$ is a root of the specific algebraic equation defined by the value of spin s.

Formula (8) generalizes the famous Sommerfeld formula for the case of arbitrary spin s [3, 4] .

2. Galilei-invariant wave equations

In addition to the Poincaré group, the Galilei group has very important applications in physics. The Galilei relativity principle is valid for the main part of physical phenomena which take place on the Earth. This makes the problem of description of Galilei-invariant equations very interesting. In papers of W. Fushchych with collaborators, the problem is obtained a consistent solution.

Starting with the first–order equations

$$(\beta_\mu p^\mu - \beta_4 m)\Psi = 0 \tag{9}$$

and requiring the invariance with respect to the Galilei transformations

$$x_a \to R_{ab}x_b + V_a t + b_a, \qquad t_0 \to t_0 + b_0,$$

($\|R_{ab}\|$ are orthogonal matrices), we come to the following purely algebraic problem:

$$\begin{aligned} &\widetilde{S}_a\beta_0 - \beta_0 S_a = 0, \qquad \widetilde{S}_a\beta_4 - \beta_4 S_a = 0, \\ &\widetilde{\eta}_a\beta_4 - \beta_4\eta_a = -i\beta_a, \qquad \widetilde{\eta}_a\beta_b - \beta_b\eta_a = -i\delta_{ab}\beta_0, \\ &\widetilde{\eta}_a\beta_0 - \beta_0\eta_a = 0, \qquad a = 1, 2, 3; \end{aligned} \tag{10}$$

where S_a, η_a and $\widetilde{S}_a, \widetilde{\eta}_a$ are matrices satisfying the algebra $AE(3)$

$$[S_a, S_0] = i\varepsilon_{abc}S_c, \quad [S_a, \eta_b] = i\varepsilon_{abc}\eta_c, \quad [\eta_a, \eta_b] = 0. \tag{11}$$

The principal result of investigation of the Galilei-invariant equations (9) is that these equations describe correctly the spin-orbit and Darwin couplings of particles with all external fields. Prior to works of W. Fushchych, it was generally accepted that these couplings are purely relativistic effects. Now we understand that these couplings are compatible with the Galilei relativity principle [3, 4].

For experts in particle physics, I present the approximate Hamiltonian for a Galilean particle interacting with an external electromagnetic field:

$$\begin{aligned} H = \frac{p^2}{2m} + m + eA_0 + \frac{e}{2ms}\vec{S}\cdot\vec{H} + \\ + \frac{e}{4m^2}\left[-\frac{1}{2}\vec{S}\cdot(\pi\times E - \vec{E}\times\vec{\pi}) + \frac{1}{b}\Theta_{ab}\frac{\partial E_a}{\partial x_b} + \frac{1}{3}s(s+1)\,\mathrm{div}\,\vec{E}\right], \end{aligned} \tag{12}$$

where

$$\Theta_{ab} = 3[S_a, S_b]_+ - 2\delta_{ab}s(s+1).$$

The approximate Hamiltonian (12), obtained by using Galilei-invariant equations, coincides for $s = \frac{1}{2}$ with the related Hamiltonian obtained from the Dirac equation [3, 4].

3. Nonlinear equations invariant with respect to the Galilei and Poincaré groups

W. Fushchych made a very large contribution into the theory of nonlinear equations with a given invariance group. Here, I present some of his results connected with Galilei and Poincaré invariant equations.

Theorem 1 [4, 5]. *The nonlinear d'Alembert equation*

$$p_\mu p^\mu \Psi + F(\Psi) = 0$$

is invariant with respect the extended Poincaré group $\widetilde{P}(1,3)$ *iff*

$$F(\Psi) = \lambda_1 \Psi^r, \qquad r \neq 1,$$

or

$$F(\Psi) = \lambda_2 \exp(\Psi).$$

Here, Ψ *is a real scalar function.*

Theorem 2 [4, 5, 8]. *The nonlinear Dirac equation*

$$[\gamma^\mu p_\mu + F(\overline{\Psi}, \Psi)]\Psi = 0$$

is invariant with respect to the Poincaré group iff

$$F(\overline{\Psi}, \Psi) = F_1 + F_2\gamma_5 + F_3\gamma^\mu\overline{\Psi}\gamma_5\gamma_\mu\Psi + F_4 S^{\nu\lambda}\overline{\Psi}\gamma_5 S_{\nu\lambda}\Psi,$$

where $F_1, \ldots, F_4$ *are arbitrary functions of* $\overline{\Psi}\Psi$ *and* $\overline{\Psi}\gamma_5\Psi$, $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$.

Theorem 3 [3, 4]. *The Maxwell's equations for electromagnetic field in a medium*

$$\frac{\partial \vec{D}}{\partial x_0} = -\vec{p}\times\vec{H}, \qquad \frac{\partial \vec{B}}{\partial x_0} = \vec{p}\times\vec{E},$$
$$\vec{p}\cdot\vec{D} = 0, \qquad \vec{p}\cdot\vec{B} = 0$$

with constitutive equations

$$\vec{E} = \vec{\Phi}(\vec{D},\vec{H}), \qquad \vec{B} = \vec{F}(\vec{D},\vec{H})$$

are invariant with respect to the group $P(1,3)$ iff

$$D = M\vec{E} + N\vec{B}, \qquad \vec{H} = M\vec{B} - N\vec{E},$$

where $M = M(C_1, C_2)$ and $N = N(C_1, C_2)$ are arbitrary functions of the invariants of electromagnetic field

$$C_1 = \vec{E}^2 - \vec{B}^2, \qquad C_2 = \vec{B}\cdot\vec{E}.$$

Theorem 4 [4]. *The nonlinear Schrödinger equation*

$$\left(p_0 - \frac{p^2}{2m}\right)u + F(x,u,u^*) = 0$$

is invariant with respect to the Galilei algebra $AG(1,3)$ iff

$$F = \Phi(|u|)u,$$

to the extended Galilei algebra (including the dilation operator) $AG_1(1,3)$ iff

$$F = \lambda|u|^k u, \qquad \lambda, k \neq 0,$$

and to the Schrödinger algebra $AG_2(1,3)$ iff

$$F = \lambda|u|^{3/4}u.$$

I present only a few fundamental theorems of W. Fushchych concerning to the description of nonlinear equations with given invariance groups. A number of other results can be found in [1–10].

By summarizing, we can say that W. Fushchych made the essential contribution to the theory of invariant wave equations. His fundamental results in this field are and will be used by numerous researchers.

2. Generalized Poincaré groups and their representations

Let us discuss briefly the series of W. Fushchych's papers devoted to representations of generalized Poincaré groups.

A generalized Poincaré group is defined as a semidirect product of the groups $SO(1,n)$ and T

$$P(1,n) = SO(1,n) \otimes T,$$

where T is an additive group of $(n+1)$-dimensional vectors $p_1, p_2, \ldots, p_n$ and $SO(1,n)$ is a connected component of the unity in the group of all linear transformations of T into T preserving the quadratic form

$$p_0^2 - p_1^2 - p_2^2 - \cdots - p_n^2.$$

Prof. W. Fushchych was one of the first who understood the importance of generalized Poincaré groups for physics. A straightforward interest to these groups can be explained, for example, by the fact that even the simplest of these, the group $P(1,4)$, includes the Poincaré, Galilei and Euclidean groups as subgroups. In other words, the group $P(1,4)$ unites the groups of motion of the relativistic and nonrelativistic quantum mechanics and the symmetry group of the Euclidean quantum field theory.

Using the Wigner induced representations method, W. Fushchych described for the first time all classes of unitary IRs of the generalized Poincaré group and the related unproper groups including reflections [3, 4].

The Lie algebra of the generalized Poincaré group $P(1,4)$ includes $\dfrac{n(n+3)+2}{2}$ basis elements $\{P_m, J_{mn}\}$ which satisfy the following commutation relations:

$$\begin{aligned}
&[P_\mu, P_\nu] = 0, \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu), \\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\rho}J_{\nu\sigma} + g_{\nu\sigma}J_{\mu\rho} - g_{\nu\rho}J_{\mu\sigma} - g_{\mu\sigma}J_{\nu\rho}), \\
&\mu, \nu, \rho, \sigma = 0, 1, \ldots, n.
\end{aligned} \tag{13}$$

W. Fushchych found realizations of algebra (13) in different bases.

3. Non-Lie symmetries

In 1974, W. Fushchych discovered that the Dirac equation admits a specific symmetry which is characterized by the following property.

1. Symmetry operators are non-Lie derivatives (i.e., do not belong to the class of first order differential operators).

2. In spite of this fact, they form a finite-dimensional Lie algebra.

This symmetry was called a non-Lie symmetry. It was proved by W. Fushchych and his collaborators that a non-Lie symmetry is not a specific property of the Dirac equation. Moreover, it is admitted by great many of equations of quantum physics and mathematical physics. Among them are the Kemmer-Duffin-Petiau, Maxwell equations, Lamé equation, relativistic and nonrelativistic wave equations for spinning particles and so on.

In order to give you an idea about "non-Lie" symmetries, I will present you an example connected with the Dirac equation. In addition to generators of the Poincaré group, this equation admits the following symmetries [1, 2]:

$$Q_{\mu\nu} = \gamma_\mu\gamma_\nu + (1 - i\gamma_5)\frac{\gamma_\mu p_\nu - \gamma_\nu p_\mu}{2m}. \tag{14}$$

Operators (14) transform solutions of the Dirac equation into solutions. They are non-Lie derivatives inasmuch as their first term includes differential operators with matrix coefficients. In spite of this fact, they form a 6-dimensional Lie algebra defined over the field of real numbers. Moreover, this algebra can be united with the Lie algebra of the Poincaré group in frames of a 16-dimensional Lie algebra. This algebra is characterized by the following commutation relations

$$\begin{aligned} &[J_{\mu\nu}, Q_{\lambda\sigma}] = [Q_{\mu\nu}, Q_{\lambda\sigma}] = 2i\left(g_{\mu\sigma}Q_{\nu\sigma} + g_{\sigma\lambda}Q_{\mu\nu} - g_{\mu\lambda}Q_{\nu\sigma} - g_{\nu\sigma}Q_{\mu\lambda}\right), \\ &[Q_{\mu\nu}, P_\lambda] = 0. \end{aligned} \tag{15}$$

Taking into account relations (15) and

$$Q_{\mu\nu}^2\Psi = \Psi, \tag{16}$$

we conclude that the Dirac equation is invariant with respect to the 16-parameter group of transformations. Generalizing symmetries (14) to the case of higher order differential operators, we come to the problem of description of complete sets of such operators (which was called higher order symmetries):

$$\Psi(x) \to B_\mu \frac{\partial\Psi}{\partial x_\mu} + D(\Lambda)\Psi(\Lambda^{-1}x - a).$$

In the papers of W. Fushchych, the complete sets of higher order symmetry operators for the main equations of classical field theory were found. Here, I present numbers of linearly independent symmetry operators of order n for the Klein-Gordon-Fock, Dirac, and Maxwell equations.

KGF equation:

$$N_n = \frac{1}{4!}(n+1)(n+2)(n+3)(n^2+3n+4).$$

The Dirac equation:

$$\widetilde{N}_n = 5N_n - \frac{1}{6}(2n+1)(13n^2+19n+18) - \frac{1}{2}[1-(-1)^n].$$

The Maxwell equation:

$$N_n = (2n+3)[2n(n-1)(n+3)(n+4) + (n+1)^2(n+2)^2]/12.$$

4. Symmetries and exact solutions of nonlinear PDE

In this fundamental field, W. Fushchych obtained a lot of excellent results. Moreover, he discovered new ways in obtaining exact solutions of very complicated systems of nonlinear PDE.

It is necessary to mention the following discoveries of W. Fushchych.

1. The ansatz method

It is proved that if a system of nonlinear differential equations

$$L(x, \Psi(x)) = 0$$

admits a Lie symmetry, it is possible to find exact solutions of this system in the form

$$\Psi = A(x)\varphi(\omega), \tag{17}$$

where $A(x)$ is a matrix, $\varphi(\omega)$ is an unknown function of group invariants $\omega = (\omega_1, \ldots, \omega_n)$.

Long ago, W. Fushchych understood that relation (17) can be treated as an ansatz which, in some sense, is a more general substance than a Lie symmetry. I should like to say that it is possible to use successfully substitutions (17) (and more general ones) even in such cases when an equation do not admit a Lie symmetry.

2. Conditional symmetry

Consider a system of nonlinear PDE of order n

$$\begin{gathered} L(x, u_1, u_2, \ldots, u_n) = 0, \qquad x \in R(1,n), \\ u_1 = \left(\frac{\partial u}{\partial x_0}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right), \qquad u_2 = \left(\frac{\partial^2 u}{\partial x_0^2}, \frac{\partial^2 u}{\partial x_0 \partial x_1}, \ldots\right). \end{gathered} \tag{18}$$

Let some operator Q do not belong to the invariance algebra of equation (18) and its prolongation satisfy the relations

$$\begin{aligned} \widetilde{Q}L &= \lambda_0 L + \lambda_1 L_1, \\ \widetilde{Q}L_1 &= \lambda_2 L + \lambda_3 L_1 \end{aligned} \tag{19}$$

with some functions $\lambda_0, \lambda_1, \lambda_3, \lambda_3$.

We say that equation (18) is *conditionally invariant* if relations (19) hold. In this case we can impose an additional condition

$$L_1 \equiv L_1(x, u_1, u_2, \ldots)$$

and system (18), (19) is invariant under Q.

We say that equation (18) is Q-invariant provided

$$\widetilde{Q}L = \lambda_0 L + \lambda_1 (Qu).$$

The essence of this definition is that we can extend a symmetry of PDE by adding some addititonal conditions on its solutions. The conditional and Q-invariance approaches make it possible to find a lot of new exact solutions for great many of important nonlinear equations. Let us enumerate some of them:

1.The nonlinear Schrödinger equation

$$i\Psi_t + \Delta\Psi = F(x, \Psi, \Psi^*).$$

2. The nonlinear wave equation

$$\Box u = F(u).$$

3. The nonlinear eikonal equation

$$u^2_{x_0} - (\vec{\nabla} u)^2 = \lambda.$$

4. The Hamilton-Jacobi equation

$$u_{x_0} - (\vec{\nabla} u)^2 = \lambda.$$

5.The nonlinear heat equation

$$u_t - \vec{\nabla}(f(u)\vec{\nabla} u) = g(u).$$

6. The Monge-Ampéré equation

$$\det\|u_{x_\mu x_\nu}\|^n_{\mu,\nu=0} = F(u).$$

7. The nonlinar Born-Infeld equation

$$(1 - u_{x_\mu} u_{x^\mu})\Box u + u_{x_\mu} u_{x_\nu} u_{x^\mu} u_{x^\nu} = 0.$$

8. The nonlinear Maxwell equations

$$\Box A_\mu - \partial_{x_\mu}\partial_{x_\nu} A_\nu = A_\mu F(A_\nu A^\nu).$$

9. The nonlinear Dirac Equations

$$i\gamma_\mu \Psi_{x_\mu} = F(\Psi^*, \Psi).$$

10. The nonlinear Lev-Leblond equations.

$$i(\gamma_0 + \gamma_4)\Psi_t + i\gamma_a \Psi_{x_a} = F(\Psi^*, \Psi).$$

11. Equations of the classical electrodynamics.

$$i\gamma_\mu \Psi_{x_\mu} + (e\gamma_\mu A^\mu - m)\Psi = 0,$$
$$\Box A_\mu - \partial_{x_\mu}\partial_{x_\nu} A_\nu = e\overline{\Psi}\gamma_\mu \Psi.$$

12. $SU(2)$ Yang-Mills Equations.

$$\partial_{x_\nu}\partial_{x^\nu}\vec{A}_\mu - \partial_{x^\mu}\partial_{x_\nu}\vec{A}_\nu + e\Big((\partial_{x_\nu}\vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_{x_\nu}\vec{A}_\mu) \times \vec{A}_\nu + +(\partial_{x^\mu}\vec{A}_\nu) \times \vec{A}^\nu\Big) + e^2\vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}.$$

Summary

In conclusion, I should like to say that the main heritage of Prof. Fushchych is a scientific school created by him. About 60 Philosophy Doctors whose theses he supervised work at many institutions of the Ukraine and abroad. And his former students will make their best to continue the ideas of Prof. Wilhelm Fushchych.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 21–33.

Differential Form Symmetry Analysis of Two Equations Cited by Fushchych

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Abstract

In Wilhelm Fushchych's address, "Ansatz '95", given to the first conference "Symmetry in Nonlinear Mathematical Physics" [1], he listed many differential equations on which he and others had done some symmetry analysis. In this talk, the present author treats two of these equations rather extensively, using differential forms to find the symmetries, based on a method by F. B. Estabrook and himself [2]. A short introduction to the differential form method will be presented.

1 Introduction

Most calculations for symmetries of differential equations are done with the classical method. However, in 1971 Frank B. Estabrook and the author published a method [2] for finding the symmetries of differential equations, using a differential form technique, with a geometrical flavor. We refer to that paper as paper I. Since that technique has not been used widely in the literature, I would like to review it, and then to apply it to two equations cited by Fushchych in his talk "Ansatz 95", which he gave here at Kyiv at the first conference on nonlinear mathematical analysis [1].

2 Differential forms

I give a brief review of differential forms here. A simple, clever definition of differential forms, due to H. Flanders [3], is that differential forms are the things found under integral signs. That gives an immediate picture of differential forms, but we need to look at their foundation.

We begin by writing out a general tensor field, in terms of components with a basis formed of tensor products of basis tangent vectors e_i and 1-forms ω_i, as shown:

$$T = T^{ik\ldots}_{\ldots mn\ldots} e_i \otimes e_k \otimes \ldots \omega^m \otimes \omega^n \otimes \ldots . \tag{1}$$

The components may be functions of position.

One may work in the "natural bases" for these spaces, written, for coordinates x^i,

$$e_i = \partial/\partial x^i, \quad \omega^i = dx^i. \tag{2}$$

Differential forms are now defined as totally antisymmetric covariant tensor fields, that is, fields in which only the ω^i appear and in which the components are totally antisymmetric. It is usual to use an antisymmetric basis written as

$$\omega^i \wedge \omega^j \wedge \omega^k \ldots = \sum_{\pi} (-1)^{\pi} \pi [\omega^i \otimes \omega^j \otimes \omega^k \ldots], \tag{3}$$

composed of antisymmetric tensor products of the ω^i. π represents a permutation of the ω^i, and the sum is over all possible permutations. The symbol $\wedge$ is called a hook or wedge product. Then a form α, say, of rank p, or p-form, may be written as

$$\alpha = \alpha_{ijk\ldots} \omega^i \wedge \omega^j \wedge \omega^k \ldots \tag{4}$$

(p factors), with sums over $i, j, k, \ldots$ (Typically the sum is written for $i < j < k \ldots$ to avoid repetition.) A 0-form is simply a function. It is common to use the natural basis and to write p-forms as sums of hook products of p of the dx^i, as

$$\beta = \beta_{ijk\ldots} dx^i \wedge dx^j \wedge dx^k \ldots \tag{5}$$

We now may work with the set of differential forms on a manifold by itself. I give a brief summary of the rules. They may be found in many references, such as paper I; a good one for mathematical physicists is Misner, Thorne, and Wheeler [4]. Vectors will be needed in defining the operations of contraction and Lie derivative.

2.1 Algebra of forms

Forms of the same rank comprise a vector space and may be added and subtracted, with coefficients as functions on the manifold. Forms may be multiplied in terms of the hook product. Multiplication satisfies a distribution rule

$$(\alpha + \beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma \tag{6}$$

and a commutation rule

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha, \tag{7}$$

where $p = \text{rank}(\alpha)$ and $q = \text{rank}(\beta)$. Thus, products of 1-forms, in particular, are antisymmetric. This implies, if the base manifold is n-dimensional, that all forms of rank greater than n vanish, since the terms would include multiples of the same 1-form, which would be zero by antisymmetry. The number of independent p-forms in n-dimensional space is $\binom{n}{p}$ and the total number of independent forms, from rank 0 to n, is 2^n.

2.2 Calculus of forms

We define the exterior derivative d as a map from p-forms to (p+1)-forms. If

$$\gamma = f dx^i \wedge dx^j \wedge \ldots \tag{8}$$

then d is defined by (sum on k)

$$d\gamma = (\partial f / \partial x^k)\, dx^k \wedge dx^i \wedge dx^j \wedge \ldots \tag{9}$$

(one simply writes $d\gamma = df \wedge dx^i \wedge dx^j \ldots$ and expands df by the chain rule.) The exterior derivative satisfies these postulates:
Linearity:

$$d(\alpha + \beta) = d\alpha + d\beta \tag{10}$$

Leibnitz rule:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{11}$$

($p = \text{rank}(\alpha)$). In particular, if f is a function,

$$d(f\alpha) = df \wedge \alpha + f d\alpha.$$

Closure rule:

$$dd\alpha = 0 \tag{12}$$

for any form α. A form β satisfying $d\beta = 0$ is said to be "closed".

In three-dimensional space, these various rules are equivalent to many familiar vector identities.

2.3 Contraction with a vector

Contraction with a vector ($\cdot$) is a map from p-forms to (p-1)-forms, defined by, if $v = v^i(\partial/\partial x^i)$,

$$v \cdot (\alpha_k dx^k) = v^i \alpha_i, \tag{13}$$

$$v \cdot (\alpha \wedge \beta) = (v \cdot \alpha) \wedge \beta + (-1)^p \alpha \wedge (v \cdot \beta), \tag{14}$$

where $p = \text{rank}(\alpha)$. Thus,

$$v \cdot dx^i = v^i. \tag{15}$$

Contractions may be added or multiplied linearly by a scalar.

2.4 Lie derivative

The Lie derivative $\pounds_v$ is a generalization of the directional derivative and requires a vector v for definition. It may be defined on tensors and geometrical objects in general, but we consider only its definition on forms here. The Lie derivative of a p-form is another p-form. If f is a function, then

$$\pounds_v\, f = v \cdot df = v^i(\partial f/\partial x^i), \tag{16}$$

$$\pounds_v\, d\alpha = d(\pounds_v\, \alpha). \tag{17}$$

Thus, $\pounds_v(x^i) = v^i$ and $\pounds_v(dx^i) = dv^i$ (dv^i is to be expanded by the chain rule.) Further identities are

$$\pounds_v(\alpha \wedge \beta) = (\pounds_v\, \alpha) \wedge \beta + \alpha \wedge (\pounds_v\, \beta) \tag{18}$$

and

$$\pounds_v\, \alpha = v \cdot d\alpha + d(v \cdot \alpha). \tag{19}$$

Applying the Lie derivative is often called *dragging* a form. If a Lie derivative is zero, the form may be said to be invariant under the transformation represented by the dragging.

3 Writing differential equations as forms

Any differential equation or a set of differential equations, ordinary or partial, may be written in terms of differential forms. The method is straightforward (see paper I). One first reduces the equation(s) to a set of first-order differential equations by introducing new variables as necessary. As an example, consider the heat equation, where subscripts indicate differentiation:

$$u_{xx} = u_t. \tag{20}$$

We define new dependent variables $z = u_x$, $w = u_t$, so that $z_t = w_x$, $z_x = w$. We consider a differentiable manifold in the five variables x, t, u, z, and w. On this manifold we introduce a set of forms by inspection. These forms are chosen such that if we (a) consider the dependent variables to be functions of the independent variables (a process called sectioning) – so that we can write their exterior derivatives in terms of the independent variables – and then (b) set the forms equal to zero (a process called annulling), we recover the original differential equations. Thus, we simply restrict the forms to the solution manifold of the differential equation(s).

For the heat equation, we first define

$$\alpha = -du + z\,dx + w\,dt. \tag{21}$$

Sectioning gives

$$\alpha = -(u_x\,dx + u_t\,dt) + z\,dx + w\,dt$$

and annulling gives $z = u_x$, $w = u_t$ back again. Now

$$d\alpha = dz\,dx + dw\,dt \tag{22}$$

where the hook product $\wedge$ is to be understood. Sectioning gives $d\alpha = (z_x\,dx + z_t\,dt)\,dx + (w_x\,dx + w_t\,dt)\,dt = z_t\,dt\,dx + w_x\,dx\,dt = (w_x - z_t)\,dx\,dt$, since $dx\,dx = dt\,dt = 0$ and $dt\,dx = -dx\,dt$, and annulling gives $w_x = z_t$ again. Finally, we write

$$\beta = dz\,dt - w\,dx\,dt = (z_x\,dx + z_t\,dt)\,dt - w\,dx\,dt, \tag{23}$$

giving $w = z_x$ when β is annulled. Thus, the set $\{\alpha, d\alpha, \beta\}$ – which we call the ideal I of forms – represents the original equation(s) when sectioned and annulled.

The forms as given are not unique. For example, we may construct an alternate set simply from $z = u_x$ and $z_x = u_t$, yielding the forms $\gamma = du\,dt - z\,dx\,dt$ and $\delta = dz\,dt + du\,dx$ and giving an alternate ideal $I' = \{\gamma, \delta\}$. These ideals should be closed, and they are: $dI \subset I$ and $dI' \subset I'$, since $d\beta = d\alpha \wedge dx$, $d\delta = 0$, and $d\gamma = \delta \wedge dx$.

4 Invariance of the differential equations

It is now simple to treat the invariance of a set of differential equations. A set of equations is invariant if a transformation leaves the equations still satisfied, provided that the original equations are satisfied. In the formalism we have introduced, this is easily stated: the Lie derivative of forms in the ideal must lie in the ideal:

$$\pounds_v I \subset I. \tag{24}$$

Then if the basis forms in the ideal are annulled, the transformed equations are also annulled. In practice, this means simply that the Lie derivative of each of the (basis) forms in I is a linear combination of the forms in I. For the heat equation, by using the ideal I, we get the equations

$$\pounds_v \alpha = \lambda_1 \alpha, \tag{25}$$

which gives

$$\pounds_v d\alpha = d\lambda_1 \wedge \alpha + \lambda_1 d\alpha, \tag{26}$$

so that $\pounds_v d\alpha$ is automatically in the ideal, and

$$\pounds_v \beta = \lambda_2 \beta + \lambda_3 d\alpha + \sigma \wedge \alpha, \tag{27}$$

where the λ_i and σ are multipliers to be eliminated. The λ_i are functions (0-forms) and σ is an arbitrary 1-form. The resulting equations, after this elimination, are simply the usual determining equations for the symmetry generators, the components v^i of v – which was called the *isovector* in paper I.

For the example considered, we consider first Eq. (25). It is simplified by putting

$$H = v \cdot \alpha = -v^u + zv^x + wv^t, \tag{28}$$

where the superscripts indicate components of v. H is to be considered a function of all variables and is as yet unspecified. Then, by Eq. (19),

$$\pounds_v \alpha = v \cdot d\alpha + d(v \cdot \alpha) = \lambda_1 \alpha \tag{29}$$

or

$$v^z dx - v^x dz + v^w dt - v^t dw + dH = \lambda_1(-du + z\,dx + w\,dt). \tag{30}$$

dH is to be expanded by the chain rule. We now set the coefficients of dx, dt, etc., to zero. From the coefficient of du, we get $\lambda_1 = -H_u$. The other coefficients give, after substitution for λ_1,

$$\begin{aligned} v^x &= H_z, \quad v^z = -H_x - zH_u \\ v^t &= H_w, \quad v^w = -H_t - wH_u. \end{aligned} \tag{31}$$

We note from Eq. (28) and Eq. (31) that

$$v^u = -H + zH_z + wH_w. \tag{32}$$

We do not need to consider $d\alpha$ separately, as noted above.

We now expand Eq. (27), using the rules for Lie differentiation given in Eqs. (16) through (18), and also substitute for the forms on the right-hand side. We get

$$\begin{aligned} dv^z\,dt &+ dz\,dv^t - v^w\,dx\,dt - w\,dv^x\,dt - w\,dx\,dv^t \\ &= \lambda_2(dz\,dt - w\,dx\,dt) + \lambda_3(dz\,dx + dw\,dt) \\ &+ (\sigma_1\,dx + \sigma_2\,dt + \sigma_3\,dz + \sigma_4\,dw) \wedge (-du + z\,dx + w\,dt), \end{aligned} \tag{33}$$

in which the σ_i (which are the components of σ), along with λ_2 and λ_3, are multipliers to be eliminated. We do not include a du in σ because it can be replaced by α.

We now expand the dv^i by the chain rule and set the coefficients of all ten possible basis 2-forms $(dx\,dt, dx\,du, \ldots, dz\,dw)$ equal to zero. After elimination of the multipliers, we get the determining equations, where commas indicate differentiation,

$$\begin{aligned} {v^t}_{,w} &= 0 \\ {v^t}_{,x} + w{v^t}_{,z} + z{v^t}_{,w} &= {v^z}_{,w} - w{v^x}_{,w} \\ {v^z}_{,x} - v^w - w{v^x}_{,x} &= -w{v^z}_{,z} + w^2{v^x}_{,z} - z{v^z}_{,u} + wz{v^x}_{,u}\,. \end{aligned} \tag{34}$$

Solution of these equations together with Eqs. (31) and (32) now gives the usual symmetry group, or *isogroup*, for the one-dimensional heat equation.

Invariant variables are now found in the usual way by solving the equation(s)

$$\frac{dt}{v^t} = \frac{dx}{v^x} = \frac{du}{v^u}. \tag{35}$$

Geometrically, this gives the characteristics for the first-order differential equation $v \cdot \alpha = 0$, in which we restore $z = u_x$, $w = u_t$.

Mathematically, this is equivalent to the traditional method of finding the invariances of differential equations. So why spend time learning a new method?

(1) It is easy to apply. One reduces the set of differential equations to first-order equations by defining appropriate variables, writes them by inspection as differential forms, writes out the Lie derivative equations and sets the coefficients of the various basis forms to zero, and eliminates the multipliers. Calculations may be long because there may be many equations and many multipliers to eliminate, but they are very straightforward. In some cases many terms in the expansion drop out because of the antisymmetry of 1-forms, simplifying the treatment. This happens in the second example discussed below.

(2) One may get some geometrical insight into the process because of the inherently geometrical nature of forms. As an example, it leads immediately to the invariant surface condition used in finding nonclassical symmetries(see paper I, also [5]). For treatments that stress this geometrical nature, see Ref. [4] and [6]. (Forms may also be used in other, related contexts, such as searching for Bäcklund transformations, conservation laws, etc. [7])

(3) It allows the possibility for the independent variable components of the isovector v to be functions of the dependent variables. Usually these components of v are automatically assumed to be functions of only the independent variables, usually without loss of generality. However, in the case of a hodograph transformation, for example, those components do depend on the dependent variables.

(4) This method is nicely adaptable to computer algebraic calculations. As an example, Paul Kersten [8] developed a very nice treatment many years ago, for use in REDUCE. It enables one to set up the forms, find the determining equations, and then interactively work on their solution. Ben T. Langton [9], a Ph.D. student of Edward Fackerell's at the University of Sydney, is just finishing work on a modification of that technique which will improve its usefulness. There is also a MAPLE code which uses this technique [10]. See a brief discussion of these by Ibragimov [11].

(5) It is easy to make Ansatzen in the variable dependence of the isovector components, simply by specifying it when one writes out their exterior derivatives in the expansion of the Lie derivatives.

5 Short wave gas dynamic equation; symmetry reduction

This equation was cited by Fushchych in his Ansatz '95 talk [1] at the last Kyiv conference. It is his Eq. (4.4),

$$2u_{tx} - 2(2x + u_x)u_{xx} + u_{yy} + 2\lambda u_x = 0. \tag{36}$$

Only a brief mention of it, with a simple Ansatz, was given at that time. We give a longer treatment here.

We define new variables

$$w = u_x,\ z = u_y; \tag{37}$$

then Eq. (36) takes the form

$$2w_t - (4x + 2w)w_x + z_y + 2\lambda w = 0. \tag{38}$$

We could write a 1-form here, to be annulled:

$$-du + w\,dx + u_t\,dt + z\,dy,$$

but it involves u_t, which is not one of our variables. So we write a 2-form by multiplying this 1-form by dt in order to remove the unwanted term:

$$\alpha = (-du + w\,dx + z\,dy)\,dt. \tag{39}$$

Then $d\alpha$ is a 3-form:

$$d\alpha = (dw\,dx + dz\,dy)\,dt \tag{40}$$

and the equation itself is expressed as the 3-form

$$\beta = 2dw\,dx\,dy + (4x + 2w)dw\,dt\,dy + dz\,dt\,dx + 2\lambda w\,dt\,dx\,dy. \tag{41}$$

We note that $d\beta$ is proportional to $d\alpha$, so that the ideal $\{\alpha, d\alpha, \beta\}$ is closed.

We now consider

$$\pounds_v\,\alpha = \lambda\alpha \tag{42}$$

(β and $d\alpha$ are 3-forms and so are not included on the right-hand side.) In expansion of this equation, most terms involve only the t-component of the isovector, v^t, and they show simply that $v^t = K(t)$, a function of t only. The other terms involving dt then provide the only other useful information; elimination of the multiplier λ leaves four equations. These are conveniently written by defining $H = v^u - wv^x - zv^y$. Then they become

$$\begin{aligned} v^x &= -H_w, \quad v^z = H_y + zH^u, \\ v^y &= -H_z, \quad v^w = H_x + wH_u. \end{aligned} \tag{43}$$

v^u then can be written as

$$v^u = H - wH_w - zH_z. \tag{44}$$

We now note that $\pounds_v\,d\alpha = d\lambda \wedge \alpha + \lambda d\alpha$, thus being in the ideal, so we need not write a separate equation for it.

The remaining equation is

$$\pounds_v \beta = \nu\beta + \sigma d\alpha + \omega \wedge \alpha, \tag{45}$$

where ν and σ are 0-form multipliers and ω is a 1-form multiplier. We now write out the terms involving all 20 possible basis 3-forms and eliminate the multipliers. We see immediately that v^w and v^y are independent of u and z and that v^x is independent of u, z and w. Use of the equations shows that

$$H = -wA(x,y,t) - zB(y,t) + C(x,y,t,u) \tag{46}$$

with $v^x = A$ and $v^y = B$; A, B, and C are as yet undetermined. Eventually we find expressions for the generators, with two remaining equations, which are polynomials in z and w. We equate the polynomial coefficients to zero and get these expressions for the generators:

$$\begin{aligned} v^t &= K, \\ v^x &= xJ - (y^2/4)(K'' + J') - yL' - N', \\ v^y &= (y/2)(K' + J) + L, \\ v^u &= u(2J - K') + G, \\ v^w &= w(J - K') + G_x, \\ v^z &= (3z/2)(J - K') + (wy/2)(K'' + J') + wL' + G_y, \end{aligned} \tag{47}$$

where J, K, L and N are functions of t and G is a function of x, y, and t. Primes indicate d/dt. We also have

$$\begin{aligned} J' &= (1/3)K'' - (4/3)(\lambda + 2)K', \\ 0 &= 2G_{xt} - 4xG_{xx} + G_{yy} + 2\lambda G_x, \\ G_x &= N'' + 2N' - x(2K' + J') + y(L'' + 2L') + (y^2/4)(K''' + J'' + 2K'' + 2J'), \end{aligned} \tag{48}$$

From these equations, we may solve for an explicit expression for G_{yy}. Then the consistency of $(G_{yy})_{,x}$ and $(G_x)_{,yy}$ yields the equation

$$(\lambda - 1)(\lambda - 4)K' = 0. \tag{49}$$

Thus, if $\lambda = 1$ or $\lambda = 4$, we may take K to be an arbitrary function of t.

We integrate the first of Eqs. (48) to get

$$J = (1/3)K' - (4/9)(\lambda + 2)K + 4a, \tag{50}$$

where a is a constant. We also define a new function, $\alpha(t)$, by

$$\frac{\alpha'}{\alpha} = -\frac{J + K'}{2K} = -\frac{2K'}{3K} + \frac{2\lambda + 4}{9} - \frac{2a}{K}. \tag{51}$$

We can also integrate for G, but the expression is long and we do not write it here.

We now find the invariant independent variables for the system. We have, as in the manner of Eq. (35),

$$\frac{dy}{dt} = \frac{v^y}{v^t} = -\frac{y\alpha'}{\alpha} + \frac{L}{K}. \tag{52}$$

Solution gives an invariant variable

$$\xi = y\alpha + \beta \tag{53}$$

where

$$\beta = -\int K^{-1} L\alpha \, dt. \tag{54}$$

From

$$\frac{dx}{dt} = \frac{v^x}{v^t} = -(\frac{2\alpha'}{\alpha} + \frac{K'}{K})x + \frac{y^2}{2K}(\frac{K\alpha'}{\alpha})' - \frac{yL' + N'}{K}, \tag{55}$$

we get the invariant variable

$$\eta = K\alpha^2 x - (1/2)K\alpha\alpha' y^2 - K\alpha\beta' y - \gamma, \tag{56}$$

where

$$\gamma = \int (-N'\alpha^2 + K\beta'^2) \, dt. \tag{57}$$

We now can use ξ and η as given in terms of K, α, β, and γ without ever referring back to L, N, etc.

From

$$\frac{du}{dt} = \frac{v^u}{v^t} = -(\frac{3K'}{K} + \frac{4\alpha'}{\alpha})u + \frac{G}{K}, \tag{58}$$

we get the invariant variable

$$F = K^3\alpha^4 u - \int K^2\alpha^4 G \, dt. \tag{59}$$

To get a solution of the original differential equation Eq. (36), we assume $F = F(\xi, \eta)$. To evaluate the integral in Eq. (59), we need to write out G, substitute for x and y in terms of ξ and η, do the t integrals while keeping ξ and η constant, and then reexpress ξ and η in terms of x and y. This is an extremely complicated procedure. One may simplify it, however, by noting, by inspection, that the completed integral will be a polynomial in ξ up to ξ^4, also with terms $\eta, \eta\xi, \eta\xi^2$, and η^2. If these are replaced by their expressions in x and y, these will yield terms in y up to y^4, also x, xy, xy^2, and x^2. So we write

$$\begin{aligned} u &= p_1 + p_2 y + p_3 y^2 + p_4 y^3 + p_5 y^4 + p_6 x \\ &\quad + p_7 xy + p_8 xy^2 + p_9 x^2 + K^{-3}\alpha^{-4} F(\xi, \eta), \end{aligned} \tag{60}$$

where the p_i are as yet undetermined functions of t, and substitute into Eq. (36). By using the expressions for ξ and η (Eqs. (53) and (56)), identifying coefficients, redefining F to include some polynomial terms in order to simplify the equation, we finally get–in which b is a new constant and a was defined in Eq. (50):

$$F_{\xi\xi} - 2F_\eta F_{\eta\eta} + 18aF_\eta + 4b\eta = 0, \tag{61}$$

$$(\lambda - 1)(\lambda - 4)K^2 = 9b + 81a^2, \tag{62}$$

(consistent with Eq. (49)), and

$$\begin{aligned}
p_9 &= -1 + \alpha'/\alpha + K'/(2K), \\
p_8 &= -(K\alpha)'/(2K\alpha), \\
p_7 &= -(K\beta')'/(K\alpha), \\
p_6 &= (\beta'^2/2 - \gamma'/K)/\alpha^2, \\
p_5 &= -(1/6)(p_8' + \lambda p_8 - 2p_8 p_9 + b\alpha'/(K^2\alpha)), \\
p_4 &= -(1/3)(p_7' + \lambda p_7 - 2p_7 p_9 + 2b\beta'/(K^2\alpha)), \\
p_3 &= -(p_6' + \lambda p_6 - 2p_6 p_9 + 2b\gamma/(K^3\alpha^2)),
\end{aligned} \tag{63}$$

and p_1 and p_2 are arbitrary. These are expressed in terms of the functions K, α, β, and γ that occur in the invariant variables ξ and η.

6 Nonlinear heat equation with additional condition

This equation also was cited by Fushchych in Ansatz '95 [1], Eqs. (3.29) and (3.30), which are given here:

$$u_t + \nabla \cdot [f(u)\nabla u] = 0, \tag{64}$$

$$u_t + (2M(u))^{-1}(\nabla u)^2 = 0. \tag{65}$$

In Theorem 5 in that treatment, he showed that the first equation (here, Eq. (64)) is conditionally invariant under Galilei operators if the second equation (Eq. (65)) holds. Here we start with Eqs. (64) and (65) and study their joint invariance, a different problem.

If we define variables

$$q = u_x,\ r = u_y,\ s = u_z, \tag{66}$$

we can write Eqs. (64) and (65) as

$$u_t = -(2M)^{-1}(q^2 + r^2 + s^2) \tag{67}$$

and

$$-(2M)^{-1}(q^2 + r^2 + s^2) + (fq)_x + (fr)_y + (fs)_z = 0. \tag{68}$$

Now we can write forms as follows:

$$\alpha = -du - (2M)^{-1}(q^2 + r^2 + s^2)\,dt + q\,dx + r\,dy + s\,dz, \tag{69}$$

$$\begin{aligned}
\beta &= d\alpha + (M'/2M^2)(q^2 + r^2 + s^2)\alpha\,dt \\
&= (M'/2M^2)(q^2 + r^2 + s^2)(q\,dx + r\,dy + s\,dz)\,dt \\
&\quad -(1/M)(qdq + rdr + sds)\,dt + dq\,dx + dr\,dy + ds\,dz
\end{aligned} \tag{70}$$

which combination gets rid of the du terms, and

$$\gamma = g(u)(q^2 + r^2 + s^2)dx\,dy\,dz\,dt + (dq\,dy\,dz + dr\,dz\,dx + ds\,dx\,dy)\,dt, \tag{71}$$

where

$$g(u) = f^{-1}(f' - (2M)^{-1}) \tag{72}$$

and in all of which a prime indicates d/du. Now we assume that v^x, v^y, v^z, and v^t are functions of x, y, z, and t; v^u is a function of u, x, y, z, and t; and that v^q, v^r, and v^s are linear in q, r, and s (with additional terms independent of those variables), and with coefficients which are functions of u, x, y, z, and t.

We now consider

$$\pounds_v\, \alpha = \lambda\alpha. \tag{73}$$

Expansion of this equation, using the conditions stated above, shows that $\lambda = v^u{}_{,u}$; it has terms up to the quadratic in q, r, and s. Setting the coefficients of the terms in these variables equal to zero and solving yields the expressions

$$\begin{aligned} v^t &= \eta(t), \\ v^x &= \xi x + \sigma_3 y - \sigma_2 z + \zeta_1 t + \nu_1, \\ v^y &= \xi y + \sigma_1 z - \sigma_3 x + \zeta_2 t + \nu_2, \\ v^z &= \xi z + \sigma_2 x - \sigma_1 y + \zeta_3 t + \nu_3, \end{aligned} \tag{74}$$

where the σ_i, ζ_i, and ν_i are constants, ξ is linear in t, and $\eta(t)$ is quadratic in t. v^q, v^r, v^s, and v^u are quadratic in x, y, and z. The u dependence is not yet entirely determined because $M(u)$ and $f(u)$ have not been specified.

From Eq. (73) we have, as before, $\pounds_v\, d\alpha = \lambda d\alpha + d\lambda \wedge \alpha$, automatically in the ideal.

The remaining calculation is the determination of $\pounds_v\, \gamma$ and setting it equal to a linear combination of $\alpha, d\alpha$ (or β), and γ, and to eliminate the multipliers. This appears to be a formidable task; however, by use of the assumptions and information we already have, it turns out to be surprisingly easy.

We expand $\pounds_v\, \gamma$ and substitute the values for the v^i that we already have. After this calculation, we find that there are terms proportional to $dx\, dy\, dz\, dt$, $du\, dy\, dz\, dt$, $du\, dz\, dx\, dt$, $du\, dx\, dy\, dt$, and $(dq\, dy\, dz + dr\, dz\, dx + ds\, dx\, dy)\, dt$. We substitute for the latter sum of three terms from Eq. (71), thus giving a term proportional to γ and one proportional to $dx\, dy\, dz\, dt$. In the three terms involving du, we substitute for du from Eq. (69). It is now seen that $\pounds_v\, \gamma$ is the sum of three terms: one proportional to γ, one proportional to α, and one proportional to $dx\, dy\, dz\, dt$. Thus, $\pounds_v\, \gamma$ is already in the desired form except for the last term! But this last term cannot be represented as a sum of terms in α, β, and γ, as is seen by inspection. Hence its coefficient must vanish, and that condition provides the remaining equations for the generators v^i.

The coefficient has terms proportional to $q^2 + r^2 + s^2$, q, r, s, and a term independent of those variables. Setting this last term equal to zero shows that $\xi_t = 0$. The q, r, and s terms give equations which can be written collectively as

$$\zeta_i(M' + gM) = 0, \quad i = 1, 2, 3. \tag{75}$$

The coefficient of $q^2 + r^2 + s^2$ gives a relation among the various functions of u. We may now summarize the results.

In addition to Eqs. (74), with ξ now constant, we get

$$\eta(t) = at + b, \tag{76}$$

where a and b are constant, and

$$\begin{aligned} v^u &= M(\zeta_1 x + \zeta_2 y + \zeta_3 z) + h(u), \\ v^q &= qJ + \sigma_3 r - \sigma_2 s + \zeta_1 M, \\ v^r &= rJ + \sigma_1 s - \sigma_3 q + \zeta_2 M, \\ v^s &= sJ + \sigma_2 q - \sigma_1 r + \zeta_3 M, \end{aligned} \tag{77}$$

where

$$J = M'(\zeta_1 x + \zeta_2 y + \zeta_3 z) + h' - \xi. \tag{78}$$

The quadratic dependence on x, y, and z has dropped out because of the condition $\xi_t = 0$. We also have these relations among the functions of u: Eqs. (72), (75) and the further equations

$$h' = hM'/M + 2\xi - a, \tag{79}$$

$$[hM^{-1}(M' + gM)]' = 0. \tag{80}$$

There are now two cases.

Case I. If

$$M' + gM = 0, \tag{81}$$

then

$$fM = u/2 + c, \tag{82}$$

where c is a constant. Integration for h from Eq. (79) now gives

$$h = kM + (2\xi - a)M \int M^{-1} du, \tag{83}$$

where k is a constant.

Case II. If $M' + gM \neq 0$, then all $\zeta_i = 0$. We may write h by Eq. (83), but now h must satisfy the additional condition Eq. (80).

Fushchych's Theorem 5 now follows: if $M = u/(2f)$, the original equations are invariant under a Galilean transformation

$$G_i = t\partial_i + Mx^i \partial_u. \tag{84}$$

The general operator is then $G = \sum_i \zeta_i G_i$, giving $v^i = \zeta_i t$ and $v^u = M(\zeta_1 x + \zeta_2 y + \zeta_3 z)$, a possible choice of generators from the above calculation. From Eq. (82) we see that his conclusion thus holds in a slightly more general case, when the constant c is not zero.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 34–47.

Implicit and Parabolic Ansatzes: Some New Ansatzes for Old Equations

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Abstract

We give a survey of some results on new types of solutions for partial differential equations. First, we describe the method of implicit ansatzes, which gives equations for functions which define implicitly solutions of some partial differential equations. In particular, we find that the family of eikonal equations (in different geometries) has the special property that the equations for implicit ansatzes are also eikonal equations. We also find that the eikonal equation defines implicitly solutions of the Hamilton-Jacobi equation. Parabolic ansatzes are ansatzes which reduce hyperbolic equations to parabolic ones (or to a Schrödinger equation). Their uses in obtaining new types of solutions for equations invariant under $AO(p, q)$ are described. We also give some results on conformally invariant nonlinear wave equations and describe some exact solutions of a conformally invariant nonlinear Schrödinger equation.

1 Introduction

In this talk, I would like to present some results obtained during the past few years in my collaboration with Willy Fushchych and some of his students. The basic themes here are *ansatz* and *symmetry algebras* for partial differential equations.

I wrote this talk after Wilhelm Fushchych' untimely death, but the results I give here were obtained jointly or as a direct result of our collaboration, so it is only right that he appears as an author.

In 1993/1994 during his visits to Linköping and my visits to Kyiv, we managed, amongst other things, to do two things: use light-cone variables to construct new solutions of some hyperbolic equations in terms of solutions of the Schrödinger or heat equations; and to develop the germ of new variation on finding ansatzes. This last piece is an indication of work in progress and it is published here for the first time. I shall begin this talk with this topic first.

2 The method of implicit ansatzes

2.1 The wave and heat equations.

Given an equation for one unknown real function (the dependent variable), u, say, and several independent ("geometric") variables, the usual approach, even in terms of symmetries, is to attempt to find ansatzes for u *explicitly*. What we asked was the following: why not try and give u *implicitly*? This means the following: look for some function $\phi(x, u)$ so that $\phi(x, u) = C$ defines u implicitly, where x represents the geometric variables and C is a constant.This is evidently natural, especially if you are used to calculating symmetry groups, because one then has to treat u on the same footing as x. If we assume, at least locally, that $\phi_u(x, u) \neq 0$, where $\phi_u = \partial\phi/\partial u$, then the implicit function theorem tells us that $\phi(x, u) = C$ defines u implicitly as a function of x, for some neighbourhood of (x, u) with $\phi_u(x, u) \neq 0$, and that $u_\mu = -\dfrac{\phi_\mu}{\phi_u}$, where $\phi_\mu = \dfrac{\partial\phi}{\partial x^\mu}$. Higher derivatives of u are then obtained by applying the correct amount of total derivatives.

The wave equation $\Box u = F(u)$ becomes

$$\phi_u^2 \Box\phi = 2\phi_u\phi_\mu\phi_{\mu u} - \phi_\mu\phi_\mu\phi_{uu} - \phi_u^3 F(u)$$

or

$$\Box\phi = \partial_u \left(\frac{\phi_\mu\phi_\mu}{\phi_u}\right) - \phi_u F(u).$$

This is quite a nonlinear equation. It has exactly the same symmetry algebra as the equation $\Box u = F(u)$, except that the parameters are now arbitrary functions of ϕ. Finding exact solutions of this equation will give u implicitly. Of course, one is entitled to ask what advantages are of this way of thinking. Certainly, it has the disadvantage of making linear equations into very nonlinear ones. The symmetry is not improved in any dramatic way that is exploitable (such as giving a conformally-invariant equation starting from a merely Poincaré invariant one). It can be advantageous when it comes to adding certain conditions. For instance, if one investigates the system

$$\Box u = 0, \quad u_\mu u_\mu = 0,$$

we find that $u_\mu u_\mu = 0$ goes over into $\phi_\mu\phi_\mu = 0$ and the system then becomes

$$\Box\phi = 0, \quad \phi_\mu\phi_\mu = 0.$$

In terms of ordinary Lie ansatzes, this is not an improvement. However, it is not difficult to see that we can make certain non-Lie ansatzes of the anti-reduction type: allow ϕ to be a polynomial in the variable u with coefficients being functions of x. For instance, assume ϕ is a quintic in u: $\phi = Au^5 + Bu + C$. Then we will have the coupled system

$$\Box A = 0, \ \ \Box B = 0, \ \ \Box C = 0,$$
$$A_\mu A_\mu = B_\mu B_\mu = C_\mu C_\mu = A_\mu B_\mu = A_\mu C_\mu = B_\mu C_\mu = 0.$$

Solutions of *this* system can be obtained using Lie symmetries. The exact solutions of

$$\Box u = 0, \quad u_\mu u_\mu = 0$$

are then obtained in an implicit form which is unobtainable by Lie symmetry analysis alone.

Similarly, we have the system

$$\Box u = 0, \quad u_\mu u_\mu = 1$$

which is transformed into

$$\Box\phi = \phi_{uu}, \quad \phi_\mu\phi_\mu = \phi_u^2$$

or

$$\Box_5\phi = 0, \quad \phi_A\phi_A = 0,$$

where $\Box_5 = \Box - \partial_u^2$ and A is summed from 0 to 4.

It is evident, however, that the extension of this method to a system of equations is complicated to say the least, and I only say that we have not contemplated going beyond the present case of just one unknown function.

We can treat the heat equation $u_t = \triangle u$ in the same way: the equation for the surface ϕ is

$$\phi_t = \triangle\phi - \frac{\partial}{\partial u}\left(\frac{\nabla\phi\cdot\nabla\phi}{\phi_u}\right).$$

If we now add the condition $\phi_u = \nabla\phi\cdot\nabla\phi$, then we obtain the system

$$\phi_t = \triangle\phi, \quad \phi_u = \nabla\phi\cdot\nabla\phi$$

so that ϕ is a solution to both the heat equation and the Hamilton-Jacobi equation, but with different propagation parameters.

If we, instead, add the condition $\phi_u^2 = \nabla\phi\cdot\nabla\phi$, we obtain the system

$$\phi_t = \triangle\phi - \phi_{uu}, \quad \phi_u^2 = \nabla\phi\cdot\nabla\phi.$$

The first of these is a new type of equation: it is a relativistic heat equation with a very large symmetry algebra which contains the Lorentz group as well as Galilei type boosts; the second equation is just the eikonal equation. The system is evidently invariant under the Lorentz group acting in the space parametrized by $(x^1, \ldots, x^n, u)$, and this is a great improvement in symmetry on the original heat equation.

It follows from this that we can obtain solutions to the heat equation using Lorentz-invariant ansatzes, albeit through a modified equation.

2.2 Eikonal equations.

Another use of this approach is seen in the following. First, let us note that there are three types of the eikonal equation

$$u_\mu u_\mu = \lambda,$$

namely the time-like eikonal equation when $\lambda = 1$, the space-like eikonal one when $\lambda = -1$, and the isotropic eikonal one when $\lambda = 0$. Representing these implicitly, we find that the time-like eikonal equation in $1+n$ time-space

$$u_\mu u_\mu = 1$$

goes over into the isotropic eikonal one in a space with the metric $(1, \underbrace{-1, \ldots, -1}_{n+1})$

$$\phi_\mu\phi_\mu = \phi_u^2.$$

The space-like eikonal equation

$$u_\mu u_\mu = -1$$

goes over into the isotropic eikonal one in a space with the metric $(1, 1, \underbrace{-1, \ldots, -1}_{n})$

$$\phi_\mu \phi_\mu = -\phi_u^2$$

whereas

$$u_\mu u_\mu = 0$$

goes over into

$$\phi_\mu \phi_\mu = 0.$$

Thus, we see that, from solutions of the isotropic eikonal equation, we can construct solutions of time- and space-like eikonal ones in a space of one dimension less. We also see the importance of studying equations in higher dimensions, in particular in spaces with the relativity groups $SO(1,4)$ and $SO(2,3)$.

It is also possible to use the isotropic eikonal to construct solutions of the Hamilton-Jacobi equation in $1+n$ dimensions

$$u_t + (\nabla u)^2 = 0$$

which goes over into

$$\phi_u \phi_t = (\nabla \phi)^2$$

and this equation can be written as

$$\left(\frac{\phi_u + \phi_t}{2}\right)^2 - \left(\frac{\phi_u - \phi_t}{2}\right)^2 = (\nabla \phi)^2$$

which, in turn, can be written as

$$g^{AB} \phi_A \phi_B = 0$$

with $A, B = 0, 1, \ldots, n+1$, $g^{AB} = \operatorname{diag}(1, -1, \ldots, -1)$ and

$$\phi_0 = \frac{\phi_u + \phi_t}{2}, \qquad \phi_{n+1} = \frac{\phi_u - \phi_t}{2}.$$

It is known that the isotropic eikonal and the Hamilton-Jacobi equations have the conformal algebra as a symmetry algebra (see [15]), and here we see the reason why this is so. It is not difficult to see that we can recover the Hamilton-Jacobi equation from the isotropic eikonal equation on reversing this procedure.

This procedure of reversal is extremely useful for hyperbolic equations of second order. As an elementary example, let us take the free wave equation for one real function u in $3+1$ space-time:

$$\partial_0^2 u = \partial_1^2 u + \partial_2^2 u + \partial_3^2 u$$

and write it now as

$$(\partial_0 + \partial_3)(\partial_0 - \partial_3)u = \partial_1^2 u + \partial_2^2 u$$

or

$$\partial_\sigma \partial_\tau u = \partial_1^2 u + \partial_2^2 u,$$

where $\sigma = \dfrac{x^0 - x^3}{2}$, $\tau = \dfrac{x^0 + x^3}{2}$. Now assume $u = e^\sigma \Psi(\tau, x^1, x^2)$. With this assumption, we find

$$\partial_\tau \Psi = \left(\partial_1^2 + \partial_2^2\right) \Psi$$

which is the heat equation. Thus, we can obtain a class of solutions of the free wave equation from solutions of the free heat equation. This was shown in [1]. The ansatz taken here seems quite arbitrary, but we were able to construct it using Lie point symmetries of the free wave equation. A similar ansatz gives a reduction of the free complex wave equation to the free Schrödinger equation. We have not found a way of reversing this procedure, to obtain the free wave equation from the free heat or Schrödinger equations. The following section gives a brief description of this work.

3 Parabolic ansatzes for hyperbolic equations: light-cone coordinates and reduction to the heat and Schrödinger equations

Although it is possible to proceed directly with the ansatz just made to give a reduction of the wave equation to the Schrödinger equation, it is useful to put it into perspective using symmetries: this will show that the ansatz can be constructed by the use of infinitesimal symmetry operators. To this end, we quote two results:

Theorem 1 *The maximal Lie point symmetry algebra of the equation*

$$\Box u = m^2 u,$$

where u is a real function, has the basis

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu$$

when $m \neq 0$, and

$$P_\mu = \partial_\mu, \quad I = u\partial_u, \quad J_{\mu\nu} = x_\mu \partial_\nu - x_\nu \partial_\mu, \quad D = x^\mu \partial_\mu, \quad K_\mu = 2x_\mu D - x^2 \partial_\mu - 2x_\mu u \partial_u$$

when $m = 0$, where

$$\partial_u = \frac{\partial}{\partial u}, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}, \quad x_\mu = g_{\mu\nu} x^\nu, \quad g_{\mu\nu} = \text{diag}(1, -1, \ldots, -1) \quad \mu,\ \nu = 0, 1, 2, \ldots, n$$

We notice that in both cases ($m = 0,\ m \neq 0$), the equation is invariant under the operator I, and is consequently invariant under $\alpha^\mu \partial_\mu + kI$ for all real constants k and real, constant four-vectors α. We choose a hybrid tetradic basis of the Minkowski space: α : $\alpha^\mu \alpha_\mu = 0$; ϵ : $\epsilon^\mu \epsilon_\mu = 0$; β : $\beta^\mu \beta_\mu = -1$; δ : $\delta^\mu \delta_\mu = -1$; and $\alpha^\mu \epsilon_\mu = 1,\ \alpha^\mu \beta_\mu = \alpha^\mu \delta_\mu = \epsilon^\mu \beta_\mu = \epsilon^\mu \delta_\mu 0$. We could take, for instance, $\alpha = \frac{1}{\sqrt{2}}(1,0,0,1),\ \epsilon = \frac{1}{\sqrt{2}}(1,0,0,-1),\ \beta = (0,1,0,0),\ \delta = (0,0,1,0)$. Then the invariance condition (the so-called invariant-surface condition),

$$(\alpha^\mu \partial_\mu + kI)u = 0,$$

gives the Lagrangian system

$$\frac{dx^\mu}{\alpha^\mu} = \frac{du}{ku}$$

which can be written as

$$\frac{d(\alpha x)}{0} = \frac{d(\beta x)}{0} = \frac{d(\delta x)}{0} = \frac{d(\epsilon x)}{1} = \frac{du}{ku}.$$

Integrating this gives us the general integral of motion of this system

$$u - e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

and, on setting this equal to zero, this gives us the ansatz

$$u = e^{k(\epsilon x)}\Phi(\alpha x, \beta x, \delta x).$$

Denoting $\tau = \alpha x,\ y_1 = \beta x,\ y_2 = \delta x$, we obtain, on substituting into the equation $\Box u = m^2 u$,

$$2k\partial_\tau \Phi = \triangle\Phi + m^2\Phi,$$

where $\triangle = \frac{\partial^2}{\partial_{y_1^2}} + \frac{\partial^2}{\partial_{y_2^2}}$. This is just the heat equation (we can gauge away the linear term by setting $\Phi = e^{\frac{m^2\tau}{2k}}\Psi$). The solutions of the wave equation we obtain in this way are given in [1].

The second result is the following:

Theorem 2 *The Lie point symmetry algebra of the equation*

$$\Box\Psi + \lambda F(|\Psi|)\Psi = 0$$

has basis vector fields as follows:
(i) when $F(|\Psi|) = \text{ const}\,|\Psi|^2$

$$\partial_\mu,\ \ J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\ \ K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right)$$
$$D = x^\nu\partial_\nu - \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

where $x^2 = x_\mu x^\mu$.

(ii) when $F(|\Psi|) = \text{ const}\,|\Psi|^k$, $k \neq 0, 2$*:*

$$\partial_\mu,\ \ J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\ \ D_{(k)} = x^\nu\partial_\nu - \frac{2}{k}\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iii) when $F(|\Psi|) \neq \text{ const}\,|\Psi|^k$ *for any* k*, but* $\dot{F} \neq 0$*:*

$$\partial_\mu,\ \ J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \quad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right).$$

(iv) when $F(|\Psi|) = \text{ const} \neq 0$*:*

$$\partial_\mu,\ \ J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu, \qquad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = F\Psi$*.*
(v) when $F(|\Psi|) = 0$*:*

$$\partial_\mu,\ \ J_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu,\ \ K_\mu = 2x_\mu x^\nu\partial_\nu - x^2\partial_\mu - 2x_\mu\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

$$D = x^\mu\partial_\mu, \qquad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$

$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}}, \quad B\partial_\Psi,$$

where B *is an arbitrary solution of* $\Box\Psi = 0$*.*

In this result, we see that in all cases we have $M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right)$ as a symmetry operator. We can obtain the ansatz

$$\Psi = e^{ik(\epsilon x)}\Phi(\alpha x, \beta x, \delta x)$$

in the same way as for the real wave equation, using M in place of I. However, now we have an improvement in that our complex wave equation may have a nonlinear term which is invariant under M (this is not the case for I). Putting the ansatz into the equation gives us a nonlinear Schrödinger equation:

$$i\partial_\tau\Phi = -\triangle\Phi + \lambda F(|\Phi|)\Phi$$

when $k = -1/2$. Solutions of the hyperbolic equation which this nonlinear Schrödinger equation gives is described in [2] (but it does not give solutions of the free Schrödinger equation).

The above two results show that one can obtain ansatzes (using symmetries) to reduce some hyperbolic equations to the heat or Schrödinger equations. The more interesting case is that of complex wave functions, as this allows some nonlinearities. There is a useful way of characterizing those complex wave equations which admit the symmetry M: if we use the amplitude-phase representation $\Psi = Re^{i\theta}$ for the wave function, then our operator M becomes ∂_θ, and we can then see that it is those equations which, written in terms of R and θ, do not contain any pure θ terms (they are present as derivatives of θ). To see this, we only need consider the nonlinear wave equation again, in this representation:

$$\Box R - R\theta_\mu\theta_\mu + \lambda F(R)R = 0,$$

$$R\Box\theta + 2R_\mu\theta_\mu = 0$$

when λ and F are real functions. The second equation is easily recognized as the continuity equation:

$$\partial_\mu(R^2\theta_\mu) = 0$$

(it is also a type of conservation of angular momentum). Clearly, the above system does not contain θ other than in terms of its derivatives, and therefore it must admit ∂_θ as a symmetry operator.

Writing an equation in this form has another advantage: one sees that the important part of the system is the continuity equation, and this allows us to consider other systems of equations which include the continuity equation, but have a different first equation. It is a form which can make calculating easier.

Having found the above reduction procedure and an operator which gives us the reducing ansatz, it is then natural to ask if there are other hyperbolic equations which are reduced down to the Schrödinger or diffusion equation. Thus, one may look at hyperbolic equations of the form

$$\Box\Psi = H(\Psi, \Psi^*)$$

which admit the operator M. An elementary calculation gives us that $H = F(|\Psi|)\Psi$. The next step is to allow H to depend upon derivatives:

$$\Box\Psi = F(\Psi, \Psi^*, \Psi_\mu, \Psi^*_\mu)\Psi$$

and we make the assumption that F is real. Now, it is convenient to do the calculations in the amplitude-phase representation, so our functions will depend on $R, \theta, R_\mu, \theta_\mu$. However, if we want the operator M to be a symmetry operator, the functions may not depend on θ although they may depend on its derivatives, so that F must be a function of $|\Psi|$, the amplitude. This leaves us with a large class of equations, which in the amplitude-phase form are

$$\Box R = F(R, R_\mu, \theta_\mu)R, \tag{1}$$

$$R\Box\theta + 2R_\mu\theta_\mu = 0 \tag{2}$$

and we easily find the solution

$$F = F(R, R_\mu R_\mu, \theta_\mu\theta_\mu, R_\mu\theta_\mu)$$

when we also require the invariance under the Poincaré algebra (we need translations for the ansatz and Lorentz transformations for the invariance of the wave operator).

We can ask for the types of systems (1), (2) invariant under the algebras of Theorem 2, and we find:

Theorem 3 *(i) System (1), (2) is invariant under the algebra* $\langle P_\mu, J_{\mu\nu}\rangle$.
(ii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D\rangle$ *with* $D = x^\sigma\partial_\sigma - \frac{2}{k}R\partial_R$, $k \neq 0$ *if and only if*

$$F = R^k G\left(\frac{R_\mu R_\mu}{R^{2+k}}, \frac{\theta_\mu\theta_\mu}{R^k}, \frac{\theta_\mu R_\mu}{R^{1+k}}\right),$$

where G is an arbitrary continuously differentiable function.
(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0\rangle$ with $D_0 = x^\sigma\partial_\sigma$ if and only if

$$F = R_\mu R_\mu G\left(R, \frac{\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{\theta_\mu R_\mu}{R_\mu R_\mu}\right),$$

where G is an arbitrary continuously differentiable function.
(iv) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, D, K_\mu\rangle$ with $D = x^\sigma\partial_\sigma - R\partial_R$ and $K_\mu = 2x_\mu D - x^2\partial_\mu$ if and only

$$F = R^2 G\left(\frac{\theta_\mu\theta_\mu}{R^2}\right),$$

where G is an arbitrary continuously differentiable function of one variable.

The last case contains, as expected, case (i) of Theorem 2 when we choose $G(\xi) = \xi - \lambda R^2$. Each of the resulting equations in the above result is invariant under the operator M and so one can use the ansatz defined by M to reduce the equation but we do not always obtain a nice Schrödinger equation. If we ask now for invariance under the operator $L = R\partial_R$ (it is the operator L of case (v), Theorem 2, expressed in the amplitude-phase form), then we obtain some other types of restrictions:

Theorem 4 *(i) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, L\rangle$ if and only if*

$$F = G\left(\frac{R_\mu R_\mu}{R^2}, \theta_\mu\theta_\mu, \frac{R_\mu\theta_\mu}{R}\right)$$

(ii) System (1),(2) is invariant under $\langle P_\mu, J_{\mu\nu}, D_0, L\rangle$ with $D_0 = x^\sigma\partial_\sigma$ if and only if

$$F = \frac{R_\mu R_\mu}{R^2} G\left(\frac{R^2\theta_\mu\theta_\mu}{R_\mu R_\mu}, \frac{R\theta_\mu R_\mu}{R_\mu R_\mu}\right)$$

(iii) System (1), (2) is invariant under $\langle P_\mu, J_{\mu\nu}, K_\mu, L\rangle$ where $K_\mu = 2x_\mu x^\sigma\partial_\sigma - x^2\partial_\mu - 2x_\mu R\partial_R$ if and only if

$$F = \kappa\theta_\mu\theta_\mu,$$

where κ is a constant.

The last case (iii) gives us the wave equation

$$\Box\Psi = (\kappa - 1)\frac{j_\mu j_\mu}{|\Psi|^4}\Psi,$$

where $j_\mu = \dfrac{1}{2i}\left[\bar{\Psi}\Psi_\mu - \Psi\bar{\Psi}_\mu\right]$, which is the current of the wave-function Ψ. For $\kappa = 1$, we recover the free complex wave equation. This equation, being invariant under both M and N, can be reduced by the ansatzes they give rise to. In fact, with the ansatz (obtained with L)

$$\Psi = e^{(\epsilon x)/2}\Phi(\alpha x, \beta x, \delta x)$$

with ϵ, α isotropic 4-vectors with $\epsilon\alpha = 1$, and β, δ two space-like orthogonal 4-vectors, the above equation reduces to the equation

$$\Phi_\tau = \triangle\Phi - (\kappa - 1)\frac{\vec{j}\cdot\vec{j}}{|\Phi|^4}\Phi,$$

where $\tau = \alpha x$ and $\triangle = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ with $y_1 = \beta x,\ y_2 = \delta x$, and we have

$$\vec{j} = \frac{1}{2i}[\bar{\Phi}\nabla\Phi - \Phi\nabla\bar{\Phi}].$$

These results show what nonlinearities are possible when we require the invariance under subalgebras of the conformal algebra in the given representation. The above equations are all related to the Schrödinger or heat equation. There are good reasons for looking at conformally invariant equations, not least physically. As mathematical reasons, we would like to give the following examples. First, note that the equation

$$\Box_{p,q}\Psi = 0, \tag{3}$$

where

$$\Box_{p,q} = g^{AB}\partial_A\partial_B, \quad A, B = 1, \ldots, p, p+1, \ldots, p+q$$

with $g^{AB} = \mathrm{diag}(\underbrace{1,\ldots,1}_{p}, \underbrace{-1,\ldots,-1}_{q})$, is invariant under the algebra generated by the operators

$$\partial_A,\ \ J_{AB} = x_A\partial_B - x_B\partial_A,\ \ K_A = 2x_Ax^B\partial_B - x^2\partial_A - 2x_A\left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$D = x^B\partial_B, \qquad M = i\left(\Psi\partial_\Psi - \overline{\Psi}\partial_{\overline{\Psi}}\right), \quad L = \left(\Psi\partial_\Psi + \overline{\Psi}\partial_{\overline{\Psi}}\right),$$
$$L_1 = i\left(\overline{\Psi}\partial_\Psi - \Psi\partial_{\overline{\Psi}}\right), \quad L_2 = \overline{\Psi}\partial_\Psi + \Psi\partial_{\overline{\Psi}},$$

namely the generalized conformal algebra $AC(p,q) \oplus \langle M, L, L_1, L_2\rangle$ which contains the algebra $ASO(p,q)$. Here, $\oplus$ denotes the direct sum. Using the ansatz which the operator M gives us, we can reduce equation (3) to the equation

$$i\partial_\tau\Phi = \Box_{p-1,q-1}\Phi. \tag{4}$$

This equation (4) is known in the literature: it was proposed by Feynman [7] in Minkowski space in the form

$$i\partial_\tau\Phi = (\partial_\mu - A_\mu)(\partial^\mu - A^\mu)\Phi.$$

It was also proposed by Aghassi, Roman and Santilli ([8]) who studied the representation theory behind the equation. Fushchych and Seheda [9] studied its symmetry properties in the Minkowski space. The solutions of equation (4) give solutions of (3) ([14]). We have that equation (4) has a symmetry algebra generated by the following operators

$$T = \partial_\tau,\ \ P_A = \partial_A,\ \ J_{AB},\ \ G_A = \tau\partial_A - x_AM$$
$$D = 2\tau\partial_\tau + x^A\partial_A - \frac{p+q-2}{2}L,\ \ M = \frac{i}{2}(\Phi\partial_\Phi - \bar{\Phi}\partial_\Phi),\ \ L = (\Phi\partial_\Phi + \bar{\Phi}\partial_\Phi)$$
$$S = \tau^2\partial_\tau + \tau x^A\partial_A - \frac{x^2}{2}M - \frac{\tau(p+q-2)}{2}L$$

and this algebra has the structure $[ASL(2,\mathbf{R}) \oplus AO(p-1,q-1)] \uplus \langle L, M, P_A, G_A\rangle$, where $\uplus$ denotes the semidirect sum of algebras. This algebra contains the subalgebra $AO(p-1,q-1) \uplus \langle T, M, P_A, G_A\rangle$ with

$$[J_{AB}, J_{CD}] = g_{BC}J_{AD} - g_{AC}J_{BD} + g_{AD}J_{BC} - g_{BD}J_{AC},$$

$$[P_A, P_B] = 0, \ \ [G_A, G_B] = 0, \ \ [P_A, G_B] = -g_{AB}M,$$

$$[P_A, J_{BC}] = g_{AB}P_C - g_{AC}P_B, \ \ [G_A, J_{BC}] = g_{AB}G_C - g_{AC}G_B,$$

$$[P_A, D] = P_A, \ \ [G_A; D] = G_A, \ \ [J_{AB}, D] = 0, \ \ [P_A, T] = 0, \ \ [G_A, T] = 0,$$

$$[J_{AB}, T] = 0, [M, T] = [M, P_A] = [M, G_A] = [M, J_{AB}] = 0,$$

It is possible to show that the algebra with these commutation relations is contained in $AO(p,q)$: define the basis by

$$T = \frac{1}{2}(P_1 - P_q), \ \ M = P_1 + P_q, \ \ G_A = J_{1A} + J_{qA}, \ \ J_{AB} \ \ (A, B = 2, \ldots, q-1),$$

and one obtains the above commutation relations. We see now that the algebra $AO(2,4)$ (the conformal algebra $AC(1,3)$) contains the algebra $AO(1,3) \uplus \langle M, P_A, G_A\rangle$ which contains the Poincaré algebra $AP(1,3) = AO(1,3) \uplus \langle P_\mu\rangle$ as well as the Galilei algebra $AG(1,3) = AO(3) \uplus \langle M, P_a, G_a\rangle$ (μ runs from 0 to 3 and a from 1 to 3). This is reflected in the possibility of reducing

$$\Box_{2,4}\Psi = 0$$

to

$$i\partial_\tau \Phi = \Box_{1,3}\Phi$$

which in turn can be reduced to

$$\Box_{1,3}\Phi = 0.$$

4 Two nonlinear equations

In this final section, I shall mention two equations in nonlinear quantum mechanics which are related to each other by our ansatz. They are

$$|\Psi|\Box\Psi - \Psi\Box|\Psi| = -\kappa|\Psi|\Psi \tag{5}$$

and

$$iu_t + \triangle u = \frac{\triangle|u|}{|u|}u. \tag{6}$$

We can obtain equation (6) from equation (5) with the ansatz

$$\Psi = e^{i(\kappa\tau - (\epsilon x)/2)}u(\tau, \beta x, \delta x),$$

where $\tau = \alpha x = \alpha_\mu x^\mu$ and ϵ, α, β, δ are constant 4-vectors with $\alpha^2 = \epsilon^2 = 0$, $\beta^2 = \delta^2 = -1$, $\alpha\beta = \alpha\delta = \epsilon\beta = \epsilon\delta = 0, \alpha\epsilon = 1$.

Equation (5), with $\kappa = m^2c^2/\hbar^2$ was proposed by Vigier and Guéret [11] and by Guerra and Pusterla [12] as an equation for de Broglie's double solution. Equation (6) was considered as a wave equation for a classical particle by Schiller [10] (see also [13]).

For equation (5), we have the following result:

Theorem 5 (Basarab-Horwath, Fushchych, Roman [3], [4]) *Equation (5) with $\kappa > 0$ has the maximal point-symmetry algebra $AC(1, n+1) \oplus Q$ generated by operators*

$$P_\mu,\ J_{\mu\nu},\ P_{n+1},\ J_{\mu n+1},\ D^{(1)},\ K_\mu^{(1)},\ K_{n+1}^{(1)},\ Q,$$

where

$$\begin{aligned}
&P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu, \quad P_{n+1} = \frac{\partial}{\partial x^{n+1}} = i(u\partial_u - u^*\partial_{u^*}),\\
&J_{\mu n+1} = x_\mu P_{n+1} - x_{n+1} P_\mu, \quad D^{(1)} = x^\mu P_\mu + x^{n+1} P_{n+1} - \frac{n}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}),\\
&K_\mu^{(1)} = 2x_\mu D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_\mu,\\
&K_{n+1}^{(1)} = 2x_{n+1} D^{(1)} - (x_\mu x^\mu + x_{n+1}x^{n+1})P_{n+1}, \quad Q = \Psi\partial_\Psi + \Psi^*\partial_{\Psi^*},
\end{aligned}$$

where the additional variable x^{n+1} is defined as

$$x^{n+1} = -x_{n+1} = \frac{i}{2\sqrt{\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa > 0.$$

For $\kappa < 0$ the maximal symmetry algebra of (9) is $AC(2, n) \oplus Q$ generated by the same operators above, but with the additional variable

$$x^{n+1} = x_{n+1} = \frac{i}{2\sqrt{-\kappa}} \ln \frac{\Psi^*}{\Psi}, \quad \kappa < 0.$$

In this result, we obtain new nonlinear representations of the conformal algebras $AC(1, n+1)$ and $AC(2, n)$. It is easily shown (after some calculation) that equation (5) is the only equation of the form

$$\Box u = F(\Psi, \Psi^*, \nabla\Psi, \nabla\Psi^*, \nabla|\Psi|\nabla|\Psi|, \Box|\Psi|)\Psi$$

invariant under the conformal algebra in the representation given in Theorem 5. This raises the question whether there are equations of the same form conformally invariant in the standard representation

$$\begin{aligned}
&P_\mu = \frac{\partial}{\partial x^\mu}, \quad J_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu,\\
&D = x^\mu P_\mu - \frac{n-1}{2}(\Psi\partial_\Psi + \Psi^*\partial_{\Psi^*}), \quad K_\mu = 2x_\mu D - x^2 P_\mu.
\end{aligned}$$

There are such equations [3] and [4], for instance:

$$\Box\Psi = |\Psi|^{4/(n-1)} F\left(|\Psi|^{(3+n)/(1-n)}\Box|\Psi|\right)\Psi, \qquad n \neq 1$$

$$\Box u = \Box|u| F\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \qquad n = 1$$

$$4\Box\Psi = \left\{\frac{\Box|\Psi|}{|\Psi|} + \lambda\frac{(\Box|\Psi|)^n}{|\Psi|^{n+4}}\right\}\Psi, \quad n \text{ arbitrary}$$

$$\Box\Psi = (1+\lambda)\frac{\Box|\Psi|}{|\Psi|}\Psi,$$

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{|\Psi|^4}\right)\Psi,$$

$$\Box\Psi = \frac{\Box|\Psi|}{|\Psi|}\left(1 + \frac{\lambda}{1+\sigma|\Psi|^4}\right)\Psi,$$

Again we see how the representation dictates the equation.

We now turn to equation (6). It is more convenient to represent it in the amplitude-phase form $u = Re^{i\theta}$:

$$\theta_t + \nabla\theta\cdot\nabla\theta = 0, \tag{7}$$

$$R_t + \triangle\theta + 2\nabla\theta\cdot\nabla R = 0. \tag{8}$$

Its symmetry properties are given in the following result:

Theorem 6 (Basarab-Horwath, Fushchych, Lyudmyla Barannyk [5], [6]) *The maximal point-symmetry algebra of the system of equations (7), (8) is the algebra with basis vector fields*

$$\begin{aligned}
&P_t = \partial_t,\ \ P_a = \partial_a,\ \ P_{n+1} = \frac{1}{2\sqrt{2}}(2\partial_t - \partial_\theta),\ \ N = \partial_R,\\
&J_{ab} = x_a\partial_b - x_b\partial_a,\ \ J_{0\,n+1} = t\partial_t - \theta\partial_\theta,\ \ J_{0a} = \frac{1}{\sqrt{2}}\left(x_a\partial_t + (t+2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\\
&J_{a\,n+1} = \frac{1}{\sqrt{2}}\left(-x_a\partial_t + (t-2\theta)\partial_{x_a} + \frac{1}{2}x_a\partial_\theta\right),\ \ D = -\left(t\partial_t + x_a\partial_a + \theta\partial_\theta - \frac{n}{2}\partial_R\right),\\
&K_0 = \sqrt{2}\left(\left(t + \frac{\vec{x}^2}{2}\right)\partial_t + (t+2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} + 2\theta^2\right)\partial_\theta - \frac{n}{2}(t+2\theta)\partial_R\right),\\
&K_{n+1} = -\sqrt{2}\left(\left(t - \frac{\vec{x}^2}{2}\right)\partial_t + (t-2\theta)x_a\partial_{x_a} + \left(\frac{\vec{x}^2}{4} - 2\theta^2\right)\partial_\theta - \frac{n}{2}(t-2\theta)\partial_R\right),\\
&K_a = 2x_aD - (4t\theta - \vec{x}^2)\partial_{x_a}.
\end{aligned}$$

The above algebra is equivalent to the extended conformal algebra $AC(1, n+1)\oplus\langle N\rangle$. In fact,with new variables

$$x_0 = \frac{1}{\sqrt{2}}(t+2\theta),\quad x_{n+1} = \frac{1}{\sqrt{2}}(t-2\theta) \tag{9}$$

the operators in Theorem 1 can be written as

$$\begin{aligned}
&P_\alpha = \partial_\alpha,\ \ J_{\alpha\beta} = x_\alpha\partial_\beta - x_\beta\partial_\alpha,\ \ N = \partial_R,\\
&D = -x_\alpha\partial_\alpha + \frac{n}{2}N,\ \ K_\alpha = -x_\alpha D - (x_\mu x^\mu)\partial_\alpha.
\end{aligned} \tag{10}$$

Exact solutions of system (7), (8) using symmetries have been given in [5] and in [6]. Some examples of solutions are the following (we give the subalgebra, ansatz, and the solutions):

$\mathbf{A_1} = \langle J_{12} + d\,N,\ P_3 + N,\ P_4\rangle$ $(d \geq 0)$

Ansatz: $\theta = -\frac{1}{2}t + f(\omega),\quad R = x_3 - d\arctan\left(\frac{x_1}{x_2}\right) + g(\omega),\quad \omega = x_1^2 + x_2^2.$

Solution:

$$\theta = -\frac{1}{2}t + \varepsilon\sqrt{\frac{x_1^2+x_2^2}{2}} + C_1, \ \ \varepsilon = \pm 1, \quad R = x_3 + d\arctan\left(\frac{x_1}{x_2}\right) - \frac{1}{4}\ln(x_1^2+x_2^2) + C_2,$$

where $C_1,\ C_2$ are constants.

$\mathbf{A_4} = \langle J_{04} + d\,N,\ J_{23} + d_2 N,\ P_2 + P_3\rangle$

Ansatz: $\theta = \dfrac{1}{t}f(\omega), \quad R = d\ln|t| + g(\omega), \quad \omega = x_1.$

Solution: $\theta = \dfrac{(x_1+C_1)^2}{4t}, \qquad R = d\ln|t| - \left(d+\dfrac{1}{2}\right)\ln|x_1+C_1| + C_2.$

$\mathbf{A_9} = \langle J_{01},\ J_{02},\ J_{03},\ J_{12},\ J_{13},\ J_{23}\rangle$

Ansatz: $\theta = \dfrac{1}{4t}f(\omega) + \dfrac{x_1^2+x_2^2+x_3^2}{4t}, \quad R = g(\omega), \quad \omega = \theta - \dfrac{1}{2}t.$

Solution: $\theta = \dfrac{\vec{x}^2 - 4C_1t + 8C_1^2}{4t - 8C_1}, \quad R = -\dfrac{3}{2}\ln\left|\dfrac{\vec{x}^2 - 2(t-2C_1)^2}{t-2C_1}\right| + C_2.$

$\mathbf{A_{14}} = \langle J_{04} + a_1 N,\ D + a_2 N,\ P_3\rangle, \ \ (a_1,\ a_2 \text{ arbitrary})$

Ansatz: $\theta = \dfrac{x_1^2}{t}f(\omega), \quad R = g(\omega) + a_1\ln|t| - \left(a_1 + a_2 + \dfrac{3}{2}\right)\ln|x_1|, \quad \omega = \dfrac{x_1}{x_2}.$

Solution: $\theta = \dfrac{x_1^2}{t}, \quad R = a_1\ln|t| + \left(a_2 - a_1 + \dfrac{1}{2}\right)\ln|x_1| - 2(a_2+1)\ln|x_2| + C.$

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 48–52.

The Conditional Symmetry and Connection Between the Equations of Mathematical Physics

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Abstract

With help of the conditional symmetry method, the connections between the linear heat equation and nonlinear heat and Burgers ones, between the generalized Harry-Deam equation and Korteweg-de Vries one are obtained. The nonlocal general formulae for solutions of the generalized Harry-Deam equation are constructed.

More than ten years ago, we proposed an idea of condition symmetry. This concept-was worked out under the leadership and with the direct participation of Wilhelm Illich Fushchych (see [1]). It is worth noting that at papers of western scientists a notion of conditional symmetry is persistently identified with nonclassical symmetry. A definition of this symmetry is given at [2]. But, in fact, these notions do not agree. The notion of conditional symmetry is more wider than one of nonclassical symmetry. A great number of our papers illustrate this fact convincingly. The conditional symmetry includes the Lie's symmetry, widening it essentially. After introducting conditional symmetry, there is a necessity to revise symmetry properties of many basic equations of theoretical and mathematical physics from this point of view.

Many articles written by W.I. Fushchych and his pupils are devoted to this problem last ten years. Conditional symmetry is used usually to build exact (conditional invariant) solutions of investigated equations. But it isn't a unique use of conditional symmetry. In this paper, we'll show a way of use conditional symmetry to give connection formulae between differential equations.

Investigating the conditional symmetry of the nonlinear heat equation

$$\frac{\partial U}{\partial t} + \vec{\nabla}(f(U)\vec{\nabla}U) = g(U) \tag{1}$$

or the equivalent equation

$$H(u)\frac{\partial u}{\partial x_0} + \triangle u = F(u), \tag{2}$$

where $u = u(x), x = (x_0, \vec{x}) \in R_{1+n}$, we receive the following result in the case $n = 1$.

Theorem 1 *The equation*

$$H(u)u_0 + u_{11} = F(u), \tag{3}$$

where $u_0 = \dfrac{\partial u}{\partial x_0}$, $u_{11} = \dfrac{\partial^2 u}{\partial x_1^2}$, $u = u(x_0, x_1)$, $H(u)$, $F(u)$ *are arbitrary smooth functions, is Q-conditionally invariant with respect to the operator*

$$Q = A(x_0, x_1, u)\partial_0 + B(x_0, x_1, u)\partial_1 + C(x_0, x_1, u)\partial_u, \tag{4}$$

if the functions A, B, C satisfy the system of differential equations in one from the following cases.

1) $A \neq 0$

(we can assume without restricting the generality that $A = 1$)

$$\begin{aligned} &B_{uu} = 0, \qquad C_{uu} = 2(B_{1u} + HBB_u), \\ &3B_uF = 2(C_{1u} + HB_uC) - (HB_0 + B_{11} + 2HBB_1 + \dot{H}BC), \\ &C\dot{F} - (C_u - 2B_1)\,F = HC_0 + C_{11} + 2HCB_1 + \dot{H}C^2; \end{aligned} \tag{5}$$

2) $A = 0$, $B = 1$.

$$C\dot{F} - C_uF = HC_0 + C_{11} + 2CC_{1u} + C^2C_{uu} + \frac{C\dot{H}}{H}\,(F - CC_u + C_1)\,. \tag{6}$$

A subscript means differentiation with respect to the corresponding argument.

If in formula (6), as the particular case , we take

$$H(u) \equiv 1, \qquad F(u) \equiv 0, \qquad C(x_0, x_1, u) = w(x_0, x_1)u, \tag{7}$$

then operator (4) has the form

$$Q = \partial_1 + w(x_0, x_1)u\partial_u, \tag{8}$$

and the function w is a solution of the Burgers equation

$$w_0 + 2ww_1 + w_{11} = 0. \tag{9}$$

With our assumptions (8), equation (3) become the linear heat equation

$$u_0 + u_{11} = 0. \tag{10}$$

We find the connection between equations (9) and (10) from the condition

$$Qu = 0. \tag{11}$$

In this case, equation (11) has the form

$$u_1 - wu = 0, \tag{12}$$

or

$$w = \partial_1(\ln u). \tag{13}$$

Thus, we receive the well-known Cole-Hopf substitution, which reduces the Burgers equation to the linear heat equation.

Let us consider other example. G. Rosen in 1969 and Blumen in 1970 showed that the nonlinear heat equation

$$w_t + \partial_x(w^{-2}w_x) = 0, \qquad w = w(t, x), \tag{14}$$

is reduced to the linear equation (10) by the substitutions

$$\begin{aligned} &1. \quad t = t, \quad x = x, \quad w = v_x; \\ &2. \quad t = x_0, \quad x = u, \quad v = x_1. \end{aligned} \tag{15}$$

Connection (15) between equations (14) and (10) is obtained from condition (11) if we take in (6)

$$H(u) \equiv 1, \quad F(u) \equiv 0, \quad C(x_0, x_1, u) = \frac{1}{w(x_0, u)}. \tag{16}$$

Equation (6) will have the form (14), and (11) will be written in the form

$$u_1 = \frac{1}{w(x_0, u)}, \tag{17}$$

that equivalent to (15).

Provided that

$$H(u) = 1, \quad F(u) = \lambda u \ln u, \quad \lambda = \text{const}, \tag{18}$$

formula (13) connects the equations

$$u_0 + u_{11} = \lambda u \ln u \quad \text{and} \quad w_0 + 2ww_1 + w_{11} = \lambda w. \tag{19}$$

When

$$H(u) = \frac{1}{f(u)}, \quad F(u) \equiv 0, \quad C(x_0, x_1, u) = \frac{1}{w(x_0, u)}, \tag{20}$$

then substitutions (15) set the connection between the equations

$$u_0 + f(u)u_{11} = 0 \quad \text{and} \quad w_t + \partial_x[f(x)w^{-2}w_x)] = 0. \tag{21}$$

It should be observed that equations (19) and (21) are widely used to decribe real physical processes.

If we assume

$$C \equiv 0, \quad B = -\frac{f_1}{f}, \quad H(u) \equiv -1, \quad F(u) \equiv 0, \tag{22}$$

in formulae (5), where $f = f(x)$ is an arbitrary solution of the equation

$$f_0 = f_{11}, \tag{23}$$

then we take the operator

$$Q = \partial_0 - \frac{f_1}{f}\partial_1.$$

From condition (11), we receive the connection between two solutions of the linear equation (23)

$$fu_0 - f_1u_1 = 0. \tag{24}$$

The characteristic equation

$$f_1dx_0 + fdx_1 = 0 \tag{25}$$

corresponds to equation (24).

Theorem 2 *If a function f is a solution of equation (23), and a function $u(x)$ is a common integral of the ordinary differential equation (25), then $u(x)$ is a solution of equation (23).*

Theorem 2 sets a generation algorithm for solutions of equation (23). Thus starting from the "old" solution $u \equiv 1$, we obtain a whole chain of solutions of equation (23):

$$1 \longrightarrow x_1 \longrightarrow x_0 + \frac{x_1^2}{2!} \longrightarrow x_0 x_1 + \frac{x_1^3}{3!} \longrightarrow \cdots$$

Prolongating this process, we find next solutions of equation (23):

$$\frac{x_1^{2m}}{(2m)!} + \frac{x_0}{1!}\frac{x_1^{2m-2}}{(2m-2)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^2}{2!} + \frac{x_0^m}{m!},$$
$$\frac{x_1^{2m+1}}{(2m+1)!} + \frac{x_0}{1!}\frac{x_1^{2m-1}}{(2m-1)!} + \cdots + \frac{x_0^{m-1}}{(m-1)!}\frac{x_1^3}{3!} + \frac{x_0^m}{m!}\frac{x_1}{1!},$$

where $m = 0, 1, 2, \cdots$

In conclusion, we give still one result, which sets the connection between the generalized Harry-Dym and Korteweg-de Vries equations.

Let us generate the Korteweg-de Vries equation

$$u_0 + \lambda u u_1 + u_{111} = 0, \qquad u = u(x_0, x_1), \tag{26}$$

and the Harry-Dym equation

$$w_t + \partial_{xx}(w^{-3/2} w_x) = 0, \qquad w = w(t, x), \tag{27}$$

by following equations

$$u_0 + f(u)u_1 + u_{111} = 0, \tag{28}$$

and

$$w_t + \partial_{xx}(F(w)w_x) = G(x, w). \tag{29}$$

Theorem 3 *The generalized Korteweg-de Vries equation (28) is Q-conditionally invariant with respect to the operator*

$$Q = \partial_1 + C(x_0, x_1, u)\partial_u, \tag{30}$$

if the function $C(x_0, x_1, u)$ is a solution of the equation

$$C_0 + C_{111} + 3CC_{11u} + 3(C_1 + CC_u)(C_{1u} + CC_{uu}) + 3C^2 C_{1uu} + C^3 C_{uuu} + C_1 f + C^2 \dot{f} = 0. \tag{31}$$

If we assume in (31) that

$$C = \frac{1}{w(x_0, u)},$$

then we obtain

$$w_0 + \partial_{uu}(w^{-3} w_u) = \dot{f}(u). \tag{32}$$

Thus, formula (17) sets the connection between the generalized Korteweg-de Vries (28) equation and Harry-Dym equation

$$w_t + \partial_{xx}(w^{-3}w_x) = \dot{f}(x). \tag{33}$$

In particular at $f(u) = \lambda u$, we obtain the connection between the following equations:

$$u_0 + \lambda u u_1 + u_{111} = 0 \quad \text{and} \quad w_t + \partial_{xx}(w^3 w_x) = \lambda, \tag{34}$$

where λ is an arbitrary constant.

So, we have shown that it is possible to obtain nonlocal connection formulae between some differential equations, using the operators of conditional symmetry. It is, for example, the well-known connection between the Burgers and linear heat equations, that is realized by the Cole-Hopf substitution. We have obtained also that there is the nonlocal connection between the generalized Harry-Dym and the Korteweg-de Vries equations.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 53–61.

On Conditional Symmetries of Multidimensional Nonlinear Equations of Quantum Field Theory

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Abstract

We give a brief review of our results on investigating conditional (non-classical) symmetries of the multidimensional nonlinear wave Dirac and $SU(2)$ Yang-Mills equations.

Below we give a brief account of results of studying conditional symmetries of multidimensional nonlinear wave, Dirac and Yang-Mills equations obtained in collaboration with W.I. Fushchych in 1989–1995. It should be noted that till our papers on exact solutions of the nonlinear Dirac equation [1]–[4], where both symmetry and conditional symmetry reductions were used to obtain its exact solutions, only two-dimensional (scalar) partial differential equations (PDEs) were studied (for more detail, see, [5]). The principal reason for this is the well-known fact that the determining equations for conditional symmetries are nonlinear (we recall that determining equations for obtaining Lie symmetries are linear). Thus, to find a conditional symmetry of a multidimensional PDE, one has to find a solution of the nonlinear system of partial differential equations whose dimension is higher that the dimension of the equation under study! In paper [3], we have suggested a powerful method enabling one to obtain wide classes of conditional symmetries of multidimensional Poincaré-invariant PDEs. Later on it was extended in order to be applicable to Galilei-invariant equations [6] which yields a number of conditionally-invariant exact solutions of the nonlinear Levi-Leblond spinor equations [7]. The modern exposition of the above-mentioned results can be found in monograph [8].

Historically, the first physically relevant example of conditional symmetry for a multidimensional PDE was obtained for the nonlinear Dirac equation. However, in this paper, we will concentrate on the nonlinear wave equation which is easier for understanding the basic techniques used to construct its conditional symmetries.

As is well known, the maximal invariance group of the nonlinear wave equation

$$\Box u \equiv u_{x_0x_0} - \Delta u = F_0(u), \quad F \in C^1(\mathbf{R}^1, \mathbf{R}^1) \tag{1}$$

is the 10-parameter Poincaré group having the generators

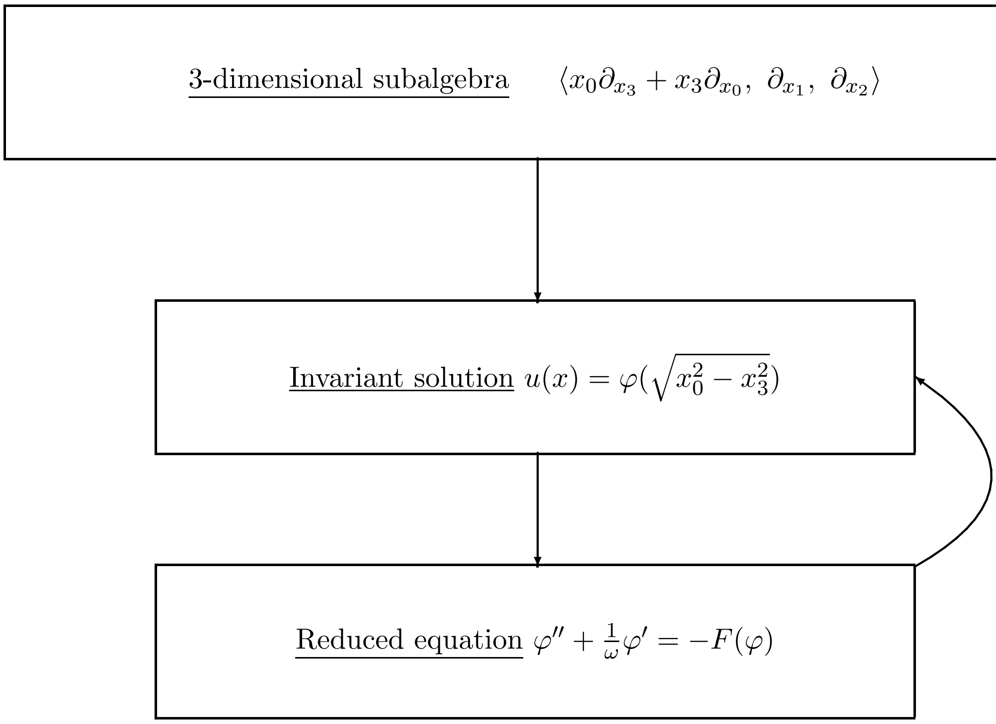
$$P_\mu = \partial_{x_\mu}, \quad J_{0a} = x_0\partial_{x_a} + x_a\partial_{x_0}, \quad J_{ab} = x_a\partial_{x_b} - x_a\partial x_a,$$

where $\mu = 0, 1, 2, 3$, $a, b = 1, 2, 3$.

The problem of symmetry reduction for the nonlinear wave equation by subgroups of the Poincaré group in its classical setting has been solved in [9]. Within the framework of the symmetry reduction approach, a solution is looked for as a function

$$u(x) = \varphi(\omega) \tag{2}$$

Fig.1. Symmetry reduction scheme



of an invariant $\omega(x)$ of a subgroup of the Poincaré algebra. Then inserting the Ansatz $\varphi(\omega)$ into (1) yields an ordinary differential equation (ODE) for the function $\varphi(\omega)$. As an illustration, we give Fig.1, where ω is the invariant of the subalgebra $\langle J_{03}, P_1, P_2\rangle \in AP(1,3)$.

The principal idea of our approach to constructing conditionally-invariant Ansätze for the nonlinear wave equation was to preserve the form of Ansatz (2) but not to fix *a priori* the function $\omega(x)$. The latter is so chosen that inserting (2) should yield an ODE for the function $\omega(x)$. This requirement leads to the following intermediate problem: we have to integrate the system of two nonlinear PDEs in four independent variables called the d'Alembert-eikonal system

$$\omega_{x_\mu}\omega_{x^\mu} = F_1(\omega), \quad \Box\omega = F_2(\omega). \tag{3}$$

Hereafter, summation over repeated indices in the Minkowski space with the metric tensor $g_{\mu\nu} = \delta_{\mu\nu} \times (1, -1, -1, -1)$ is understood, i.e.,

$$\omega_{x_\mu}\omega_{x^\mu} \equiv \omega_{x_0}^2 - \omega_{x_1}^2 - \omega_{x_2}^2 - \omega_{x_3}^2.$$

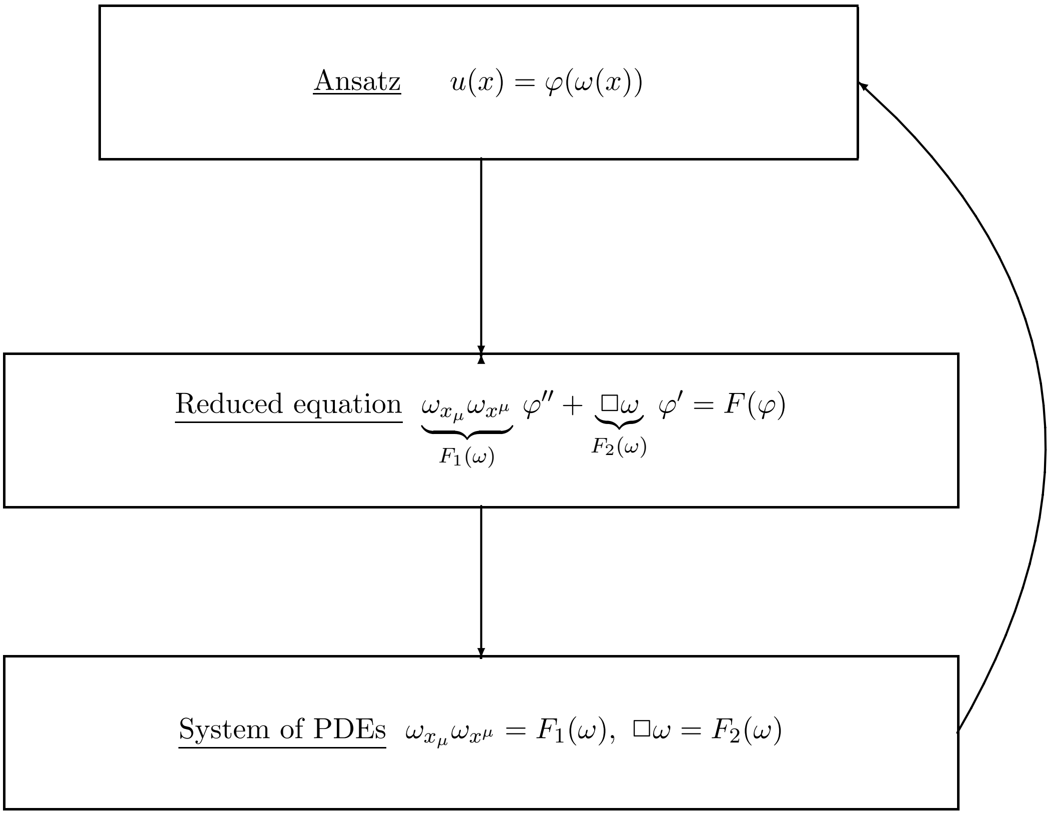
As an illustration, we give below Fig.2.

According to [10], the compatible system of PDEs (3) is equivalent to the following one:

$$\omega_{x_\mu}\omega_{x^\mu} = \lambda, \quad \Box\omega = \frac{\lambda N}{\omega}, \tag{4}$$

where $\lambda = 0, \pm 1$ and $N = 0, 1, 2, 3$. In papers [11, 12], we have constructed general solutions of the above system for all possible values λ, N. Here, we restrict ourselves to giving the general solution of the d'Alembert-eikonal system for the case $N = 3, \lambda = 1$.

Fig.2. Conditional symmetry reduction scheme



Theorem. *The general solution of the system of nonlinear PDEs*

$$\omega_{x_\mu}\omega_{x^\mu} = 1, \quad \Box\omega = \frac{3}{\omega} \tag{5}$$

is given by the following formula:

$$u^2 = \Big(x_\mu + A_\mu(\tau)\Big)\Big(x^\mu + A^\mu(\tau)\Big),$$

where the function $\tau = \tau(x)$ is determined in implicit way

$$\Big(x_\mu + A_\mu(\tau)\Big)B^\mu(\tau) = 0$$

and the functions $A_\mu(\tau)$, $B_\mu(\tau)$ satisfy the relations

$$A'_\mu B^\mu = 0, \qquad B_\mu B^\mu = 0.$$

This solution contains **five** arbitrary functions of one variable. Choosing $A_\mu = C_\mu =$ const, $B_\mu = 0$, $\mu = 0, 1, 2, 3$, yields an invariant of the Poincaré group $\omega(x) = (x_\mu + C_\mu)(x^\mu + C^\mu)$. All other choices of the functions A_μ, B_μ lead to Ansätze that correspond to conditional symmetry of the nonlinear wave equation. Conditional symmetry generators can be constructed in explicit form, however, the resulting formulae are rather cumbersome. That is why we will consider a more simple example of a non-Lie Ansatz, namely

$$u(x) = \varphi\Big(x_1 + w(x_0 + x_3)\Big).$$

As a direct computation shows, the above function is the general solution of the system of linear PDEs

$$Q_a u(x) = 0, \quad a = 1, 2, 3,$$

where

$$Q_1 = \partial_{x_0} - \partial_{x_3}, \quad Q_2 = \partial_{x_0} + \partial_{x_3} - 2w'\partial_{x_1}, \quad Q_3 = \partial_{x_2}.$$

The operator Q_2 cannot be represented as a linear combination of the basis elements of the Lie algebra of the Poincaré group, since it contains an arbitrary function. Furthermore, the operators Q_1, Q_2, Q_3 are commuting and fulfill the relations

$$\widehat{Q}_1 L = 0, \quad \widehat{Q}_2 L = \underline{4w''\partial_{x_1}(Q_1 u)}, \quad \widehat{Q}_3 L = 0,$$

where $L = \Box u - F(u)$, whence it follows that the system

$$\Box u = F(u), \quad Q_1 u = 0, \quad Q_2 u = 0, \quad Q_3 u = 0$$

is invariant in Lie's sense with respect to the commutative Lie algebra $\mathcal{A} = \langle Q_1, Q_2, Q_3 \rangle$. This means, in its turn, that the nonlinear wave equation $\Box u = F(u)$ is conditionally-invariant with respect to the algebra $\mathcal{A}$. The geometric interpretation of these reasonings is given in Fig.3.

We recall that a PDE

$$U(x, u) = 0$$

is conditionally-invariant under the (involutive) set of Lie vector fields $\langle Q_1, \ldots, Q_n \rangle$ if there exist PDEs

$$U_1(x, u) = 0, \quad U_2(x, u) = 0, \ldots, \quad U_N(x, u) = 0$$

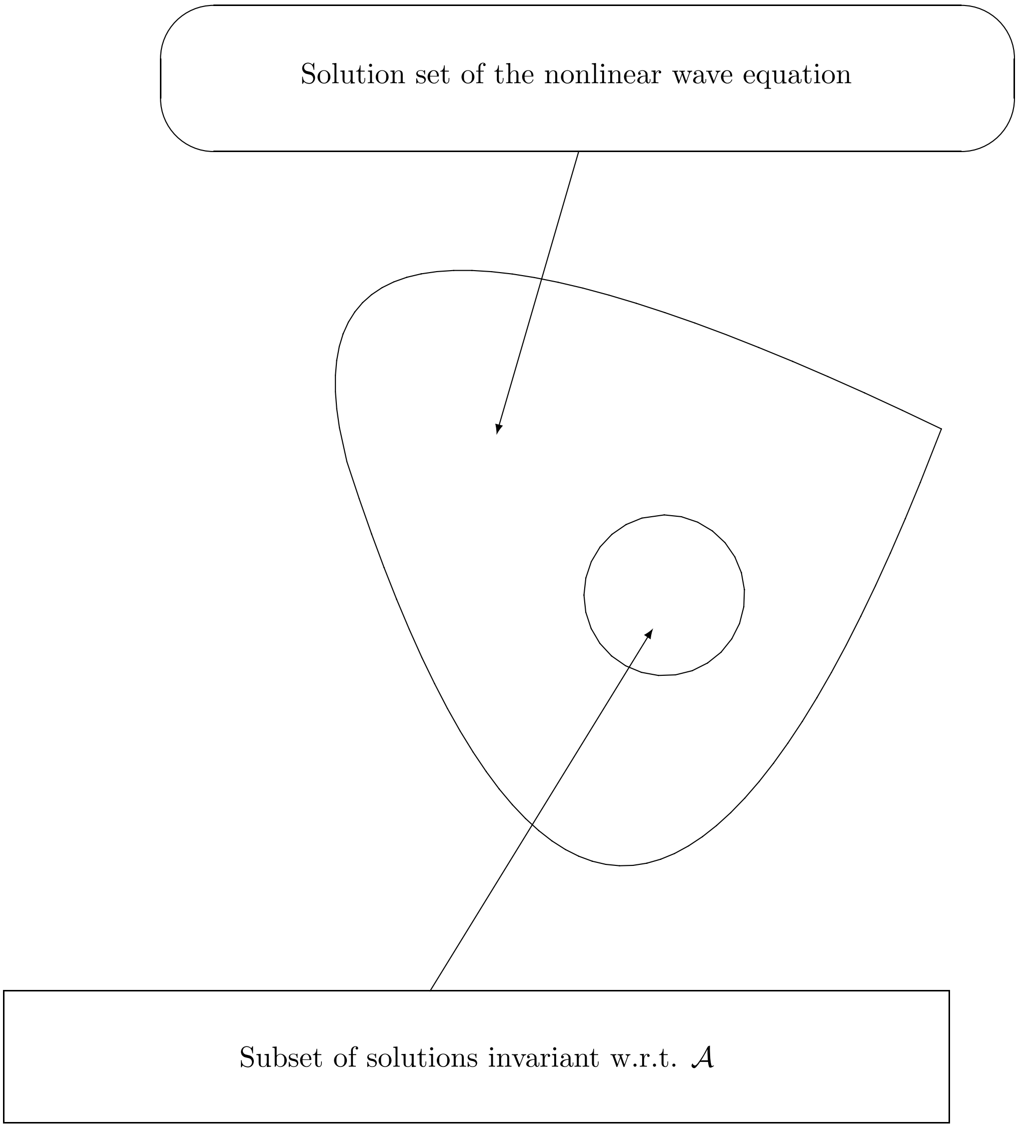
such that the system

$$\begin{cases} U(x, u) = 0, \\ U_1(x, u) = 0, \\ \cdots \\ U_N(x, u) = 0 \end{cases}$$

is invariant in Lie's sense with respect to the operators Q_a, $\forall a$.

In particular, we may take $n = N$, $U_i(x, u) = Q_i u$ which yields a particular form of the conditional symmetry called sometimes Q-conditional symmetry (for more detail, see [8, 14]).

Fig.3



Next we give without derivation examples of new conditionally-invariant solutions of the nonlinear wave equation with polynomial nonlinearities $F(u)$ obtained in [15].

1. $F(u) = \lambda u^3$

1) $u(x) = \dfrac{1}{a\sqrt{2}}(x_1^2 + x_2^2 + x_3^2 - x_0^2)^{-1/2} \tan\left\{ -\dfrac{\sqrt{2}}{4} \ln\Big(C(\omega)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)\right\},$

where $\lambda = -2a^2 < 0$,

2) $u(x) = \dfrac{2\sqrt{2}}{a} C(\omega)\Big(1 \pm C^2(\omega)(x_1^2 + x_2^2 + x_3^2 - x_0^2)\Big)^{-1},$

where $\lambda = \pm a^2$;

2. $F(u) = \lambda u^5$

1) $$u(x) = a^{-1}(x_1^2 + x_2^2 - x_0^2)^{-1/4}\Big\{\sin\ln\Big(C(\omega)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) + 1\Big\}^{1/2} \times \Big\{2\sin\ln\Big(C(\omega)(x_1^2 + x_2^2 - x_0^2)^{-1/2}\Big) - 4\Big\}^{-1/2},$$

where $\lambda = a^4 > 0$,

2) $$u(x) = \frac{3^{1/4}}{\sqrt{a}} C(\omega)\Big(1 \pm C^4(\omega)(x_1^2 + x_2^2 - x_0^2)\Big)^{-1/2},$$

where $\lambda = \pm a^2$.

In the above formulae, $C(\omega)$ is an arbitrary twice continuously differentiable function of

$$\omega(x) = \frac{x_0 x_1 \pm x_2\sqrt{x_1^2 + x_2^2 - x_0^2}}{x_1^2 + x_2^2},$$

$a \neq 0$ is an arbitrary real parameter.

Let us turn now to the following class of Poincaré-invariant nonlinear generalizations of the Dirac equation:

$$\Big(i\gamma_\mu \partial_{x_\mu} - f_1(\bar\psi\psi, \bar\psi\gamma_4\psi) - f_2(\bar\psi\psi, \bar\psi\gamma_4\psi)\gamma_4\Big)\psi = 0. \tag{6}$$

Here, γ_μ are 4×4 Dirac matrices, $\mu = 0, 1, 2, 3$, $\gamma_4 = \gamma_0\gamma_1\gamma_2\gamma_3$, ψ is the 4-component complex-valued function, $\bar\psi = (\psi^*)^T\gamma_0$, f_1, f_2 are arbitrary continuous functions.

The class of nonlinear Dirac equations (6) contains as particular cases the nonlinear spinor models suggested by Ivanenko, Heisenberg and Gürsey.

We have completely solved the problem of symmetry reduction of system (6) to systems of ODEs by subgroups of the Poincaré group. An analysis of invariant solutions obtained shows that the most general form of a Poincaré-invariant solution reads

$$\psi(x) = \exp\{\theta_j\gamma_j(\gamma_0 + \gamma_3)\}\exp\{(1/2)\theta_0\gamma_0\gamma_3 + (1/2)\theta_3\gamma_1\gamma_2\}\varphi(\omega), \tag{7}$$

where $\theta_0(x), \ldots, \theta_3(x), \omega(x)$ are some smooth functions and $\varphi(\omega)$ is a new unknown four-component function.

Now we make use of the same idea as above. Namely, we do not fix *a priori* the form of the functions θ_μ, ω. They are chosen in such a way that inserting Ansatz (7) into system (6) yields a system of ODEs for $\varphi(\omega)$. After some tedious calculations, we get a system of 12 nonlinear first-order PDEs for five functions θ_μ, ω. We have succeeded in constructing its general solution which gives rise to 11 classes of Ansätze (7) that reduce the system of PDEs (6) to ODEs. And what is more, only five of them correspond to the Lie symmetry of (6). Other six classes are due to the conditional symmetry of the nonlinear Dirac equation.

Below we give an example of a conditionally-invariant Ansatz for the nonlinear Dirac equation

$$\theta_j = \frac{1}{2}w_j' + \left(\frac{a\sqrt{z_1^2 + z_2^2}}{x_0 + x_3}\arctan\frac{z_1}{z_2} + w_3\right)\partial_{x_j}\left(\arctan\frac{z_1}{z_2}\right),$$

$$\theta_0 = \ln(x_0 + x_3), \quad \theta_3 = -\arctan\frac{z_1}{z_2}, \quad \omega = z_1^2 + z_2^2,$$

where $z_j = x_j + w_j$, $j = 1, 2$, w_1, w_2, w_3 are arbitrary smooth functions of $x_0 + x_3$ and a is an arbitrary real constant. Provided,

$$a = 0, \quad w_1 = \text{const}, \quad w_2 = \text{const}, \quad w_3 = 0$$

the above Ansatz reduces to a solution invariant under the 3-parameter subgroup of the Poincaré group. However if, at least, one of these restrictions is not satisfied, the Ansatz cannot in principle be obtained by the symmetry reduction method.

Next we will turn to the $SU(2)$ Yang-Mills equations

$$\begin{aligned} &\Box \vec{A}_\mu - \partial_{x^\mu}\partial_{x_\nu}\vec{A}_\nu + e\Big((\partial_{x_\nu}\vec{A}_\nu) \times \vec{A}_\mu - 2(\partial_{x_\nu}\vec{A}_\mu) \times \vec{A}_\nu \\ &\quad +(\partial_{x^\mu}\vec{A}_\nu) \times \vec{A}^\nu\Big) + e^2\vec{A}_\nu \times (\vec{A}^\nu \times \vec{A}_\mu) = \vec{0}. \end{aligned} \tag{8}$$

Here, $\vec{A}_\mu = \vec{A}_\mu(x_0, x_1, x_2, x_3)$ is the three-component vector-potential of the Yang-Mills field, symbol $\times$ denotes vector product, e is a coupling constant.

It is a common knowledge that the maximal symmetry group of the Yang-Mills equations contains as a subgroup the ten-parameter Poincaré group $P(1, 3)$. In our joint paper with Fushchych and Lahno, we have obtained an exhaustive description of the Poincaré-invariant Ansätze that reduce the Yang-Mills equations to ODEs [16]. An analysis of the results obtained shows that these Ansätze, being distinct at the first sight, have the same general structure, namely

$$\vec{A}_\mu = R_{\mu\nu}(x)\vec{B}_\nu(\omega(x)), \quad \mu = 0, 1, 2, 3, \tag{9}$$

where

$$\begin{aligned} R_{\mu\nu}(x) &= (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (a_\mu d_\nu - d_\mu a_\nu)\sinh\theta_0 \\ &\quad +2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu \\ &\quad +(\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 \\ &\quad -(c_\mu b_\nu - b_\mu c_\nu)\sin\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu). \end{aligned}$$

In (9), $\omega(x)$, $\theta_\mu(x)$ are some smooth functions and what is more

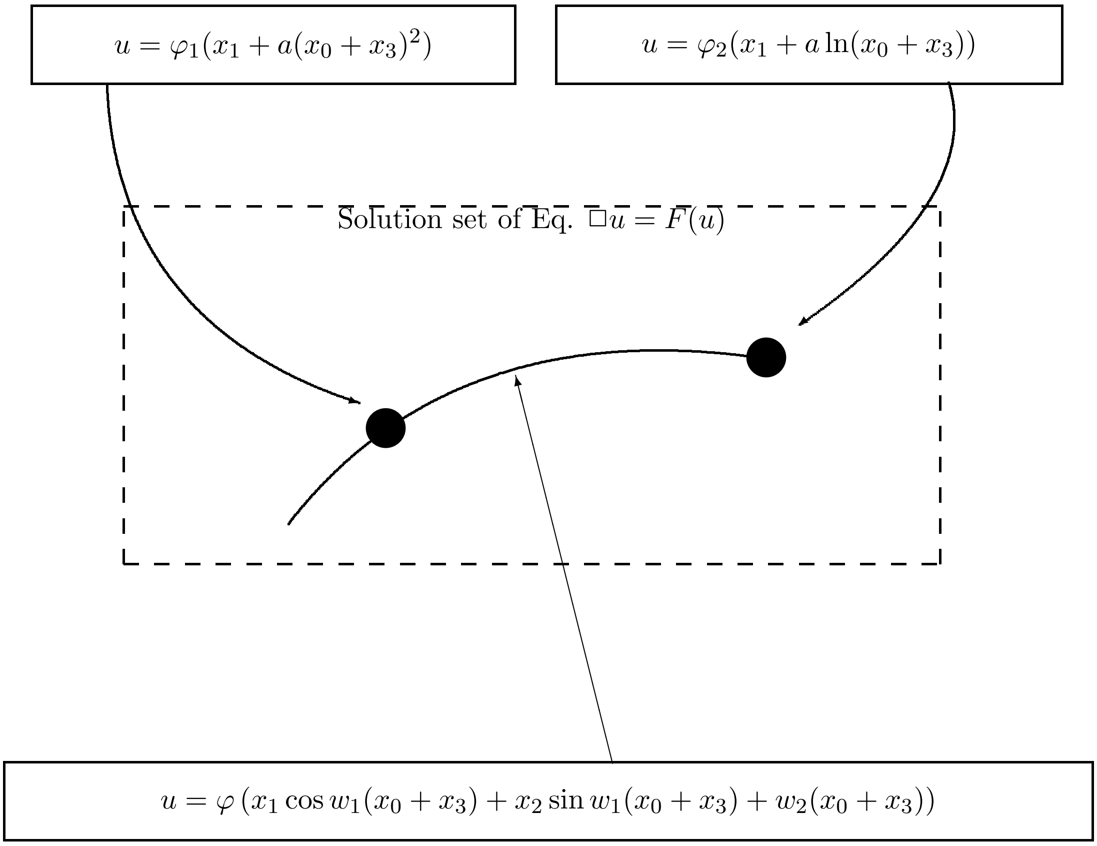
$$\theta_j = \theta_j(a_\mu x^\mu + d_\mu x^\mu,\, b_\mu x^\mu,\, c_\mu x^\mu), \quad j = 1, 2,$$

$a_\mu,\ b_\mu,\ c_\mu,\ d_\mu$ are arbitrary constants satisfying the following relations:

$$\begin{aligned} &a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \\ &a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0. \end{aligned}$$

We have succeeded in constructing three classes of conditionally-invariant Ansätze of the form (9) which yield five new classes of exact solutions of the $SU(2)$ Yang-Mills equations [17].

Fig.4



In conclusion, we would like to point out a remarkable property of conditionally-invariant solutions obtained with the help of the above-presented approach. As noted in [15], a majority of solutions of the wave and Dirac equations constructed by virtue of the symmetry reduction routine are particular cases of the conditionally-invariant solutions. They are obtained by a proper specifying of arbitrary functions and constants contained in the latter. As an illustration, we give in Fig.4, where we demonstrate the correspondence between two invariant solutions $u(x) = \varphi_1(x_1 + a(x_0 + x_3)^2)$, $u(x) = \varphi_2(x_1 + a(x_0 + x_3)^2)$ and the more general conditionally-invariant solution of the form:

$$u(x) = \varphi\Big(x_1 \cos w_1(x_0 + x_3) + x_2 \sin w_1(x_0 + x_3) + w_2(x_0 + x_3)\Big),$$

where w_1, w_2 are arbitrary functions.

Acknowledgments

This work is supported by the DFFD of Ukraine (project 1.4/356).

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Invariant Solutions of the Multidimensional Boussinesq Equation

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Abstract

The reduction of the n-dimensional Boussinesq equation with respect to all subalgebras of rank n of the invariance algebra of this equation is carried out. Some exact solutions of this equation are obtained.

1 Introduction

In this paper, we make research of the Boussinesq equation

$$\frac{\partial u}{\partial x_0} + \nabla\left[(au+b)\nabla u\right] + cu + d = 0, \tag{1}$$

where

$$u = u\left(x_0, x_1, \ldots, x_n\right), \qquad \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right),$$

a, b, c, d are real numbers, $a \neq 0$. This equation has applications in hydrology [1, 2] and heat conduction [3]. Group properties of (1) were discussed in [4] for $n = 1$, in [5] for $n = 2, 3$, and in [6, 7] for each n. In the case $n \leq 3$, $c = d = 0$, some invariant solutions of (1) have been obtained in [1, 6–9]. The aim of the present paper is to perform the symmetry reduction of (1) for each n to ordinary differential equations. Using this reduction we find invariant solutions of this equation.

2 Classification of subalgebras of the invariance algebra

The substitution $(au + b) = v^{\frac{1}{2}}$ reduces equation (1) to

$$\frac{\partial v}{\partial x_0} + v^{\frac{1}{2}}\Delta v + \delta v + \gamma v^{\frac{1}{2}} = 0, \tag{2}$$

where $\delta = 2c$, $\gamma = 2ad - bc$. If $\gamma = \delta = 0$, then equation (2) is invariant under the direct sum of the extended Euclidean algebras $A\widetilde{E}\,(1) = \langle P_0, D_1\rangle$ and $A\widetilde{E}\,(n) = \langle P_1, \ldots, P_n\rangle \Supset \left(AO\,(n) \oplus \langle D_2\rangle\right)$, $AO\,(n) = \langle J_{ab}:\ a, b = 1, \ldots, n\rangle$ generated by the vector fields [7]:

$$\begin{aligned} &P_0 = \frac{\partial}{\partial x_0}, \quad D_1 = x_0\frac{\partial}{\partial x_0} - 2v\frac{\partial}{\partial v}, \quad P_b = \frac{\partial}{\partial x_b}, \\ &J_{bc} = x_b\frac{\partial}{\partial x_c} - x_c\frac{\partial}{\partial x_b}, \quad D_2 = x_b\frac{\partial}{\partial x_b} + 4v\frac{\partial}{\partial v}, \end{aligned} \tag{3}$$

where $b, c = 1, \ldots, n$.

If $\gamma = 0$, $\delta \neq 0$, then equation (2) is invariant under $A\widetilde{E}\,(1) \oplus A\widetilde{E}\,(n)$, where [5]

$$P_0 = e^{\delta x_0}\frac{\partial}{\partial x_0} - 2\delta\, e^{\delta x_0}\, v\frac{\partial}{\partial v}, \quad D_1 = -\frac{1}{\delta}\frac{\partial}{\partial x_0},$$

whereas the remaining operators are of the form (3). For $\gamma \neq 0$ equation (2) is invariant under the direct sum of $AE\,(1) = \langle P_0\rangle$ and $AE\,(n) = \langle P_1, \ldots, P_n\rangle \supset\!\!\!\!+ AO\,(n)$ generated by the vector fields (3). From here we assume that $\gamma = 0$.

Let $v = v\,(x_0, x_1, \ldots, x_n)$ be a solution of equation (2) invariant under P_0. In this case if $\delta = 0$, then $v = v\,(x_1, \ldots, x_n)$ is a solution of the Laplace equation $\Delta v = 0$. If $\delta \neq 0$, then

$$v = e^{-2\delta x_0}\,\varphi\,(x_1, \ldots, x_n)\,, \tag{4}$$

where $\Delta\varphi = 0$. Furthermore for each solution of the Laplace equation, function (4) satisfies equation (2). In this connection, let us restrict ourselves to those subalgebras of the algebra $F = A\widetilde{E}\,(1) \oplus A\widetilde{E}\,(n)$ that do not contain P_0. Among subalgebras possessing the same invariants, there exists a subalgebra containing all the other subalgebras. We call it by the I-maximal subalgebra. To carry out the symmetry reduction of equation (2), it is sufficient to classify I-maximal subalgebras of the algebra F up to conjugacy under the group of inner automorphisms of the algebra F.

Denote by $AO\,[r,s]$, $r \leq s$, a subalgebra of the algebra $AO\,(n)$. It is generated by operators J_{ab}, where $a, b = r, r+1, \ldots, s$. If $r > s$, then we suppose that $AO\,[r,s] = 0$. Let $AE\,[r,s] = \langle P_r, \ldots, P_s\rangle \supset\!\!\!\!+ AO\,[r,s]$ for $r \leq s$ and $AE\,[r,s] = 0$ for $r > s$.

Let us restrict ourselves to those subalgebras of the algebra F whose projections onto $AO\,(n)$ be subdirect sums of algebras of the form $AO\,[r,s]$.

Theorem 1 *Up to conjugacy under the group of inner automorphisms, the algebra F has 7 types of I-maximal subalgebras of rank n which do not contain P_0 and satisfy the above condition for projections:*

$$L_0 = AE(n);$$

$$L_1 = (AO\,(m) \oplus AO\,[m+1,q] \oplus AE\,[q+1,n]) \supset\!\!\!\!+ \langle D_1, D_2\rangle\,,$$

$$\textit{where}\quad 1 \leq m \leq n-1,\ m+1 \leq q \leq n;$$

$$L_2 = (AO\,(m) \oplus AE\,[m+1,n]) \supset\!\!\!\!+ \langle D_1 + \alpha D_2\rangle \quad (\alpha \in \mathbb{R},\ 1 \leq m \leq n)\,;$$

$$L_3 = AO\,(m-1) \oplus \{(\langle P_0 + P_m\rangle \oplus AE\,[m+1,n]) \supset\!\!\!\!+ \langle D_1 + D_2\rangle\}\ \ (2 \leq m \leq n)\,;$$

$$L_4 = \langle P_0 + P_1\rangle \oplus AE\,[2,n]\,;$$

$$L_5 = AO\,(m) \oplus (AE\,[m+1,n] \supset\!\!\!\!+ \langle D_2 + \alpha P_0\rangle) \quad (\alpha = 0, \pm 1;\ 1 \leq m \leq n)\,;$$

$$L_6 = \langle J_{12} + P_0,\ D_2 + \alpha P_0\rangle \Subset AE\,[3,n] \quad (\alpha \in \mathbb{R})\,.$$

Proof. Let K be an I-maximal rank n subalgebra of the algebra F, $\pi(F)$ be a projection of K onto $AO(n)$ and $W = \langle P_0, P_1, \ldots, P_n\rangle \cap K$. If a projection of W onto $\langle P_0\rangle$ is nonzero, then W is conjugate to $\langle P_0 + P_m, P_{m+1}, \ldots, P_n\rangle$. In this case $\pi(K) = AO(m-1) \oplus AO[m+1,n]$ and a projection of K onto $\langle D_1, D_2\rangle$ is zero or it coincides with $\langle D_1 + D_2\rangle$. Therefore $K = L_3$ or $K = L_4$.

Let a projection of W onto $\langle P_0\rangle$ be zero. If $\dim W = n-q$, then up to conjugacy $W=\langle P_{q+1},\dots,P_n\rangle$ and $AE[q+1,n]\subset K$. In this case $\pi(K)=Q\oplus AO[q+1,n]$, where Q is a subalgebra of the algebra $AO(q)$. For $q=0$ we have the algebra L_0. Let $Q\neq 0$. Then Q is the subdirect sum of the algebras $AO[1,m_1]$, $AO[m_1+1,m_2],\dots,AO[m_{s-1}+1,m_s]$, where $m_s\le q$ and in this case a projection of K onto $\langle P_1,\dots,P_n\rangle$ is contained in $\langle P_{m_s+1},\dots,P_n\rangle$. Since the rank of Q doesn't exceed m_s-s, we have $s\le 2$. If $s=2$, then $K=L_1$.

Let $s=1$. If a projection of K onto $\langle P_0\rangle$ is zero, then K is conjugate to L_2 or L_5, where $\alpha=0$. If the projection of K onto $\langle P_0\rangle$ is nonzero, then $K=L_6$ or $K=L_5$, where $\alpha=\pm 1$. The theorem is proved.

3 Reduction of the Boussinesq equation without source

For each of the subalgebras L_1–L_6 obtained in Theorem 1, we point out the corresponding ansatz $\omega'=\varphi(\omega)$ solved for v, the invariant ω, as well as the reduced equation which is obtained by means of this ansatz. In those cases where the reduced equation can be solved, we point out the corresponding invariant solutions of the Boussinesq equation:

3.1. $v=\left(\dfrac{x_1^2+\cdots+x_m^2}{x_0}\right)^2\varphi(\omega),\quad \omega=\dfrac{x_1^2+\cdots+x_m^2}{x_{m+1}^2+\cdots+x_q^2},$ then

$$2\omega^2(1+\omega)\ \ddot\varphi+\left[(8+m)\omega-(q-m-4)\omega^2\right]\dot\varphi+2(2+m)\varphi-\varphi^{\frac12}=0.$$

The reduced equation has the solution $\varphi=\dfrac{1}{4(2+m)^2}$. The corresponding invariant solution of equation (1) is of the form

$$u=\frac{x_1^2+\cdots+x_m^2}{2(2+m)ax_o}-\frac{b}{a}. \tag{5}$$

3.2. $v=x_0^{4\alpha-2}\varphi(\omega),\quad \omega=\dfrac{x_1^2+\cdots+x_m^2}{x_0^{2\alpha}},$ then

$$2\omega\ddot\varphi+\left(m-\alpha\omega\varphi^{-\frac12}\right)\dot\varphi+(2\alpha-1)\varphi^{\frac12}=0. \tag{6}$$

For $\alpha=\dfrac14$, the reduced equation is equivalent to the equation

$$4\omega\dot\varphi+(2m-4)\varphi-\omega\varphi^{\frac12}=\widetilde{C},$$

where $\tilde C$ is an arbitrary constant. If $\tilde C$ =0, then

$$\varphi=\left[\frac{\omega}{2(m+2)}+C\omega^{\frac{2-m}{4}}\right]^2.$$

The corresponding invariant solution of the Boussinesq equation is of the form

$$u=\frac{x_1^2+\cdots+x_m^2}{2(m+2)ax_0}+C\left(x_1^2+\cdots+x_m^2\right)^{\frac{2-m}{4}}x_0^{\frac{m-6}{8}}-\frac{b}{a}.$$

The nonzero function $\varphi = (A\omega + B)^2$, where A and B are constants, satisfies equation (6) if and only if one of the following conditions holds:

1. $\alpha = \dfrac{1}{2}, \quad A = 0;$
2. $A = \dfrac{1}{4+2m}, \quad B = 0;$
3. $\alpha = \dfrac{1}{m+2}, \quad A = \dfrac{1}{4+2m}.$

By means of φ obtained above, we find the invariant solution (5) and solution

$$u = \frac{x_1^2 + \cdots + x_m^2}{2\,(2+m)\,a x_0} + B x_0^{-\frac{m}{m+2}} - \frac{b}{a}. \tag{7}$$

3.3. $v = (x_0 - x_m)^2\,\varphi(\omega)\,, \quad \omega = \dfrac{x_1^2 + \cdots + x_{m-1}^2}{(x_0 - x_m)^2}$, then

$$2\omega\,(\omega+1)\;\ddot{\varphi} + \left(m - 1 - \omega - \omega\varphi^{-\frac{1}{2}}\right)\dot{\varphi} + \varphi + \varphi^{\frac{1}{2}} = 0.$$

3.4. $v = \varphi(\omega)$, $\omega = x_0 - x_1$, then $\varphi^{\frac{1}{2}}\ddot{\varphi} + \dot{\varphi} = 0$. The general solution of the reduced equation is of the form

$$\varphi^{\frac{1}{2}} + \frac{C}{2}\ln\left|C - 2\varphi^{\frac{1}{2}}\right| = -\omega + C'.$$

The corresponding invariant solution of equation (1) is the function $u = u\,(x_0, x_1)$ given implicitly by

$$au + b + \frac{C}{2}\ln|C - 2au - 2b| = x_1 - x_0 + C',$$

where C and C' are arbitrary constants.

3.5. $v = \left(x_1^2 + \cdots + x_m^2\right)^2\varphi(\omega)\,, \quad \omega = 2x_0 - \alpha\ln\left(x_1^2 + \cdots + x_m^2\right)$, then

$$2\alpha^2\ddot{\varphi} + \left[\varphi^{-\frac{1}{2}} + (m+6)\,\alpha\right]\dot{\varphi} + 2\,(m+2)\,\varphi = 0.$$

3.6. $v = \left(x_1^2 + x_2^2\right)^2\varphi(\omega)\,, \quad \omega = \left(x_1^2 + x_2^2\right)^{-\frac{\alpha}{2}}\exp\left(x_0 + \arctan\dfrac{x_1}{x_2}\right)$, then

$$\left(1+\alpha^2\right)\omega^2\ddot{\varphi} + \left[\omega\varphi^{-\frac{1}{2}} + \left(\alpha^2 - 8\alpha + 1\right)\omega\right]\dot{\varphi} + 16\varphi = 0.$$

Notation. Case 3.1 corresponds to the subalgebra L_1; 3.2 – to L_2 and so on.

4 Reduction of the Boussinesq equation with source

Case 4.1 corresponds to the subalgebra L_1; 4.2 – to L_2 and so on. Let $2ad - bc = 0$.

4.1. $v = \left(x_1^2 + \dots + x_m^2\right)^2 \varphi(\omega)$, $\quad \omega = \left(x_1^2 + \dots + x_m^2\right)\left(x_{m+1}^2 + \dots + x_q^2\right)^{-1}$, then

$$2\omega^2(1+\omega)\ddot{\varphi} + \left[(8+m)\omega - (q-m-4)\omega^2\right]\dot{\varphi} + 2(2+m)\varphi + \delta\varphi^{\frac{1}{2}} = 0.$$

4.2. $v = \exp(-4\alpha\delta x_0)\varphi(\omega)$, $\quad \omega = \left(x_1^2 + \dots + x_m^2\right)\exp(-4\alpha\delta x_0)$, then

$$2\omega\ddot{\varphi} + \left[m + \alpha\delta\omega\varphi^{-\frac{1}{2}}\right]\dot{\varphi} + \delta(1-2\alpha)\varphi^{\frac{1}{2}} = 0. \qquad (8)$$

Integrating this reduced equation for $\alpha = \dfrac{1}{4}$, we obtain the following equation:

$$2\omega\dot{\varphi} + (m-2)\varphi + \frac{1}{2}\delta\omega\varphi^{\frac{1}{2}} = C.$$

For $C = 0$ we have

$$\varphi^{\frac{1}{2}} = \tilde{C}\omega^{\frac{2-m}{4}} - \frac{\delta}{2(m+2)}\omega.$$

The corresponding invariant solution of the Boussinesq equation is of the form

$$u = -\frac{c}{(m+2)a}\left(x_1^2 + \dots + x_m^2\right) + \tilde{C}\left(x_1^2 + \dots + x_m^2\right)^{\frac{2-m}{4}}\exp\left(-\frac{2+m}{4}cx_0\right) - \frac{b}{a},$$

where $\tilde{C}$ is an arbitrary constant.

If $\alpha = \dfrac{1}{2+m}$, then $\varphi^{\frac{1}{2}} = -\dfrac{\delta}{2(m+2)}\omega + B$ is a solution of equation (7). Thus, we find the exact solution

$$u = B'\exp\left(-\frac{4}{2+m}cx_0\right) - \frac{c}{(m+2)a}\left(x_1^2 + \dots + x_m^2\right) - \frac{b}{a}.$$

4.3. $v = \left(x_m \mathrm{e}^{-\delta x_0} + \dfrac{1}{\delta}\mathrm{e}^{-2\delta x_0}\right)^2 \varphi(\omega)$, $\quad \omega = \dfrac{x_1^2 + \dots + x_{m-1}^2}{\left(x_m + \frac{1}{\delta}\mathrm{e}^{-\delta x_0}\right)^2}$, then

$$2\omega(1+\omega)\ddot{\varphi} + (m-1-6\omega)\dot{\varphi} = 0.$$

Integrating this reduced equation we obtain

$$\varphi = C_1\int \omega^{\frac{1-m}{2}}(1+\omega)^{\frac{m+5}{2}}\,d\omega + C_2.$$

For $m = 3$ we have the invariant solution of equation (1):

$$u = \frac{1}{a}\left(x_3 + \frac{1}{2c}\mathrm{e}^{-2cx_0}\right)\mathrm{e}^{-2cx_0}\left\{C_1\left[\ln\omega + 4\omega + 3\omega^2 + \frac{4}{3}\omega^3 + \frac{\omega^4}{4}\right] + C_2\right\} - \frac{b}{a},$$

with $C_1 \neq 0$, C_2 being arbitrary constants and $\omega = \dfrac{x_1^2 + x_2^2}{\left(x_3 + \frac{1}{2c}\mathrm{e}^{-2cx_0}\right)^2}$.

4.4. $v = \mathrm{e}^{-2\delta x_0}\,\varphi(\omega)\,, \quad \omega = x_1 + \dfrac{1}{\delta}\,\mathrm{e}^{-\delta x_0}$, then $\ddot{\varphi} - \varphi^2\dot{\varphi} = 0$. This equation is equivalent to the equation

$$\dot{\varphi} - \frac{\varphi^3}{3} = C'.$$

If $C' = 0$, then $\varphi = \dfrac{\sqrt{3}}{(C - 2\omega)^{\frac{1}{2}}}$. Thus, we find the exact solution

$$u = \frac{3^{\frac{1}{6}}}{a}\frac{\mathrm{e}^{-2cx_0}}{\left(\tilde{C} - 2x_1 - \frac{1}{c}\,\mathrm{e}^{-2cx_0}\right)^{\frac{1}{4}}} - \frac{b}{a}.$$

4.5. $v = \dfrac{\left(x_1^2 + \cdots + x_m^2\right)^2}{\mathrm{e}^{2\delta x_0}}\varphi(\omega)\,, \quad \omega = \alpha\delta\ln\left(x_1^2 + \cdots + x_m^2\right) + 2\,\mathrm{e}^{-\delta x_0}$, then

$$4\alpha^2\delta^2\ddot{\varphi} + 2\alpha\delta\,(m+6)\,\dot{\varphi} - 2\delta\varphi^{\frac{1}{2}}\dot{\varphi} + 4\,(m+2)\,\varphi = 0.$$

For $\alpha = 0$ we find the exact solution of equation (1):

$$u = \left(x_1^2 + \cdots + x_m^2\right)\mathrm{e}^{-2cx_0}\left[\tilde{C} + \frac{m+2}{ac}\,\mathrm{e}^{-2cx_0}\right] - \frac{b}{a}.$$

4.6. $v = \left(x_1^2 + x_2^2\right)^2\mathrm{e}^{-2\delta x_0}\,\varphi(\omega)\,, \quad \omega = \left(x_1^2 + x_2^2\right)^{\frac{\alpha}{2}}\exp\left(\dfrac{1}{\delta}\,\mathrm{e}^{-\delta x_0} - \arctan\dfrac{x_1}{x_2}\right)$, then

$$\left(1 + \alpha^2\right)\omega^2\ddot{\varphi} + \left[\left(\alpha^2 + 8\alpha + 1\right)\omega - \omega\varphi^{-\frac{1}{2}}\right]\dot{\varphi} + 16\varphi = 0.$$

5 Application to heat conduction of non-linear materials

The exact solution of the Boussinesq equation obtained in the previous section can be applied to calculate the temperature distribution in metals.

Heat conduction of platinum is described by the coefficient of heat conduction [3]

$$\lambda_{PT}(u) = 0,0156u + 68,75$$

depending on the temperature u. Function (7) written for $m = 3$ and $B = -1$ as

$$u(t,r) = 64,103r^2t^{-1} - t^{-\frac{3}{5}} - 4,407\cdot 10^3$$

describes the temperature distribution in a platinum ball

$$r^2 \le 1 \qquad \left(x_1^2 + x_2^2 + x_3^2 \le 1\right),$$

when the external boundary temperature is of the form

$$u(t,1) = 64,103t^{-1} - t^{-\frac{3}{5}} - 4,407\cdot 10^3.$$

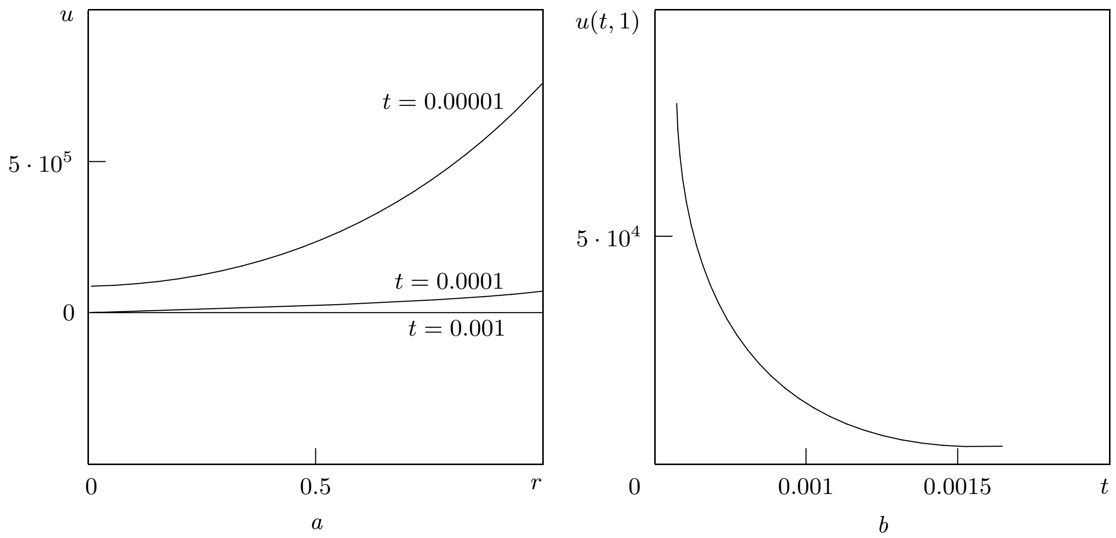

Fig.1. Temperature distribution $u(t,r)$ of a platinum ball: *a)* temperatures $u(t,r)$ at the times t; *b)* boundary temperature $u(t,r)$ at the time $[0; 0, 002]$.

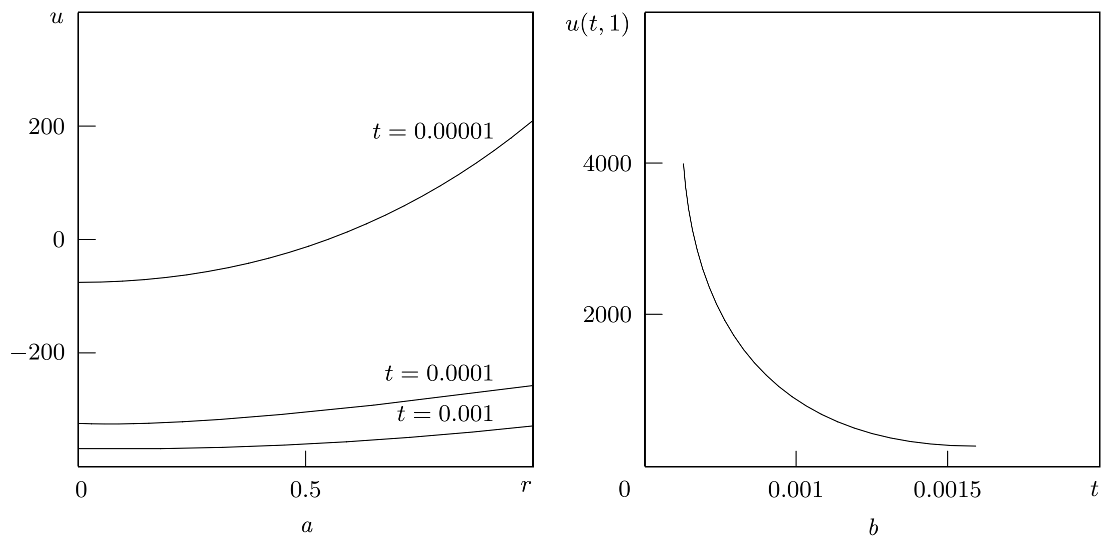

Fig.2. Temperature distribution $u(t,r)$ of a beryllium ball: *a)* temperatures $u(t,r)$ at the times t; *b)* boundary temperature $u(t,r)$ at the time $[0; 0, 002]$.

Heat conduction of beryllium is described by the heat conduction coefficient [3]

$$\lambda_B(u) = \frac{1}{3}u + 158$$

depending on the temperature u. Function (7) written for $m = 3$ and $B = -2$ as

$$u(t,r) = \frac{3r^2}{10t} - 2t^{-\frac{3}{5}} - 474$$

describes the temperature distribution in a beryllium ball

$$r^2 \leq 1 \qquad \left(r^2 = x_1^2 + x_2^2 + x_3^2\right),$$

when the external boundary temperature is of the form

$$u\,(t,1) = \frac{3}{10}t^{-1} - 2t^{-\frac{3}{5}} - 474.$$

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Symmetries for a Class of Explicitly Space- and Time-Dependent (1+1)-Dimensional Wave Equations

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Abstract

The general d'Alembert equation $\Box u + f(t, x, u) = 0$ is considered, where $\Box$ is the two-dimensional d'Alembert operator. We classify the equation for functions f by which it admits several Lie symmetry algebras, which include the Lorentz symmetry generator. The conditional symmetry properties of the equation are discussed.

1 Introduction

In the present paper, we derive some results on the invariants of the nonlinear wave equation

$$\Box u + f(x_0, x_1, u) = 0, \qquad (1.1)$$

where $\Box := \partial^2/\partial x_0^2 - \partial^2/\partial x_1^2$ and f is an arbitrary smooth function of its arguments, to be determined under some invariance conditions.

It is well known that Lie transformation groups play an important role in the investigation of nonlinear partial differential equations (PDEs) in modern mathematical physics. If a transformation leaves a PDE invariant, the PDE is said to possess a symmetry. A particular class of symmetries, known as the Lie point symmetries, has been studied by several authors (see, for example, the books of Ovsyannikov [10], Olver [9], Fushchych *et al.* [7], Ibragimov [8], Steeb [11]). Lie symmetries of nonlinear PDEs may be used to construct exact solutions and conservation laws for the equations (Fushchych *et al.* [7]). The classification of PDEs with respect to their Lie symmetry properties is an important direction in nonlinear mathematical physics. In particular the book of Fushchych, Shtelen and Serov [7] is devoted to the classification of several classes of nonlinear PDEs and systems of PDEs admitting several fundamental Lie symmetry algebras, such as the Poincaré algebra, the Euclidean and Galilean algebras, and the Schrödinger algebra. They mostly consider equations in (1+3)-dimensions as well as arbitrary-dimensional equations, usually excluding the $(1+1)$-dimensional cases. The classification of the $(1+1)$-dimensional wave equation (1.1) is the main theme in the present paper. The invariance of (1.1) with respect to the most general Lie point symmetry generator, Lie symmetry algebras of relativistic invariance, and conditional invariance is considered. We present the theorems without proofs. The proofs are given in Euler *et al.* [2].

2 The General Lie point symmetry generator

Before we classify (1.1) with respect to a particular set of Lie symmetry generators, we establish the general invariance properties of (1.1).

Theorem 1. *The most general Lie point symmetry generator for (1.1) is of the form*

$$Z = \{g_1(y_1) + g_2(y_2)\}\frac{\partial}{\partial x_0} + \{g_1(y_1) - g_2(y_2)\}\frac{\partial}{\partial x_1} + \{ku + h(x_0, x_1)\}\frac{\partial}{\partial u}, \tag{2.1}$$

where g_1, g_2, and h are arbitrary smooth functions of their arguments and $k \in \mathcal{R}$. One must distinguish between three cases:

a) For $g_1 \neq 0$ and $g_2 \neq 0$, the following form of (1.1) admits (2.1):

$$\Box u + \frac{\exp(k\varepsilon)}{g_1(y_1)g_2(y_2)}\left\{-4\int g_1(y_1)g_2(y_2)\frac{\partial^2 h}{\partial y_1 \partial y_2}\exp(-k\varepsilon)\,d\varepsilon + G(Y_1, Y_2)\right\} = 0. \tag{2.2}$$

Here, G is an arbitrary smooth function of its arguments, and

$$\frac{dy_1}{d\varepsilon} = 2g_1(y_1), \qquad \frac{dy_2}{d\varepsilon} = 2g_2(y_2),$$
$$Y_1 = \int \frac{dy_1}{g_1(y_1)} - \int \frac{dy_2}{g_2(y_2)},$$
$$Y_2 = u\exp(-k\varepsilon) - \int h(\varepsilon)\exp(-k\varepsilon)\,d\varepsilon,$$
$$y_1 = x_0 + x_1, \qquad y_2 = x_0 - x_1.$$

b) For $g_1 = 0$ and $g_2 \neq 0$, the following form of (1.1) admits (2.1):

$$\Box u + G(Y_1, Y_2)\, g_2(y_2)^{-1}\exp(k\varepsilon) = 0, \tag{2.3}$$

where G is an arbitrary smooth function of its arguments, and

$$Y_1 = x_0 + x_1, \qquad Y_2 = u\exp(-k\varepsilon) - \int h(\varepsilon)\exp(-k\varepsilon)d\varepsilon,$$
$$\frac{dy_2}{d\varepsilon} = 2g_2(y_2), \qquad y_2 = x_0 - x_1.$$

c) For $g_1 \neq 0$ and $g_2 = 0$, the following form of (1.1) admits (2.1):

$$\Box u + G(Y_1, Y_2)\, g_1(y_1)^{-1}\exp(k\varepsilon) = 0, \tag{2.4}$$

where G is an arbitrary smooth function of its arguments, and

$$Y_1 = x_0 - x_1, \qquad Y_2 = u\exp(-k\varepsilon) - \int h(\varepsilon)\exp(-k\varepsilon)d\varepsilon,$$
$$\frac{dy_1}{d\varepsilon} = 2g_1(y_1), \qquad y_1 = x_0 + x_1.$$

3 A particular Lie symmetry algebra

As a special case of the above general invariance properties, we now turn to the classification of (1.1) with respect to the invariance under the Lorentz, scaling, and conformal transformations, the Lie generators of which are given by

$$\begin{aligned}
&L_{01} = x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1}, \qquad S = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1} + \lambda u \frac{\partial}{\partial u},\\
&K_0 = (x_0^2 + x_1^2) \frac{\partial}{\partial x_0} + 2x_0 x_1 \frac{\partial}{\partial x_1} + \alpha(x_0, x_1) \frac{\partial}{\partial u},\\
&K_1 = -2x_0 x_1 \frac{\partial}{\partial x_0} - (x_0^2 + x_1^2) \frac{\partial}{\partial x_1} - \beta(x_0, x_1) \frac{\partial}{\partial u}.
\end{aligned} \tag{3.1}$$

Here, α and β are arbitrary smooth functions and $\lambda \in \mathcal{R}$, to be determined for the particular Lie symmetry algebras. We are interested in the 4-dimensional Lie symmetry algebra spanned by $\{L_{01}, S, K_0, K_1\}$, the 3-dimensional Lie symmetry algebra spanned by $\{L_{01}, K_0, K_1\}$, the 2-dimensional case $\{L_{01}, S\}$ as well as the invariance of (1.1) under the Lorentz transformation generated by $\{L_{01}\}$. The following Lemma gives the conditions on α and β for the closure of the Lie algebras:

Lemma.
a) The generators $\{L_{01}, S, K_0, K_1\}$ span the 4-dimensional Lie algebra with commutation relations as given in the commutator table below if and only if

$$\alpha(x_0, x_1) = c x_0 (x_0^2 - x_1^2)^{\lambda/2}, \qquad \beta(x_0, x_1) = c x_1 (x_0^2 - x_1^2)^{\lambda/2}, \tag{3.2}$$

where c is an arbitrary real constant.

b) The generators $\{L_{01}, K_0, K_1\}$ span the 3-dimensional Lie algebra with commutations as given in the commutator table below if and only if

$$\begin{aligned}
&\alpha(x_0, x_1) = (x_0 + x_1)\phi(y) + (x_0 + x_1)^{-1}\psi(y),\\
&\beta(x_0, x_1) = (x_0 + x_1)\phi(y) - (x_0 + x_1)^{-1}\psi(y),
\end{aligned} \tag{3.3}$$

where ϕ and ψ are restricted by the condition

$$y^2 \frac{d\phi}{dy} - y \frac{d\psi}{dy} + \psi = 0, \tag{3.4}$$

with $y = x_0^2 - x_1^2$.

Commutator Table

	L_{01}	S	K_0	K_1
L_{01}	0	0	$-K_1$	$-K_0$
S	0	0	K_0	K_1
K_0	K_1	$-K_0$	0	0
K_1	K_0	$-K_1$	0	0

Using the Lemma, we can prove the following four theorems:

Theorem 2. *Equation (1.1) admits the 4-dimensional Lie symmetry algebra spanned by the Lie generators* $\{L_{01}, S, K_0, K_1\}$ *given by (3.1) if and only if* α, β, *and equation (1.1) are of the following forms:*

a) For $\lambda \neq 0$,

$$\alpha(x_0, x_1) = c_1 x_0 (x_0^2 - x_1^2)^{\lambda/2}, \qquad \beta(x_0, x_1) = c_1 x_1 (x_0^2 - x_1^2)^{\lambda/2},$$

whereby (1.1) takes the form

$$\Box u - \lambda c_1 y^{(\lambda-2)/2} + y^{-2} c_2 \left(u - \frac{c_1}{\lambda} y^{\lambda/2} \right)^{(\lambda+2)/\lambda} = 0 \tag{3.5}$$

with $c_1, c_2 \in \mathcal{R}$ *and* $y = x_0^2 - x_1^2$.

b) For $\lambda = 0$,

$$\alpha(x_0, x_1) = c_1 x_0, \qquad \beta(x_0, x_1) = c_1 x_1,$$

whereby (1.1) takes the form

$$\Box u + y^{-1} \exp\left(-\frac{2}{c_1} u \right) = 0 \tag{3.6}$$

with $c_1 \in \mathcal{R} \backslash \{0\}$ *and* $y = x_0^2 - x_1^2$.

Theorem 3. *Equation (1.1) admits the 3-dimensional Lie symmetry algebra spanned by the Lie generators* $\{L_{01}, K_0, K_1\}$ *given by (3.1) if and only if* α, β, *and equation (1.1) are of the following forms:*

a) For f *linear in* u, *we yield*

$$\alpha(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_1 y^{-1} \ln y + k_4 \right\} + (x_0 + x_1)^{-1} \left\{ k_1 \ln y + k_2 y + k_3 \right\},$$

$$\beta(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_1 y^{-1} \ln y + k_2 \right\} - (x_0 + x_1)^{-1} \left\{ k_1 \ln y + k_2 y + k_3 \right\},$$

and (1.1) takes the form

$$\Box u - \frac{1}{y^2} \left(\frac{2k_1}{k_4 - k_2} u + \frac{2k_1(k_3 + k_1)}{k_4 - k_2} y^{-1} + \frac{2k_1^2}{k_4 - k_2} y^{-1} \ln y - 4k_1 \ln y + k_5 \right) = 0,$$

where $y = x_0^2 - x_1^2$ *and* $k_1, \ldots, k_5$ *are arbitrary real constants with* $k_1 \neq 0$, $k_4 \neq k_2$.

b) For f *independent of* u, *we have*

$$\alpha(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_4 \right\} + (x_0 + x_1)^{-1} \left\{ k_2 y + k_3 \right\},$$

$$\beta(x_0, x_1) = (x_0 + x_1) \left\{ k_3 y^{-1} + k_4 \right\} - (x_0 + x_1)^{-1} \left\{ k_2 y + k_3 \right\},$$

and (1.1) takes the form

$$\Box u + c y^{-2} = 0,$$

where k_2, k_3, k_4 *are arbitrary real constants and* $y = x_0^2 - x_1^2$.

c) For f nonlinear in u, it holds that

$$\alpha(x_0, x_1) = (x_0 + x_1)\{k_3 y^{-1} + k_2\} + (x_0 + x_1)^{-1}\{k_2 y + k_3\},$$

$$\beta(x_0, x_1) = (x_0 + x_1)\{k_3 y^{-1} + k_2\} - (x_0 + x_1)^{-1}\{k_2 y + k_3\},$$

whereby (1.1) takes the form

$$\Box u + y^{-2} g\left(u - k_2 \ln y + k_3 y^{-1}\right) = 0. \tag{3.7}$$

Here, k_2 and k_3 are arbitrary real constants, $y = x_0^2 - x_1^2$, and g is an arbitrary smooth function of its argument.

Theorem 4. *Equation (1.1) admits the 2-dimensional Lie symmetry algebra spanned by the Lie generators $\{L_{01}, S\}$ given by (3.1) if and only if (1.1) takes the following forms:*

a) For $\lambda = 0$, (1.1) takes the form

$$\Box u + y^{-1} g(u) = 0,$$

where g is an arbitrary function of its argument and $y = x_0^2 - x_1^2$.

b) For $\lambda \neq 0$, (1.1) takes the form

$$\Box u + u^{(\lambda-2)/\lambda} g\left(y^{-\lambda/2} u\right) = 0, \tag{3.8}$$

where g is an arbitrary function of its argument and $y = x_0^2 - x_1^2$.

Theorem 5. *Equation (1.1) admits the Lorentz transformation generated by $\{L_{01}\}$ if and only if (1.1) takes the form*

$$\Box u + g(y, u) = 0, \tag{3.9}$$

where g is an arbitrary function of its arguments and $y = x_0^2 - x_1^2$.

4 Lie symmetry reductions

In this section, we reduce the nonlinear equations stated in the above theorems to ordinary differential equations. This is accomplished by the symmetry Ansätze which are obtained from the first integrals of the Lie equations.

The invariants and Ansätze of interest are listed in Table 1 and the corresponding reductions in Table 2.

Remark. The properties of the reduced equations may, for example, be studied by the use of Lie point transformations and the Painlevé analysis. Some of the equations listed in Table 2 were considered by Euler [3]. In particular, the transformation properties of the equation

$$\ddot{\varphi} + f_1(\omega)\dot{\varphi} + f_2(\omega)\varphi + f_3(\omega)\varphi^n = 0,$$

where f_1, f_2, and f_3 are smooth functions and $n \in \mathcal{Q}$, were studied in detail by Euler [3].

Table 1

Generator	ω	$u(x_0,x_1)=f_1(x_0,x_1)\varphi(\omega)+f_2(x_0,x_1)$
L_{01}	$\omega=x_0^2-x_1^2$	$f_1=1, \quad f_2=0$
S	$\omega=\dfrac{x_0}{x_1}$	$f_1=x_0^\lambda, \quad f_2=0$
K_0	$\omega=\dfrac{x_0^2-x_1^2}{x_1}$	Theorem 2a: $f_1=1, \quad f_2=\dfrac{c_1}{\lambda}\omega^{\lambda/2}x_1^{\lambda/2}$ Theorem 2b: $f_1=1, \quad f_2=\dfrac{c_1}{2}\ln x_1$ Theorem 3c: $f_1=1, \quad f_2=k_2\ln x_1-\dfrac{k_3}{\omega x_1}$
K_1	$\omega=\dfrac{x_0^2-x_1^2}{x_0}$	Theorem 2a: $f_1=1, \quad f_2=\dfrac{c_1}{\lambda}\omega^{\lambda/2}x_0^{\lambda/2}$ Theorem 2b: $f_1=1, \quad f_2=\dfrac{c_1}{2}\ln x_0$ Theorem 3c: $f_1=1, \quad f_2=k_2\ln x_0-\dfrac{k_3}{\omega x_0}$

Table 2

We refer to ...	Reduced Equation
Theorem 2a	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}-c_1\lambda\omega^{(\lambda-2)/2}+c_2\omega^{-2}\left(\varphi-\dfrac{c_1}{\lambda}\omega^{\lambda/2}\right)^{(\lambda+2)/\lambda}=0$ S : $\omega^2(\omega^2+1)\ddot{\varphi}-2\omega(\omega^2-\lambda)\dot{\varphi}+\lambda(\lambda-1)\varphi-c_1\lambda(1-\omega^{-2})^{(\lambda-2)/2}$ $+c_2(1-\omega^{-2})^{-2}\left(\varphi-\dfrac{c_1}{\lambda}(1-\omega^{-2})^{\lambda/2}\right)^{(\lambda+2)/\lambda}=0$ K_0^- and K_1^+ : $\omega^2\ddot{\varphi}+2\omega\dot{\varphi}\mp c_2\omega^{-2}\varphi^{(\lambda+2)/\lambda}=0$
Theorem 2b	L_{01} : $4\omega\ddot{\varphi}+4\dot{\varphi}+\omega^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$ S : $(\omega^2-1)\ddot{\varphi}+2\omega\dot{\varphi}+(1-\omega^2)^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$ K_0^+ and K_1^- : $\omega^2\ddot{\varphi}+2\omega\dot{\varphi}\pm\dfrac{c_1}{2}+\omega^{-1}\exp\left(-\dfrac{2\varphi}{c_1}\right)=0$

Table 2 (Continued)

We refer to ...	Reduced Equation
Theorem 3c	L_{01} : $4\omega\ddot{\varphi} + 4\dot{\varphi} + \omega^{-2} g\left(\varphi - k_2 \ln\omega + k_3\omega^{-1}\right) = 0$ K_0^- and K_1^+ : $\omega^2\ddot{\varphi} + 2\omega\dot{\varphi} + 4k_3\omega^{-2} - k_2 \mp \omega^{-2} g\left(\varphi - k_2\ln\omega\right) = 0$
Theorem 4a	L_{01} : $4\omega\ddot{\varphi} + 4\dot{\varphi} + \omega^{-1} g(\varphi) = 0$ S : $(1-\omega^2)\ddot{\varphi} - 2\omega\dot{\varphi} - (1-\omega^2)^{-1} g(\varphi) = 0$
Theorem 4b	L_{01} : $4\omega\ddot{\varphi} + 4\dot{\varphi} + \varphi^{(\lambda-2)/\lambda} g\left(\omega^{-\lambda/2}\varphi\right) = 0$ S : $2\omega^2(1-\omega^2)\ddot{\varphi} + 2\omega(\lambda-\omega^2)\dot{\varphi} + \lambda(\lambda-1)\varphi$ $+\varphi^{(\lambda-2)/\lambda} g\left((\omega^2-1)^{-\lambda/2}\varphi\right) = 0$
Theorem 5	L_{01} : $4\omega\ddot{\varphi} + 4\dot{\varphi} + g(\omega,\varphi) = 0$

5 Conditional symmetries

An extension of the classical Lie symmetry reduction of PDEs may be realized as follows: Consider the compatibility problem posed by the following two equations

$$F \equiv \Box u + f(x_0, x_1, u) = 0, \tag{5.1}$$

$$Q \equiv \xi_0(x_0, x_1, u)\frac{\partial u}{\partial x_0} + \xi_1(x_0, x_1, u)\frac{\partial u}{\partial x_1} - \eta(x_0, x_1, u) = 0. \tag{5.2}$$

Here, (5.1) is the invariant surface condition for the symmetry generator

$$Z = \xi_0(x_0, x_1, u)\frac{\partial}{\partial x_0} + \xi_1(x_0, x_1, u)\frac{\partial}{\partial x_1} + \eta(x_0, x_1, u)\frac{\partial}{\partial u}.$$

A necessary and sufficient condition of compatibility on ξ_0, ξ_1, and η is given by the following invariance condition (Fushchych *et al.* [7], Euler *et al.* [4], Ibragimov [8])

$$Z^{(2)}\Big|_{F=0,\ Q=0} = 0. \tag{5.3}$$

A generator Z satisfying (5.3) is known as a Q-conditional Lie symmetry generator (Fushchych *et al.* [7]). Note that conditional symmetries were first introduced by Bluman and Cole [1] in their study of the heat equation.

Let us now study the Q-symmetries of (1.1). It turns out that it is more convenient to transform (1.1) in light-cone coordinates, i.e., the transformation

$$x_1 \to \frac{1}{2}(x_0 + x_1), \qquad x_0 \to \frac{1}{2}(x_0 - x_1), \qquad u \to u.$$

Without changing the notation, we now consider the system (written in jet coordinates)

$$F \equiv u_{01} + f(x_0, x_1, u) = 0,$$
$$Q \equiv u_0 + \xi_1(x_0, x_1, u)u_1 - \eta(x_0, x_1, u) = 0,$$

where we have normalized ξ_0. After applying the invariance condition (5.3) and equating to zero the coefficients of the jet coordinates 1, u_1, u_1^2, u_1^3, u_{11}, and $u_1 u_{11}$, we obtain the nonlinear determining equations:

$$\frac{\partial \xi_1}{\partial u} = 0, \qquad \frac{\partial \xi_1}{\partial x_0} = 0, \qquad \frac{\partial^2 \eta}{\partial u^2}\xi_1 = 0, \tag{5.4}$$

$$\frac{\partial^2 \eta}{\partial x_0 \partial u} - \frac{\partial^2 \xi_1}{\partial x_0 \partial x_1} + \frac{\partial^2 \eta}{\partial u^2}\eta - \frac{\partial^2 \eta}{\partial x_1 \partial u}\xi_1 = 0, \tag{5.5}$$

$$\frac{\partial f}{\partial x_0} + \xi_1 \frac{\partial f}{\partial x_1} + \eta \frac{\partial f}{\partial u} + f\left(\frac{\partial \xi_1}{\partial x_1} - \frac{\partial \eta}{\partial u}\right) + \frac{\partial^2 \eta}{\partial x_1 \partial u}\eta + \frac{\partial^2 \eta}{\partial x_0 \partial x_1} = 0. \tag{5.6}$$

According to (5.4), we need to consider two cases:

Case 1. $\dfrac{\partial^2 \eta}{\partial u^2} = 0$ and $\xi_1 = \xi_1(x_1)$.
By solving (5.5), η takes on the form

$$\eta(x_0, x_1, u) = \phi(z)u + h(x_0, x_1), \qquad z = x_0 + \int \frac{dx_1}{\xi_1(x_1)}, \tag{5.7}$$

where ϕ and h are arbitrary smooth functions of their arguments. The condition on f is given by (5.6), i.e., the following linear first order PDE

$$\begin{aligned} &\frac{\partial f}{\partial x_0} + \xi(x_1)\frac{\partial f}{\partial x_1} + (\phi(z)u + h(x_0, x_1))\frac{\partial f}{\partial u} + \left(\frac{d\xi_1}{dx_1} - \phi(z)\right) f \\ &\qquad + \frac{u}{\xi_1(x_1)}\left(\phi'(z)\phi(z) + \phi''(z)\right) + \frac{h(x_0, x_1)}{\xi(x_1)} + \frac{\partial^2 h}{\partial x_0 \partial x_1} = 0. \end{aligned} \tag{5.8}$$

Since ϕ is not a constant, as in the case of a Lie symmetry generator (see Theorem 1), it is clear that there exist non-trivial Q-symmetry generators of the form

$$Z = \frac{\partial}{\partial x_0} + \xi_1(x_1)\frac{\partial}{\partial x_1} + \{\phi(z)u + h(x_0, x_1)\}\frac{\partial}{\partial u}.$$

For given functions ϕ, h, and ξ_1, the form of f may be determined by solving (5.8).

Case 2. $\dfrac{\partial^2 \eta}{\partial u^2} \neq 0$ and $\xi_1 = 0$.
The determining equations reduce to

$$\frac{\partial^2 \eta}{\partial x_0 \partial u} + \frac{\partial^2 \eta}{\partial u^2}\eta = 0, \qquad \frac{\partial f}{\partial x_0} + \eta\frac{\partial f}{\partial u} - \frac{\partial \eta}{\partial u}f + \frac{\partial^2 \eta}{\partial x_1 \partial u}\eta + \frac{\partial^2 \eta}{\partial x_0 \partial x_1} = 0 \tag{5.9}$$

Any solution of (5.9) determines f and η for which system (5.1)–(5.2) is compatible. In this case, the non-trivial Q-symmetry generators are of the form

$$Z = \frac{\partial}{\partial x_0} + \eta(x_0, x_1, u)\frac{\partial}{\partial u}.$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 79–88.

Symmetry in Perturbation Problems

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Abstract

The work is devoted to a new branch of application of continuous group's techniques in the investigation of nonlinear differential equations.

The principal stages of the development of perturbation theory of nonlinear differential equations are considered in short. It is shown that its characteristic features make it possible a fruitful usage of continuous group's techniques in problems of perturbation theory.

1. Introduction

The idea of introducing coordinate transformations to simplify the analytic expression of a general problem is a powerful one. Symmetry and differential equations have been close partners since the time of the founding masters, namely, Sophus Lie (1842–1899), and his disciples. To these days, symmetry has continued to play a strong role. The ideas of symmetry penetrated deep into various branches of science: mathematical physics, mechanics and so on.

The role of symmetry in perturbation problems of nonlinear mechanics, which was already used by many investigators since the 70-th years (J. Moser, G. Hori, A. Kamel, U. Kirchgraber), has been developed considerably in recent time to gain further understanding and development of such constructive and powerful methods as averaging and normal form methods.

The principal stages of the development of perturbation theory of nonlinear differential equations connected with the fundamental works by A. Poincaré, A.M. Lyapunov, Van der Pol, N.M. Krylov, N.N. Bogolyubov, and their followers are considered. The growing role of symmetry (and, accordingly, of continuous group's techniques) is shown by the example of perturbation theory's problems.

2. Short survey of perturbation theory

2.1. The basic problem

Let us consider the system

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= x + \varepsilon x^3, \end{aligned} \tag{1}$$

which is equivalent to Duffing's equation, ε is a small positive parameter.

When $\varepsilon = 0$, the system of *"zero approximation"* has the periodic solution

$$x = \cos t, \quad y = \sin t.$$

The main question is: has system (1) also a periodic solution when $\varepsilon \neq 0$ *and is sufficiently small?*

It is naturally to try to find a periodic solution of nonlinear system (1) as a series

$$\begin{aligned} x &= \cos t + \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \cdots, \\ y &= \sin t + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \cdots. \end{aligned} \tag{2}$$

Substitution of series (2) in equation (1) replaces the original system by an infinite sequence of simple systems of equations

$$\begin{aligned} &\dot{u}_1 = -v_1, \\ &\dot{v}_1 = u_1 + \cos^3 t, \\ &u_1(0) = 1, \quad v_1(0) = 0; \\ &\dot{u}_k = -v_k, \\ &\dot{v}_k = u_k + f_k(t, u_1(t), \ldots, u_{k-1}(t)), \\ &u_k(0) = 0, \quad v_k(0) = 0, \qquad k = 2, 3, \ldots. \end{aligned} \tag{3}$$

Let us solve system (3). Excepting the function v_1, one comes to the equation of the second order

$$\ddot{u}_1 + u_1 = \cos^3 t = \frac{4}{3}\cos t + \frac{1}{4}\cos 3t. \tag{4}$$

The general solution of (4) is

$$u_1 = A\cos t + B\sin t + \frac{3}{8}t\sin t - \frac{1}{9}\cos 3t.$$

Constants A and B are to be found to satisfy an initial value.

So, already the first member of series (2) has an addend $t\sin t$ and, as a result, the functions x, y are not periodic. It is easy to see that among subsequent members of series (2) there will be also terms of kind $t^n \sin t$. Such terms in celestial mechanics are called *"secular terms"*.

2.2. Principal stages of the development of perturbation theory

All the long-standing history of solution of nonlinear problems of form (1) (and more complicate ones) was connected with the construction of solutions as series (2) which do not contain secular terms. Here, let us go into three most important stages of the development of perturbation theory.

– Works by A. Poincaré and A.M. Lyapunov.

– Works by Van der Pol, N.M. Krylov and N.N. Bogoliubov.

– Group-theoretic methods in perturbation theory.

Works by A. Poincaré and A.M. Lyapunov

The problem formulated in section 2.1 was completely solved in works of the authors cited above. Really, they have received more general results, but there is no necessity of considering them here.

Theoretical bases for solving problem (1) are given by the following two theorems.

Let us formulate Poincaré's theorem suitably to the system of form (1). (The complete formulation see, for example, in [1], p.39.)

A. Poincaré's theorem. *The solutions of system (1) are analytic functions of the parameter ε, i.e., series converge when the absolute value of ε is sufficiently small and, hence, they are solutions of system (1), expanded in an infinite power series in the parameter ε.*

Therefore, the presentation of solutions of system (1) as series (2) is quite true. Its default is that the periodic solution of system (1) is expanded into series in nonperiodic functions.

Let us consider a more general system of the second order

$$\begin{aligned} \dot{x} &= -y + X(x, y), \\ \dot{y} &= x + Y(x, y), \end{aligned} \tag{5}$$

where $X(x, y), Y(x, y)$ are analytic functions.

A.M. Lyapunov's theorem. *If system (5) has the analytic first integral*

$$H(x, y) = x^2 + y^2 + R(x, y) = \mu \tag{6}$$

and μ is sufficiently small, then it has a family of solutions periodic in t.

The period of these functions tends to 2π when $\mu \to 0$. The solutions of system (5) are analytic functions of a quantity c, the initial deviation of variables x, y.

A.M. Lyapunov had proved also the inverse statement. Hence, the existence of integral (6) for system (5) is necessary and sufficient condition for the existence for system (5) in the neighborhood of the origin of the coordinate system of periodic solutions which depend upon an arbitrary constant c.

On making in (5) change of variables

$$x = \varepsilon\bar{x}, \qquad y = \varepsilon\bar{y},$$

one easy comes from system (5) to the one with a small parameter of the form (1).

System (1) has the first integral in elliptic functions. Hence, A.M. Lyapunov's theorem can be applied to it: there exists a periodic solution in the neighborhood of the origin of the coordinate system.

A.M. Lyapunov had gave an effective algorithm of construction of solutions of system (5) as series. The algorithm uses the change of variable

$$t = \tau(1 + c^2 h_2 + c^3 h_3 + \cdots),$$

where $h_2, h_3, \ldots$ are some constants which are to be find in the process of calculations.

A. Poincaré's method of defining the autoperiodic oscillations of equations of the form

$$\ddot{x} + \lambda y = \varepsilon F(x, \dot{x})$$

uses the change of variable

$$t = \frac{\tau}{\lambda}(1 + g_1\varepsilon + g_2\varepsilon^2 + \cdots),$$

where $g_1, g_2, \ldots$ are some constants which are to be find in the process of calculations.

In conclusion, let us note the typical features of A. Poincaré's and A.M. Lyapunov's methods.

- The creation of constructive algorithms of the producing of periodic solutions as series, which do not contain secular terms.
- The active transformation of the initial system: the introduction of arbitrary variables, which are to be find in the process of calculations.
- The base of the developed algorithms consists in the proof of analyticity of the series which represent a desired periodic solution.

For more details about questions touched here, see, for example, [1].

Researches by Van der Pol, N.M. Krylov and N.N. Bogolyubov

The next stage in perturbation theory is connected with names of the scientists cited above. A typical object of their investigations is the system of nonlinear equations

$$\begin{aligned} \dot{x} &= \varepsilon X(\varepsilon, x, y), \\ \dot{y} &= \omega(x) + \varepsilon Y(\varepsilon, x, y), \end{aligned} \tag{7}$$

where $x \in R^n, y \in R^1$, functions $X(\varepsilon, x, y), Y(\varepsilon, x, y)$ are supposed to be periodical in y of the period T.

One looks for solutions of system (7) as series

$$\begin{aligned} x &= \bar{x} + \varepsilon u_1(\bar{x}, \bar{y}) + \varepsilon^2 u_2(\bar{x}, \bar{y}) + \cdots, \\ y &= \bar{y} + \varepsilon v_1(\bar{x}, \bar{y}) + \varepsilon^2 v_2(\bar{x}, \bar{y}) + \cdots. \end{aligned} \tag{8}$$

Functions $u_i(\bar{x}, \bar{y})$, $v_i(\bar{x}, \bar{y})$ are unknown yet and are to be found in the process of solving the problem. Variables $\bar{x}(t), \bar{y}(t))$ must satisfy a system

$$\begin{aligned} \dot{\bar{x}} &= \varepsilon A_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \cdots, \\ \dot{\bar{y}} &= \omega(\bar{x}) + \varepsilon B_1(\bar{x}) + \varepsilon^2 B_2(\bar{x}) + \cdots. \end{aligned} \tag{9}$$

Functions A, B in (9) are also unknown and are to be found in the process of calculations.

Therefore, there is a problem of transformation of the original system (7) to a new one (9) more simple for investigation. This transformation actively influences the system as it contains uncertain functions to be found.

An original algorithm close to the above described scheme for $n = 1$ was first suggested by the Dutch engineer Van der Pol in the 20-th years. His method had beautiful clearness and was convenient for design calculations. It very quickly became popular among engineers. But no proof of the method existed. That is why it was out of mathematics for a long time (like Heaviside's method).

In the 30-th years N.M. Krylov and N.N. Bogolyubov suggested the just cited above general scheme (7)–(9) for investigation of systems like (7). It started the creation of a rigorous theory of nonlinear oscillations developed in the subsequent decades.

N.N. Bogolyubov and his pupils also investigated systems

$$\dot{x} = \varepsilon X(\varepsilon, x, y),$$

which were called *standard form's systems.* They created the strictly proved *method of averaging*, which is successfully applied for investigation of nonlinear systems with a small parameter.

In conclusion, let us note typical features of the considered period of the theory of perturbations.

- The transformation of the original system to a simplified one. This transformation is active as it contains unknown functions to be found.
- The convergence of series of the form (8) is not investigated. Instead, the asymptotic nearness is investigated, i.e., the existence of relations

  $$x \to \bar{x}, y \to \bar{y}, \quad \text{when} \quad \varepsilon \to 0.$$

- The essential weakening of the demands on the analytic characteristics of the right-hand sides of (8).
- The essential extension of classes of the problems under consideration: searching for periodic solutions, limit cycles, the description of transition processes, resonances and so on.

One can find the detailed exposition of the questions touched here, for example, in [2], [3].

3. Group-theoretical methods in perturbation theory

3.1. Short survey

J. Moser [4] used the group-theoretic approach in the investigation of quasiperiodic solutions of nonlinear systems. Lie's rows and transformations in the perturbation problems were used by G. Hori [5], [6], A. Kamel [7], U. Kirchgraber and E. Steifel [8], U. Kirchgraber [9], Bogaevsky V.N., Povzner A.Ya. [10], and Zhuravlev V.F., Klimov D.N. [11].

Asymptotic methods of nonlinear mechanics developed by N.M. Krylov, N.N. Bogolyubov and Yu.A. Mitropolsky known as the KBM method (see, for example, Bogolyubov N.N. and Mitropolsky Yu.A. [2]) is a powerful tool for the investigation of nonlinear vibrations.

The further development of these methods took place due to work by Yu.A. Mitropolsky, A.K. Lopatin [12]–[14], A.K. Lopatin [15]. In their works, a new method was proposed for investigating systems of differential equations with small parameters. It was a further development of Bogolyubov's averaging method referred to by the authors as "the asymptotic decomposition method". The idea of a new approach originates from Bogolyubov's averaging method (see [2]) but its realization needs to use essentially a new apparatus – the theory of continuous transformation groups.

3.2. Generalization of Bogolyubov's averaging method through symmetry

The asymptotic decomposition method is based on the group-theoretic interpretation of the averaging method. Consider the system of ordinary differential equations

$$\frac{dx}{dt} = \omega(x) + \varepsilon\tilde{\omega}(x), \tag{10}$$

where

$$\omega(x) = \mathrm{col}\,[\omega_1(x), \ldots, \omega_n(x)], \quad \tilde{\omega}(x) = \mathrm{col}\,[\tilde{\omega}_1(x), \ldots, \tilde{\omega}_n(x)].$$

The differential operator associated with the perturbed system (10) can be represented as

$$U_0 = U + \varepsilon\tilde{U},$$

where

$$U = \omega_1\frac{\partial}{\partial x_1} + \cdots + \omega_n\frac{\partial}{\partial x_n}, \quad \tilde{U} = \tilde{\omega}_1\frac{\partial}{\partial x_1} + \cdots + \tilde{\omega}_n\frac{\partial}{\partial x_n}.$$

By using a certain change of variables in the form of a series in ε

$$x = \varphi(\bar{x}, \varepsilon), \tag{11}$$

system (10) is transformed into a new system

$$\frac{d\bar{x}}{dt} = \omega(\bar{x}) + \sum_{\nu=1}^{\infty}\varepsilon^\nu b^{(\nu)}(\bar{x}), \tag{12}$$

which is referred to as *a centralized system*. For this system, $\bar{U}_0 = \bar{U} + \varepsilon\tilde{\bar{U}}$, where

$$\begin{aligned}\bar{U} &= \omega_1(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \cdots + \omega_n(\bar{x})\frac{\partial}{\partial \bar{x}_n},\\ \tilde{\bar{U}} &= \sum_{\nu=1}^{\infty}\varepsilon^\nu N_\nu, \quad N_\nu = b_1^{(\nu)}(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \cdots + b_n^{(\nu)}(\bar{x})\frac{\partial}{\partial \bar{x}_n}.\end{aligned} \tag{13}$$

We impose a condition on the choice of transformations (11) saying that the centralized system (12) should be invariant with respect to the one-parameter transformation group

$$\bar{x} = e^{s\bar{\mathrm{U}}(\bar{x}_0)}\bar{x}_0, \tag{14}$$

where $\bar{x}_0$ is the vector of new variables. Therefore, after the change of variables (14), system (12) turns into

$$\frac{d\bar{x}_0}{dt} = \omega(\bar{x}_0) + \sum_{\nu=1}^{\infty}\varepsilon^\nu b^{(\nu)}(\bar{x}_0),$$

which coincides with the original one up to the notations. This means that we have the identities $[\bar{U}, N_\nu] \equiv 0$ for $\bar{U}, N_\nu, \nu = 1, 2, \ldots$.

The essential point in realizing the above-mentioned scheme of the asymptotic decomposition algorithm is that transformations (11) are chosen in the form of a series

$$x = e^{\varepsilon \mathrm{S}}\bar{x}, \tag{15}$$

where

$$S = S_1 + \varepsilon S_2 + \cdots,$$

$$S_j = \gamma_{j1}(\bar{x})\frac{\partial}{\partial \bar{x}_1} + \cdots + \gamma_{jn}(\bar{x})\frac{\partial}{\partial \bar{x}_n}.$$

Coefficients of $S_j, \gamma_{j1}(\bar{x}), \dots, \gamma_{jn}(\bar{x})$ are unknown functions. They should be determined by the recurrent sequence of operator equations

$$[U, S_\nu] = F_\nu. \tag{16}$$

The operator F_ν is a known function of U and $S_1, \dots, S_{\nu-1}$ are obtained on previous steps $(\nu = 1, 2, \dots)$.

In the case where S depends upon ε, the Lie series (15) is called a *Lie transformation*. Thus, the application of a Lie transformation as a change of variables enables us to use the technique of continuous transformation groups.

From the theory of linear operators, it is known that the solvability of the nonhomogeneous operator equation (16) depends on the properties of solutions of the homogeneous equation

$$[U, S_\nu] = 0. \tag{17}$$

Operator (13) N_ν is a projection of the right-hand side of the equation onto the kernel of operator (17), which is determined from the condition of solvability in the sense of the nonhomogeneous equation

$$[U, S_\nu] = F_\nu - N_\nu, \quad \nu = 1, 2, \dots . \tag{18}$$

Depending on the way for solving equations (16)–(18), various modifications of the algorithm of the method are obtained.

The above indicated result can be summarized as the following theorem.

Theorem 1. *There exists a formal change of variables as Lie's transform (15) which transforms the initial system (10) into the centralized one (12), invariant with respect to the one-parameter transformation group (14) generated by zero approximation system's vector field.*

One can find the detailed exposition of the questions touched here in [12]–[15].

4. Some examples

A further investigation of the structure of the centralized system (12) gives the possibility to receive some nontrivial conclusions. Let us illustrate it by the examples.

4.1. Example 1. Perturbed motion on $SO(2)$

Let us consider a system of the second order

$$\begin{aligned} \dot{\bar{x}} &= -\bar{y} + \varepsilon Q(\varepsilon, \bar{x}, \bar{y}), \\ \dot{\bar{y}} &= \bar{x} + \varepsilon R(\varepsilon, \bar{x}, \bar{y}), \end{aligned} \tag{19}$$

where $Q(\varepsilon, \bar{x}, \bar{y})$, $R(\varepsilon, \bar{x}, \bar{y})$ are the known analytical functions of variables $\varepsilon, \bar{x}, \bar{y}$.

When $\varepsilon = 0$, the structure of the solution of system (19) is quite simple: it is the movement on the circle of radius $R = \sqrt{\bar{x}^2 + \bar{y}^2}$ with the proportional angular velocity $\omega = 1$.

The following statement is true.

Theorem 2. *System (19) in the neighborhood of the point $\varepsilon = 0$ has a family of periodic solutions which depends upon an arbitrary constant if and only if it can be transformed by the analytic change of the variables*

$$\bar{x} = e^{\varepsilon S}u, \quad \bar{y} = e^{\varepsilon S}v,$$

where $S = S_1 + S_2 + \cdots$, S_j are known operators with the analytic coefficients,

to the system

$$\dot{u} = -(1 + \varepsilon G_1(\varepsilon, u^2 + v^2))v,$$
$$\dot{v} = (1 + \varepsilon G_1(\varepsilon, u^2 + v^2))u,$$

where $G_1(\varepsilon, u^2 + v^2)$ is the known analytic function of $\varepsilon, u^2 + v^2$,

which is invariant in respect to the following one-parameter transformation groups: $SO(2)$

$$\bar{u} = e^{\varepsilon U}u, \quad \bar{v} = e^{\varepsilon U}v, \quad U = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$

and the group defined via

$$\bar{u} = e^{\varepsilon W}u, \quad \bar{v} = e^{\varepsilon W}v, \quad W = f(u^2 + v^2)U,$$

where $f(\rho)$ is an arbitrary analytic function of $\rho = \sqrt{u^2 + v^2}$.

Theorem 2 generalizes the well-known result by A.M. Lyapunov about the existence of a family of periodic solutions in the neighborhood of the point $|\bar{x}| = 0, |\bar{y}| = 0$ of system (19) when $\varepsilon = 1$. (See A.M. Lyapunov's theorem above).

4.2. Example 2. Perturbed motion on $SO(3)$

Let us consider the system of the third order (in the spherical coordinates)

$$\begin{aligned}
\dot{\rho} &= \varepsilon F_1(\varepsilon, \rho, , \theta, \varphi), \\
\dot{\theta} &= \sin\varphi + \varepsilon F_2(\varepsilon, \rho, \theta, \varphi), \\
\dot{\varphi} &= -1 + \operatorname{ctg}\theta\cos\varphi + \varepsilon F_3(\varepsilon, \rho, \theta, \varphi),
\end{aligned} \tag{20}$$

where $F_j(\varepsilon, \rho, \theta, \varphi)$ are the known analytic functions of the variables $\varepsilon, \rho, \theta, \varphi$, $j = 1, 2, 3$.

The system of zero approximation, which is received from (20) if one supposes $\varepsilon = 0$, has a quite complicate structure. (See Fig.1.) The following statement is true.

Theorem 3. *System (20) in the neighborhood of the point $\varepsilon = 0$ has a family of solutions which depends upon an arbitrary constant and saves the topological structure of the system of zero approximation, if and only if it can be transformed by the analytic change of the variables*

$$\bar{\rho} = e^{\varepsilon S}\rho, \quad \bar{\theta} = e^{\varepsilon S}\theta, \quad \bar{\varphi} = e^{\varepsilon S}\varphi,$$

where $S = S_1 + S_2 + \cdots$, S_j are the known operators with analytic coefficients,

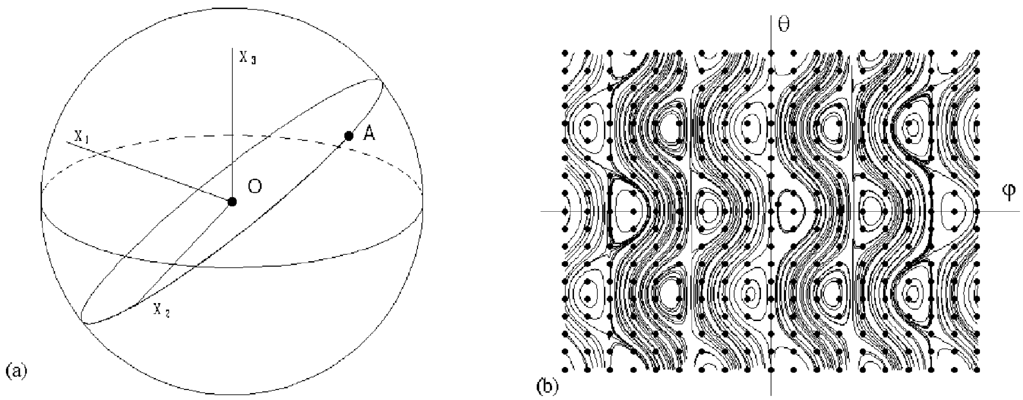

Fig. 1. (a) Solution for linear system of the movement of a point on a sphere.
(b) Solution in the phase plane for angle spherical variables, governing the movement of a point on a sphere.

to the system

$$\dot{\bar\rho} = 0,$$
$$\dot{\bar\theta} = \sin\bar\varphi(1 + \varepsilon G(\varepsilon, \bar\rho, \bar\theta, \bar\varphi)),$$
$$\dot{\bar\varphi} = (-1 + \operatorname{ctg}\bar\theta \cos\bar\varphi)(1 + \varepsilon G(\varepsilon, \bar\rho, \bar\theta, \bar\varphi)),$$

where $G(\varepsilon, \bar\rho, \bar\theta, \bar\varphi)$ *is the known analytic function of* $\varepsilon, \bar\rho, \bar\theta, \bar\varphi$,
which is invariant in respect to the one-parameter transformation groups:

$$\bar\rho = e^{\varepsilon U}\rho, \quad \bar\theta = e^{\varepsilon U}\theta, \quad \bar\varphi = e^{\varepsilon U}\varphi, \quad U = (-1 + \operatorname{ctg}\theta\cos\varphi)\frac{\partial}{\partial\varphi} + \sin\varphi\frac{\partial}{\partial\theta}$$

and

$$\bar\rho = e^{\varepsilon W}\rho, \quad \bar\theta = e^{\varepsilon W}\theta, \quad \bar\varphi = e^{\varepsilon W}\varphi, \quad W = f(\rho, \theta, \varphi)U,$$

where $f(\rho, \theta, \varphi)$ *is an arbitrary analytic function of* ρ, θ, φ *which is an integral of the equation* $Uf = 0$.

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Lie Symmetries and Preliminary Classification of $u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b})$

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Abstract

Differential-difference equations (DDEs) of the form $u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b})$ with $k \geq 2$ are studied for Lie symmetries and preliminary classification. Explicit forms of equations are given for those admitting at least one intrinsic Lie symmetry. An algorithmic mechanism is also proposed to automate the symmetry calculation for fairly general DDEs via computer algebras.

1. Introduction

The Lie symmetry method [1, 2] for differential equations has by now been well established, though the same theory for differential-difference equations (DDEs) [3–7] or difference equations [8, 9] is much less studied or understood. Symmetry Lie algebras often go a long way to explain the behaviour of the corresponding system, just like the algebra on which its Lax pair (if any) lives would through the use of ISMs [10, 11]. To overcome the difficulties set by the infinite number of variables in DDEs, the concept of *intrinsic* symmetries [3] has been introduced to simplify the task of symmetry calculations. Our purpose here is to study the DDEs of the form

$$u_n^{(k)}(t) = F_n(t, u_{n+a}, \ldots, u_{n+b}), \qquad n \in \mathbf{Z}, \quad a \leq b, \quad k \geq 2 \tag{1.1}$$

where we use the superindex $^{(i)}$ to denote the i-th partial derivative with respect to (w.r.t.) t. We note that both a and b in (1.1) are integers and $\mathbf{Z}$ will always denote the set of all integers. We first look for the symmetries

$$\mathbf{X} = \xi(t, u_i : \ i \in \mathbf{Z})\partial_t + \sum_{n \in \mathbf{Z}} \phi_n(t, u_i : \ i \in \mathbf{Z})\partial_{u_n} \tag{1.2}$$

for system (1.1) and then consider the subsequent task of classification by means of the intrinsic Lie symmetries in the form

$$\mathbf{X} = \xi(t)\partial_t + \phi_n(t, u_n)\partial_{u_n} . \tag{1.3}$$

We note that systems (1.1) are so far only studied [7] for $k = 2$ with $b = a = -1$ w.r.t. the *intrinsic* Lie symmetries, and our objective here in this respect is to give an explicit list of DDEs (1.1) admitting at least one intrinsic Lie symmetry. This thus serves as a preliminary or semi classification. Complete classification of some more specific form of (1.1) is still under investigation. Another objective of ours is to devise an algorithmic mechanism to automate the calculation of Lie symmetries for DDEs so that the symmetries could be efficiently calculated for practical problems.

The paper is organized as follows. We first give in Section 2 the general Lie symmetry for (1.1). We will explain why the study of intrinsic Lie symmetries, particularly with regard to the classification, will not cause a significant loss of generality. Section 3 serves as a preliminary classification: the forms of system (1.1) are explicitly given for those bearing at least one intrinsic Lie symmetry. In Section 4, we will briefly propose an algorithmic mechanism for calculating intrinsic symmetries by means of computer algebras, along with several illustrative examples.

2. The general Lie symmetries for (1.1)

We call DDE (1.1) *nontrivial* if there exists at least one $n_0 \in \mathbf{Z}$ such that $F_{n_0}(t, u_{n_0+a}, \ldots, u_{n_0+b})$ is not a function of only variables t and u_{n_0}. Suppose system (1.1) is nontrivial for at least k=2 and $\mathbf{X}$ in (1.2) is a Lie symmetry of the system. Then ξ and ϕ_n in (1.2) must have the following form

$$\begin{aligned} &\xi = \xi(t), \quad \phi_n = \Big(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n\Big)u_n + \sum_{i\in\mathbf{Z}} c_{n,i}u_i + \beta_n(t)\ , \quad k \geq 2 \\ &\xi(t) = \alpha_2 t^2 + \alpha_1 t + \alpha_0\ , \qquad k \geq 3 \end{aligned} \tag{2.1}$$

where $c_{n,i}$, α_i and γ_n are all constants. Moreover, (1.2) with (2.1) is a Lie symmetry of (1.1) iff

$$\phi_n^{(k)} - k\dot{\xi}F_n - \xi\dot{F}_n + \sum_{i\in\mathbf{Z}} \phi_{n,u_i}F_i - \sum_{j=a}^{b} \phi_{n+j}F_{n,u_{n+j}} = 0 \tag{2.2}$$

is further satisfied.

We omit the derivation details of (2.1) and (2.2) as they are too lengthy to be put here. We note that nonintrinsic symmetries do exist for system (1). For example, the system $u_n^{(k)}(t) = \sum_{j=a}^{b} \mu_j e^{j\lambda t}u_{n+j}$ has the nonintrinsic Lie symmetry $\mathbf{X} = \partial_t + \sum_{n\in\mathbf{Z}} \Big\{ \sum_{i\in\mathbf{Z}} \alpha_i u_{n+i} - n\lambda u_n + \beta_n(t) \Big\}\partial_{u_n}$, and the system $u_n^{(k)}(t) = \sum_{j=a}^{b} \mu_j \lambda^{nj} u_{n+j}$ has the nonintrinsic Lie symmetry $\mathbf{X} = \partial_t + \sum_{n\in\mathbf{Z}} \Big\{ \sum_{i\in\mathbf{Z}} \alpha_i \lambda^{ni} u_{n+i} + \beta_n(t)\Big\}\partial_{u_n}$ if $\beta_n(t)$ satisfies the original DDEs respectively. In fact we can show that system (1.1) can only have intrinsic Lie symmetries unless it is linear or 'essentially' linear. We will not dwelt upon this here. Instead we quote without proof one of the results in [12] that *all the Lie symmetries of the system* $u_n^{(k)} = f(t, u_{n+c}) + g(t, u_{n+a}, \ldots, u_{n+c-1}, u_{n+c+1}, \ldots, u_{n+b})$ *for* $k \geq 2$ *and* $a \leq c \leq b$ *are intrinsic if* f *is nonlinear in* u_{n+c} *and the system is nontrivial.*

3. Systems bearing intrinsic Lie symmetries

The classification is in general a very laborious task. Due to the difficulties of handling nonintrinsic symmetries and their scarceness implied in Section 2 anyway, only intrinsic symmetries are known to be ever considered in this respect. For systems of type (1.1), the only known results in this connection have been for $k = 2$ along with $b = -a = 1$ [7]. We shall thus mostly consider the case of $k \geq 3$ in this section. Apart from two systems related to two special symmetry Lie algebras, we shall be mainly concerned with systems that bear at least one intrinsic Lie symmetry rather than the classification via the *complete* symmetry Lie algebras. For this purpose, we first show that, for any $k \geq 3$, a fiber-preserving transformation

$$u_n(t) = \Omega_n(y_n(\tilde{t}\,), t), \qquad \tilde{t} = \theta(t), \tag{3.1}$$

will transform system (1.1) into

$$\frac{d^k}{d\,\tilde{t}^k} y_n(\tilde{t}\,) = \tilde{F}_n(\tilde{t}, y_{n+a}, \ldots, y_{n+b})\ , \tag{3.2}$$

iff transformation (3.1) is given by

$$u_n(t) = A_n(\gamma t + \delta)^{k-1} y_n(\tilde{t}\,) + B_n(t), \quad \tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \tag{3.3}$$

where $B_n(t)$ are arbitrary functions and A_n, α, β, γ and δ are arbitrary constants satisfying

$$\alpha\delta - \beta\gamma = \pm 1, \quad A_n \neq 0\ . \tag{3.4}$$

We now prove the above statement. First, since it is easy to show inductively

$$\begin{aligned} u_n^{(m)} = \Omega_{n,y_n} \dot{\theta}^m y_n^{(m)} + \tbinom{m}{1} \Omega_{n,y_n y_n} \dot{\theta}^m y_n^{(m-1)} \dot{y}_n + \\ +\text{terms of other powers of } y_n\text{'s derivatives}, \qquad m \geq 3\ , \end{aligned} \tag{3.5}$$

we must have $\Omega_{n,y_n y_n} = 0$ if (3.1) is to transform (1.1) into (3.2). Hence, (3.1) must take the form $u_n(t) = f_n(t) y_n(\tilde{t}) + B_n(t)$. Since it is not difficult to show inductively $u_n^{(m)}(t) = \lambda_m(t) y_n^{(m)} + \mu_m(t) y_n^{(m-1)} + \nu_m(t) y_n^{(m-2)} + \cdots$ for $m \geq 3$ in which λ_m, μ_m and ν_m are given correspondingly by the following more explicit formula

$$\begin{aligned} u_n^{(m)}(t) = f_n(t) \dot{\theta}^m y_n^{(m)} + \left[\binom{m}{1} \dot{f}_n \dot{\theta}^{m-1} + \binom{m}{2} f_n \dot{\theta}^{m-2} \ddot{\theta} \right] y_n^{(m-1)} + \\ \left[\tbinom{m}{2} \ddot{f}_n \dot{\theta}^{m-2} + 3\tbinom{m}{3} \dot{f}_n \dot{\theta}^{m-3} \ddot{\theta} + 3\tbinom{m}{4} f_n \dot{\theta}^{m-4} \ddot{\theta}^2 + \tbinom{m}{3} f_n \dot{\theta}^{m-3} \dddot{\theta} \right] y_n^{(m-2)} + \ \cdots\ , \end{aligned} \tag{3.6}$$

in order (3.1) transforms (1.1) into (3.2), it is clearly necessary that $\mu_k(t) = \nu_k(t) = 0$, read off from (3.6), be satisfied. Removing $\ddot{f}_n$ from these two equations, we have $\dot{f}_n \dot{\theta} \ddot{\theta} + \dfrac{k-2}{2} f_n \ddot{\theta}^2 + \dfrac{1}{3} f_n \dot{\theta} \dddot{\theta} = 0$, $\dot{f}_n \dot{\theta} + \dfrac{k-1}{2} f_n \ddot{\theta} = 0$ which are equivalent to $3\ddot{\theta}^2 = 2\dot{\theta}\dddot{\theta}$ and $\dot{f}_n \dot{\theta} + \dfrac{k-1}{2} f_n \ddot{\theta} = 0$. The solution of the above equations (with $\dot{\theta} \neq 0$) is elementary: it is given by

$$u_n(t) = A_n(t + \alpha_1)^{k-1} y_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \frac{\beta_1}{t + \alpha_1} + \gamma_1, \tag{3.7}$$

for $\ddot{\theta} \neq 0$, and is otherwise by

$$u_n(t) = A_n y_n(\tilde{t}) + B_n(t), \quad \tilde{t} = \beta_1 t + \alpha_1,\ . \tag{3.8}$$

In (3.7) and (3.8), the function $B_n(t)$ and constants A_n, α_1, β_1 and γ_1 are arbitrary and satisfy $\beta_1 A_n \neq 0$. It is easy to verify that (3.7) and (3.8) together are equivalent to (3.3) with (3.4).

We now proceed to prove the sufficiency. For this purpose, we can show inductively for $m \geq 1$

$$x^{(m)} = A \sum_{i=0}^{m} \binom{m}{i} (-\beta_1)^i \mathbf{A}_{k-1-i}^{m-i} (t+\alpha_1)^{k-1-m-i} y^{(i)}, \tag{3.9}$$

where

$$x(t) = A(t+\alpha_1)^{k-1} y(\theta(t)), \quad \theta(t) = \frac{\beta_1}{t+\alpha_1} + \gamma_1, \quad k \geq 2$$
$$\mathbf{A}_n^i = n(n-1)\cdots(n-i+1), \quad \mathbf{A}_n^0 = 1 \,.$$

The proof is again meticulous but straightforward, and is thus skipped here for brevity. If we choose $m = k$, then each coefficient in front of $y^{(i)}$ in (3.9) for $0 \leq i < m$ has the factor $(m-k)$ and is thus zero. Hence, we obtain simply

$$x^{(k)} = A(-\beta_1)^k (t+\alpha_1)^{-k-1} y^{(k)} \,. \tag{3.10}$$

It is now clear from (3.10) that (3.7) will transform (1.1) into (3.2). Since the sufficiency regarding to (3.8) is almost trivial, we have completed the proof of the form-invariance of (1.1) under (3.1) and (3.3).

Notice that under $u_n(t) = y_n(\tilde{t})/\sigma_n(t) + B_n(t)$ with $\tilde{t} = \theta(t)$, the symmetry

$$\mathbf{X} = \xi(t)\partial_t + \Big[(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n) u_n + \beta_n(t) \Big] \partial_{u_n} \tag{3.11}$$

will read as

$$\mathbf{X} = \xi\dot{\theta}\partial_{\tilde{t}} + \Big\{ \Big[\frac{k-1}{2}\dot{\xi} + \gamma_n + \frac{\dot{\sigma}_n}{\sigma_n}\xi \Big] y_n + \sigma_n \Big[(\frac{k-1}{2}\dot{\xi} + \gamma_n) B_n - \xi \dot{B}_n + \beta_n \Big] \Big\} \partial_{y_n} \,. \tag{3.12}$$

Of course, for (1.1) to become (3.2), we need $\sigma_n(t)$ to be given for $k \geq 3$ by (3.3) and for $k = 2$ [7] by $\sigma_n(t) = \dot{\theta}^{\frac{1}{2}}/A_n$ with $A_n \neq 0$. We can thus conclude from (2.1), (3.3) and (3.4) that there are exactly three canonical forms of $\xi(t) \neq 0$ given by

$$\text{(i)} \quad \xi(t) = 1, \quad \text{(ii)} \quad \xi(t) = t, \quad \text{(iii)} \quad \xi(t) = t^2 + 1 \tag{3.13}$$

because all other cases can be transformed via (3.3) into one of above three. In all these three cases, we may choose $B_n(t)$ such that

$$\xi(t)\dot{B}_n(t) = \Big(\frac{k-1}{2}\dot{\xi}(t) + \gamma_n \Big) B_n + \beta_n(t) \,. \tag{3.14}$$

This means we may choose simply $\beta_n(t) = 0$ in all the three canonical cases in (3.13). If $\xi(t) = 0$, then (3.14) still has a solution $B_n = -\beta_n/\gamma_n$ if $\gamma_n \neq 0$. Hence the fourth and the last canonical cases are

$$\text{(iv)} \quad \xi(t) = 0,\, \gamma_n \neq 0,\, \beta_n(t) = 0, \qquad \text{(v)} \quad \xi(t) = 0,\, \gamma_n = 0,\, \beta_n(t) \neq 0 \,. \tag{3.15}$$

We note from (3.12) that systems in case (v) with β_n and $\tilde{\beta}_n$ are equivalent if $\tilde{\beta}_n(t) = \sigma_n(t)\beta_n(t)$. Also that in the case of $k = 2$, cases (ii) and (iii) are transformable to case

(i) due to allowed more general fiber transformations. Note furthermore that although in principle one should replace cases (iv) and (v) with the following more general form

$$\xi(t)=0,\ \gamma_m\neq 0,\ \beta_m(t)=0,\ \gamma_n=0,\ \beta_n(t)\neq 0,\quad m\in\mathbf{S},\ n\in\mathbf{Z}\backslash\mathbf{S}$$

for some subset $\mathbf{S}$ of $\mathbf{Z}$, such a mixture (which won't exist if the continuity in n is imposed on F_n as adopted in [7]) will lead to only $F_n=F_{d,n}$ for $n\in\mathbf{S}$ and $=F_{e,n}$ for $n\in\mathbf{Z}\backslash\mathbf{S}$, where $F_{d,n}$ and $F_{e,n}$ are just the F_n given in $(3.17)_d$ and $(3.17)_e$, respectively.

With the above preliminaries, we are ready to give the general forms of F_n such that (1.1) has at least one intrinsic Lie symmetry. The form of F_n is thus to be determined from (2.2), or more explicitly for $k\geq 2$,

$$\begin{aligned}&\frac{k-1}{2}\xi^{(k+1)}u_n+\beta_n^{(k)}+\Big[\gamma_n-\frac{k+1}{2}\dot{\xi}\Big]F_n-\xi\dot{F}_n-\\&\qquad\sum_{i=n+a}^{n+b}\Big[\Big(\frac{k-1}{2}\dot{\xi}+\gamma_i\Big)u_i+\beta_i\Big]F_{n,u_i}=0\,.\end{aligned}\tag{3.16}$$

In fact, for three canonical cases in (3.13) with $\beta_n=0$ and another two in (3.15), the solutions of (3.16) are

$$\begin{aligned}&\text{(i)}\quad F_n=e^{\gamma_n t}f_n(\zeta_{n+a},\dots,\zeta_{n+b}),\quad \zeta_i=u_ie^{-\gamma_i t}\\&\text{(ii)}\quad F_n=t^{\gamma_n-(k+1)/2}f_n(\zeta_{n+a},\dots,\zeta_{n+b}),\quad \zeta_i=u_it^{-\gamma_i-(k-1)/2}\\&\text{(iii)}\quad F_n=\frac{\exp\left(\gamma_n\tan^{-1}(t)\right)}{(t^2+1)^{(k+1)/2}}f_n(\zeta_{n+a},\dots,\zeta_{n+b}),\quad \zeta_i=u_i\frac{(t^2+1)^{(1-k)/2}}{\exp\left(\gamma_n\tan^{-1}(t)\right)}\\&\text{(iv)}\quad F_n=u_{n+c}^{\gamma_n/\gamma_{n+c}}f_n(t,\zeta_{n+a},\dots,\zeta_{n+c-1},\zeta_{n+c+1},\dots,\zeta_{n+b}),\quad \zeta_i=\frac{u_i^{\gamma_{n+c}}}{u_{n+c}^{\gamma_i}}\\&\text{(v)}\quad F_n=\frac{\beta_n^{(k)}}{\beta_{n+c}}u_{n+c}+f_n(t,\zeta_{n+a},\dots,\zeta_{n+c-1},\zeta_{n+c+1},\dots,\zeta_{n+b}),\\&\qquad \zeta_i=u_i\beta_{n+c}-u_{n+c}\beta_i\end{aligned}\tag{3.17}$$

where c is any given integer in $[a,\,b]$, and the equivalent classes in case (v) are related by $\beta_n\sim\beta_n\sigma_n$ for the same σ_n as in (3.12). If $0\in[a,\,b]$, then case (iv) in $(3.17)_d$ may also be rewritten as

$$F_n=u_nf_n(t,\zeta_{n+a},\dots,\zeta_{n-1},\zeta_{n+1},\dots,\zeta_{n+b}),\quad \zeta_i=u_i^{\gamma_n}u_n^{-\gamma_i}\,.$$

Since (ii) and (iii) for $k=2$ in (3.13) are transformable to (i) there, formulas in (3.17), minus the cases of (ii)–(iii), also give the complete list for $k=2$. We also note that the form of F_n in (3.17) can be further refined by extending the symmetry algebras. Since the details of such undertakings would belong to the scope of complete classification which will be considered elsewhere in future, we shall limit our attention to providing two cases which are related to the Toda lattice [13] and the FPU system [14], respectively.

For this purpose, let us choose $\beta_n(t)=t^m$ in $(3.17)_e$ for $0\leq m<k$ and $k\geq 2$. Then F_n can be written as $F_n=f_n(t,\zeta_{n+a},\dots,\zeta_{n+c-1},\zeta_{n+c+1},\dots,\zeta_{n+b})$ with $\zeta_i=u_i-u_{n+c}$. It is obvious that this F_n also fits the form of $(3.17)_a$ with $\gamma_i=0$ if the function f_n is independent of t. Hence, for

$$F_n=f_n(\zeta_{n+a},\dots,\zeta_{n+c-1},\zeta_{n+c+1},\dots,\zeta_{n+b}),\quad \zeta_i=u_i-u_{n+c}\,,\tag{3.18}$$

system (1.1) has the following $(k+1)$-dimensional nilpotent symmetry Lie algebra

$$\mathbf{X}_i = t^i \partial_{u_n},\ i = 0, 1, \dots, k-1; \quad \mathbf{X}_k = \partial_t\ . \tag{3.19}$$

We now look for an additional symmetry $\mathbf{Y}$ of the form (3.11) such that the new Lie algebra formed by $\{\mathbf{X}_i, 0 \le i \le k,\ \mathbf{Y}\}$ contains $\{\mathbf{X}_i, 0 \le i \le k\}$ as its nilradical, i.e., $[\mathbf{X}_i, \mathbf{Y}] = \sum_{j=0}^{k} \alpha_{i,j} \mathbf{X}_j$. Since, for $0 \le i < k$,

$$[\mathbf{X}_k, \mathbf{Y}] = \dot{\xi}\partial_t + \Big(\frac{k-1}{2}\ddot{\xi}u_n + \dot{\beta}_n\Big)\partial_{u_n}, \quad [\mathbf{X}_i, \mathbf{Y}] = \Big(\gamma_n t^i + \frac{k-1}{2} t^i \dot{\xi} - i t^{i-1}\xi\Big)\partial_{u_n}\ , \tag{3.20}$$

we conclude that both ξ and $\gamma_n t + (k-1)t\dot{\xi}/2 - i\xi$ must be linear in t. Hence (3.20) implies $[\mathbf{X}_i, \mathbf{Y}]$ is linear in $\mathbf{X}_i$ for $0 \le i < k$ and $[\mathbf{X}_k, \mathbf{Y}] = \dot{\xi}\partial_t + \dot{\beta}_n \partial_{u_n}$ which in turn induces

$$\frac{d^{k+1}}{d\,t^{k+1}}\,\beta_n(t)\ =\ 0\ . \tag{3.21}$$

In order the structure constants in $[\mathbf{X}_i, \mathbf{Y}]$ be independent of n, we set $\gamma_n = \gamma$ and $\dot{\beta}_n(t) = \beta(t)$. In this way it is easy to see that $\mathbf{Y}$ can be any linear combination of

$$\begin{aligned} &\mathbf{Y}_1 = t\partial_t + \Big(\frac{k-1}{2} + \gamma\Big) u_n \partial_{u_n},\ \gamma \ne -\frac{k-1}{2}; \\ &\mathbf{Y}_2 = t\partial_t + \omega_n \partial_{u_n},\ \omega_i \ne \omega_j\ \forall i \ne j\ ; \\ &\mathbf{Y}_3 = t\partial_t + (k u_n + t^k)\partial_{u_n}; \quad \mathbf{Y}_4 = u_n \partial_{u_n} \end{aligned} \tag{3.22}$$

modulus some linear combinations of $\mathbf{X}_i$. The FPU and Toda systems turn out to be related to $\mathbf{Y}_1$ and $\mathbf{Y}_2$, respectively. For symmetry operator $\mathbf{Y}_1$, (3.16) reduces to

$$\Big(\gamma - \frac{k+1}{2}\Big)F_n - t\dot{F}_n - \sum_i \Big(\frac{k-1}{2} + \gamma\Big) u_i F_{n,u_i} = 0$$

which is solved by (and is consistent with (3.18))

$$\begin{aligned} &F_n = (u_{n+c} - u_{n+d})^{\frac{\gamma - (k+1)/2}{\gamma + (k-1)/2}} \times \\ &\qquad \times f_n(\zeta_{n+a}, \dots, \zeta_{n+c-1}, \zeta_{n+c+1}, \dots, \zeta_{n+d-1}, \zeta_{n+d+1}, \dots, \zeta_{n+b}), \\ &\zeta_i = (u_i - u_{n+d})/(u_{n+c} - u_{n+d}), \qquad a \le c \ne d \le b\ . \end{aligned} \tag{3.23}$$

When $b = -a = 1$, $\gamma = (1-3k)/2$ and $f_n(\zeta_{n+a}) = 1 - \zeta_{n+a}^2$, the FPU type system reads with $c = 1$ and $d = 0$ as

$$u_n^{(k)} = (u_{n+1} - u_n)^2 - (u_n - u_{n-1})^2\ . \tag{3.24}$$

We note that for $k = 2$ eq.(3.24) can be rewritten as the original form of the FPU [14] system $\ddot{y}_n - (y_{n+1} + y_{n-1} - 2y_n)[1 + \omega(y_{n+1} - y_{n-1})] = 0$ under the transformation $y_n = u_n/\omega - n/(2\omega)$.

For $\mathbf{Y}_2$ in (3.22), eq.(3.16) is reduced to $kF_n + t\dot{F}_n + \sum_i \omega_i F_{n,u_i} = 0$ whose solution reads for any given $a \le c \ne d \le b$ as

$$F_n = \exp\Big[- k\frac{u_{n+c} - u_{n+d}}{\omega_{n+c} - \omega_{n+d}}\Big] \times \\ \times f_n(\zeta_{n+a}, \dots, \zeta_{n+c-1}, \zeta_{n+c+1}, \dots, \zeta_{n+d-1}, \zeta_{n+d+1}, \dots, \zeta_{n+b}), \tag{3.25}$$

$$\zeta_i = (\omega_{n+c} - \omega_{n+d})u_i + (\omega_{n+d} - \omega_i)u_{n+c} + (\omega_i - \omega_{n+c})u_{n+d}.$$

When $b = -a = -c = 1$ and $d = 0$, then (3.25) reduces to

$$F_n = \exp\Big[- k\frac{u_{n-1} - u_n}{\omega_{n-1} - \omega_n}\Big] f_n(\zeta_{n+1}), \quad \zeta_{n+1} = \sum_{\substack{n-1,n,n+1 \\ \text{cyclic}}} (\omega_{n-1} - \omega_n)u_{n+1} \ . \tag{3.26}$$

Let furthermore $\omega_n = kn$ and $f_n(\zeta_{n+1}) = 1 - \exp(\zeta_{n+1}/k)$. Then (3.26) gives rise to the following Toda type system

$$u_n^{(k)}(t) = \exp(u_{n-1} - u_n) - \exp(u_n - u_{n+1}) \ .$$

4. Local overdeterminacy in explicit symmetry calculation

In this section, we shall consider a fairly large class of DDEs of the form

$$G_n(t, \partial^k u_i : \ |k| + |i - n| \le M) = 0, \qquad \forall n \in \mathbf{I} \ , \tag{4.1}$$

where $t = (t_1, \dots, t_m)$, $\partial^k = \partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_m}$, $|k| = k_1 + \cdots + k_m$, $\mathbf{I}$ is the index grid, and G_n is uniformly defined w.r.t. n in the sense that all partial differentiations of G_n commute with the index n. The purpose here is to propose a mechanism to find intrinsic Lie symmetries for uniformly defined DDEs *through* the use of such computer algebras that can deal with [15] systems of *finite* variables. Let the intrinsic symmetry be given by

$$\mathbf{X} = \xi(t)\partial_t + \eta(t, n, u_n)\partial_{n_n} \equiv \sum_{j=1}^{m} \xi_j \partial_{t_j} + \sum_{n \in \mathbf{I}} \eta_n \partial_{u_n} \ , \tag{4.2}$$

and let the Lie symmetry for (4.1) over $n \in \mathbf{J}$, a finite subset of $\mathbf{I}$, be denoted by

$$\mathbf{X}^{\mathbf{J}} = \xi^{\mathbf{J}} \partial_t + \sum_{i \in \mathbf{J}} \eta_i^{\mathbf{J}} \partial_{u_i} \ , \tag{4.3}$$

then our mechanism is based on the following observation:

If $\eta_n^{\mathbf{J}} = \eta^{\mathbf{J}}(t, n, u_n)$ is uniformly defined w.r.t. n with $\xi^{\mathbf{J}} = \xi^{\mathbf{J}}(t)$, then (4.2) with $\xi = \xi^{\mathbf{J}}$ and $\eta_i = \eta_i^{\mathbf{J}}$ ($\forall i \in \mathbf{I}$) is the intrinsic Lie symmetry of (4.1).

For convenience, we shall always denote by R_N the subsystem of N equations $G_j = 0$ for $j = n, \dots, n + N - 1$. The algorithm proposed in this section for finding intrinsic Lie symmetries has in fact been applied to various DDEs, and *all* results have been consistent with the analytic ones whenever the later ones do exist. For instance, our consideration for the inhomogeneous Toda lattice [16]

$$\ddot{u}_n - \frac{1}{2}\dot{u}_n + (\frac{1}{4} - \frac{n}{2}) + \Big[\frac{1}{4}(n-1)^2 + 1\Big] e^{u_{n-1} - u_n} - \Big[\frac{1}{4}n^2 + 1\Big] e^{u_n - u_{n+1}} = 0$$

for R_4 will lead to the symmetry generators ∂_t, ∂_{u_n}, $e^{t/2}\partial_{u_n}$ and $e^{-t/2}\partial_t + (\frac{1}{2} - n)e^{-t/2}\partial_{u_n}$ which are exactly those obtained in [6] analytically. New but straightforward cases include applying the procedure to the discretized KZ equation

$$u_{n,xt} + \partial_x(u_{n,x}u_n) + u_{n+1} + u_{n-1} - 2u_n = 0$$

for R_3, which gives the symmetry $(\alpha t + \beta)\partial_t + (f(t) - \alpha x)\partial_x + (\dot{f}(t) - 2\alpha u_n)\partial_{u_n}$, and to the 2-dimensional system

$$\partial_x\partial_t u_n(x,t) = \exp\left(u_{n+1}(x,t) + u_{n-1}(x,t) - 2u_n(x,t)\right)$$

which gives the symmetry $(\alpha_1(t) + \beta_1(x) + (\alpha_2(t) + \beta_2(x))n - (1/2)n^2(\dot{f}(t) + g'(x)))\partial_{u_n} + f(t)\partial_t + g(x)\partial_x$ for arbitrary $f(t)$, $g(x)$, $\alpha_i(t)$ and $\beta_i(x)$. Likewise the 1-dimensional

$$\ddot{u}_n(t) = \exp\left(u_{n+1} + u_{n-1} - 2u_n\right), \qquad n \in \mathbf{Z}, \tag{4.4}$$

via R_5 will lead to the intrinsic Lie symmetry $(c_1 + c_2 t)\partial_t + (c_3 n + c_4 + (c_5 n + c_6)t - c_2 n^2)\partial_{u_n}$ for (4.4), which is again consistent with the analytic results in [7]. As for the similarity solutions, we only note that the reduction $u_n(t) = -(n^2 + a^2)\log t + w_n$ with $a \neq 0$, induced by the symmetry $-t\partial_t + (n^2 + a^2)\partial_{u_n}$, reduces (4.4) into $w_{n+1} + w_{n-1} - 2w_n = \log(n^2 + a^2)$ which has a solution $(n \neq 1, n \neq 0)$

$$w_n = \sum_{i=1}^{|n|-1} (|n| - i)\log(i^2 + a^2) + nw_1 - (n-1)w_0 + \chi(-n)|n|\log a^2,$$

where w_0 and w_1 are treated as arbitrary constants and χ is the step function defined by $\chi(t) = 0$ if $t \leq 0$ and =1 otherwise.

Acknowledgment

The author wishes to thank Professor Pavel Winternitz for the interesting discussions and constructive comments. The author is also grateful to the CRM, University of Montreal for the hospitality – the main work in this paper is done in the period of his sabbatical leave there.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 98–107.

On Some Exact Solutions of Nonlinear Wave Equations

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Abstract

A new simple method for constructing solutions of multidimensional nonlinear wave equations is proposed

1 Introduction

The method of the symmetry reduction of an equation to equations with fewer variables, in particularly, to ordinary differential equations [1–3] is among efficient methods for constructing solutions of nonlinear equations of mathematical physics. This method is based on investigation of the subgroup structure of an invariance group of a given differential equation. Solutions being obtained in this way are invariant with respect to a subgroup of the invariance group of the equation. It is worth to note that the invariance imposes very severe constraints on solutions. For this reason, the symmetry reduction doesn't allow to obtain in many cases sufficiently wide classes of solutions.

At last time, the idea of the conditional invariance of differential equations, proposed in [3–6], draws intent attention to itself. By conditional symmetry of an equation, one means the symmetry of some solution set. For a lot of important nonlinear equations of mathematical physics, there exist solution subsets, the symmetry of which is essentially different from that of the whole solution set. One chooses such solution subsets, as a rule, with the help of additional conditions representing partial differential equations. The description of these additional conditions in the explicit form is a difficult problem and unfortunately there are no efficient methods to solve it.

In this paper, we propose a constructive and simple method for constructing some classes of exact solutions to nonlinear equations of mathematical physics. The essence of the method is the following. Let we have a partial differential equation

$$F\left(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}\right) = 0, \tag{1}$$

where $u = u(x)$, $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}_{1,n}$, $\underset{m}{u}$ is a collection of all possible derivatives of order m, and let equation (1) have a nontrivial symmetry algebra. To construct solutions of equation (1), we use the symmetry (or conditional symmetry) ansatz [3]. Suppose that it is of the form

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k) + g(x), \tag{2}$$

where $\omega_1 = \omega_1(x_0, x_1, \ldots, x_n), \ldots, \omega_k = \omega_k(x_0, x_1, \ldots, x_n)$ are new independent variables. Ansatz (2) singles out some subset S from the whole solution set of equation (1). Construct (if it is possible) a new ansatz

$$u = f(x)\varphi(\omega_1, \ldots, \omega_k, \omega_{k+1}, \ldots, \omega_l) + g(x), \tag{3}$$

being a generalization of ansatz (2). Here $\omega_{k+1}, \ldots, \omega_l$ are new variables that should be determined. We choose the variables $\omega_{k+1}, \ldots, \omega_l$ from the condition that the reduced equation corresponding to ansatz (3) coincides with the reduced equation corresponding to ansatz (2). Ansatz (3) singles out a subset S_1 of solutions to equation (1), being an extension of the subset S. If solutions of the subset S are known, then one also can construct solutions of the subset S_1. These solutions are constructed in the following way. Let $u = u(x, C_1, \ldots, C_t)$ be a multiparameter solution set of the form (2) of equation (1), where $C_1, \ldots, C_t$ are arbitrary constants. We shall obtain a more general solution set of equation (1) if we take constants C_i in the solution $u = u(x, C_1, \ldots, C_t)$ to be arbitrary smooth functions of $\omega_{k+1}, \ldots, \omega_l$.

Basic aspects of our approach are presented by the examples of d'Alembert, Liouville and eikonal equations.

2 Nonlinear d'Alembert equations

Let us consider a nonlinear Poincaré-invariant d'Alembert equation

$$\Box u + F(u) = 0, \tag{4}$$

where

$$\Box u = \frac{\partial^u}{\partial x_0^2} - \frac{\partial^u}{\partial x_1^2} - \cdots - \frac{\partial^u}{\partial x_n^2},$$

$F(u)$ is an arbitrary smooth function. Papers [3, 7–9] are devoted to the construction of exact solutions to equation (4) for different restrictions on the function $F(x)$. Majority of these solutions is invariant with respect to a subgroup of the invariance group of equation (4), i.e., they are Lie solutions. One of the methods for constructing solutions is the method of symmetry reduction of equation (4) to ordinary differential equations. The essence of this method for equation (4) consists in the following.

Equation (4) is invariant under the Poincaré algebra $AP(1,n)$ with the basis elements

$$J_{0a} = x_0\partial_a + x_a\partial_0, \quad J_{ab} = x_b\partial_a - x_a\partial_b,$$
$$P_0 = \partial_0, \quad P_a = \partial_a \quad (a, b = 1, 2, \ldots, n).$$

Let L be an arbitrary rank n subalgebra of the algebra $AP(1,n)$. The subalgebra L has two main invariants u, $\omega = \omega(x_0, x_1, \ldots, x_n)$. The ansatz $u = \varphi(\omega)$ corresponding to the subalgebra L reduces equation (4) to the ordinary differential equation

$$\ddot{\varphi}(\nabla\omega)^2 + \dot{\varphi}\Box\omega + F(\varphi) = 0, \tag{5}$$

where

$$(\nabla\omega)^2 \equiv \left(\frac{\partial\omega}{\partial x_0}\right)^2 - \left(\frac{\partial\omega}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial\omega}{\partial x_n}\right)^2.$$

Such a reduction is called the *symmetry reduction*, and the ansatz is called the *symmetry ansatz*. There exist eight types of nonequivalent rank n subalgebras of the algebra $AP(1,n)$ [7]. In Table 1, we write out these subalgebras, their invariants and values of $(\nabla\omega)^2$, $\Box\omega$ for each invariant.

Table 1.

N	Algebra	Invariant ω	$(\nabla\omega)^2$	$\Box\omega$
1.	$P_1,\dots,P_n$	x_0	1	0
2.	$P_0,P_1,\dots,P_{n-1}$	x_n	-1	0
3.	$P_1,\dots,P_{n-1},\ J_{0n}$	$(x_0^2-x_n^2)^2$	1	$\frac{1}{\omega}$
4.	$J_{ab}\ (a,b=1,\dots,k)$, $P_{k+1},\dots,P_n,\ P_0\ \ (k\ge 2)$	$(x_1^2+\cdots+x_k^2)^{1/2}$	-1	$-\frac{k-1}{\omega}$
5.	$G_a=J_{0a}-J_{ak},\ J_{ab}$ $(a,b=1,\dots,k-1)$ $J_{0k},\ P_{k+1},\dots,P_n\ (k\ge 1)$	$(x_0^2-x_1^2-\cdots-x_k^2)^{1/2}$	1	$\frac{k}{\omega}$
6.	$P_1,\dots,P_{n-2},\ P_0+P_n$ $J_{0n}+\alpha P_{n-1}$	$\alpha\ln(x_0-x_n)+x_{n-1}$	-1	0
7.	$P_0+P_n,\ P_1,\dots,P_{n-1}$	x_0-x_n	0	0
8.	$P_a\ (a=1,\dots,n-2)$, $G_{n-1}+P_0-P_n,\ P_0+P_n$	$(x_0-x_n)^2-4x_{n-1}$	-1	0

The method proposed in [11] of reduction of equation (4) to ODE is a generalization of the symmetry reduction method. Equation (4) is reduced to ODE with the help of the ansatz $u=\varphi(\omega)$, where $\omega=\omega(x)$ is a new variable, if $\omega(x)$ satisfies the equations

$$\Box\omega = F_1(\omega), \qquad (\nabla\omega)^2 = F_2(\omega). \tag{6}$$

Here F_1, F_2 are arbitrary smooth functions depending only on ω.

Thus, if we construct all solutions to system (6), hence we get the set of all values of the variable ω, for which the ansatz $u=\varphi(\omega)$ reduces equation (4) to ODE in the variable ω. Papers [10–11] are devoted to the investigation of system (6).

Note, however, that ansatzes obtained by solving system (6), don't exaust the set of all ansatzes reducing equation (4) to ordinary differential equations. For this purpose, let us consider the process of finding generalized ansatzes (3) on the known symmetry ansatzes (2) of equation (4).

a) Consider the symmetry ansatz $u = \varphi(\omega_1)$ for equation (4), where $\omega_1 = (x_0^2 - x_1^2 - \cdots - x_k^2)$, $k \geq 2$. The ansatz reduces equation (4) to the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\omega_1) = 0, \tag{7}$$

where $\varphi_{11} = \dfrac{d^2\varphi}{d\omega_1^2}$, $\varphi_1 = \dfrac{d\varphi}{d\omega_1}$. This ansatz should be regarded as a partial case of the more general ansatz $u = \varphi(\omega_1, \omega_2)$, where ω_2 is an unknown variable. The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces equation (4) to the equation

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + 2\varphi_{12}(\nabla\omega_1 \cdot \nabla\omega_2) + \varphi_2\Box\omega_2 + \varphi_{22}(\nabla\omega_2)^2 + F(\varphi) = 0, \tag{8}$$

where

$$\nabla\omega_1 \cdot \nabla\omega_2 = \frac{\partial\omega_1}{\partial x_0}\cdot\frac{\partial\omega_2}{\partial x_0} - \frac{\partial\omega_1}{\partial x_1}\cdot\frac{\partial\omega_2}{\partial x_1} - \cdots - \frac{\partial\omega_1}{\partial x_n}\cdot\frac{\partial\omega_2}{\partial x_n}.$$

Let us impose the condition on equation (8), under which equation (8) coincides with the reduced equation (7). Under such assumption, equation (8) decomposes into two equations

$$\varphi_{11} + \frac{k}{\omega_1}\varphi_1 + F(\varphi) = 0, \tag{9}$$

$$2\varphi_{12}(\nabla\omega_1 \cdot \nabla\omega_2) + \varphi_{22}(\nabla\omega_2)^2 + \varphi_{12}\Box\omega_2 = 0. \tag{10}$$

Equation (10) will be fulfilled for an arbitrary function φ if we impose the conditions

$$\Box\omega_2 = 0, \quad (\nabla\omega_2)^2 = 0, \tag{11}$$

$$\nabla\omega_1 \cdot \nabla\omega_2 = 0 \tag{12}$$

on the variable ω_2. Therefore, if we choose the variable ω_2 such that conditions (11), (12) are satisfied, then the multidimensional equation (4) is reduced to the ordinary differential equation (7) and solutions of the latter equation give us solutions of equation (4). So, the problem of reduction is reduced to the construction of general or partial solutions to system (11), (12).

The overdetermined system (11) is studied in detail in papers [12–13]. A wide class of solutions to system (11) is constructed in papers [12–13]. These solutions are constructed in the following way. Let us consider a linear algebraic equation in variables $x_0, x_1, \ldots, x_n$ with coefficients depending on the unknown ω_2:

$$a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \cdots - a_n(\omega_2)x_n - b(\omega_2) = 0. \tag{13}$$

Let the coefficients of this equation represent analytic functions of ω_2 satisfying the condition

$$[a_0(\omega_2)]^2 - [a_1(\omega_2)]^2 - \cdots - [a_n(\omega_2)]^2 = 0.$$

Suppose that equation (13) is solvable for ω_2 and let a solution of this equation represent some real or complex function

$$\omega_2(x_0, x_1, \ldots, x_n). \tag{14}$$

Then function (14) is a solution to system (11). Single out those solutions (14), that possess the additional property $\nabla\omega_1 \cdot \nabla\omega_2 = 0$. It is obvious that

$$\frac{\partial \omega_2}{\partial x_0} = -\frac{a_0}{\delta'}, \quad \frac{\partial \omega_2}{\partial x_1} = \frac{a_1}{\delta'}, \quad \ldots, \quad \frac{\partial \omega_2}{\partial x_n} = \frac{a_n}{\delta'},$$

where

$$\delta(\omega_2) \equiv a_0(\omega_2)x_0 - a_1(\omega_2)x_1 - \cdots - a_n(\omega_2)x_n - b(\omega_2)$$

and δ' is the derivative of δ with respect to ω_2. Since

$$\frac{\partial \omega_1}{\partial x_0} = \frac{x_0}{\omega_1}, \quad \frac{\partial \omega_1}{\partial x_1} = -\frac{x_1}{\omega_1}, \quad \ldots, \quad \frac{\partial \omega_1}{\partial x_n} = -\frac{x_n}{\omega_1},$$

we have

$$\nabla\omega_1 \cdot \nabla\omega_2 = -\frac{1}{\omega_1 \delta'}(a_0 x_0 - a_1 x_1 - \cdots - a_n x_n).$$

Hence, with regard for (13), the equality $\nabla\omega_1 \cdot \nabla\omega_2 = 0$ is fulfilled if and only if $b(\omega_2) = 0$. Therefore, we have constructed the wide class of ansatzes reducing the d'Alembert equation to ordinary differential equations. The arbitrariness in choosing the function ω_2 may be used to satisfy some additional conditions (initial, boundary and so on).

b) The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = (x_1^2 + \cdots + x_l^2)^{1/2}$, $1 \le l < n-1$, is generalized in the following way. Let ω_2 be an arbitrary solution to the system of equations

$$\begin{aligned} &\frac{\partial^2 \omega}{\partial x_0^2} - \frac{\partial^2 \omega}{\partial x_{l+1}^2} - \cdots - \frac{\partial^2 \omega}{\partial x_n^2} = 0, \\ &\left(\frac{\partial \omega}{\partial x_0}\right)^2 - \left(\frac{\partial \omega}{\partial x_{l+1}}\right)^2 - \cdots - \left(\frac{\partial \omega}{\partial x_n}\right)^2 = 0. \end{aligned} \tag{15}$$

The ansatz $u = \varphi(\omega_1, \omega_2)$ reduces equation (4) to the equation

$$-\frac{d^2\varphi}{d\omega_1^2} - \frac{k-1}{\omega_1}\frac{d\varphi}{d\omega_1} + F(\varphi) = 0.$$

If $l = n-1$, then the ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_2 = x_0 - x_n$ is a generalization of the symmetry ansatz $u = \varphi(\omega_1)$.

Ansatzes corresponding to subalgebras 2, 6 and 8 in Table 1, are particular cases of the ansatz constructed above. Doing in a similar way, one can obtain wide classes of ansatzes reducing equation (4) to two-dimensional, three-dimensional and so on equations. Let us present some of them.

c) The ansatz $u = \varphi(\omega_1, \ldots, \omega_l, \omega_{l+1})$, where $\omega_1 = x_1, \ldots, \omega_l = x_l$, ω_{l+1} is an arbitary solution of system (15), $l \le n-1$, is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_l)$ and reduces equation (4) to the equation

$$-\frac{\partial^2\varphi}{\partial\omega_1^2} - \frac{\partial^2\varphi}{\partial\omega_2^2} - \cdots - \frac{\partial^2\varphi}{\partial\omega_l^2} + F(\varphi) = 0.$$

d) The ansatz $u = \varphi(\omega_1, \ldots, \omega_s, \omega_{s+1})$, where $\omega_1 = (x_0^2 - x_1^2 - \cdots - x_l^2)^{1/2}$, $\omega_2 = x_{l+1}, \ldots, \omega_s = x_{l+s-1}$, $l \geq 2$, $l + s - 1 \leq n$, ω_{s+1} is an arbitrary solution of the system

$$\Box\omega_{s+1} = 0, \quad (\nabla\omega_{s+1})^2 = 0, \quad \nabla\omega_i \cdot \nabla\omega_{s+1} = 0, \quad i = 1, 2, \ldots, s, \tag{16}$$

is a generalization of the symmetry ansatz $u = \varphi(\omega_1, \ldots, \omega_s)$ and reduces equation (4) to the equation

$$\varphi_{11} - \frac{l}{\omega_1}\varphi_1 - \varphi_{22} - \cdots - \varphi_{ss} + F(\varphi) = 0.$$

Let us construct in the way described above some classes of exact solutions of the equation

$$\Box u + \lambda u^k = 0, \quad k \neq 1. \tag{17}$$

The following solution of equation (17) is obtained in paper [9]:

$$u^{1-k} = \sigma(k, l)(x_1^2 + \cdots + x_l^2), \tag{18}$$

where

$$\sigma(k, l) = \frac{\lambda(1-k)^2}{2(l - lk + 2k)}, \quad l = 1, 2, \ldots, n.$$

Solution (18) defines a multiparameter solution set

$$u^{1-k} = \sigma(k, l)\left[(x_1 + C_1)^2 + \cdots + (x_l + C_l)^2\right],$$

where $C_1, \ldots, C_l$ are arbitrary constants. Hence, according to c), we obtain the following set of solutions to equation (17) for $l \leq n - 1$:

$$u^{1-k} = \sigma(k, l)\left[(x_1 + h_1(\omega))^2 + \cdots + (x_l + h_l(\omega))^2\right], \quad k \neq \frac{l}{l-2},$$

where ω is an arbitrary solution of system (15) and $h_1(\omega), \ldots, h_l(\omega)$ are arbitrary twice differentiable functions of ω. In particular, if $n = 3$ and $l = 1$, then equation (17) possesses in the space $\mathbb{R}_{1,3}$ the solution set

$$u^{1-k} = \frac{\lambda(1-k)^2}{2(1+k)}\left[x_1 + h_1(\omega)\right]^2, \quad k \neq -1.$$

Next, let us consider the following solution of equation (4) [9]:

$$u^{1-k} = \sigma(k, s)(x_0^2 - x_1^2 - \cdots - x_s^2), \quad s = 2, \ldots, n, \tag{19}$$

where

$$\sigma(k, s) = -\frac{\lambda(1-k)^2}{2(s - ks + k + 1)}, \quad k \neq \frac{s+1}{s-1}.$$

Solution (19) defines the multiparameter solution set

$$u^{1-k} = \sigma(k, s)\left[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + C_{l+1})^2 - \cdots - (x_s + C_s)^2\right],$$

where $C_{l+1}, \dots, C_s$ are arbitrary constants. According to d) we obtain the following solution set for $l \geq 2$

$$u^{1-k} = \sigma(k,s)\left[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + h_{l+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2\right],$$

where ω is an arbitrary solution of system (16), and $h_{l+1}(\omega), \dots, h_s(\omega)$ are arbitrary twice differentiable functions. In particular, if $l = 2$ and $s = 3$, then equation (4) possesses in the space $\mathbb{R}_{1,3}$ the following solution set

$$u^{1-k} = \frac{\lambda(1-k)^2}{4(k-2)}\left[x_0^2 - x_1^2 - x_2^2 - (x_3 - h_3(\omega))^2\right], \quad k \neq 2.$$

The equation

$$\Box u + 6u^2 = 0 \tag{20}$$

possesses the solution $u = \mathcal{P}(x_3 + C_2)$, where $\mathcal{P}(x_3 + C_2)$ is an elliptic Weierstrass function with the invariants $g_2 = 0$ and $g_3 = C_1$. Therefore, according to c) we get the following set of solutions of equation (20):

$$u = \mathcal{P}(x_3 + h(\omega)),$$

where ω is an arbitrary solution to system (15) and $h(\omega)$ is an arbitrary twice differentiable function of ω.

Next consider the Liouville equation

$$\Box u + \lambda \exp u = 0. \tag{21}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_3$, reduces equation (21) to the equation

$$\frac{d^2\varphi}{d\omega_1^2} = \lambda \exp \varphi(\omega_1).$$

Integrating this equation, we obtain that φ coincides with one of the following functions:

$$\ln\left\{\left(-\frac{C_1}{2\lambda}\sec^2\left[\frac{\sqrt{-C_1}}{2}(\omega_1 + C_2)\right]\right)\right\} \quad (C_1 < 0,\ \lambda > 0,\ C_2 \in \mathbb{R});$$

$$\ln\left\{\frac{2C_1C_2\exp(\sqrt{C_1}\omega_1)}{\lambda[1 - C_2\exp(\sqrt{C_1}\omega_1)]^2}\right\} \quad (C_1 > 0,\ \lambda C_2 > 0);$$

$$-\ln\left(\sqrt{\frac{\lambda}{2}}\omega_1 + C\right)^2.$$

Hence, according to c) we get the following solutions set for equation (21):

$$u = \ln\left\{\left(-\frac{h_1(\omega)}{2\lambda}\sec^2\left[\frac{\sqrt{-h_1(\omega)}}{2}(\omega_1 + h_2(\omega))\right]\right)\right\} \quad (h_1(\omega) < 0,\ \lambda > 0);$$

$$u = \ln\left\{\frac{2h_1(\omega)h_2(\omega)\exp(\sqrt{h_1(\omega)}\omega_1)}{\lambda[1 - h_2(\omega)\exp(\sqrt{h_1(\omega)}\omega_1)]^2}\right\} \quad (h_1(\omega) > 0,\ \lambda h_2(\omega) > 0);$$

$$u = -\ln\left(\sqrt{\frac{\lambda}{2}}\omega_1 + h(\omega)\right)^2,$$

where $h_1(\omega)$, $h_2(\omega)$, $h(\omega)$ are arbitrary twice differentaible functions; ω is an arbitrary solution to system (15).

Using, for example, the solution to the Liouville equation (21) [9]

$$u = \ln\frac{2(s-2)}{\lambda[x_0^2 - x_1^2 - \cdots - x_s^2]}, \quad s \neq 2,$$

we obtain the wide class of solutions to the Liouville equation

$$u = \ln\frac{2(s-2)}{\lambda[x_0^2 - x_1^2 - \cdots - x_l^2 - (x_{l+1} + h_{l+1}(\omega))^2 - \cdots - (x_s + h_s(\omega))^2]},$$

where ω is an arbitrary solution to system (16), and $h_{l+1}(\omega), \ldots, h_s(\omega)$ are arbitrary twice differentaible functions. If $s = 3$, then equation (21) possesses in the space $\mathbb{R}_{1,3}$ the following solution set

$$u = \ln\frac{2}{\lambda[x_0^2 - x_1^2 - x_2^2 - (x_3 + h_3(\omega))^2]}.$$

Let us consider now the sine–Gordon equation

$$\Box u + \sin u = 0.$$

Doing in an analogous way, we get the following solutions:

$$u = 4\arctan h_1(\omega)\exp(\varepsilon_0 x_3) - \frac{1}{2}(1-\varepsilon)\pi, \quad \varepsilon_0 = \pm 1, \ \varepsilon = \pm 1;$$

$$u = 2\arccos[\mathrm{dn}\,(x_3 + h_1(\omega)), m] + \frac{1}{2}(1+\varepsilon)\pi, \quad 0 < m < 1;$$

$$u = 2\arccos\left[\mathrm{cn}\left(\frac{x_3 + h_1(\omega)}{m}\right), m\right] + \frac{1}{2}(1+\varepsilon)\pi, \quad 0 < m < 1,$$

where $h_1(\omega)$ is an arbitrary twice differentiable function, ω is is an arbitrary solution to system (15).

3 Eikonal equation

Consider the eikonal equation

$$\left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{22}$$

The symmetry ansatz $u = \varphi(\omega_1)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, reduces equation (22) to the equation

$$4\omega_1\left(\frac{\partial \varphi}{\partial \omega_1}\right)^2 - 1 = 0. \tag{23}$$

We shall look for a generalized ansatz in the form $u = \varphi(\omega_1, \omega_2)$. This ansatz reduces equation (22) to the equation

$$4\omega_1 \left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 + 2(\nabla\omega_1 \cdot \nabla\omega_2)\frac{\partial\varphi}{\partial\omega_1} + (\nabla\omega_2)^2 \left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 = 1. \qquad (24)$$

Impose the condition on equation (24), under which equation (24) coincides with equation (23). It is obvious that this condition will be fulfilled if we impose the conditions

$$(\nabla\omega_2)^2 = 0, \quad \nabla\omega_1 \cdot \nabla\omega_2 = 0 \qquad (25)$$

on the variable ω_2. Having solved system (25), we get the explicit form of the variable ω_2. It is easy to see that an arbitrary function of a solution to system (25) is also a solution to this system.

Having integrated equation (23), we obtain $(u+C)^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$, where C is an arbitrary constant. We shall obtain a more general solution set for the eikonal equation if we take C to be an arbitrary solution to system (25).

The symmetry ansatz $u = \varphi(\omega_1, \omega_2)$, $\omega_1 = x_0^2 - x_1^2 - x_2^2$, $\omega_2 = x_3$ can be generalized in the following way. Let ω_3 be an arbitrary solution to the system of equations

$$\begin{aligned} &\left(\frac{\partial\omega_3}{\partial x_0}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_1}\right)^2 - \left(\frac{\partial\omega_3}{\partial x_2}\right)^2 = 0, \\ &x_0\frac{\partial\omega_3}{\partial x_0} + x_1\frac{\partial\omega_3}{\partial x_1} + x_3\frac{\partial\omega_3}{\partial x_2} = 0. \end{aligned} \qquad (26)$$

Then the ansatz $u = \varphi(\omega_1, \omega_2, \omega_3)$ reduces the eikonal equation to the equation

$$4\omega_1 \left(\frac{\partial\varphi}{\partial\omega_1}\right)^2 - \left(\frac{\partial\varphi}{\partial\omega_2}\right)^2 - 1 = 0. \qquad (27)$$

Equation (27) possesses the solution [9]

$$\varphi = \frac{C_1^2 + 1}{2C_1}(x_0^2 - x_1^2 - x_2^2)^{1/2} + \frac{C_1^2 - 1}{2C_1}x_3 + C_2,$$

$$(\varphi + C_2)^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + C_1)^2,$$

that can be easily found by using the symmetry reduction method of equation (27) to ordinary differential equation. Having replaced arbitary constants C_1 and C_2 by arbitrary functions $h_1(\omega)$ and $h_2(\omega)$, we get the more wide classes of exact solutions to the eikonal equation:

$$u = \frac{h_1(\omega_3)^2 + 1}{2h_1(\omega_3)}(x_0^2 - x_1^2 - x_2^2)^{1/2} + \frac{h_1(\omega_3)^2 - 1}{2h_1(\omega_3)}x_3 + h_2(\omega_3),$$

$$(u + h_2(\omega_3))^2 = x_0^2 - x_1^2 - x_2^2 - (x_3 + h_1(\omega_3))^2.$$

Let us note, since the Born–Infeld equation is a differential consequence of the eikonal equation [3], hence we also constructed wide classes of exact solutions of the Born–Infeld equation.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 108–114.

On Specific Symmetries of the KdV Equation and on New Representations of Nonlinear *sl*(2)-algebras

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Abstract

On the one hand, we put in evidence new symmetry operators of the nonlinear Korteweg-de Vries equation by exploiting its Lax form expressed in terms of a pair of linear equations. A KdV supersymmetric version is also studied in order to determine its symmetry Lie superalgebra. On the other hand, nonlinear $sl(2)$-algebras are then visited and new unitary irreducible representations are characterized.

This talk is dedicated to the Memory of my Colleague ***Wilhelm Fushchych****, a man I have mainly appreciated* ***outside*** *our common interests in mathematical physics.*

1 Introduction

I would like to discuss in this talk *two* subjects which both are concerned with fundamental *symmetries* in theoretical *physics* and both are developed through *mathematical methods*. Moreover, these two subjects deal with *nonlinear* characteristics so that the invitation of the organisers of this Conference was welcome and I take this opportunity to thank them cordially.

The *first* subject (reported in Sections 2 and 3) deals with new symmetries and supersymmetries of the famous (nonlinear) Korteweg-de Vries equation [1] by exploiting its formulation(s) through the corresponding Lax form(s) [2, 3]. These results have already been collected in a not yet published recent work [4] to which I refer for details if necessary.

The *second* subject (developed in Sections 4 and 5) concerns the so-called nonlinear $sl(2,R)$-algebras containing, in particular, the linear $sl(2)$-case evidently, but also the quantum $sl_q(2)$-case. These two particular cases are respectively very interesting in connection with the so-important theory of angular momemtum [5, 6] (developed in quantum physics at all the levels, i.e., the molecular, atomic, nuclear and subnuclear levels) *and* with the famous quantum deformations [7] applied to one of the simplest Lie algebras with fundamental interest in quantum physics [8, 9]. But my main purpose is to study here the nonlinear $sl(2)$-algebras which are, in a specific sense, just *between* these two cases: in

Talk presented in Kyiv (July 1997): 2nd International Conference on "Symmetry in Nonlinear Mathematical Physics".

such categories we find the nonlinear cases corresponding to *finite* powers of Lie generators differing from the first power evidently but being possibly of second (the *quadratic* context) or third powers (the *cubic* context) for example. Both of these quadratic and cubic cases [10] have already been exploited in *physical* models or theories so that a very interesting amount of original informations is the knowledge of the unitary irreducible representations (unirreps) of these nonlinear algebras: such results can be found in different recent papers [10, 11, 12] that I want to refer to in the following by dealing more particularly here with the so-called Higgs algebra [13]. This algebra is an example of a *cubic* $sl(2)$-algebra with much physical attraction: it corresponds to physical descriptions in *curved* spaces (its original appearance [13]) or in *flat* spaces (its more interesting recent discover in quantum optics, for example [14]).

2 On the KdV equation and its symmetries

Let us remember the famous nonlinear Korteweg-de Vries (KdV) equation [1] introduced in 1885 in the form

$$u_t = 6uu_x - u_{xxx}, \tag{1}$$

where the unknown function $u(x,t)$ also admits time and space partial derivatives with the usual notations

$$u_t \equiv \frac{\partial u(x,t)}{\partial t}, \qquad u_x \equiv \frac{\partial u(x,t)}{\partial x} = \partial_x u(x,t), \tag{2}$$

the space ones going to the maximal third order (u_{xxx}). Its Lax form [2] is usually denoted by (L, A) where

$$L \equiv -\partial_x^2 + u, \qquad A \equiv -4\partial_x^3 + 6u\partial_x + 3u_x, \tag{3}$$

so that

$$\partial_t L = [A, L]. \tag{4}$$

This is equivalent to a pair of *linear* equations

$$L\psi(x,t) = \lambda\psi(x,t), \qquad \psi_t = A\psi \tag{5}$$

which can be rewritten as a system of the following form

$$\begin{aligned} L_1\psi(x,t) &= 0, \qquad L_1 \equiv L - \lambda, \\ L_2\psi(x,t) &= 0, \qquad L_1 \equiv \partial_t - A, \end{aligned} \tag{6}$$

with the compatibility condition

$$[L_1, L_2]\psi = 0. \tag{7}$$

By searching for *symmetry* operators X of such a system, we ask for operators X such that

$$[\triangle, X] = \lambda X \tag{8}$$

ensuring that, if ψ is a solution of $\triangle\psi = 0$, we know that $X\psi$ is still a solution of the same equation, i.e., $\triangle(X\psi) = 0$. We have solved [4] such an exercise with two equations (6) so that the general conditions (8) here reduce to

$$[L_1, X] = \lambda_1 L_1 \quad \text{and} \quad [L_2, X] = \lambda_2 L_1, \tag{9}$$

where λ_1 and λ_2 are arbitrary functions of x and t. This system leads to nine partial differential equations and to a resulting set of *three* (nontrivial) independent symmetry operators according to different u-values [4]. They are given by the explicit expressions

$$\begin{aligned} X_1 &\equiv \partial_x, \\ X_2 &\equiv \partial_x^3 - \frac{3}{2}u\partial_x + \lambda\partial_x - \frac{3}{4}u_x, \\ X_3 &\equiv t\partial_x^3 - \frac{1}{12}x\partial_x - \frac{3}{2}tu\partial_x + \lambda t\partial_x - \frac{3}{4}tu_x. \end{aligned} \tag{10}$$

They generate a (closed) invariance (Lie) algebra characterized by the commutation relations

$$[X_1, X_2] = 0, \quad [X_1, X_3] = \frac{1}{3}X_1, \quad [X_2, X_3] = X_2 - \frac{4}{3}\lambda X_1. \tag{11}$$

In fact, such results are readily obtained if one requires that the X-operators are given by

$$X = \sum_{i=0}^{3} a_i(x,t)\partial_x^i, \tag{12}$$

where $i = 0$ refers to time and $i = 1, 2, 3$ refer to first, second and third respective space derivatives. Conditions (9) relate among themselves the corresponding arbitrary $a_i(x,t)$-functions so that the discussion appears to be restricted to the following contexts:

$$u_{xx} = 0 \quad \text{and} \quad u_{xx} \neq 0. \tag{13}$$

Another simple equation leads to the possible values

$$u = \lambda \neq 0 \quad \text{or} \quad 6ut = -x \tag{14}$$

so that these different contexts allow the existence of new symmetries of the KdV equation.

3 The supersymmetric context

Let us remember the Mathieu supersymmetric extension [3] of the KdV equation characterized by

$$u_t = 6uu_x - u_{xxx} - a\xi\xi_{xx} \tag{15}$$

and

$$\xi_t = -\xi_{xxx} + au_x\xi + (6-a)u\xi_x, \tag{16}$$

where a is a constant taken equal to 3 for obtaining a nontrivial Lax representation and ξ, ξ_t , ξ_x, ... are "fermionic" quantities as usual. Now the differential operators of the Lax pair take the explicit forms

$$L \equiv -\partial_x^2 + u + \theta\xi_x + \theta\xi\partial_x - \xi\partial_\theta - u\theta\partial_\theta \tag{17}$$

and

$$\begin{aligned} A \equiv & -4\partial_x^3 + 6u\partial_x + 3u_x + 3\theta\xi_{xx} - 3\theta u_x\partial_\theta + 9\theta\xi_x\partial_x \\ & -3\xi_x\partial_\theta + 6\theta\xi\partial_x^2 - 6\xi\partial_x\partial_\theta - 6\theta u\partial_x\partial_\theta. \end{aligned} \tag{18}$$

Here θ is the necessary Grassmannian variable permitting to distinguish *even* and *odd* terms in these developments, or let us say "bosonic" and "fermionic" symmetry operators.

By taking care of the grading in the operators as developed in the study of supersymmetric Schrödinger equations [15], we can determine the even and odd symmetries of the context after a relatively elaborated discussion. We refer the interested reader to the original work [4]. Here evidently we get invariance Lie *superalgebras* [16] for the KdV equation whose orders are 11, 9 or 4 in the complete discussion we skip here.

4 On nonlinear $sl(2)$-algebras

Let us remember the linear $sl(2)$-algebra subtending all the ingredients of the angular momentum theory [5, 6] , i.e., the set of commutation relations (defining this Lie algebra) between the three Cartan-Weyl generators $(J_\pm, J_3)$ given by

$$[J_3, J_\pm] = \pm J_\pm, \tag{19}$$

$$[J_+, J_-] = 2J_3. \tag{20}$$

Here J_+ and J_- are the respective raising and lowering operators acting on the real orthogonal basis $\{ \mid j, m > \}$ in the following well-known way:

$$J_+ \mid j, m >= \sqrt{(j-m)(j+m+1)} \mid j, m+1 >, \tag{21}$$

$$J_- \mid j, m >= \sqrt{(j+m)(j-m+1)} \mid j, m-1 > \tag{22}$$

while J_3 is the diagonal generator giving to $m(= -j, -j+1, ..., j-1, j)$ its meaning through the relation

$$J_3 \mid j, m >= m \mid j, m > . \tag{23}$$

Moreover, by recalling the role of the Casimir operator

$$C \equiv \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2 \tag{24}$$

such that

$$C \mid j, m >= j(j+1) \mid j, m >, \qquad j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, \tag{25}$$

it is easy to define our nonlinear $sl(2)$-algebras by asking that the commutation relation (20) has to be replaced by the following one:

$$[J_+, J_-] = f(J_3) = \sum_{p=0}^{N} \beta_p \, (2J_3)^{2p+1}. \tag{26}$$

In particular, if $N = p = 0$, $\beta_0 = 1$, we recover the linear (above-mentioned) context. If $N = 1, p = 0, 1$, $\beta_0 = 1, \beta_1 = 8\beta$ where β is a real continuous parameter, we obtain the remarkable Higgs algebra [13] already studied [10, 11, 12] in connection with specific physical contexts in curved or flat spaces [13, 14]. If $N \to \infty$ and if we choose ad hoc β_p-coefficients [12] , we easily recover the quantum $sl_q(2)$-algebra [8].

Here, let us more specifically consider finite values of $N(\neq 0)$ in order to include (in particular) the Higgs algebra. Our main purpose is to determine the unirreps of such a family of algebras by putting in evidence the effect of the deformed generators satisfying Eqs. (19) and (26). In terms of the old $sl(2)$-generators, we have obtained the whole answer [12] by noticing that relation (23) has to play a very important role in the discovery of new unirreps having a physical interest. In order to summarize our improvements, let us mention that equation (23) can be generalized through two steps.

The first step is to modify it in the following way as proposed by Abdesselam et al. [12], i.e.,

$$J_3 \mid j, m >= (m + \gamma) \mid j, m >, \tag{27}$$

where γ is a real scalar parameter. This proposal leads us to

$$\begin{aligned} J_+ \mid j, m >= &((j-m)(j+m+1+2\gamma)(1+2\beta(j(j+1)+m(m+1)+ \\ &2\gamma(j+m+1+\gamma))))^{\frac{1}{2}} \mid j, m+1 >, \end{aligned} \tag{28}$$

$$\begin{aligned} J_- \mid j, m >= &((j-m+1)(j+m+2\gamma)(1+2\beta(j(j+1)+m(m-1)+ \\ &2\gamma(j+m+\gamma))))^{\frac{1}{2}} \mid j, m-1 > . \end{aligned} \tag{29}$$

This context (interesting to consider at the limits $\gamma = 0$, or $\beta = 0$, or $\beta = \gamma = 0$) is such that three types of unirreps can be characterized, each of them corresponding to specific families. Restricting our discussion to the Higgs algebra corresponding to $N = 1$, $\beta_0 = 1$, $\beta_1 = \beta$, it is possible to show [12] that, if $\gamma = 0$, it corresponds a *class I* of unirreps permitting

$$\beta \geq -\frac{1}{4j^2} \qquad \forall j(\neq 0) \tag{30}$$

but also that, if $\gamma \neq 0$, we get two other families of unirreps characterized by

$$\gamma_+ = \frac{1}{2\beta}\sqrt{-\beta - 4\beta^2 j(j+1)}, \qquad (class\ II), \tag{31}$$

or by

$$\gamma_- = -\frac{1}{2\beta}\sqrt{-\beta - 4\beta^2 j(j+1)}, \qquad (class\ III), \tag{32}$$

both values being constrained by the parameter β according to

$$-\frac{1}{4j(j+1)} < \beta \leq \frac{1}{4j(j+1)+1}. \tag{33}$$

The second step is another improvement proposed by N. Debergh [12] so that Eqs. (23) or (27) are now replaced by

$$J_3 \mid j,m >= \left(\frac{m}{c} + \gamma\right) \mid j,m > \tag{34}$$

where c is a nonnegative and nonvanishing integer. The previous contexts evidently correspond to $c = 1$, but for other values, we get new unirreps of specific interest as we will see. Let us also mention that actions of the ladder operators are also c-dependent. We have

$$J_+ \mid j,m >= \sqrt{f(m)} \mid j,m+c > \tag{35}$$

and

$$J_- \mid j,m >= \sqrt{f(m-c)} \mid j,m-c >, \tag{36}$$

where the f-functions can be found elsewhere [12]. The Casimir operator of this deformed (Higgs) structure appears also as being c-dependent. In conclusion, this second step leads for $c = 2, 3, \ldots$ to new unirreps of the Higgs algebra.

5 On the Higgs algebra and some previous unirreps

The special context characterizing the Higgs algebra is summarized by the following commutation relation replacing Eq.(20)

$$[J_+, J_-] = 2J_3 + 8\beta J_3^3 \tag{37}$$

besides the unchanged ones

$$[J_3, J_\pm] = \pm J_\pm. \tag{38}$$

After Higgs [13, 17], we know that this is an invariance algebra for different physical systems (the harmonic oscillator in two dimensions, in particular) but described in *curved* spaces. It has also been recognized very recently as an interesting structure for physical models in *flat* spaces and I just want to close this talk by mentioning that multiphoton processes of scattering described in quantum optics [14, 18] are also subtended (in a 2-dimensional flat space) by such a nonlinear algebra. The important point to be mentioned here is that these developments deal with the very recent last step (34) and the corresponding unirreps, this fact showing their immediate interest.

Acknowledgment
It is a real pleasure to thank my collaborator *Dr. Nathalie Debergh* for her participation to this programme and for a careful reading of this manuscript.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 115–121.

To the Classification of Integrable Systems in 1+1 Dimensions

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Abstract

The aim of this article is to classify completely integrable systems of the following form $u_t = u_3 + f(u, v, u_1, v_1, u_2, v_2,)$, $v_t = g(u, v, u_1, v_1)$. Here, $u_i = \partial u/\partial x^i$, $u_t = \partial u/\partial t$. The popular symmetry approach to the classification of integrable partial differential systems requires large calculations. That is why we applied the simpler "Chinese" method that deals with canonical conserved densities. Moreover, we proved and applied some additional integrability conditions. These conditions follow from the assumption that the Noether operator exists.

1 Introduction

This article contains our recent results on a classification of the following completely integrable systems

$$u_t = u_3 + f(u, v, u_1, v_1, u_2, v_2), \qquad v_t = g(u, v, u_1, v_1) \tag{1}$$

Here, $u_t = \partial u/\partial t$, $u_1 = \partial u/\partial x$, $u_2 = \partial^2 u/\partial x^2$ and so on. We used the so-called "Chinese" method of classification [1] that was developed in the recent years (see [2], [3], for example).

Let us consider the evolutionary partial differential system

$$u_t = K(u_1, u_2, \ldots, u_q) \tag{2}$$

with two independent variables t, x and the m-dependent $u = \{u^1, u^2, \ldots, u^m\}$. Let K' be a Frechet derivative of the operator K and K'^+ be the formally conjugate operator for K':

$$(K')^\alpha_\beta(D) = \frac{\partial K^\alpha}{\partial u^\beta_n} D^n, \qquad (K'^+)^\alpha_\beta(D) = (-D)^n \frac{\partial K^\beta}{\partial u^\alpha_n}.$$

Here D is the total differentiation operator with respect to x and the summation over the index n is implied.

The "Chinese" method deals with the following equation

$$\left[D_t + \theta + K'^+(D + \rho)\right] a = 0, \tag{3}$$

where D_t is the derivative along trajectories of system (2), the functions θ and ρ satisfy the continuity equation

$$D_t\rho = D\theta \tag{4}$$

and the vector function a satisfies the normalization condition $(c, a) = 1$ with a constant vector c.

It is proved in [3] that if system (2) admits the Lax representation and satisfies some additional conditions, then equation (3) generates a sequence of *local* conservation laws for system (2). The notion *local function* F means that F depends on $t, x, u^\alpha, u_1^\alpha, \ldots, u_n^\alpha$ only and $n < \infty$. (Local function does not depend on any integrals in the form $\int h(t, x, u)\, dx$.) The conserved densities ρ_k and the currents θ_k follow from the formal series expansions

$$\rho = \sum_{k=0}^{\infty} \rho_k z^{k-n}, \quad \theta = \sum_{k=l}^{\infty} \theta_k z^{k-n}, \quad a = \sum_{k=0}^{\infty} a_k z^k, \tag{5}$$

where z is a parameter and n is a positive integer. Substituting expansions (5) into equation (3), one can obtain ρ_k as differential polynomials K. Then equation (4) provides the infinity of the local conservation laws $D_t\rho_k = D\theta_k$. As $\rho_i = \rho_i(K)$, constraints for the function K are obtained. The explicit form of these constraints are $\delta(D_t\rho_i)/\delta u^\alpha = 0$, where $\delta/\delta u^\alpha$ is variational derivative. The conserved densities ρ_i arising from equation (3) are called canonical densities.

If system (3) has a formal local solution in the form (5) satisfying equation (4), then we call system (2) formally integrable.

2 Systems with a Noether operator

According to the definition [4], a Noether operator N satisfies the following equation

$$(D_t - K')N = N(D_t + K'^{+}) \tag{6}$$

Theorem. *If the formally integrable system (2) admits the Noether operator N, then the equation*

$$[D_t + \tilde{\theta} - K'(D + \tilde{\rho})]\tilde{a} = 0 \tag{7}$$

generates the same canonical densities ρ_i as equation (3).

Proof. Let us set $\omega = \int \rho\, dx + \theta\, dt$ and $\gamma = a \exp(\omega)$, where a is a solution of equation (3). Then the following obvious equation is valid

$$[D_t + K'^{+}(D)]\gamma = \mathrm{e}^{\omega}[D_t + \theta + K'^{+}(D + \rho)]a = 0.$$

Therefore, equation (6) implies

$$[D_t + \theta - K'(D + \rho)]\tilde{a} = 0, \tag{8}$$

where $\tilde{a} = \mathrm{e}^{-\omega}N(D)\mathrm{e}^{\omega}a = N(D + \rho)a$. It was proved in [3] that one may require the constraint $(\tilde{a}, \tilde{c}) = 1$. This is equivalent to the gauge transformation

$$\rho(u, z) = \tilde{\rho}(u, z) + D\xi(u, z), \qquad \theta(u, z) = \tilde{\theta}(u, z) + D_t\xi(u, z),$$

where ξ is a holomorphic function of z. This completes the proof.

The known today Noether operators take the following form

$$N^{\alpha\beta} = \sum_{k=0}^{p} N_k^{\alpha\beta} D^k + \sum_{i=1}^{r} A_i^{\alpha} D^{-1} B_i^{\beta} \tag{9}$$

and we consider below this case only.

Proposition. *If a Noether operator takes the form (9), then the vector function $\tilde{a} = N(D+\rho)a$ can be represented by the Laurent series in a parameter z.*

Proof. If ρ and a are given by series (5), then the expression $(D+\rho)^k a = D^k a + kD^{k-1}\rho a + \cdots$ is the Laurent series obviously. Let us consider the last term in expression (9) and denote $\mathrm{e}^{-\omega} D^{-1} \mathrm{e}^{\omega}(B_i, a) \equiv h$. This is equivalent to the following equation for h:

$$(D+\rho)h = (B_i, a) = \sum_{k=0}^{\infty} (B_i, a_k) z^k.$$

Setting $h = \sum h_i z^i$, we obtain $h_i = 0$ for $i < n,\ h_n = \rho_0^{-1}(B_i, a_0),\ h_{n+1} = \rho_0^{-1}[(B_i, a_1) - \rho_1 h_n - \delta_n^1 D h_n]$ and so on. So, the last term in expression (9) gives the Taylor series and this completes the proof.

As equation (8) contains $\tilde{a}$ in the first power, we can multiply $\tilde{a}$ by any power of z. Hence, we can consider $\tilde{a}$ the Taylor series. It was mentioned above that we can submit the vector $\tilde{a}$ to a normalization condition $(\tilde{c}, \tilde{a}) = 1$ by the gauge transformation. It is important to stress that this gauge transformation does not change the densities $\rho_0, \ldots, \rho_{n-1}$ as $\xi(z)$ is a holomorphic function. Hence, the conserved densities ρ_i and $\tilde{\rho}_i$ obtained from equations (3) and (7), respectively, satisfy the following conditions

$$\rho_i = \tilde{\rho}_i \text{ for } i < n, \quad \rho_i - \tilde{\rho}_i \in \mathrm{Im} D \quad \text{for} \quad i \geq n. \tag{10}$$

These conditions give very strong constraints for system (2) and we must explain why they are relevant. It is well known that systems integrable by the inverse scattering transform method possess the Hamiltonian structures. It is also known that any Hamiltonian (or implectic) operator is a Noether operator for the associated evolution system [4]. Hence, for a wide class of integrable systems, conditions (10) are valid.

To use conditions (10), we must choose the correct normalization vector $\tilde{c}$ for $\tilde{a}$. Let us write the matrix operator K' in the form $K' = K_q D^q + K_{q-1} D^{q-1} + \cdots$, then the vector a_0 is a eigenvector of the matrix K_q^T

$$K_q^T a_0 = \lambda a_0, \qquad \lambda = (-1)^{q+1} \theta_l / \rho_0^q, \tag{11}$$

where $l = n(1-q)$ [3]. As the series expansions for $\tilde{\rho},\ \tilde{\theta}$ and $\tilde{a}$ take the same form (5) and $\tilde{\theta}_l = \theta_l$, $\tilde{\rho}_0 = \rho_0$ according to (10), then we easily obtain

$$K_q \tilde{a}_0 = (-1)^{q+1} \lambda \tilde{a}_0. \tag{12}$$

Equations (11) and (12) define the normalization of the vector $\tilde{a}$, but some ambiguity is always possible [3].

3 Classification results

For system (1), we set $u^1 = u,\ u^2 = v.$ Then

$$K_3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \lambda = \tilde{\lambda} = 1.$$

One can see now that $a_0 = \tilde{a}_0 = (1,0)^T$. This means that we can choose $c = \tilde{c} = (1,0)^T$ or equivalently $a = (1,b)^T,\ \tilde{a} = (1,\tilde{b})^T$. The series expansions (5) take now the following form

$$\rho = z^{-1} + \sum_{i=0}^{\infty} \rho_i z^i, \qquad \theta = z^{-3} + \sum_{i=0}^{\infty} \theta_i z^i,$$

$\tilde{\rho}$ and $\tilde{\theta}$ has the same form. Substituting these series into equations (3) and (7), we obtain the recursion relations for ρ_i and $\tilde{\rho}_i$. We can not present these relations here because they take large room. Here are the first terms of the sequences of ρ_i and $\tilde{\rho}_i$:

$$\rho_0 = -\tilde{\rho}_0 = \frac{1}{3}\frac{\partial f}{\partial u_2}, \qquad \rho_1 = \tilde{\rho}_1 = \rho_0^2 - \frac{1}{3}\frac{\partial f}{\partial u_1},$$
$$\rho_2 = -\tilde{\rho}_2 = \frac{1}{3}\left(\theta_0 - \rho_0^3 + 3\rho_0\rho_1 + \frac{\partial f}{\partial u} + \frac{\partial f}{\partial v_2}\frac{\partial g}{\partial u_1}\right).$$

We find with the help of a computer that sufficiently many integrability conditions (4) and (10) (8 or 12 sometimes) are satisfied in the following four cases only: (I) system (1) admits nontrivial higher conserved densities; (II) the system is reducible to the triangular or linear form with the help of a contact transformation; (III) the system is linear or triangular. The last case is not interesting and we omit it.

We present here the complete classification the systems (I) and (II).

List I. Systems admitting higher conserved densities

$$u_t = u_3 - 3u_2^2/(4u_1) + v_2u_1 + c_1u_1, \quad v_t = u + c_2v_1. \tag{13}$$

$$u_t = u_3 - 3u_2^2/(4u_1) + v_1u_1 + c_1u_1, \quad v_t = u_1 + c_2v_1. \tag{14}$$

$$\begin{aligned} &u_t = w_2 + 6c_0\mathrm{e}^u u_1 - 1/2w^3 + c_1w + c_2u_1, \quad w = u_1 - v, \\ &v_t = 3c_0\mathrm{e}^u u_1^2 + 4c_0\mathrm{e}^u v_1 + c_0\mathrm{e}^u v^2 - 2c_0c_1\mathrm{e}^u + c_2v_1, \quad c_0 \neq 0. \end{aligned} \tag{15}$$

$$u_t = u_3 + uv_2 + u_1v_1 + c_1u_1, \quad v_t = u + c_0v_1. \tag{16}$$

$$u_t = u_3 + u_1v + uv_1 + c_1u_1 + c_2v_1, \quad v_t = u_1 + c_0v_1. \tag{17}$$

$$u_t = u_3 + u_1v_1 + c_1u_1 + c_2v_1, \quad v_t = u_1 + c_0v_1. \tag{18}$$

$$u_t = u_3 + u_1v_2 + c_1u_1 + c_2v_2, \quad v_t = u + c_0v_1. \tag{19}$$

$$u_t = u_3 + 3u_1v + 2uv_1 + v_1v_2 + c_1v_1 + 2c_2vv_1 - 2v^2v_1, \quad v_t = u_1 + vv_1 + c_2v_1. \tag{20}$$

$$u_t = u_3 + 2u_1v_1 + uv_2 + c_1u_1, \quad v_t = c_2u^2 + c_3v_1. \tag{21}$$

$$u_t = u_3 + 2u_1v + uv_1 + c_1u_1, \quad v_t = 2c_2uu_1 + c_3v_1. \tag{22}$$

$$\begin{aligned} u_t = {} & u_3 - u_1v_2/v - 3u_2v_1/(2v) + 3u_1v_1^2/(2v^2) + u^2u_1/v+ \\ & +c_1u_1/(2v) + 2c_2v^2u_1 + 3c_2uvv_1, \quad v_t = 2uu_1. \end{aligned} \tag{23}$$

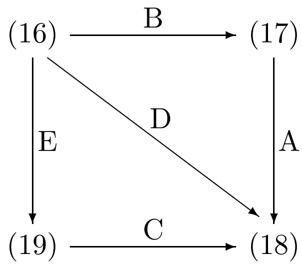
$$\begin{aligned} u_t = {} & u_3 - u_1v_2/v - 3u_2v_1/(2v) + 3u_1v_1^2/(2v^2) + 3c_1uv_1^2/(2v^2)+ \\ & +u^2u_1/v - c_1(u_1v_1 + uv_2 - u^3)/v + c_1^2uv_1/(2v) - c_1^2u_1 - c_2u, \\ v_t = {} & 2uu_1 + 2c_1u^2 - 2c_2v. \end{aligned} \tag{24}$$

$$u_t = u_3 + 3/2u1^2 + c_1v_1^2 + c_2u_1, \quad v_t = u_1v_1 + c_3v_1. \tag{25}$$

$$u_t = u_3 + 3uu1 + 2c_1v_1v_2 + c_2u_1, \quad v_t = uv_1. \tag{26}$$

$$u_t = u_3 + 3uu1 + 2c_1vv_1 + c_2u_1, \quad v_t = uv_1 + u_1v. \tag{27}$$

System (14) follows from system (13) under the substitution $v_1 \to v$. System (15) is triangular if $c_0 = 0$, and moreover the transformation $(u, v) \to (v, w)$ gives in this case the pair of independent equations. Systems (16)–(19) are connected by the contact transformations A, B, C, D and E according to the following diagram



A: $(u, v) \to (u_1, v_1)$

B: $(u, v_1) \to (u + c_2, v)$

C: $(u, v_1) \to (u, v)$

D: $(u, v) \to (u_1 + c_2, v + c_2t)$

E: $(u, v) \to (u_1 + c_2, v_1 + c_2t)$

System (22) follows from system (21) under the substitution $v_1 \to v$. The systems (25)–(27) are connected each other according to the following diagram

where the maps F and G take the following form: $F : (u_1, v, c_2) \to (u - c_3, v, c_2 + 3c_3)$, $G : (u, v_1) \to (u, v)$.

List II. Systems reducible to the triangular form

$$\begin{aligned} u_t &= w_2 - 3w_1^2/(4w) - h(u, v) + f_1(w), \\ v_t &= h_uu_1 + h_vv_1 + f_2(w), \quad w = v + u_1. \end{aligned} \tag{28}$$

The reduction to the triangular form: $(u, v) \to (u, w)$.

$$\begin{aligned} u_t &= u_3 + v_2 + h(u, v) - \xi(u)v_1 + 3/2\xi' u_1^2, \\ v_t &= \xi^2v_1 - h_vv_1 - h_uu_1 + \xi' u_1v_1 - 3/2\xi\xi' u_1^2 - 1/2\xi'' u_1^3 - h\xi + c. \end{aligned} \tag{29}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=v+u_1+\int\xi(u)\,du$.

$$\begin{aligned} u_t &= u_3-2/3(v_2+u_1v^2+c_1v_1\mathrm{e}^{-u}-c_2v_1\mathrm{e}^{u})-3/2(c_1^2\mathrm{e}^{-2u}+c_2^2\mathrm{e}^{2u})u_1\\ &\quad +2(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})u_1v+u_1^2v-1/2u_1^3+h(u,v)+c_0u_1,\\ v_t &= 3/2u_1^2v(c_2\mathrm{e}^{u}-c_1\mathrm{e}^{-u})+3/2(h_uu_1+h_vv_1-c_1h\mathrm{e}^{-u}+c_2h\mathrm{e}^{u})-2/3v^2v_1\\ &\quad +(u_1v_1+2vv_1)(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})-1/2v_1(c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u})^2+(c_0-c_1c_2)v_1. \end{aligned} \tag{30}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=u_1-2v/3+c_1\mathrm{e}^{-u}+c_2\mathrm{e}^{u}$.

$$u_t=w_2-3w_1^2/(4w)-c_1v^2+c_2w,\quad v_t=c_3\sqrt{w}-c_1v_1,\quad w=u_1+v^2. \tag{31}$$

The reduction to the linear form: $(u,v)\to(y,v)$, where $y=\sqrt{w}$.

$$u_t=w_2-h(u,v)-3{w_1}^2/(2w)+c_1w^3+c_2/w,\quad v_t=h_vv_1+h_uu_1+c_3w, \tag{32}$$

where $w=u_1+v$. The reduction to the triangular form: $(u,v)\to(u,w)$.

$$u_t=w_2-\frac{3}{2}\frac{w{w_1}^2}{w^2+c}-h(u,v)+f_1(w),\quad v_t=h_vv_1+h_uu_1+f_2(w),\quad w=u_1+v. \tag{33}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$u_t=u_3+u_1\phi(v,v_2-u),\quad v_t=u_1,\quad \frac{\partial\phi}{\partial v}=\phi\frac{\partial\phi}{\partial v_2}. \tag{34}$$

One can check that $D_t(\phi)=0$, hence, $\phi=F_1(x)$ on solutions of system (34). Denoting $v_2-u=w$, we can integrate the equation $\phi_v=\phi\phi_w$ in the following implicit form $w+v\phi=H(\phi)$, where H is arbitrary function. This implies the following equations

$$u=v_{xx}+vF_1(x)+F_2(x),\quad v_t=v_{xxx}+(vF_1)_x+F_{2,x}, \tag{34$'$}$$

where F_1 is an arbitrary function and $F_2=H(F_1)$.

$$\begin{aligned} u_t &= w_2+kw_1+h(u,v)+f_1(w)+k^2u_1,\quad w=u_1+v-ku,\\ v_t &= kh-h_uu_1-h_vv_1+kc_1w+kc_2+k^2v_1. \end{aligned} \tag{35}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$\begin{aligned} u_t &= u_3+c_3u_1+c_4w+c_5w^2+kw^3+c_6,\quad w=v_2-u-c_2v_1+c_1v,\\ v_t &= u_1+c_2u-c_1c_2v+(c_3+c_2^2-c_1)v_1+c_7. \end{aligned} \tag{36}$$

The reduction to the triangular form: $(u,v)\to(u,w)$.

$$\begin{aligned} u_t &= u_3+v_2+6c_0(c_1uu_1-vu_1-c_0u^2u_1)-2c_0uv_1+c_1v_1+c_2u_1-h(u,v),\\ v_t &= (c_1^2+c_2)v_1-6c_0vv_1+2c_0(u_1v_1-c_0u^2v_1+c_1uv_1)\\ &\quad +2c_0uh-c_1h+h_uu_1+h_vv_1. \end{aligned} \tag{37}$$

The reduction to the triangular form: $(u,v)\to(u,w)$, where $w=u_1+v-c_1u+c_0u^2$.

In formulas (13)–(37), c_i and k are arbitrary constants, h, ξ, ϕ and f_i are arbitrary functions, $h_u=\partial h/\partial u$.

Conclusion

We believe that any system from the list (I) is integrable in the frame of the inverse scattering transform method.

Let us note that some equations from the list (I) admit the reduction to a single integrable equation. For example, excluding the function u from system (13) and setting $c_2 = 0$ for simplicity, we obtain

$$v_{tx}v_{tt} - v_{tx}v_{txxx} + \frac{3}{4}v_{txx}^2 - v_{xx}v_{tx}^2 - c_1 v_{tx}^2 = 0. \tag{13'}$$

For system (16), the same operation gives

$$z_{tt} - z_x z_{tx} + (2z_x c_0 - z_t)z_{xx} - z_{txxx} + c_0 z_{xxxx} = 0, \tag{16'}$$

where $v = z - (c_1 + c_0)x - c_0(c_1 + 2c_0)t$. And system (20) is reduced to the following form

$$\begin{gathered} z_{tt} + (3c_2 - 4z_x)z_{tx} + z_x z_{xxxx} - z_{txxx} + 2z_{xx}z_{xxx} + \\ +3z_x z_{xx}(2z_x - 3c_2) - 2z_t z_{xx} = 0, \end{gathered} \tag{20'}$$

where $v = z_x - c_2$. Let us also notice another forms for systems (23) and (24) that arise under the exponential substitution $v \to \mathrm{e}^v$.

$$\begin{aligned} &u_t = u_3 - u_1 v_2 + 1/2 u_1 v_1^2 - 3/2 u_2 v_1 + c_2 \mathrm{e}^{2v}(2u_1 + 3uv_1) + \mathrm{e}^{-v}(u^2 + c_1/2)u_1, \\ &v_t = 2uu_1 \mathrm{e}^{-v}. \end{aligned} \tag{23'}$$

$$\begin{gathered} u_t = u_3 - 3/2 u_2 v_1 - u_1 v_2 + c_1(uv_1^2/2 - uv_2 - u_1 v_1 - c_1 u_1 + c_1 uv_1/2) + \\ +1/2 u_1 v_1^2 - c_2 u + \mathrm{e}^{-v}(c_1 u^3 + u^2 u_1), \quad v_t = 2(uu_1 + c_1 u^2)\mathrm{e}^{-v} - 2c_2. \end{gathered} \tag{24'}$$

For the linearizable system (34), we get

$$u_t = u_3 + u_1(u - v_2)/v, \quad v_t = u_1.$$

This system is equivalent to system (34′), where $F_2 = 0$.

Acknowledgement

The authors were supported by the research grant 96–01–01384 from Russian Foundation for Fundamental Reseach.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 122–129.

On New Generalizations of the Burgers and Korteweg-de Vries Equations

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Abstract

We describe new classes of nonlinear Galilean–invariant equations of Burgers and Korteweg–de Vries type and study symmetry properties of these equations.

1 Introduction

Equations which are writen below, simple wave, Burgers, Korteweg–de Vries, Korteweg-de Vries-Burgers and Kuramoto-Sivashinski equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} = 0, \tag{2}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{3}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0, \tag{4}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{5}$$

are widely used for the mathematical modelling of various physical and hydrodynamic processes [4, 7, 9].

These equations possess very important properties:

1. All these equations have the same nonlinearity $u\frac{\partial u}{\partial x}$, and the operator $\frac{\partial}{\partial t}+u\frac{\partial}{\partial x} \equiv \frac{d}{dt}$ is "the material derivative".
2. All these equations are compatible with the Galilean relativity principle.

This talk is based on the results obtained in collaboration with Prof. W. Fushchych [2] and dedicated to his memory.

The last assertion means that equations (1)–(5) are invariant with respect to the Galilean transformations

$$t \to t' = t, \qquad x \to x' = x + vt, \qquad u \to u' = u + v, \tag{6}$$

where v is the group parameter (velocity of an inertial system with respect to another inertial system).

It is evident that equations (1)–(5) are invariant also with respect to the transformations group

$$t \to t' = t + a, \qquad x \to x' = x + b, \qquad u \to u' = u, \tag{7}$$

where a and b are group parameters.

In terms of the Lie algebra, invariance of equations (1)–(5) with respect to transformations (6)–(7) means that the Galilean algebra, which will be designated as $AG(1,1) = \langle P_0, P_1, G\rangle$ [3] with basis elements

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad G = t\partial_x + \partial_u. \tag{8}$$

is an invariance algebra of the given equations.

Let us recall some well-known facts on symmetry properties of equations (1)–(5). The equation of a simple wave (1) has general solutions of the form $u = f(x - ut)$ [9] and admits an infinite invariance algebra.

Equation (2) admits a five-dimensional invariance algebra [3], besides, let us note that this equation can be reduced to the heat equation by means of the Cole-Hopf transformation [9].

The Korteweg-de Vries equation (3) admits a four-dimensional invariance algebra [5], besides equation (2) is the classical example of an integrable equation [1].

Unfortunately, the symmetry of equations (4) and (5) is rather poor (the maximal invariance algebra is a three-dimensional algebra (8)), though, the presence of members which contain u_{xxx}, u_{xxxx} in these equations, is very important from the physical point of view.

Thus, linearity of equations (3)–(5) with respect to u_{xxx}, u_{xxxx} is bad from the point of view of symmetry, linearity of these equations causes the essential narrowing of symmetry the compared to the Burgers equation (2). The question arises how we can "correct" equations (3)–(5) so as at least to preserve the symmetry of the Burgers equation or to obtain some new generalization of the Galilean algebra (8).

To solve the problem, let us consider a natural generalization of all adduced equations, namely, the equation of the form

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right), \tag{9}$$

and, as particular case, the equation

$$u_{(0)} + uu_{(1)} = F\left(u_{(n)}\right). \tag{10}$$

Here and further, we use the following designations: $u = u(t,x)$; $u_{(0)} = \dfrac{\partial u}{\partial t}$; $u_{(n)} = \dfrac{\partial^n u}{\partial x^n}$; $F\left(u_{(2)}, u_{(3)}, \ldots, u_{(n)}\right)$, $F\left(u_{(n)}\right)$ are arbitrary smooth functions.

Evidently, equations (9)–(10) will be invariant with respect to transformation (6)–(7), so they are compatible with the Galilean relativity princeple, and thus equations (9), (10) with an arbitrary function F will be invariant with respect to the Galilean algebra (8).

To have a hope to construct at least partial solutions of equations (9), (10), we need to specify (to fix) the function F. One of approaches to this problem is based on description of equations (9), (10) which admit wider invariance algebras than the Galilean algebra $AG(1,1)$ [3]. Wide symmetries of nonlinear equations, as is well known [3, 5, 6], enable to describe ansatzes reducing partial differential equations to ordinary differential equations which can often be solved exactly or approximately, or for which qualitative properties of solutions, asymptotic properties, etc. can be studied.

The principal aim of our work is as follows: to give a description of equations (9), (10) which have wider symmetry properties than the algebra $AG(1,1)$ or to describe nonlinear smooth functions F for which these equations are invariant with respect to Lie algebras which are extensions of the Galilean algebra $AG(1,1)$; using the symmetry of equations, to construct ansatzes and to reduce partial differential equations to ordinary differential equations.

The paper is organized as follows. In Section 2, we present all principal theorems and corollaries on symmetry classification of equations (9), (10) which admit wider symmetry than the Galilean algebra $AG(1,1)$. We do not give proofs of theorems, because they are extremely cumbersome, though simple from the point of view of ideas. In Section 3, we adduce finite group transformations, construct anzatzes and some classes of exact solutions.

2 Symmetry classification

Let us first formulate the statements on the Lie symmetry of certain equations of the type (10). Consider the following equations:

$$u_{(0)} + uu_{(1)} = F\left(u_{(2)}\right), \tag{11}$$

$$u_{(0)} + uu_{(1)} = F\left(u_{(3)}\right), \tag{12}$$

$$u_{(0)} + uu_{(1)} = F\left(u_{(4)}\right). \tag{13}$$

Theorem 1 *The maximal invariance algebras of equation (11) depending on $F\left(u_{(2)}\right)$ are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(2)}\right)$ *is arbitrary;*
2. $\langle P_0, P_1, G, Y_1\rangle$ *if* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq 1$; $k \neq \dfrac{1}{3}$;
3. $\langle P_0, P_1, G, Y_2\rangle$ *if* $F\left(u_{(2)}\right) = \ln u_{(2)}$;
4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(2)}\right) = \lambda u_{(2)}$;
5. $\langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$ *if* $F\left(u_{(2)}\right) = \lambda\left(u_{(2)}\right)^{1/3}$.

Here, $\lambda = \mathrm{const}$, $\lambda \neq 0$, *and basis elements of the Lie algebras have the following repre-*

sentation:

$$Y_1 = (k+1)t\partial_t + (2-k)x\partial_x + (1-2k)u\partial_u, \quad Y_2 = t\partial_t + \left(2x - \frac{3}{2}t^2\right)\partial_x + (u-3t)\partial_u,$$
$$D = 2t\partial_t + x\partial_x - u\partial_u, \quad \Pi = t^2\partial_t + tx\partial_x + (x-tu)\,\partial_u, \quad R_1 = 4t\partial_t + 5x\partial_x + u\partial_u,$$
$$R_2 = u\partial_x, \quad R_3 = (2tu - x)\,\partial_x + u\partial_u, \quad R_4 = (tu - x)\,(t\partial_x + \partial_u)\,.$$

Theorem 1 makes the result obtained in [8] more precise. The Burgers equation (2) as a particular case of (11) is includes in Case 4 of Theorem 1.

Note that the following equation has the widest symmetry in the class of equations (11) (7–dimensional algebra):

$$u_{(0)} + uu_{(1)} = \lambda\left(u_{(2)}\right)^{1/3}. \tag{14}$$

Theorem 2 *The maximal invariance algebras of equation (7) depending on* $F\left(u_{(3)}\right)$ *are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(3)}\right)$ *is arbitrary;*

2. $\langle P_0, P_1, G, Y_3\rangle$ *if* $F\left(u_{(3)}\right) = \lambda\left(u_{(3)}\right)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq \frac{3}{4}$;

3. $\langle P_0, P_1, G, Y_4\rangle$ *if* $F\left(u_{(3)}\right) = \ln u_{(3)}$;

4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(3)}\right) = \lambda\left(u_{(3)}\right)^{3/4}$.

Here, $\lambda = \mathrm{const}$, $\lambda \neq 0$,

$$Y_3 = (2k+1)t\partial_t + (2-k)x\partial_x + (1-3k)u\partial_u, \quad Y_4 = t\partial_t + \left(2x - \frac{5}{2}t^2\right)\partial_x + (u-5t)\partial_u.$$

Case 2 of Theorem 2 for $k = 1$ includes the Korteweg–de Vries equation (3) as a particular case of (12).

Theorem 3 *The maximal invariance algebras of equation (8) depending on* $F\left(u_{(4)}\right)$ *are the following Lie algebras:*

1. $\langle P_0, P_1, G\rangle$ *if* $F\left(u_{(4)}\right)$ *is arbitrary;*

2. $\langle P_0, P_1, G, Y_5\rangle$ *if* $F\left(u_{(4)}\right) = \lambda\left(u_{(4)}\right)^k$, $k = \mathrm{const}$; $k \neq 0$; $k \neq \frac{3}{5}$;

3. $\langle P_0, P_1, G, Y_6\rangle$ *if* $F\left(u_{(4)}\right) = \ln u_{(4)}$;

4. $\langle P_0, P_1, G, D, \Pi\rangle$ *if* $F\left(u_{(4)}\right) = \lambda\left(u_{(4)}\right)^{3/5}$.

Here, $\lambda = \mathrm{const}$, $\lambda \neq 0$,

$$Y_5 = (3k+1)t\partial_t + (2-k)x\partial_x + (1-4k)u\partial_u, \quad Y_6 = t\partial_t + \left(2x - \frac{7}{2}t^2\right)\partial_x + (u-7t)\partial_u.$$

Theorems 1–3 give the exhaustive symmetry classification of equations (11)–(13).

On the basis of Theorems 1–3, let us formulate some generalizations concerning the symmetry of equations (10), namely, investigate symmetry properties of equation (10) with fixed functions $F(u_{(n)})$.

Theorem 4 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \ln u_{(n)} \tag{15}$$

is the four–dimensional algebra $\langle P_0, P_1, G, A_1\rangle$, where

$$A_1 = t\partial_t + \left(2x - \frac{2n-1}{2}t^2\right)\partial_x + \Big(u - (2n-1)t\Big)\partial_u.$$

Theorem 5 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(n)}\Big)^k \tag{16}$$

is the four–dimensional algebra $\langle P_0, P_1, G, A_2\rangle$, where

$$A_2 = ((n-1)k+1)\, t\partial_t + (2-k)x\partial_x + (1-nk)u\partial_u,$$

k, λ are real constants, $k \neq 0$, $k \neq \dfrac{3}{n+1}$, $\lambda \neq 0$; for $n = 2$, there is the additional condition: $k \neq \dfrac{1}{3}$ (see Case 5 of Theorem 1).

Theorem 6 *For any integer $n \geq 2$, the maximal invariance algebra of the equation*

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(n)}\Big)^{3/(n+1)}, \quad \lambda = \mathrm{const}, \lambda \neq 0 \tag{17}$$

is the five–dimensional algebra

$$\langle P_0, P_1, G, D, \Pi\rangle. \tag{18}$$

Remark. If $n = 1$ in (17), then we get the equation

$$u_{(0)} + uu_{(1)} = \lambda\Big(u_{(1)}\Big)^{3/2}. \tag{19}$$

Theorem 7 *The maximal invariance algebra of equation (19) is the four–dimensional algebra $\langle P_0, P_1, G, D\rangle$.*

Remark. It is interesting that (18) defines an invariance algebra for equation (17) for any natural $n \geq 2$. With $n = 2$, (17) is the classical Burgers equation (2). Let us note that operators (18) determine a representation of the generalized Galilean algebra $AG_2(1,1)$ [3].

Now let us investigate the invariance of equation (9) with respect to representation (18) or point out from the class of equations (9) those which are invariant with respect of the invariance algebra of the classical Burgers equation. The following statement is true:

Theorem 8 *Equation (9) is invariant under the generalized Galilean algebra $AG_2(1,1)$ (18) iff it has the form*

$$u_{(0)} + uu_{(1)} = u_{(2)}\Phi\Big(\omega_3, \omega_4, \ldots, \omega_n\Big), \tag{20}$$

where Φ is an arbitrary smooth function,

$$\omega_k = \frac{1}{u_{(2)}}\Big(u_{(k)}\Big)^{3/(k+1)}, \quad u_{(k)} = \frac{\partial^k u}{\partial x^k}, \quad k = 3, \ldots, n.$$

The class of equations (20) contains the Burgers equation (2) (for $\Phi = \text{const}$) and equation (17). Equation (20) includes as a particular case the following equation which can be interpreted as a generalization of the Burgers equation and used for description of wave processes:

$$u_{(0)} + uu_{(1)} = \lambda_2 u_{(2)} + \lambda_3 \left(u_{(3)}\right)^{3/4} + \cdots + \lambda_n \left(u_{(n)}\right)^{3/(n+1)}, \tag{21}$$

$\lambda_2, \lambda_3, \ldots, \lambda_n$ are an arbitrary constant.

Let us note that the maximal invariance algebra of equation (21) is a generalized Galilean algebra (18).

Below we describe all second–order equations invariant under the generalized Galilean algebra (18). The following assertions are true:

Theorem 9 *A second–order equation is invariant under the generalized Galilean algebra $AG_2(1,1)$ iff it has the form*

$$\Phi\left(\frac{\left(u_{00}u_{11} - (u_{01})^2 + 4u_0u_1u_{11} + 2uu_{11}(u_1)^2 - 2u_{01}(u_1)^2 - (u_1)^4\right)^3}{(u_{11})^8};\right.$$
$$\left.\frac{u_0 + uu_1}{u_{11}}; \frac{\left(u_{01} + uu_{11} + (u_1)^2\right)^3}{(u_{11})^4}\right) = 0, \tag{22}$$

where Φ is an arbitrary function.

3 Finite group transformations, ansatzes, solutions

Operators of the algebra $L = \langle P_0, P_1, G, R_1, R_2, R_3, R_4\rangle$ which define the invariance algebra equation (14), satisfy the following group relations:

	P_0	P_1	G	R_1	R_2	R_3	R_4
P_0	0	0	P_1	$4P_0$	0	$2R_2$	R_3
P_1	0	0	0	$5P_1$	0	$-P_1$	$-G$
G	$-P_1$	0	0	G	P_1	G	0
R_1	$-4P_0$	$-5P_1$	$-G$	0	$-4R_2$	0	$4R_4$
R_2	0	0	$-P_1$	$4R_2$	0	$-2R_2$	$-R_3$
R_3	$-2R_2$	P_1	$-G$	0	$2R_2$	0	$-2R_4$
R_4	$-R_3$	$-G$	0	$-4R_4$	R_3	$2R_4$	0

Let us note that it is possible to specify three subalgebras of the algebra L, which are Galilean algebras: $\langle P_0, P_1, G\rangle$, $\langle P_1, G, -R_4\rangle$, $\langle -R_2, P_1, G\rangle$.

The finite transformations corresponding to the operators R_1, R_2, R_3, R_4 are the following:

$$R_1: \quad t \to \tilde{t} = t\exp(4\theta), \quad x \to \tilde{x} = x\exp(5\theta), \quad u \to \tilde{u} = u\exp(\theta),$$

$$R_2: \quad t \to \tilde{t} = t \quad x \to \tilde{x} = x + \theta u, \quad u \to \tilde{u} = u,$$

$$R_3: \quad t \to \tilde{t} = t, \quad x \to \tilde{x} = x\exp(-\theta) + tu\exp(\theta), \quad u \to \tilde{u} = u\exp(\theta),$$

$$R_4: \quad t \to \tilde{t} = t, \quad x \to \tilde{x} = x + \theta t\,(ut - x), \quad u \to \tilde{u} = u + \theta(ut - x),$$

where θ is the group parameter.

Let us represent the exact solution of (14) (below, we point out the operator, the ansatz, the reduced equation, and the solution obtained by means of reduction and integration of the reduced equation)

the operator: $R_3 = (2tu - x)\,\partial_x + u\partial_u$,

the ansatz: $xu - tu^2 = \varphi(t)$,

the reduced equation: $\varphi' = \lambda(2\varphi)^{1/3}$,

the solution:

$$xu - tu^2 = \frac{1}{2}\left(\frac{4}{3}\lambda t + C\right)^{3/2}. \tag{23}$$

Relation (23) determines the set of exact solutions of equation (14) in implicit form.

The following Table contains the commutation relations for operators (18):

	P_0	P_1	G	D	Π
P_0	0	0	P_1	$2P_0$	D
P_1	0	0	0	P_1	G
G	$-P_1$	0	0	$-G$	0
D	$-2P_0$	$-P_1$	G	0	2Π
Π	$-D$	$-G$	0	-2Π	0

The finite group transformations corresponding to the operators D, Π in representation (18) are the following:

$$\begin{aligned} D:\quad & t \to \tilde t = t\exp(2\theta), \\ & x \to \tilde x = x\exp(\theta), \\ & u \to \tilde u = u\exp(-\theta), \end{aligned} \qquad \begin{aligned} \Pi:\quad & t \to \tilde t = \frac{t}{1-\theta t}, \\ & x \to \tilde x = \frac{x}{1-\theta t}, \\ & u \to \tilde u = u + (x - ut)\,\theta, \end{aligned}$$

where θ is the group parameter.

The ansatz

$$u = t^{-1}\varphi(\omega) + \frac{x}{t}, \qquad \omega = 2xt^{-1}$$

constructed by the operator Π reduces equation (17) to the following ordinary differential equation

$$\varphi\varphi' = \lambda_1 2^{(2n-1)/(n+1)}\left(\varphi^{(n)}\right)^{3/(n+1)}. \tag{24}$$

A partial solution (24) has the form

$$\varphi = -2\left(\lambda_1{}^{(n+1)}(n!)^3\right)^{1/(2n-1)}\omega^{-1},$$

and whence we get the following exact solution of equation (17)

$$u = -2\left(\lambda_1{}^{(n+1)}(n!)^3\right)^{1/(2n-1)}\frac{1}{2x} + \frac{x}{t}.$$

In general case, it is necessary to use nonequvilent one-dimensional subalgebras to obtain solutions. In Table, nonequivalent one–dimensional subalgebras for algebra (18) and corresponding ansatzes are adduced. (Classification of one–dimensional subalgebras is carried out according to the scheme adduced in [5].)

	Ansatz
P_1	$u = \varphi(t)$
G	$u = \varphi(t) + xt^{-1}$
$P_0 + \alpha G,\ \alpha \in \mathbf{R}$	$u = \varphi\left(x - \frac{\alpha}{2}t^2\right) + \alpha t$
D	$u = t^{-1/2}\varphi\left(xt^{-1/2}\right)$
$P_0 + \Pi$	$u = (t^2+1)^{-1/2}\varphi\left(\frac{x}{(t^2+1)^{1/2}}\right) + \frac{tx}{t^2+1}$

The ansatzes constructed can be used for symmetry reduction and for construction of solutions for equations (17), (20), (21).

The author is grateful to the DFFD of Ukraine (project 1.4/356) for financial support.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 130–137.

Nonclassical Potential Symmetries of the Burgers Equation

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Abstract

In this paper, new classes of symmetries for partial differential equations (PDE) which can be written in a conserved form are introduced. These new symmetries called nonclassical potential symmetries, are neither potential symmetries nor nonclassical symmetries. Some of these symmetries are carried out for the Burgers equation

$$u_t + uu_x - u_{xx} = 0. \tag{1}$$

by studying the nonclassical symmetries of the integrated equation

$$v_t + \frac{v_x^2}{2} - v_{xx} = 0. \tag{2}$$

By comparing the classical symmetries of the associated system

$$\begin{aligned} &v_x = u, \\ &v_t = u_x - \frac{u^2}{2} \end{aligned} \tag{3}$$

with those of the integrated equation (2), we deduce the condition for the symmetries of (2) to yield potential symmetries of (1). The nonclassical potential symmetries are realized as local nonclassical symmetries of (2). Similarity solutions are also discussed in terms of the integrated equation and yield solutions of the Burgers equation which are neither nonclassical solutions of the Burgers equation nor solutions arising from potential symmetries.

1 Introduction

Local symmetries admitted by a PDE are useful for finding invariant solutions. These solutions are obtained by using group invariants to reduce the number of independent variables.

The fundamental basis of the technique is that, when a differential equation is invariant under a Lie group of transformations, a reduction transformation exists. The machinery of the Lie group theory provides a systematic method to search for these special group-invariant solutions. For PDE's with two independent variables, as it is equation (1), a single group reduction transforms the PDE into ODE's, which are generally easier to solve than the original PDE. Most of the required theory and description of the method can be found in [6, 12, 13].

Local symmetries admitted by a nonlinear PDE are also useful to discover whether or not the equation can be linearized by an invertible mapping and construct an explicit linearization when one exists. A nonlinear scalar PDE is linearizable by an invertible contact (point) transformation if and only if it admits an infinite-parameter Lie group of contact transformations satisfying specific criteria [5, 6, 7, 11].

An obvious limitation of group-theoretic methods based on local symmetries, in their utility for particular PDE's, is that many of these equations does not have local symmetries. It turns out that PDE's can admit nonlocal symmetries whose infinitesimal generators depend on the integrals of the dependent variables in some specific manner. It also happens that if a nonlinear scalar PDE does not admit an infinite-parameter Lie group of contact transformations, it is not linearizable by an invertible contact transformation. However, most of the interesting linearizations involve noninvertible transformations, such linearizations can be found by embedding given nonlinear PDE's in auxiliary systems of PDE's. [6].

Krasil'shchik and Vinogrod [15, 10] gave criteria which must be satisfied by nonlocal symmetries of a PDE when realized as local symmetries of a system of PDE's which 'covers' the given PDE. Akhatov, Gazizov and Ibragimov [1] gave nontrivial examples of nonlocal symmetries generated by heuristic procedures.

In [5, 6], Bluman introduced a method to find a new class of symmetries for a PDE. By writing a given PDE, denoted by R{x,t,u} in a conserved form, a related system denoted by S{x,t,u,v} with potentials as additional dependent variables is obtained. If $u(x,t), v(x,t)$ satisfies S{x,t,u,v}, then $u(x,t)$ solves R{x,t,u} and $v(x,t)$ solves an integrated related equation T{x,t,v}. Any Lie group of point transformations admitted by S{x,t,u,v} induces a symmetry for R{x,t,u}; when at least one of the generators of a group depends explicitly of on potential, then the corresponding symmetry is neither a point nor a Lie-Bäcklund symmetry. These symmetries of R{x,t,u} are called *potential* symmetries.

The nature of potential symmetries allows one to extend the uses of point symmetries to such nonlocal symmetries. In particular:

1. Invariant solutions of S{x,t,u,v}, respectively T{x,t,v}, yield solutions of R{x,t,u} which are not invariant solutions for any local symmetry admitted by R{x,t,u}.
2. If R{x,t,u} admits a potential symmetry leading to the linearization of S{x,t,u,v}, respectively T{x,t,v}, then R{x,t,u} is linearized by a noninvertible mapping.

Suppose S{x,t,u,v} admits a local Lie group of transformations with the infinitesimal generator

$$X_S = p(x,t,u,v)\frac{\partial}{\partial x} + q(x,t,u,v)\frac{\partial}{\partial t} + r(x,t,u,v)\frac{\partial}{\partial u} + s(x,t,u,v)\frac{\partial}{\partial v}, \tag{4}$$

this group maps any solution of S{x,t,u,v} to another solution of S{x,t,u,v} and hence induces a mapping of any solution of R{x,t,u} to another solution of R{x,t,u}. Thus, (4) defines a symmetry group of R{x,t,u}. If

$$\left(\frac{\partial p}{\partial v}\right)^2 + \left(\frac{\partial q}{\partial v}\right)^2 + \left(\frac{\partial r}{\partial v}\right)^2 \neq 0, \tag{5}$$

then (4) yields a nonlocal symmetry of R {x,t,u}, such a nonlocal symmetry is called a *potential* symmetry of R{x,t,u}, otherwise X_S projects onto a point symmetry of R{x,t,u}.

Suppose

$$X_T = p^T(x,t,v)\frac{\partial}{\partial x} + q^T(x,t,v)\frac{\partial}{\partial t} + s^T(x,t,v)\frac{\partial}{\partial v} \tag{6}$$

defines a point symmetry of the related integrated equation T{x,t,v}. Then X_T yields a nonlocal potential symmetry of R{x,t,u} if and only if

$$X_S = X_T + r(x,t,u,v)\frac{\partial}{\partial u} \tag{7}$$

yields a nonlocal potential symmetry of R{x,t,u}.

Motivated by the fact that symmetry reductions for many PDE's are known that are not obtained using the classical Lie group method, there have been several generalizations of the classical Lie group method for symmetry reductions. Bluman and Cole developed the nonclassical method to study the symmetry reductions of the heat equation; Clarkson and Mansfield [8] presented an algorithm for calculating the determining equations associated with nonclassical symmetries.

The basic idea of the nonclassical method is that PDE is augmented with the invariance surface condition

$$pu_x + qu_t - r = 0 \tag{8}$$

which is associated with the vector field

$$X_R = p(x,t,u)\frac{\partial}{\partial x} + q(x,t,u)\frac{\partial}{\partial t} + r(x,t,u)\frac{\partial}{\partial u}. \tag{9}$$

By requiring that both (1) and (8) be invariant under the transformation with the infinitesimal generator (9), one obtains an overdetermined nonlinear system of equations for the infinitesimals $p(x,t,u)$, $q(x,t,u)$, $r(x,t,u)$. The number of determining equations arising in the nonclassical method is smaller than for the classical method, consequently the set of solutions is larger than for the classical method, as in this method one requires only the subset of solutions of (1) and (8) be invariant under the infinitesimal generator (9). However, associated vector fields do not form a vector space.

The determining equations, by applying the nonclassical method to the Burgers equation were first derived by Ames. This set of determining equations was partially solved by Pucci [14], Pucci also obtained, by using the nonclassical method due to Bluman, some new solutions of the Burgers equation which cannot be obtained by the direct method of Clarkson and Kruskal. In [2], Arrigo *et al.* formulate a criterion for determining when a solution obtained from a symmetry reduction of any equation calculated by Bluman and Cole's method is recoverable by the Clarkson and Kruskal approach and obtained some new solutions for the nonclassical determining equations of the Burgers equation as well as some new solutions for the Burgers equation.

Knowing that an associated system S{x,t,u,v} to the Boussinesq equation has the same classical symmetries as the Boussinesq equation, Clarkson proposed as an open problem if an auxiliary system S{x,t,u,v} of the Boussinesq equation does possess more or less nonclassical symmetries as compared with the equation itself.

Bluman says [3] that the ansatz to generate nonclassical solutions of S{x,t,u,v} could yield solutions of R{x,t,u} which are neither nonclassical solutions of R{x,t,u} nor solutions arising from potential symmetries. However, as far as we know, none of these new symmetries have been obtained.

The aim of this work is to obtain new symmetries that we will call *nonclassical potential* symmetries for the Burgers equation.

The basic idea is that the related integrated equation T{x,t,v} is augmented with the invariance surface condition

$$pv_x + qv_t - s = 0 \tag{10}$$

which is associated with the vector field (6).

By requiring that both (1) and (10) be invariant under the transformation with infinitesimal generator (6), one obtains an overdetermined, nonlinear system of equations for the infinitesimals $p(x,t,v)$, $q(x,t,v)$, $s(x,t,v)$.

Then X_T yields a nonclassical potential symmetry of R{x,t,u} if and only if (7) yields a nonlocal symmetry of R{x,t,u} which is not a classical potential symmetry.

This new symmetry is a potential symmetry of R{x,t,u} which does not arise from a Lie symmetry of T{x,t,v} but from a nonclassical symmetry of T{x,t,v}.

2 Potential symmetries for the Burgers equation

Let R{x,t,u} be the Burgers equation (1)

$$u_t + uu_x - u_{xx} = 0.$$

In order to find the potential symmetries of (1), we write the equation in a conserved form

$$u_t + \left(\frac{u^2}{2} - u_x\right)_x = 0.$$

From this conserved form, the associated auxiliary system S{x,t,u,v} is given by (3). If (u(x),v(x)) satisfies (3), then u(x) solves the Burgers equation and v(x) solves the integrated Burgers equation (2).

If (6) is the infinitesimal generator that leaves (2) invariant, then (7) is the infinitesimal generator that leaves (3) invariant if and only if $p(x,t) = p^T(x,t)$, $q(t) = q^T(t)$, $s(x,t,v) = s^T(x,t,v)$, and $r(x,t,u,v) = (s_v - p_x)u + s_x$.

Hence we obtain that X_T yields a potential symmetry of (1) if and only if

$$s_{vv}u + s_{xv} \neq 0. \tag{11}$$

Bluman [4] derived that (3) admits an infinite-parameter Lie group of point symmetries corresponding to the infinitesimal generator

$$X_s = e^{\frac{v}{2}} \left[\left(\frac{s_1(x,t)u}{2} + \frac{\partial s_1(x,t)}{\partial x} \right) \frac{\partial}{\partial u} + s_1 \frac{\partial}{\partial v} \right]. \tag{12}$$

We can see (7) that (2) admits an infinite-parameter Lie group of point symmetries corresponding to the infinitesimal generator

$$X^T = e^{\frac{v}{2}} s_1(x,t) \frac{\partial}{\partial v}. \tag{13}$$

These infinite-parameter Lie groups of point symmetries yield a potential symmetry for (1).

3 Nonclassical symmetries of the integrated equation

To obtain potential nonclassical symmetries of the Burgers equation, we apply the nonclassical method to the integrated equation (2). To apply the nonclassical method to (2), we require (2) and (10) to be invariant under the infinitesimal generator (6). In the case $q \neq 0$, without loss of generality, we may set $q(x,t,u) = 1$. The nonclassical method applied to (2) gives rise to the following determining equations for the infinitesimals

$$2\,\frac{\partial^2 p}{\partial v^2} + \frac{\partial p}{\partial v} = 0,$$

$$-2\,\frac{\partial^2 s}{\partial v^2} + \frac{\partial s}{\partial v} + 4\,\frac{\partial^2 p}{\partial v \partial x} - 4\,p\,\frac{\partial p}{\partial v} = 0,$$

$$\frac{\partial s}{\partial x} - 2\,\frac{\partial^2 s}{\partial v \partial x} + 2\,\frac{\partial p}{\partial v}\,s + \frac{\partial^2 p}{\partial x^2} - 2\,p\,\frac{\partial p}{\partial x} - \frac{\partial p}{\partial t} = 0,$$

$$\begin{aligned} &4\,\frac{\partial^2 p}{\partial v^2}\,\frac{\partial s}{\partial x} + 2\,\frac{\partial p}{\partial v}\,\frac{\partial s}{\partial x} - 4\,\frac{\partial^4 s}{\partial v^2 \partial x^2} + 4\,\frac{\partial p}{\partial v}\,\frac{\partial^2 s}{\partial v \partial x} + 4\,\frac{\partial^2 p}{\partial v \partial x}\,\frac{\partial s}{\partial v} + \frac{\partial s}{\partial t} + 2\,\frac{\partial p}{\partial x}\,s + \\ &\quad 4\,\frac{\partial^3 p}{\partial v^2 \partial x}\,s + 2\,\frac{\partial^2 p}{\partial v \partial x}\,s + \frac{\partial^3 p}{\partial x^3} - 4\,\frac{\partial p}{\partial v}\,\frac{\partial^2 p}{\partial x^2} - 2\,p\,\frac{\partial^2 p}{\partial x^2} - 2(\frac{\partial p}{\partial x})^2 - \\ &\quad 8\,\frac{\partial^2 p}{\partial v \partial x}\,\frac{\partial p}{\partial x} + 2\,\frac{\partial^4 p}{\partial v \partial x^3} - 4\,p\,\frac{\partial^3 p}{\partial v \partial x^2} - \frac{\partial^2 p}{\partial t \partial x} - 2\,\frac{\partial^3 p}{\partial t \partial v \partial x} = 0. \end{aligned}$$

Solving these equations, we obtain

$$p = \mathrm{p}_1(\mathrm{x},\mathrm{t})\,e^{-\frac{v}{2}} + \mathrm{p}_2(\mathrm{x},\mathrm{t}),$$

$$s = \left(s_1 e^{\frac{v}{2}} + \left(2p_1 p_2 - 2\frac{\partial p_1}{\partial x}\right) e^{-\frac{v}{2}} + \frac{2}{3} p_1^2\right) e^{-v} + s_2,$$

with $s_1 = s_1(x,t)$ and $s_2 = s_2(x,t)$. Substituting into the determining equations leads to

$$p_1 = 0, \quad p = p_2(x,t), \quad s = \mathrm{s}_1(\mathrm{x},\mathrm{t})e^{-\frac{v}{2}} + \mathrm{s}_2(\mathrm{x},\mathrm{t}), \tag{14}$$

where p_2, s_1, and s_2 are related by

$$\frac{\partial \mathrm{s}_2}{\partial x} + \frac{\partial^2 \mathrm{p}_2}{\partial x^2} - 2\,\mathrm{p}_2\,\frac{\partial \mathrm{p}_2}{\partial x} - \frac{\partial \mathrm{p}_2}{\partial t} = 0, \tag{15}$$

$$-\frac{\partial^2 \mathrm{s}_1}{\partial x^2} + \frac{\partial \mathrm{s}_1}{\partial t} + 2\,\frac{\partial \mathrm{p}_2}{\partial x}\,\mathrm{s}_1 = 0, \tag{16}$$

$$\frac{\partial \mathrm{s}_2}{\partial t} + 2\,\frac{\partial \mathrm{p}_2}{\partial x}\,\mathrm{s}_2 + \frac{\partial^3 \mathrm{p}_2}{\partial x^3} - 2\,\mathrm{p}_2\,\frac{\partial^2 \mathrm{p}_2}{\partial x^2} - 2\,\frac{\partial \mathrm{p}_2}{\partial x} - \frac{\partial^2 \mathrm{p}_2}{\partial t \partial x} = 0. \tag{17}$$

It has been shown that (1) admits a potential symmetry when (11) is satisfied. As s is given by (14), (11) is satisfied if and only if $s_1 \neq 0$. If $s_1 = 0$, the symmetries obtained for (2) project on to point symmetries of (1).

Although the previous equations are too complicated to be solved in general, some solutions can be obtained. Choosing $p_2 = p_2(x)$, $s_1 = s_1(x)$ and $s_2 = s_2(x)$, we can distinguish the following cases:

1. For $s_2 \neq 0$ from (15), we obtain

$$s_2 = -\frac{\partial p_2}{\partial x} + p_2^2 + k_1.$$

Substituting into (17), we have

$$\frac{d^3 \mathrm{p}_2}{dx^3} - 2\,\mathrm{p}_2 \frac{d^2 \mathrm{p}_2}{dx^2} + 2\,{\mathrm{p}_2}^2 \frac{d\mathrm{p}_2}{dx} + 2\,\mathrm{k}_1 \frac{d\mathrm{p}_2}{dx} - 4\left(\frac{d\mathrm{p}_2}{dx}\right)^2 = 0.$$

Multiplying by p_2 and integrating with respect to x, we have

$$\mathrm{p}_2 \frac{d^2 \mathrm{p}_2}{dx^2} - 2\,{\mathrm{p}_2}^2 \frac{d\mathrm{p}_2}{dx} - \frac{1}{2}\left(\frac{\partial p_2}{\partial x}\right)^2 + \frac{{\mathrm{p}_2}^4}{2} + \mathrm{k}_1\,{\mathrm{p}_2}^2 + \mathrm{k}_2 = 0.$$

Dividing by $2p_2^2$, setting $k_1 = k_2 = 0$ and making

$$p_2 = -\frac{w^2}{\int w^2}, \tag{18}$$

(17) can be written as

$$w'' = 0.$$

Consequently,

$$w = ax + b \qquad \text{and} \qquad p_2 = -\frac{a^2\,x^2 + 2\,a\,b\,x + b^2}{\frac{a^2\,x^3}{3} + a\,b\,x^2 + b^2\,x}.$$

Substituting p_2 into (16), we obtain

$$s_1 = b_0\left(\frac{a^5\,x^7\,\log x}{45\,b^5} - \frac{a^4\,x^6\,\log x}{45\,b^4} + \frac{a^2\,x^4\,\log x}{15\,b^2} + x^2\,\log x - \frac{13\,a^5\,x^7}{1800\,b^5} + \frac{4\,a^4\,x^6}{525\,b^4} - \frac{7\,a^2\,x^4}{150\,b^2} + \cdots\right) + a_0\left(\frac{a^5\,x^7}{45\,b^5} - \frac{a^4\,x^6}{45\,b^4} + \frac{a^2\,x^4}{15\,b^2} + x^2 + \cdots\right),$$

$$s_2 = \frac{2\,a^2\,x + 2\,a\,b}{\frac{a^2\,x^3}{3} + a\,b\,x^2 + b^2\,x}.$$

Setting $b = 0$,

$$X = -\frac{3}{x}\partial_x + \partial_t + \left[e^{\frac{\log x + v}{2}}\left(i\,\mathrm{k}_3\,\sinh\left(\frac{5\,\log x}{2}\right) + \mathrm{k}_4\,\cosh\left(\frac{5\,\log x}{2}\right)\right) + \frac{6}{x^2}\right]\partial_v.$$

2. For $s_2 = 0$, by solving (15), we obtain

$$p_2 = \sqrt{c_1}\tan(\sqrt{c_1}(x + c_2)) \quad \text{if} \quad c_1 > 0,$$

$$p_2 = -\frac{\sqrt{-c_1}[c_2\exp(2\sqrt{-c_1}x) + 1]}{c_2\exp(2\sqrt{-c_1}x) - 1} \quad \text{if} \quad c_1 < 0,$$

$$p_2 = -\frac{1}{x + c_2} \quad \text{if} \quad c_1 = 0.$$

2.1. For $c_1 > 0$, setting $c_1 = 1$, $c_2 = 0$ and solving (16), we obtain

$$p_2 = \tan(x), \qquad s_1 = k_1 x \tan(x) + k_2 \tan(x) + k_1.$$

Solving the invariant surface condition, we obtain the nonclassical symmetry reduction

$$z = t - \log(\sin(x)), \qquad v = 2\,\log\left(-\frac{4}{2\,\mathrm{k}_1\,\log\sin x + \mathrm{k}_1\,x^2 + 2\,\mathrm{k}_2\,x + 2\,H(z)}\right). \tag{19}$$

Substitution of (19) into the integrated Burgers equation (2) leads to the ODE

$$H'' + H' - k_1 = 0,$$

whose solution is

$$H(z) = k_4 e^{-z} + k_1 z + k,$$

from which we obtain that an exact solution for (2) is

$$v = -2\,\left(\log\left(-2\,\mathrm{k}_4\,\sin x - \mathrm{k}_1\,e^t\,x^2 - 2\,\mathrm{k}_2\,e^t\,x - (2\,\mathrm{k}_1\,t + 2\,k)\,e^t\right) - t - 2\,\log 2\right)$$

and by (3) a new exact solution of the Burgers equation is

$$u = -\frac{2\,\left(-2\,\mathrm{k}_4\,\cos x - 2\,\mathrm{k}_1\,e^t\,x - 2\,\mathrm{k}_2\,e^t\right)}{-2\,\mathrm{k}_4\,\sin x - \mathrm{k}_1\,e^t\,x^2 - 2\,\mathrm{k}_2\,e^t\,x - (2\,\mathrm{k}_1\,t + 2\,k)\,e^t}.$$

2.2. For $c_1 = 0$, setting $c_2 = 0$,

$$X = -\frac{1}{x}\partial_x + \partial_t + e^{\frac{v}{2}}\,\left(\mathrm{k}_2\,x^2 - \frac{\mathrm{k}_1}{3\,x}\right)\partial_v.$$

Solving the invariant surface condition leads to the similarity variable and to the implicit solution ansatz

$$z = \frac{x^2}{2} + t, \qquad H(z) = \frac{\frac{3\,\mathrm{k}_2\,x^4}{4} - \mathrm{k}_1\,x}{3} + 2\,e^{-\frac{v}{2}}.$$

Substituting into the integrated Burgers equation leads to the ODE

$$H'' - 3k_2 = 0,$$

whose solution is

$$H(z) = \frac{3\,\mathrm{k}_2\,z^2}{2} + \mathrm{k}_3\,z + \mathrm{k}_4$$

from which we obtain the exact solution

$$v = 2\,\log\left(-\frac{48}{3\,\mathrm{k}_2\,x^4 + (36\,\mathrm{k}_2\,t - 12\,\mathrm{k}_3)\,x^2 + 8\,\mathrm{k}_1\,x + 36\,\mathrm{k}_2\,t^2 - 24\,\mathrm{k}_3\,t + 24\,\mathrm{k}_4}\right)$$

and by (3) a new exact solution of the Burgers equation is

$$u = -\frac{2\,\left(12\,\mathrm{k}_2\,x^3 + 2\,(36\,\mathrm{k}_2\,t - 12\,\mathrm{k}_3)\,x + 8\,\mathrm{k}_1\right)}{3\,\mathrm{k}_2\,x^4 + (36\,\mathrm{k}_2\,t - 12\,\mathrm{k}_3)\,x^2 + 8\,\mathrm{k}_1\,x + 36\,\mathrm{k}_2\,t^2 - 24\,\mathrm{k}_3\,t + 24\,\mathrm{k}_4}.$$

4 Concluding remarks

In this work, we have introduced new classes of symmetries for the Burgers equation. If the Burgers equation is written in a conserved form, then a related system (3) and a related integrated equation (2) may be obtained. The ansatz to generate nonclassical solutions of the associated integrated equation (2) yields solutions of (1) which are neither nonclassical solutions of (1) nor solutions arising from potential symmetries.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 138–146.

New Ansätze and Exact Solutions for Nonlinear Reaction-Diffusion Equations Arising in Mathematical Biology

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Abstract

The method of additional generating conditions is applied for finding new non-Lie ansätze and exact solutions of nonlinear generalizations of the Fisher equation.

1. Introduction

In the present paper, I consider nonlinear reaction-diffusion equations with convection term of the form

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U), \tag{1}$$

where $U = U(t, x)$ is an unknown function, $A(U), B(U), C(U)$ are arbitrary smooth functions. The indices t and x denote differentiating with respect to these variables. Equation (1) generalizes a great number of the well-known nonlinear second-order evolution equations describing various processes in biology [1]–[3].

Equation (1) contains as a particular case the classical Burgers equation

$$U_t = U_{xx} + \lambda_1 U U_x \tag{2}$$

and the well-known Fisher equation [4]

$$U_t = U_{xx} + \lambda_2 U - \lambda_3 U^2, \tag{3}$$

where λ_1, λ_2, and $\lambda_3 \in \mathbf{R}$. A particular case of equation (1) is also the Murray equation [1]–[2]

$$U_t = U_{xx} + \lambda_1 U U_x + \lambda_2 U - \lambda_3 U^2, \tag{4}$$

which can be considered as a generalization of the Fisher and Burgers equations.

Construction of particular exact solutions for nonlinear equations of the form (1) remains an important problem. Finding exact solutions that have a biological interpretation is of fundamental importance.

On the other hand, the well-known principle of linear superposition cannot be applied to generate new exact solutions to *nonlinear* partial differential equations (PDEs). Thus, the classical methods are not applicable for solving nonlinear partial differential equations.

Of course, a change of variables can sometimes be found that transforms a nonlinear partial differential equation into a linear equation, but finding exact solutions of most nonlinear partial differential equations generally requires new methods.

Now, the very popular method for construction of exact solutions to nonlinear PDEs is the Lie method [5, 6]. However it is well known that some very popular nonlinear PDEs have a poor Lie symmetry. For example, the Fisher equation (3) and the Murray equation (4) are invariant only under the time and space translations. The Lie method is not efficient for such PDEs since in these cases it enables us to construct ansätze and exact solutions, which can be obtained without using this cumbersome method.

A constructive method for obtaining non-Lie solutions of nonlinear PDEs and a system of PDEs has been suggested in [7, 8]. The method (see Section 2) is based on the consideration of a fixed nonlinear PDE (a system of PDEs) together with an *additional generating condition* in the form of a linear high-order ODE (a system of ODEs). Using this method, new exact solutions are obtained for nonlinear equations of the form (1) (Section 3). These solutions are applied for solving some nonlinear boundary-value problems.

2. A constructive method for finding new exact solutions of nonlinear evolution equations

Here, the above-mentioned method to the construction of exact solutions is briefly presented. Consider the following class of nonlinear evolution second-order PDEs

$$U_t = (\lambda + \lambda_0 U)U_{xx} + rU_x^2 + pUU_x + qU^2 + sU + s_0, \tag{5}$$

where coefficients $\lambda_0, \lambda, r, p, q, s$, and s_0 are arbitrary constants or arbitrary smooth functions of t. It is easily seen that the class of PDEs (1) contains this equation as a particular case. On the other hand equation (5) is a generalization of the known nonlinear equations (2)–(4).

If coefficients in (5) are constants, then this equation is invariant with respect to the transformations

$$x' = x + x_0 \qquad t' = t + t_0, \tag{6}$$

and one can find plane wave solutions of the form

$$U = U(kx + vt), \quad v, k \in \mathbf{R}. \tag{7}$$

But here, such solutions are not constructed since great number papers are devoted to the construction of plane wave solutions for nonlinear PDEs of the form (1) and (5) (see references in [9], for instance).

Hereinafter, I consider (5) together with the *additional generating conditions* in the form of linear high-order homogeneous equations, namely:

$$\alpha_0(t,x)U + \alpha_1(t,x)\frac{dU}{dx} + \cdots + \frac{d^m U}{dx^m} = 0, \tag{8}$$

where $\alpha_0(t,x), \ldots, \alpha_{m-1}(t,x)$ are arbitrary smooth functions and the variable t is considered as a parameter. It is known that the general solution of (8) has the form

$$U = \varphi_0(t)g_0(t,x) + \cdots + \varphi_{m-1}(t)g_{m-1}(t,x), \tag{9}$$

where $\varphi_0(t), \varphi_1(t), \ldots, \varphi_{m-1}(t)$ are arbitrary functions and $g_0(t,x) = 1$, $g_1(t,x), \ldots$, $g_{m-1}(t,x)$ are fixed functions that form a fundamental system of solutions of (8). Note that in many cases the functions $g_0(t,x)$, …, $g_{m-1}(t,x)$ can be expressed in an explicit form in terms of elementary ones.

Consider relation (9) as an ansatz for PDEs of the form (5). It is important to note that this ansatz contains m unknown functions $\varphi_i, i = 1, \ldots, m$ that yet-to-be determined. This enables us to reduce a given PDE of the form (5) to a quasilinear system of ODEs of the first order for the unknown functions φ_i. It is well known that such systems have been investigated in detail.

Let us apply ansatz (9) to equation (5). Indeed, calculating with the help of ansatz (9) the derivatives U_t, U_x, U_{xx} and substituting them into PDE (5) , one obtains a very cumbersome expression. But, if one groups similar terms in accordance with powers of the functions $\varphi_i(t)$, then sufficient conditions for reduction of this expression to a system of ODEs can be found. These sufficient conditions have the following form:

$$\lambda g_{i,xx} + s g_i - g_{i,t} = g_{i_1} Q_{ii_1}(t), \tag{10}$$

$$\lambda_0 g_i g_{i,xx} + r(g_{i,x})^2 + p g_i g_{i,x} + q(g_i)^2 = g_{i_1} R_{ii_1}(t), \tag{11}$$

$$\lambda_0(g_i g_{i_1,xx} + g_{i_1} g_{i,xx}) + 2r g_{i,x} g_{i_1,x} + p(g_i g_{i_1,x} + g_{i_1} g_{i,x}) + 2q g_i g_{i_1} = g_j T^j_{ii_1}(t),\ i < i_1, \tag{12}$$

where $Q_{ii_1}, R_{ii_1}, T^j_{ii_1}$ on the right-hand side are defined by the expressions on the left-hand side. The indices t and x of functions $g_i(t,x)$ and $g_{i_1}(t,x)$, $i, i_1 = 0, \ldots, m-1$, denote differentiating with respect to t and x.

With help of conditions (10)–(12), the following system of ODEs is obtained

$$\frac{d\varphi_i}{dt} = Q_{i_1 i}\varphi_{i_1} + R_{i_1 i}(\varphi_{i_1})^2 + T^i_{i_1 i_2}\varphi_{i_1}\varphi_{i_2} + \delta_{i,0} s_0, \quad i = 0, \ldots, m-1 \tag{13}$$

to find the unknown functions $\varphi_i, i = 0, \ldots, m-1$ ($\delta_{i,0} = 0, 1$ is the Kronecker symbol). On the right-hand sides of relations (10)–(12) and (13), a summation is assumed from 0 to m-1 over the repeated indices i_1, i_2, j. So, we have obtained the following statement.

Theorem 1. *Any solution of system (13) generates the exact solution of the form (9) for the nonlinear PDE (5) if the functions $g_i, i = 0, \ldots, m-1$, satisfy conditions (10)–(12).*

Remark 1. The suggested method can be realized for systems of PDEs (see [7, 8], [10]) and for PDEs with derivatives of second and higher orders with respect to t. In the last case, one will obtain systems of ODEs of second and higher orders.

Remark 2. If the coefficients $\lambda_0, \lambda, r, p, q, s$, and s_0 in equation (5) are smooth functions of the variable t, then Theorem 1 is true too. But in this case, the systems of ODEs with time-dependent coefficients are obtained.

3. Construction of the families of non-Lie exact solutions of some nonlinear equations.

Since a *constructive method* for finding new ansätze and exact solutions is suggested, its efficiency will be shown by the examples below. In fact, let us use Theorem 1 for the construction of new exact solutions.

Consider an additional generating condition of third order of the form

$$\alpha_1(t)\frac{dU}{dx} + \alpha_2(t)\frac{d^2U}{dx^2} + \frac{d^3U}{dx^3} = 0, \tag{14}$$

which is the particular case of (8) for $m = 3$. Condition (14) generates the following chain of the ansätze:

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma_1(t)x) + \varphi_2(t)\exp(\gamma_2(t)x) \tag{15}$$

if $\gamma_{1,2}(t) = \frac{1}{2}(\pm(\alpha_2^2 - 4\alpha_1)^{1/2} - \alpha_2)$ and $\gamma_1 \neq \gamma_2$;

$$U = \varphi_0(t) + \varphi_1(t)\exp(\gamma(t)x) + x\varphi_2(t)\exp(\gamma(t)x) \tag{16}$$

if $\gamma_1 = \gamma_2 = \gamma \neq 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)\exp(\gamma(t)x) \tag{17}$$

if $\alpha_1 = 0$;

$$U = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2 \tag{18}$$

if $\alpha_1 = \alpha_2 = 0$.

Remark 3. In the case $D = \alpha_2^2 - 4\alpha_1 < 0$, one obtains complex functions $\gamma_1 = \gamma_2^* = \frac{1}{2}(\pm i(-D)^{1/2} - \alpha_2)$, $i^2 = -1$ and then ansatz (15) is reduced to the form

$$U = \varphi_0(t) + \left[\psi_1(t)\cos\left(\frac{1}{2}(-D)^{1/2}x\right) + \psi_2(t)\sin\left(\frac{1}{2}(-D)^{1/2}x\right)\right]\exp\left(-\frac{\alpha_2 x}{2}\right), \tag{19}$$

where $\varphi_0(t), \psi_1(t), \psi_2(t)$ are yet-to-be determined functions.

Example 1. Consider the following equation

$$U_t = (\lambda + \lambda_0 U)U_{xx} + \lambda_1 U U_x + \lambda_2 U - \lambda_3 U^2 \tag{20}$$

which in the case $\lambda = 1$, $\lambda_0 = 0$ coincides with the Murray equation (4). Hereinafter, it is supposed that $\lambda_2 \neq 0$ since the case $\lambda_2 = 0$ is very especial and was considered in [8]. By substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma_1(t)x)$, $g_2 = \exp(\gamma_2(t)x)$ from ansatz (15) into relations (10)–(12), one can obtain

$$\begin{cases} Q_{00} = \lambda_2, \quad Q_{11} = \lambda\gamma_1^2 + \lambda_2, \quad Q_{22} = \lambda\gamma_2^2 + \lambda_2, \\ R_{00} = -\lambda_3, \quad T^1_{01} = -\lambda_3, \quad T^2_{02} = -\lambda_3 \end{cases} \tag{21}$$

and the following relations

$$R_{ii_1} = Q_{ii_1} = T^j_{ii_1} = 0 \tag{22}$$

for different combinations of the indices i, i_1, j. With the help of relations (21)–(22), system (13) is reduced to the form

$$\begin{cases} \dfrac{d\varphi_0}{dt} = \lambda_2\varphi_0 - \lambda_3\varphi_0^2, \\ \dfrac{d\varphi_1}{dt} = (\lambda\gamma_1^2 + \lambda_2)\varphi_1 - \lambda_3\varphi_0\varphi_1, \\ \dfrac{d\varphi_2}{dt} = (\lambda\gamma_2^2 + \lambda_2)\varphi_2 - \lambda_3\varphi_0\varphi_2. \end{cases} \tag{23}$$

The system of ODEs (23) is nonlinear, but it is easily integrated and yields the general solutions

$$\varphi_0 = \frac{\lambda_2}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \quad \varphi_1 = \frac{c_1\exp(\lambda\gamma_1^2 t)}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \quad \varphi_2 = \frac{c_2\exp(\lambda\gamma_2^2 t)}{\lambda_3 + c_0\exp(-\lambda_2 t)}. \tag{24}$$

In (24) and hereinafter, c_0, c_1, c_2 are arbitrary constants . So, by substituting relations (24) into ansatz (15) the three-parameter family of exact solutions of the nonlinear equation (20) for $\lambda_1^2 + 4\lambda_0\lambda_3 \neq 0$ is obtained, namely:

$$U = \frac{\lambda_2 + c_1\exp(\lambda\gamma_1^2 t + \gamma_1 x) + c_2\exp(\lambda\gamma_2^2 t + \gamma_2 x)}{\lambda_3 + c_0\exp(-\lambda_2 t)} \tag{25}$$

where γ_1 and γ_2 are roots of the algebraic equation

$$\lambda_0\gamma^2 + \lambda_1\gamma - \lambda_3 = 0, \qquad \lambda_1^2 + 4\lambda_0\lambda_3 \neq 0. \tag{26}$$

Remark 4. In the case $\lambda_1^2+4\lambda_0\lambda_3 = -\gamma_0^2 < 0$, the complex values γ_1 and γ_2 are obtained, and then the following family of solutions

$$U = \frac{\lambda_2 + \exp\left[\frac{\lambda}{4\lambda_0^2}(\lambda_1^2 - \gamma_0^2)t - \frac{\lambda_1}{2\lambda_0}x)\right](c_1\cos\omega + c_2\sin\omega)}{\lambda_3 + c_0\exp(-\lambda_2 t)}, \tag{27}$$

where $\omega = \dfrac{\gamma_0}{2\lambda_0^2}(\lambda\lambda_1 t - \lambda_0 x)$, are constructed (see ansatz (19)).

Similarly, by substituting the functions $g_0 = 1$, $g_1 = \exp(\gamma(t)x)$, $g_2 = x\exp(\gamma(t)x)$ from ansatz (16) into relations (10)–(12), the corresponding values of the functions R_{ii_1}, Q_{ii_1}, $T_{ii_1}^j$ are obtained, for which system (13) generates the three-parameter family of exact solutions of the nonlinear equation (20) for $\lambda_1^2 + 4\lambda_0\lambda_3 = 0$, namely:

$$U = \frac{\lambda_2 + (c_1 + 2c_2\lambda\gamma t)\exp(\lambda\gamma^2 t + \gamma x) + c_2 x\exp(\lambda\gamma^2 t + \gamma x)}{\lambda_3 + c_0\exp(-\lambda_2 t)}. \tag{28}$$

Analogously, we obtain with the help of ansatz (17) the following family of solutions of equation (20) (at $\lambda_3 = 0$)

$$U = \frac{c_1 + \lambda_2 x + c_2\exp(\lambda\gamma^2 t + \gamma x)}{-\lambda_1 + c_0\exp(-\lambda_2 t)}, \tag{29}$$

where $\gamma = -\dfrac{\lambda_1}{\lambda_0}$.

Finally, ansatz (18) gives the three-parameter family of exact solutions

$$U = \frac{c_2 + 2\lambda\lambda_2 t + c_1 x + \lambda_2 x^2}{-2\lambda_0 + c_0\exp(-\lambda_2 t)} \tag{30}$$

of equation (20) for the case $\lambda_1 = \lambda_3 = 0$, i.e.,

$$U_t = (\lambda + \lambda_0 U)U_{xx} + \lambda_2 U. \tag{31}$$

As is noted, equation (20) for $\lambda = 1$ and $\lambda_0 = 0$ yields the Murray equation (4). If we apply Theorem 1 and ansatz (15) for constructing exact solutions of the Murray equation, then the consraint $\varphi_2 = 0$ is obtained and the two-parameter family of solutions

$$U = \frac{\lambda_2 + c_1 \exp(\gamma^2 t + \gamma x)}{\lambda_3 + c_0 \exp(-\lambda_2 t)}, \tag{32}$$

where $\gamma = \dfrac{\lambda_3}{\lambda_1}$, is found. It is easily seen that the family of exact solutions (25) generates this family if one puts formally $c_2 = 0$, $\lambda = 1$, and $\lambda_0 = 0$ in (25) and (26).

Solutions of the form (32) are not of the plane wave form, but in the case $\lambda_1 < 0$ and $\lambda_3 > 0$, they have similar properties to the plane wave solutions, which were illustrated in [1, 2] in Figures. So, they describe similar processes. In the case $c_0 = 0$, a one-parameter family of plane wave solutions is obtained from (32).

Taking into account solution (32), one obtains the following theorem:

Theorem 2. *The exact solution of the boundary-value problem for the Murray equation (4) with the conditions*

$$U(0, x) = \frac{\lambda_2 + c_1 \exp(\gamma x)}{\lambda_3 + c_0}, \tag{33}$$

$$U(t, 0) = \frac{\lambda_2 + c_1 \exp(\gamma^2 t)}{\lambda_3 + c_0 \exp(-\lambda_2 t)}, \tag{34}$$

and

$$U_x(t, +\infty) = 0, \quad \gamma = \frac{\lambda_3}{\lambda_1} < 0 \tag{35}$$

is given in the domain $(t, x) \in [0, +\infty) \times [0, +\infty)$ by formula (32), and, for $\lambda_2 < -\gamma^2$, it is bounded.

Note that the Neumann condition (35) (the zero flux on the boundary) is a typical request in the mathematical biology (see, e.g., [1, 2]).

Example 2. Let us consider the following equation

$$U_t = [(\lambda + \lambda_0 U) U_x]_x + \lambda_2 U - \lambda_3 U^2 \tag{36}$$

that, in the case where $\lambda_0 = 0$ and $\lambda = 1$, coincides with the Fisher equation (3). The known soliton-like solution of the Fisher equation was obtained in [11]. Note that this solution can be found using the suggested method too. It turns out that the case $\lambda_0 \neq 0$ is very special.

Let us apply Theorem 1 to construction of exact solutions of equation (36) in the case of ansatz (15). Similarly to Example 1, one can find the following two-parameter families of solutions

$$U = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t - c_0)}{2}\right] + c_2 \frac{\exp\frac{(2\lambda\lambda_3 + \lambda_0\lambda_2)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}} \exp\left(\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right) \tag{37}$$

and

$$U = \frac{\lambda_2}{2\lambda_3}\left[1 + \tanh\frac{\lambda_2(t-c_0)}{2}\right] + c_1\frac{\exp\frac{(2\lambda\lambda_3+\lambda_0\lambda_2)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}}\exp\left(-\sqrt{\frac{\lambda_3}{2\lambda_0}}x\right), \tag{38}$$

where c_0, c_1, c_2 are arbitrary constants. The solutions from (38) have nice properties. Indeed, any solution U^* of the form (38) holds the conditions $U^* \to \dfrac{\lambda_2}{\lambda_3}$ if $t \to \infty$ and $\lambda\lambda_3 < \lambda_0\lambda_2$; $U^* \to \dfrac{\lambda_2}{2\lambda_3}\left[1 + \tanh\dfrac{\lambda_2(t-c_0)}{2}\right] < 1$ if $x \to +\infty, \lambda_0\lambda_3 > 0$. Taking into account these properties, we obtain the following theorem.

Theorem 3. *The bounded exact solution of the boundary-value problem for the generalized Fisher equation*

$$U_t = [(1+\lambda_0 U)U_x]_x + \lambda_2 U - \lambda_2 U^2, \qquad \lambda_0 > 1,\ \lambda_2 > 0, \tag{39}$$

with the initial condition

$$U(0,x) = C_0 + C_1\exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right), \tag{40}$$

and the Neumann conditions

$$U_x(t,-\infty) = 0, \quad U_x(t,+\infty) = 0, \tag{41}$$

is given in the domain $(t,x) \in [0,+\infty) \times (-\infty,+\infty)$ by the formula

$$U = \frac{1}{2}\left[1 + \tanh\frac{\lambda_2(t-c_0)}{2}\right] + c_1\frac{\exp\frac{\lambda_2(2+\lambda_0)t}{4\lambda_0}}{\left(\cosh\frac{\lambda_2(t-c_0)}{2}\right)^{3/2}}\exp\left(-\sqrt{\frac{\lambda_2}{2\lambda_0}}|x|\right), \tag{42}$$

where $C_0 = \dfrac{1}{2}\left[1 + \tanh\dfrac{-\lambda_2 c_0}{2}\right]$, $C_1 = c_1\left(\cosh\dfrac{-\lambda_2 c_0}{2}\right)^{-3/2}$, *and* $c_1 > 0$.

Example 3. Consider the nonlinear reaction-diffusion equation with a convection term

$$Y_t = [Y^\alpha Y_x]_x + \lambda_1(t)Y^\alpha Y_x + \lambda_2 Y - \lambda_3 Y^{1-\alpha}, \qquad \alpha \neq 0, \tag{43}$$

that can be interpreted as a generalization of the Fisher and Murray equations. One can reduce this equation to the form

$$U_t = UU_{xx} + \frac{1}{\alpha}U_x^2 + \lambda_1(t)UU_x + \alpha\lambda_2 U - \alpha\lambda_3 \tag{44}$$

using the substitution $U = Y^\alpha$. It turns out that equation (44) for $\lambda_1(t) = -\left(1+\dfrac{1}{\alpha}\right)\gamma(t)$ is reduced by ansatz (17) to the following system of ODEs:

$$\begin{cases} \dfrac{d\gamma}{dt} = -\dfrac{1}{\alpha}\gamma^2\varphi_1, \\ \dfrac{d\varphi_0}{dt} = -\left(1+\dfrac{1}{\alpha}\right)\gamma\varphi_0\varphi_1 + \alpha\lambda_2\varphi_0 + \dfrac{1}{\alpha}\varphi_1^2 - \alpha\lambda_3, \\ \dfrac{d\varphi_1}{dt} = \alpha\lambda_2\varphi_1 - \left(1+\dfrac{1}{\alpha}\right)\gamma\varphi_1^2, \\ \dfrac{d\varphi_2}{dt} = \left[-\dfrac{1}{\alpha}\gamma^2\varphi_0 + \left(\dfrac{1}{\alpha}-1\right)\gamma\varphi_1 + \alpha\lambda_2\right]\varphi_2 \end{cases} \tag{45}$$

for finding the unknown functions $\gamma(t)$ and φ_i, $i = 0, 1, 2$. It is easily seen that in this case the function $\gamma(t) \neq \mathrm{const}$ if $\varphi_1 \neq 0$. Solving the system of ODEs (45), the family of exact solutions is found that are not the ones with separated variables, i.e.,

$$U = \varphi_0(t)g_0(x) + \cdots + \varphi_{m-1}(t)g_{m-1}(x). \tag{46}$$

It is easily seen that in the case $\alpha = -1$, the system of ODEs (45) is integrated in terms of elementary functions and one finds $\gamma(t) = [\lambda_2 c_0 + c_1 \exp(-\lambda_2 t)]^{-1}$.

Remark 5. The family of exact solutions with $\gamma(t) \neq \mathrm{const}$ (see ansatz (17)) has an essential difference from the ones obtained above since they contain the function $\gamma(t)$. So this family cannot be obtained using the method of linear invariant subspaces recently proposed in [12, 13] (note that the basic ideas of the method used in [12, 13] were suggested in [14]) because that method is reduced to finding solutions in the form (46).

Finally, it is necessary to observe that all found solutions are not of the form (7), i.e., they are not plane wave solutions. Moreover, all these solutions except (30) can not be obtained using the Lie method. One can prove this statement using theorems that have been obtained in [15].

4. Discussion

In this paper, a constructive method for obtaining exact solutions of certain classes of nonlinear equations arising in mathematical biology was applied. The method is based on the consideration of a fixed nonlinear partial differential equation together with *additional generating condition* in the form of a linear high-order ODE. With the help of this method, new exact solutions were obtained for nonlinear equations of the form (20), which are generalizations of the Fisher and Murray equations.

As follows from Theorems 2 and 3, the found solutions can be used for construction of exact solutions of some boundary-value problems with zero flux on the boundaries. Similarly to Theorem 3, it is possible to obtain theorems for construction of periodic solutions and blow-up solutions of some boundary-value problems with zero flux on the boundaries.

The efficiency of the suggested method can be shown also by construction of exact solutions to nonlinear reaction-diffusion systems of partial differential equations. For example, it is possible to find ones for the well-known systems of the form (see, e.g., [16])

$$\begin{cases} \lambda_1 U_t = \Delta U + U\dfrac{f_1(U,V)}{f_2(U,V)}, \\ \lambda_2 V_t = \Delta V + V\dfrac{g_1(U,V)}{g_2(U,V)}, \end{cases} \tag{47}$$

where the f_k and g_k, $k = 1, 2$, are linear functions of U and V. The form of the found solutions will be essentially depend on coefficients in the functions f_k and g_k, $k = 1, 2$. Note that, in the particular case, system (47) gives the nonlinear system

$$\begin{cases} \lambda_1 U_t = \Delta U + \beta_1 U^2 V^{-1}, \\ \lambda_2 V_t = \Delta V + \beta_2 U, \quad \beta_1 \neq \beta_2. \end{cases} \tag{48}$$

As was shown in [17, 18], system (48) has the wide Lie symmetry. Indeed, it is invariant with respect to the same transformations as *the linear heat equation.* This fact gives additional wide possibilities for finding families of exact solutions.

Acknowledgments

The author acknowledges the financial support by DFFD of Ukraine (project 1.4/356).

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 147–154.

Nonlinear Conformally Invariant Wave Equations and Their Exact Solutions

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Abstract

New nonlinear representations of the conformal algebra and the extended Poincaré algebra are found and new classes of conformally invariant wave equations are constructed. Exact solutions of the equations in question containing arbitrary functions are obtained.

1 Introduction

The classical and quantum scalar fields are known to be described by the Poincaré-invariant wave equations for the complex function. Therefore, according to the symmetry selection principle, it is interesting to construct classes of nonlinear wave equations admitting wider symmetry, in particular invariant under the different representations of the extended Poincaré algebra and the conformal one, which include the Poincaré algebra as a subalgebra.

It has been stated [1, 2], that the Poincaré-invariant wave equation

$$\Box u = F(|u|)u$$

(F is an arbitrary smooth function, $u = u(x^0 \equiv ct, x^1, \dots, x^n)$, $\Box \equiv p_\mu p^\mu$ is the d'Alembertian in the $(n+1)$–dimensional pseudo-Euclidean space $R(1,n)$, $|u| = \sqrt{uu^*}$, the asterisk designates the complex conjugation) admits the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(2)}\rangle$ iff it is of the form:

$$\Box u = \lambda_1 |u|^k u, \quad k \neq 0, \tag{1}$$

and admits the conformal algebra $AC(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(1)}, K^{(1)}_\mu\rangle$ iff it is of the form:

$$\Box u = \lambda_2 |u|^{4/(n-1)} u, \quad n \neq 1. \tag{2}$$

Here $\lambda_1,\ \lambda_2,\ k$ are real parameters,

$$P_\mu = p_\mu \equiv i\frac{\partial}{\partial x^\mu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{3}$$

$$D^{(1)} = x^\mu p_\mu + \frac{1-n}{2}[up_u + u^* p_{u^*}], \qquad K^{(1)}_\mu = 2x_\mu D^{(1)} - (x_\nu x^\nu)p_\mu, \tag{4}$$

$$D^{(2)} = x^\mu p_\mu - \frac{2}{k} u p_u - \frac{2}{k} u^* p_{u^*}; \qquad p_u = i\frac{\partial}{\partial u}, \qquad p_{u^*} = i\frac{\partial}{\partial u^*},$$

sumation under repeated idices from 0 to n is understood, raising or lowering of the vector indices is performed by means of the metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, \ldots, -1)$, i.e. $p_\mu = g_{\mu\nu} p^\nu$, $p^\mu = g^{\mu\nu} p_\nu$.

Some equations of the form

$$\Box u = F(u, u^*, p_\mu u, p_\mu u^*),$$

that are invariant under *linear* representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$ for $n \geq 2$ are described in [3]. Some equations of the second order, invariant under linear representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$ are adduced in [4], where for those in $(1+n)$–dimensional space ($n \geq 3$) the functional basis of the differential invariants has been constructed. It should be noted that equations (1) and (2) are invariant under linear representations of $A\widetilde{P}(1,n)$ and $AC(1,n)$, correspondingly.

The following natural question arises: do there exist nonlinear representations of the conformal algebra and the extended Poincaré algebra for a complex scalar field? Our answer to this question is that there exist such representations.

Here we present some classes of nonlinear wave equations invariant under unusual representations of the extended Poincaré algebra and the conformal algebra. In particular, we have found [5], that the equation

$$\Box u = \frac{\Box |u|}{|u|} u + m^2 c^2 u$$

is invariant under nonlinear representation of the conformal algebra $AC(1, n+1)$. It should be noted that this equation is proposed by Gueret and Vigier [6] and by Guerra and Pusterla [7]. It arises in the modelling of the equation for de Broglie's theory of double solution [8].

Also we describe some equations admitting both the standard linear representation of the conformal algebra and a nonlinear representation of the extended Poincaré algebra and find their exact solutions.

2 Different representations of the extended Poincaré algebra and the conformal algebra

Let us investigate symmetry properties of the more general wave equation with nonlinearities containing the second order derivatives, namely:

$$\Box u = F\left(|u|, (\nabla |u|)^2, \Box |u|\right) u. \tag{5}$$

Here, $F(\cdot,\cdot,\cdot)$ is an arbitrary smooth function, $u = u(x^0 \equiv ct, x^1, \ldots, x^n)$, $\nabla \equiv (p_0, p_1, \ldots, p_n)$, $(\nabla |u|)^2 \equiv (\nabla |u|)(\nabla |u|) \equiv (p_\mu |u|)(p^\mu |u|)$.

Theorem 1. *The maximal invariance algebra (MIA) of equation (5) for an arbitrary function F is the Poincaré algebra $AP(1,n) \oplus Q \equiv \langle P_\mu, J_{\mu\nu}, Q\rangle$ with the basis operators (3) and*

$$Q = i[u p_u - u^* p_{u^*}].$$

Let us introduce the following notations: R designates an arbitrary function, λ, k, l are arbitrary real parameters, and λ is not equal to zero.

Theorem 2. *Equation (5) is invariant under the extended Poincaré algebra $A\widetilde{P}(1,n)$ iff it is of the form:*

$$\Box u = \left\{ \frac{\Box|u|}{|u|} + \left(\frac{\Box|u|}{|u|}\right)^{1-2l} R\left(\frac{|u|\Box|u|}{(\nabla|u|)^2}, |u|^2\left(\frac{\Box|u|}{|u|}\right)^k\right)\right\} u, \tag{6}$$

$$MIA: \quad \langle P_\mu, J_{\mu\nu}, D, Q\rangle, \qquad D = x^\mu p_\mu + l\ln(u/u^*)[up_u - u^* p_{u^*}] + k[up_u + u^* p_{u^*}].$$

We can see that, when $l \neq 0$, equation 6 is invariant under the nonlinear representation of $A\widetilde{P}(1,n)$, because in this case the operator D generates the following nonlinear finite transformations of variables x and u:

$$x'_\mu = x_\mu \exp(\tau), \qquad u' = |u|\exp(k\tau)(u/u^*)^{\exp(2l\tau)/2},$$

where τ is a group parameter.

Theorem 3. *Equation (5) is invariant under the conformal algebra iff it takes one of the following forms:*

1. $$\Box u = |u|^{4/(n-1)} R\left(|u|^{(3+n)/(1-n)}\Box|u|\right) u, \qquad n \neq 1, \tag{7}$$

$$MIA: \quad \langle P_\mu, J_{\mu\nu}, D^{(1)}, K^{(1)}_\mu, Q\rangle;$$

2. $$\Box u = \Box|u| R\left(\frac{\Box|u|}{(\nabla|u|)^2}, |u|\right) u, \qquad n = 1, \qquad MIA: \quad \langle Z^{(1)}, Q\rangle, \tag{8}$$

$$Z^{(1)} = \left[s^{(1)}_1(x^0 + x^1) + s^{(1)}_2(x^0 - x^1)\right] p_0 + \left[s^{(1)}_1(x^0 + x^1) - s^{(1)}_2(x^0 - x^1)\right] p_1,$$

$s^{(1)}_1$, $s^{(1)}_2$ *are real smooth functions;*

3. [5] $$\Box u = \frac{\Box|u|}{|u|} u + \lambda u, \tag{9}$$

$$MIA: \quad \langle P_\mu, P_{n+1}, J_{\mu\nu}, J_{\mu n+1}, D^{(3)}, K^{(3)}_\mu, K^{(3)}_{n+1}, Q_3\rangle, \tag{10}$$

$$P_{n+1} \equiv p_{n+1} \equiv i\frac{\partial}{\partial x^{n+1}} = i\sqrt{|\lambda|}[up_u - u^* p_{u^*}], \qquad J_{\mu n+1} = x_\mu p_{n+1} - x_{n+1} p_\mu,$$

$$D^{(3)} = x^\mu p_\mu + x^{n+1} p_{n+1} - \frac{n}{2}[up_u + u^* p_{u^*}], \qquad Q_3 = up_u + u^* p_{u^*},$$

$$K^{(3)}_\mu = 2x_\mu D^{(3)} - (x_\nu x^\nu + x_{n+1} x^{n+1}) p_\mu,$$

$$K^{(3)}_{n+1} = 2x_{n+1} D^{(3)} - (x_\nu x^\nu + x_{n+1} x^{n+1}) p_{n+1},$$

where the additional variable x^{n+1} *is determined in the following way:*

$$x^{n+1} = \frac{i}{2\sqrt{|\lambda|}} \ln(u^*/u),$$

and a new metric tensor

$$\begin{array}{ll} g_{\mathrm{ij}} = \mathrm{diag}(1, -1, \dots, -1, -1), & \lambda > 0 \\ g_{\mathrm{ij}} = \mathrm{diag}(1, -1, \dots, -1, 1), & \lambda < 0, \qquad \mathrm{i}, \mathrm{j} = \overline{0, n+1} \end{array} \tag{11}$$

is introduced in the space of variables $(x_0, x_1, \dots, x_n, x_{n+1})$.

Direct verification shows that the symmetry operators of equation 9, namely

$$\langle P_\mu, P_{n+1}, J_{\mu\nu}, J_{\mu n+1}, D^{(3)}, K_\mu^{(3)}, K_{n+1}^{(3)} \rangle, \tag{12}$$

satisfy the commutational relations of the conformal algebra $AC(1, n+1)$ when $\lambda > 0$ and of $AC(2, n)$ when $\lambda < 0$.

To give a geometric interpretation of these operators, we rewrite equation 9 in the amplitude-phase terms:

$$A\Box\theta + 2(\nabla A)(\nabla\theta) = 0, \tag{13}$$

$$(\nabla\theta)^2 = -\lambda, \tag{14}$$

where $A = |u| = \sqrt{uu^*}$ and $\theta = (i/2)\ln(u^*/u)$.

The symmetry algebra of 9 is actually obtained by, first, calculating the symmetry algebra of system 13, 14. The maximal invariance algebra of this system is described by operators 10, where

$$\begin{aligned} & P_{n+1} = \sqrt{|\lambda|} p_\theta, \qquad J_{\mu n+1} = \sqrt{|\lambda|}(x_\mu p_\theta + (\theta/\lambda) p_\mu), \\ & D^{(3)} = x^\mu p_\mu + \theta p_\theta - (n/2) A p_A, \qquad K_\mu^{(3)} = 2x_\mu D^{(3)} - (x_\nu x^\nu - \theta^2/\lambda) p_\mu, \\ & K_{n+1}^{(3)} = \sqrt{|\lambda|}\left(2(\theta/\lambda) D^{(3)} + (x_\nu x^\nu - \theta^2/\lambda) p_\theta\right), \qquad Q_3 = A p_A. \end{aligned} \tag{15}$$

Here, we have introduced the following notations: $p_A = i\dfrac{\partial}{\partial A}$, $p_\theta = i\dfrac{\partial}{\partial \theta}$.

From 15 we see that, in the symmetry operators of system 13, 14, the phase variable $x^{n+1} = \theta/\sqrt{|\lambda|}$ is added to the $(n+1)$–dimensional geometric space $(x_0, x_1, \dots, x_n)$. In addition, the metric tensor 11 is introduced. This is the same effect we see for the eikonal equation [1]. The symmetry of equation 9 has the same property because 14 is the eikonal equation for the function θ, and equation 13, which is the continuity one, does not reduce the symmetry of 14.

It should be noted that the Lie algebra 12 realizes the nonlinear representation of the conformal algebra. Solving the corresponding Lie equations, we obtain that the operators $K_\mu^{(3)}, K_{n+1}^{(3)}$ generate the following nonlinear finite transformations of variables x_μ, A, θ:

$$\begin{aligned} x'^\mu &= \frac{x^\mu - b^\mu(x_\delta x^\delta - \theta^2/\lambda)}{1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)}, \\ A' &= A[1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)]^{n/2}, \\ \theta' &= \frac{\theta - \sqrt{|\lambda|} b^{n+1}(x_\delta x^\delta - \theta^2/\lambda)}{1 - 2x_\nu b^\nu - 2b_{n+1}\theta/\sqrt{|\lambda|} + b\cdot b(x_\delta x^\delta - \theta^2/\lambda)}, \end{aligned}$$

where b is the vector of group parameters in the $(n+2)$–dimensional space with metric the tensor 11.

The expression for these transformations differs from the standard one because the variable θ is considered as a geometric variable on the same footing as the variables x_μ and the amplitude A transforms as a dependent variable [5].

Thus, we see that the wave equation 9 which has a nonlinear quantum–potential term $(\Box|u|)/|u|$ has an unusually wide symmetry, namely it is invariant under a nonlinear representation of the conformal algebra.

It should be noted that the maximal invariance algebra of equation 9 for $\lambda \neq 0$ is a infinite-dimensional algebra with the following operators:

$$\langle Z^{(2)}, Q_1, Q_2\rangle, \quad n \neq 1; \qquad \langle Z^{(3)}, Q_1, Q_2\rangle, \quad n = 1,$$

where

$$Z^{(2)} = a^\mu\left(i\ln(u/u^*)\right)P_\mu + b^{\mu\nu}\left(i\ln(u/u^*)\right)J_{\mu\nu} + d\left(i\ln(u/u^*)\right)D^{(1)} + \\ f^\mu\left(i\ln(u/u^*)\right)K_\mu^{(1)},$$

$$Q_1 = q_1\left(i\ln(u/u^*)\right)[up_u + u^*p_{u^*}], \qquad Q_2 = q_2\left(i\ln(u/u^*)\right)[up_u - u^*p_{u^*}],$$

$$Z^{(3)} = \left\{s_1^{(3)}\left(x^0+x^1, i\ln(u/u^*)\right) + s_2^{(3)}\left(x^0-x^1, i\ln(u/u^*)\right)\right\}p_0 + \\ \left\{s_1^{(3)}\left(x^0+x^1, i\ln(u/u^*)\right) - s_2^{(3)}\left(x^0-x^1, i\ln(u/u^*)\right)\right\}p_1.$$

Here, a^μ, $b^{\mu\nu}$, d, f^μ, q_1, q_2, $s_1^{(3)}$, $s_2^{(3)}$ are real smooth functions.

As stated in Theorem 3, the standard representation of the conformal algebra 3, 4 is realized on the set of solutions of equations 7, 8. Moreover, equation 8 admits a wider symmetry, namely an infinite–dimensional algebra. The invariance under the infinite–dimensional algebra $\langle Z^{(1)}\rangle$ dictates the invariance under the conformal algebra $AC(1,1) = \langle P_\mu, J_{\mu\nu}, D^{(1)}, K_\mu^{(1)}\rangle$, $n = 1$, as long as the latter is a subalgebra of the former.

It is interesting to note that the class of equations 7, 8 contains ones invariant under both the standard representation of the conformal algebra and under a nonlinear representation of the extended Poincaré algebra.

It follows from Theorems 2 and 3 that the equation

$$\Box u = \left\{\frac{\Box|u|}{|u|} + \lambda|u|^{l(n+3)-1}(\Box|u|)^{l(1-n)+1}\right\}u, \qquad n \neq 1,$$

admitting the conformal algebra 3, 4 is invariant under a nonlinear representation of the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(4)}\rangle$, and the equation

$$\Box u = \left\{\frac{\Box|u|}{|u|} + |u|^{4l-1}\Box|u| R\left(\frac{|u|\Box|u|}{(\nabla|u|)^2}\right)\right\}u, \qquad n = 1,$$

admitting the infinite–dimensional algebra $\langle Z^{(1)}\rangle$ which includes the conformal algebra as a subalgebra, is invariant under a nonlinear representation of the extended Poincaré algebra $A\widetilde{P}(1,n) = \langle P_\mu, J_{\mu\nu}, D^{(5)}\rangle$.

Here,

$$D^{(4)} = x^\mu p_\mu + l\frac{n-1}{2}\ln(u/u^*)[up_u - u^*p_{u^*}],$$

$$D^{(5)} = x^\mu p_\mu + l\ln(u/u^*)[up_u - u^*p_{u^*}] + [up_u + u^*p_{u^*}].$$

3 Exact solutions of conformally invariant equations

Given an equation, its symmetry algebra can be exploited to construct ansatzes (see, for example [1]) for the equation, which reduce the problem of solving the equation to one of solving the equation of lower order, even ordinary differential equations. We examine this question for the three-dimensional conformally invariant equation

$$\Box u = \left\{ \frac{\Box|u|}{|u|} + \lambda |u|^{6l-1} (\Box|u|)^{1-2l} \right\} u. \tag{16}$$

As follows from Theorems 2 and 3, when $n = 3$, equation 16 is invariant under the conformal algebra $AC(1,3)= \langle P_\mu, J_{\mu\nu}, D^{(1)}, K_\mu^{(1)} \rangle$ and the operators Q, $D^{(4)}$. Making use of this symmetry, we construct exact solutions of equation 16. To this end, we rewrite equation 16 in the amplitude–phase representation:

$$\begin{cases} A\Box\theta + 2(\nabla A)(\nabla\theta) = 0, \\ (\nabla\theta)^2 + \lambda A^{6l-1} (\Box A)^{1-2l} = 0. \end{cases} \tag{17}$$

To construct solutions containing arbitrary functions, we consider the following ansatz:

$$\begin{array}{l} A = \varphi(\omega_1, \omega_2, \omega_3), \\ \theta = \psi(\omega_1, \omega_2, \omega_3), \end{array} \qquad \omega_1 = \beta x, \ \omega_2 = \gamma x, \ \omega_3 = \alpha x, \tag{18}$$

where α, β, γ are $(n+1)$–dimensional constant vectors in $R(1,n)$, satisfying the following conditions:

$$\alpha^2 = \alpha\beta = \alpha\gamma = \beta\gamma = 0, \quad \beta^2 = \gamma^2 = -1.$$

Substituting 18 into 17, we get the system

$$\begin{cases} \varphi(\psi_{11} + \psi_{22}) + 2(\varphi_1\psi_1 + \varphi_2\psi_2) = 0, \\ -\varphi_1^2 - \varphi_2^2 + \lambda\varphi^{6l-1}(-\varphi_{11} - \varphi_{22})^{1-2l} = 0, \end{cases} \tag{19}$$

containing the variable ω_3 as a parameter. System 19 admits the infinite–dimensional symmetry operator:

$$\widehat{X} = \tau_1 \widehat{P}_1 + \tau_2 \widehat{P}_2 + \tau_3 \widehat{J}_{12} + \tau_4 \widehat{D}_1 + \tau_5 \widehat{D}_2 + \tau_6 \widehat{Q}, \tag{20}$$

where $\tau_1, \ldots, \tau_6$ are arbitrary functions of ω_3, and

$$\widehat{P}_1 = \frac{\partial}{\partial\omega_1}, \qquad \widehat{P}_2 = \frac{\partial}{\partial\omega_2}, \qquad \widehat{J}_{12} = \omega_1 \frac{\partial}{\partial\omega_2} - \omega_2 \frac{\partial}{\partial\omega_1}, \qquad \widehat{Q} = \frac{\partial}{\partial\psi},$$
$$\widehat{D}_1 = \omega_1 \frac{\partial}{\partial\omega_1} + \omega_2 \frac{\partial}{\partial\omega_2} - \varphi \frac{\partial}{\partial\varphi}, \qquad \widehat{D}_2 = \omega_1 \frac{\partial}{\partial\omega_1} + \omega_2 \frac{\partial}{\partial\omega_2} + 2r\psi \frac{\partial}{\partial\psi}.$$

Making use of ansatzes constructed via nonequivalent one-dimensional subalgebras of algebra 20, one can reduce system 19 to ordinary differential equations and find their exact solutions. Let us consider the following subalgebras:

$$A_1 = \langle \widehat{P}_1 \rangle, \quad A_2 = \langle \widehat{D}_1 \rangle, A_4 = \langle \widehat{J}_{12} + a(\alpha x)\widehat{D}_1 \rangle,$$

where a is an arbitrary real function of αx.

Ansatzes corresponding to these subalgebras are of the form:

$$1.\quad \begin{cases} \varphi = v(\omega), \\ \psi = w(\omega), \end{cases} \qquad \omega = \gamma x;$$

$$2.\quad \begin{cases} \varphi = [(\beta x)^2 + (\gamma x)^2]^{-1/2} v(\omega), \\ \psi = w(\omega), \end{cases} \qquad \omega = \arctan \tfrac{\beta x}{\gamma x}; \tag{21}$$

$$3.\quad \begin{cases} \varphi = [(\beta x)^2 + (\gamma x)^2]^{-1/2} v(\omega), \\ \psi = w(\omega), \end{cases} \qquad \omega = 2a(\alpha x) \arctan \frac{\beta x}{\gamma x} - \ln[(\beta x)^2 + (\gamma x)^2].$$

Group parameters of the group with generator 20 are arbitrary real functions of ω_3. Therefore, acting 21 by the finite transformations of this group, we obtain 3 classes of ansatzes containing arbitrary functions of $\omega_3 = \alpha x$:

$$1.\quad \begin{cases} A = \varphi = \rho_1 v(\omega), \\ \theta = \psi = \rho_4^{-2l} w(\omega) + \rho_5, \end{cases}$$
$$\omega = \rho_1 \rho_4 (\gamma x \cos\rho_2 - \beta x \sin\rho_2) + \rho_3;$$

$$2.\quad \begin{cases} A = \varphi = \rho_4 [(\beta x + \rho_1)^2 + (\gamma x + \rho_2)^2]^{-1/2} v(\omega), \\ \theta = \psi = \rho_4^{2l} w(\omega) + \rho_5, \end{cases}$$
$$\omega = \arctan \frac{(\beta x + \rho_1)\cos\rho_3 + (\gamma x + \rho_2)\sin\rho_3}{(\gamma x + \rho_2)\cos\rho_3 - (\beta x + \rho_1)\sin\rho_3};$$

$$3.\quad \begin{cases} A = \varphi = \rho_4 [(\beta x + \rho_1)^2 + (\gamma x + \rho_2)^2]^{-1/2} v(\omega), \\ \theta = \psi = \rho_4^{2l} w(\omega) + \rho_6, \end{cases}$$
$$\omega = 2a(\alpha x) \arctan \frac{(\beta x + \rho_1)\cos\rho_3 + (\gamma x + \rho_2)\sin\rho_3}{(\gamma x + \rho_2)\cos\rho_3 - (\beta x + \rho_1)\sin\rho_3} -$$
$$\ln[(\beta x + \rho_1)^2 + (\gamma x + \rho_2)^2] + \rho_5.$$

Here, $\rho_1, \ldots, \rho_6$ are arbitrary real functions of αx.

The ansatzes constructed reduce system 19 as well as system 17 to the systems of ordinary differential equations. Finding their partial solutions, we obtained the following exact solutions of equation 16 when $l = 1/2$:

$$1.\ u = \pm\sigma \exp\left\{ i\sqrt{\lambda}|\sigma|(\gamma x \cos\rho_2 - \beta x \sin\rho_2) \right\};$$

$$2.\ u = \pm\sigma[(\beta x + \rho_1)^2 + (\gamma x + \rho_2)^2]^{-1/2} \times$$
$$\exp\left\{ i\sqrt{\lambda}|\sigma| \arctan \frac{(\beta x + \rho_1)\cos\rho_3 + (\gamma x + \rho_2)\sin\rho_3}{(\gamma x + \rho_2)\cos\rho_3 - (\beta x + \rho_1)\sin\rho_3} \right\};$$

$$3.\ u = \pm 2\sigma[(\beta x + \rho_1)^2 + (\gamma x + \rho_2)^2]^{(-1-3a^2)/(6a^2+6)} \times$$

$$\exp\left\{-\frac{2a}{3(a^2+1)}\arctan\frac{(\beta x+\rho_1)\cos\rho_3+(\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3-(\beta x+\rho_1)\sin\rho_3}+\right.$$

$$3i\sqrt{\lambda}\sqrt{a^2+1}|\sigma|[(\beta x+\rho_1)^2+(\gamma x+\rho_2)^2]^{1/(3a^2+3)}\times$$

$$\left.\exp\left\{-\frac{2a}{3(a^2+1)}\arctan\frac{(\beta x+\rho_1)\cos\rho_3+(\gamma x+\rho_2)\sin\rho_3}{(\gamma x+\rho_2)\cos\rho_3-(\beta x+\rho_1)\sin\rho_3}\right\}\right\}.$$

Here, $\rho_1, \rho_2, \rho_3, a, q$ are arbitrary real functions of αx, and σ is an arbitrary complex function of αx. It should be noted that all the solutions obtained contain arbitrary functions.

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The Generalized Emden-Fowler Equation

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Abstract

We give the description of nonlinear nonautonomous ordinary differential equations of order n with a so-called *reducible* linear part. The group classification of generalized Emden-Fowler equations of the mentioned class is done. We have found such laws of the variation of $f(x)$ that the equation admits one, two, or tree one-parameter Lie groups.

1. Introduction: the method of autonomization [1, 2]

Nonlinear nonautonomous equations with a reducible linear part form a wide class of ordinary differential equations (ODE) that have both theoretical and applied significance. We can write

$$(NLNA)y \equiv \sum_{k=0}^{n} \binom{n}{k} a_k y^{(n-k)} = \Phi(x, y, y', \ldots, y^{(m)}), \quad a_k \in \mathbf{C^{n-k}}(I), \tag{1.1}$$

$I = \{x|,\ a \leq x \leq b\}$, where the corresponding linear equation

$$L_n y \equiv \sum_{k=0}^{n} \binom{n}{k} a_k y^{(n-k)} = 0,$$

can be reduced by the Kummer-Liouville (KL) transformation

$$y = v(x)z, \quad dt = u(x)dx, \quad v,\ u \in \mathbf{C^n}(I), \quad uv \neq 0,\ \forall x \in I, \tag{1.2}$$

to the equation with constant coefficients

$$M_n z \equiv \sum_{k=0}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \quad b_k = \text{const}.$$

Theorem 1.1. *For the reduction of (1.1) to the nonlinear autonomous form*

$$(NLA)z \equiv \sum_{k=0}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = aF(z, z'(t), \ldots, z^{(m)}), \quad a = \text{const},$$

by the KL transformation (1.2), it is necessary and sufficient that $L_n y = 0$ is reducible and the nonlinear part Φ can be represented in the form:

$$\Phi(x, y, y', \ldots, y^{(m)}) = au^n vF\left[\frac{y}{v}, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)y, \ldots, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)^m y\right],$$

Supported in part by the Grant 96-01-01997 of the RFBR

where $D = d/dx$, $\left(\frac{1}{u}D - \frac{v'}{vu}\right)^k y$ *is the* k*-th iteration of differential expression* $\left(\frac{1}{u}D - \frac{v'}{vu}\right) y$*, and* $u(x)$ *and* $v(x)$ *satisfy the equations*

$$\begin{aligned} &\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \frac{3}{n+1}B_2u^2 = \frac{3}{n+1}A_2, \\ &v(x) = |u(x)|^{(1-n)/2}\exp\left(-\int a_1 dx\right)\exp\left(b_1\int u dx\right) \end{aligned} \tag{1.3}$$

respectively; $A_2 = a_2 - a_1^2 - a_1'$, $B_2 = b_2 - b_1^2$*, i.e., (1.1) is invariant under a one-parameter group with the generator*

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \quad \xi(x,y) = \frac{1}{u(x)}, \quad \eta(x,y) = \frac{v'}{uv}y. \tag{1.4}$$

In this case, (1.1) assumes partial solutions of the kind

$$y = \rho v(x), \ b_n = aF(\rho, 0, \ldots, 0). \tag{1.5}$$

Theorem 1.2. *1) If the linear part* $L_n y$ *of the equation*

$$N_n(y) \equiv L_n y + \sum_{s=1}^{l} f_s(x) y^{m_s} = F(x), \quad 1 \le m_1 < m_2 < \ldots < m_l, \tag{1.6}$$

can be reduced by the KL transformation and, in addition, the following conditions

$$p_s u^n = f_s(x) v^{m_s - 1}, \quad p_s = \text{const},$$

are fulfilled, then equation (1.6) can be transformed to the equation

$$M_n(z) + \sum_{s=1}^{l} p_s z^{m_s} = v^{-1}(x(t)) u^{-n}(x(t)) F(x(t));$$

2) the equation

$$L_n y + \sum_{s=1}^{l} f_s(x) y^{m_s} = 0,$$

corresponding (1.6) assumes the solutions of the form (1.5), where $v(x)$ *not only satisfies to relation (1.3) but it is also a solution of the linear equation*

$$(L_n - b_n u^n)v = 0,$$

and ρ *satisfies to the algebraic equation*

$$b_n \rho + \sum_{s=1}^{l} p_s \rho^{m_s} = 0.$$

2. The Emden-Fowler equation and the method of autonomization [3]

Let us consider the Emden-Fowler equation

$$y'' + \frac{a}{x}y' + bx^{m-1}y^n = 0, n \neq 0, \qquad n \neq 1,\ m, a, b \text{ are parameters }, \tag{2.1}$$

which is used in mathematical physics, theoretical physics, and chemical physics. Equation (2.1) has interesting mathematical and physical properties, and it has been investigated from various points of view. In this paper, we are interested in it from the point of view of autonomization.

Proposition 2.1. *1) Equation (2.1) can be reduced to the autonomous form*

$$\ddot{z} - \frac{(1-a)(n-1)+2(1+m)}{n-1}\dot{z} + \frac{[(1-a)(n-1)+1+m](1+m)}{(n-1)^2}z + bz^n = 0$$

by the transformation $y = x^{(1+m)/(1-n)}z,\ dt = x^{-1}dx$ *and has the invariant solutions*

$$y = \rho x^{(1+m)/(1-n)}, \qquad \frac{[(1-a)(n-1)+1+m](1+m)}{(n-1)^2}\rho + b\rho^n = 0.$$

2) (2.1) admits the one-parameter group $x_1 = e^{\epsilon}x,\ y_1 = e^{-2\epsilon(1+m)/(n-1)}y,\ \epsilon$ *is a parameter, with the generator*

$$X = x\frac{\partial}{\partial x} + \frac{1+m}{1-n}y\frac{\partial}{\partial y}.$$

3. The generalized Emden-Fowler equation

We consider the group analysis and exact solutions of the equation

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^n = 0, \quad n \neq 0,\ n \neq 1. \tag{3.1}$$

Equation (3.1) can be reduced to the autonomous form

$$\ddot{z} \pm b_1\dot{z} + b_0z + cz^n = 0 \tag{3.2}$$

by the KL transformation (1.3) under specific laws of variation of $f(x)$.

We have found such laws of variation of $f(x)$ that equation (3.1) admits one, two, or three-parameter Lie groups. It can't admit a larger number of pointwise symmetries.

We call the equation

$$y'' + g(x)y^n = 0, \tag{3.3}$$

a *canonical* generalized Emden-Fowler equation.

Equation (3.1) can always be reduced to the form (3.3) by a KL transformation.

Lemma 3.1. *In order that (3.1) can be reduced to (3.2) by the KL transformation (1.3), it is necessary and sufficient that the following equivalent conditions be satisfied:*

1°. *The kernel* $u(x)$ *of transformation (1.3) satisfies the Kummer-Schwartz equation*

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 - \frac{1}{4}\delta u^2 = A_0(x),$$

where $\delta = b_1^2 - 4b_0$ *is the discriminant of the characteristic equation* $r^2 \pm b_1 r + b_0 = 0$, *and* $A_0(x) = a_0 - \frac{1}{4}a_1^2 - \frac{1}{2}a_1'$ *is the semiinvariant of the adjoint linear equation*

$$y'' + a_1(x)y' + a_0(x)y = 0. \tag{3.4}$$

The factor $v(x)$ *of transformation (1.2) has the form*

$$v(x) = |u(x)|^{-1/2} \exp\left(-\frac{1}{2}\int a_1 dx\right) \exp\left(\pm\frac{1}{2}b_1 \int u dx\right). \tag{3.5}$$

Here, the function $f(x)$ *can be represented in the form*

$$f(x) = cu^2(x)v^{1-n}(x), \quad c = \text{const}.$$

2°. *Equation (3.1) admits a one-parameter group Lie group with generator (1.4).*

Theorem 3.1. *All laws of variation* $f(x)$ *in (3.1), admitting a one-parameter Lie group with generator (1.4), have one of the following forms:*

$$f_1 = F^2(\alpha_1 y_1 + \beta_1 y_2)^{-\frac{n+3}{2} \pm \frac{b_1(1-n)}{2\sqrt{\delta_1}}} (\alpha_2 y_1 + \beta_2 y_2)^{-\frac{n+3}{2} \mp \frac{b_1(1-n)}{2\sqrt{\delta_1}}}, \ \delta_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 0;$$

$$f_2 = F^2(Ay_2^2 + By_2y_1 + Cy_1^2)^{-\frac{n+3}{2}} \exp\left(\pm\frac{1-n}{2}\frac{b_1}{\sqrt{-\delta_2}} \arctan\frac{2Ay_2 + By_1}{\sqrt{-\delta_2}y_1}\right),$$
$$\delta_2 = B^2 - 4AC < 0;$$

$$f_3 = F^2(\alpha y_1 + \beta y_2)^{-(n+3)} \exp\left(\mp\frac{1-n}{2\alpha}\frac{b_1 y_1}{\alpha y_1 + \beta y_2}\right), \quad \delta_3 = 0;$$

$$f_4 = F^2(\alpha y_1 + \beta y_2)^{-\frac{n+3}{2} \pm \frac{b_1(1-n)}{2\alpha}} y_i^{-\frac{n+3}{2} \mp \frac{b_1(1-n)}{2\alpha}}, \quad \delta_4 = \alpha^2 > 0;$$

$$f_5 = F^2 y_i^{-(n+3)} \exp\left(\pm\frac{1-n}{2}b_1\frac{y_2}{y_1}\right), \quad \delta_5 = 0, \ i = 1, 2,$$

where $F = \exp\left(-\int a_1 dx\right)$, *and* $y_1, \ y_2 = y_1 \int F y_1^{-2} dx$ *generate the fundamental system of solutions (FSS) of the linear equation (3.4).*

Here, (3.1) assumes the exact solution

$$y = \rho v(x), \qquad b_0\rho + c\rho^n = 0,$$

where $v(x)$ *satisfies relation (3.5).*

Theorem 3.2. *If* $f(x)$ *is a factor of the nonlinear term of the equation (3.1), admitting symmetry (1.4), then* $f(x)$ *satisfies to one of the following equations:*

$$f'' - \frac{n+4}{n+3}\frac{f'^2}{f} + \frac{n-1}{n+3}a_1 f' - (n+3\left(a_0 - \frac{2(n+1)}{(n+3)^2}a_1^2 - \frac{2}{n+3}a_1'\right) f +$$
$$+(n+3)b_0 \exp\left(\frac{2(1-n)}{3+n}\int a_1 dx\right) f^{\frac{n+7}{n+3}} = 0, \quad b_1 = 0, \ n \neq -3;$$

or the equation

$$f'' - \frac{n+4}{n+3}\frac{f'^2}{f} + \frac{n-1}{n+3}a_1 f' - (n+3\left(a_0 - \frac{2(n+1)}{(n+3)^2}a_1^2 - \frac{2}{n+3}a_1'\right) f +$$
$$+\frac{[(n+3)b_0 \mp \frac{2(n+1)}{n+3}b_1^2] f^{\frac{n+7}{n+3}} \exp[\frac{2(1-n)}{3+n}\int a_1 dx]}{\left(k \pm \frac{1-n}{n+3} b_1 \int f^{\frac{2}{n+3}} \exp\left(\frac{1-n}{n+3}\int a_1 dx\right) dx\right)^2} = 0, \quad b_1 \neq 0,\ n \neq -3;$$

or the equation

$$2(f' + 2a_1 f)f''' - 3f''^2 - 12(a_1' f + a_1 f')f'' + \left(1 - \frac{\delta}{4b_1^2}\right)\frac{f'^4}{f^2} + 8\left(1 - \frac{\delta}{4b_1^2}\right)a_1\frac{f'^3}{f} +$$
$$+\left[\left(1 - 4\frac{\delta}{b_1^2}\right)a_1^2 + 14a_1' - 4a_0\right] f'^2 + 4\left[a_1'' - 4a_0 a_1 + 2a_1 a_1' + \left(1 - 2\frac{\delta}{b_1^2}\right)a_1^3\right] f f' +$$
$$+4\left[2a_0 a_1'' - 3a_1' 2 - 4a_0 a_1^2 + \left(1 - \frac{\delta}{b_1^2}\right)a_1^4 + 2a_1^2 a_1'\right] f^2 = 0, \quad n = -3,\ b_1 \neq 0;$$

or $f(x) = c\exp(-2\int a_1 dx)$, $n = -3$, $b_1 = 0$, $a_1 \neq 0$*; or* $f(x) = \text{const}$, $n = -3$, $b_1 = 0$, $a_1(x) = 0$.

4. The case $f(x) = \text{const} = p$

Consideer the equation

$$y'' + a_1(x)y' + a_0(x)y + py^n = 0, \qquad n \neq -3. \tag{4.1}$$

If $b_1 = 0$, we have

$$a_0(x) = \frac{2(n+1)}{(n+3)^2}a_1^2 + \frac{2}{n+3}a_1' + k\exp\left(\frac{2(1-n)}{3+n}\int a_1 dx\right), \quad k = \text{const},$$

or

$$a_0(x) = \frac{2(n+1)}{(n+3)^2}a_1^2 + \frac{2}{n+3}a_1' + q\frac{[(n+3)b_0 \mp \frac{2(n+1)}{n+3}b_1^2]\exp[\frac{2(1-n)}{3+n}\int a_1 dx]}{\left(k \pm \frac{1-n}{n+3}b_1\int \exp\left(\frac{1-n}{n+3}\int a_1 dx\right)dx\right)^2} = 0,$$

$b_1 \neq 0$. $n \neq -3$; $q = \text{const}$.

Theorem 4.1. *In order that the equation*

$$y'' + a_1 y' + a_0 y + py^n = 0, \quad a_1, a_0 = \text{const} \tag{4.2}$$

have the set of elementary exact solutions depending from one arbitrary constant (besides $a_1 = 0$*), it is sufficient that condition of its factorization,*

$$(n+3)^2 a_0 = 2(n+1)a_1^2 \tag{4.3}$$

hold.

In fact, in this case, equations (4.2), (4.3) admit the factorization:

$$\left(D + \frac{n+1}{n+3}a_1 \mp \frac{n+1}{2}ky^{(n-1)/2}\right)\left(D + \frac{2}{n+3}a_1 \pm ky^{(n-1)/2}\right)y = 0,$$

$k = \sqrt{-2p/(n+1)}$.

In this specific case (at $n = 3$) for some classes of anharmonic oscillators, the exact solutions were obtained in [4] by the Kowalewsky-Painlevé asymptotic method.

Theorem 4.2. *In order that the equation*

$$y'' + a_1(x)y' + py^n = 0, \quad n \neq -3,$$

admit the group with generator (1.4), it is necessary and sufficient that the function $a_1(x)$ satisfy the equation

$$a_1'' + \frac{4n}{n+3}a_1a_1' + \frac{2(n^2-1)}{(n+3)^2}a_1^3 = 0, \tag{4.4}$$

where (4.4) is integrated in elementary functions or quadratures (elliptic integrals). Equation (4.4) can be linearized by the method of the exact linearization (see [5]). Namely, by the substitution $A = a_1^2$, $dt = a_1(x)dx$, it can be reduced to the form

$$\ddot{A} + \frac{4n}{n+3}\dot{A} + \frac{4(n^2-1)}{(n+3)^2}A = 0, \quad (\cdot) = \frac{d}{dt}.$$

It possesses a one-parameter set of solutions

$$a_1(x) = \frac{n+3}{(n-1)(x+c)}, \quad a_1(x) = \frac{n+3}{(n+1)(x+c)} \tag{4.5}$$

and has a general solution of the following parameter kind:

$$a_1 = s^{n-1}(c_1 + s^4)^{1/2}, \quad x = -(n+3)\int s^{-n}(c_1+s^4)^{-1/2}ds + c_2. \tag{4.6}$$

Then it follows from the Chebyshev theorem (see [6])

Corollary 4.1. Equation (4.4), (4.6) (besides $c_1 = 0$, i.e., (4.5)) has elementary solutions for $n = \pm 1 - 4l$, $l \in \mathbf{Z}$.

Corollary 4.2. The equation

$$y'' + a_0(x)y + py^n = 0, \quad n \neq -3,$$

admits pointwise Lie symmetries only for $a_0(x) = \text{const}$, ($b_1 = 0$) or $a_0(x) = \dfrac{\nu}{(\lambda+\mu x)^2}$, ($b_1 \neq 0$).

Corollary 4.3. The Painlevé equation

$$y'' \pm xy = y^3$$

can't be reduced to the autonomous kind by a KL transformation KL (it doesn't admit pointwise Lie symmetries).

Theorem 4.3. *The Ermakov equation (Ermakov V.P., 1880; Pinney, 1951, see, for example, [1, 5])*

$$y'' + a_0(x)y + py^{-3} = 0$$

admits a three-dimensional Lie algebra with the generators

$$X_1 = y_1^2(x)\frac{\partial}{\partial x} + y_1(x)y_1'(x)y\frac{\partial}{\partial y}, \quad X_3 = y_2^2(x)\frac{\partial}{\partial x} + y_2(x)y_2'(x)y\frac{\partial}{\partial y},$$
$$X_2 = y_1y_2\frac{\partial}{\partial x} + \frac{1}{2}(y_1y_2' + y_2y_1')y\frac{\partial}{\partial y},$$

which has the commutators

$$[X_1, X_2] = X_1, \quad [X_2, X_3] = X_3, \quad [X_3, X_1] = -2X_2,$$

and is isomorphic to the algebra $sl(2, R)$ (type G_3 VIII according to the classification of Lie-Bianchi).

5. The special case $n = 2$

Theorem 5.1. *(see [7]). The equation*

$$y'' + a_1(x)y' + a_0(x)y + f(x)y^2 = 0 \tag{5.1}$$

has only point symmetries of the kind

$$X = \xi(x)\frac{\partial}{\partial x} + [\eta_1(x)y + \eta_2(x)]\frac{\partial}{\partial y}, \tag{5.2}$$

where

$$\eta_2'' + a_1\eta_2' + a_0\eta_2 = 0,$$
$$\xi''' - (2a_1' + a_1^2 - 4a_0)\xi' - \left(a_1' + \frac{1}{2}a_1^2 - 2a_0\right)''\xi = 4k\eta_2\xi^{-5/2}\exp\left[\frac{1}{2}\int\left(a_1 \mp \frac{b_1}{\xi}\right)dx\right],$$
$$\eta_1(x) = \frac{1}{2}(\xi' - a_1\xi \pm b_1), \quad f(x) = k\xi^{-5/2}\exp\left[\frac{1}{2}\int\left(a_1 \mp \frac{b_1}{\xi}\right)dx\right], \quad k = \text{const.}$$

Lemma 5.1. *The equation*

$$\xi''' - (2a_1' + a_1^2 - 4a_0)\xi' - \left(a_1' + \frac{1}{2}a_1^2 - 2a_0\right)''\xi = 4k\eta_2\xi^{-5/2}\exp\left(\frac{1}{2}\int a_1dx\right), \quad b_1 = 0,$$

can be reduced to the form

$$\zeta'''(s) = 4k\zeta^{-5/2} \tag{5.3}$$

by the transformation $\xi = u^{-1}\zeta, \quad ds = udx$, where

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = A_0(x).$$

Lemma 5.2. *Equation (5.3) assumes an exact linearization by the transformation $Z = \zeta^{-1}, \quad dt = \zeta^{-3/2}ds$, namely, $Z'''(t) + 4k = 0$.*

Theorem 5.2. *Equation (5.1) can be reduced to the autonomous form*

$$\ddot{z} \pm b_1\dot{z} + b_0z + c + kz^2 = 0, \quad c = \frac{1}{4k}\left(b_0^2 - \frac{36}{625}b_1^4\right)$$

by the substitution $y = v(x)z + w(x)$, $dt = u(x)dx$,

$$u(x) = \frac{1}{\xi}, \quad v(x) = \exp\left(\int \frac{\eta_1}{\xi}dx\right), \quad w = k\exp\left(\int \frac{\eta_1}{\xi}dx\right)\int \frac{\eta_2}{\xi}\exp\left(-\int \frac{\eta_1}{\xi}dx\right),$$

and has the exact solutions

$$y = \rho v(x) + w(x), \quad \rho = \frac{1}{2}\left(\frac{b_0}{2} \pm \frac{3b_1^2}{25}\right).$$

Theorem 5.3. *If equation (5.1) admits a symmetry of the kind (5.2), then the function* $f(x)$ *satisfies to the system of equations*

$$\varphi'' + a_1\varphi' + a_0\varphi + \frac{1}{2}\varphi^2 = \frac{1}{2}\left(b_0^2 - \frac{36}{625}b_1^4\right)u^4; \tag{5.4}$$

$$\varphi = \frac{1}{5}\frac{f''}{f} - \frac{6}{25}\frac{f'^2}{f^2} + \frac{1}{25}a_1\frac{f'}{f} - \left(a_0 - \frac{6}{25}a_1^2 - \frac{2}{5}a_1'\right) + \left(b_0 - \frac{6}{25}b_1^2\right)u^2; \tag{5.5}$$

$$u = \frac{f^{2/5}\exp(-1/5\int a_1dx)}{C_1 \mp \frac{1}{5}b_1\int f^{2/5}\exp(-1/5\int a_1dx)dx}. \tag{5.6}$$

Corollary 5.1. Let $a_1 = 0$, $a_0 = 0$, and $b_0 = \dfrac{6}{25}b_1^2$. Equation (5.4)–(5.6) takes the form

$$f^{iv} - \frac{32}{5}\frac{f'f'''}{f} - \frac{43}{10}\frac{f''^2}{f} + \frac{594}{25}\frac{f'^2}{f^2}f'' - \frac{1782}{125}\frac{f'^4}{f^3} = 0. \tag{5.7}$$

Equation (5.7) admits solutions of the kind $f(x) = \lambda x^\mu$, where μ satisfies to the algebraic equation

$$49\mu^4 + 490\mu^3 + 1525\mu^2 + 1500\mu = 0, \quad \{\mu = -5,\ -20/7,\ -15/7,\ 0\}.$$

Theorem 5.4. *Equation (5.4)–(5.6) in respect of* $f(x)$ *(at* $b_1 = 0$*) has the following general solution represented in the parameter form:*

$$f(x)\exp\left(2\int a_1dx\right)y_1^5 = k\psi^{5/2}, \quad y_2y_1^{-1} = \int \psi^{-3/2}dt$$

or

$$f(x) = \exp\left(2\int a_1dx\right)y_2^5 = k\psi^{5/2}, \quad y_2y_1^{-1} = -\left(\int \psi^{-3/2}dt\right)^{-1},$$

$$\psi = -\frac{2}{3}kt^3 + c_1t^2 + c_2t + c_3,$$

where $F = \exp\left(-\int a_1dx\right)$, *and* y_1, $y_2 = y_1\int Fy_1^{-2}dx$ *generate the FSS of the linear equation (3.4).*

Thus, even under the restriction $b_1 = 0$, the function $f(x)$ can be expressed via elliptic integrals. These expressions can be simplified in the case of pseudoelliptic integrals that takes place for the discriminant $\Delta = 0$. Namely,

$$\Delta = c_1^2c_2^2 + \frac{8}{3}kc_2^3 - 4c_1^3c_3 - 12k^2c_3^2 - 12kc_1c_2c_3 = 0.$$

Let, in particular, $c_1 = c_2 = c_3 = 0$. Then $f(x)$ has one of the following forms:

$$f(x) = \lambda \exp\left(-2\int a_1 dx\right) y_1^{-5} \left(\int \exp\left(-\int a_1 dx\right) y_1^{-2} dx\right)^{-15/7},$$

$$f(x) = \lambda \exp\left(-2\int a_1 dx\right) y_1^{-5} \left(\int \exp\left(-\int a_1 dx\right) y_1^{-2} dx\right)^{-20/7},$$

where $y_1(x)$ is a partial solution of equation (3.4).

Example. The equation $y'' + f(x)y^2 = 0$ can be reduced to the autonomous form for $f(x) = \lambda x^{-15/7}$, $f(x) = \lambda x^{-20/7}$, and $f(x) = \lambda x^{-5}$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 164–171.

Group Analysis of Ordinary Differential Equations of the Order $n > 2$

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Abstract

This paper deals with three strategies of integration of an n-th order ordinary differential equation, which admits the r-dimensional Lie algebra of point symmetries. These strategies were proposed by Lie but at present they are not well known. The "first" and "second" integration strategies are based on the following main idea: to start from an n-th order differential equation with r symmetries and try to reduce it to an $(n-1)$-th order differential equation with $r-1$ symmetries. Whether this is possible or not depends on the structure of the Lie algebra of symmetries. These two approaches use the normal forms of operators in the space of variables ("first") or in the space of first integrals ("second"). A different way of looking at the problem is based on the using of differential invariants of a given Lie algebra.

1. Introduction

The experience of an ordinary differential equation (ODE) with one symmetry which could be reduced in order by one and of a second order differential equation with two symmetries which could be solved may lead us to the following question: *Is it possible to reduce a differential equation with r symmetries in order by r?* In full generality, the answer is "no". This paper deals with three integration strategies which are based on the group analisys of an n-th order ordinary differential equation (ODE-n, $n > 2$) with r symmetries ($r > 1$). These strategies were proposed by S. Lie (see [1–2]) but at present they are not well known. We studied the connection between the structure of a Lie algebra of point symmetries and the integrability conditions of a differential equation. We refer readers to the literature where these approaches are described (see [3], [6], [7]).

Suppose we have an n-th order ordinary differential equation (ODE-n, $n > 2$)

$$y^{(n)} = \omega\left(x, y, y', \ldots, y^{(n-1)}\right), \tag{1.1}$$

which admits r point symmetries $X_1, X_2, \ldots, X_r$. It is well known (due to Lie) that $r \leq n + 4$.

Definition. *The infinitesimal generator*

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{1.2}$$

Present work was partially financed by RFBR, the grant 96–01–01997.

is called a point symmetry of ODE-n (1.1) if

$$X^{(n-1)}\omega\left(x, y, y', \ldots, y^{(n-1)}\right) \equiv \eta^{(n)}\left(x, y, y', \ldots, y^{(n)}\right) \quad (\text{mod } y^{(n)} = \omega) \tag{1.3}$$

holds; here,

$$\begin{aligned} X^{(n-1)} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}(x,y,y')\frac{\partial}{\partial y'} + \cdots \\ +\eta^{(n-1)}\left(x, y, y', \ldots, y^{(n-1)}\right)\frac{\partial}{\partial y^{(n-1)}} \end{aligned} \tag{1.4}$$

is an extension (prolongation) X up to the n-th derivative.

Consider the differential operator

$$A = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + \cdots + \omega\left(x, y, y', \ldots, y^{(n-1)}\right)\frac{\partial}{\partial y^{(n-1)}}. \tag{1.5}$$

It is not difficult to see that A can formally be written as

$$A \equiv \frac{d}{dx} \quad (\text{mod } y^{(n)} = \omega).$$

Proposition 1. *Differential equation (1.1) admits the infinitesimal generator* $X = \xi(x,y)\dfrac{\partial}{\partial x} + \eta(x,y)\dfrac{\partial}{\partial y}$ *iff* $[X^{(n-1)}, A] = -(A(\xi(x,y)))A$ *holds.*

Concept of proof. Let $\varphi_i(x, y, y', ..., y^{(n-1)}),\ i = \overline{1,n}$ be the set of functionally independent first integrals of ODE-n (1.1); then $\{\varphi_i\}_{i=1}^n$ are functionally independent solutions of the partial differential equation

$$A\varphi = 0. \tag{1.6}$$

It's easy to show that ODE-n (1.1) admits the infinitesimal generator (1.2) iff $X^{(n-1)}\varphi_i$ is a first integral of ODE-n (1.1) for all $i = \overline{1,n}$.

So, on the one hand, we have that the partial differential equations (1.6) and

$$\left[X^{(n-1)}, A\right]\varphi = 0 \tag{1.7}$$

are equivalent iff X is a symmetry of (1.1). On the other hand, we have that (1.6) and (1.7) are equivalent iff

$$\left[X^{(n-1)}, A\right] = \lambda\left(x, y, y', \ldots, y^{(n-1)}\right)A \tag{1.8}$$

holds.

Comparing the coefficient of $\dfrac{\partial}{\partial x}$ on two sides of (1.8) yields $\lambda = -A(\xi(x,y))$ ■

2. First integration strategy: normal forms of generators in the space of variables

Take one of the generators, say, X_1 and transform it to its normal form $X_1 = \dfrac{\partial}{\partial s}$, i.e., introduce new coordinates t (independent) and s (dependent),where the functions $t(x,y)$, $s(x,y)$ satisfy the equations $X_1 t = 0$, $X_1 s = 1$. This procedure allows us to transform the differential equation (1.1) into

$$s^{(n)} = \Omega\left(t, s', \dots, s^{(n-1)}\right), \tag{2.1}$$

which, in fact, is a differential equation of order $n-1$ (we take s' as a new dependent variable). Now we interest in the following question:

Does (2.1) really inherit $r-1$ symmetries from (1.1), which are given by (2.2)?

$$Y_i = X_i^{(n-1)} - \overline{\eta}(t,s)\frac{\partial}{\partial s}. \tag{2.2}$$

The next theorem answers this question.

Theorem 1. *The infinitesimal generators* $Y_i = X_i^{(n-1)} - \overline{\eta}(t,s)\dfrac{\partial}{\partial s}$ *are the symmetries of ODE-(n-1) (2.1) if and only if*

$$[X_1, X_i] = \lambda_i X_1, \qquad \lambda_i = \text{const}, \quad i = \overline{2, r}, \tag{2.3}$$

hold.

So, if we want to follow this first integration strategy for a given algebra of generators, we should choose a generator X_1 at the first step (as a linear combination of the given basis), for which we can find as many generators X_a satisfying (2.3) as possible; choose Y_a and try to do everything again.

At each step, we can reduce the order of a given differential equation by one.

Example 1. The third-order ordinary differential equation $4y^2 y''' = 18yy'y'' - 15y'^3$ admits the symmetries

$$X_1 = \frac{\partial}{\partial x}; \quad X_2^{(2)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''}; \quad X_3^{(2)} = y\frac{\partial}{\partial y} + y'\frac{\partial}{\partial y'} + y''\frac{\partial}{\partial y''}.$$

Remark. All examples presented in this paper only illustrate how one can use these integration strategies.

Note the relations $[X_1, X_2] = X_1$, $[X_1, X_3] = 0$.

Transform X_1 to its normal form by introducing new coordinates: $t = y; s = x$. Now we have the ODE-2

$$s''' = \frac{3s''^2}{s'} + \frac{18ts's'' + 15s'^2}{4t^2 s'}$$

with symmetries

$$Y_2 = s'\frac{\partial}{\partial s'} + s''\frac{\partial}{\partial s''}; \quad Y_3 = t\frac{\partial}{\partial t} - s''\frac{\partial}{\partial s''}; \quad [Y_2, Y_3] = 0.$$

Transform Y_2 to its normal form: $v = \log t$; $u = \log s'$

$$u'' = 2u'^2 + \frac{11}{2}u' + \frac{9}{2},$$

$$x = c_2 \int \frac{dy}{y^{\frac{11}{8}} \left(\cos\left(\frac{\sqrt{23}}{4} \log c_1 y\right)\right)^{1/2}} + c_3.$$

3. Second integration strategy: the normal form of a generator in the space of first integrals

We begin with the assumptions:

a) $r = n$;

b) $X_i, i = \overline{1,n}$, act transitively in the space of first integrals, i.e., there is no linear dependence between $X_i^{(n-1)}$, $i = \overline{1,n}$, and A.

We'll try to answer the following question:

Does a solution to the system of equations

$$X_1^{(n-1)}\varphi = \left(\xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial y} + \ldots + \eta_1^{(n-1)} \frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 1, \tag{3.1}$$

$$X_i^{(n-1)}\varphi = \left(\xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \ldots + \eta_i^{(n-1)} \frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 0, \qquad i = \overline{2,n}, \tag{3.2}$$

$$A\varphi = \left(\frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \ldots + \omega \frac{\partial}{\partial y^{(n-1)}}\right)\varphi = 0, \tag{3.3}$$

exist?

A system of n homogeneous linear partial differential equations in $n+1$ variables $(x, y, \ldots, y^{(n-1)})$ (3.2)–(3.3) has a solution if all commutators between $X_i^{(n-1)}$, $i = \overline{2,n}$, and A are linear combinations of the same operators. It's easy to check that these integrability conditions are fulfilled iff $X_i, i = \overline{2,n}$, generate an $(n-1)$-dimensional Lie subalgebra in the given Lie algebra of point symmetries.

Let φ be a solution to system (3.1)–(3.3), then

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]\varphi = X_1^{(n-1)}\left(X_i^{(n-1)}\varphi\right) - X_i^{(n-1)}\left(X_1^{(n-1)}\varphi\right) = 0 \tag{3.4}$$

necessarily holds. On the other hand, we have

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]\varphi = C_{1i}^1 X_1^{(n-1)}(\varphi) + C_{1i}^k X_k^{(n-1)}(\varphi) = C_{1i}^1, \qquad i, k = \overline{2,n}. \tag{3.5}$$

(3.4) and (3.5) do not contradict each other if and only if

$$C_{1i}^1 = 0, \qquad i = \overline{2,n}. \tag{3.6}$$

All preceding reasonings lead us to the necessary condition of existence of the function φ. This condition is also sufficient. Now we prove it. Let $u \neq \text{const}$ be a solution to system (3.2)–(3.3), then we have

$$\left[X_1^{(n-1)}, X_i^{(n-1)}\right]u = X_1^{(n-1)}\left(X_i^{(n-1)}u\right) - X_i^{(n-1)}\left(X_1^{(n-1)}u\right) =$$
$$= -X_i^{(n-1)}\left(X_1^{(n-1)}u\right) = 0, \quad i = \overline{2,n},$$

that is, $X_1^{(n-1)}u$ is a nonzero solution to system (3.2)–(3.3). Hence, $X_1^{(n-1)}u = f(u)$. It is not difficult to check that the function $\int \frac{du}{f(u)}$ is a solution to system (3.1)–(3.3).

Suppose that the integrability conditions for system (3.1)–(3.3) are fulfilled. Now we consider this system as a system of linear algebraic equations in $\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \ldots, \frac{\partial\varphi}{\partial y^{(n-1)}}$. We can solve this system using Cramer's rule:

$$\mathbf{\Delta} = \begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \ldots & \eta_1^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \ldots & \omega \end{vmatrix} \neq 0; \quad \frac{\partial\varphi}{\partial x} = \mathbf{\Delta}^{-1} \begin{vmatrix} 1 & \eta_1 & \eta_1^{(1)} & \ldots & \eta_1^{(n-1)} \\ 0 & \eta_2 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \eta_n & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 0 & y' & y'' & \ldots & \omega \end{vmatrix};$$

$$\frac{\partial\varphi}{\partial y} = \mathbf{\Delta}^{-1} \begin{vmatrix} \xi_1 & 1 & \eta_1^{(1)} & \ldots & \eta_1^{(n-1)} \\ \xi_2 & 0 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & 0 & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 1 & 0 & y'' & \ldots & \omega \end{vmatrix}; \ldots; \frac{\partial\varphi}{\partial y^{(n-1)}} = \mathbf{\Delta}^{-1} \begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \ldots & 1 \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \ldots & 0 \\ 1 & y' & y'' & \ldots & 0 \end{vmatrix}.$$

The differential form

$$d\varphi = \mathbf{\Delta}^{-1} \begin{vmatrix} dx & dy & dy' & \ldots & dy^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \ldots & \omega \end{vmatrix}$$

is a differential of the solution φ to system (3.1)–(3.3).

Theorem 2. *Suppose point symmetries $X_i, i = \overline{1,n}$, act transitively in the space of first integrals; then there exists a solution to the system $X_1^{(n-1)}\varphi = 1$; $X_i^{(n-1)}\varphi = 0$, $i = \overline{2,n}$; $A\varphi = 0$ if and only if X_i, $i = \overline{2,n}$, generate an $(n-1)$-dimensional ideal in the given Lie algebra of point symmetries. This solution is as follows:*

$$\varphi = \int \frac{\begin{vmatrix} dx & dy & dy' & \ldots & dy^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \ldots & \omega \end{vmatrix}}{\begin{vmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \ldots & \eta_1^{(n-1)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \ldots & \eta_2^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \xi_n & \eta_n & \eta_n^{(1)} & \ldots & \eta_n^{(n-1)} \\ 1 & y' & y'' & \ldots & \omega \end{vmatrix}}. \tag{3.7}$$

Now we can use $\varphi\left(x, y, y', \ldots, y^{(n-1)}\right)$ instead of $y^{(n-1)}$ as a new variable. In new variables, we have

$$y^{(n-1)} = y^{(n-1)}\left(x, y, y', \ldots, y^{(n-2)}; \varphi\right), \tag{3.8}$$

$$X_i^{(n-2)} = \xi_i \frac{\partial}{\partial x} + \eta_i \frac{\partial}{\partial y} + \cdots + \eta_i^{(n-2)} \frac{\partial}{\partial y^{(n-2)}}, \qquad i = \overline{2, n}, \tag{3.9}$$

$$A = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + \cdots + y^{(n-1)}\left(x, y, y', \ldots, y^{(n-2)}; \varphi\right) \frac{\partial}{\partial y^{(n-2)}}. \tag{3.10}$$

System (3.8)–(3.10) is exactly what we want to achieve. Now we can establish an iterative procedure.

Example 2. $2y'y''' = 3y''^2$. This equation admits the 3-dimensional Lie algebra of point symmetries with the basis.

$$X_1 = \frac{\partial}{\partial y}; \quad X_2^{(2)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'} - 2y''\frac{\partial}{\partial y''}; \quad X_3 = \frac{\partial}{\partial x}$$

and commutator relations: $[X_1, X_2] = 0,\ [X_1, X_3] = 0,\ [X_2, X_3] = -X_3$.
The given ODE-3 is equivalent to the equation $\{y, x\} = 0$, where

$$\{y, x\} = 1/2\frac{y'''}{y'} - 3/4\frac{y''^2}{y'^2}$$

is Schwarz's derivative.

$$\boldsymbol{\Delta} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ x & 0 & -y' & -2y'' \\ 1 & 0 & 0 & 0 \\ 1 & y' & y'' & \dfrac{3y''^2}{2y'} \end{vmatrix} = y''^2/2 \neq 0, \quad \varphi_1 = \int \frac{\begin{vmatrix} dx & dy & dy' & dy'' \\ x & 0 & -y' & -2y'' \\ 1 & 0 & 0 & 0 \\ 1 & y' & y'' & \dfrac{3y''^2}{2y'} \end{vmatrix}}{\boldsymbol{\Delta}} = y - \frac{2y'^2}{y''}.$$

Now we have the ODE-2: $y'' = \dfrac{2y'^2}{y - \varphi_1}$, which admits the generators

$$X_2^{(1)} = x\frac{\partial}{\partial x} - y'\frac{\partial}{\partial y'}; \quad X_3 = \frac{\partial}{\partial x};$$

$$\boldsymbol{\Delta_1} = \begin{vmatrix} x & 0 & -y' \\ 1 & 0 & 0 \\ 1 & y' & \dfrac{2y'^2}{y - \varphi_1} \end{vmatrix} = -y'^2. \quad \varphi_2 = \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & 0 & 0 \\ 1 & y' & \dfrac{2y'^2}{y - \varphi_1} \end{vmatrix}}{\boldsymbol{\Delta_1}} = \log\frac{(y - \varphi_1)^2}{y'},$$

$$y' = \frac{(y - \varphi_1)^2}{\exp \varphi_2}.$$

A general solution of the differential equation is given by the next function: $y = \dfrac{ax + b}{cx + d}$.

4. Third integration strategy: differential invariants

Definition. *Differential invariants of order k (DI-k) are functions*

$$\psi\left(x, y, y', ..., y^{(k)}\right), \qquad \frac{\partial \psi}{\partial y^{(k)}} \not\equiv 0,$$

that are invariant under the action of $X_1, \ldots, X_r$*, that is, satisfy* r *equations* ($i = \overline{1, r}$)*:*

$$X_i^{(k)}\psi = \left(\xi_i(x, y)\frac{\partial}{\partial x} + \eta_i(x, y)\frac{\partial}{\partial y} + \cdots + \eta_i^{(k)}\left(x, y, y', \ldots, y^{(k)}\right)\frac{\partial}{\partial y^{(k)}}\right)\psi = 0. \quad (4.1)$$

How can one find differential invariants? To do it, we must know two lowest order invariants φ and ψ.

Theorem 3. *If* $\psi\left(x, y, y', \ldots, y^{(l)}\right)$ *and* $\varphi\left(x, y, y', \ldots, y^{(s)}\right)$ $(l \leq s)$ *are two lowest order differential invariants, then*

1) $s \leq r$ *; 2) List of all functionally independent differential invariants is given by the following sequence:*

$$\psi, \varphi, \frac{d\varphi}{d\psi}, \ldots, \frac{d^n\varphi}{d\psi^n}, \ldots$$

Let $X_i,\ i = \overline{1, r}$, be symmetries of (1.1), then we have the list of functionally independent DI up to the n-th order:

$$\psi, \varphi, \frac{d\varphi}{d\psi}, \ldots, \frac{d^{(n-s)}\varphi}{d\psi^{(n-s)}}.$$

Express all derivatives $y^{(k)},\ \ k \geq s$, in terms $\psi,\ \varphi$, and derivatives $\varphi^{(m)}$ beginning with the highest order. That will give ODE-$(n-s)$:

$$H\left(\psi, \varphi, \varphi', \ldots, \frac{d^{(n-i)}\varphi}{d\psi^{(n-s)}}\right) = 0.$$

Unfortunately, this equation does not inherit any group information from (1.1). If it's possible to solve (4.2), then we have ODE-s

$$\varphi\left(x, y, y', \ldots, y^{(s)}\right) = f\left(\psi(x, y, \ldots, y^{(l)})\right) \quad (4.2)$$

with r symmetries.

Example 3. ODE-3 $yy'y''' = y'^2y'' + yy''^2$ admits the 2-dimensional Lie algebra of point symmetries $X_1 = \dfrac{\partial}{\partial x}$; $X_2 = x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$.

DI-0 does not exist. DI-1: $\psi = \dfrac{y'}{y^2}$, DI-2: $\varphi = \dfrac{y''}{y^3}$. Now we can find DI-3:

$$\frac{d\varphi}{d\psi} = \varphi' = \frac{y'''y - 3y''y'}{y(yy'' - 2y'^2)}.$$

Express y''', y'', y' in terms of ψ, φ, φ'. This procedure leads us to ODE-1: $\varphi' = \dfrac{\varphi}{\psi}$. Hence, we obtain $\varphi = C\psi$ or $y'' = Cyy'$, that is, we obtain ODE-2 with 2 symmetries, which can be solved as

$$x = \int \frac{2dy}{Cy^2 + a} + b.$$

In conclusion, we have to note that we have discussed only some simple strategies. A different way of looking at the problem is described in [9] and based on using both point and nonpoint symmetries.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 172–176.

Non-local Symmetry of the 3-Dimensional Burgers-Type Equation

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Abstract

Non-local transformation, which connects the 3-dimensional Burgers-type equation with a linear heat equation, is constructed. Via this transformation, nonlinear superposition formulae for solutions are obtained and the conditional non-local symmetry of this equation is studied.

The multidimensional generalization of the Burgers equation

$$L_1(u) = u_0 - u|\nabla u| - \triangle u = 0, \tag{1}$$

is called further the Burgers-type equation. This equation was suggested by W. Fushchych in [1]. We use here such notations:

$$\partial_\mu u = \frac{\partial u}{\partial x_\mu}, \qquad \{x_\mu\} = (x_0, x_1, ..., x_{n-1}),$$
$$\nabla u = \|u_1, u_2, ..., u_n\|^T, \quad (\mu = 0, n-1),$$
$$|\nabla u| = \sqrt{(\nabla u)^2}, \qquad \triangle = (\nabla)^2 = \partial_1^2 + \partial_2^2 + \cdots + \partial_{n-1}^2.$$

In the present paper, we construct the non-local transformation, which connects the 3-dimensional equation (1) with a linear heat equation. Via this transformation, we obtain nonlinear superposition formulae for solutions of equation (1). Also we investigate the conditional non-local symmetry of equation (1) and obtain formulae generating solutions of this equation.

1. Conditional non-local superposition

Let us consider the 3-dimensional scalar heat equation

$$L_2(w) = w_0 - \triangle w = 0. \tag{2}$$

For the vector-function H, such equations are fulfiled:

$$H = 2\nabla \ln w, \qquad H = \|h^1, h^2, h^3\|^T, \tag{3}$$

$$\frac{1}{2}(H)^2 + (\nabla \cdot H) = \partial_0 \ln w, \qquad \nabla \times H = 0. \tag{4}$$

From the integrability condition for equations (3), (4), it follows that equation (2) is connected with the vector equations

$$H_0 - \frac{1}{2}(H)^2 - \nabla(\nabla \cdot H) = 0, \qquad \nabla \times H = 0. \tag{5}$$

Let $|H|^2 = (h^1)^2 + (h^2)^2 + (h^3)^2 = u^2$ for $H = \theta \cdot u$, where $|\theta| = 1$, $\theta = |\theta^1, \theta^2, \theta^3|^T$. Then we obtain from (5) that $\nabla \times \theta = 0$ and the equality

$$\theta \left[u_0 - u|\nabla u| - \triangle u\right] = u \left[\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta)\right].$$

Let relations

$$L_1(u) = u_0 - u|\nabla u| - \triangle u = 0, \tag{6}$$

$$\begin{aligned} &\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0, \\ &\nabla \times \theta = 0. \end{aligned} \tag{7}$$

be fulfiled on the some subset of solutions of equation (5). System (3), (4) in new variables, which connect equations (2) and (6), has the form

$$\nabla \ln w = \frac{1}{2}\theta u, \tag{8}$$

$$\partial_0 \ln w = \frac{1}{4}u^2 + \frac{1}{2}(\nabla \cdot \theta u). \tag{9}$$

So via transformations (8), (9), PDE (6) is reduced to the linear equation (2). The corresponding generalization of the Cole-Hopf transformation (substitution) is obtained in the form

$$u = 2\sqrt{(\nabla \ln w)^2}. \tag{10}$$

Theorem 1. *The non-local substitution (10) is a linearization of the 3-dimensional Burgers-type equation (1), and the operator equality*

$$\partial_0(\nabla \ln w) - 2(\nabla \ln w) \cdot \triangle \ln w - \nabla(\triangle \ln w) = [w^{-1}\nabla - w^{-2} \cdot \nabla w] \cdot (w_0 - \triangle w) = 0$$

is fulfiled.

Example 1. The function

$$u = 2\sqrt{[\nabla \ln \varphi(x_0, \omega)]^2}, \qquad \omega = \alpha \cdot x = \alpha_a x_a, \quad (a = 1, 2, 3), \tag{11}$$

is a non-Lie solution of the nonlinear equation (1). Let us substitute this non-local ansatz into equation (1). We find the condition on φ:

$$[\varphi^{-1} \cdot \partial_\omega - \varphi^{-2} \cdot \varphi_\omega] \cdot [\varphi_0 - \varphi_{\omega\omega}] = 0. \tag{12}$$

So, we obtain the non-local reduction of equation (1) to the form (12). Analogously, with the ansatz

$$u = 2\sqrt{[\nabla \ln\{2(n-1)x_0 + x\}]^2}, \quad x \equiv (x_1, x_2, x_3), \quad (n = 4), \tag{13}$$

one obtains

$$[w^{-1}\nabla - w^{-2}\nabla w] \cdot [\partial_0 - \triangle] \cdot \{2(n-1)x_0 + x\}. \tag{14}$$

Here, $w \equiv 2(n-1)x_0 + x$.

The linearization of the differential equation (1) makes it possible via a linear superposition of solutions of equation (2) and the non-local transformation (8), (9) to construct the principle of nonlinear superposition for solutions of equation (1).

Theorem 2. *The superposition formula for a subset of solutions* $\overset{(k)}{u}(x)$, $(k=1,2)$ *of equation (1) has the form*

$$\begin{aligned}
&\overset{(3)}{u}(x) = 2\left[\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right],\\
&\overset{(3)}{\theta}\overset{(3)}{u}(x) = 2\left(\overset{(1)}{\tau} + \overset{(2)}{\tau}\right)^{-1}\left(\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}\right),\\
&\overset{(k)}{\tau_0} = \frac{1}{2}\overset{(k)}{\tau}\left[\overset{(k)}{u}{}^2 + (\nabla\cdot\overset{(k)}{\theta}\overset{(k)}{u})\right],\\
&\nabla\overset{(k)}{\tau} = \frac{1}{2}\overset{(k)}{\tau}\overset{(k)}{u}\overset{(k)}{\theta}, \nabla\times\overset{(k)}{\theta},\\
&\partial_0\ln(\overset{(1)}{\tau} + \overset{(2)}{\tau}) = \left[\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right] + \left[\nabla\frac{\overset{(1)}{\tau}\overset{(1)}{u}\overset{(1)}{\theta} + \overset{(2)}{\tau}\overset{(2)}{u}\overset{(2)}{\theta}}{\overset{(1)}{\tau} + \overset{(2)}{\tau}}\right].
\end{aligned} \tag{15}$$

Here, $\overset{(k)}{u}$, $(k=1,2)$ *are known solutions of equation (1),* $\overset{(s)}{u}$, $(s=1,2,3)$ *and* $\overset{(k)}{\tau}, (k=1,2)$ *are functional parameters, and* $\overset{(k)}{\tau_0} - \triangle\overset{(k)}{\tau} = 0$.

2. Conditional non-local invariance

Let us consider the potential hydrodynamic-type system in $R(1,3)$ of independent variables

$$H_0 + (H,\nabla)H - \triangle H = 0, \qquad \nabla\times H = 0, \tag{16}$$

or, in the form (5),

$$H_0 + \frac{1}{2}\nabla(H)^2 - \nabla(\nabla\cdot H) = 0, \qquad \nabla\times H = 0. \tag{17}$$

Let $|H| = u$ or, in other words

$$|H|^2 = (H^1)^2 + (H^2)^2 + (H^3)^2 = u^2.$$

Then we have such equalities:

$$H = \theta\cdot|H| = \theta\cdot u, \qquad (\theta = H\cdot|H|^{-1}). \tag{18}$$

So, θ is a unit vector collinear with H. Let $\nabla \times \theta = 0$. It makes one possible to obtain

$$\begin{aligned}
&[\nabla \times \theta u] = u \cdot [\nabla \times \theta] + [\nabla u \times \theta] \longrightarrow \nabla u \times \theta = 0,\\
&\nabla u = \theta \cdot |\nabla u|, \qquad (\nabla \cdot \theta u) = u(\nabla \cdot \theta) + \theta \cdot \nabla u,\\
&\nabla(\nabla \cdot \theta u) = u \cdot \nabla(\nabla \cdot \theta) + 2\nabla u \cdot (\nabla \cdot \theta) + \theta \triangle u.
\end{aligned} \tag{19}$$

Substituting H of the form (18) into equation (17), we find the operator equation

$$\theta[u_{0+}u|\nabla u| - \triangle u] = -u[\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta)].$$

Let now pick out the subset of solutions of this equation, which consists of solutions of the system

$$u_{0+}u|\nabla u| - \triangle u = 0, \tag{20}$$

$$\theta_0 - 2\nabla \ln u(\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0. \tag{21}$$

Variables $\theta = \| \theta^1, \theta^2, \theta^3 \|^T$ are playing the role of supplementary parameters. Notice that, for a known solution u of equation (20), equation (21) is linear.

Let us take another copy of equation (17):

$$Q_0 + \frac{1}{2}\nabla(Q)^2 - \nabla(\nabla \cdot Q) = 0, \qquad \nabla \times Q = 0. \tag{22}$$

Let $Q = \tau \cdot w$, where $w = |Q|$ and $\nabla \times \tau = 0$. The differential equation (22) can be substituted now by the system

$$w_0 + w|\nabla w| - \triangle w = 0, \tag{23}$$

$$\tau_0 - 2\nabla \ln w \cdot (\nabla \cdot \tau) - \nabla(\nabla \cdot \tau) = 0. \tag{24}$$

Assume now such equalities:

$$\begin{aligned}
&-2\nabla \ln w = H - Q,\\
&-\partial_0 \ln w = \frac{1}{2}(H \cdot Q) - \left[\frac{1}{2}(H)^2 - (\nabla \cdot H)\right].
\end{aligned} \tag{25}$$

Excluding w from this system by cross-differentiation, we obtain

$$H_0 + \frac{1}{2}\nabla(H)^2 - \nabla(\nabla \cdot H) = Q_0 + \frac{1}{2}\nabla(H \cdot Q). \tag{26}$$

So, with the condition

$$Q_0 + \frac{1}{2}\nabla(H \cdot Q) = 0, \tag{27}$$

we get equation (17). Let

$$\frac{1}{2}(H \cdot Q) = \frac{1}{2}(Q)^2 - (\nabla \cdot Q). \tag{28}$$

Then (27) becomes equation (22). As one can see, condition (28) is the necessary condition for reducing equations (20), (21) via the non-local transformation (25), (28) to system (23), (24). Substituting

$$H = \tau \cdot w - 2\nabla \ln w$$

into (28), one obtains

$$w_0 + w \cdot |\nabla w| - \triangle w = \frac{1}{2} w\{|\nabla w| - (\nabla \cdot Q)\}.$$

Let us put the condition

$$|\nabla w| = (\nabla \cdot Q). \qquad (29)$$

Since

$$(\nabla \cdot \tau w) = w(\nabla \cdot \tau) + |\nabla w|,$$

it follows from condition (29) that

$$(\nabla \tau) = -4|\nabla \ln w|.$$

Introducing notations $u \equiv \overset{(2)}{u}$, $w \equiv \overset{(1)}{u}$, we can formulate such a theorem:

Theorem 3. *Let* $\overset{(1)}{u}$ *be a known solution of equation (1)*

$$u_0 + u \cdot |\nabla u| - \triangle u = 0.$$

Then its new solution $\overset{(2)}{u}$ *is defined by the formulae*

$$\overset{(2)}{u} = (\theta \cdot \tau) \overset{(1)}{u} - 2 \cdot \theta \cdot \nabla \ln \overset{(1)}{u},$$

$$\frac{1}{2}(\theta \cdot \tau) \overset{(1)}{u}\overset{(2)}{u} + 2\partial_0 \ln \overset{(1)}{u} = \frac{1}{2} \overset{(2)}{u}{}^2 - (\nabla \cdot \theta \overset{(2)}{u}),$$

$$\theta \overset{(2)}{u} = \tau \overset{(1)}{u} - 2 \cdot \nabla \ln \overset{(1)}{u},$$

$$\theta_0 - 2 \cdot \nabla \ln \overset{(2)}{u} (\nabla \cdot \theta) - \nabla(\nabla \cdot \theta) = 0,$$

$$\tau_0 - 2 \cdot \nabla \ln \overset{(1)}{u} (\nabla \cdot \tau) - \nabla(\nabla \cdot \tau) = 0,$$

$$\nabla \times \theta = 0, \quad \nabla \times \tau = 0, \quad (\nabla \cdot \tau) = -4|\nabla \ln w|.$$

The last equalities are additional conditions for the non-local invariance of equation (1) with respect to the non-local transformation (25), (26).

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 177–184.

Equivalence Transformations and Symmetry of the Schrödinger Equation with Variable Potential

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Abstract

We study symmetry of the Schrödinger equation with potential as a new dependent variable, i.e., transformations which do not change the form of a class of equations, which are called equivalence trasformations. We consider the systems comprising a Schrödinger equation and a certain condition for the potential. Symmetry properties of the equation with convection term are investigated.

This talk is based on the results obtained by the authors in collaboration with Prof. W. Fushchych and dedicated to his memory.

1. Introduction

Consider the following generalization of the Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x},|\psi|)\psi + V_a(t,\vec{x})\frac{\partial\psi}{\partial x_a} = 0, \tag{1}$$

where $\Delta = \dfrac{\partial^2}{\partial x_a \partial x_a}$, $a = \overline{1,n}$, $\psi = \psi(t,\vec{x})$ is an unknown complex function, $W = W(t,\vec{x},|\psi|)$ and $V_a = V_a(t,\vec{x})$ are potentials of interaction (for convenience, we set $m = \frac{1}{2}$).

In the case where $V_a = 0$ in (1), we have the standard Schrödinger equation. Symmetry properties of this equation were thoroughly investigated (see, e.g., [1]–[4]). For arbitrary $W(t,\vec{x},|\psi|)$, equation (1) admits only the trivial group of identical transformations $\vec{x} \to \vec{x'} = \vec{x}$, $t \to t' = t$, $\psi \to \psi' = \psi$ [1], [3].

In [5]–[7], the method of extension of the symmetry group of equation (1) was suggested. The idea lies in the fact that, in equation (1), we assume that $W(t,\vec{x},|\psi|)$ and $V_a(t,\vec{x})$ are new dependent variables. This means that equation (1) is regarded as a nonlinear equation even in the case where the potential W does not depend on ψ. Symmetry operators of this type generate transformations called equivalence transformations.

2. Symmetry of the Schrödinger equation with potential

Using the above idea, we obtain the invariance algebra of the Schrödinger equation with potential

$$i\frac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x},|\psi|)\psi = 0. \tag{2}$$

Theorem 1. *Equation (2) is invariant under the infinite-dimensional Lie algebra with basis operators of the form*

$$\begin{aligned}
&J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a},\\
&Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{2}\ddot{U}_a x_a\partial_W,\\
&Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} + \frac{i}{4}\ddot{A}x_c x_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) -\\
&\qquad - \frac{n\dot{A}}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) + \left(\frac{1}{4}\dddot{A}\,x_c x_c - 2W\dot{A}\right)\partial_W,\\
&Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W, \quad Z_1 = \psi\partial_\psi, \qquad Z_2 = \psi^*\partial_{\psi^*},
\end{aligned} \tag{3}$$

where $U_a(t), A(t), B(t)$ are arbitrary smooth functions of t, there is summation from 1 to n over the repeated index c and no summation over the repeated index a, $a,b = \overline{1,n}$, the upper dot means the derivative with respect to time.

Note that the invariance algebra (3) includes the operators of space ($U_a = 1$) and time ($A = 1/2$) translations, the Galilei operator ($U_a = t$), the dilation ($A = t$) and projective ($A = t^2/2$) operators.

Proof of Theorem 1. We seek the symmetry operators of equation (2) in the class of first-order differential operators of the form:

$$X = \xi^\mu(t,\vec{x},\psi,\psi^*)\partial_{x_\mu} + \eta(t,\vec{x},\psi,\psi^*)\partial_\psi + \eta^*(t,\vec{x},\psi,\psi^*)\partial_{\psi^*} + \rho(t,\vec{x},\psi,\psi^*,W)\partial_W. \tag{4}$$

Using the invariance condition [1], [8], [9] of equation (2) under the operator (4) and the fact that $W = W(t,\vec{x},|\psi|)$, i.e., $\psi\dfrac{\partial W}{\partial\psi} = \psi^*\dfrac{\partial W}{\partial\psi^*}$, we obtain the system of determining equations:

$$\begin{aligned}
&\xi^j_\psi = \xi^j_{\psi^*} = 0, \quad \xi^0_a = 0, \quad \xi^a_a = \xi^b_b, \quad \xi^a_b + \xi^b_a = 0, \quad \xi^0_0 = 2\xi^a_a,\\
&\eta_{\psi^*} = 0, \quad \eta_{\psi\psi} = 0, \quad \eta_{\psi a} = (i/2)\xi^a_0,\\
&\eta^*_\psi = 0, \quad \eta^*_{\psi^*\psi^*} = 0, \quad \eta^*_{\psi^* a} = -(i/2)\xi^a_0,\\
&i\eta_0 + \eta_{cc} - \eta_\psi W\psi + 2W\xi^n_n\psi + W\eta + \rho\psi = 0,\\
&-i\eta^*_0 + \eta^*_{cc} - \eta^*_{\psi^*}W\psi^* + 2W\xi^n_n\psi^* + W\eta^* + \rho\psi^* = 0,\\
&\rho_\psi = \rho_{\psi^*} = 0,
\end{aligned} \tag{5}$$

where the index j varies from 0 to n, $a,b = \overline{1,n}$, we mean summation from 1 to n over the repeated index c and no summation over the indices a,b.

We solve system (5) and obtain the following result:

$$\xi^0 = 2A, \quad \xi^a = \dot{A}x_a + C^{ab}x_b + U_a, \quad a = \overline{1,n},$$

$$\eta = \frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + B\right)\psi,$$

$$\eta^* = -\frac{i}{2}\left(\frac{1}{2}\ddot{A}x_c x_c + \dot{U}_c x_c + E\right)\psi^*,$$

$$\rho = \frac{1}{2}\left(\frac{1}{2}\dddot{A}\,x_c x_c + \ddot{U}_c x_c + \dot{B}\right) - \frac{n}{2}i\ddot{A} - 2W\dot{A},$$

where A, U_a, B are arbitrary functions of t, $E = B - 2in\dot{A} + C_1$, $C^{ab} = -C^{ba}$ and C_1 are arbitrary constants. Theorem 1 is proved.

The operators Q_B generate the finite transformations

$$\begin{cases} t' = t, \quad \vec{x}' = \vec{x}, \\ \psi' = \psi\exp(iB(t)\alpha), \quad \psi^{*'} = \psi^*\exp(-iB(t)\alpha), \\ W' = W + \dot{B}(t)\alpha, \end{cases} \tag{6}$$

where α is a group parameter, $B(t)$ is an arbitrary smooth function.

Using the Lie equations, we obtain that the following transformations correspond to the operators Q_a:

$$\begin{cases} t' = t, \; x'_b = U_a(t)\beta_a\delta_{ab} + x_b, \\ \psi' = \psi\exp\left(\frac{i}{4}\dot{U}_a U_a\beta_a^2 + \frac{i}{2}\dot{U}_a x_a\beta_a\right), \\ \psi^{*'} = \psi^*\exp\left(-\frac{i}{4}\dot{U}_a U_a\beta_a^2 - \frac{i}{2}\dot{U}_a x_a\beta_a\right), \\ W' = W + \frac{1}{2}\ddot{U}_a x_a\beta_a + \frac{1}{4}\ddot{U}_a U_a\beta_a^2, \end{cases} \tag{7}$$

where $\beta_a (a = \overline{1,n})$ are group parameters, $U_a = U_a(t)$ are arbitrary smooth functions, there is no summation over the index a, δ_{ab} is a Kronecker symbol. In particular, if $U_a(t) = t$, then the operators Q_a are the standard Galilei operators

$$G_a = t\partial_{x_a} + \frac{i}{2}x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}). \tag{8}$$

For the operators Q_A, it is difficult to write out finite transformations in the general form. We consider several particular cases:
(a) $A(t) = t$.
Then $Q_A = 2t\partial_t + x_c\partial_{x_c} - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W$ is a dilation operator generating the transformations

$$\begin{cases} t' = t\exp(2\lambda), \quad x'_c = x_c\exp(\lambda), \\ \psi' = \exp\left(-\frac{n}{2}\lambda\right)\psi, \quad \psi^{*'} = \exp\left(-\frac{n}{2}\lambda\right)\psi^*, \\ W' = W\exp(-2\lambda), \end{cases} \tag{9}$$

where λ is a group parameter.

(b) $A(t) = t^2/2$.

Then $Q_A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_cx_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) - \frac{n}{2}t(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2tW\partial_W$ is the operator of projective transformations:

$$\begin{cases} t' = \dfrac{t}{1-\mu t}, \; x'_c = \dfrac{x_c}{1-\mu t}, \\ \psi' = \psi(1-\mu t)^{n/2}\exp\left(\dfrac{ix_cx_c\mu}{4(1-\mu t)}\right), \\ \psi^{*'} = \psi^*(1-\mu t)^{n/2}\exp\left(\dfrac{-ix_cx_c\mu}{4(1-\mu t)}\right), \\ W' = W(1-\mu t)^2, \end{cases} \tag{10}$$

μ is an arbitrary parameter.

Consider the example. Let

$$W = \frac{1}{\vec{x}^2} = \frac{1}{x_cx_c}. \tag{11}$$

We find how new potentials are generated from potential (11) under transformations (6), (7), (9), (10).

(i) Q_B:

$$W = \frac{1}{x_cx_c} \to W' = \frac{1}{x_cx_c} + B(t)\alpha \to W'' = \frac{1}{x_cx_c} + B(t)(\alpha+\alpha') \to \cdots,$$

where $B(t)$ is an arbitrary smooth function, α and α' are arbitrary real parameters.

(ii) Q_a:

$$W = \frac{1}{x_cx_c} \to W',$$

$$W' = \frac{1}{(x_a - U_a(t)\beta_a)^2 + x_bx_b} + \frac{1}{4}\ddot{U}_aU_a\beta_a^2 + \frac{1}{2}\ddot{U}_a\beta_a(x_a - U_a\beta_a),$$

$$W' \to W'',$$

$$W'' = \frac{1}{(x_a - U_a(t)(\beta_a+\beta'_a))^2 + x_bx_b} + \frac{1}{4}\ddot{U}_aU_a(\beta_a^2 + \beta_a'^2) + $$
$$+ \frac{1}{2}\ddot{U}_a(\beta_a+\beta'_a)(x_a - U_a(\beta_a+\beta'_a)) + \frac{1}{2}\ddot{U}_aU_a\beta_a\beta'_a \to \cdots,$$

where U_a are arbitrary smooth functions, β_a and β'_a are real parameters, there is no summation over a but there is summation over b ($b \neq a$). In particular, if $U_a(t) = t$, then we have the standard Galilei operator (8) and

$$W = \frac{1}{x_cx_c} \to W' = \frac{1}{(x_a - t\beta_a)^2 + x_bx_b} \to W'' = \frac{1}{(x_a - t(\beta_a+\beta'_a))^2 + x_bx_b} \to \cdots$$

(iii) Q_A for $A(t) = t$ or $A(t) = t^2/2$ do not change the potential, i.e.,

$$W = \frac{1}{x_cx_c} \to W' = \frac{1}{x_cx_c} \to W'' = \frac{1}{x_cx_c} \to \cdots$$

3. The Schrödinger equation and conditions for the potential

Consider several examples of the systems in which one of the equations is equation (2) with potential $W = W(t,\vec{x})$, and the second equation is a certain condition for the potential W. We find the invariance algebras of these systems in the class of operators

$$X = \xi^\mu(t,\vec{x},\psi,\psi^*,W)\partial_{x_\mu} + \eta(t,\vec{x},\psi,\psi^*,W)\partial_\psi + \eta^*(t,\vec{x},\psi,\psi^*,W)\partial_{\psi^*} + \\ + \rho(t,\vec{x},\psi,\psi^*,W)\partial_W.$$

(i) Consider equation (2) with the additional condition for the potential, namely, the Laplace equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x})\psi = 0, \\ \Delta W = 0. \end{cases} \tag{12}$$

System (12) admits the infinite-dimensional Lie algebra with the basis operators

$$\begin{aligned} &P_0 = \partial_t,\ P_a = \partial_{x_a},\ J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a}, \\ &Q_a = U_a\partial_{x_a} + \frac{i}{2}\dot{U}_a x_a(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{2}\ddot{U}_a x_a\partial_W,\ a = \overline{1,n}, \\ &D = x_c\partial_{x_c} + 2t\partial_t - \frac{n}{2}(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2W\partial_W, \\ &A = t^2\partial_t + tx_c\partial_{x_c} + \frac{i}{4}x_c x_c(\psi\partial_\psi - \psi^*\partial_{\psi^*}) - \frac{n}{2}t(\psi\partial_\psi + \psi^*\partial_{\psi^*}) - 2Wt\partial_W, \\ &Q_B = iB(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \dot{B}\partial_W,\ Z_1 = \psi\partial_\psi,\ Z_2 = \psi^*\partial_{\psi^*}, \end{aligned} \tag{13}$$

where $U_a(t)$ $(a = \overline{1,n})$ and $B(t)$ are arbitrary smooth functions. In particular, algebra (13) includes the Galilei operator (8).

(ii) The condition for the potential is the wave equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi + W(t,\vec{x})\psi = 0, \\ \Box W = 0. \end{cases} \tag{14}$$

The maximal invariance algebra of system (14) is $\langle P_0,\ P_a,\ J_{ab},\ Z_1,\ Z_2,\ Z_3, Z_4\rangle$, where P_0, P_a, J_{ab}, Z_1, Z_2 have the form (13) and

$$Z_3 = it(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \partial_W, \qquad Z_4 = it^2(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + 2t\partial_W.$$

(iii) Consider the important case in a $(1+1)$-dimensional space–time where the condition for the potential is the KdV equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \dfrac{\partial^2\psi}{\partial x^2} + W(t,x)\psi = 0, \\ \dfrac{\partial W}{\partial t} + \lambda_1 W\dfrac{\partial W}{\partial x} + \lambda_2\dfrac{\partial^3 W}{\partial x^3} = F(|\psi|), \quad \lambda_1 \neq 0. \end{cases} \tag{15}$$

For an arbitrary $F(|\psi|)$, system (15) is invariant under the Galilei operator and the maximal invariance algebra is the following:

$$\begin{aligned} &P_0 = \partial_t,\ P_1 = \partial_x,\ Z = i(\psi\partial_\psi - \psi^*\partial_{\psi^*}), \\ &G = t\partial_x + \frac{i}{2}\left(x + \frac{2}{\lambda_1}t\right)(\psi\partial_\psi - \psi^*\partial_{\psi^*}) + \frac{1}{\lambda_1}\partial_W. \end{aligned} \tag{16}$$

For $F = C = \text{const}$, system (15) admits the extension, namely, it is invariant under the algebra $\langle P_0, P_1, G, Z_1, Z_2\rangle$, where G has the form (16).

The Galilei operator G generates the following transformations:

$$\begin{cases} t' = t,\ x' = x + \theta t, \quad W' = W + \dfrac{1}{\lambda_1}\theta, \\ \psi' = \psi \exp\left(\dfrac{i}{2}\theta x + \dfrac{i}{\lambda_1}\theta t + \dfrac{i}{4}\theta^2 t\right), \\ \psi^{*'} = \psi^* \exp\left(-\dfrac{i}{2}\theta x - \dfrac{i}{\lambda_1}\theta t - \dfrac{i}{4}\theta^2 t\right), \end{cases}$$

where θ is a group parameter. Here, it is important that $\lambda_1 \neq 0$, since otherwise, system (15) does not admit the Galilei operator.

4. The Schrödinger equation with convection term

Consider equation (1) for $W = 0$, i.e., the Schrödinger equation with convection term

$$i\frac{\partial \psi}{\partial t} + \Delta\psi = V_a \frac{\partial \psi}{\partial x_a}, \tag{17}$$

where ψ and V_a $(a = \overline{1,n})$ are complex functions of t and $\vec{x}$. For extension of symmetry, we again regard the functions V_a as dependent variables. Note that the requirement that the functions V_a be complex is essential for symmetry of (17).

Let us investigate symmetry of (17) in the class of first-order differential operators

$$X = \xi^\mu \partial_{x_\mu} + \eta\partial_\psi + \eta^*\partial_{\psi^*} + \rho^a \partial_{V_a} + \rho^{*a}\partial_{V_a^*},$$

where $\xi^\mu, \eta, \eta^*, \rho^a, \rho^{*a}$ are functions of $t, \vec{x}, \psi, \psi^*, \vec{V}, \vec{V}^*$.

Theorem 2. *[10] Equation (17) is invariant under the infinite-dimensional Lie algebra with the basis operators*

$$\begin{aligned} &Q_A = 2A\partial_t + \dot{A}x_c\partial_{x_c} - i\ddot{A}x_c(\partial_{V_c} - \partial_{V_c^*}) - \dot{A}(V_c\partial_{V_c} + V_c^*\partial_{V_c^*}), \\ &Q_{ab} = E_{ab}(x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*}) - \\ &\qquad - i\dot{E}_{ab}(x_a\partial_{V_b} - x_b\partial_{V_a} - x_a\partial_{V_b^*} + x_b\partial_{V_a^*}), \\ &Q_a = U_a\partial_{x_a} - i\dot{U}_a(\partial_{V_a} - \partial_{V_a^*}),\ Z_1 = \psi\partial_\psi,\ Z_2 = \psi^*\partial_{\psi^*},\ Z_3 = \partial_\psi,\ Z_4 = \partial_{\psi^*}, \end{aligned} \tag{18}$$

where A, E_{ab}, U_a are arbitrary smooth functions of t, we mean summation over the index c and no summation over indices a and b.

This theorem is proved by analogy with the previous one.

Note that algebra (18) includes as a particular case the Galilei operator of the form:

$$\widetilde{G}_a = t\partial_{x_a} - i\partial_{V_a} + i\partial_{V_a^*}. \tag{19}$$

This operator generates the following finite transformations:

$$\begin{cases} x_b' = x_b + \beta_a t\delta_{ab}, \quad t' = t, \\ \psi' = \psi,\ \psi^{*'} = \psi^*, \quad V_b' = V_b - i\beta_a\delta_{ab}, \quad V_b^{*'} = V_a^* + i\beta_a\delta_{ab}, \end{cases}$$

where β_a is an arbitrary real parameter. Operator (19) is essentially different from the standard Galilei operator (8) of the Schrödinger equation, and we cannot derive operator (8) from algebra (18).

Consider now the system of equation (17) with the additional condition for the potentials V_a, namely, the complex Euler equation:

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} + \Delta\psi = V_a\dfrac{\partial\psi}{\partial x_a}, \\ i\dfrac{\partial V_a}{\partial t} - V_b\dfrac{\partial V_a}{\partial x_b} = F(|\psi|)\dfrac{\partial\psi}{\partial x_a}, \end{cases} \tag{20}$$

Here, ψ and V_a are complex dependent variables of t and $\vec{x}$, F is a function of $|\psi|$. The coefficients of the second equation of the system provide the broad symmetry of this system.

Let us investigate the symmetry classification of system (20). Consider the following five cases:

1. F is an arbitrary smooth function.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a\rangle$, where

$$J_{ab} = x_a\partial_{x_b} - x_b\partial_{x_a} + V_a\partial_{V_b} - V_b\partial_{V_a} + V_a^*\partial_{V_b^*} - V_b^*\partial_{V_a^*},$$

$\widetilde{G}_a$ has the form (19).

2. $F = C|\psi|^k$, where C is an arbitrary complex constant, $C \neq 0$, k is an arbitrary real number, $k \neq 0$ and $k \neq -1$.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}\rangle$, where

$$D^{(1)} = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*} - \frac{2}{1+k}(\psi\partial_\psi + \psi^*\partial_{\psi^*}).$$

3. $F = \dfrac{C}{|\psi|}$, where C is an arbitrary complex constant, $C \neq 0$.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, Z = Z_1 + Z_2\rangle$, where

$$Z = \psi\partial_\psi + \psi^*\partial_{\psi^*}, \quad Z_1 = \psi\partial_\psi, \quad Z_2 = \psi^*\partial_{\psi^*}.$$

4. $F = C \neq 0$, where C is an arbitrary complex constant.
The maximal invariance algebra is $\langle P_0, P_a, J_{ab}, \widetilde{G}_a, D^{(1)}, Z_3, Z_4\rangle$, where

$$Z_3 = \partial_\psi, \quad Z_4 = \partial_{\psi^*}.$$

5. $F = 0$.
In this case, system (20) admits the widest maximal invariance algebra, namely, $\langle P_0$, P_a, J_{ab}, $\widetilde{G}_a$, D, A, Z_1, Z_2, Z_3, $Z_4\rangle$, where

$$D = 2t\partial_t + x_c\partial_{x_c} - V_c\partial_{V_c} - V_c^*\partial_{V_c^*},$$
$$A = t^2\partial_t + tx_c\partial_{x_c} - (ix_c + tV_c)\partial_{V_c} + (ix_c - tV_c^*)\partial_{V_c^*}.$$

In conclusion, we note that the equivalence groups can be successfully used for construction of exact solutions of the nonlinear Schrödinger equation.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 185–192.

The Higher Dimensional Bateman Equation and Painlevé Analysis of Nonintegrable Wave Equations

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Abstract

In performing the Painlevé test for nonintegrable partial differential equations, one obtains differential constraints describing a movable singularity manifold. We show, for a class of wave equations, that the constraints are in the form of Bateman equations. In particular, for some higher dimensional wave equations, we derive the exact relations, and show that the singularity manifold condition is equivalent to the higher dimensional Bateman equation. The equations under consideration are: the sine-Gordon, Liouville, Mikhailov, and double sine-Gordon equations as well as two polynomial field theory equations.

1 Introduction

The Painlevé analysis, as a test for integrability of PDEs, was proposed by Weiss, Tabor and Carnevale in 1983 [20]. It is an generalization of the singular point analysis for ODEs, which dates back to the work of S. Kovalevsky in 1888. A PDE is said to possess the Painlevé property if solutions of the PDE are single-valued in the neighbourhood of non-characteristic, movable singularity manifolds (Ward [17], Steeb and Euler [15], Ablowitz and Clarkson [1]). Weiss, Tabor and Carnevale [20] proposed a test of integrability (which may be viewed as a necessary condition of integrability), analogous to the algorithm given by Ablowitz, Ramani and Segur [2] to determine whether a given ODE has the Painlevé property. One seeks a solution of a given PDE (in rational form) in the form of a Laurent series (also known as the Painlevé series)

$$u(\boldsymbol{x}) = \phi^{-m}(\boldsymbol{x}) \sum_{j=0}^{\infty} u_j(\boldsymbol{x}) \phi^j(\boldsymbol{x}), \qquad (1.1)$$

where $u_j(\boldsymbol{x})$ are analytic functions of the complex variables $\boldsymbol{x} = (x_0, x_1, \ldots, x_{n-1})$ (we do not change the notation for the complex domain), with $u_0 \neq 0$, in the neighbourhood of a non-characteristic, movable singularity manifold defined by $\phi(\boldsymbol{x}) = 0$ (the pole manifold), where $\phi(\boldsymbol{x})$ is an analytic function of $x_0, x_1, \ldots, x_{n-1}$. The PDE is said to pass the Painlevé test if, on substituting (1.1) in the PDE, one obtains the correct number of arbitrary functions as required by the Cauchy-Kovalevsky theorem, given by the expansion coefficients in (1.1), whereby ϕ should be one of arbitrary functions. The positions in the Painlevé expansion where arbitrary functions are to appear, are known as the resonances. If a PDE passes the Painlevé test, it is usually (Newell *et al* [13]) possible to construct

Bäcklund transformations and Lax pairs (Weiss [18], Steeb and Euler [15]), which proves the necessary condition of integrability.

Recently much attention was given to the construction of exact solutions of nonintegrable PDEs by the use of a truncated Painlevé series (Cariello and Tabor [3], Euler *et al.* [10], Webb and Zank [17], Euler [5]). On applying the Painlevé test to nonintegrable PDEs, one usually obtains conditions on ϕ at resonances; the singular manifold conditions. With a truncated series, one usually obtains additional constraints on the singularity manifolds, leading to a compatibility problem for the solution of ϕ.

In the present paper, we show that the general solution of the Bateman equation, as generalized by Fairlie [11], solves the singularity manifold condition at the resonance for a particular class of wave equations. For the n-dimensional ($n \geq 3$) sine-Gordon, Liouville, and Mikhailov equations, the n-dimensional Bateman equation is the general solution of the singularity manifold condition, whereas, the Bateman equation is only a special solution of the polynomial field theory equations which were only studied in two dimensions. For the n-dimensional ($n \geq 2$) double sine-Gordon equation, the Bateman equation also solves the constraint at the resonance in general.

2 The Bateman equation for the singularity manifold

The Bateman equation in two dimensions has the following form:

$$\phi_{x_0x_0}\phi_{x_1}^2 + \phi_{x_1x_1}\phi_{x_0}^2 - 2\phi_{x_0}\phi_{x_1}\phi_{x_0x_1} = 0. \tag{2.2}$$

In the Painlevé analysis of PDEs, (2.2) was first obtained by Weiss [19] in his study of the double sine-Gordon equation. As pointed out by Weiss [19], the Bateman equation (2.2) may be linearized by a Legendre transformation. Moreover, it is invariant under the Moebius group. The general implicit solution of (2.2) is

$$x_0 f_0(\phi) + x_1 f_1(\phi) = c, \tag{2.3}$$

where f_0 and f_1 are arbitrary smooth functions and c an arbitrary constant. Fairlie [11] proposed the following generalization of (2.2) for n dimensions:

$$\det \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \cdots & \phi_{x_0x_{n-1}} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \cdots & \phi_{x_1x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0x_{n-1}} & \phi_{x_1x_{n-1}} & \cdots & \phi_{x_{n-1}x_{n-1}} \end{pmatrix} = 0. \tag{2.4}$$

We call (2.4) the n-dimensional Bateman equation. It admits the following general implicit solution

$$\sum_{j=0}^{n-1} x_j\, f_j(\phi) = c, \tag{2.5}$$

where f_j are n arbitrary functions.

Let us consider the following direct n-dimensional generalization of the well-known sine-Gordon, Liouville, and Mikhailov equations, as given respectively by

$$\begin{aligned}
&\Box_n u + \sin u = 0, \\
&\Box_n u + \exp(u) = 0, \qquad (2.6)\\
&\Box_n u + \exp(u) + \exp(-2u) = 0.
\end{aligned}$$

By a direct n-dimensional generalization, we mean that we merely consider the d'Alembert operator $\Box$ in the n-dimensional Minkowski space, i.e.,

$$\Box_n := \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}.$$

It is well known that the above given wave equations are integrable if $n = 2$, i.e., time and one space coordinates. We call PDEs integrable if they can be solved by an inverse scattering transform and there exists a nontrivial Lax pair (see, for example, the book of Ablowitz and Clarkson [1] for more details). For such integrable equations, the Painlevé test is passed and there are no conditions at the resonance, so that ϕ is an arbitrary function.

Before we state our proposition for the singularity manifold of the above given wave equations, we have to introduce some notations and a lemma. We call the $(n+1)\times(n+1)$-matrix, of which the determinant is the general Bateman equation, the Bateman matrix and denote this matrix by B, i.e.,

$$B := \begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \cdots & \phi_{x_{n-1}} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \cdots & \phi_{x_0x_{n-1}} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \cdots & \phi_{x_1x_{n-1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \phi_{x_{n-1}} & \phi_{x_0x_{n-1}} & \phi_{x_1x_{n-1}} & \cdots & \phi_{x_{n-1}x_{n-1}} \end{pmatrix}. \qquad (2.7)$$

Definition. *Let*

$$M_{x_{j_1}x_{j_2}\dots x_{j_r}}$$

denote the determinant of the submatrix that remains after the rows and columns containing the derivatives $\phi_{x_{j_1}}$, $\phi_{x_{j_2}}$, …, $\phi_{x_{j_r}}$ have been deleted from the Bateman matrix (2.7). If

$$j_1, \dots, j_r \in \{0, 1, \dots, n-1\}, \quad j_1 < j_2 < \cdots < j_r, \quad r \le n-2, \quad \text{for } n \ge 3,$$

then $M_{x_{j_1}x_{j_2}\dots x_{j_r}}$ are the determinants of Bateman matrices, and we call the equations

$$M_{x_{j_1}x_{j_2}\dots x_{j_r}} = 0 \qquad (2.8)$$

the minor Bateman equations of (2.4).

Note that the n-dimensional Bateman equation (2.4) has $n!/[r!(n-r)!]$ minor Bateman equations. Consider an example. If $n = 5$ and $r = 2$, then there exist 10 minor Bateman equations, one of which is given by $M_{x_2x_3}$, i.e.,

$$\det\begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_4} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \phi_{x_0x_4} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \phi_{x_1x_4} \\ \phi_{x_4} & \phi_{x_0x_4} & \phi_{x_1x_4} & \phi_{x_4x_4} \end{pmatrix} = 0. \qquad (2.9)$$

Note that the subscript of M, namely x_2 and x_3, indicates that the derivatives of ϕ w.r.t. x_2 or x_3 do not appear in the minor Bateman equation.

We can now state the following

Lemma. *The Bateman equation (2.4) is equivalent to the following sum of minor Bateman equations*

$$\sum_{j_1,j_2,\dots,j_r=1}^{n-1} M_{x_{j_1}x_{j_2}\dots x_{j_r}} - \sum_{j_1,j_2,\dots,j_{r-1}=1}^{n-1} M_{x_0 x_{j_1}x_{j_2}\dots x_{j_{r-1}}} = 0, \tag{2.10}$$

where $j_1,\dots,j_r \in \{1,\dots,n-1\}, \quad j_1 < j_2 < \dots < j_r, \quad r \le n-2, \quad n \ge 3.$

Proof. It is easy to show that the general solution of the n-dimensional Bateman equation satisfies every minor Bateman equation in n dimensions identically. Thus, equations (2.4) and (2.10) have the same general solution and are therefore equivalent. □

Theorem 1. *For $n \ge 3$, the singularity manifold condition of the direct n-dimensional generalization of the sine-Gordon, Liouville and Mikhailov equations (2.6), is given by the n-dimensional Bateman equation (2.4).*

The detailed proof will be presented elsewhere. Let us sketch the proof for the sine-Gordon equation. By the substitution

$$v(\boldsymbol{x}) = \exp[iu(\boldsymbol{x})],$$

the n-dimensional sine-Gordon equation takes the following form:

$$v\Box_n v - (\nabla_n v)^2 + \frac{1}{2}\left(v^3 - v\right) = 0, \tag{2.11}$$

where

$$(\nabla_n v)^2 := \left(\frac{\partial v}{\partial x_0}\right)^2 - \sum_{j=1}^{n-1}\left(\frac{\partial v}{\partial x_j}\right)^2.$$

The dominant behaviour of (2.11) is 2, so that the Painlevé expansion is

$$v(\boldsymbol{x}) = \sum_{j=0}^{\infty} v_j(\boldsymbol{x})\phi^{j-2}(\boldsymbol{x}).$$

The resonance is at 2 and the first two expansion coefficients have the following form:

$$v_0 = -4\left(\nabla_n\phi\right)^2, \qquad v_1 = 4\Box_n\phi.$$

We first consider $n = 3$. The singularity manifold condition at the resonance is then given by

$$\det\begin{pmatrix} 0 & \phi_{x_0} & \phi_{x_1} & \phi_{x_2} \\ \phi_{x_0} & \phi_{x_0x_0} & \phi_{x_0x_1} & \phi_{x_0x_2} \\ \phi_{x_1} & \phi_{x_0x_1} & \phi_{x_1x_1} & \phi_{x_1x_2} \\ \phi_{x_2} & \phi_{x_0x_2} & \phi_{x_1x_2} & \phi_{x_2x_2} \end{pmatrix} = 0,$$

which is exactly the 2-dimensional Bateman equation as defined by (2.4).

Consider now $n \geq 4$. At the resonance, we then obtain the following condition

$$\sum_{j_1, j_2, \dots, j_{n-3}=1}^{n-1} M_{x_{j_1} x_{j_2} \dots x_{j_{n-3}}} - \sum_{j_1, j_2, \dots, j_{n-4}=1}^{n-1} M_{x_0 x_{j_1} x_{j_2} \dots x_{j_{n-4}}} = 0, \tag{2.12}$$

where

$$j_1, \dots, j_{n-3} \in \{1, \dots, n-1\}, \quad j_1 < j_2 < \dots < j_{n-3},$$

i.e., $M_{x_{j_1} x_{j_2} \dots x_{j_{n-3}}}$ and $M_{x_0 x_{j_1} x_{j_2} \dots x_{j_{n-4}}}$ are the determinants of all possible 4×4 Bateman matrices. By the Lemma give above, equation (2.12) is equivalent to the n-dimensional Bateman equation (2.4).

The proof for the Liouville and Mikhailov equations is similar.

The wave equations studied above have the common feature that they are integrable in two dimensions. Let us consider the double sine-Gordon equation in n dimensions:

$$\Box_n u + \sin \frac{u}{2} + \sin u = 0. \tag{2.13}$$

It was shown by Weiss [19] that this equation does not pass the Painlevé test for $n = 2$, and that the singularity manifold condition is given by the Bateman equation (2.2). For n dimensions, we can state the following

Theorem 2. *For $n \geq 2$, the singularity manifold condition of the n-dimensional double sine-Gordon equation (2.13) is given by the n-dimensional Bateman equation (2.4).*

The proof will be presented elsewhere.

In Euler *et al.* [4], we studied the above wave equations with explicitly space- and time-dependence in one space dimension.

3 Higher order singularity manifold conditions

It is well known that in one and more space dimensions, polynomial field equations such as the nonlinear Klein-Gordon equation

$$\Box_2 u + m^2 u + \lambda u^n = 0 \tag{3.14}$$

cannot be solved exactly for $n = 3$, even for the case $m = 0$. In light-cone coordinates, i.e.,

$$x_0 \longrightarrow \frac{1}{2}(x_0 - x_1), \qquad x_1 \longrightarrow \frac{1}{2}(x_0 + x_1),$$

(3.14) takes the form

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^n = 0, \tag{3.15}$$

where we let $m = 0$ and $\lambda = 1$. The Painlevé test for the case $n = 3$ was performed by Euler *et al.* [10]. We are now interested in the relation between the Bateman equation and the singularity maifold condition of (3.15) for the case $n = 3$ as well as $n = 2$.

First, we consider equation (3.15) with $n = 3$. Performing the Painlevé test (Euler *et al.* [10]), we find that the dominant behaviour is -1, the resonance is 4, and the first three expansion coefficients in the Painlevé expansion are

$$u_0^2 = 2\phi_{x_0}\phi_{x_1},$$
$$u_1 = -\frac{1}{3u_0^2}\left(u_0\phi_{x_0x_1} + u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1}\right),$$
$$u_2 = \frac{1}{3u_0^2}\left(u_{0x_0x_1} - 3u_0u_1^2\right),$$
$$u_3 = \frac{1}{u_0^2}\left(u_2\phi_{x_0x_1} + u_{2x_1}\phi_{x_0} + u_{2x_0}\phi_{x_1} + u_{1x_0x_1} - 6u_0u_1u_2\right).$$

At the resonance, we obtain the following singularity manifold condition:

$$\Phi\sigma - \left(\phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0}\right)^2 = 0, \qquad (3.16)$$

where Φ is the two-dimensional Bateman equation given by (2.2) and

$$\begin{aligned}
\sigma = (&24\phi_{x_0}\phi_{x_1}^6\phi_{x_0x_0x_0}\phi_{x_0x_0} - 54\phi_{x_0}^2\phi_{x_1}^5\phi_{x_0x_0}\phi_{x_0x_0x_1} - 18\phi_{x_0}^2\phi_{x_1}^5\phi_{x_0x_1}\phi_{x_0x_0x_0}\\
&+18\phi_{x_0}^3\phi_{x_1}^4\phi_{x_0x_1}\phi_{x_0x_0x_1} + 36\phi_{x_0}^3\phi_{x_1}^4\phi_{x_0x_0}\phi_{x_0x_1x_1} - 3\phi_{x_0}^2\phi_{x_1}^6\phi_{x_0x_0x_0x_0}\\
&+36\phi_{x_0}^4\phi_{x_1x_1}\phi_{x_1}^3\phi_{x_0x_0x_1} - 6\phi_{x_0}^4\phi_{x_0x_0}\phi_{x_1}^3\phi_{x_1x_1x_1} + 18\phi_{x_0}^4\phi_{x_1}^3\phi_{x_0x_1}\phi_{x_0x_1x_1}\\
&-6\phi_{x_0}^3\phi_{x_1x_1}\phi_{x_1}^4\phi_{x_0x_0x_0} + 24\phi_{x_0}^6\phi_{x_1}\phi_{x_1x_1}\phi_{x_1x_1x_1} - 54\phi_{x_0}^5\phi_{x_1}^2\phi_{x_1x_1}\phi_{x_0x_1x_1}\\
&-18\phi_{x_0}^5\phi_{x_1}^2\phi_{x_0x_1}\phi_{x_1x_1x_1} - 3\phi_{x_0}^6\phi_{x_1}^2\phi_{x_1x_1x_1x_1} + 12\phi_{x_0}^5\phi_{x_1}^3\phi_{x_0x_1x_1x_1}\\
&-18\phi_{x_0}^4\phi_{x_1}^4\phi_{x_0x_0x_1x_1} + 12\phi_{x_0}^3\phi_{x_1}^5\phi_{x_0x_0x_0x_1} + 48\phi_{x_1}\phi_{x_0x_1}\phi_{x_0}^5\phi_{x_1x_1}^2\\
&-30\phi_{x_0}^3\phi_{x_1}^3\phi_{x_0x_0}\phi_{x_0x_1}\phi_{x_1x_1} + 3\phi_{x_0}^2\phi_{x_0x_0}^2\phi_{x_1}^4\phi_{x_1x_1} - 2\phi_{x_0}^3\phi_{x_1}^3\phi_{x_0x_1}^3\\
&+3\phi_{x_0}^4\phi_{x_1}^2\phi_{x_0x_0}\phi_{x_1x_1}^2 - 15\phi_{x_0}^4\phi_{x_1}^2\phi_{x_0x_1}^2\phi_{x_1x_1} - 20\phi_{x_0}^6\phi_{x_1x_1}^3\\
&+48\phi_{x_0}\phi_{x_1}^5\phi_{x_0x_1}\phi_{x_0x_0}^2 - 20\phi_{x_1}^6\phi_{x_0x_0}^3 - 15\phi_{x_0}^2\phi_{x_1}^4\phi_{x_0x_1}^2\phi_{x_0x_0})/(3\phi_{x_0}^2\phi_{x_1}^2).
\end{aligned}$$

Clearly, the general solution of the two-dimensional Bateman equation solves (3.16).

For the equation

$$\frac{\partial^2 u}{\partial x_0 \partial x_1} + u^2 = 0, \qquad (3.17)$$

the singularity manifold condition is even more complicated. However, also in this case, we are able to express the singularity manifold condition in terms of Φ. The dominant behaviour of (3.17) is -2 and the resonance is at 6. The first five expansion coefficients in the Painlevé expansion are as follows:

$$u_0 = -6\phi_{x_0}\phi_{x_1},$$
$$u_1 = \frac{1}{\phi_{x_0}\phi_{x_1} + u_0}\left(u_{0x_1}\phi_{x_0} + u_{0x_0}\phi_{x_1} + u_0\phi_{x_0x_1}\right),$$
$$u_2 = -\frac{1}{2u_0}\left(u_{0x_0x_1} + u_1^2 - u_{1x_1}\phi_{x_0} - u_{1x_0}\phi_{x_1} - u_1\phi_{x_0x_1}\right),$$
$$u_3 = -\frac{1}{2u_0}\left(u_{1x_0x_1} + 2u_1u_2\right),$$

$$u_4 = -\frac{1}{\phi_{x_1}\phi_{x_0} + u_0}\left(u_3\phi_{x_0x_1} + u_{2x_0x_1} + 2u_1u_3 + u_{3x_1}\phi_{x_0} + u_{3x_0}\phi_{x_1} + u_2^2\right),$$
$$u_5 = -\frac{1}{6\phi_{x_0}\phi_{x_1} + 2u_0}\left(2u_1u_4 + 2u_4\phi_{x_0x_1} + 2u_{4x_0}\phi_{x_1} + 2u_{4x_1}\phi_{x_0} + 2u_2u_3 + u_{3x_0x_1}\right).$$

At the resonance, the singularity manifold condition is a PDE of order six, which consists of 372 terms (!) all of which are derivatives of ϕ with respect to x_0 and x_1. This condition may be written in the following form:

$$\sigma_1\Phi + \sigma_2\Psi + (\phi_{x_0}\Psi_{x_1} - \phi_{x_1}\Psi_{x_0} - \sigma_3\Psi - \sigma_4\Phi)^2 = 0, \tag{3.18}$$

where Φ is the two-dimensional Bateman equation (2.2), and

$$\Psi = \phi_{x_0}\Phi_{x_1} - \phi_{x_1}\Phi_{x_0}.$$

The σ's are huge expressions consisting of derivatives of ϕ with respect to x_0 and x_1. We do not present these expressions here. Thus, the general solution of the Bateman equation satisfies the full singularity manifold condition for (3.17).

Solution (2.5) may now be exploited in the construction of exact solutions for the above wave equations, by truncating their Painlevé series. A similar method, as was used in the papers of Webb and Zank [17] and Euler [5], may be applied. This will be the subject of a future paper.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 193–199.

Subgroup Structure of the Poincaré Group *P(1,4)* and Symmetry Reduction of Five-Dimensional Equations of Mathematical Physics

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Abstract

Using the subgroup structure of the generalized Poincaré group $P(1,4)$, the symmetry reduction of the five-dimensional wave and Dirac equations and Euler–Lagrange–Born–Infeld, multidimensional Monge–Ampere, eikonal equations to differential equations with a smaller number of independent variables is done. Some classes of exact solutions of the investigated equations are constructed.

1 Introduction

The knowledge of the nonconjugate subgroups of local Lie groups of point transformations and construction in explicit form of the invariants of these subgroups is important in order to solve many problems of mathematics. In particular, in mathematical physics, the subgroup structure of the invariance groups of partial differential equations (PDEs) and the invariants of these subgroups allow us to solve many problems. Let me mention some of them.

1. The symmetry reduction of PDEs to differential equations with fewer independent variables [1–10].
2. The construction of PDEs with a given (nontrivial) symmetry group [6,8,9,11–14].
3. The construction of systems of coordinates in which the linear PDEs which are invariant under given groups admit partial or full separation of variables [15–20].

The development of theoretical and mathematical physics has required various extensions of the four-dimensional Minkowski space and, correspondingly, various extensions of the Poincaré group $P(1,3)$. One extension of the group $P(1,3)$ is the generalized Poincaré group $P(1,4)$. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Minkowski space $M(1,4)$. This group has many applications in theoretical and mathematical physics [21–24].

The purpose of the present paper is to give a survey of results obtained in [25–37] as well as some new results.

2 The subgroup structure of the group $P(1,4)$

The Lie algebra of the group $P(1,4)$ is given by 15 basis elements $M_{\mu\nu} = -M_{\nu\mu}$ $(\mu, \nu = 0, 1, 2, 3, 4)$ and P'_μ $(\mu = 0, 1, 2, 3, 4)$ satisfying the commutation relations

$$\left[P'_\mu, P'_\nu\right] = 0, \qquad \left[M'_{\mu\nu}, P'_\sigma\right] = g_{\mu\sigma}P'_\nu - g_{\nu\sigma}P'_\mu,$$
$$\left[M'_{\mu\nu}, M'_{\rho\sigma}\right] = g_{\mu\rho}M'_{\nu\sigma} + g_{\nu\sigma}M'_{\mu\rho} - g_{\nu\rho}M'_{\mu\sigma} - g_{\mu\sigma}M'_{\nu\rho},$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\mu\nu} = 0$, if $\mu \neq \nu$. Here and in what follows, $M'_{\mu\nu} = iM_{\mu\nu}$.

In order to study the subgroup structure of the group $P(1,4)$, we use the method proposed in [38]. Continuous subgroups of the group $P(1,4)$ were found in [25–29].

One of nontrivial consequences of the description of the continuous subalgebras of the Lie algebra of the group $P(1,4)$ is that the Lie algebra of the group $P(1,4)$ contains as subalgebras the Lie algebra of the Poincaré group $P(1,3)$ and the Lie algebra of the extended Galilei group $\widetilde{G}(1,3)$ [24], i.e., it naturally unites the Lie algebras of the symmetry groups of relativistic and nonrelativistic quantum mechanics.

3 Invariants of subgroups of the group $P(1,4)$

In this Section, we will say something about the invariants of subgroups of the group $P(1,4)$. For all continuous subgroups of the group $P(1,4)$, we have constructed the invariants in the five-dimensional Minkowski space. The majority of these invariants are presented in [30, 31]. Among the invariants obtained, there are one-, two-, three- and four-dimensional ones.

Let us note that the invariants of continuous subgroups of the group $P(1,4)$ play an important role in solution of the problem of reduction for many equations of theoretical and mathematical physics which are invariant under the group $P(1,4)$ or its continuous subgroups.

In the following, we will consider the application of the subgroup structure of the group $P(1,4)$ and the invariants of these subgroups for the symmetry reduction and construction of classes of exact solutions for some important equations of theoretical and mathematical physics.

4 The nonlinear five-dimensional wave equation

Let us consider the equation

$$\frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} - \frac{\partial^2 u}{\partial x_4^2} = F(u), \tag{4.1}$$

where $u = u(x)$, $x = (x_0, x_1, x_2, x_3, x_4)$, F is a sufficiently smooth function. The invariance group of equation (4.1) is the generalized Poincaré group $P(1,4)$. Using the invariants of subgroups of the group $P(1,4)$, we have constructed ansatzes, which reduce the investigated equation to differential equations with a smaller number of independent variables. Taking into account the solutions of the reduced equations, some classes of exact solutions of the equations investigated are been found. The majority of these results have been published in [30–33].

Let us consider ansatzes of the form

$$u(x) = \varphi(\omega), \tag{4.2}$$

where $\omega(x)$ is a one-dimensional invariant of subgroups of the group $P(1,4)$. In many cases, these ansatzes reduce equation (4.1) to ordinary differential equations (ODEs) of the form

$$\frac{d^2\varphi}{d\omega^2} + \frac{d\varphi}{d\omega} k\omega^{-1} = \varepsilon F(\varphi), \tag{4.3}$$

where some of the new variables ω with the k and ε corresponding to them have the form

1. $\omega = \left(x_4^2 + x_2^2 + x_1^2 - x_0^2\right)^{1/2}, \qquad k = 3, \quad \varepsilon = -1;$

2. $\omega = \left(x_3^2 + x_1^2 + x_2^2 + x_4^2 - x_0^2\right)^{1/2}, \qquad k = 4, \quad \varepsilon = -1.$

Let $F(\varphi) = \lambda\varphi^n \quad (n \neq 1)$, then equations (4.3) have the form

$$\frac{d^2\varphi}{d\omega^2} + \frac{d\varphi}{d\omega} k\omega^{-1} = \varepsilon\lambda\varphi^n \quad (n \neq 1). \tag{4.4}$$

This equation is of the Fowler-Emden type. Particular solutions of equation (4.4) have the form:

$$\varphi = \alpha\omega^\nu,$$

where

$$\alpha = \left[\frac{2[1 + k + n(1-k)]}{\varepsilon\lambda(1-n)^2}\right]^{\frac{1}{n-1}}, \qquad \nu = \frac{2}{1-n}.$$

On the basis of the solutions of the reduced equations, we have obtained particular solutions of equation (4.1) with the right-hand side $F(u) = \lambda u^n \quad (n \neq 1)$.

Let us consider ansatzes of the form

$$u(x) = \varphi(\omega_1, \omega_2),$$

where $\omega_1(x)$ and $\omega_2(x)$ are invariants of subgroups of the group $P(1,4)$. These ansatzes reduce the equation at hand to two-dimensional PDEs. In 22 cases, we obtained ODEs instead of two-dimensional PDEs.

Let us note that the same situation for nonlinear relativistically invariant equations was noted and studied in [10].

Let me mention ansatzes of the form

$$u(x) = \varphi(\omega_1, \omega_2, \omega_3),$$

where $\omega_1(x)$, $\omega_2(x)$, $\omega_3(x)$ are invariants of subgroups of the group $P(1,4)$. Among the reduced equations, there are 10 two-dimensional PDEs.

5 Separation of variables in $M(1,4)$

Using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, we have constructed systems of coordinates in the space $M(1,4)$, in which a large class of linear PDEs invariant under the group $P(1,4)$ or under its Abelian subgroups admits partial or full separation of variables. Using this, we have studied the five-dimensional wave equation of the form

$$\Box u = -\kappa^2 u, \tag{5.1}$$

where

$$\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

$u(x) = u(x_0, x_1, x_2, x_3, x_4)$ is a scalar $\mathbf{C}^2$ function, $\kappa = \text{const}$. As a result of partial (or full) separation of variables, we obtained PDEs with a smaller number of independent variables (or ODEs) which replace the original equation. Some exact solutions of the five-dimensional wave equation have been constructed. More details about these results can be found in [34].

6 The Dirac equation in $M(1,4)$

Let us consider the equation

$$\left(\gamma_k p^k - m\right) \psi(x) = 0, \tag{6.1}$$

where $x = (x_0, x_1, x_2, x_3, x_4)$, $p_k = i\dfrac{\partial}{\partial x_k}$, $k = 0, 1, 2, 3, 4$; γ_k are the (4×4) Dirac matrices. Equation (6.1) is invariant under the generalized Poincaré group $P(1,4)$.

Following [39] and using the subgroup structure of the group $P(1,4)$, the ansatzes which reduce equation (6.1) to systems of differential equations with a lesser number of independent variables are constructed. The corresponding symmetry reduction has been done. Some classes of exact solutions of the investigated equation have been found. The part of the results obtained is presented in [35].

7 The eikonal equation

We consider the equation

$$u^\mu u_\mu \equiv (u_0)^2 - (u_1)^2 - (u_2)^2 - (u_3)^2 = 1, \tag{7.1}$$

where $u = u(x),\ x = (x_0, x_1, x_2, x_3) \in M(1,3)$, $u_\mu \equiv \dfrac{\partial u}{\partial x_\mu}$, $u^\mu = g^{\mu\nu} u_\nu$, $\mu, \nu = 0, 1, 2, 3$.

From the results of [40], it follows that the symmetry group of equation (7.1) contains the group $P(1,4)$ as a subgroup. Using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, we have constructed ansatzes which reduce the investigated equation to differential equations with a smaller number of independent variables, and the corresponding symmetry reduction has been carried out. Having solved some of the reduced equations, we have found classes of exact solutions of the investigated equation. Some of these results are given in [30, 31, 36].

It should be noted that, among the ansatzes obtained, there are those which reduce the investigated equation to linear ODEs.

8 The Euler-Lagrange-Born-Infeld equation

Let us consider the equation

$$\Box u\,(1-u_\nu u^\nu)+u^\mu u^\nu u_{\mu\nu}=0, \tag{8.1}$$

where $u=u(x), x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_\mu\equiv\dfrac{\partial u}{\partial x^\mu}$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x^\mu\partial x^\nu}$, $u^\mu=g^{\mu\nu}u_\nu$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu=0,1,2,3$, $\Box$ is the d'Alembert operator.

The symmetry group [40] of equation (8.1) contains the group $P(1,4)$ as a subgroup.

On the basis of the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups, the symmetry reduction of the investigated equation to differential equations with a lesser number of independent variables has been done. In many cases the reduced equations are linear ODEs. Taking into account the solutions of reduced equations, we have found multiparametric families of exact solutions of the considered equation. The part of these results can be found in [36].

9 The multidimensional homogeneous Monge-Ampère equation

Consider the equation

$$\det(u_{\mu\nu})=0, \tag{9.1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $\mu,\nu=0,1,2,3$.

The symmetry group of equation (9.1) was found in [40].

We have made the symmetry reduction of the investigated equation to differential equations with a smaller number of independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. Among the reduced equations, there are linear ODEs. Having solved some of the reduced equations, we have obtained classes of exact solutions of the investigated equation. Some of these results are presented in [36].

10 The multidimensional inhomogeneous Monge-Ampère equation

In this section, we consider the equation

$$\det(u_{\mu\nu})=\lambda\,(1-u_\nu u^\nu)^3,\quad \lambda\neq 0, \tag{10.1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2,x_3)\in M(1,3)$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $u^\nu=g^{\nu\alpha}u_\alpha$, $u_\alpha\equiv\dfrac{\partial u}{\partial x_\alpha}$, $g_{\mu\nu}=(1,-1,-1,-1)\delta_{\mu\nu}$, $\mu,\nu,\alpha=0,1,2,3$.

Equation (10.1) is invariant [40] under the group $P(1,4)$.

We have constructed ansatzes which reduce the investigated equation to differential equations with a less number of independent variables, using the subgroup structure of the group $P(1,4)$ and the invariants of its subgroups. The corresponding symmetry reduction has been done. Some classes of exact solutions of the considered equation are found. The majority of these results was published in [37].

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 200–205.

Differential Invariants for a Nonlinear Representation of the Poincaré Algebra. Invariant Equations

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Abstract

We study scalar representations of the Poincaré algebra $p(1,n)$ with $n \geq 2$. We present functional bases of the first- and second-order differential invariants for a nonlinear representation of the Poincaré algebra $p(1,2)$ and describe new nonlinear Poincaré-invariant equations.

0. Introduction

The classical linear Poincaré algebra $p(1,n)$ can be represented by basis operators

$$p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \tag{1}$$

where μ, ν take values $0, 1, \ldots, n$; summation is implied over repeated indices (if they are small Greek letters) in the following way:

$$\begin{aligned} &x_\nu x_\nu \equiv x_\nu x^\nu \equiv x^\nu x_\nu = x_0^2 - x_1^2 - \cdots - x_n^2, \\ &g_{\mu\nu} = \mathrm{diag}\,(1, -1, \ldots, -1). \end{aligned} \tag{2}$$

We consider x_ν and x^ν equivalent with respect to summation. Algebra (1) is an invariance algebra of many important equations of mathematical physics, such as the nonlinear wave equation

$$\Box u = F(u)$$

or the eikonal equation

$$u_\alpha u_\alpha = 0,$$

and such invariance reflects compliance with the Poincaré relativity principle. Poincaré-invariant equations can be used for construction of meaningful mathematical models of relativistic processes. For more detail, see [11].

Submitted to Proceedings of II International Conference 'Symmetry in Nonlinear Mathematical Physics', Kyiv, Ukraine, July 7-13, 1997

In paper [8], scalar representations of the algebra $p(1,1)$ were studied, and it appeared that there exist nonlinear representations which are not equivalent to (1). Here, we investigate the same possibility for the algebra $p(1,n)$ with $n \geq 2$. We prove that there is one representation for $p(1,2)$ which is nonlinear and non-equivalent to (1). For $n > 2$, there are no scalar representations non-equivalent to (1).

To describe invariant equations with respect to our new representation, we need its differential invariants. A functional basis of these invariants is presented below together with examples of new nonlinear Poincaré-invariant equations.

1. Construction of a new representation for the scalar Poincaré algebra

The Poincaré algebra $p(1,2)$ is defined by the commutational relations

$$[J_{01}, J_{02}] = iJ_{12}, \quad [J_{01}, J_{12}] = iJ_{02}, \quad [J_{02}, J_{12}] = -iJ_{01}; \tag{3}$$

$$[P_\mu, J_{\mu\nu}] = iP_\nu, \qquad \mu, \nu = 0, 1, 2; \tag{4}$$

$$[P_\mu, P_\nu] = 0. \tag{5}$$

We look for new representations of the operators $P_\mu, J_{\mu\nu}$ in the form

$$X = \xi^\mu(x_\mu, u)\partial_{x_\mu} + \eta(x_\mu, u)\partial_u \tag{6}$$

We get from (3), (4), (5) that up to equivalence with respect to local transformations of x_μ and u, we can take $P_\mu, J_{\mu\nu}$ in the following form:

$$\begin{aligned} &p_\mu = ig_{\mu\nu}\frac{\partial}{\partial x_\nu}, \\ &J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + if_{\mu\nu}(u)\partial_u, \qquad \left(\partial_u \equiv \frac{\partial}{\partial u}\right). \end{aligned} \tag{7}$$

We designate $f_{01} = a$, $f_{02} = b$, $f_{12} = c$ and get from (2) the conditions on these functions:

$$ab_u - ba_u = c, \quad ac_u - ca_u = b, \quad bc_u - cb_u = -a.$$

Whence

$$c^2 = a^2 + b^2, \quad a = br, \quad c = b(1 + r^2)^{1/2}, \quad b = \frac{1}{\left(\ln\left(r + \sqrt{1 + r^2}\right)\right)_u},$$

where r is an arbitrary function of u.

Up to a transformation $u \to \varphi(u)$, we can consider the following nonlinear representation of the operators $J_{\mu\nu}$:

$$\begin{aligned} &J_{01} = -i\left(x_0\partial_1 + x_1\partial_0 + \sin u\partial_u\right), \\ &J_{02} = -i\left(x_0\partial_2 + x_2\partial_0 + \cos u\partial_u\right), \\ &J_{12} = -i\left(x_1\partial_2 - x_2\partial_1 + \partial_u\right). \end{aligned} \tag{8}$$

It is easily checked that the representation P_μ (1), $J_{\mu\nu}$ (8) is not equivalent to the representation (1).

To prove that there are no representations of $p(1, n)$, $n > 2$, which would be non-equivalent to (1), we take $P_\mu, J_{\mu\nu}$ in the form (6) and use the commutational relations of the algebra $p(1, n)$. Similarly to the previous case, we can take $P_\mu, J_{\mu\nu}$ in the form (7), and get from commutation relations for $J_{\mu\nu}$ that

$$f_{0a}^2 + f_{0b}^2 = f_{ab}^2, \quad f_{ab}^2 + f_{bc}^2 + f_{ac}^2 = 0, \quad a, b, c = 1, \ldots, n.$$

Whence all $f_{\mu\nu} = 0$, what was to be proved.

2. First-order differential invariants for a nonlinear representation

Definition. *The function $F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u})$, where $x = (x_0, x_1, \ldots, x_n)$, $\underset{k}{u}$ is the set of all k-th order partial derivatives of the function u, is called a differential invariant for the Lie algebra L with basis elements X_s of the form (6) if it is an invariant of the m-th prolongation of this algebra:*

$$\overset{m}{X}_s F(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) = \lambda_s(x, u, \underset{1}{u}, \underset{2}{u}, \ldots, \underset{m}{u}) F. \tag{9}$$

Theoretical studies of differential invariants and their applications can be found in [1–4].

Here, only absolute differential invariants are considered, for which all $\lambda_s = 0$. We look for a first-order absolute differential invariant in the form $F = F(u, \underset{1}{u})$. We use designations $u_0 = x, u_1 = y, u_2 = z$, and from (9) get the following defining conditions for F:

$$\begin{aligned}
&\sin u F_u - x F_y - y F_x + \cos u (x F_x + y F_y + z F_z) = 0, \\
&\cos u F_u - x F_z - z F_x - \sin u (x F_x + y F_y + z F_z) = 0, \\
&F_u - y F_z + z F_y = 0.
\end{aligned}$$

From the above equations, we get the only non-equivalent absolute differential invariant of the first order for the representation P_μ (1), $J_{\mu\nu}$ (8):

$$I_1 = \frac{u_0 - u_1 \cos u + u_2 \sin u}{u_0^2 - u_1^2 - u_2^2}. \tag{10}$$

The expressions

$$u_0 - u_1 \cos u + u_2 \sin u \tag{11}$$

and

$$u_0^2 - u_1^2 - u_2^2 = u_\mu u_\mu \tag{12}$$

are relative differential invariants for the representation P_μ (1), $J_{\mu\nu}$ (8).

3. Second-order differential invariants for a nonlinear representation

Here, we adduce a functional basis of second-order differential invariants for the representation P_μ (1), $J_{\mu\nu}$ (8). These invariants are found using the system of partial differential equations, obtained from the definition of differential invariants (9). The basis we present contains six more invariants in addition to I_1 (10).

$$I_2 = \frac{F_1}{(u_0^2 - u_1^2 - u_2^2)^{3/2}}, \tag{13}$$

where

$$\begin{aligned} F_1 = \lambda(u_0 - u_1 \cos u + u_2 \sin u) = u_{00} - 2u_{01}\cos u + 2u_{02}\sin u + u_{11}\cos^2 u - \\ -2u_{12}\sin u \cos u + u_{12}\sin^2 u - u_0 u_1 \sin u - u_0 u_2 \cos u - u_2^2 \sin u \cos u + \\ +u_1 u_2(\cos^2 u - \sin^2 u) + u_1^2 \sin u \cos u. \end{aligned} \tag{14}$$

L is an operator of invariant differentiation for the algebra $p(1,2)$, P_μ (1), $J_{\mu\nu}$ (8):

$$L = \partial_0 - \cos u \partial_1 + \sin u \partial_2. \tag{15}$$

Its first Lie prolongation has the form

$$\overset{1}{L} = L - (u_1 \cos u + u_2 \sin u) u_\alpha \partial_{u_\alpha}.$$

$$I_3 = \frac{F_2}{(u_0^2 - u_1^2 - u_2^2)^2}, \tag{16}$$

where

$$F_2 = \overset{1}{L}(u_0^2 - u_1^2 - u_2^2). \tag{17}$$

The remaining invariants from our chosen basis do not contain trigonometric functions. To construct them, we use second-order differential invariants of the standard linear scalar representation of $p(1,2)$ (1) [5]. The notations used for these invariants are as follows:

$$\begin{aligned} &r = u_0^2 - u_1^2 - u_2^2, \quad S_1 = \Box u, \quad S_2 = u_{\mu\nu}u_{\mu\nu}, \quad S_3 = u_\mu u_{\mu\nu} u_\nu, \\ &S_4 = u_{\mu\nu}u_{\mu\alpha}u_{\nu\alpha}, \quad S_5 = u_\mu u_{\mu\nu} u_{\nu\alpha} u_\alpha. \end{aligned} \tag{18}$$

It is easy to see that the expressions $r, S_1, S_2, S_3, S_4, S_5$, where $\mu, \nu, \alpha = 0, 1, 2$, are functionally independent. The absolute differential invariants of the nonlinear representation of $p(1,2)$ I_4, I_5, I_6, I_7 look as follows:

$$\begin{aligned} &I_4 = (S_3 - rS_1)r^{-3/2}, \quad I_5 = (S_2 r^2 - S_3^2)r^{-3}, \quad I_6 = (S_5 r - S_3^2)r^{-3}, \\ &I_7 = \left(rS_4 - 3(S_1 S_5 - S_1^2 S_3 + \frac{1}{3} r S_1^3\right) r^{-5/2}. \end{aligned} \tag{19}$$

Proof of the fact that the invariants $I_1, I_2, \ldots, I_7$ present a functional basis of absolute differential invariants of the nonlinear representation of $p(1,2)$ consists of the following steps:

1. Proof of functional independence.
2. Proof of completeness of the set of invariants.

The first step is made by direct verification. The second requires calculation of the rank of the basis of the non-linear representation of $p(1,2)$. The rank of the set $\langle J_{01}, J_{02}, J_{12} \rangle$ (8) is equal to 2, and $F\left(u, \underset{1}{u}, \underset{2}{u}\right)$ depends on 9 variables. So, a complete set has to consist of 7 invariants.

4. Examples of invariant equations and their symmetry

The equation

$$u_0 - u_1 \cos u + u_2 \sin u = 0 \tag{20}$$

is invariant with respect to the algebra $p(1,2)$ with basis operators P_μ (1), $J_{\mu\nu}$ (8). The following theorem describes its maximal symmetry:

Theorem 1. *Equation (20) is invariant with respect to an infinite-dimensional algebra generated by operators*

$$X = \xi^0(x,u)\partial_0 + \xi^1(x,u)\partial_1 + \xi^2(x,u)\partial_2 + \eta(x,u)\partial_u,$$

where

$$\begin{aligned}
&\eta = \eta(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u),\\
&\xi^0 = \xi^0(x_0, u, x_1 + x_0 \cos u, x_2 - x_0 \sin u),\\
&\xi^1 = \varphi^1(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u) + \eta x_0 \sin u - \xi_0 \cos u,\\
&\xi^2 = \varphi^2(u, x_1 + x_0 \cos u, x_2 - x_0 \sin u) + \eta x_0 \cos u + \xi_0 \sin u;
\end{aligned}$$

η, ξ^0, φ^1, φ^2 *are arbitrary functions of their arguments.*

The theorem is proved by means of the Lie algorithm [6, 7, 12].

The infinite-dimensional algebra described above contains as subalgebras the Poincaré algebra $p(1,2)$ (operators P_μ of the form (1), $J_{\mu\nu}$ of the form(8)) and also its extension - a nonlinear representation of the conformal algebra $c(1,2)$. For details on nonlinear representations of $c(1,2)$, see [10]. A basis of this algebra is formed by operators P_μ (1), $J_{\mu\nu}$ (8),

$$D = x_\mu \partial_\mu, \quad K_\nu = 2x_\nu x_\mu \partial_\mu - x^2 \partial_\nu + i\eta^\nu(x,u)\partial_u; \quad \mu,\nu = 0,1,2, \tag{21}$$

where

$$\eta^0 = 2(x_1 \sin u + x_2 \cos u), \quad \eta^1 = -2(x_2 - x_0 \sin u), \quad \eta^2 = 2(x_1 + x_0 \cos u).$$

Equation (20) has the general solution

$$u = \Phi(x_1 + x_0 \cos u, x_2 - x_0 \sin u).$$

The transformation

$$\widetilde{u} = u, \quad \widetilde{x}_0 = x_0, \quad \widetilde{x}_1 = x_1 + x_0 \cos u, \quad \widetilde{x}_2 = x_2 - x_0 \sin u$$

applied to (20) yields the equation

$$\widetilde{u}_{\widetilde{x}_0} = 0. \tag{22}$$

A simplest linear equation (22) appears to be invariant with respect to the following nonlinear representation of $p(1,2)$:

$$\widetilde{P}_0 = i(\cos u\partial_1 - \sin u\partial_2 + \partial_0), \quad \widetilde{P}_1 = -i\partial_1, \quad \widetilde{P}_2 = -i\partial_2,$$

$$\widetilde{J}_{01} = -i\Big((x_0 \sin^2 u - x_1 \cos u)\partial_1 - \sin u(x_1 - x_0 \cos u)\partial_2 + \\ +(x_1 - x_0 \cos u)\partial_0 + \sin u\partial_u\Big),$$

$$\widetilde{J_{02}} = -i\Big((x_0 \cos^2 u - x_2 \cos u)\partial_2 + \cos u(x_2 + x_0 \cos u)\partial_1 + \\ +(x_2 + x_0 \sin u)\partial_0 - \cos u\partial_u\Big),$$

$$\widetilde{J}_{12} = -i(x_1\partial_2 - x_2\partial_1 + \partial_u).$$

Examples of explicit solutions for equation (20):

$$u = \frac{\cos^{-1} c}{(x_1^2 + x_2^2)^{-1/2}} + \tan^{-1}\left(\frac{x_1}{x_2}\right); \qquad u = \frac{\cos^{-1} x_1}{c - x_0};$$

$$u = \tan^{-1}\left(\frac{c - x_1}{x_2}\right); \qquad u = 2\tan^{-1}\left(\frac{c + x_0 + x_1}{x_2}\right).$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 206–210.

Representations of Subalgebras of a Subdirect Sum of the Extended Euclid Algebras and Invariant Equations

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Abstract

Representations of subdirect sums of the extended Euclid algebras $A\tilde{E}(3)$ and $A\tilde{E}(1)$ in the class of Lie vector fields are constructed. Differential invariants of these algebras are obtained.

Description of general classes of partial differential equations invariant with respect to a given group is one of the central problems of the group analysis of differential equations. As is well known, to get the most general solution of this problem, we have to construct the complete set of functionally independent differential invariants of some fixed order for all possible realizations of the local transformation group under study. Fushchych and Yehorchenko [2–4] found the complete set of first- and second-order differential invariants for the standard representations of the groups $P(1,n)$, $E(n)$, $G(1,n)$. Rideau and Winternitz [5, 6] have obtained new realizations of the Poincaré and Galilei group in two-dimensional space-time. New (nonlinear) realizations of the Poincaré groups $P(1,2)$, $P(2,2)$ and Euclid group $E(3)$ were found by Yehorchenko [7] and Fushchych, Zhdanov & Lahno [8, 9, 10].

In this paper, we consider the problem of constructing the complete set of the second-order differential invariants of subalgebras of a subdirect sum of extended Euclid algebras. These algebras are invariance algebras of a number of important differential equations (for example, the Boussinesq equation, equations for the polytropic gas [11]).

1. Let $V = X \times U \cong R^4 \times R^1$ be the space of real variables x_0, $x = (x_1, x_2, x_3)$ and u, G be a local transformation group acting in V and having the generators

$$Q = \tau(x_0, x, u)\partial_{x_0} + \xi^a(x_0, x, u)\partial_{x_a} + \eta(x_0, x, u)\partial_u. \tag{1}$$

By definition the operators $\langle Q_1, \ldots, Q_N\rangle$ form the Lie algebra AG and fulfill the commutation relations

$$[Q_a, Q_b] = C^c_{ab} Q_c, \quad a, b, c = 1, \ldots, N. \tag{2}$$

The problem of classification of realizations of the transformation group G is reduced to classifying realizations of its Lie algebra AG within the class of Lie vector fields.

Introduce the binary relation on the set of realizations of the Lie algebra AG. Two realizations are called equivalent if there exists a nondegenerate change of variables

$$x_0' = h(x_0, x, u), \quad x_a' = g_a(x_0, x, u), \quad u' = f(x_0, x, u), \quad a = 1, 2, 3, \tag{3}$$

transforming them one into another. Note that the introduced equivalence relation does not affect the form of relations (2) [1]. Furthermore, it divides the set of all possible representations into equivalence classes.

We consider covariant realizations of subalgebras of a subdirect sum of the extended Euclid algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$. Saying about a subdirect sum of these algebras, we mean two algebras. If L is a direct sum of the algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$, then its basis operators satisfy the following commutation relations:

$$\begin{aligned}
&[P_0, P_a] = [P_0, J_{ab}] = [P_a, P_b] = 0;\\
&[P_a, J_{bc}] = \delta_{ab} P_c - \delta_{ac} P_b;\\
&[J_{ab}, J_{cd}] = \delta_{ad} J_{bc} + \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac};
\end{aligned} \tag{4}$$

$$\begin{aligned}
&[P_0, D_1] = P_0, \quad [P_a, D_2] = P_a,\\
&[P_0, D_2] = [D_1, D_2] = [P_a, D_1] = [J_{ab}, D_1] = [J_{ab}, D_2] = 0,
\end{aligned} \tag{5}$$

where $a, b, c, = 1, 2, 3,\ \delta_{ab}$ is Kronecker symbol.

Next, if K is a subdirect sum of the algebras $A\tilde{E}(1)$ and $A\tilde{E}(3)$ and K is not isomorphic to L, then its basis is formed by the operators P_0, P_a, J_{ab}, D that satisfy the commutation relations (4), and

$$[P_0, D] = kP_0, \quad [P_a, D] = P_a, \quad [J_{ab}, D] = 0, \tag{6}$$

where $a, b = 1, 2, 3,\ k \neq 0,\ k \in R$.

Lemma 1. *An arbitrary covariant representation of the algebra $AE(1) \oplus AE(3)$ in the class of vector fields is reduced by transformations (3) to the following representation:*

$$P_0 = \partial_{x_0}, \quad P_a = \partial_{x_a}, \quad J_{ab} = x_a \partial_{x_b} - x_b \partial_{x_a}, \quad a, b = 1, 2, 3. \tag{7}$$

The proff of Lemma 1 follows from the results of Theorem 1 [8].

Theorem 1. *Nonequivalent covariant representations of the Lie algebra L in the class of vector fields are exhausted by representations (7) of the translation and rotation generators and one of the following representations of dilatation operators:*

$$D_1 = x_0 \partial_{x_0}, \quad D_2 = x_a \partial_{x_a}; \tag{8}$$

$$D_1 = x_0 \partial_{x_0}, \quad D_2 = x_a \partial_{x_a} + 2u\partial_u; \tag{9}$$

$$D_1 = x_0 \partial_{x_0} - u\partial_u, \quad D_2 = x_a \partial_{x_a} + ku\partial_u,\ k \neq 0; \tag{10}$$

$$D_1 = x_0 \partial_{x_0} - u\partial_u, \quad D_2 = x_a \partial_{x_a}; \tag{11}$$

$$D_1 = x_0 \partial_{x_0} + u\partial_u, \quad D_2 = u\partial_{x_0} + x_a \partial_{x_a}. \tag{12}$$

Theorem 2. *Nonequivalent covariant representations of the algebra K in the class of vector fields are exhausted by representations (7) of the translation and rotation generators and one of the following representations of dilatation operators:*

$$D = kx_0 \partial_{x_0} + x_a \partial_{x_a}, \quad k \neq 0, \tag{13}$$

$$D = kx_0\partial_{x_0} + x_a\partial_{x_a} + u\partial_u, \quad k \neq 0. \tag{14}$$

To prove these theorems, it is sufficient to complete the representation of the algebra $AE(1) \oplus AE(3)$ obtained in Lemma 1 by dilatation operators of the form (1) and to verify that the commutation relations (5) or (6) are true.

2. Consider the problem of description of second-order partial differential equations of the most general form

$$F\left(x, u, \underset{1}{u}, \underset{2}{u}\right) = 0, \tag{15}$$

invariant with respect to the obtained realizations of the algebras L and K in the class of the first-order differential operators.

As is known [1], this problem can be reduced to obtaining the differential invariants of the given algebras, i.e., solving the following first-order partial differential equations:

$$\underset{2}{X_i}\,\Psi\left(x_0, x, u, \underset{1}{u}, \underset{2}{u}\right) = 0, \quad i = 1, \ldots, N,$$

where $\underset{2}{X_i}$ $(i = 1, \ldots, N)$ are basis operators of the algebras L or K.

Lemma 2. *The functions*

$$\begin{array}{lll} u, & u_0, & u_{00} \\ S_1 = u_a u_a, & S_2 = u_{aa}, & S_3 = u_a u_{ab} u_b, \\ S_4 = u_{ab}u_{ab}, & S_5 = u_a u_{ab} u_{bc} u_c, & S_6 = u_{ab}u_{bc}u_{ca}, \\ S_7 = u_a u_{0a}, & S_8 = u_{0a}u_{0a}, & S_9 = u_{0a}u_{0b}u_{ab} \end{array} \tag{16}$$

form the fundamental system of second–order differential invariants of the algebra $AE(1) \oplus AE(3)$. Here, $a, b, c = 1, 2, 3$, we mean summation from 1 to 3 over the repeated indices, $u_a = \dfrac{\partial u}{\partial x_a}$, $u_{ab} = \dfrac{\partial^2 u}{\partial x_a \partial x_b}$.

The lemma is proved in the same way as it is done in the paper by Fushchych and Yegorchenko [4].

Theorem 3. *The functions Λ_j $(j = 1, 2, \ldots, 10)$ given below form the basis of the fundamental system of differential invariants of the second-order of the algebra L:*

1) $\Lambda_1 = u_{00}u_0^{-2}$, $\Lambda_2 = S_2S_1^{-1}$, $\Lambda_3 = S_3S_1^{-2}$, $\Lambda_4 = S_4S_1^{-2}$, $\Lambda_5 = S_5S_1^{-3}$, $\Lambda_6 = S_6S_1^{-3}$, $\Lambda_7 = S_7u_0^{-1}S_1^{-1}$, $\Lambda_8 = S_8u_0^{-2}S_1^{-1}$, $\Lambda_9 = S_9u_0^{-2}S_1^{-2}$, $\Lambda_{10} = u$,

if the generators D_1, D_2 are of the form (8).

2) $\Lambda_1 = S_1u^{-1}$, $\Lambda_2 = S_2$, $\Lambda_3 = S_3u^{-1}$, $\Lambda_4 = S_4$, $\Lambda_5 = S_5u^{-1}$, $\Lambda_6 = S_6$, $\Lambda_7 = S_7u_0^{-1}$, $\Lambda_8 = S_8uu_0^{-2}$, $\Lambda_9 = S_9uu_0^{-2}$, $\Lambda_{10} = u_{00}uu_0^{-2}$,

if the generators D_1, D_2 are of the form (9).

3) $\Lambda_1 = u^{4-2k}u_0^{-2}S_1^k$, $\Lambda_2 = u^2u_0^{-2}S_2^k$, $\Lambda_3 = u^{8-3k}u_0^{-4}S_3^k$, $\Lambda_4 = u^{8-2k}u_0^{-4}S_4^k$, $\Lambda_5 = u^{12-4k}u_0^{-6}S_5^k$, $\Lambda_6 = u^{12-3k}u_0^{-6}S_6^k$, $\Lambda_7 = u^{-k+4}u_0^{-k-2}S_7^k$, $\Lambda_8 = u^4u^{-2k-2}S_8^k$, $\Lambda_9 = u^{-k+8}u^{-2k-2}S_9^k$, $\Lambda_{10} = uu_0^{-2}u_{00}$,

if the generators D_1, D_2 are of the form (10).

4) $\Lambda_1 = u^{-3}u_{00}$, $\Lambda_2 = uS_2S_1^{-1}$, $\Lambda_3 = uS_3S_1^{-2}$, $\Lambda_4 = u^2S_4S_1^{-2}$, $\Lambda_5 = u^2S_5S_1^{-3}$, $\Lambda_6 = u^3S_6S_1^{-3}$, $\Lambda_7 = u^{-1}S_7S_1^{-1}$, $\Lambda_8 = u^{-2}S_8S_1^{-1}$, $\Lambda_9 = u^{-1}S_9S_1^{-2}$, $\Lambda_{10} = u^{-2}u_0$,

if the generators D_1, D_2 are of the form (11).

5)
$$\begin{aligned}
&\Lambda_1 = u_0^{-6}u_{00}^2\exp(2u_0^{-1})S_1,\\
&\Lambda_2 = u_0^{-6}u_{00}\exp(2u_0^{-1})[u_{00}S_1 + u_0^2S_2 - 2u_0S_7],\\
&\Lambda_3 = u_0^{-14}u_{00}^3\exp(4u_0^{-1})[u_{00}S_1^2 - 2u_0S_1S_7 + u_0^2S_3],\\
&\Lambda_4 = u_0^{-12}u_{00}^2\exp(4u_0^{-1})[2u_0^2(S_1S_8 + S_7^2) + u_{00}^2S_1^2 +\\
&\qquad u_0^4S_4 - 4u_0u_{00}S_1S_7 - 4u_0^3S_1^{-1}S_3S_7 + 2u_0^2u_{00}S_3],\\
&\Lambda_5 = u_0^{-12}u_{00}^4\exp(6u_0^{-1})[u_0^{-4}S_5 - 4u^{-5}S_3S_7 +\\
&\qquad u_0^{-6}(3S_1S_7^2 + S_1^2S_8 + 2u_{00}S_1S_3) - 4u_0^{-7}u_{00}S_1^2S_7 + u_0^{-8}u_{00}^2S_1^3,\\
&\Lambda_6 = u_0^{-18}u_{00}^3\exp(6u_0^{-1})[u_0^6S_6 - 2u_0^5(S_3^2S_7S_1^{-2} + 2S_5S_7S_1^{-1}) +\\
&\qquad 3u_0^4(S_1S_9 + S_3S_8 + 2S_7^2S_3S_1^{-1} + u_{00}S_5) - 2u_0^3(3S_1S_7S_8 + S_7^3 + 6u_{00}S_3S_7) +\\
&\qquad 3u_0^2u_{00}(S_1^2S_8 + 3S_1S_7^2 + u_{00}S_1S_3) - 6u_0u_{00}^2S_1^2S_7 + u_{00}^3S_1^3],\\
&\Lambda_7 = u_0^{-7}u_{00}\exp(2u_0^{-1})(u_{00}S_1 - u_0S_7),\\
&\Lambda_8 = u_0^{-6}\exp(2u_0^{-1})(u_{00}^2S_1 - 2u_0u_{00}S_7 + u_0^2S_8),\\
&\Lambda_9 = u_0^{-12}u_{00}\exp(4u_0^{-1})[u_0^4S_9 - 2u_0^3(S_7S_8 + u_{00}S_3S_7S_1^{-1}) +\\
&\qquad u_0^2(3u_{00}S_7^2 + 2u_{00}S_1S_8 + u_{00}^2S_3) - 4u_0u_{00}^2S_1S_7 + u_{00}^3S_1^2],\\
&\Lambda_{10} = uu_0^{-3}u_{00},
\end{aligned}$$

if the generators D_1, D_2 are of the form (12).

Here $S_1, S_2, \dots, S_9$ are of the form (16).

Theorem 4. *The functions Ω_j $(j = 1, 2, \dots, 11)$ given below form the basis of the fundamental system of differential invariants of the second-order of the algebra K:*

1) $\Omega_1 = S_1^ku_0^{-2}$, $\Omega_2 = S_2^ku_0^{-2}$, $\Omega_3 = S_3^ku_0^{-4}$, $\Omega_4 = S_4^ku_0^{-4}$, $\Omega_5 = S_5^ku_0^{-6}$, $\Omega_6 = S_6^ku_0^{-6}$, $\Omega_7 = S_7^ku_0^{-(2+k)}$, $\Omega_8 = S_8^ku_0^{-2(1+k)}$, $\Omega_9 = S_9^ku_0^{-2(k+2)}$, $\Omega_{10} = u_{00}u_0^{-2}$, $\Omega_{11} = u$,

if the generator D is of the form (13).

2) $\Omega_1 = S_1$, $\Omega_2 = S_2u$, $\Omega_3 = S_3u$, $\Omega_4 = S_4u^2$, $\Omega_5 = S_5u^2$, $\Omega_6 = S_6u^3$, $\Omega_7 = S_7u^k$, $\Omega_8 = S_8u^{2k}$, $\Omega_9 = S_9u^{2k+1}$, $\Omega_{10} = u_{00}u^{2k-1}$, $\Omega_{11} = u_0u^{k-1}$,

if the generator D is of the form (14).

It follows from Theorems 3, 4 that for the case of the algebra L (15), reads as

$$F(\Lambda_1, \Lambda_2, \dots, \Lambda_{10}) = 0,$$

and for the case of the algebra K, (15) takes the form

$$\Phi(\Omega_1, \Omega_2, \dots, \Omega_{11}) = 0.$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 211–213.

To Separation of Variables in a (1+2)-Dimensional Fokker-Planck Equation

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Abstract

The problem of separation of variables in a (1+2)-dimensional Fokker-Planck equation is considered. For the case of constant coefficients of second derivatives, the classification of coordinate systems, where the variables in the Fokker-Planck equation are separable, is made and new coordinate systems for one class of this equation, which provide a separation of variables, are obtained.

Let us consider the Fokker-Planck equation

$$u_t = (d_{ij}u_{x_j})_{x_i} - (a_iu)_{x_i} \qquad i,j=\overline{1,n}.$$

Here, $u_t = \dfrac{\partial u}{\partial t}$ and $u_{x_i} = \dfrac{\partial u}{\partial x_i}$. Coefficients d_{ij} and a_i depend on $t, \vec{x}$, and d_{ij} are non-negative definite quadratic form coefficients. The summation is done over indeices i and j. If $d_{ij} = \text{const}$, then the two-dimensional Fokker-Planck equation is reduced to

$$u_t = u_{x_1x_1} + u_{x_2x_2} - (a_1u)_{x_1} - (a_2u)_{x_2} \tag{1}$$

by a linear change of variables.

To separate variables in the given equation means to find new variables $\omega_0, \omega_1, \omega_2$ and a function $Q(t, x_1, x_2)$ such that equation (1) splits after the substitution of

$$u = Q(t, x_1, x_2)\varphi_0(\omega_0, \lambda_1, \lambda_2)\varphi_1(\omega_1, \lambda_1, \lambda_2)\varphi_2(\omega_2, \lambda_1, \lambda_2) \tag{2}$$

into three ordinary differential equations for the functions $\varphi_0, \varphi_1, \varphi_2$. Here, λ_1, λ_2 are separation constants.

A method of classification of coordinate systems, which allow a separation of variables for the Schrödinger equation, is realized in [1, 2]. It allows one also to obtain explicitly potentials for which a separation of variables is possible. Let us use this method for the classification of coordinate systems, which allow us to separate variables in equation (1). Let

$$\omega_0 = t, \quad \omega_i(t, x_1, x_2), \quad i = 1, 2,$$

and suppose that solution (2) and

$$u' = Q'\varphi_0'(t, \lambda_1', \lambda_2')\varphi_1'(\omega_1', \lambda_1', \lambda_2')\varphi_2'(\omega_2', \lambda_1', \lambda_2')$$

are equivalent if

$$Q' = Q\Phi_1(\omega_1)\Phi_2(\omega_2), \quad \omega_i' = \Omega_i(\omega_i), \quad \lambda_i' = \Lambda_i(\lambda_1, \lambda_2), \qquad i = 1, 2. \tag{3}$$

Substituting (2) to (1), splitting the obtained equation with respect to φ_0, φ_1, φ_2, $\dfrac{d\varphi_1}{d\omega_1}$, $\dfrac{d\varphi_2}{d\omega_2}$, λ_1, λ_2, and considering the relations of equivalency (3), we have that (1) splits into the system of three linear ordinary differential equations

$$\begin{aligned}
&\frac{d\varphi_0}{dt} = (\lambda_1 r_1(t) + \lambda_2 r_2(t) + r_0(t))\varphi_0,\\
&\frac{d^2\varphi_1}{d\omega_1^2} = (\lambda_1 b_{11}(\omega_1) + \lambda_2 b_{21}(\omega_1) + b_{01}(\omega_1))\varphi_1\\
&\frac{d^2\varphi_2}{d\omega_2^2} = (\lambda_1 b_{12}(\omega_2) + \lambda_2 b_{22}(\omega_2) + b_{02}(\omega_2))\varphi_2,
\end{aligned} \tag{4}$$

where

$$\operatorname{rank}\begin{pmatrix} r_1 & r_2\\ b_{11} & b_{21}\\ b_{12} & b_{22}\end{pmatrix} = 2$$

and the functions ω_1, ω_2, Q, a_1, a_2 satisfy the system of equations

$$\omega_{1x_1}\omega_{2x_1} + \omega_{1x_2}\omega_{2x_2} = 0, \tag{5}$$

$$b_{i1}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + b_{i2}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) + r_1(t) = 0, \quad i = 1, 2, \tag{6}$$

$$Q\omega_{it} - Q\triangle\omega_i - 2Q_{x_1}\omega_{ix_1} - 2Q_{x_2}\omega_{ix_2} + a_1 Q\omega_{ix_1} + a_2 Q\omega_{ix_2} = 0, \quad i = 1, 2, \tag{7}$$

$$\begin{aligned}
&Q_t - \triangle Q - Qb_{01}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) - Qb_{02}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) -\\
&\quad - Qr_0(t) + Q_{x_1}a_1 + Q_{x_2}a_2 + Q(a_{1x_1} + a_{2x_2}) = 0.
\end{aligned} \tag{8}$$

While solving equations (5) and (6), we define coordinate systems providing a separation of variables in equation (1), determined up to the equivalency relation (3). Substituting the obtained expressions for ω_1, ω_2 into equation (7), we have functions $Q(t, x_1, x_2)$ explicitly for each of the found coordinate systems and also the condition, imposed on a_1, a_2 for a providing of system compatibility. The substitution of the obtained results into (8) gives the equation for coefficients a_1, a_2. If we succeed in solution of this equation, we obtain the functions a_1, a_2, with which the variables in equation (1) separate in the corresponding coordinate systems. Here, we give the list of coordinate systems, which provide the separation of variables in equation (1) and the functions $Q(t, x_1, x_2)$ explicitly, which correspond to these variables.

1) $\omega_1 = A(t)(x_1\cos\alpha t + x_2\sin\alpha t) + W_1(t)$,
$\omega_2 = B(t)(x_1\sin\alpha t - x_2\cos\alpha t) + W_2(t)$,

$$\begin{aligned}
&Q(t, x_1, x_2) = \exp\left(\frac{\dot A}{4A}(x_1^2\cos^2\alpha t + x_2^2\sin^2\alpha t) + \frac{\dot B}{4B}(x_1^2\sin^2\alpha t + x_2^2\cos^2\alpha t)\right)\times\\
&\times\exp\left(\frac{1}{2}\left(\frac{\dot A}{A} - \frac{\dot B}{B}\right)x_1x_2\sin\alpha t\cos\alpha t + \frac{\dot W_1}{2A}(x_1\cos\alpha t + x_2\sin\alpha t)\right)\times\\
&\times\exp\left(\frac{\dot W_2}{2B}(x_1\sin\alpha t - x_2\cos\alpha t) + \frac{F}{2}\right).
\end{aligned}$$

2) $x_1 = W(t)e^{\omega_1}\cos(\omega_2+\alpha t)+W_1(t), \quad x_2 = W(t)e^{\omega_1}\sin(\omega_2+\alpha t)+W_2(t),$
$Q(t,x_1,x_2) = R(t,x_1,x_2)$

3) $$x_1 = W(t)\left(\omega_1\omega_2\cos\alpha t+\frac{1}{2}(\omega_2^2-\omega_1^2)\sin\alpha t\right)+W_1(t),$$
$$x_2 = W(t)\left(\omega_1\omega_2\sin\alpha t+\frac{1}{2}(\omega_2^2-\omega_1^2)\cos\alpha t\right)+W_2(t),$$
$$Q(t,x_1,x_2) = R(t,x_1,x_2).$$

4) $$x_1 = W(t)(\cosh\omega_1\cos\omega_2\cos\alpha t+\sinh\omega_1\sin\omega_2\sin\alpha t)+W_1(t),$$
$$x_2 = W(t)(\sinh\omega_1\sin\omega_2\cos\alpha t-\cosh\omega_1\cos\omega_2\sin\alpha t)+W_2(t),$$
$$Q(t,x_1,x_2) = R(t,x_1,x_2).$$

Here, $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, and $W_2(t)$ are arbitrary sufficiently smooth functions,

$$\frac{\partial F}{\partial x_1} = \alpha x_2+a_1, \qquad \frac{\partial F}{\partial x_2} = -\alpha x_1+a_2, \qquad a_{1x_2}-a_{2x_1} = -2\alpha,$$

$$R(t,x_1,x_2) = \exp\left(-\frac{\dot W}{4W}((x_1-W_1)^2+(x_2-W_2)^2)-\frac{1}{2}\dot W_1x_1-\frac{1}{2}\dot W_2x_2\right)\times$$
$$\times\exp\left(-\frac{\alpha}{2}W_2x_1+\frac{\alpha}{2}W_1x_2+\frac{F}{2}\right).$$

We note that equation (1) reduces for $a_{2x_1}-a_{1x_2}=0$ to the heat equation with potential (see [2]).

As an example , let us consider the equation:

$$u_t = u_{x_1x_1}+u_{x_2x_2}-((a_{11}x_1+a_{12}x_2)u)_{x_1}-((a_{21}x_1+a_{22}x_2)u)_{x_2}. \tag{9}$$

If $a_{12}=a_{21}$ ($\alpha=0$), it is reduced [2] to heat equation and provides the separation of variables in five coordinate systems. Equation (9) proves to allow the separation of variables also when $a_{12}=-a_{21}$, $a_{11}=a_{22}$. In this case, four coordinate systems are obtained, in which equation (9) is separated:

$$\omega_1 = \begin{cases} (\sin(a_{11}(t+C_1)))^{-1}(x_1\cos\alpha t+x_2\sin\alpha t)+C_2(\sin(a_{11}(t+C_1)))^{-2}, \\ (C_3+\sin(2a_{11}(t+C_1)))^{-1/2}(x_1\cos\alpha t+x_2\sin\alpha t), \end{cases}$$

$$\omega_2 = \begin{cases} (\sin(a_{11}(t+C_4)))^{-1}(x_1\sin\alpha t-x_2\cos\alpha t)+C_5(\sin(a_{11}(t+C_4)))^{-2}, \\ (C_6+\sin(2a_{11}(t+C_4)))^{-1/2}(x_1\sin\alpha t-x_2\cos\alpha t), \end{cases}$$

$C_1,\ldots,C_6$ are arbitrary real constants.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 214–216.

On Exact Solutions of the Lorentz-Dirac-Maxwell Equations

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Abstract

Exact solulions of the Lorentz-Dirac-Maxwell equations are constructed.

Motion of a classical spinless particle moving in electromagnetic field is described by the system of ordinary differential equations (Lorentz-Dirac) and partial differential equations (Maxwell) [1]:

$$m\dot{u}_\mu = eF_{\mu\nu}u^\nu + \frac{2}{3}e^2\left(\ddot{u}_\mu + u_\mu\dot{u}_\nu\dot{u}^\nu\right), \tag{1}$$

where $F_{\mu\nu} = \dfrac{\partial A_\nu}{\partial x^\mu} - \dfrac{\partial A_\mu}{\partial x^\nu}$ is the tensor of electromagnetic field

$$\frac{\partial F_{\mu\nu}}{\partial x^\nu} = 0, \qquad \frac{\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}}{\partial x^\nu} = 0, \tag{2}$$

$$u_\mu \equiv \dot{x}_\mu = \frac{dx_\mu}{d\tau}, \qquad u_\mu u^\mu = 1. \tag{3}$$

Some exact solutions of system (1), (2) can be found in [2].

In the present paper, we have obtained new classes of exact solutions of the Lorentz-Dirac-Maxwell system using $P(1,3)$ symmetry properties of (1), (2).

1. Let us show that if system (1) is invariant with respect to the algebra

$$\langle\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_3}, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\rangle, \tag{4}$$

then its particular solutions can be looked for in the form

$$x_1 = a\tau, \qquad x_1 = R\cos d\tau, \qquad x_2 = R\sin d\tau, \qquad x_3 = b\tau, \tag{5}$$

where a, b, d, R are constants.

Indeed, equation (1) admits algebra (4) if and only if

$$\begin{aligned}
&\frac{\partial\vec{E}}{\partial x_0} = \frac{\partial\vec{H}}{\partial x_0} = \frac{\partial\vec{E}}{\partial x_3} = \frac{\partial\vec{H}}{\partial x_3} = 0,\\
&\frac{\partial E_a}{\partial x_2}x_1 - \frac{\partial E_a}{\partial x_1}x_2 = \delta_{a2}E_1 - \delta_{a1}E_2,\\
&\frac{\partial H_a}{\partial x_2}x_1 - \frac{\partial H_a}{\partial x_1}x_2 = \delta_{a2}H_1 - \delta_{a1}H_2.
\end{aligned} \tag{6}$$

The general solutions of equations (6) read:

$$\begin{aligned} &H_1 = f_1x_1 + f_2x_2, \qquad H_2 = f_1x_2 - f_2x_1, \qquad H_3 = f_5,\\ &E_1 = f_3x_1 + f_4x_2, \qquad E_2 = f_3x_2 - f_4x_1, \qquad E_3 = f_6, \end{aligned} \tag{7}$$

where $f_i = f_i(x_1^2 + x_2^2)$.

Substituting (7) into (2), we find functions f_i

$$f_i = \frac{\lambda_i}{x_1^2 + x_2^2}, \quad i = \overline{1,4}, \qquad f_i = \lambda_i, \quad i = \overline{5,6}. \tag{8}$$

Substituting expessions (5) into (1), where $F_{\mu\nu}$ is given by (7), (8), we obtain the additional condition for the constants a, b, d, R, λ_i:

$$\begin{aligned} &a^2 - b^2 - R^2d^2 = 1,\\ &aR^2d^4 + \frac{3}{2e}\{\lambda_6 b - \lambda_4 d\} = 0,\\ &R^2d^3 = -R^4d^5 + \frac{3}{2e}\{a\lambda_4 - b\lambda_1\},\\ &\frac{m}{e}R^2d^2 + a\lambda_3 + b\lambda_2 + R^2d\lambda_5 = 0. \end{aligned} \tag{9}$$

2. To construct another class of exact solutions of equations (1), (2), we require the invariance of (1) with respect to the algebra

$$\langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + \alpha\frac{\partial}{\partial x_0}\rangle, \qquad \alpha \neq 0. \tag{10}$$

In this case, solutions (1) can be looked for in the form:

$$x_0 = a\tau, \qquad x_1 = R\cos\frac{a\tau}{\alpha}, \qquad x_2 = R\sin\frac{a\tau}{\alpha}, \qquad x_3 = 0, \tag{11}$$

where a, R, α are constants.

Indeed, a requirement of the invariance of (1) with respect to algebra (10) yields the following form of the functions E_i, H_i:

$$\begin{aligned} &E_1 = f_1\cos\frac{x_0}{\alpha} + f_2\sin\frac{x_0}{\alpha}, \qquad E_2 = f_2\sin\frac{x_0}{\alpha} - f_1\cos\frac{x_0}{\alpha}, \qquad E_3 = f_6,\\ &H_1 = f_3\cos\frac{x_0}{\alpha} + f_4\sin\frac{x_0}{\alpha}, \qquad H_2 = f_3\sin\frac{x_0}{\alpha} - f_4\cos\frac{x_0}{\alpha}, \qquad H_3 = f_5, \end{aligned} \tag{12}$$

where $f_i = f_i(x_3)$.

The electromagnetic field $\{E_i, H_i\}$ (12) satisfies the Maxwell equations if the functions f_i are of the form

$$\begin{aligned} &f_1 = \lambda_1\cos\frac{x_3}{\alpha} + \lambda_2\sin\frac{x_3}{\alpha}, \qquad f_2 = \lambda_3\cos\frac{x_3}{\alpha} + \lambda_4\sin\frac{x_3}{\alpha},\\ &f_3 = -\lambda_1\sin\frac{x_3}{\alpha} + \lambda_2\cos\frac{x_3}{\alpha}, \qquad f_4 = -\lambda_3\sin\frac{x_3}{\alpha} + \lambda_4\cos\frac{x_3}{\alpha},\\ &f_5 = \lambda_5, \qquad f_6 = \lambda_6, \end{aligned} \tag{13}$$

where $\lambda_i = \text{const}$.

After substituting expressions (11) into (2), where E_i, H_i are determined from (12), (13), we obtain the following expressions for constants:

$$a^2\left(1-\frac{R^2}{\alpha^2}\right)=1, \qquad \lambda_5-\frac{R}{\alpha}\lambda_3, \qquad m\frac{a}{\alpha^2}R=e\lambda_1,$$
$$R\left(\frac{a}{\alpha}\right)^3+\left(\frac{a}{\alpha}\right)^5R^5+\frac{3}{2e}a\lambda_2=0. \tag{14}$$

Thus, exact solutions of (1), (2) are given by formulae (11)–(14).

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 217–221.

Evolution Equations Invariant under the Conformal Algebra

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Abstract

The evolution equation of the following form $u_0 = f(u, \triangle u)$ which is invariant under the conformal algebra are investigated. The symmetry property of this equation is used for the construction of its exact solutions.

Analyzing the symmetry of the nonlinear heat equation

$$u_0 = \partial_1(u_1 f(u)) \tag{1}$$

in paper [1], L. Ovsyannikov showd that, in case

$$F(u) = \lambda u^{-\frac{4}{3}},$$

the widest invariance algebra consists of the operators:

$$\langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 4x_0\partial_0 + 3u\partial_u, D_2 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u\rangle$$

We shall consider a generalization of equation (1) in the form:

$$u_0 = F(u, u_{11}) \tag{2}$$

and define functions F with which equation (2) is invariant with respect to the algebra:

$$A = \langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u\rangle. \tag{3}$$

Theorem 1. *Equation (2) is invariant with respect to algebra (3) under the condition:*

$$F(u, u_{11}) = uf(u^3 u_{11}), \tag{4}$$

where f is an arbitrary function.

Proof. Coordinates ξ^0, ξ^1, η of the infinitesimal operator

$$X = \xi^0\partial_0 + \xi^a\partial_a + \eta\partial_u,$$

of algebra (3) have a form

$$\xi^0 = 0, \quad \xi^1 = x_1^2, \quad \eta = x_1 u,$$

Using the invariance condition of equation (2) with respect to algebra (3) (see, for example, [2]) we have:

$$u\frac{\partial F}{\partial u} - 3u_{11}\frac{\partial F}{\partial u_{11}} = F. \tag{5}$$

The general solution of equation (5) has the form (4). Theorem is prooved.

When the function F is given by formula (4), equation (2) has the form:

$$u_0 = uf(u^3 u_{11}). \tag{6}$$

We research the Lie's symmetry of equation (6).

Theorem 2. *Equation (6) is invariant with respect to the algebra:*

$$1.\quad A = \langle \partial_0 = \frac{\partial}{\partial x_0}, \partial_1 = \frac{\partial}{\partial x_1}, D_1 = 2x_1\partial_1 + u\partial_u, K = x_1^2\partial_1 + x_1 u\partial_u\rangle \tag{7}$$

if f is an arbitrary smooth function,

$$2.\quad A_1 = \langle A, D_2 = -\frac{4}{3}x_0\partial_0 + u\partial_u, G = u\partial_1, K_1 = x_1 u\partial_1 + u^2\partial_u\rangle \tag{8}$$

if $f = \lambda_1 u u_{11}^{\frac{1}{3}}$, ($\lambda_1$ is an arbitrary constant),

$$3.\quad A_2 = \langle A, Q_1 = e^{-\frac{4}{3}\lambda_2 x_0}\partial_1, Q_2 = e^{-\frac{4}{3}\lambda_2 x_0}(\partial_0 + \lambda_2 u\partial_u)\rangle, \tag{9}$$

if $f = \lambda_1 u u_{11}^{\frac{1}{3}} + \lambda_2$, ($\lambda_2$ is an arbitrary constant $\lambda_2 \neq 0$),

Theorem 2 is proved by the standard Lie's method [2].

Remark 1. *From results of Theorem 2, it is follows that equation (6) has the widest symmetry in case where it has a form:*

$$u_0 = \lambda u^2 u_{11}^{\frac{1}{3}}. \tag{10}$$

We use the symmetry of equation (10) to find its exact solutions. The invariant solutions of this equation have such a form:

$$W = \varphi(\omega), \tag{11}$$

where ω, W are first integrals of the system of differential equations:

$$\begin{cases} \dot{x_0} = 4\kappa_1 x_0 + d_0, \\ \dot{x_1} = a_1 x_1^2 + (a_2 x_1 + gu + 2\kappa x_1 + d_1, \\ \dot{u} = a_1 x_1 u + a_2 u^2 + (\kappa_2 - 3\kappa_1)u; \end{cases} \tag{12}$$

Depending on values of the parameters $\kappa_1, \kappa_2, d_0, d_1, a_1, a_2, g$, we receive different solutions system (12).

Substituting anzatses built with the help of invariants of algebra (8) in equation (10), we obtain reduced equations. The solutions of system (12), corresponding anzatzes, and reduced equations are given in Table 1.

Remark 2. *All the received results can be generated by the formulae:*

$$\begin{cases} x_0' = e^{-\frac{4}{3}\theta_2} x_0 + \theta_1 \\ x_1' = \dfrac{e^{2\theta_3} x_1 + \theta_7 e^{\theta_2} u + \theta_5}{1 + \theta_5\theta_4 x_1 e^{2\theta_3} + \theta_6 e_2^{\theta} u + \theta_4} \\ u' = \dfrac{e^{\theta_3+\theta_2} u}{1 + \theta_5\theta_4 x_1 e^{2\theta_3} + \theta_6 e_2^{\theta} u + \theta_4}, \end{cases} \tag{13}$$

where $\theta_1, \ldots, \theta_7$ are arbitrary constants. The way of generation is described in [3].

We shall consider the case of variables $x = (x_0, \vec{x}) \in \mathrm{R}_{1+n}$. We generalize equation (1) in such a way

$$u_0 = F(u, \triangle u), \tag{14}$$

and the operator K is replaced by the conformal operators

$$K_a = 2x_a D - \vec{x}^2 \partial_a, \tag{15}$$

where $D = x_b\partial_b + k(u)\partial_u$, $k = k(u)$ is an arbitrary smooth function.

Theorem 3. *Equation (3) is invariant with respect to the conformal algebra $AC(n)$ with the basic operators*

$$\begin{aligned} &1.\ \ \partial_a = \frac{\partial}{\partial x_a}, \quad J_{ab} = x_a\partial_b - x_b\partial_a, \quad D = x_a\partial_a + \frac{2-n}{2} u\partial_u, \\ &K_a = 2x_a D - \vec{x}^2\partial_a; \quad a, b = 1, \ldots, n, \end{aligned} \tag{16}$$

if

$$F = u\Phi(u^{\frac{2+n}{2-n}} \triangle u), \qquad n \neq 2. \tag{17}$$

$$2.\ \ \partial_a, \quad J_{12} = x_1\partial_2 - x_2\partial_1, \quad D = x_a\partial_a + 2\partial_u, K_a = 2x_a D - \vec{x}^2\partial_a; \quad a = 1, 2, \tag{18}$$

if

$$F = \Phi(e^u \triangle u), \qquad n = 2. \tag{19}$$

Proof. Using the invariance condition of equation (14) with respect to operators (15) to find function F, we have the following system

$$\begin{cases} k'F - kF_1 - (k' - 2)\triangle u F_2 = 0, \\ (2k' + n - 2)F_2 = 0. \end{cases} \tag{20}$$

From the second equation of system (20), we receive

$$k = \frac{2-n}{2} u, \quad n \neq 2, \tag{21}$$

$$k = \mathrm{const}, \quad n = 2. \tag{22}$$

Table 1.

N	ω	Anzatzes	Reduced equations
1	x_0	$u = \varphi(\omega)$	$\varphi' = 0$
2	x_0	$u = x_1\varphi(\omega)$	$\varphi' = 0$
3	x_0	$u = \sqrt{x_1}\varphi(\omega)$	$\varphi' = -\frac{\lambda}{\sqrt[3]{4}}\varphi^{\frac{7}{3}}$
4	x_0	$u = \sqrt{x_1^2+1}\varphi(\omega)$	$\varphi' = \lambda\sqrt[3]{k^2}\varphi^{\frac{7}{3}}$
5	x_0	$u = \sqrt{x_1^2-1}\varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{k^2}\varphi^{\frac{7}{3}}$
6	x_0	$\frac{x_1}{u} + \frac{k}{u^2} = \varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{2k}$
7	$x_1 + mx_0$	$u = \varphi(\omega)$	$\varphi' = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
8	$\frac{1}{x_1} + mx_0$	$u = x_1\varphi(\omega)$	$\varphi' = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
9	$\ln x_1 + mx_0$	$u = \sqrt{x_1}\varphi(\omega)$	$\varphi' = \lambda\varphi^2\left(\varphi'' - \frac{1}{4}\varphi\right)^{\frac{1}{3}}$
10	$\mathrm{arctg}\, x_1 + mx_0$	$u = \sqrt{x_1^2+1}\varphi(\omega)$	$\varphi' = -\lambda\sqrt[3]{k}\varphi^2(\varphi'' + \varphi)^{\frac{1}{3}}$
11	$\mathrm{arcth}\, x_1 + mx_0$	$u = \sqrt{x_1^2-1}\varphi(\omega)$	$\varphi' = \lambda\varphi^2\left(\varphi'' - \frac{1}{4}\varphi\right)^{\frac{1}{3}}$
12	$\frac{1}{u} + mx_0$	$\frac{x_1}{u} + \frac{k}{u^2} = \varphi(\omega)$	$m\varphi' = \lambda\varphi(\varphi\varphi'' - 2\varphi')^{\frac{1}{3}}$
13	$\frac{1}{u} + mx_0$	$\frac{u}{x_1} = \varphi(\omega)$	$m\varphi' = -\lambda(2k - \varphi'')^{\frac{1}{3}}$
14	$x_1 + kx_0u + mx_0$	$u = \varphi(\omega)$	$k_1 + m\omega = -\lambda\omega^2\varphi''^{\frac{1}{3}}$
15	$x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
16	$\frac{1}{x_1} + m\ln x_0$	$u = x_0^{-\frac{3}{4}}x_1\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\varphi''^{\frac{1}{3}}$
17	$\ln x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda\varphi^2\left(-\frac{1}{4}\varphi - \varphi''\right)^{\frac{1}{3}}$
18	$\mathrm{arctg}\, x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1^2+1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda k^{\frac{2}{3}}\varphi^2(\varphi + \varphi'')^{\frac{1}{3}}$
19	$\mathrm{arcth}\, x_1 + m\ln x_0$	$u = x_0^{-\frac{3}{4}}\sqrt{x_1^2-1}\varphi(\omega)$	$m\varphi' - \frac{3}{4}\varphi = \lambda k^{\frac{2}{3}}\varphi^2(\varphi'' - \varphi)^{\frac{1}{3}}$
20	$\frac{1}{u} + m\ln x_0$	$x_0^{-\frac{3}{2}}x_1u^{-1} = \varphi(\omega)$	$\varphi = \frac{2}{3}\lambda\omega(2\varphi' + \omega\varphi'')^{\frac{1}{3}}$
21	$\frac{x_1}{u} + m\ln x_0$	$x_0^{-\frac{3}{2}}u^{-1} = \varphi(\omega)$	$m - \frac{3}{2}\omega\varphi' = -\lambda\omega(\omega\varphi'' + 2\varphi')^{\frac{1}{3}}$

Substituting (21) and (22) into the first equation of system (20), we have

$$1)\ \ F - uF_1 + \frac{2+n}{2-n}\triangle uF_2 = 0, \qquad \text{if} \quad n \neq 2. \tag{23}$$

$$2)\ \ -kF_1 + 2\triangle uF_2 = 0, \qquad \text{if} \quad n = 2. \tag{24}$$

Without restricting the generality, we assume $k = 2$. Solving equations (23) and (24), we obtain formulas (17), (19), and algebra (16), (18). Theorem is proved.

The following theorems are proved by the standard Lie's method.

Theorem 4. *The widest Lie's algebra of invariance of equation (17) for $n \neq 2$ consists of the operators:*

1. *(16), ∂_0, when Φ is an arbitrary smooth function;*
2. *(16), ∂_0, $D_0 = 2mx_0\partial_0 + x_a\partial_a$, when $\Phi(w) = \lambda w^m$, (λ, m are arbitrary constants).*

Theorem 5. *The widest Lie's algebra of invariance of equation (19) for $n = 2$ consists of the operators:*

1. *(18), ∂_0, when Φ is an arbitrary smooth function;*
2. *(18), ∂_0, $D_0 = 2mx_0\partial_0 + x_a\partial_a$, when $\Phi(w) = \lambda w^m$, (λ, m are arbitrary constants).*

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 222–226.

Generalization of Translation Flows of an Ideal Incompressible Fluid: a Modification of the "Ansatz" Method

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Abstract

New exact solutions of the Euler equations describing flows of an ideal homogeneous incompressible fluid are obtained by means of a modification of the "ansatz" method.

In recent years, several new methods for finding exact solutions of the partial differential equations have been developed. Often these methods are generalizations of older ones and are reduced to either appending additional differential equations (the method of differential constraints, side conditions, conditional symmetry, and so on) or to assuming a general form for the solution (the "ansatz" method called often the direct method, and the generalization of the usual "separation of variables" technique). Both approaches are closely connected with each other.

In our paper [1], the Euler equations (the EEs)

$$\vec{u}_t + (\vec{u}\cdot\nabla)\vec{u} + \nabla p = \vec{0}, \quad \mathrm{div}\,\vec{u} = 0 \tag{1}$$

which describe flows of an ideal homogeneous incompressible fluid were considered with the following additional condition:

$$u^1_1 = u^3 = 0. \tag{2}$$

All the solutions of system (1)–(2) were found. They can be interpreted as a particular case of translation flows. In this paper, we construct more general classes of exact solutions for EEs (1) by means of a modification of the "ansatz" method.

Let us transform the variables in (1):

$$\begin{aligned} &\vec{u} = O(t)\vec{w}(\tau,\vec{y}) - O(t)O_t^T(t)\vec{x}, \quad p = q(\tau,\vec{y}) + \frac{1}{2}|O_t^T(t)\vec{x}|^2,\\ &\tau = t, \quad \vec{y} = O^T(t)\vec{x}, \end{aligned} \tag{3}$$

where $O = O(t)$ is an orthogonal matrix function depending on t, i.e., transformation (3) defines time-depending space rotation. It can be noted that the non-Lie invariance of hydrodynamics equations under transformations of the type (3) was investigated, for instance, in [2, 3]. As a result of transformation (3), we obtain equations in new unknown functions $\vec{w}$ and q and new independent variables τ and $\vec{y}$:

$$\vec{w}_\tau + (\vec{w}\cdot\nabla)\vec{w} + \nabla q - 2\vec{\gamma}\times\vec{w} - \vec{\gamma}_t\times\vec{y} = \vec{0}, \tag{4}$$

$$\mathrm{div}\,\vec{w} = 0, \tag{5}$$

where the vector function $\vec{\gamma} = \vec{\gamma}(t)$ is defined by means of the formula

$$\vec{\gamma}\times\vec{z} = O_t^T O\vec{z} \quad \forall z, t.$$

(The matrix $O_t^T O$ being antisymmetric, the vector $\vec{\gamma}$ exists.)

In fact, instead of equation (4), we investigate its differential consequence

$$(\mathrm{rot}\ \vec{w})_\tau + (\vec{w}\cdot\nabla)\mathrm{rot}\ \vec{w} - (\mathrm{rot}\ \vec{w}\cdot\nabla)\vec{w} + 2(\vec{\gamma}\cdot\nabla)\vec{w} - 2\vec{\gamma}_\tau = \vec{0}. \tag{6}$$

Equation (4) will be used only to find the expression for p. To simplify solutions of (4)–(5), we make transformations generated by a Lie symmetry operator of the form

$$\tilde{R}(\vec{n}) = n^a\partial_{y_a} + n^a_\tau\partial_{w^a} - (\vec{n}_{\tau\tau} - 2\gamma\times\vec{n}_\tau - \vec{\gamma}_\tau\times\vec{n})\cdot\vec{y}\,\partial_q, \tag{7}$$

where $\vec{n}$ is an arbitrary smooth vector function of τ. The vector function $\vec{w}$ is to be found in the form

$$\begin{aligned} w^1 &= v^1(\tau, y_2, y_3) + \alpha^1(\tau)y_1,\\ w^2 &= v^2(\tau, y_1, y_3) + \alpha^2(\tau)y_2,\\ w^3 &= \beta^i(\tau)y_i + \alpha^3(\tau)y_3, \end{aligned} \tag{8}$$

where $v^1_{22}v^2_{11}\neq 0$, i.e., it is to satisfy the additional conditions

$$w^1_{1a} = w^2_{2a} = w^3_{ab} = 0, \quad w^1_{22}w^2_{11}\neq 0.$$

Note that the functions v^i depend on the different "similarity" variables.

Here and below, we sum over repeated indices. Subscript of a function denotes differentiation with respect to the corresponding variables. The indices a, b take values in $\{1,2,3\}$ and the indices i, j in $\{1,2\}$.

It follows from (5) that $\alpha^3 = -(\alpha^1+\alpha^2)$. Substituting (8) into (6), we obtain the equations to find the functions v^i, β^i, α^i and γ^a:

$$\begin{aligned} &\beta^2_\tau - v^2_{3\tau} - v^2_{31}(v^1+\alpha^1y_1) - (\beta^iy_i - (\alpha^1+\alpha^2)y_3)v^2_{33} - (\beta^2 - v^2_3)\alpha^1 + \beta^1v^1_2 -\\ &\quad v^2_1v^1_3 + 2\gamma^1\alpha^1 + 2\gamma^2v^1_2 + 2\gamma^3v^1_3 - 2\gamma^1_t = 0,\\ &v^1_{3\tau} - \beta^1_\tau + (v^2+\alpha^2y_2)v^1_{32} + (\beta^iy_i - (\alpha^1+\alpha^2)y_3)v^1_{33} - \beta^2v^2_1 - (v^1_3 - \beta^1)\alpha^2 +\\ &\quad v^1_2v^2_3 + 2\gamma^1v^2_1 + 2\gamma^2\alpha^2 + 2\gamma^3v^2_3 - 2\gamma^2_t = 0,\\ &v^2_{1\tau} - v^1_{2\tau} + (v^1+\alpha^1y_1)v^2_{11} - (v^2+\alpha^2y_2)v^1_{22} + v^2_3\beta^1 - v^1_3\beta^2 +\\ &\quad (v^2_1 - v^1_2)(\alpha^1+\alpha^2) + (\beta^iy_i - (\alpha^1+\alpha^2)y_3)(v^2_{13} - v^1_{23}) +\\ &\quad 2\gamma^1\beta^1 + 2\gamma^2\beta^2 - 2\gamma^3(\alpha^1+\alpha^2) - 2\gamma^3_t = 0. \end{aligned} \tag{9}$$

Unlike the "ansatz" method, we do not demand realizing the reduction conditions in system (9). The differential consequences of system (9) are the equations

$$v^1_{22}v^2_{1111} = v^2_{11}v^1_{2222},$$

i.e.,

$$\frac{v^1_{2222}}{v^1_{22}} = \frac{v^2_{1111}}{v^2_{11}} := h = h(t, x_3), \tag{10}$$

and

$$(v^1_{22}v^2_{11})_3 = 0. \tag{11}$$

Consider the particular cases.

Case I. $h>0$. Let $k:=h^{1/2}$. Then equation (10) gives that

$$\begin{aligned} v^1 &= f^1e^{ky_2}+f^2e^{-ky_2}+f^3y_2+f^4,\\ v^2 &= g^1e^{ky_1}+g^2e^{-ky_1}+g^3y_1+g^4, \end{aligned} \tag{12}$$

where $f^m=f^m(\tau,y_3)$, $g^m=g^m(\tau,y_3)$, $m=\overline{1,4}$, $f^if^i\neq 0$, $g^ig^i\neq 0$. It follows from (11) that

$$k_3=0,\quad (f^ig^j)_3=0.$$

Therefore, there exist the functions $\mu^i=\mu^i(\tau)$, $\nu^i=\nu^i(\tau)$, $f=f(\tau,y_3)$, and $g=g(\tau,y_3)$ such that

$$f^i=\mu^if,\quad g^i=\nu^ig.$$

Substituting expression (12) for v^i into system (9) and using a linear independence of the functions

$$y_ie^{k(\pm y_2\pm y_1)},\quad e^{k(\pm y_1\pm y_2)},\quad y_i^2,\quad y_1y_2,\quad y_i,\quad \text{and}\quad 1,$$

we obtain the complicated system for the rest of functions:

$$\begin{aligned}
&\nu^i(\beta^2g_3-(-1)^if^3g)=0,\quad \mu^i(\beta^1f_3-(-1)^ikg^3f)=0,\\
&\nu^i((k_\tau+\alpha^1k)g-(-1)^i\beta^1g_3)=0,\quad \mu^i((k_\tau+\alpha^2k)f-(-1)^i\beta^2f_3)=0,\\
&\nu^i(\beta^2g_{33}+(\beta^2-2\gamma^1)k^2g+(-1)^i\gamma^3kg_3)=0,\\
&\mu^i(\beta^1f_{33}+(\beta^1+2\gamma^2)k^2f-(-1)^i\gamma^3kf_3)=0,\\
&\nu^i_\tau g+\nu^i(g_\tau-(-1)^ikf^4g-(\alpha^1+\alpha^2)y_3g_3+\alpha^2g)=0,\\
&\mu^i_\tau f+\mu^i(f_\tau-(-1)^ikg^4f-(\alpha^1+\alpha^2)y_3f_3+\alpha^1f)=0,\\
&-(f^3g^3)_3-\beta^2g^4_{33}+2\gamma^3f^3_3=0,\quad (f^3g^3)_3+\beta^1f^4_{33}+2\gamma^3g^3_3=0,\\
&f^3_{3\tau}-(\alpha^1+\alpha^2)y_3f^3_{33}+\beta^2f^4_{33}=0,\quad g^3_{3\tau}-(\alpha^1+\alpha^2)y_3g^3_{33}+\beta^1g^4_{33}=0,\\
&\beta^1(f^3_3-2g^3_3)=0,\quad \beta^2(2f^3_3-g^3_3)=0,\quad \beta^if^3_{33}=\beta^ig^3_{33}=0,\\
&\beta^2_\tau-g^4_{3\tau}-(f^4g^3)_3+(\alpha^1+\alpha^2)y_3g^4_{33}+\alpha^1g^4_3+(\beta^1+2\gamma^2)f^3+\\
&\quad 2\gamma^3f^4_3-2\gamma^1_\tau-\alpha^1(\beta^2-2\gamma^1)=0,\\
&f^4_{3\tau}-\beta^1_\tau+(g^4f^3)_3-(\alpha^1+\alpha^2)y_3f^4_{33}-\alpha^1f^4_3-(\beta^2-2\gamma^2)g^3+\\
&\quad 2\gamma^3g^4_3+\alpha^2(\beta^1+2\gamma^2)-2\gamma^2_t=0,\\
&g^3_\tau-f^3_\tau-(\alpha^1+\alpha^2)y_3(g^3_3-f^3_3)+\beta^1g^4_3-\beta^2f^4_3+(\alpha^1+\alpha^2)(g^3-f^3)+\\
&2\gamma^i\beta^i-2\gamma^3(\alpha^1+\alpha^2)-2\gamma^3_t=0.
\end{aligned} \tag{13}$$

Here we do not sum over the index i.

We integrate system (13), substitute the obtained expressions for the functions f^m, g^m, $m=\overline{1,4}$, k, α^i, β^i, and γ^a into (12) and (8). Then, integrating equation (4) to find the function q, we get solutions of (4)–(5) that are simplified by means of transformations generated by operators of the form (8)

Depending on different means of integrating system (13), the following solutions of (4)–(5) can be obtained in such a way:

1. $(\nu^1\nu^2)^2 + (\mu^1\mu^2)^2 \neq 0$, $\vec{\gamma} \neq \vec{0}$:

$$\begin{aligned}
w^1 &= C_{11}ke^{ky_2} + C_{12}ke^{-ky_2} - k_\tau k^{-1}y_1,\\
w^2 &= C_{21}ke^{ky_1} + C_{22}ke^{-ky_1} - k_\tau k^{-1}y_2,\\
w^3 &= -2\gamma^2 y_1 + 2\gamma^1 y_2 + 2k_\tau k^{-1}y_3,\\
q &= -k^2(C_{11}e^{ky_2} - C_{12}e^{-ky_2} + 2\gamma^3k^{-2})(C_{21}e^{ky_1} - C_{22}e^{-ky_1} - 2\gamma^3k^{-2}) +\\
&\quad \tfrac{1}{2}k_{\tau\tau}k^{-1}(y_1^2 + y_2^2 - 2y_3^2) - (k_\tau k^{-1})^2|\vec{y}|^2 - 2(\gamma^1y_2 - \gamma^2y_1)^2 +\\
&\quad (\gamma^2_\tau + 4\gamma^2k_\tau k^{-1})y_1y_3 - (\gamma^1_\tau + 4\gamma^1k_\tau k^{-1})y_2y_3.
\end{aligned}$$

2. $(\nu^1\nu^2)^2 + (\mu^1\mu^2)^2 \neq 0$, $\vec{\gamma} = \vec{0}$. Then the matrix O can be considered to be equal to the unit matrix, and $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$\begin{aligned}
u^1 &= ke^{\zeta(\omega)}(C_{11}e^{kx_2} + C_{12}e^{-kx_2}) - k_tk^{-1}x_1,\\
u^2 &= ke^{-\zeta(\omega)}(C_{21}e^{kx_1} + C_{22}e^{-kx_1}) - k_tk^{-1}x_2,\\
u^3 &= 2k_tk^{-1}x_3,\\
p &= -k^2(C_{11}e^{kx_2} - C_{12}e^{-kx_2})(C_{21}e^{kx_1} - C_{22}e^{-kx_1}) +\\
&\quad \tfrac{1}{2}k_{tt}k^{-1}(x_1^2 + x_2^2 - 2x_3^2) - (k_tk^{-1})^2|\vec{x}|^2,
\end{aligned}$$

where $\omega = k^{-2}(t)x_3$, k is an arbitrary function of t which does not vanish, ζ is an arbitrary function of ω.

3. $\mu^1\mu^2 = \nu^1\nu^2 = 0$, $\beta^i = \gamma^3 = 0$. Then $\gamma^i = 0$ and, as above, we can assume that $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$\begin{aligned}
u^1 &= C_1k\exp\{(-1)^ikx_2 + H(\tau,\omega)\} + (-1)^ikF(\omega) - k_tk^{-1}x_1,\\
u^2 &= C_2k\exp\{(-1)^jkx_1 - H(\tau,\omega)\} - (-1)^jkF(\omega) - k_tk^{-1}x_2,\\
u^3 &= 2k_tk^{-1}x_3,\\
p &= -C_1C_2(-1)^{i+j}k^2e^{(-1)^ikx_2+(-1)^jkx_1} + \tfrac{1}{2}k_{tt}k^{-1}(x_1^2 + x_2^2 - 2x_3^2) - (k_tk^{-1})^2|\vec{x}|^2,
\end{aligned}$$

where $\omega = k^{-2}(\tau)x_3$, k is an arbitrary function of t which does not vanish.

$$H = (-1)^{i+j}F(\omega)\int k^2(t)dt + G(\omega).$$

F and G are arbitrary functions of ω, i and j assumed to be fixed from $\{1; 2\}$.

4. $\mu^1\mu^2 = \nu^1\nu^2 = 0$, $\beta^i\beta^i + (\gamma^3)^2 \neq 0$. The solution obtained in this case is very complicated, and we omit it.

Case 2. $h < 0$. Let $k := (-h)^{1/2}$. Then equation (10) gives that

$$v^1 = f^1 \cos(ky_1) + f^2 \sin(ky_1) + f^3 y_1 + f^4,$$
$$v^2 = g^1 \cos(ky_2) + g^2 \sin(ky_2) + g^3 y_2 + g^4,$$

where $f^m = f^m(\tau, y_3)$, $g^m = g^m(\tau, y_3)$, $m = \overline{1,4}$, $f^i f^i \neq 0$, $g^i g^i \neq 0$. In a way being analogous to Case 1, we obtain the following solutions of equations (4)–(5):

1. $\vec{\gamma} \neq \vec{0}$:

$$w^1 = C_{11} k \cos(ky_2) + C_{12} k \sin(ky_2) - k_\tau k^{-1} y_1,$$
$$w^2 = C_{21} k \cos(ky_1) + C_{22} k \sin(ky_1) - k_\tau k^{-1} y_2,$$
$$w^3 = -2\gamma^2 y_1 + 2\gamma^1 y_2 + 2k_\tau k^{-1} y_3,$$
$$q = k^2 (C_{11} \sin(ky_2) - C_{12} \cos(ky_2) - 2\gamma^3 k^{-2})(C_{21} \sin(ky_1) - C_{22} \cos(ky_1) + 2\gamma^3 k^{-2}) + \frac{1}{2} k_{\tau\tau} k^{-1} (y_1^2 + y_2^2 - 2y_3^2) - (k_\tau k^{-1})^2 |\vec{y}|^2 - 2(\gamma^1 y_2 - \gamma^2 y_1)^2 + (\gamma^2_\tau + 4\gamma^2 k_\tau k^{-1}) y_1 y_3 - (\gamma^1_\tau + 4\gamma^1 k_\tau k^{-1}) y_2 y_3,$$

where $\vec{\gamma}$ is an arbitrary vector function of τ, $k = C|\gamma^3|^{\frac{1}{2}}$ if $\gamma^3 \neq 0$ and k is an arbitrary function of τ if $\gamma^3 = 0$.

2. $\vec{\gamma} = \vec{0}$. As above, we can consider that $\vec{w} = \vec{u}$, $q = p$, $\vec{y} = \vec{x}$, $\tau = t$.

$$u^1 = k e^{\zeta(\omega)} (C_{11} \cos(kx_2) + C_{12} \sin(kx_2)) - k_t k^{-1} x_1,$$
$$u^2 = k e^{-\zeta(\omega)} (C_{21} \cos(kx_1) + C_{22} \sin(kx_1)) - k_t k^{-1} x_2,$$
$$u^3 = 2k_t k^{-1} x_3,$$
$$p = k^2 (C_{11} \sin(kx_2) - C_{12} \cos(kx_2))(C_{21} \sin(kx_1) - C_{22} \cos(kx_1)) + \frac{1}{2} k_{tt} k^{-1} (x_1^2 + x_2^2 - 2x_3^2) - (k_t k^{-1})^2 |\vec{x}|^2,$$

where $\omega = k^{-2}(t) x_3$, $\zeta = \zeta(\omega)$ and $k = k(\tau)$ are arbitrary functions of their arguments, $k \neq 0$.

Case $h = 0$ is impossible.

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On Lie Reduction of the MHD Equations to Ordinary Differential Equations

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Abstract

The MHD equations describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity are reduced to ordinary differential equations by means of Lie symmetries.

The MHD equations (the MHDEs) describing flows of a viscous homogeneous incompressible fluid of finite electrical conductivity have the following form:

$$\begin{aligned}&\vec{u}_t + (\vec{u}\cdot\vec{\nabla})\vec{u} - \triangle\vec{u} + \vec{\nabla}p + \vec{H}\times\mathrm{rot}\,\vec{H} = \vec{0},\\ &\vec{H}_t - \mathrm{rot}\,(\vec{u}\times\vec{H}) - \nu_m\triangle\vec{H} = \vec{0}, \qquad \mathrm{div}\,\vec{u} = 0, \qquad \mathrm{div}\,\vec{H} = 0.\end{aligned} \tag{1}$$

System (1) is very complicated and the construction of new exact solutions is a difficult problem. Following [1], in this paper we reduce the MHDEs (1) to ordinary differential equations by means of three-dimensional subalgebras of the maximal Lie invariance algebra of the MHDEs.

In (1) and below, $\vec{u} = \{u^a(t,\vec{x})\}$ denotes the velocity field of a fluid, $p = p(t,\vec{x})$ denotes the pressure, $\vec{H} = \{H^a(t,\vec{x})\}$ denotes the magnetic intensity, ν_m is the coefficient of magnetic viscosity, $\vec{x} = \{x_a\}$, $\partial_t = \partial/\partial t$, $\partial_a = \partial/\partial x_a$, $\vec{\nabla} = \{\partial_a\}$, $\triangle = \vec{\nabla}\cdot\vec{\nabla}$ is the Laplacian. The kinematic coefficient of viscosity and fluid density are set equal to unity, permeability is done $(4\pi)^{-1}$. Subscript of a function denotes differentiation with respect to the corresponding variables. The maximal Lie invariance algebra of the MHDEs (1) is an infinite-dimensional algebra $A(\mathrm{MHD})$ with the basis elements (see [2])

$$\begin{aligned}&\partial_t, \quad D = t\partial_t + \tfrac{1}{2}x_a\partial_a - \tfrac{1}{2}u^a\partial_{u^a} - \tfrac{1}{2}H^a\partial_{H^a} - p\partial_p,\\ &J_{ab} = x_a\partial_b - x_b\partial_a + u^a\partial_{u^b} - u^b\partial_{u^a} + H^a\partial_{H^b} - H^b\partial_{H^a},\ a<b,\\ &R(\vec{m}) = R(\vec{m}(t)) = m^a\partial_a + m^a_t\partial_{u^a} - m^a_{tt}x_a\partial_p, \quad Z(\eta) = Z(\eta(t)) = \eta\partial_p,\end{aligned} \tag{2}$$

where $m^a = m^a(t)$ and $\eta = \eta(t)$ are arbitrary smooth functions of t (for example, from $C^\infty((t_0,t_1),\mathbf{R})$). We sum over repeated indices. The indices a, b take values in $\{1,2,3\}$ and the indices i, j in $\{1,2\}$. The algebra $A(\mathrm{MHD})$ is isomorphic to the maximal Lie invariance algebra $A(\mathrm{NS})$ of the Navier-Stokes equations [3, 4, 5].

Besides continuous transformations generated by operators (2), the MHDEs admit discrete transformations I_b of the form

$$\begin{aligned}&\tilde{t} = t, \qquad \tilde{x}_b = -x_b, \qquad \tilde{x}_a = x_a,\\ &\tilde{p} = p, \quad \tilde{u}^b = -u^b, \quad \tilde{H}^b = -H^b, \quad \tilde{u}^a = u^a, \quad \tilde{H}^a = H^a,\ a\neq b,\end{aligned}$$

where b is fixed.

We construct a complete set of A(MHD)-inequivalent three-dimensional subalgebras of A(MHD) and choose those algebras from this set which can be used to construct ansatzes for the MHD field. The list of the classes of these algebras is given below.

1. $A^3_1 = \langle D,\ \partial_t,\ J_{12}\rangle$.

2. $A^3_2 = \langle D + \frac{1}{2}\kappa J_{12},\ \partial_t,\ R(0,0,1)\rangle$,

where $\kappa \ge 0$. Here and below, κ, σ_{ij}, ε_i, μ, and ν are real constants.

3. $A^3_3 = \langle D,\ J_{12} + R(0,0,\nu|t|^{1/2}\ln|t|) + Z(\nu\varepsilon_2|t|^{-1}\ln|t| + \varepsilon_1|t|^{-1}),$
$R(0,0,|t|^{\sigma+1/2}) + Z(\varepsilon_2|t|^{\sigma-1})\rangle$,

where $\sigma\nu = 0$, $\sigma\varepsilon_2 = 0$, $\varepsilon_1 \ge 0$, and $\nu \ge 0$.

4. $A^3_4 = \langle \partial_t,\ J_{12} + R(0,0,\nu t) + Z(\nu\varepsilon_2 t + \varepsilon_1),\ R(0,0,e^{\sigma t}) + Z(\varepsilon_2 e^{\sigma t})\rangle$,

where $\sigma\nu = 0$, $\sigma\varepsilon_2 = 0$, $\sigma \in \{-1;0;1\}$, and, if $\sigma = 0$, the constants ν, ε_1, and ε_2 satisfy one of the following conditions:

$$\nu = 1,\ \varepsilon_1 \ge 0; \quad \nu = 0,\ \varepsilon_1 = 1,\ \varepsilon_2 \ge 0; \quad \nu = \varepsilon_1 = 0,\ \varepsilon_2 \in \{0;1\}.$$

5. $A^3_5 = \langle D + \kappa J_{12},\ R(|t|^{1/2} f^{ij}(t)\hat{A}(t)\vec{e}_j) + Z(|t|^{-1} f^{ij}(t)\varepsilon_j), i = 1,2\rangle$,

6. $A^3_6 = \langle \partial_t + \kappa J_{12},\ R(f^{ij}(t)\check{A}(t)\vec{e}_j) + Z(f^{ij}(t)\varepsilon_j), i = 1,2\rangle$,

Here, $(f^{1j}(t), f^{2j}(t)), j = 1,2$, are solutions of the Cauchy problems

$$f^{ij}_\tau = \sigma_{ik} f^{kj}, \quad f^{ij}(0) = \delta_{ij},$$

where $\tau = \ln|t|$ in the case of the algebra A^3_5 and $\tau = t$ in the case of the algebra A^3_6. $\vec{e}^i = \mathrm{const} : \vec{e}^i \cdot \vec{e}^j = \delta_{ij}$, δ_{ij} is the Kronecker delta,

$$\sigma_{ii}(\sigma_{12} - \sigma_{21} - 2\kappa K\vec{e}_1 \cdot \vec{e}_2) = 0, \quad \varepsilon_i(\sigma_{ik}\sigma_{kj}\vec{e}_j - 2\kappa\sigma_{ij}K\vec{e}_j) = \vec{0}, \quad \sigma_{12} + \sigma_{21} = 0.$$

$$A(\zeta) = \begin{pmatrix} \cos\zeta & -\sin\zeta & 0 \\ \sin\zeta & \cos\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{array}{l} \hat{A}(t) = A(\kappa\ln|t|), \\ \check{A}(t) = A(\kappa t). \end{array}$$

$\kappa\varepsilon_l K\vec{e}_k \cdot \vec{e}_l = 0$ if $\sigma_{kj} = 0$ and $\varepsilon_k(2\kappa K\vec{e}_k \cdot \vec{e}_l - \sigma_{kl}) = 0$ if $\sigma_{kj} = \sigma_{ll} = 0$, where k and l take fixed values from $\{1;2\}$, $k \ne l$. To simplify parameters in A^3_5 and A^3_6, one can also use the adjoint actions generated by I_b, D, J_{12}, and, if $\kappa = 0$, J_{23} and J_{31}.

7. $A^3_7 = \langle J_{12} + R(0,0,\eta^3),\ R(\eta^1,\eta^2,0),\ R(-\eta^2,\eta^1,0)\rangle$,

where η^a are smooth functions of t, $\eta^i\eta^i \ne 0$, $\eta^3 \ne 0$, $\eta^1_{tt}\eta^2 - \eta^1\eta^2_{tt} = 0$. The algebras $A^3_7(\eta^1,\eta^2,\eta^3)$ and $A^3_7(\tilde\eta^1,\tilde\eta^2,\tilde\eta^3)$ are equivalent if $\exists E_i \in \mathbf{R}\backslash\{0\}$, $\exists\delta \in \mathbf{R}$, $\exists(b_{ij}) \in O(2)$:

$$\tilde\eta^i(\tilde t) = E_2 b_{ij}\eta^i(t), \quad \tilde\eta^3(\tilde t) = E_1\eta^3(t),$$

where $t = E_1^2\tilde t + \delta$.

8. $A^3_8 = \langle R(\vec{m}^a),\ a = \overline{1,3}\rangle$,

where $\vec{m}^a$ are smooth functions of t, rank $(\vec{m}^1, \vec{m}^2, \vec{m}^3) = 3$, $\vec{m}^a_{tt} \cdot \vec{m}^b - \vec{m}^a \cdot \vec{m}^b_{tt} = 0$. The algebras $A^3_8(\vec{m}^1,\vec{m}^2,\vec{m}^3)$ and $A^3_8(\tilde{\vec{m}}^1,\tilde{\vec{m}}^2,\tilde{\vec{m}}^3)$ are equivalent if $\exists E_1 \in \mathbf{R}\backslash\{0\}$, $\exists\delta \in \mathbf{R}$, $\exists B \in O(3)$, and $\exists(d_{ab}) : \det(d_{ab}) \ne 0$ such that

$$\tilde{\vec{m}}^a(\tilde t) = d_{ab}B\vec{m}^b(t),$$

where $t = E_1^2\tilde t + \delta$.

The way of obtaining (and the form of writing down) the algebras $A^3_1 - A^3_8$ differs slightly from the one used in [1].

By means of subalgebras $A^3_1 - A^3_8$, one can construct the following ansatzes that reduce the MHDEs to ODEs:

1. $$\begin{aligned} u^1 &= x_1R^{-2}\varphi^1 - x_2(Rr)^{-1}\varphi^2 + x_1x_3r^{-1}R^{-2}\varphi^3,\\ u^2 &= x_2R^{-2}\varphi^1 + x_1(Rr)^{-1}\varphi^2 + x_2x_3r^{-1}R^{-2}\varphi^3,\\ u^3 &= x_3R^{-2}\varphi^1 - rR^{-2}\varphi^3,\\ p &= R^{-2}h, \end{aligned}$$

where $\omega = \arctan r/x_3$, the expressions for H^a are obtained by means of the substitution of ψ^a for φ^a in the expressions for u^a.

Here and below, $\varphi^a = \varphi^a(\omega)$, $\psi^a = \psi^a(\omega)$, $h = h(\omega)$, $R = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $r = (x_1^2 + x_2^2)^{1/2}$. The numeration of ansatzes and reduced systems corresponds to that of the algebras above. All the parameters satisfy the equations given for these algebras.

2. $$u^1 = r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2), \quad u^3 = r^{-1}\varphi^3, \quad p = r^{-2}h,$$

where $\omega = \arctan x_2/x_1 - \kappa \ln r$, the expressions for H^a are obtained by means of substituting ψ^a for φ^a in the expressions for u^a.

3. $$\begin{aligned} u^1 &= r^{-2}(x_1\varphi^1 - x_2\varphi^2) + \tfrac{1}{2}x_1t^{-1},\\ u^2 &= r^{-2}(x_2\varphi^1 + x_1\varphi^2) + \tfrac{1}{2}x_2t^{-1},\\ u^3 &= |t|^{-1/2}\varphi^3 + (\sigma + \tfrac{1}{2})x_3t^{-1} + \nu|t|^{1/2}t^{-1}\arctan x_2/x_1,\\ H^1 &= r^{-2}(x_1\psi^1 - x_2\psi^2), \quad H^2 = r^{-2}(x_2\psi^1 + x_1\psi^2), \quad H^3 = |t|^{-1/2}\psi^3,\\ p &= |t|^{-1}h + \tfrac{1}{8}x_ax_at^{-2} - \tfrac{1}{2}\sigma^2x_3^2t^{-2} + \varepsilon_1|t|^{-1}\arctan x_2/x_1 + \varepsilon_2x_3|t|^{-3/2}, \end{aligned}$$

where $\omega = |t|^{-1/2}r$.

4. $$\begin{aligned} u^1 &= r^{-2}(x_1\varphi^1 - x_2\varphi^2), \quad u^2 = r^{-2}(x_2\varphi^1 + x_1\varphi^2),\\ u^3 &= \varphi^3 + \sigma x_3 + \nu\arctan x_2/x_1,\\ H^1 &= r^{-2}(x_1\psi^1 - x_2\psi^2), \quad H^2 = r^{-2}(x_2\psi^1 + x_1\psi^2), \quad H^3 = \psi^3,\\ p &= h - \tfrac{1}{2}\sigma^2x_3^2 + \varepsilon_1\arctan x_2/x_1 + \varepsilon_2x_3, \end{aligned}$$

where $\omega = r$.

5. $$\begin{aligned} &\vec{u} = |t|^{1/2}t^{-1}\hat{A}(t)\vec{v} + \tfrac{1}{2}\vec{x}t^{-1} - \kappa K\vec{x}t^{-1}, \quad \vec{H} = |t|^{1/2}t^{-1}\hat{A}(t)\vec{G},\\ &p = |t|^{-1}q + \tfrac{1}{8}x_ax_at^{-2} + \tfrac{1}{2}\kappa^2x_ix_it^{-2}, \end{aligned} \tag{3}$$

where $\vec{y} = |t|^{-1/2}\hat{A}(t)^T\vec{x}$.

6. $$\vec{u} = \check{A}(t)\vec{v} - \kappa K\vec{x}, \quad \vec{H} = \check{A}(t)\vec{G}, \quad p = q + \tfrac{1}{2}\kappa^2x_ix_i, \tag{4}$$

where $\vec{y} = \check{A}(t)^T\vec{x}$.

In (3) and (4) $\vec v$, $\vec G$, q, and ω are defined by means of the following formulas:

$$\vec v = \vec\varphi(\omega) + \sigma_{ij}(\vec e_i\cdot\vec y)\vec e_j, \quad \vec G = \vec\psi(\omega),$$
$$q = h(\omega) + \varepsilon_i(\vec e_i\cdot\vec y) - \tfrac12(\vec d_i\cdot\vec y)(\vec e_i\cdot\vec y) - \tfrac12(\vec d_i\cdot\vec e_3)\omega(\vec e_i\cdot\vec y),$$

where $\omega = (\vec e_3\cdot\vec y)$, $\vec e_3 = \vec e_1\times\vec e_2$, $\vec d_i = \sigma_{ik}\sigma_{kj}\vec e_j - 2\kappa\sigma_{ij}K\vec e_j$.

7. $$u^1 = \varphi^1\cos z - \varphi^2\sin z + x_1\theta^1 + x_2\theta^2,$$
$$u^2 = \varphi^1\sin z + \varphi^2\cos z - x_1\theta^2 + x_2\theta^1,$$
$$u^3 = \varphi^3 + \eta^3_t(\eta^3)^{-1}x_3,$$
$$H^1 = \psi^1\cos z - \psi^2\sin z, \quad H^2 = \psi^1\sin z + \psi^2\cos z, \quad H^3 = \psi^3,$$
$$p\ = h - \tfrac12\eta^3_{tt}(\eta^3)^{-1}x_3^2 - \tfrac12\eta^j_{tt}\eta^j(\eta^i\eta^i)^{-1}r^2,$$

where $\omega = t$, $\theta^1 = \eta^i_t\eta^i(\eta^j\eta^j)^{-1}$, $\theta^2 = (\eta^1_t\eta^2 - \eta^1\eta^2_t)(\eta^j\eta^j)^{-1}$, $z = x_3/\eta^3$.

8. $$\vec u = \vec\varphi + \lambda^{-1}(\vec n^a\cdot\vec x)\vec m^a_t, \quad \vec H = \vec\psi,$$
$$p = h - \lambda^{-1}(\vec m^a_{tt}\cdot\vec x)(\vec n^a\cdot\vec x) + \tfrac12\lambda^{-2}(\vec m^b_{tt}\cdot\vec m^a)(\vec n^a\cdot\vec x)(\vec n^b\cdot\vec x),$$

where $\omega = t$, $\vec m^a_{tt}\cdot\vec m^b - \vec m^a\cdot\vec m^b_{tt} = 0$, $\lambda = \vec m^1\vec m^2\vec m^3 \neq 0$,

$$\vec n^1 = \vec m^2\times\vec m^3, \quad \vec n^2 = \vec m^3\times\vec m^1, \quad \vec n^3 = \vec m^1\times\vec m^2.$$

Substituting the ansatzes 1–8 into the MHDEs, we obtain the following systems of ODE in the functions φ^a, ψ^a, and h:

1. $$\varphi^3\varphi^1_\omega - \psi^3\psi^1_\omega - \varphi^1_{\omega\omega} - \varphi^1_\omega\cot\omega - \varphi^a\varphi^a - 2h = 0,$$
$$\varphi^3\varphi^2_\omega - \psi^3\psi^2_\omega - \varphi^2_{\omega\omega} - \varphi^2_\omega\cot\omega + \varphi^2\sin^{-2}\omega + (\varphi^3\varphi^2 - \psi^3\psi^2)\cot\omega = 0,$$
$$\varphi^3\varphi^3_\omega - \psi^3\psi^3_\omega - \varphi^3_{\omega\omega} - \varphi^3_\omega\cot\omega + \varphi^3\sin^{-2}\omega - 2\varphi^1_\omega + h_\omega + \psi^a_\omega\psi^a -$$
$$((\varphi^2)^2 - (\psi^2)^2)\cot\omega = 0,$$
$$\varphi^3\psi^1_\omega - \psi^3\varphi^1_\omega - \nu_m(\psi^1_{\omega\omega} + \psi^1_\omega\cot\omega) = 0,$$
$$\varphi^3\psi^2_\omega - \psi^3\varphi^2_\omega - \nu_m(\psi^2_{\omega\omega} + \psi^2_\omega\cot\omega - \psi^2\sin^{-2}\omega) + 2(\psi^1\varphi^2 - \psi^2\varphi^1) +$$
$$(\psi^3\varphi^2 - \psi^2\varphi^3)\cot\omega = 0,$$
$$\varphi^3\psi^3_\omega - \psi^3\varphi^3_\omega - \nu_m(\psi^3_{\omega\omega} + \psi^3_\omega\cot\omega - \psi^3\sin^{-2}\omega + 2\psi^1_\omega) + 2(\psi^1\varphi^3 - \psi^3\varphi^1) = 0,$$
$$\varphi^3_\omega + \varphi^3\cot\omega + \varphi^1 = 0, \quad \psi^3_\omega + \psi^3\cot\omega + \psi^1 = 0.$$

2. $$\tilde\varphi^2\varphi^1_\omega - \tilde\psi^2\psi^1_\omega - \nu\varphi^1_{\omega\omega} - \varphi^i\varphi^i + \psi^i\psi^i - 2\tilde h - \kappa\tilde h_\omega = 0,$$
$$\tilde\varphi^2\varphi^2_\omega - \tilde\psi^2\psi^2_\omega - \nu\varphi^2_{\omega\omega} - 2(\varphi^1_\omega + \kappa\varphi^2_\omega) + \tilde h_\omega = 0,$$
$$\tilde\varphi^2\varphi^3_\omega - \tilde\psi^2\psi^3_\omega - \nu\varphi^3_{\omega\omega} - \varphi^1\varphi^3 + \psi^1\psi^3 - 2\kappa\varphi^3_\omega - \varphi^3 = 0,$$
$$\tilde\varphi^2\psi^1_\omega - \tilde\psi^2\varphi^1_\omega - \tilde\nu\psi^1_{\omega\omega} = 0,$$
$$\tilde\varphi^2\psi^2_\omega - \tilde\psi^2\varphi^2_\omega - \tilde\nu\psi^2_{\omega\omega} + 2(\psi^1\varphi^2 - \psi^2\varphi^1) - 2\nu_m(\psi^1_\omega + \kappa\psi^2_\omega) = 0,$$
$$\tilde\varphi^2\psi^3_\omega - \tilde\psi^2\varphi^3_\omega - \tilde\nu\psi^3_{\omega\omega} + \psi^1\varphi^3 - \psi^3\varphi^1 - \nu_m(2\kappa\psi^3_\omega + \psi^3) = 0,$$
$$\tilde\varphi^2_\omega = 0, \quad \tilde\psi^2_\omega = 0,$$

where $\tilde\varphi^2 = \varphi^2 - \kappa\varphi^1$, $\tilde\psi^2 = \psi^2 - \kappa\psi^1$, $\tilde h = h + \frac{1}{2}\psi^a\psi^a$, $\nu = 1 + \kappa^2$, $\tilde\nu = \nu_m(1 + \kappa^2)$.

3-4. $\omega^{-1}(\varphi^1\varphi^1_w - \psi^1\psi^1_\omega) - \varphi^1_{\omega\omega} + \omega^{-1}\varphi^1_\omega - \omega^{-2}(\varphi^i\varphi^i - \psi^i\psi^i) +$
$$\omega(h + \tfrac{1}{2}\omega^{-2}\psi^i\psi^i + \tfrac{1}{2}(\psi^3)^2)_\omega = 0,$$
$$\omega^{-1}(\varphi^1\varphi^2_w - \psi^1\psi^2_\omega) - \varphi^2_{\omega\omega} + \omega^{-1}\varphi^2_\omega + \varepsilon_1 = 0,$$
$$\omega^{-1}(\varphi^1\varphi^3_w - \psi^1\psi^3_\omega) - \varphi^3_{\omega\omega} - \omega^{-1}\varphi^3_\omega + \nu\varepsilon\omega^{-2}\varphi^2 + \varepsilon\sigma\varphi^3 + \varepsilon_2 = 0,$$
$$\omega^{-1}(\varphi^1\psi^1_w - \psi^1\varphi^1_\omega) - \nu_m(\psi^1_{\omega\omega} - \omega^{-1}\psi^1_\omega) - \varepsilon\delta\psi^1 = 0,$$
$$\omega^{-1}(\varphi^1\psi^2_w - \psi^1\varphi^2_\omega) - \nu_m(\psi^2_{\omega\omega} - \omega^{-1}\psi^2_\omega) + 2\omega^{-2}(\psi^1\varphi^2 - \psi^2\varphi^1) - \varepsilon\delta\psi^2 = 0,$$
$$\omega^{-1}(\varphi^1\psi^3_w - \psi^1\varphi^3_\omega) - \nu_m(\psi^3_{\omega\omega} + \omega^{-1}\psi^3_\omega) - \nu\varepsilon\omega^{-2}\psi^2 - (\sigma + \delta)\varepsilon\psi^3 = 0,$$
$$\varphi^1_\omega + (\sigma + \tfrac{3}{2}\delta)\varepsilon\omega = 0, \quad \psi^1_\omega = 0,$$
where $\varepsilon = \operatorname{sign} t$ and $\delta = 1$ in case 3 and $\varepsilon = 1$ and $\delta = 0$ in case 4.

5-6. $\varphi^3\vec\varphi_\omega - \psi^3\vec\psi_\omega + \sigma_{ij}\varphi^i\vec e_j - 2\kappa K\vec\varphi - \varepsilon\vec\varphi_{\omega\omega} + (h_\omega + \vec\psi\cdot\vec\psi_\omega)\vec e_3 +$
$$\varepsilon_i\vec e_i + 2\kappa\omega\sigma_{ij}(K\vec e_j\cdot\vec e_3)\vec e_i = \vec 0,$$
$$\varphi^3\vec\psi_\omega - \psi^3\vec\varphi_\omega - \sigma_{ij}\psi^i\vec e_j - \nu_m\varepsilon\vec\varphi_{\omega\omega} - \delta\vec\psi = \vec 0,$$
$$\varphi^3_\omega + \sigma_{ii} + \tfrac{3}{2}\delta = 0, \quad \psi^3_\omega = 0,$$
where $\varphi^a = \vec e_a\cdot\vec\varphi$, $\psi^a = \vec e_a\cdot\vec\psi$;
$\varepsilon = \operatorname{sign} t$ and $\delta = 1$ in case 5 and $\varepsilon = 1$ and $\delta = 0$ in case 6.

7. $\varphi^1_\omega + \varphi^1(\theta^1 + (\eta^3)^{-2}) + \varphi^2(\theta^2 - (\eta^3)^{-1}\varphi^3) + \psi^3(\eta^3)^{-1}\psi^2 = 0,$
$$\varphi^2_\omega - \varphi^1(\theta^2 - (\eta^3)^{-1}\varphi^3) + \varphi^2(\theta^1 + (\eta^3)^{-2}) - \psi^3(\eta^3)^{-1}\psi^1 = 0,$$
$$\varphi^3_\omega + \eta^3_w(\eta^3)^{-1}\varphi^3 = 0,$$
$$\psi^1_\omega + \psi^1(-\theta^1 + \nu_m(\eta^3)^{-2}) - \psi^2(\theta^2 + (\eta^3)^{-1}\varphi^3) + \psi^3(\eta^3)^{-1}\varphi^2 = 0,$$
$$\psi^2_\omega + \psi^1(\theta^2 + (\eta^3)^{-1}\varphi^3) + \psi^2(-\theta^1 + \nu_m(\eta^3)^{-2}) - \psi^3(\eta^3)^{-1}\varphi^1 = 0,$$
$$\psi^3_w - \psi^3\eta^3_w(\eta^3)^{-1} = 0, \quad 2\theta^1 + \eta^3_w(\eta^3)^{-1} = 0.$$

8. $\vec\varphi_\omega + \lambda^{-1}(\vec n^a\cdot\vec\varphi)\vec m^a_\omega = \vec 0, \quad \vec\psi_\omega - \lambda^{-1}(\vec n^a\cdot\vec\psi)\vec m^a_\omega = \vec 0, \quad \vec n^a\cdot\vec m^a_\omega = 0.$

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 232–236.

Symmetry Reduction of Nonlinear Equations of Classical Electrodynamics

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Abstract

Symmetry reduction of generalized Maxwell equations is carried out on the three-dimensional subgroup of the extended Poincaré group. Some their exact solutions are constructed.

1. Electromagnetic field is described by the familiar Maxwell equations that, with the help of a real covariant vector of electromagnetic potential $A = (A_0, A_1, A_2, A_3)$, can be presented in the form (see, e.g., [1])

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = 0, \quad \mu, \nu = 0, 1, 2, 3. \tag{1}$$

We use the notation:

$$\Box u = \frac{\partial^2 u}{\partial x_0^2} - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2}, \qquad \partial_\mu = \frac{\partial}{\partial x_\mu},$$

and we sum over repeated indices (from 0 to 3). The raising and lowering of indices is performed with the help of the metric tensor $g = g_{\alpha\beta}$, where $g_{\alpha\beta} = \mathrm{diag}\,[1, -1, -1, -1]$.

The equation

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = F(A_\nu A^\nu)A_\mu \tag{2}$$

is a natural generalization of system (1) [2].

If $F = \lambda(A_\nu A^\nu)$, then equation (2) is invariant with respect to the algebra $AC(1,3)$ [2], [3] with generators

$$\begin{gathered} P_\mu = \partial_\mu, \quad J_{\mu\nu} = x^\mu\partial_\nu - x^\nu\partial_\mu + A^\mu\frac{\partial}{\partial A_\nu} - A^\nu\frac{\partial}{\partial A_\mu}, \quad D = x_\mu\partial_\mu - A_\mu\frac{\partial}{\partial A_\mu}, \\ K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^\mu x_\nu\frac{\partial}{\partial A_\nu} - 2A_\nu x^\nu\frac{\partial}{\partial A_\mu}. \end{gathered} \tag{3}$$

If $F = -m^2 + \lambda(A_\nu A^\nu)$ and $m \neq 0$, then the maximal invariance algebra of equation (2) is the Poincaré algebra $AP(1,3) = \langle P_\mu, J_{\mu\nu} \mid \mu, \nu = 0, 1, 2, 3\rangle$. Note, that system (1) is also invariant with respect to the algebra $AC(1,3)$ with basis (3).

Yehorchenko [4] considered the problem of symmetry reduction of equations (1), (2) by subalgebras of the Poincaré algebra $AP(1,3)$. In this paper, we consider the problem of symmetry reduction of the system

$$\Box A_\mu - \partial^\mu(\partial_\nu A_\nu) = \lambda(A_\nu A^\nu)A_\mu, \quad (\mu, \nu = 0, 1, 2, 3) \tag{4}$$

on subalgebras of the algebra $A\tilde{P}(1,3)$ to a system of ordinary differential equations.

2. The symmetry reduction of equation (4) to ordinary differential equations is carried out by subalgebras of the algebra $A\tilde{P}(1,3)$ of rank 3. The list of such subalgebras is known [7]:

$L_1 = \langle D, P_0, P_3 >$, $L_2 = \langle J_{12} + \alpha D, P_0, P_3 >$, $L_3 = \langle J_{12}, D, P_0\rangle$,
$L_4 = \langle J_{12}, D, P_3\rangle$, $L_5 = \langle J_{03} + \alpha D, P_0, P_3\rangle$, $L_6 = \langle J_{03} + \alpha D, P_1, P_2\rangle$,
$L_7 = \langle J_{03} + \alpha D, M, P_1\rangle$ $(\alpha \neq 0)$, $L_8 = \langle J_{03} + D + 2T, P_1, P_2\rangle$,
$L_9 = \langle J_{03} + D + 2T, M, P_1\rangle$, $L_{10} = \langle J_{03}, D, P_1\rangle$, $L_{11} = \langle J_{03}, D, M\rangle$,
$L_{12} = \langle J_{12} + \alpha J_{03} + \beta D, P_0, P_3\rangle$, $L_{13} = \langle J_{12} + \alpha J_{03} + \beta D, P_1, P_2\rangle$,
$L_{14} = \langle J_{12} + \alpha(J_{03} + D + 2T), P_1, P_2\rangle$, $L_{15} = \langle J_{12} + \alpha J_{03}, D, M\rangle$,
$L_{16} = \langle J_{03} + \alpha D, J_{12} + \beta D, M\rangle(0 \leq |\alpha| \leq 1,\ \beta \geq 0, |\alpha| + |\beta| \neq 0)$,
$L_{17} = \langle J_{03} + D + 2T, J_{12} + \alpha T, M\rangle$ $(\alpha \geq 0)$, $L_{18} = \langle J_{03} + D, J_{12} + 2T, M\rangle$,
$L_{19} = \langle J_{03}, J_{12}, D\rangle$, $L_{20} = \langle G_1, J_{03} + \alpha D, P_2\rangle$ $(0 < |\alpha| \leq 1)$,
$L_{21} = \langle J_{03} + D, G_1 + P_2, M\rangle$, $L_{22} = \langle J_{03} - D + M, G_1, P_2\rangle$,
$L_{23} = \langle J_{03} + 2D, G_1 + 2T, M\rangle$, $L_{24} = \langle J_{03} + 2D, G_1 + 2T, P_2\rangle$,

where $M = P_0 + P_3$, $G_1 = J_{01} - J_{13}$, $T = \dfrac{1}{2}(P_0 - P_3)$, unless otherwise stated, $\alpha, \beta > 0$.

The structure of generators of the algebra $AP(1,3)$ (3) allows one to construct linear invariant ansatzes that correspond to subalgebras of the algebra $AP(1,3)$, ([5], [6])

$$A = \Lambda(x)B(\omega), \tag{5}$$

where $\Lambda(x)$ is a known nondegenerate square matrix of order 4, and $B(\omega)$ is a new unknown vector function for invariants of the subalgebra $\omega = \omega(x)$, $x = (x_0, x_1, x_2, x_3)$.

Using the approach suggested by Fushchych, Zhdanov, and Lahno [8], [9], ansatzes (5) for subalgebras of the extended Poincaré algebra $AP(1,3)$ can be represented in the form

$$A_\mu(x) = \theta(x)a_{\mu\nu}(x)B^\nu(\omega), \tag{6}$$

where $B^\nu = B^\nu(\omega)$ are new unknown functions of ω,

$$\begin{aligned} a_{\mu\nu} = {} & (a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 + \\ & 2(a_\mu + d_\mu)[\theta_2\cos\theta_1 b_\nu - \theta_2\sin\theta_1 c_\nu + \theta_2^2\exp(-\theta_0)(a_\nu + d_\nu)] + \\ & (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_1 - (c_\mu c_\nu + b_\mu b_\nu)\cos\theta_1 - 2\exp(-\theta_0)\theta_2 b_\mu(a_\nu + d_\nu). \end{aligned} \tag{7}$$

Here $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary constants satisfying the following equalities:

$$a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1, \qquad a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0,$$

$\mu, \nu = 0, 1, 2, 3$. The form of the (non-zero) functions θ, θ_i $(i = 0, 1, 2)$, ω is determined by subalgebras L_j, $(j = \overline{1, 24})$ of the algebra $A\tilde{P}(1,3)$, and we given them below for each of these subalgebras.

$L_1 : \theta = |bx|^{-1}$, $\omega = cx(bx)^{-1}$;
$L_2 : \theta = \Psi_1^{-\frac{1}{2}}$, $\theta_1 = \Phi$, $\omega = \ln\Psi_1 + 2\Phi$;
$L_3 : \theta = |dx|^{-1}$, $\theta_1 = \Phi$, $\omega = \Psi_1(dx)^{-2}$;
$L_4 : \theta = |ax|^{-1}$, $\theta_1 = \Phi$, $\omega = \Psi_1(ax)^{-2}$;
$L_5 : \theta = |bx|^{-1}$, $\theta_0 = \alpha^{-1}\ln|bx|$, $\omega = cx(bx)^{-1}$;

$L_6 : \theta = |\Psi_2|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|(ax-dx)(kx)^{-1}|, \quad \omega = |ax-dx|^{1-\alpha}|kx|^{1+\alpha};$
$L_7 : \theta = |cx|^{-1},\ \theta_0 = \alpha^{-1}\ln|cx|, \quad \omega = |kx|^{\alpha}|cx|^{1-\alpha};$
$L_8 : \theta = |ax-dx|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|ax-dx|, \quad \omega = kx - \ln|ax-dx|;$
$L_9 : \theta = |cx|^{-1},\ \theta_0 = \ln|cx|, \quad \omega = kx - 2\ln|cx|;$
$L_{10} : \theta = |cx|^{-1},\ \theta_0 = \ln|(ax-dx)(cx)^{-1}|, \quad \omega = \Psi_2(cx)^{-2};$
$L_{11} : \theta = |cx|^{-1},\ \theta_0 = -\ln|kx(cx)^{-1}|, \quad \omega = cx(bx)^{-1};$
$L_{12} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\alpha\Phi,\ \theta_1 = \Phi,\ \omega = \ln\Psi_1 + 2\beta\Phi;$
$L_{13} : \theta = |\Psi_2|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|(ax-dx)(kx)^{-1}|,$
$\theta_1 = -\frac{1}{2\alpha}\ln|(ax-dx)(kx)^{-1}|, \quad \omega = |ax-dx|^{\alpha-\beta}|kx|^{\alpha+\beta};$
$L_{14} : \theta = |ax-dx|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln|ax-dx|,\ \theta_1 = -\frac{1}{2}\ln|ax-dx|,\ \omega = kx - \ln|ax-dx|;$
$L_{15} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\alpha\Phi, \quad \theta_1 = \Phi, \quad \omega = \ln[\Psi_1(kx)^{-2}] + 2\alpha\Phi;$
$L_{16} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln[\Psi_1(kx)^{-2}], \quad \theta_1 = \Phi, \quad \omega = \ln[\Psi_1^{1-\alpha}(kx)^{2\alpha}] + 2\beta\Phi;$
$L_{17} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln\Psi_1, \quad \theta_1 = \Phi, \quad \omega = kx - \ln\Psi_1 + 2\alpha\Phi;$
$L_{18} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2}\ln\Psi_1, \quad \theta_1 = \Phi,\ \omega = kx + 2\Phi;$
$L_{19} : \theta = \Psi_1^{-\frac{1}{2}},\ \theta_0 = -\frac{1}{2}\ln|kx(ax-dx)^{-1}|, \quad \theta_1 = \Phi, \quad \omega = \Psi_1|\Psi_2|^{-1};$
$L_{20} : \theta = |\Psi_3|^{-\frac{1}{2}},\ \theta_0 = \frac{1}{2\alpha}\ln|\Psi_3|,\ \theta_2 = \frac{1}{2}bx(kx)^{-1},\ \omega = |kx|^{2\alpha}|\Psi_3|^{1-\alpha};$
$L_{21} : \theta = |cx\,kx - bx|^{-1},\ \theta_0 = \ln|cx\,kx - bx|, \quad \theta_2 = \frac{1}{2}cx,\ \omega = kx;$
$L_{22} : \theta = |kx|^{-\frac{1}{2}}, \quad \theta_0 = -\frac{1}{2}\ln|kx|, \quad \theta_2 = \frac{1}{2}bx(kx)^{-1},$
$\omega = ax - dx + \ln|kx| - (bx)^2(kx)^{-1};$
$L_{23} : \theta = |cx|^{-1},\ \theta_0 = \frac{1}{2}\ln|cx|, \quad \theta_2 = -\frac{1}{4}kx,\ \omega = [4bx + (kx)^2](cx)^{-1};$
$L_{24} : \theta = |4bx + (kx)^2|^{-1},\ \theta_0 = \frac{1}{2}\ln|4bx + (kx)^2|,$
$\theta_2 = -\frac{1}{4}kx,\ \omega = [ax - dx + bxkx + \frac{1}{6}(kx)^3]^2[4bx + (kx)^2]^{-3}.$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = c_\mu x^\mu$, $dx = d_\mu x^\mu$, $kx = ax + dx$, $\Phi = \arctan\dfrac{cx}{bx}$, $\Psi_1 = (bx)^2 + (cx)^2$, $\Psi_2 = (ax)^2 - (dx)^2$, $\Psi_3 = (ax)^2 - (bx)^2 - (dx)^2$.

3. The covariant form of ansatz (6), (7) which we have obtained enables us to perform the $\tilde{P}(1,3)$-invariant reduction of equation (4) in general form.

Theorem. *Ansatz (6), (7) reduces equation (4) to the system of ODEs*

$$k_{\mu\gamma}\ddot{B}^\gamma + l_{\mu\gamma}\dot{B}^\gamma + m_{\mu\gamma}B^\gamma = \lambda(B^\gamma B_\gamma)B_\mu, \tag{8}$$

where

$$k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \qquad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2Q_{\mu\gamma} - G_\mu H_\gamma - G_\mu\dot{G}_\gamma,$$
$$m_{\mu\gamma} = R_{\mu\gamma}, \qquad \dot{B}^\gamma = \frac{dB^\gamma}{d\omega}, \qquad \ddot{B}^\gamma = \frac{d^2B^\gamma}{d\omega^2}, \qquad \dot{G}_\gamma = \frac{dG_\gamma}{d\omega}. \tag{9}$$

In (9), $F_1 = F_1(\omega)$, $F_2 = F_2(\omega)$, $G_\mu = G_\mu(\omega)$, $Q_{\mu\gamma} = Q_{\mu\gamma}(\omega)$, $H_\gamma = H_\gamma(\omega)$, $R_{\mu\gamma} = R_{\mu\gamma}(\omega)$ are smooth functions of ω and are determined from the relations

$$\frac{\partial\omega}{\partial x_\mu}\cdot\frac{\partial\omega}{\partial x^\mu} = F_1(\omega)\theta^2; \quad \theta\,\Box\omega + 2\frac{\partial\theta}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu} = F_2(\omega)\theta^3; \quad a_{\nu\mu}\frac{\partial\omega}{\partial x_\nu} = G_\mu(\omega)\theta;$$

$$\theta\frac{\partial a_{\nu\mu}}{\partial x_\nu}+3a_{\nu\mu}\frac{\partial\theta}{\partial x_\nu}=H_\mu(\omega)\theta^2;\quad a_\mu^\gamma\frac{\partial a_{\gamma\nu}}{\partial x^\delta}\frac{\partial\omega}{\partial x_\delta}+G_\mu(\omega)a_{\delta\nu}\frac{\partial\theta}{\partial x_\delta}-G_\nu(\omega)a_{\delta\mu}\frac{\partial\theta}{\partial x_\delta}=Q_{\mu\nu}(\omega)\theta^2;$$

$$g_{\mu\nu}\Box\theta+2a_\mu^\gamma\frac{\partial a_{\gamma\nu}}{\partial x^\delta}\frac{\partial\theta}{\partial x_\delta}-a_\mu^\gamma a_{\delta\nu}\frac{\partial^2\theta}{\partial x^\gamma\partial x_\delta}-a_\mu^\gamma\frac{\partial a_{\delta\nu}}{\partial x_\delta}\frac{\partial\theta}{\partial x^\gamma}-a_\mu^\gamma\frac{\partial a_{\delta\nu}}{\partial x^\gamma}\frac{\partial\theta}{\partial x_\delta}$$

$$+\theta(a_\mu^\gamma\Box a_{\gamma\nu}-a_\mu^\gamma\frac{\partial^2 a_{\delta\nu}}{\partial x^\gamma\partial x_\delta})=R_{\mu\nu}(\omega)\theta^3;$$

where $\mu,\nu,\gamma,\delta=0,1,2,3$; $\Box$ *is the d'Alembertian.*

Using the results of the theorem for each subalgebra L_j ($j=\overline{1,24}$), we obtain the corresponding systems of ordinary differential equations which for the case of equation (4) are, in general, nonlinear. Integration of the reduced equations we obtain and substitution of their solutions into ansatz (6), (7) lead to solutions of the original system. We give below some typical solutions of both the linear Maxwell system (1) and the nonlinear equations (4).

Solutions of equations (1)

$$\mathbf{1)}A_\mu=\frac{a_\mu}{2}\left\{G_1+\frac{E}{(bx)^2+(cx)^2}(AF_1+BF_2)\right\}+\frac{d_\mu}{2}\left\{G_1-\frac{E}{(bx)^2+(cx)^2}(AF_1+BF_2)\right\}+$$

$$\frac{c_\mu}{(bx)^2+(cx)^2}\left\{\frac{E}{2(1+\alpha^2)}[(\zeta B+\chi A)F_1+(\chi B-\zeta A)F_2]+\chi(G_1+C_1\omega_1+C_2)+\zeta C_3\right\}+$$

$$\frac{b_\mu}{(bx)^2+(cx)^2}\left\{\frac{E}{2(1+\alpha^2)}[(\zeta A-\chi B)F_1+(\zeta B+\chi A)F_2]+\zeta(G_1+C_1\omega_1+C_2)-\chi C_3\right\}.$$

Here

$$E=\exp\left(-\frac{\omega_1}{1+\alpha^2}\right);\quad F_1=\sin\left(\frac{\alpha\omega_1}{1+\alpha^2}\right);\quad F_2=\cos\left(\frac{\alpha\omega_1}{1+\alpha^2}\right).$$

$$\mathbf{2)}\quad A_\mu=C(a_\mu-d_\mu)\frac{(kx)^2}{\Psi}+k_\mu\left\{2C\varepsilon(kx)^{-2}+C_1\frac{bx}{kx}|\Psi|^{-\frac12}+C(bx)^2|\Psi|^{-1}\right\}+c_\mu C_2|\Psi|^{-\frac12}-$$

$$b_\mu\left\{2C\ bx\ kx\ |\Psi|^{-1}+C_1|\Psi|^{-\frac12}\right\},$$

3)

$$A_\mu=a_\mu\left\{\frac{\epsilon}{kx}(G_2+C_1\omega_2+C_2)-\epsilon\left[1+\frac{(bx)^2}{(kx)^2}\right](C_1\omega_2+C_2-G_2)-\frac{bx}{|kx|^{\frac32}}(C_3\omega_2+C_4)\right\}+$$

$$d_\mu\left\{\frac{\epsilon}{kx}(G_2+C_1\omega_2+C_2)+\epsilon\left[1-\frac{(bx)^2}{(kx)^2}\right](C_1\omega_2+C_2-G_2)-\frac{bx}{|kx|^{\frac32}}(C_3\omega_2+C_4)\right\}+$$

$$b_\mu\left\{2\epsilon\frac{bx}{kx}(C_1\omega_2+C_2-G_2)+|kx|^{-\frac12}(C_3\omega_2+C_4)\right\}+c_\mu|kx|^{-\frac12}(C_5\omega_2+C_6).$$

Solutions of equations (4)

$$A_\mu(x)=2\delta\ A_\mu^*(x);$$

where

$$\delta=\sqrt{-\frac{2}{3\lambda}}\quad\text{for }\epsilon=1\qquad\text{and}\quad\delta=\sqrt{\frac{2}{\lambda}}\quad\text{for }\epsilon=-1,$$

$$A^*_\mu = -\frac{\epsilon}{2}(a_\mu - d_\mu)\frac{1}{\omega_2 + C_0} + k_\mu \left\{ \frac{1}{2} \cdot \frac{1}{\omega_2 + C_0} \left(|kx|^{-1} - \epsilon \frac{(bx)^2}{(kx)^2} \right) - \epsilon |kx|^{-\frac{3}{2}} \cdot \frac{1}{\omega_2 + C_1} \right\} +$$
$$c_\mu \cdot \frac{|kx|^{-\frac{1}{2}}}{\omega_2 + C_2} + b_\mu \left\{ \frac{bx}{kx} \cdot \frac{\epsilon}{\omega_2 + C_0} + \frac{|kx|^{-\frac{1}{2}}}{\omega_2 + C_1} \right\}$$

We use the following notation:
$\zeta = \alpha cx + bx$; $\chi = \alpha bx - cx$; $\Psi = (ax)^2 - (bx)^2 - (dx)^2$; $kx = ax + dx$; $\varepsilon = 1$ for $\Psi > 0$ and $\varepsilon = -1$ for $\Psi < 0$; $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$;
$\omega_1 = kx - \ln[(bx)^2 + (cx)^2] + 2\alpha \arctan \dfrac{cx}{bx}$, $\omega_2 = ax - dx + \ln|kx| - (bx)^2(kx)^{-1}$.

$G_1 = G_1(\omega_1)$; $G_2 = G_2(\omega_2)$ are arbitrary smooth functions of ω_1 and ω_2, respectively. A, B, C, C_i $(i = \overline{0,6})$ are constants of integration.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 237–240.

On Symmetries of a Generalized Diffusion Equation

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Abstract

The non-linear diffusion eqaution, describing the vertical transfer of both heat and moisture in the absence of solutes are considered. Lie symmetries of the equation are obtained for some specific form of diffusion coefficient.

The group properties of diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial \vec{x}}\left[D(u)\frac{\partial u}{\partial \vec{x}}\right] = 0, \tag{1}$$

(where $u = u(\vec{x}, t)$, $D(u)$ are real functions) describing nonlinear processes of heat conductivity are investigated by many authors [1–3].

In the present paper we investigate a particular case of eq.(1), where u is vector function and $D(u)$ is a non-singular matrix, defining the diffusive properties of the soil. Recently, such systems have been extensively studied, from both mathematical and biological viewpoints [4, 5]. It is motivated by a successful application of these models to a wide range of developmental and ecological systems. We will analyze symmetry properties of given models.

Let us consider non-linear diffusion equation with sources

$$\frac{\partial u^a}{\partial t} - \frac{\partial}{\partial x}\left[K^{ab}(\vec{u})\frac{\partial u^b}{\partial \vec{x}}\right] = M^a(\vec{u}), \qquad a, b = 1, 2, \ldots, n, \tag{2}$$

where K^{ab} is a n-dimensional matrix.

Classical symmetry groups for coupled non-linear diffusion equation with $M^a = 0$ were found in [6].

Let us rewrite eq.(2) in the form:

$$\frac{\partial u^a}{\partial t} - \frac{K^{ab}}{\partial u^c}\frac{\partial u^c}{\partial x}\frac{\partial u^b}{\partial x} - K^{ab}\frac{\partial^2 u^b}{\partial x^2} - M^a = 0. \tag{3}$$

The symmetry operator Q is defined by

$$Q = \xi^1(x, t, \vec{u})\frac{\partial}{\partial t} + \xi^2(x, t, \vec{u})\frac{\partial}{\partial x} + \eta^a(x, t, \vec{u})\frac{\partial}{\partial u^a}. \tag{4}$$

The prolongation operator of eq.(3) has the form

$$\underset{2}{Q} = Q + \eta_1^a\frac{\partial}{\partial u_t^a} + \eta_2^a\frac{\partial}{\partial u_x^a} + \eta_{11}^a\frac{\partial}{\partial u_{tt}^a} + \eta_{12}^a\frac{\partial}{\partial u_{xt}^a} + \eta_{22}^a\frac{\partial}{\partial u_{xx}^a}, \tag{5}$$

where

$$\eta_1^a = \eta_t^a + u_t\eta_u - u_t\xi_t^1 - u_x\xi_t^2 - (u_t)^2\xi_u^1 - u_tu_x\xi_u^2,$$
$$\eta_2^a = \eta_x^a + u_x\eta_u - u_t\xi_x^1 - u_x\xi_x^2 - u_xu_t\xi_u^1 - (u_x)^2\xi_u^2,$$
$$\begin{aligned}\eta_{22}^a = {} & \eta_{xx} + 2u_x\eta_{xu} + u_{xx}\eta_u + (u_x)^2\eta_{uu} - 2u_{xt}\xi_x^1 - 2u_{xx}\xi_x^2 - \\ & 2u_{xt}u_x\xi_u^1 - 2u_{xx}u_x\xi_u^2 - u_{xx}u_t\xi_u^1 - u_{xx}u_x\xi_u^2 - 2u_tu_x\xi_{xu}^1 - \\ & 2(u_x)^2\xi_{xu}^2 - u_x\xi_{xx}^2 - u_t\xi_{xx}^1 - (u_x)^2u_t\xi_{uu}^1 - (u_x)^3\xi_{uu}^2.\end{aligned} \tag{6}$$

To find the invariance condition we act by $\underset{2}{Q}$ on the eq.(3).

As a result we obtain:

$$\begin{aligned}& -\eta^a K_{u^cu^a}^{ab}u_x^cu_x^b - \eta^a K_{u^a}^{db}u_{xx}^b - M_{u^a}^d\eta^a + \eta_1^a - \\ & \quad \eta_2^a K_{u^c}^{da}u_x^c - \eta_2^a K_{u^a}^{db}u_x^b + \eta_{22}^a K^{da}\Big|_{u_t^a = K^{ab}u_{xx}^b + K_{u^c}^{ab}u_x^cu_x^b + M^a} = 0.\end{aligned} \tag{7}$$

Substituting into (7) explicit forms of η_1^a, η_2^a, η_{22}^a and equating the coefficients of the various partial derivatives we obtain the following defining equations:

$$\xi_u^1 = \xi_x^1 = 0, \tag{8}$$

$$K^{da}\xi_{u^c}^2 + 2K^{dc}\xi_{u^a}^2 - \xi_{u^f}^2K^{fc}\delta_{da} = 0, \tag{9}$$

$$\begin{aligned}& -2K^{da}\eta_{u^bu^c}^a - K_{u^a}^{dc}\eta_{u^b}^a - K_{u^a}^{db}\eta_{u^c}^a + 2K_{u^b}^{dc}\xi_x^2 + 2K_{u^c}^{db}\xi_x^2 - K_{u^c}^{da}\eta_{u^b}^a - K_{u^b}^{da}\eta_{u^c}^a + \\ & \eta_{u^f}^dK_{u^c}^{fb} + \eta_{u^f}^dK_{u^b}^{fc} - \xi_t^1K_{u^c}^{db} - \xi_t^1K_{u^b}^{dc} - \eta^aK_{u^cu^a}^{db} - \eta^aK_{u^au^b}^{dc} = 0,\end{aligned} \tag{10}$$

$$-K^{da}\eta_{u^b}^a + 2K^{db}\xi_x^2 + \eta_{u^c}^dK^{cb} - \xi_t^1K^{db} - \eta^aK_{u^a}^{db} = 0, \tag{11}$$

$$K^{da}\xi_{xx}^2 - 2K^{db}\eta_{xu^a}^b - K_{u^b}^{da}\eta_x^b - \xi_t^2\delta_{ad} = 0, \tag{12}$$

$$-M_{u^a}^d\eta^a + \eta_t^d - \xi_t^1M^d + \eta_{u^b}^dM^b - \eta_{xx}^aK^{da} = 0. \tag{13}$$

It follows from (8), (9) that

$$\xi^1 = \xi^1(t), \qquad \xi^2 = \xi^2(x,t).$$

Solving of eqs.(8)–(13) is a complicated problem. We consider some particular cases which can find applications in mathematical biology.

Choosing

$$K^{11} = D, \qquad K^{12} = -B(u^1)u^1, \qquad K^{21} = 0, \qquad K^{22} = D, \qquad M^a = 0, \tag{14}$$

we reduce eq.(3) to the following system:

$$\begin{cases} u_t^1 = Du_{xx}^1 - B_{u^1}u_x^1u^1u_x^2 - Bu_x^1u_x^2 - Bu_1u_{xx}^2, \\ u_t^2 = Du_{xx}^2. \end{cases} \tag{15}$$

Theorem 1.* *The invariance algebra of system (15) is a 5-parameter algebra Lie whose basis elements have a following form:*

$$\begin{aligned} &Q_1 = \partial_t, \qquad Q_2 = \partial_x, \qquad Q_3 = \partial_{u^2}, \\ &Q_4 = B(u_1)\partial_{u^1}, \qquad Q_5 = x\partial_x + 2t\partial_t. \end{aligned} \tag{16}$$

Proof. Substituting (14) into determing eqs.(8)–(13) we find the general solutions of η^a, ξ^a $(a = 1, 2)$:

$$\begin{aligned} &\eta^1 = C^1 B(u^1)u^1, \qquad \eta^2 = C_2, \\ &\xi^1 = 2C_3 t + C_4, \qquad \xi^2 = C_3 x + C_5. \end{aligned} \tag{17}$$

It is easily to verify using (17) and (4) that basis elements have the form (16).
In the case

$$M^a = 0, \qquad K = K(x,t) \tag{18}$$

we come to

Theorem 2. *Eq.(3), (18) invariant under 5-parameter Lie algebra, whose operator are:*

$$\begin{aligned} &Q_1 = \partial_t, \qquad Q_2 = \partial_x, \qquad Q_3 = x\partial_x + 2t\partial_t, \\ &Q_4 = (K^{-1})^{ab}u^b x\partial_{u^a} - 2t\partial_x, \qquad Q_5 = (K^{-1})^{ab}u^b\partial_{u^a}, \end{aligned} \tag{19}$$

iff a diffusion matrix K has a specific dependence on t, x:

$$K = K\left[\frac{\gamma t + \sigma}{\left(\frac{\gamma}{2}x + \alpha\right)^2}\right], \tag{20}$$

where γ, α, δ are constants.

In the conclusion we consider a cell equation of the following general form:

$$\begin{aligned} &n_t + (nu_t)_x = 0, \\ &\rho_t + (\rho u_t)_x = 0, \\ &\mu u_{xxt} + u_{xx} + \Big[\tau n(\rho + \gamma\rho_{xx})\Big]_x = s\rho u, \end{aligned} \tag{21}$$

where μ, τ, s are constants, $n = n(t,x)$, $u = u(t,x)$, $\rho = \rho(t,x)$.

Using classical Lie methods [7] we obtained that system (21) is invariant under Heisenberg algebra $H(1)$.

Acknowledgment

Author acknowledges a grant from the Royal Society of United Kingdom and would like to thank the University of Glamorgan for its warm hospitality where this work being completed. Useful discussions and comments of Prof. Wiltshire and Prof. Nikitin are also acknowledged.

*This result was also obtained by program package "Lie" in a collaboration with Dr. O.Roman

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 241–244.

Reduction and Some Exact Solutions of the Eikonal Equation

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Abstract

Using the nonsplitting subgroups of the generalized Poincaré group $P(1,4)$, ansatzes which reduce the eikonal equation to differential equations with a lesser number of independent variables are constructed. The corresponding symmetry reduction is made. Some classes of exact solutions of the investigated equation are presented.

The relativistic eikonal (the relativistic Hamilton-Jacobi) equation is fundamental in theoretical and mathematical physics. We consider the equation

$$\frac{\partial u}{\partial x_\mu}\frac{\partial u}{\partial x^\mu} \equiv \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \left(\frac{\partial u}{\partial x_2}\right)^2 - \left(\frac{\partial u}{\partial x_3}\right)^2 = 1. \tag{1}$$

In [1], it was shown that the maximal local (in the sense of Lie) invariance group of equation (1) is the conformal group $C(1,4)$ of the five-dimensional Poincaré-Minkowski space. Using special ansatzes, the multiparametric families of exact solutions of the eikonal equation were constructed [1–4].

The conformal group $C(1,4)$ contains the generalized Poincaré group $P(1,4)$ as a subgroup. The group $P(1,4)$ is the group of rotations and translations of the five-dimensional Poincaré–Minkowski space. For the investigation of equation (1), we have used the nonsplitting subgroups [5–7] of the group $P(1,4)$. We have constructed ansatzes which reduce equation (1) to differential equations with a smaller number of independent variables using invariants [8] of the nonsplitting subgroups of the group $P(1,4)$. The corresponding symmetry reduction is performed. Using solutions of the reduced equations, we have found some classes of exact solutions of the eikonal equation.

Below we write ansatzes which reduce equation (1) to ordinary differential equations (ODEs), and we list the ODEs obtained as well as some solutions of the eikonal equation.

1. $\frac{c}{\alpha}x_3 + \ln(x_0+u) = \varphi(\omega), \quad \omega = (x_1^2+x_2^2+u^2-x_0^2)^{1/2},$

   $(\varphi')^2 - \frac{2}{\omega}\varphi' + \frac{c^2}{\alpha^2} = 0;$

2. $(x_0^2 - x_1^2 - x_2^2 - u^2)^{1/2} = \varphi(\omega), \quad \omega = x_3 + \alpha\ln(x_0+u),$

   $(\varphi')^2 - \frac{2\alpha}{\varphi}\varphi' - 1 = 0;$

3. $(u^2 + x_3^2 - x_0^2)^{1/2} = \varphi(\omega), \quad \omega = x_2 + \alpha\ln(x_0+u),$

$$(\varphi')^2 - \frac{2\alpha}{\varphi}\varphi' - 1 = 0;$$

4. $u = \exp\left(\varphi(\omega) + \frac{x_2}{\widetilde{a}_2}\right) - x_0, \quad \omega = x_3,$

$$(\varphi')^2 = -\frac{1}{\widetilde{a}_2^2};$$

$$u = \exp\left(\frac{\varepsilon i x_3 + x_2}{\widetilde{a}_2} + c\right) - x_0;$$

5. $u = -\exp\left(\varphi(\omega) - \frac{x_2}{\widetilde{a}_2}\right) + x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i x_3 - x_2}{\widetilde{a}_2} + c\right) + x_0;$$

6. $u = \exp\left(\varphi(\omega) - \frac{x_1}{a}\right) - x_0, \quad \omega = x_2,$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = \exp\left(\frac{\varepsilon i x_2}{a^2} + \frac{x_1}{a} + c\right) - x_0;$$

7. $u = -\exp\left(\varphi(\omega) - \frac{x_3}{a}\right) + x_0, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i}{a}(x_1^2 + x_2^2)^{1/2} - \frac{x_3}{a} + c\right) + x_0;$$

8. $u = \exp\left(\frac{x_3}{a} - \varphi(\omega)\right) - x_0, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$$(\varphi')^2 + \frac{1}{a^2} = 0;$$

$$u = \exp\left(\frac{x_3}{a} - \frac{\varepsilon i}{a}(x_1^2 + x_2^2)^{1/2} - c\right) - x_0;$$

9. $u = -\exp\left(\varphi(\omega) - \frac{x_2}{\widetilde{a}_2}\right) + x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = -\exp\left(\frac{\varepsilon i}{\widetilde{a}_2}x_3 - \frac{x_2}{\widetilde{a}_2} + c\right) + x_0;$$

10. $u = \exp\left(\frac{x_2}{\widetilde{a}_2} - \varphi(\omega)\right) - x_0, \quad \omega = x_3,$

$$(\varphi')^2 + \frac{1}{\widetilde{a}_2^2} = 0;$$

$$u = \exp\left(\frac{x_2}{\widetilde{a}_2} - \frac{\varepsilon i}{\widetilde{a}_2}x_3 + c\right) - x_0.$$

Ansatzes (1)–(10) can be written in the following form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x), \tag{2}$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function. $\omega=\omega(x)$ is a one-dimensional invariant of the nonsplitting subgroups of the group $P(1,4)$.

11. $\arcsin\frac{x_3}{\omega}=\varphi(\omega)-\frac{x_0}{\alpha},\quad \omega=(x_3^2+u^2)^{1/2},$

$$(\varphi')^2+\frac{1}{\omega^2}-\frac{1}{\alpha^2}=0;$$

12. $\arcsin\frac{x_3}{\omega}=\varphi(\omega)-\frac{c}{\alpha}x_0,\quad \omega=(x_3^2+u^2)^{1/2}\ \ (0<c<1,\ \alpha>0),$

$$(\varphi')^2+\frac{1}{\omega^2}-\frac{c^2}{\alpha^2}=0;$$

13. $\operatorname{arch}\frac{x_0}{\omega}=\varphi(\omega)-\frac{c}{\alpha}x_3,\quad \omega=(x_0^2-u^2)^{1/2}\ \ (\alpha>0),$

$$(\varphi')^2-\left(\frac{1}{\omega^2}+\frac{c^2}{\alpha^2}\right)=0;$$

$$\frac{c}{\alpha}x_3+\operatorname{arch}\frac{x_0}{\sqrt{x_0^2-u^2}}=(x_0^2-u^2)^{1/2}\left(\frac{c^2}{\alpha^2}+\frac{1}{x_0^2-u^2}\right)^{1/2}-$$

$$-\ln\left[\frac{1}{(x_0^2-u^2)^{1/2}}+\left(\frac{c^2}{\alpha^2}+\frac{1}{x_0^2-u^2}\right)^{1/2}\right],\quad (c>0);$$

14. $\frac{\varepsilon}{3}\left(2(\omega-x_3)\right)^{3/2}+\varepsilon x_3\left(2(\omega-x_3)\right)^{1/2}=\varphi(\omega)-x_0,$

$$\omega=x_3+\frac{(x_0+u)^2}{2},$$

$$(\varphi')^2-2\omega-1=0;$$

15. $x_0+\frac{x_3}{\alpha}(x_0+u)+\frac{(x_0+u)^3}{3\alpha^2}=\varphi(\omega),\quad \omega=\alpha x_3+\frac{(x_0+u)^2}{2},$

$$(\varphi')^2-\frac{1}{\alpha^2}-\frac{2\omega}{\alpha^4}=0;$$

$$x_0+\frac{x_0+u}{\alpha}x_3+\frac{(x_0+u)^3}{3\alpha^2}=\frac{2\sqrt{2}}{3\alpha^2}\left[\alpha x_3+\frac{(x_0+u)^2}{2}+\frac{\alpha^2}{2}\right]^{3/2};$$

16. $\frac{(x_0+u)^3}{3}+\varepsilon x_3(x_0+u)+\frac{1}{2}(x_0-u)=\varphi(\omega),$

$$\omega=\frac{(x_0+u)^2}{2}+\varepsilon x_3\ \ (\varepsilon=\pm1),\qquad (\varphi')^2-2\omega=0;$$

$$\frac{(x_0+u)^3}{3}+\varepsilon x_3(x_0+u)+\frac{1}{2}(x_0-u)=\frac{2\sqrt{2}}{3}\left[\frac{(x_0+u)^2}{2}+\varepsilon x_3\right]^{3/2}+c.$$

Ansatzes (11)–(16) can be written in the following form:

$$h(\omega,x)=f(x)\cdot\varphi(\omega)+g(x), \tag{3}$$

where $h(\omega,x)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega=\omega(x)$ is a one-dimensional invariant of the nonsplitting subgroups of the group $P(1,4)$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.1, 245–248.

On Reduction and Some Exact Solutions of the Euler-Lagrange-Born-Infeld Equation

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Abstract

The ansatzes which reduce the Euler-Lagrange-Born-Infeld equation to differential equations with a smaller number of independent variables are constructed using the subgroup structure of the Poincaré group $P(1,3)$. The corresponding symmetry reduction is made. Some classes of exact solutions of the investigated equation are found.

In [1, 2], the symmetry properties of the Euler-Lagrange-Born-Infeld equation were studied and multiparametric families of exact solutions of the equation have been found using special ansatzes.

Consider the equation

$$\Box u\,(1-u_\nu u^\nu)+u_{\mu\nu}u^\mu u^\nu=0, \tag{1}$$

where $u=u(x)$, $x=(x_0,x_1,x_2)\in\mathbf{R}_3$, $u_{\mu\nu}\equiv\dfrac{\partial^2 u}{\partial x_\mu\partial x_\nu}$, $u_\nu\equiv\dfrac{\partial u}{\partial x_\nu}$, $\mu,\nu=0,1,2$, $\Box$ is the d'Alembertian.

The symmetry group [1, 2] of equation (1) includes the Poincaré group $P(1,3)$ as a subgroup. We construct ansatzes which reduce equation (1) to differential equations with a smaller number of independent variables, using the invariants [3, 4] of the subgroups of the group $P(1,3)$. The corresponding symmetry reduction is performed. Using the solutions of the reduced equations, we have found some classes of exact solutions of the Euler-Lagrange-Born-Infeld equation.

1. Below we write ansatzes which reduce equation (1) to ordinary differential equations (ODEs), and we list the ODEs obtained as well as some exact solutions of the Euler-Lagrange-Born-Infeld equation.

1. $u=\varphi(\omega),\quad \omega=\left(x_1^2+x_2^2\right)^{1/2},\quad \omega\varphi''+\varphi'^3+\varphi'=0;$

   $$u=c\ln\left((x_1^2+x_2^2)^2+\sqrt{x_1^2+x_2^2-c^2}\right);$$

2. $u=-\varphi(\omega)+x_0,\ \ \omega=\left(x_1^2+x_2^2\right)^{1/2},\ \ \varphi'=0;\ \ u=x_0-c;$

3. $u^2=-\varphi^2(\omega)+x_0^2,\quad \omega=x_1,\quad \varphi''\varphi-\varphi'^2+1=0;$

   $$u^2=-\frac{1}{c_1^2}\sin^2(c_1x_1+c_2)+x_0^2;\quad u^2=-(\varepsilon x_1+c)^2+x_0^2,\quad \varepsilon=\pm1;$$

4. $u = -\varphi(\omega) + x_0, \quad \omega = (x_0^2 - x_1^2 - u^2)^{1/2}$,

$\omega\varphi^2\varphi'' - 2\omega^2\varphi'^3 + 4\omega\varphi\varphi'^2 - \varphi^2\varphi' = 0;$

$u = \frac{1}{2c}\left(1 + \varepsilon\sqrt{1 - 4c(x_0 - c(x_0^2 - x_1^2))}\right); \quad u = x_0 - c(x_0^2 - x_1^2 - u^2)^{1/4};$

5. $u^2 = -\varphi^2(\omega) + x_0^2, \quad \omega = (x_1^2 + x_2^2)^{1/2}$,

$\omega\varphi''\varphi - \varphi'^3\varphi - \omega\varphi'^2 + \varphi'\varphi + \omega = 0;$

$u^2 = x_0^2 - \left(\varepsilon(x_1^2 + x_2^2)^{1/2} + c\right)^2, \quad \varepsilon = \pm 1;$

6. $u^2 = -\varphi^2(\omega) + x_0^2 - x_1^2 - x_2^2, \quad \omega = x_0 - u,$

$\omega^2\varphi''\varphi + \omega^2\varphi'^2 - 6\omega\varphi\varphi' + 3\varphi^2 = 0;$

$u^2 = x_0^2 - x_1^2 - x_2^2 - c^2(x_0 - u)^6; \; u = \frac{1}{2}\left(c + \varepsilon\sqrt{c^2 - 4(cx_0 - x_0^2 + x_1^2 + x_2^2)}\right),$

$\varepsilon = \pm 1;$

7. $u = \varphi(\omega), \quad \omega = (x_0^2 - x_1^2 - x_2^2)^{1/2}, \quad \omega\varphi'' - 2\varphi'^3 + 2\varphi' = 0;$

$u = \varepsilon(x_0^2 - x_1^2 - x_2^2)^{1/2} + c, \quad \varepsilon = \pm 1;$

8. $u^2 = \varphi^2(\omega) - x_1^2 - x_2^2, \quad \omega = x_0, \quad \varphi''\varphi + 2\varphi'^2 - 2 = 0;$

$u^2 = (\varepsilon x_0 + c)^2 - x_1^2 - x_2^2, \quad \varepsilon = \pm 1;$

9. $u = \varphi(\omega) - \alpha \arctan\frac{x_2}{x_1}, \quad \omega = (x_1^2 + x_2^2)^{1/2},$

$(\alpha^2 + 1)\omega^3\varphi'' + \omega^2\varphi'^3 + (2\alpha^2 + \omega^2)\varphi' = 0;$

10. $u = -\varphi(\omega) + x_0 + \arctan\frac{x_2}{x_1}, \quad \omega = (x_1^2 + x_2^2)^{1/2}, \quad \omega^3\varphi'' + \omega^2\varphi'^3 + 2\varphi' = 0.$

Ansatzes (1)–(10) can be written in the following form:

$$h(u) = f(x) \cdot \varphi(\omega) + g(x),$$

where $h(u)$, $f(x)$, $g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

11. $\omega x_2^2 - (1 - \omega)\left((2x_0 + \omega)\omega + x_1^2\right) = \varphi(\omega), \quad \omega = u - x_0,$

$\omega^2(1 - \omega)^2\varphi'' - 4\omega(1 - \omega)(1 - 2\omega)\varphi' + 2(7\omega^2 - 7\omega + 2)\varphi = 0;$

$(u - x_0)x_2^2 + (x_0 - u + 1)(x_0^2 - x_1^2 - u^2) = \widetilde{c}_1(u - x_0)^4 \times$

$(6(u - x_0)^3 - 21(u - x_0)^2 + 25(u - x_0) + 10) + c_2(u - x_0)(u - x_0 - 1).$

Ansatz (11) can be written in the following form:

$$h(\omega, x) = f(x) \cdot \varphi(\omega) + g(x),$$

where $h(\omega, x), f(x), g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

12. $\alpha \ln(x_0 - u) = \varphi(\omega) - x_1, \quad \omega = (x_0^2 - u^2)^{1/2},$

$\omega(\alpha^2+\omega^2)\varphi'' - \omega^2\varphi'^3 + \alpha(2+\alpha)\omega\varphi'^2 + (\omega^2-\alpha^2)\varphi' = 0;$

13. $\alpha\ln(x_0-u) = \varphi(\omega) - x_1, \quad \omega = x_2, \quad \varphi'' = 0;$

    $u = x_0 - \exp\left((c_1x_2 - x_1 + c_2)/\alpha\right);$

14. $x_0 + u - x_1(x_0-u) + \frac{1}{6}(x_0-u)^3 = \varphi(\omega), \quad \omega = \frac{1}{4}(x_0-u)^2 - x_1,$

    $2\omega\varphi'' - \varphi' = 0;$

    $x_0 + u - x_1(x_0-u) + \frac{1}{6}(x_0-u)^3 = \frac{2c_1}{3}\left(\frac{1}{4}(x_0-u)^2 - x_1\right)^{3/2} + c_2;$

15. $(x_0-u)^2 = 4\varphi(\omega) + 4x_1, \quad \omega = x_2, \quad \varphi'' = 0;$

    $u = x_0 + 2\varepsilon\sqrt{x_1 + c_1x_2 + c_2}, \quad \varepsilon = \pm 1;$

16. $\alpha\ln(x_0-u) = \varphi(\omega) - x_2, \quad \omega = (x_0^2 - x_1^2 - u^2)^{1/2},$

    $\omega(\alpha^2+\omega^2)\varphi'' - 2\omega^2\varphi'^3 + 5\alpha\omega\varphi'^2 + (2\omega^2-\alpha^2)\varphi' = 0.$

Ansatzes (12)–(16) can be written in the following form:

$$h(u,x) = f(x)\cdot\varphi(\omega) + g(x),$$

where $h(u,x), f(x), g(x)$ are given functions, $\varphi(\omega)$ is an unknown function, $\omega = \omega(x)$ is a one-dimensional invariant of the subgroups of the group $P(1,3)$.

2. Next we consider the reduction of the investigated equation to two-dimensional partial differential equations (PDEs). The PDEs obtained can be written in the form:

$$A\left(\varphi_{11}\varphi_2^2 + \varphi_{22}\varphi_1^2 - 2\varphi_{12}\varphi_1\varphi_2\right) + B_1\varphi_{11} + B_2\varphi_{22} + 2B_3\varphi_{12} + V = 0,$$

$$\varphi_i \equiv \frac{\partial\varphi}{\partial\omega_i}, \quad \varphi_{ij} \equiv \frac{\partial^2\varphi}{\partial\omega_i\partial\omega_j}, \quad i = 1,2.$$

Below, we present the ansatzes, which reduce equation (1) to two-dimensional PDEs, and the corresponding coefficients $A,\ B_1,\ B_2,\ B_3,\ V$ of the reduced equation.

1. $u = \varphi(\omega_1,\omega_2), \quad \omega_1 = x_1, \quad \omega_2 = x_2;$

   $A = B_1 = B_2 = 1, \quad B_3 = V = 0.$

2. $u = -\varphi(\omega_1,\omega_2) + x_0, \quad \omega_1 = x_1, \quad \omega_2 = x_2;$

   $A = 1, \quad B_1 = B_2 = B_3 = V = 0.$

3. $u = \varphi(\omega_1,\omega_2), \quad \omega_1 = x_0, \quad \omega_2 = \left(x_1^2 + x_2^2\right)^{1/2};$

   $A = B_1 = -B_2 = \omega_2, \quad B_3 = 0, \quad V = \varphi_2\left(\varphi_1^2 - \varphi_2^2 - 1\right).$

4. $u^2 = -\varphi^2(\omega_1,\omega_2) + x_0^2, \quad \omega_1 = x_1, \quad \omega_2 = x_2;$

   $A = -B_1 = -B_2 = \varphi^3, \quad B_3 = 0, \quad V = \varphi^2\left(\varphi_1^2 + \varphi_2^2 - 1\right).$

5. $u^2 = -\varphi^2(\omega_1,\omega_2) + x_0^2 - x_1^2, \quad \omega_1 = x_0 - u, \quad \omega_2 = x_2;$

   $A = 0, \quad B_1 = \omega_1^2\varphi, \quad B_2 = \varphi^2\left(\varphi - 2\omega_1\varphi_1\right), \quad B_3 = \omega_1\varphi^2\varphi_2,$

   $V = \omega_1\varphi_1\left(\omega_1\varphi_1 - 4\varphi\right) - 2\varphi^2\left(\varphi_2^2 - 1\right).$

6. $u=\varphi(\omega_1,\omega_2),\quad \omega_1=\alpha\arctan\frac{x_2}{x_1}+x_0,\quad \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$A=B_1=\omega_2\left(\omega_2^2-\alpha^2\right),\quad B_2=-\omega_2^3,\quad B_3=0,$

$V=\omega_2^2\varphi_2\left(\varphi_1^2-\varphi_2^2-1\right)-2\alpha^2\varphi_1^2\varphi_2.$

7. $u=\varphi(\omega_1,\omega_2)-\alpha\arctan\frac{x_2}{x_1},\quad \omega_1=x_0,\quad \omega_2=\left(x_1^2+x_2^2\right)^{1/2};$

$A=\omega_2^3,\quad B_1=-B_2=\omega_2\left(\omega_2^2+\alpha^2\right),\quad B_3=0,$

$V=\omega_2^2\varphi_2\left(\varphi_1^2-\varphi_2^2-1\right)-2\alpha^2\varphi_2.$

8. $u^2=-\varphi^2(\omega_1,\omega_2)+x_0^2-x_1^2,\quad \omega_1=x_0-u,\quad \omega_2=x_1-x_2\left(x_0-u\right);$

$A=0,\quad B_1=\omega_1^2\varphi,\quad B_2=\varphi\left(\omega_2^2+\varphi\left(\omega_1^2+1\right)\left(\varphi-2\omega_1\varphi_1\right)\right),$

$B_3=\omega_1\varphi\left(\omega_2+\left(\omega_1^2+1\right)\varphi_2\varphi\right),$

$V=\left(\omega_1\varphi_1+\omega_2\varphi_2\right)^2-\varphi^2\varphi_2^2-4\varphi\left(\omega_1\varphi_1+\omega_2\varphi_2\right)+2\varphi^2.$

9. $\alpha\ln(x_0+u)=\varphi(\omega_1,\omega_2)-\arctan\frac{x_2}{x_1},\quad \omega_1=\left(x_1^2+x_2^2\right)^{1/2},\quad \omega_2=\left(x_0^2-u^2\right)^{1/2};$

$A=\omega_1^3\omega_2^2,\quad B_1=-\omega_1^3\omega_2^2\left(\omega_2+2\alpha\varphi_2\right),\quad B_2=\omega_1\omega_2\left(\alpha^2\omega_1^2+\omega_2^2\right),$

$B_3=\alpha\omega_1^3\omega_2^2\varphi_1,\quad V=\omega_1^2\omega_2^2\left(\varphi_1^2-\varphi_2^2\right)\left(\omega_1\varphi_2-\omega_2\varphi_1\right)+3\alpha\omega_1^3\omega_2\varphi_2^2-$

$-2\alpha\omega_1^2\omega_2^2\varphi_1\varphi_2+\omega_1\left(\omega_2^2-\alpha^2\omega_1^2\right)\varphi_2-2\omega_2^3\varphi_1.$

10. $\left(x_1^2+x_2^2\right)^{1/2}=\varphi(\omega_1,\omega_2),\quad \omega_1=x_0+u,\quad \omega_2=\arctan\frac{x_2}{x_1}+x_0-u;$

$A=4\varphi^3,\quad B_1=0,\quad B_2=-\varphi,\quad B_3=2\varphi^3,\quad V=2\varphi_2^2-\varphi^2\left(4\varphi_1\varphi_2-1\right).$

11. $\alpha\ln(x_0-u)=\varphi(\omega_1,\omega_2)-x_1,\quad \omega_1=\left(x_0^2-u^2\right)^{1/2},\quad \omega_2=x_2;$

$A=\omega_1^3,\quad B_1=\omega_1\left(\alpha^2+\omega_1^2\right),\quad B_2=-2\alpha\omega_1^2\varphi_1,\quad B_3=\alpha\omega_1^2\varphi_2,$

$V=-\omega_1^2\varphi_1\left(\varphi_1^2-\varphi_2^2\right)+3\alpha\omega_1\varphi_1^2+\left(\omega_1^2-\alpha^2\right)\varphi_1.$

12. $u+x_0-x_1(x_0-u)+\frac{1}{6}(x_0-u)^3=\varphi(\omega_1,\omega_2),\quad \omega_1=\frac{1}{4}(x_0-u)^3-x_1,\quad \omega_2=x_2;$

$A=1,\quad B_1=B_2=-4\omega_1,\quad B_3=0,\quad V=2\varphi_1.$

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Proceedings

of the Second International Conference

SYMMETRY IN NONLINEAR MATHEMATICAL PHYSICS

Memorial Prof. W. FUSHCHYCH Conference

July 7–13, 1997, Kyiv, Ukraine

Organized by

Institute of Mathematics of the National Academy of Sciences of Ukraine
Ukrainian Pedagogical University

Editors

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Volume 2

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ISBN 966–02–0342–X

Symmetry in Nonlinear Mathematical Physics,
Editors: M. Shkil, A. Nikitin and V. Boyko.

ISBN 966–02–0343–8 (Volume 1) and **ISBN 966–02–0344–6** (Volume 2)

Printed in Ukraine, Kyiv, 1997.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 254–261.

Quantum Mechanics in Noninertial Reference Frames and Representations of the Euclidean Line Group

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Abstract

The Galilean covariance of nonrelativistic quantum mechanics is generalized to an infinite parameter group of acceleration transformations called the Euclidean line group. Projective representations of the Euclidean line group are constructed and the resulting unitary operators are shown to implement arbitrary accelerations. These unitary operators are used to modify the time-dependent Schrödinger equation and produce the quantum mechanical analog of fictitious forces. The relationship of accelerating systems to gravitational forces is discussed. Solutions of the time-dependent Schrödinger equation for time varying, spatially constant external fields are obtained by transforming to appropriate accelerating reference frames. Generalizations to relativistic quantum mechanics are briefly discussed.

Infinite-dimensional groups and algebras continue to play an important role in quantum physics. The goal of this work, which is dedicated to the memory of Wilhelm Fushchych, is to look at some special representations of a group $\mathcal{E}(3)$, called the Euclidean line group in three dimensions, the group of maps from the real line to the three-dimensional Euclidean group. The motivation for studying such a group arises from a long-standing question in quantum mechanics, namely how to do quantum mechanics in noninertial reference frames. Now to "do" quantum mechanics in noninertial frames means constructing unitary operators that implement acceleration transformations. We will show that representations of $\mathcal{E}(3)$ on an appropriate Hilbert space provide the unitary operators that are needed to implement acceleration transformations.

The natural Hilbert space on which to construct representations of $\mathcal{E}(3)$ is the Hilbert space of a single particle of mass m and spin s, $\mathcal{H}_{m,s}$, which itself is the representation space for the central extension of the Galilei group [1]. Thus, the central extension of $\mathcal{E}(3)$ should contain the central extension of the Galilei group as a subgroup, and moreover, under the restriction of the representation of $\mathcal{E}(3)$ to that of the Galilei group, the Hilbert space should remain irreducible. The representations of $\mathcal{E}(3)$ will be obtained from the generating functions of the corresponding classical mechanics problem. These generating functions carry a representation of the Lie algebra of $\mathcal{E}(3)$ under Poisson bracket operations; using the correspondence between Poisson brackets in classical mechanics and commutators in

[1]Supported in part by DOE.

quantum mechanics gives the desired representation, which is readily exponentiated to give the representation of $\mathcal{E}(3)$. Moreover, this procedure automatically gives a representation of the central extension of $\mathcal{E}(3)$.

Once the unitary operators implementing acceleration transformations on $\mathcal{H}_{m,s}$ are known, it is possible to compute the analog of fictitious forces in quantum mechanics by applying the acceleration operators to the time-dependent Schrödinger equation. The resulting fictitious forces are proportional to the mass of the particle, and for linear accelerations, proportional to the position operator. This means that in a position representation, fictitious forces can simulate constant gravitational forces, which is the principle of equivalence in nonrelativistic physics. As will be shown, it is also possible to explicitly solve the time-dependent Schrödinger equation for such potentials by transforming to an appropriate noninertial frame.

We begin by looking at acceleration transformations that form the Euclidean line group $\mathcal{E}(3)$. Consider the acceleration transformations

$$\begin{aligned} &\vec{x} \to \vec{x}\,' = \vec{x} + \vec{a}(t) \quad \text{(linear acceleration)}, \\ &\vec{x} \to \vec{x}\,' = \vec{R}(t)\vec{x} \quad \text{(rotational acceleration)}, \end{aligned} \tag{1}$$

where $R(t) \in SO(3)$ is a rotation and $\vec{a}(t) \in \mathbb{R}^3$ is a three-dimensional translation. Both of these types of transformations are indexed by the time parameter t, so that the Euclidean line group consists of maps $\mathbb{R} \to E(3)$, from the real line to the Euclidean group in three dimensions. Such an infinite-dimensional group contains all transformations that preserve the distance between two points in the three-dimensional space $(\vec{x} - \vec{y}\,)^2$.

Linear accelerations contain translations and Galilei boosts of the Galilei group,

$$\begin{aligned} &\vec{x}\,' = \vec{x} + \vec{a} \quad \text{(translations)}, \\ &\vec{x}\,' = \vec{x} + \vec{v}t \quad \text{(Galilei boosts)}, \end{aligned} \tag{2}$$

as well as such acceleration transformations as constant accelerations,

$$\vec{x}\,' = \vec{x} + \frac{1}{2}\,\vec{a}t^2 \quad \text{(constant accelerations)}. \tag{3}$$

Similarly, rotational accelerations contain constant rotations of the Galilei group as well as constant angular velocity rotations,

$$R(t) = R(\hat{n}, \omega t), \tag{4}$$

where $\hat{n}$ is the axis of rotation and ωt is the angle of rotation.

Given the group $\mathcal{E}(3)$, we wish to find its unitary projective representations on the Hilbert space for a nonrelativistic particle of mass m and spin s, namely $\mathcal{H}_{m,s} = L^2(\mathbb{R}^3) \times V^s$, where V^s is the $2s+1$ dimensional complex vector space for a particle of spin s [1]. In momentum space, the wave functions $\varphi(\vec{p}, m_s) \in \mathcal{H}_{m,s}$ transform under the Galilei group elements as

$$\begin{aligned} &(U_{\vec{a}}\varphi)(\vec{p}, m_s) = e^{-i\vec{P}\cdot\vec{a}/\hbar}\varphi(\vec{p}, m_s) = e^{-i\vec{p}\cdot\vec{a}/\hbar}\varphi(\vec{p}, m_s), \\ &(U_{\vec{v}}\varphi)(\vec{p}, m_s) = e^{-i\vec{X}\cdot m\vec{v}/\hbar}\varphi(\vec{p}, m_s) = \varphi(\vec{p} + m\vec{v}\,), \\ &(U_R\varphi)(\vec{p}, m_s) = e^{-i\vec{J}\cdot\hat{n}\theta}\varphi(\vec{p}, m_s) = \sum_{m_s'=-s}^{+s} D^s_{m_s m_s'}(R)\varphi(R^{-1}\vec{p}, m_s'), \end{aligned} \tag{5}$$

where $R \in SO(3)$ is a rotation, and $D^s_{m_s m'_s}(R)$ is a Wigner D function. $\vec{P}$, $\vec{X}$, and $\vec{J}$ are, respectively, the momentum, position, and angular momentum operators. The goal is to find $U_{\vec{a}(t)}$ and $U_{R(t)}$ as unitary operators on $\mathcal{H}_{m,s}$, where $\vec{a}(t)$ and $R(t)$ are elements of $\mathcal{E}(3)$.

In Ref. [2], we have shown how to calculate $U_{\vec{a}(t)}$ and $U_{R(t)}$ by looking at the generating functions for a particle in classical mechanics and then changing Poisson brackets to commutators in quantum mechanics. Here we will illustrate the basic idea by considering linear accelerations in only one spatial dimension, namely

$$x' = x + a(t). \tag{6}$$

A generating function for such a transformation is given by

$$\begin{aligned} &F(x,p') = (x + a(t))p' - mx\dot{a}(t), \\ &x' = \frac{\partial F}{\partial p'} = x + a(t), \\ &p = \frac{\partial F}{\partial x} = p' - m\dot{a}(t), \qquad \dot{a}(t) := \frac{da}{dt}, \\ &H'(x',p') = H(x,p) + \frac{\partial F}{\partial t}. \end{aligned} \tag{7}$$

To get the operators that generate various acceleration transformations, we write $a(t)$ as a power series in t,

$$a(t) = \sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \ , \tag{8}$$

where the expansion coefficients a_n play the role of group parameters for the one-parameter subgroups of linear accelerations. Note that a_0 generates spatial translations, while a_1 generates Galilei boost transformations [see Eq. (2)].

For each one-parameter subgroup specified by a_n, we compute the infinitesimal generating functions $A_n(x,p)$, which, because of the group properties of $\mathcal{E}(3)$, will close under Poisson bracket operations. $A_n(x,p)$ comes from the generating function $F_n(x,p')$ relative to the group element a_n:

$$\begin{aligned} &F_n(x,p') = \left(x + \frac{a_n}{n!}\, t^n\right) p' - mx\, \frac{a_n t^{n-1}}{(n-1)!}, \\ &x' = x + \frac{a_n t^n}{n!}, \\ &p' = p + m\,\frac{a_n}{(n-1)!}\, t^{n-1}, \\ &A_n(x,p) = \frac{t^n}{n!}\, p - \frac{m t^{n-1}}{(n-1)!}\, x\ , \quad n = 1, 2, \ldots \\ &A_0(x,p) = p. \end{aligned} \tag{9}$$

Then the Poisson brackets of A_n with $A_{n'}$ close to give

$$\begin{aligned} &\{A_n, A_{n'}\} = \frac{m t^{n+n'-1}}{(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'}\right)\ , \quad n, n' \neq 0, \\ &\{A_n, A_0\} = -\frac{m t^{n-1}}{(n-1)!}. \end{aligned} \tag{10}$$

Since A_0 generates spatial translations, which are related to the momentum operator, we want the canonical commutation relations $\{x, p\} = 1$. But from Eq. (10) it is seen that $\{A_1, A_0\} = -m$. So define

$$\begin{aligned} B_n &:= -\frac{1}{m} A_n , \quad n = 1, 2, \ldots \\ B_0 &:= A_0, \end{aligned} \tag{11}$$

which gives

$$\begin{aligned} \{B_n, B_{n'}\} &= \frac{t^{n+n'-1}}{m(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'}\right) , \quad n, n' \neq 0, \\ \{B_n, B_0\} &= \frac{t^{n-1}}{(n-1)!}. \end{aligned} \tag{12}$$

In particular, $\{B_1, B_0\} = 1$. These equations are the starting point for introducing acceleration operators into quantum mechanics, for they provide a projective representation of the Lie algebra of the one-dimensional linear accelerations, in which commutators replace Poisson brackets. That is

$$\begin{aligned} &B_n \to X_n := \tfrac{t^{n-1}}{(n-1)!}\, i\hbar\, \tfrac{\partial}{\partial p} - \tfrac{t^n}{mn!}\, p , \quad n = 1, 2, \ldots \\ &B_0 \to P = p, \\ &[X_n, X_{n'}] = \frac{i\hbar t^{n+n'-1}}{m(n-1)!(n'-1)!} \left(\frac{1}{n} - \frac{1}{n'}\right) I , \quad n, n' \neq 0, \\ &[X_n, P] = \frac{i\hbar t^{n-1}}{(n-1)!}\, I, \\ &[X_1, P] = i\hbar I, \end{aligned} \tag{13}$$

where I is the identity operator.

But $X_1 = i\hbar(\partial/\partial p) - (t/m)p$ is not the usual position operator, $X = i\hbar(\partial/\partial p)$. The appendix of Ref. [2] shows that X_1 is unitarily equivalent to X. X_1 is a perfectly good position operator and we continue to use it because the form of $U_{a(t)}$ is particularly simple.

The operators X_n are readily exponentiated; as shown in Ref. [2], the unitary operator implementing the acceleration transformation $a(t)$ is then

$$(U_{a(t)}\varphi)(p) = e^{i(a(t)p/\hbar)}\varphi(p + m\dot{a}(t)) . \tag{14}$$

This can be generalized to the full $\mathcal{E}(3)$ group to give

$$\begin{aligned} (U_{\vec{a}(t)}\varphi)(\vec{p}, m_s) &= e^{i(\vec{a}(t)\cdot\vec{p}/\hbar)}\varphi(\vec{p} + m\dot{\vec{a}}(t)), \\ (U_{R(t)}\varphi)(\vec{p}, m_s) &= \sum_{m'_s} D^s_{m_s m'_s}(R(t))\varphi(R^{-1}(t)\vec{p}, m'_s), \end{aligned} \tag{15}$$

which are the unitary operators implementing acceleration transformations on $\mathcal{H}_{m,s}$. These operators form a unitary projective representation of $\mathcal{E}(3)$ with multiplier

$$\omega(a_1, a_2) = \frac{m}{\hbar}\, \dot{\vec{a_1}}(t) \cdot \vec{a}_2(t) .$$

The unitary operators, Eq. (15), can be used to derive the form of fictitious potentials that arise in noninertial reference frames. Let $g(t)$ denote either $\vec{a}(t)$ or $R(t)$ in $\mathcal{E}(3)$ and let $\psi' = U_{g(t)}\psi$ be the wave function in the noninertial frame obtained from the wave function ψ in the inertial frame under the transformation $g(t)$. Applying $U_{g(t)}$ to the time-dependent Schrödinger equation valid in an inertial frame gives

$$\begin{aligned} i\hbar U_{g(t)} \frac{\partial \psi_t}{\partial t} &= U_{g(t)} H \psi_t, \\ i\hbar \left[\frac{\partial}{\partial t}(U_{g(t)}\psi_t) - \frac{\partial U_{g(t)}}{\partial t}\, \psi_t \right] &= U_{g(t)} H U^{-1}_{g(t)} U_{g(t)} \psi_t, \\ i\hbar \frac{\partial}{\partial t}\, \psi'_t &= \left[H' + i\hbar \frac{\partial U_{g(t)}}{\partial t}\, U^{-1}_{g(t)} \right] \psi'_t; \end{aligned} \tag{16}$$

here $H' := U_{g(t)} H U^{-1}_{g(t)}$ is the transformed Hamiltonian in the noninertial reference frame, and

$$\frac{\partial U_{g(t)}}{\partial t} := \lim_{\epsilon \to 0} \frac{U_{g(t+\epsilon)} - U_{g(t)}}{\epsilon}\ . \tag{17}$$

Since $U_{g(t)}$ is known explicitly, the quantum fictitious potential can be computed; the result is that

$$\begin{aligned} i\hbar \frac{\partial U_{\vec{a}(t)}}{\partial t}\, U^{-1}_{\vec{a}(t)} &= m\ddot{\vec{a}}(t) \cdot \vec{X} - \dot{\vec{a}}(t)\vec{P} + m\vec{a}(t) \cdot \ddot{\vec{a}}(t) I, \\ i\hbar \frac{\partial U_{R(t)}}{\partial t}\, U^{-1}_{R(t)}\Big|_{m_s, m'_s} &= -\hbar\vec{\omega}(t) \cdot [\vec{L}\delta_{m_s m'_s} + \vec{S}^s_{m_s m'_s}]; \end{aligned} \tag{18}$$

the angular velocity $\vec{\omega}(t)$ is obtained from the angular momentum matrix $\Omega(t) := \dot{R}(t)\, R^{-1}(t)$, which is antisymmetric,

$$\omega_i(t) = \frac{1}{2}\, \epsilon_{ijk} \Omega_{jk}(t)\ . \tag{19}$$

$\vec{S}^s_{m_s m'_s}$ are the angular momentum matrices for spin s,

$$\vec{S}^s_{m_s m'_s} := \langle s m_s | \vec{J} | s m'_s \rangle\ . \tag{20}$$

If the Hamiltonian is the free particle Hamiltonian, $H_0 = \vec{P}^2/2m$, then under linear accelerations,

$$\begin{aligned} U_{\vec{a}(t)} H_0 U^{-1}_{\vec{a}(t)} &= \frac{1}{2m} U_{\vec{a}(t)} \vec{P} \cdot \vec{P} U^{-1}_{\vec{a}(t)} = H_0 + \dot{\vec{a}}(t) \cdot \vec{P} + \frac{m}{2}\, \dot{\vec{a}}(t) I, \\ H_{\text{accel}} &:= U_{\vec{a}(t)} H_0 U^{-1}_{\vec{a}(t)} + i\hbar \frac{\partial U_{\vec{a}(t)}}{\partial t} U^{-1}_{\vec{a}(t)} \\ &= H_0 + m \left[\ddot{\vec{a}}(t) \cdot \vec{X} + \tfrac{\dot{\vec{a}}(t) \cdot \dot{\vec{a}}(t)}{2}\, I + \vec{a}(t) \cdot \ddot{\vec{a}}(t) I \right] \end{aligned} \tag{21}$$

and the fictitious potential is proportional to m, as expected.

If it were not known how to couple an external gravitational field to a quantum mechanical particle, the nonrelativistic version of the principle of equivalence could be used to link a constant acceleration with a constant gravitational field. That is, for $\vec{a}(t) = 1/2\vec{a}t^2$, $\vec{a}$ constant, the fictitious potential (in a position representation and neglecting the terms related to I) is

$$V_{\text{fic}}(\vec{x}\,) = m\vec{a} \cdot \vec{x} \tag{22}$$

which is the potential for a constant gravitational field.

This reasoning can be turned around. Say, we are given a potential (for simplicity in one dimension) of the form $V_t = f(t)X$, a potential linear in X, but varying arbitrarily in time. An example is a spatially constant external electric field that varies in an arbitrary manner in time. The time-dependent Schrödinger equation for such a system in a momentum representation is

$$i\hbar \frac{\partial \varphi_t}{\partial t} = \left(\frac{p^2}{2m} + f(t) i\hbar \frac{\partial}{\partial p} \right) \varphi_t. \tag{23}$$

If this equation is transformed to a noninertial reference frame, the acceleration $a(t)$ can be chosen so as to cancel off the effect of $f(t)$. That is, if $m\ddot{a}(t) + f(t) = 0$, the $\partial/\partial p$ terms cancel and the Schrodinger equation becomes

$$i\hbar \frac{\partial \varphi'_t}{\partial t} = \left(\frac{p^2}{2m} + \frac{m\dot{a}^2}{2} \right) \varphi'_t,$$

which can readily be solved. Transforming back to the inertial frame gives the solution

$$\begin{aligned} \varphi_t(p) &= U^{-1}_{a(t)} \varphi'_t(p) = \varphi_{t=0}(p - m\dot{a}(t)) \\ &\times \exp\left\{ -\frac{i}{\hbar} \left[\frac{p^2}{2m} t + (a(t) - t\dot{a}(t))p + \frac{m}{2} \Big(t\dot{a}^2(t) + \int^t (\dot{a}^2 - 2a(t)\ddot{a}(t)) \Big) \right] \right\}, \end{aligned} \tag{24}$$

with $\varphi_{t=0}(p) \in L^2(\mathbb{R})$ the wave function at $t = 0$.

To conclude this paper, we briefly discuss the question of how to generalize these nonrelativistic results to relativistic quantum mechanics. Relativistic means that the Galilei group is replaced by the Poincaré group, consisting of Lorentz transformations and space-time translations. The first problem that arises is that there are a number of different ways of formulating relativistic quantum mechanics, the most prominent being quantum field theory. These different formulations all carry representations of the Poincaré group in one way or another, but they differ in how interactions are introduced. One way in which interactions can be introduced is through the Poincaré generators. In such a formulation, some generators contain interactions, others not. Dirac [3] classified three such possibilities as instant, front, and point forms. The instant form is the most familiar form, in that the Poincaré generators not containing interactions are the Euclidean subalgebra of rotations and spatial translation generators. More recently, the front form of relativistic quantum mechanics has been of great interest [4].

However, to analyze accelerations in relativistic quantum mechanics, the point form [5], wherein all interactions are put into four-momentum generators, is the natural form to use, for it is the only form which is manifestly covariant under Lorentz transformations (the generators of Lorentz transformations do not contain interactions).

In the point form of relativistic quantum mechanics, the interacting four-momentum operators must satisfy the (Poincaré) conditions

$$\begin{aligned} &[P^\mu, P^\nu] = 0, \\ &U_\Lambda P^\mu U_\Lambda^{-1} = (\Lambda^{-1})^\mu{}_\nu P^\nu, \end{aligned} \tag{25}$$

where U_Λ is the unitary operator representing the Lorentz transformation Λ on an appropriate Hilbert space $\mathcal{H}$ (which may be a Fock space or a subspace of a Fock space). Since the four-momentum operators all commute with one another, they can be simultaneously diagonalized, and used to construct the invariant mass operator,

$$M := \sqrt{P^\mu P_\mu}\ . \tag{26}$$

The spectrum of M must be bounded from below; the discrete part of the spectrum of M corresponds to bound states and the continuous part to scattering states.

However, the most important feature of the point form is that the time-dependent Schrödinger equation naturally generalizes to

$$i\hbar \frac{\partial \psi_x}{\partial x_\mu} = P^\mu \psi_x, \tag{27}$$

where x is the space-time point $(ct, \vec{x}\,)$. This relativistic Schrödinger equation simply states that the interacting four-momentum operators act as generators of space-time translations, in a Lorentz covariant manner. But its importance with regard to acceleration transformations is that relativistic fictitious forces will arise in exactly the same way as the nonrelativistic ones arose in Eq. (16).

This leads to a second problem, namely finding the generalization of $\mathcal{E}(3)$ for relativistic acceleration. Since, in relativistic mechanics space and time are on an equal footing, time cannot be an independent parameter, as was the case for $\mathcal{E}(3)$. But since the principle of equivalence links acceleration to general relativity [6], the natural group to consider is the diffeomorphism group on the Minkowski space, the group of invertible differentiable maps from the Minkowski manifold to itself, Diff($\mathcal{M}$). This group has both $\mathcal{E}(3)$ and the Poincaré group as subgroups.

If the transformation

$$x^\mu \to x'^\mu = f^\mu(x^\nu) \tag{28}$$

is an element of Diff($\mathcal{M}$), then what is needed is a unitary representation of Diff($\mathcal{M}$) on $\mathcal{H}$,

$$f \in \mathrm{Diff}(\mathcal{M}) \to U_f \quad \text{on} \quad \mathcal{H}, \tag{29}$$

such that in the limit as $c \to \infty$, the representation of Diff($\mathcal{M}$) contracts to a representation of $\mathcal{E}(3)$. For such U_f, the relativistic fictitious force is given by

$$i\hbar \frac{\partial U_f}{\partial x_\mu} U_f^{-1}, \tag{30}$$

and, from the principle of equivalence, shows how to couple an external gravitational field to a relativistic particle of arbitrary mass and spin. Details of these ideas will be carried out in a future paper.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 262–269.

On Relativistic Non-linear Quantum Mechanics

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Abstract

The possibility that quantum mechanics itself could be non-linear has run up against difficulties with relativistic covariance. Most of the schemes proposed up to now engender superluminal communication, and those that don't, have been equivalent to linear theories. We show in a simplified model that a proposal based on the consistent histories approach to quantum mechanics avoids the usual difficulties and a relativistic quantum theory with non-linearly defined histories is possible.

1 Introduction

There is presently a growing interest in non-linear quantum mechanics resulting from a variety of motivations: fundamental speculation, presence of gravity, string theory, representations of current algebras, etc. Although apparently well motivated, it became apparent that non-linear theories suffer from some *prima facie* serious difficulties. These are of various types, but the most notable is conflict with relativity or causality. N. Gisin [1, 2] and G. Svetlichny [3] pointed out that non-linearity allows us to use EPR-type correlations and the instantaneous nature of state-vector collapse to send a signal across a space-like interval. Analyzing further, one finds that one has in fact a contradiction with relativity [4]. Certain progress has been made in overcoming these difficulties. One of the proposals is based on the idea that since the difficulty stems from the instantaneous state-vector collapse in measurement, a modification of measurement algorithms could allow for non-linear processes without superluminal signals. G. A. Goldin, H.-D. Doebner and P. Nattermann [5, 6, 7] have argued that non-linearity *per se* does not lead to superluminal signals (this was also pointed out by Svetlichny [4]), as with the *prima facie* reasonable assumptions that all measurements are in the end expressible in terms of measurements of position, certain non-linear Schrödinger equations are then observationally equivalent, via a non-linear "gauge transformation", to the free linear equation. We shall call these the GDN theories. Unfortunately, these theories, and others studied by these authors, are non-relativistic, and we are still far from understanding the true relation of linearity to relativity.

Here we adopt an even more radical view and reconsider the question from the point of view of a quantum theory without measurements, as the complete absence of the "measurement process" will eliminate any obstruction to non-linearities from the manifest non-covariance of this process. Of the several "measurementless" theories, the one most adaptable to relativistic considerations is the consistent histories approach already widely discussed in the literature [8, 9].

We argue that in the histories approach, non-linearity and relativistic covariance can indeed coexist peacefully and present a simple model to support this view. Such a model cannot yet be taken as a proposal for a realistic theory but does establish the logical point and suggests where one should look for experimental evidence.

2 Linear and non-linear histories

Let $\mathcal{H}$ be a Hilbert space and $\Psi \in \mathcal{H}$ a normalized vector. For each $i = 1, \ldots, n$, let $P_j^{(i)}$ where $j = 1, 2, \ldots, n_i$, be a finite resolution of the identity. We call each state vector of the form

$$P_\alpha \Psi = P_{\alpha_n}^{(n)} \cdots P_{\alpha_j}^{(j)} \cdots P_{\alpha_2}^{(2)} P_{\alpha_1}^{(1)} \Psi \tag{1}$$

a *history*. Let

$$p_\alpha = ||P_\alpha \Psi||^2. \tag{2}$$

One interpretation of the above quantities is that Ψ is a Heisenberg state and that $P_j^{(i)}$ is the spectral resolution of a Heisenberg observable $A^{(i)} = \sum \lambda_j^{(i)} P_j^{(i)}$ at time t_i where $t_1 < t_2 < \cdots < t_{n-1} < t_n$. In this case, p_α is the joint probability of getting the sequence of outcomes $\lambda_{\alpha_1}^{(1)}, \ldots, \lambda_{\alpha_n}^{(n)}$ in a sequence of measurements that correspond to the observables $A^{(1)}, \ldots, A^{(n)}$. The coherent histories interpretation of quantum mechanics however goes beyond this viewpoint and in certain special cases interprets p_α as the probability of the history Q_α even if no actual measurements are made. It is a way of assigning probabilities to alternate views of the quantum state Ψ, corresponding to the possible different sequences $\alpha = (\alpha_1, \ldots, \alpha_n)$. Such an attitude is maintained only if a condition, called *consistency*, or even a stronger condition called *decoherence*, is satisfied by the set of alternative histories.

The notion of consistent histories forms the basis of a new interpretation of quantum mechanics that in a certain sense transcends at the same time the Copenhagen interpretation and the Everett many-worlds one. As such it has attracted the attention of cosmologists. Its main feature that makes it attractive to the present case is that it does not rely on the notion of measurement nor on the collapse of the wave-function. Thus even though (1) can be interpreted as a sequence of evolutions and collapses, this is not essential, and (2) can be viewed as merely a formula for a joint probability. In a more generalized setting, the evolution-collapse picture is not even possible for some sets of histories. This view of quantum mechanics thus transcends the notions of instantaneous state, its evolution, and its collapse, which means that it is well suited for formulating theories in which these notions are troublesome, such as non-linear quantum mechanics.

The most naive way to adapt the consistent histories approach to non-linear quantum mechanics is to replace in (1) the linear projectors $P_j^{(i)}$ by non-linear operators $B_j^{(i)}$ and so introduce the *non-linear histories*

$$B_\alpha \Psi = B_{\alpha_n}^{(n)} \cdots B_{\alpha_j}^{(j)} \cdots B_{\alpha_2}^{(2)} B_{\alpha_1}^{(1)} \Psi,$$

with the corresponding probability function

$$b_\alpha = ||B_\alpha \Psi||^2.$$

Such expressions are in fact the correct ones for a succession of measurements for the GDN theories.

The most primitive property that the operators B should satisfy is that $\sum_\alpha b_\alpha = 1$. This is true in particular if one has $\sum_j ||B_j^{(i)}\Phi||^2 = 1$ for every i and all Φ , which is the case of GDN.

To complete the rest of the program and have an *interpretation* of this non-linear quantum mechanics, similar to the consistent histories approach of linear quantum mechanics, one needs to address the notions of consistency in the nonlinear context. There is no *a priori* difficulty in formulating such a notion, though the stronger notion of decoherence may not survive the passage to nonlinearity.

We shall not address the interpretational issues in this paper and only limit ourselves to showing that one can pass on to non-linearity while maintaining Lorentz covariance. Again, the most naive way to envisage Lorentz covariance is to assume that there is a unitary representation $U(g)$ of the Poincaré group along with an action ϕ_g of the same on suitable non-linear operators such that it makes sense to talk of the transformed histories

$$\tilde{B}_\alpha \tilde{\Psi} = \tilde{B}^{(n)}_{\alpha_n} \cdots \tilde{B}^{(j)}_{\alpha_j} \cdots \tilde{B}^{(2)}_{\alpha_2} \tilde{B}^{(1)}_{\alpha_1} \tilde{\Psi}.$$

where $\tilde{B} = \phi_g(B)$ and $\tilde{\Psi} = U(g)\Psi$. Lorentz covariance would then be expressed through the statement $\tilde{b}_\alpha = b_\alpha$. Such a scheme holds in the GDN case for Euclidean and Galileian covariances.

Now it should not be very hard to implement the above scheme without further constraints, but for an interesting theory, one should require a locality condition that would preclude superluminal signals. It would only be then that one could say that one has overcome the relativistic objections to non-linear theories. This is the concern of the next sections.

3 Free quantum fields

This section is based on the suggestion presented in [4]. Consider a free neutral scalar relativistic quantum field. For each limited space-time region $\mathcal{O}$, let $\mathcal{A}(\mathcal{O})$ be the von-Neuman algebra of observables associated to $\mathcal{O}$. Consider now a set of limited space-time regions $\mathcal{O}_1, \ldots \mathcal{O}_n$ which are so disposed that for any two, either all points of one are space-like in relation to all points of the other, or they are time-like. Assume the regions are numbered so that whenever one is in the time-like future of another, then the first one has a greater index. Let $P_i \in \mathcal{A}(\mathcal{O}_i)$ be orthogonal projections that correspond to outcomes of measurements made in the corresponding regions. Let Ψ represent a Heisenberg state in some reference frame and prior to all measurements. According to the usual rules, the probability to obtain all the outcomes represented by the projections is: $||P_n \cdots P_2 P_1 \Psi||^2$. We will modify this formula by replacing P_i by B_i, a possibly non-linear operator, likewise somehow associated to the region $\mathcal{O}_i$, whenever there is a region $\mathcal{O}_j$ that is time-like past to the given one. This effectively differentiates between space-like and time-like conditional probabilities. For this to be consistent, relativistic, and causal, the (in general non-linear) operators B_i have to satisfy certain constraints. There are several ambiguities in the above construction. The relative order of the $\mathcal{O}_i$ is not determined except for the case of time-like separation. Presumably, the ambiguity of the order corresponds to the possible choices of the time-like reference direction. Even for some of these, two space-like separated regions may not be separated by the time coordinate, in such cases we shall suppose that the relative order does not matter. This means that we should allow certain permutations of the sequence $\mathcal{O}_1, \ldots, \mathcal{O}_n$ and the choice of such a permutation must not affect the final assignment of probabilities. We shall call each allowed permutation an *admissible sequence*. There is also the ambiguity in the relation $P_i \in \mathcal{O}_i$, resulting from the inclusion $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$ whenever $\mathcal{O}_1 \subset \mathcal{O}_2$. This too must not affect the final assignment. The operators B_i can depend on several aspects of the construction. In principle, each B_i can depend on the full set of constituents $\{\Psi, P_1, \mathcal{O}_1, \ldots, P_n, \mathcal{O}_n\}$ and even, in a self-consistent manner, on the choices of the other B_j. Such generality leaves little room for insight. Given that, we are trying to establish here a point of principle, that non-linear relativistic quantum mechanics *is* possible, and not propose what would be a realistic theory, we shall look for the simplest type of modification. For a typical datum $P \in \mathcal{A}(\mathcal{O})$ in an admissible sequence, we shall at times write $B_{\mathcal{O}}$ for the corresponding B.

1. If $\mathcal{O}_i$ and $\mathcal{O}_j$ are space-like separated, then
$$[B_i, P_j] = [P_i, B_j] = [B_i, B_j] = 0 \tag{3}$$
2. If $\mathcal{O} \subset \mathcal{O}'$ and $P \in \mathcal{A}(\mathcal{O})$ is a projector, then in any sequence in which P and $\mathcal{O}$ take place for which changing $\mathcal{O}$ to $\mathcal{O}'$ results in a new admissible sequence (with, of course, identical spatial-temporal relation between the regions), one has $B_{\mathcal{O}'} = B_{\mathcal{O}}$
3. If $U(g)$ is a unitary operator representing the element g of the Poincaré group, then $B_{g\mathcal{O}} = U(g) B_{\mathcal{O}} U(g)^*$
4. For all resolutions of identity $P_j \in \mathcal{A}(\mathcal{O})$ one has $\sum ||B_j \Phi||^2 = ||\Phi||^2$ for all states Φ.

The bracket in (3) is a commutator, for example, $[B_i, P_j] = B_i P_j - P_j B_i$ and not the Lie bracket of the two operators interpreted as vector fields on Hilbert space, which for non-linear operators would be different.

We leave the problem of finding operators B_i to the next section and first discuss some of the consequences.

Let us now pick in each space-time region $\mathcal{O}_i$ a finite resolution of the identity $P_j^{(i)}$, where $j = 1, 2, \ldots, n_i$ with $P_j^{(i)} \in \mathcal{A}(\mathcal{O}_i)$. We have $\sum_j P_j^{(i)} = I$.

Consider now the expression
$$p_\alpha = ||B_{\alpha_1}^{(n)} \cdots B_{\alpha_j}^{(j)} \cdots B_{\alpha_2}^{(2)} B_{\alpha_1}^{(1)} \Psi||^2, \tag{4}$$
where $B_j^{(i)} = P_j^{(i)}$ if there is no region $\mathcal{O}_k$ to the time-like past of $\mathcal{O}_j$ and a possibly different operator if there is. Expression (4) is to be interpreted as the joint probability distribution of alternate histories as discussed in the previous section. From property (4) it follows that $\sum_\alpha p_\alpha = 1$ so that the interpretation as a joint probability is consistent. Property (1) assures us that the mentioned ambiguity in the temporal order of space-like separated regions does not affect the numerical values of the probabilities p_α, but only the way they are labeled. Property (3) assures us that the mentioned ambiguity in associating

a region to a projector leaves the resulting probabilities the same. Finally, property (3) assures us that if we replace the data

$$\{\Psi, P_1, \mathcal{O}_1, \ldots, P_n, O_n\}$$

by

$$\{U(g)\Psi, U(g)P_1U(g)^*, g(\mathcal{O}_1), \ldots, U(g)P_nU(g)^*, g(O_n)\},$$

the resulting probabilities don't change, that is, the theory is relativistically covariant. One still does not know how to compute joint probabilities for events in regions that are neither space-like nor time-like to each other, nor exactly how to interpret the formalism based on the consistent histories approach. We leave this question for posterior investigation. In any case, when only space-like separated regions occur in the histories, then the situation is the same as in conventional quantum mechanics.

To show that joint probabilities as defined above do not lead to superluminal signals, suppose one region $\mathcal{O}$ is space-like separated in relation to all the others. Then by the considerations above, one can label the regions so that $\mathcal{O} = \mathcal{O}_1$. The probability of observing the event that corresponds to $P_j^{(1)}$ is, if no other observations are made, given by $||P_j^{(1)}\Psi||^2$, and by condition (4) it is also $\sum_{\alpha_2,\ldots,\alpha_n} p_{\alpha_n,\ldots,\alpha_2,j}$ that is, the marginal probability if the other observations *are* made. This means that the probability of an event is independent of what happens in space-like separated regions, and so no signals using long-range correlations are possible, just as in the linear case.

One would thus have a non-linear relativistic quantum mechanics if conditions (1–4) can be realized.

4 A simple explicit model

Free fields can be realized in appropriate Fock spaces. One has $\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_0 = \mathbb{C}$ is the subspace spanned by the vacuum, and each $\mathcal{H}_n$ is the n-particle subspace. We consider a free scalar field $\phi(x)$ of mass m. In configuration space, H_n now consists of symmetric functions $\Phi(x_1, \ldots x_n)$ of n space-time coordinates which obey a Klein-Gordon equation in each space-time variable and contain only positive-frequency Fourier components in each momentum variable.

A very simple way of satisfying (1–4) is to set in the time-like case $B_j^{(i)} = BP_j^{(i)}B^{-1}$, where B is an invertible not-necessarily linear operator that is Poincaré invariant, $U(g)BU(g)^* = B$. In this case, it is easily seen that all the conditions are automatically satisfied except possibly for the case of condition (1) which involves a modified and a non-modified projector, that is,

$$[BPB^{-1}, Q] = 0 \tag{5}$$

if P and Q are orthogonal projectors belonging to space-like separated regions, and condition (4) which would be satisfied if B were norm-preserving $||B\Psi|| = ||\Psi||$. If B is a real homogeneous operator $Br\Psi = rB\Psi$ for real r, then one can define a new operator $(||\Psi||B\Psi)/||B\psi||$ which is now norm-preserving and continues to satisfy all the other desired properties, so we shall not concern ourselves anymore with norm-preservation.

A stronger condition than (5) would be to assume that

$$[BAB^{-1}, C] = 0 \tag{6}$$

if A and C are operators belonging to von-Neuman algebras of space-like separated regions. Such a condition may seem somewhat strong given that B is supposed to be non-linear, but one sees similar situations in the GDN theories, in which the non-linear operators are in fact linear on spaces generated by functions with disjoint supports. In the GDN theories such a property follows essentially from homogeneity and the local character of differential operators. As such we can hope to achieve it in our case also. In particular, one should have for smeared fields

$$[B\phi(f)B^{-1}, \phi(g)] = 0 \tag{7}$$

for f and g with space-like separated supports.

We must now face the task of finding an appropriate B. Now it is not hard to find Poincaré invariant non-linear operators. As an example, for each n, let M_n be a permutation and Lorentz invariant non-linear differential operator acting on a function $g(x_1, \ldots, x_n)$ of n space-time points. One can apply M_n to the n-particle component Φ_n of a vector in a Fock space. Now $M_n\Phi_n$ is not necessarily a positive-frequency solution of the Klein-Gordon equation, but we can then convolute it with an appropriate Green's function. Define the operator C by $(C\Phi)_n = \Delta^{(+)\otimes n} \star M_n\Phi_n$ where $\Delta^{(+)\otimes n}$ is the n-fold tensor product of $\Delta^{(+)}(x)$, the positive frequency invariant Green's function for the Klein-Gordon equation, and $\star$ denotes convolution. Much more elaborate operators in which the various n-particle sectors get coupled can also be constructed.

The difficulty in (7) is of course the presence of B^{-1}. We shall try to overcome this by assuming that B is a part of a one-parameter group $B(r)$ generated by a non-linear operator K. Thus, the equation $\frac{d}{dr}\Phi(r) = K\Phi(r)$ is solved by $\Phi(r) = B(r)\Phi(0)$. We assume $B = B(1)$ and that (7) holds for each $B(r)$. To the first order, one then has

$$[[K, \phi(f)], \phi(g)] = 0 \tag{8}$$

for f and g with space-like separated supports. This equation imposes a recursive series of constraints on the n-particle operators K_n for which, however, there are no formal obstructions. We shall not here go into a full analysis of these constraints as the typical situation already arises when we apply (8) to the vacuum state. We assume that the K_n operators do not change the number of particles. One must have $K_0 = 0$ as the vacuum is the unique Lorentz invariant state. For a one particle function $g(x)$, let $\tilde{g} = i\Delta^{(+)} \star g$, and let $\hat{\otimes}$ denote the symmetric tensor product. One then derives from (8), applying $[[K, \phi(f)], \phi(g)]$ to the vacuum, that if f and g have space-like separate supports, then

$$K_2\tilde{f}\hat{\otimes}\tilde{g} = (K_1\tilde{f})\hat{\otimes}\tilde{g} + \tilde{f}\hat{\otimes}(K_1\tilde{g}), \tag{9}$$

$$0 = (g, K_1\tilde{f}) + (f, K_1\tilde{g}), \tag{10}$$

where $(g, f) = \int g(x)f(x)\,dx$. Now (9) defines K_2 on the symmetric tensor product of two functions in terms of K_1. This is similar to the tensor derivation property for separating non-linear Schrödinger equations [10]. We can thus assume that the hierarchy K_n is in fact a tensor derivation with respect to the symmetric tensor product. Equation (10) must now hold for functions with space-like separated supports. This will hold, just as it does in the linear free quantum field theory if K_1 does not change the support of a function on which it acts, which is not hard to achieve. The conclusion now is that in fact, at least formally, a causal non-linear relativistic quantum mechanics of the type described in the initial sections of this paper is possible.

5 Conclusions

The previous two sections have argued the logical point that, indeed, relativistic nonlinear histories without superluminal signals are possible. Because of it's *ad hoc* nature, the model presented above cannot be considered realistic. In particular, if the space-time regions involved constitute a time-like chain, then the joint probabilities in the non-relativistic limit would not be of the type governed by a non-linear Schrödinger equation. Thus, we have not yet shown that the GDN theories can be obtained as non-relativistic limits. All the schemes based on non-linear histories which differentiate between space-like and time-like joint-probabilities in principle should exhibit physical effects as one crosses the light cone. Thus, in a typical photon correlation experiment, one can delay the light-ray on one side so that at a certain point the detector events become time-like. In crossing the light cone, an effect should be present that was not forseen by the linear theory. This happens in the models above as $||PQ\Phi||^2$ suddenly becomes $||BQ\Phi||^2$. There are strong plausibility arguments [4, 11] that theories that do not suffer such discontinuities at the future light-cone are necessarily linear, so a true verification of non-linearity would involve light-cone experiments. Theories of the type here considered are thus *light cone singular* and, for such theories, the notion of non-relativistic limit has to be modified. Whereas in usual theories the non-relativistic regime is one for which all relevant velocities are small compared to the velocity of light, in light-cone singular theories one must add the requirement that all relevant space-time intervals be time-like. This further requirement removes the paradox that a causal relativistic theory may have a non-relativistic limit that seems manifestly acausal by allowing instantaneous signals through long-range correlations, and may thus remove a major objection to formulation of non-linear quantum mechanics.

6 Acknowledgement

This research was partially financed by the Ministério de Ciência e Tecnologia (MCT) and by the Conselho Nacional de Desenvolvimento Tecnológico e Científico (CNPq), agencies of the Brazilian government. The author wishes to thank Professor Sheldon Goldstein of Rutgers University for helpful discussions on the consistent histories formalism.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 270–278.

On (Non)Linear Quantum Mechanics

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Abstract

We review a possible framework for (non)linear quantum theories, into which linear quantum mechanics fits as well, and discuss the notion of “equivalence” in this setting. Finally, we draw the attention to persisting severe problems of nonlinear quantum theories.

1 Nonlinearity in quantum mechanics

Nonlinearity can enter quantum mechanics in various ways, so there are a number of associations a physicist can have with the term “nonlinear quantum mechanics”. Because of this, we shall start with a (certainly incomplete) list of those ways that we shall not deal with here.

In quantum field theory, nonlinearity occurs in the equations of interacting field operators. These equations may be quantizations of nonlinear classical field equations (see, e.g., [1]) or mathematically tractable models as in ϕ^4-theory. Here, however, the field operators remain linear, as does the whole quantum mechanical setup for these quantum field theories.

On a first quantized level, nonlinear terms have been proposed very early for a phenomenological and semi-classical description of self-interactions, e.g., of electrons in their own electromagnetic field (see, e.g., [2]). Being phenomenological, these approaches are build on linear quantum mechanics and use the standard notion of observables and states. For complex systems, the linear multi-particle Schrödinger equation is often replaced by a nonlinear single-particle Schrödinger equation as in the density functional theory of solid state physics.

There have also been attempts to incorporate friction on a microscopic level using nonlinear Schrödinger equations. Many of these approaches incorporate stochastic frictional forces in the nonlinear evolution equation for wavefunctions (see, e.g., [3]).

Contrary to these, we are concerned with a more fundamental role of nonlinearity in quantum mechanics. Notable efforts in this direction have been launched, for example, by Bialinycki-Birula and Mycielski [4], Weinberg [5], and Doebner and Goldin [6, 7].

2 Problems of a fundamentally nonlinear nature

There are evident problems if we merely replace (naively) the evolution equation of quantum mechanics, i.e., the linear Schrödinger equation, by a nonlinear variant, but stick to the usual definitions of linear quantum mechanics, like observables being represented by self-adjoint operators, and states being represented by density matrices.

Density matrices $\mathbf{W} \in \mathcal{T}_1^+(\mathcal{H})$ represent in general a couple of different, but *indistinguishable* mixtures of pure states,

$$\sum_j \lambda_j |\psi_j\rangle\langle\psi_j| = \mathbf{W} = \sum_j \lambda'_j |\psi'_j\rangle\langle\psi'_j|, \tag{1}$$

where $\{(\lambda_j, \psi_j)\}_{j=1,\ldots}$ and $\{(\lambda'_j, \psi'_j)\}_{j=1,\ldots}$ are different mixtures of pure states ψ_j and ψ'_j with weights $\sum_j \lambda_j = 1$ and $\sum_j \lambda'_j = 1$, respectively. This identification of different mixtures is evidently *not* invariant under a nonlinear time-evolution Φ_t of the wavefunctions,

$$\sum_j \lambda_j |\Phi_t(\psi_j)\rangle\langle\Phi_t(\psi_j)| \overset{i.g.}{\neq} \sum_j \lambda'_j |\Phi_t(\psi'_j)\rangle\langle\Phi_t(\psi'_j)|. \tag{2}$$

This apparent contradiction has been used by Gisin, Polchinski, and others [8, 9] to predict superluminal communications in an EPR-like experiment for *any* nonlinear quantum theory.

Rather than taking this observation as an inconsistency of a nonlinear quantum theory (e.g., as in [10, 11]), we take it as an indication that the notions of observables and states in a nonlinear quantum theory have to be adopted appropriately [12]. If nonlinear quantum mechanics is to remain a statistical theory, we need a consistent and complete statistical interpretation of the wave function and the observables, and therefore a consistent description of mixed states.

3 Generalized quantum mechanics

In view of the intensive studies on nonlinear Schrödinger equations in the last decade, it is astonishing to note that a framework for a consistent framework of nonlinear quantum theories has already been given by Mielnik in 1974 [13]. We shall adopt this approach here and develop the main ingredients of a quantum theory with nonlinear time evolutions of wavefunctions.

Our considerations will be based on a fundamental hypothesis on physical experiments:

All measurements can in principle be reduced to a change of the dynamics of the system (e.g., by invoking external fields) and positional measurements.

In fact, this point of view, which has been taken by a number of theoretical physicists [13, 14, 15, 16], becomes most evident in scattering experiments, where the localization of particles is detected (asymptotically) after interaction.

Based on this hypothesis, we build our framework for a nonlinear quantum theory on three main "ingredients" [17]:

First, a *topological space* $\mathcal{T}$ of wavefunctions. In the one-particle examples below, this topological space is a Hilbert space of square integrable functions, but we may also think of other function spaces [18].

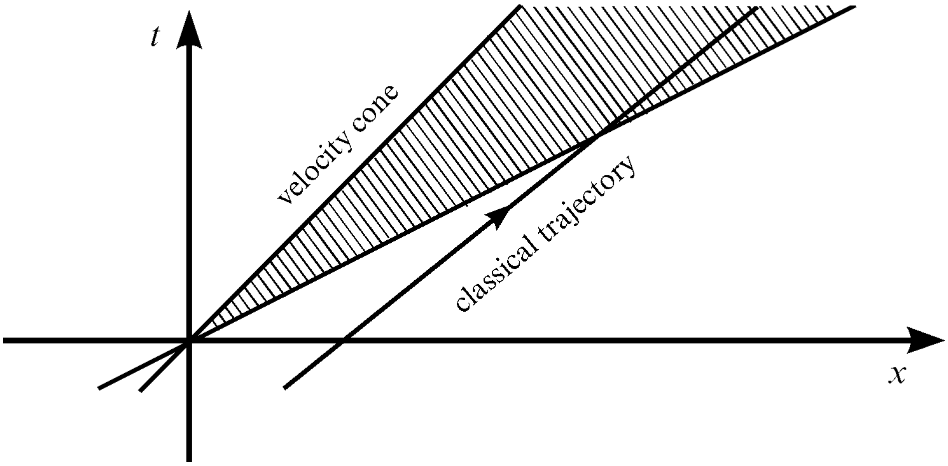

Figure 1: Velocity cone for the asymptotic measurement of momentum.

Secondly, the *time evolutions* are given by homeomorphisms of $\mathcal{T}$,

$$\Phi_t^{(Ext)}\colon \mathcal{T} \to \mathcal{T}\,, \tag{3}$$

which depend on the time interval t and the external conditions (e.g., external fields) $Ext \in \mathcal{C}$. Mielnik's *motion group* $\mathcal{M}$ [13] is the smallest (semi-) group containing *all* time evolutions $\Phi_t^{(Ext)}$, close in the topology of pointwise convergence.

Finally, *positional observables* $\mathcal{P}$ are represented by probability measures on the physical space M, which depend on the wavefunction $\phi \in \mathcal{T}$, i.e., $\mathcal{P} = \{p_B \mid B \in \mathcal{B}(M)\}$, where

$$p_B : \mathcal{T} \to [0,1]\,, \qquad \sum_{k=1}^{\infty} p_{B_k} = p_B\,, \quad B = \bigcup_{k=1}^{\infty} B_k, \tag{4}$$

for disjoint $B_k \in \mathcal{B}(M)$.

We shall call the triple $(\mathcal{T}, \mathcal{M}, \mathcal{P})$ a *quantum system*. Using these basic ingredients, we can define effects and states of the quantum system $(\mathcal{T}, \mathcal{M}, \mathcal{P})$ as derived concepts. An *effect* (or a *counter*) is (at least approximately in the sense of pointwise convergence) a combination of evolutions $T \in \mathcal{M}$ and positional measurements $p \in \mathcal{P}$, i.e.,

$$\mathcal{E} := \overline{\{p \circ T \,|p \in \mathcal{P}, T \in \mathcal{M}\}}^{p.c.} \tag{5}$$

is the *set of effects*. A general *observable* A is an $\mathcal{E}$-valued measure on the set M^A of its classical values,

$$p^A\colon \mathcal{B}(M^A) \to \mathcal{E}\,, \quad p^A_{M^A}[\phi] = 1\,. \tag{6}$$

The standard example of such an asymptotic observable is the (dynamical) momentum of a single particle of mass m in $\mathbb{R}^3$. Let $B \in \mathcal{B}(\mathbb{R}^3_p)$ be an open subset of the momentum space $\mathbb{R}^3_p$, then

$$B_t := \left\{ \frac{t}{m}\vec{p} \;\middle|\; \vec{p} \in B \right\} \tag{7}$$

defines the corresponding velocity cone, see Figure 1.

If $\Phi_t^{(0)}$ denotes the free quantum mechanical time evolution of our theory — provided, of course, there is such a distinguished evolution — the limit

$$p_B^P[\phi] := \lim_{t\to\infty} p_{B_t}\left[\Phi_t^{(0)}[\phi]\right] \tag{8}$$

defines a probability measure on $\mathbb{R}_p^3$, so that the momentum observable is given by the $\mathcal{E}$-valued measure

$$P = \left\{ p_B^P \in \mathcal{E} \;\middle|\; B \in \mathcal{B}(\mathbb{R}_p^3) \right\}. \tag{9}$$

Coming back to the general framework, once we have determined the set of effects, we can define the *states* of the quantum system as equivalence classes of mixtures of wavefunctions.

Different mixtures of wave functions

$$\pi = \{(\lambda_j, \phi_j)\}_{j=1,\dots}, \quad \sum_j \lambda_j = 1, \tag{10}$$

with the corresponding effects $f[\pi] := \sum_j \lambda_j f(\phi_j)$ may be *indistinguishable* with respect to the effects $\mathcal{E}$,

$$\pi_1 \sim \pi_2 \quad \Leftrightarrow \quad \Big(f[\pi_1] = f[\pi_2] \quad \forall\, f \in \mathcal{E} \Big). \tag{11}$$

Hence, the *state space*

$$\mathcal{S} := {}^{\Pi(\mathcal{T})}\!/_{\sim} \tag{12}$$

is a convex set with *pure states* as extremal points $\mathcal{E}(\mathcal{S})$.

4 Linear quantum mechanics

Generalized quantum mechanics is indeed a generalization of linear quantum mechanics, as the latter is contained in the general framework as a special case. To see this, we consider a non-relativistic particle of mass m in $\mathbb{R}^3$ (Schrödinger particle), defined in our setting as a quantum system $(\mathcal{H}, \mathcal{M}_S, \mathcal{P}_\chi)$ with the topological space of wave functions as the Hilbert-space

$$\mathcal{T} \equiv \mathcal{H} \equiv L^2(\mathbb{R}^3, d^3x), \tag{13}$$

the Born interpretation of ${}^{|\psi(\vec{x})|^2}\!/_{\|\psi\|^2}$ as a positional probability density on $\mathbb{R}^3$, i.e.,

$$p_B[\psi] := \frac{\langle\psi|\mathbf{E}(B)\psi\rangle}{\|\psi\|^2} = \frac{\|\mathbf{E}(B)\psi\|^2}{\|\psi\|^2} \tag{14}$$

defines the positional observables, and unitary time evolutions generated by the linear Schrödinger equations

$$i\hbar\partial_t\psi_t = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi_t \equiv \mathbf{H}_V\psi_t\,, \tag{15}$$

with a class of suitable potentials V representing the external conditions of the system.

Starting with these three objects, we recover indeed the full structure of (linear) quantum mechanics. First, the motion group of a Schrödinger particle is the whole unitary group [19]

$$\mathcal{M}_S \simeq U(\mathcal{H})\,. \tag{16}$$

Furthermore, the (decision) effects are given precisely by orthogonal projection operators [20],

$$\mathcal{E} \simeq \mathit{Proj}(\mathcal{H})\,, \quad f_{\mathbf{E}}[\psi] = \frac{\|\mathbf{E}\psi\|^2}{\|\psi\|^2}\,, \tag{17}$$

so that the logical structure of quantum mechanics is recovered; observables occur naturally through their spectral measures in this scheme. For example, the asymptotic definition of momentum along the lines given above is well known in linear quantum mechanics [21] and leads through standard Fourier transform to the usual spectral measure of the momentum operator $\mathbf{P}$.

Finally, as a consequence of the above set of effects, the *state space* coincides with the space of normalized, positive trace class operators,

$$\mathcal{S} \simeq \mathcal{T}_1^+(\mathcal{H})\,, \qquad \mathcal{E}(\mathcal{S}) \simeq P(\mathcal{H})\,. \tag{18}$$

5 Equivalent quantum systems

Having based our discussion on a fundamental hypothesis on the distinguished role of positional measurements in quantum mechanics, the notion of *gauge equivalence* has to be reconsidered within the generalized framework of the previous section.

As our framework is based on topological spaces, two quantum systems $(\mathcal{T}, \mathcal{M}, \mathcal{P})$ and $(\widehat{\mathcal{T}}, \widehat{\mathcal{M}}, \widehat{\mathcal{P}})$ are *topologically equivalent*, if $\mathcal{P}$ and $\widehat{\mathcal{P}}$ are positional observables on the same physical space M, the time evolutions depend on the same external conditions $\mathcal{C}$, and there is a homeomorphism $N\colon \mathcal{T} \to \widehat{\mathcal{T}}$, such that

$$\begin{array}{rcll} p_B &=& \widehat{p}_B \circ N\,, & \forall B \in \mathcal{B}(M)\,, \\ \Phi_t^{(Ext)} &=& N^{-1} \circ \widehat{\Phi}_t^{(Ext)} \circ N\,, & \forall t \in R,\ Ext \in \mathcal{C}\,. \end{array} \tag{19}$$

For the linear quantum systems of the previous section, this notion of topological equivalence reduces naturally to ordinary unitary equivalence.

A particular case arises if we consider automorphisms of the same topological space of wavefunctions $\mathcal{T}$ that leave the positional observables invariant,

$$N\colon \mathcal{T} \to \mathcal{T}\,, \qquad p_B = p_B \circ N \quad \forall B \in \mathcal{B}(M)\,. \tag{20}$$

We call these automorphisms *generalized gauge transformations*. For linear quantum systems, these reduce to ordinary gauge transformations of the second kind, $(\mathbf{U}_{\theta_t}\psi_t)\,(\vec{x}) = e^{i\theta_t(\vec{x})}\psi_t(\vec{x})$. As in this linear case, the automorphisms N may be (explicitly) time-dependent.

6 Quantum mechanics in a nonlinear disguise

As we have seen in Section 4, the framework of Section 3 can indeed be filled in the case of linear evolution equations; but are there also nonlinear models? Mielnik has listed a number of nonlinear toy models for his framework [13] and has furthermore considered finite-dimensional nonlinear systems [22]; Haag and Bannier have given an interesting example of a quantum system with linear and nonlinear time evolutions [23].

Here, however, we shall proceed differently in order to obtain a nonlinear quantum system: We use the generalized gauge transformations introduced in the previous section in order to construct nonlinear quantum systems $(\mathcal{H}, \mathcal{M}, \mathcal{P})$ with L^2-wavefunctions that are *gauge equivalent* to linear quantum mechanics.

To simplify matters, we assume that the time evolution of a nonlinear quantum system is still given by a local, (quasi-)homogeneous nonlinear Schrödinger equation. This leads us to consider *strictly local, projective generalized gauge transformations* [20]

$$N_{\gamma_t}(\psi_t) = \psi_t \exp\left(i\gamma_t \ln|\psi_t|\right) , \tag{21}$$

where γ_t is a time-dependent parameter. As these automorphisms of $L^2(R^3, d^3x)$ are extremely similar to local linear gauge transformations of the second kind, they have been called *nonlinear gauge transformations* [24] or gauge transformations of the third kind [25].

Using these transformations, the evolution equations for $\psi'_t := N_{\gamma_t}(\psi_t)$, where ψ_t is a solution of the linear Schrödinger equation 15 are easily calculated:

$$\begin{aligned} i\hbar\partial_t\psi_t \quad = \quad & \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi_t - i\frac{\hbar^2\gamma_t}{4m}R_2[\psi_t]\psi_t - \frac{\hbar^2\gamma_t}{4m}\left(R_1[\psi_t] - R_4[\psi_t]\right)\psi_t \\ & + \frac{\hbar^2\gamma_t^2}{16m}\left(2R_2[\psi_t] - R_5[\psi_t]\right)\psi_t - \frac{1}{2}\dot{\gamma}_t \ln|\psi_t|^2\,\psi_t \,, \end{aligned} \tag{22}$$

where

$$R_1[\psi] := \frac{\nabla\cdot\vec{J}}{\rho}, \quad R_2[\psi] := \frac{\Delta\rho}{\rho}, \quad R_4[\psi] := \frac{\vec{J}\cdot\nabla\rho}{\rho^2}, \quad R_5[\psi] := \frac{\nabla\rho\cdot\nabla\rho}{\rho^2}\,. \tag{23}$$

These equations contain typical functionals R_j of the Doebner–Goldin equations [7] as well as the logarithmic term of Bialynicki-Birula–Mycielski [4]. Note that the form of Eq. 22 does not immediately reveal its linearizability, the underlying linear structure of this model is disguised.

In fact, through an iterated process of gauge generalization and gauge closure – similar to the minimal coupling scheme of linear quantum mechanics – we could obtain a unified family of nonlinear Schrödinger equations [25, 26] ($R_3[\psi] := \frac{\vec{J}^2}{\rho^2}$):

$$i\partial_t\psi_t = i\sum_{j=1}^{2}\nu_j R_j[\psi_t]\,\psi_t + \mu_0 V + \sum_{k=1}^{5}\mu_k R_k[\psi_t]\,\psi_t + \alpha_1 \ln|\psi_t|^2\,\psi_t\,. \tag{24}$$

7 Final Remarks: Histories and Locality

In this contribution, we have sketched a framework for nonlinear quantum theories that generalizes the usual linear one. We close with three remarks.

The first is concerned with the definition of effects (and positional observables) in our framework. Since we have used real-valued measures, our observables do not allow for an idealization of measurements as in the linear theory, where a projection onto certain parts of the spectrum is possible using the projection-valued measure. Combined subsequent measurements (*histories*) have to be described by quite complicated time evolutions. However, in case of a linearizable quantum system, *generalized projections*

$$E := N \circ \mathbf{E} \circ N^{-1}\,, \qquad \mathbf{E} \in \mathit{Proj}(\mathcal{H}) \tag{25}$$

onto nonlinear submanifolds of $\mathcal{H}$ can be realized as an idealization of measurements, and yield a nonlinear realization of the standard quantum logic [12].

Secondly, we should emphasize that we have not been able to describe a complete and satisfactory nonlinear theory that is *not* gauge equivalent to linear quantum mechanics. One of the obstacles of quantum mechanical evolution equations like 24 is the difficulty of the (global) Cauchy problem for partial differential equations. Whereas there is a solution of the logarithmic nonlinear Schrödinger equation [27], there are only local solutions of (non-linearizable) Doebner–Goldin equations [28].

Another problem of nonlinear Schrödinger equations in quantum mechanics is the locality of the corresponding quantum theory: EPR-like experiments could indeed lead to superluminal communications, though not in the naive (and irrelevant) fashion described in Section 2, relevant Gisin-effects [29] can occur if changes of the external conditions in spatially separated regions have instantaneous effects. Since the nonlinear equations we have considered here are *separable*, this effect can only occur for *entangled* initial wavefunctions, i.e.,

$$V(\vec{x}_1,\vec{x}_2) = V_1(\vec{x}_1) + V_2(\vec{x}_2)\,, \qquad \psi_0(\vec{x}_1,\vec{x}_2) \neq \varphi_1(\vec{x}_1)\varphi_2(\vec{x}_2)\,. \tag{26}$$

For the Doebner-Goldin equations, for instance, such effects indeed occur (at least) for certain subfamilies that are not Galilei invariant [29]; (higher order) calculations for the Galilei invariant case and the logarithmic Schrödinger equation are not yet completed.

In the title of this contribution, we have put the prefix “non” in parentheses; the remarks above may have indicated why. Finally, one might be forced to find *different* ways of extending a nonlinear single-particle theory to many particles (see, e.g., [30, 31]).

Acknowledgments

Thanks are due to H.-D. Doebner, G.A. Goldin, W. Lücke, and R.F. Werner for interesting and fruitful discussions on the subject. I am also greatful to the organizers for their invitation to the “Second International Conference on Symmetry in Nonlinear Mathematical Physics”. It is a pleasure to thank Renat Zhdanov, his family and friends for their warm hospitality during our stay in Kyiv.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 279–286.

A Fiber Bundle Model of the Ice I_h Structure

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Abstract

A three-dimensional model of ice I_h based on the fiber bundle approach is presented. A hybrid structure of ice consisting of the oxygen lattice with the $P6_3$/mmc crystallographic symmetry and the hydrogen subsystem satisfying the Bernal-Fowler rules is considered. Controllable change of the protons position at hydrogen bounds by optoacoustic perturbation is discussed. Both classic and bispinor Hamiltonians are proposed.

Forecasting intensive progress in microphysics as early as in 1959, R. Feynman has made a presentation at the annual session of the American Physical Society under the symbolic title "There's plenty of room at the bottom" [1]. He underlined that microcosm can give in future practically unlimited possibilities for material technology and information processing. But for achievement of practical results, it is necessary to overcome not a few obstacles. And one of them is the gap between micro- and macrolevels which prevents a direct contact without information losses and order distraction. Over the year, it was clearly understood that control and information theory should play an important role in microphysics progress as well as in molecular and quantum computing [2–7]. First of all, proper modelling of hybrid systems must be developed. One of possible models is presented here.

Hexagonal ice I_h was chosen as an object thanks to its wide spread in nature and because it allows information processing at the molecular level. We start from the brief description of the usual ice structure following the short but very consistent book of N. Maeno [8]. Hexagonal modification of ice exists under normal pressure and temperature $T < -6$ °C. It has the crystallographic symmetry $P6_3$/mmc but only for the oxygen lattice, the hydrogen subsystem is characterized by relative arbitrariness in protons' distribution at hydrogen bonds. So if the Bernal-Fowler rules are fulfilled, the protons have many variants of distribution. These rules are the next:

1) exactly two protons are situated beside every oxygen atom,

2) exactly one proton is present at every hydrogen bond.

Mathematically, they can be expressed with some equations binding binary variables for bistable proton positions at hydrogen bonds. This and another technique will be demonstrated below.

One of possible dispositions of protons in the frame of an ice I_h elementary cell is shown in Fig.1. Framework of the cell is composed of two mirror symmetric strata which are based on local orthogonal triplets $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ (lower,right) and $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ (upper, left) with the origins situated at the endpoints of the vertical hydrogen bond. The triplets are oriented towards middle points of tetrahedron edges which are connected with the

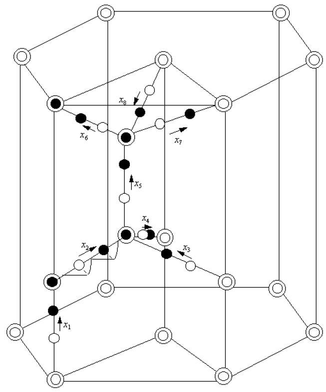

Fig.1.

correspondent oxygen atoms. Hydrogen bonds geometrically are represented by vectors $\mathbf{f}_i$, $\mathbf{f}'_i$ $(i = 0,1,2,3)$, so that

$$\mathbf{e}_1 = \frac{1}{2}(\mathbf{f}_1 + \mathbf{f}_0), \quad \mathbf{e}_2 = \frac{1}{2}(\mathbf{f}_2 + \mathbf{f}_0), \quad \mathbf{e}_3 = \frac{1}{2}(\mathbf{f}_3 + \mathbf{f}_0);$$

$$\mathbf{e}'_1 = -\frac{1}{2}(\mathbf{f}'_1 + \mathbf{f}'_0), \quad \mathbf{e}'_2 = -\frac{1}{2}(\mathbf{f}'_2 + \mathbf{f}'_0), \quad \mathbf{e}'_3 = -\frac{1}{2}(\mathbf{f}'_3 + \mathbf{f}'_0);$$

$$\mathbf{f}_0 + \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 = 0, \quad \mathbf{f}'_0 + \mathbf{f}'_1 + \mathbf{f}'_2 + \mathbf{f}'_3 = 0;$$

$$\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 = \mathbf{f}_0 = \mathbf{f}'_0 = -(\mathbf{e}'_1 + \mathbf{e}'_2 + \mathbf{e}'_3);$$

$$\mathbf{f}_1 = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{f}_2 = -\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{f}_3 = -\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3;$$

$$\mathbf{f}'_1 = -(\mathbf{e}'_1 - \mathbf{e}'_2 - \mathbf{e}'_3) = -\frac{5}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3 = -\left(\mathbf{f}_1 + \frac{2}{3}\mathbf{f}_0\right),$$

$$\mathbf{f}'_2 = -(-\mathbf{e}'_1 + \mathbf{e}'_2 - \mathbf{e}'_3) = \frac{1}{3}\mathbf{e}_1 - \frac{5}{3}\mathbf{e}_2 + \frac{1}{3}\mathbf{e}_3 = -\left(\mathbf{f}_2 + \frac{2}{3}\mathbf{f}_0\right),$$

$$\mathbf{f}'_3 = -(-\mathbf{e}'_1 - \mathbf{e}'_2 + \mathbf{e}'_3) = \frac{1}{3}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 - \frac{5}{3}\mathbf{e}_3 = -\left(\mathbf{f}_3 + \frac{2}{3}\mathbf{f}_0\right).$$

Following the relations between vectors and having used the length of hydrogen bonds

$$d = |\mathbf{f}_i| = \left|\mathbf{f}'_i\right| = 0.276 \text{ nm}, \qquad (i = 0,1,2,3)$$

one can determine horizontal a and vertical h moduli of the hexagonal lattice:

$$a = |\mathbf{f}_2 - \mathbf{f}_3| = |\mathbf{a}_1| = |\mathbf{f}_3 - \mathbf{f}_1| = |\mathbf{a}_2| = \frac{2}{3}\sqrt{6}d = 0.452 \text{ nm},$$

$$h = |\mathbf{a}_3| = |2\mathbf{f}_0 - \mathbf{f}_3 - \mathbf{f}'_3| = \frac{8}{3}d = 0.736 \text{ nm}.$$

Proton configuration can be specified by different methods. One of them was applied in [6] to cubic ice I_c where protons' positions have been determined by binary variables $z_\alpha(\mathbf{n}) \in GF(2)$ $(\alpha = 0,1,2,3; \mathbf{n} \in Z^3)$. Displacement of protons from middles of hydrogen bonds could be expressed as follows:

$$x_\alpha(\mathbf{n}) = \frac{1}{2}(-1)^{z_\alpha(\mathbf{n})} = \pm\frac{1}{2}.$$

It was found that the Bernal-Fowler rules could be written in the form of homogenous equations

$$\sum_{\alpha=0}^{3} x_\alpha(\mathbf{n}) = 0, \qquad \sum_{\alpha=0}^{3} x_\alpha(\mathbf{n}+\mathbf{e}_\alpha) = 0 \qquad (\mathbf{n} \in Z^3),$$

where $\mathbf{e}_0 = \{0,0,0\}, \mathbf{e}_1 = \{1,0,0\}$, $\mathbf{e}_2 = \{0,1,0\}$, $\mathbf{e}_3 = \{0,0,1\}$.

These equations are linear relatively to $x_\alpha(\mathbf{n})$ but not to $z_\alpha(\mathbf{n})$. So definite selection rules should be applied after all. This hidden nonlinearity becomes explicit when another method is attracted.

Beforehand, let us make some notifications:

– near every node of the oxygen lattice, exactly 1 pair of protons can be found which may occupy 4 hydrogen bonds (outcoming from the given node) by $C_4^2 = 6$ different possible variants;

– hydrogen dihedron $D_\alpha(\mathbf{n})$ ($\alpha-$ node's number, $\mathbf{n}-$ elementary cell number) formed by two unitary vectors $\mathbf{r}_\alpha^{(1)}(\mathbf{n})$, $\mathbf{r}_\alpha^{(2)}(\mathbf{n})$ directed from the oxygen node to protons remains scalar product $\langle \mathbf{r}_\alpha^{(1)}(\mathbf{n}), \mathbf{r}_\alpha^{(2)}(\mathbf{n})\rangle = \text{const}$ in all 6 possible positions of its bisectrix unitary vector $\mathbf{d}_\alpha(\mathbf{n}) = \dfrac{\sqrt{3}}{2}\left[\mathbf{r}_\alpha^{(1)}(\mathbf{n}) + \mathbf{r}_\alpha^{(2)}(\mathbf{n})\right]$;

– dihedron bisectrix unitary vector $\mathbf{d}_\alpha(\mathbf{n})$ can be oriented at every vertex of an octahedron inscribed in the tetrahedron surrounding a given node of the lattice if this is not forbidden by neighboring dihedron positions;

– dihedron's position is identically determined by the index $\lambda_\alpha(\mathbf{n}) = 0, 1, \ldots, 5$ of vector's $\mathbf{d}_\alpha(\mathbf{n})$ projection on the horizontal plane.

Let us introduce in residue class ring Z_6 characteristic functions for some of its subsets:

$$\begin{aligned}
&\Phi(\lambda_\alpha) = \begin{cases} 0 \ (\lambda = 5,0,1) \\ 1 \ (\lambda = 2,3,4) \end{cases} = \begin{cases} 0 \text{ when } |\lambda| \le 1 \ (\text{mod}6) \\ 1 \text{ when } |\lambda| > 1 \ (\text{mod}6) \end{cases} = \\
&\qquad = \{\lambda(\text{mod}2) + [\lambda(\text{mod}3)]^2\}(\text{mod}2); \\
&\Theta(\lambda_\alpha) = \begin{cases} 0 (\lambda = 0,2,4) \\ 1 \ (\lambda = 1,3,5) \end{cases} = \lambda(\text{mod}2); \\
&\Phi^+(\lambda_\alpha) = \Phi(\lambda_\alpha + 2), \qquad \Phi^-(\lambda_\alpha) = \Phi(\lambda_\alpha - 2).
\end{aligned} \tag{1}$$

Now the Bernal-Fowler rules have an explicitly nonlinear character as follows:

$$\begin{aligned}
&\Phi[\lambda_1(\mathbf{n})] - \Phi[\lambda_2(\mathbf{n})] = 0, \\
&\Phi^+[\lambda_1(\mathbf{n}-\nu_1)] - \Phi^+[\lambda_2(\mathbf{n})] = 0, \\
&\Phi^-[\lambda_1(\mathbf{n})] - \Phi^-[\lambda_2(\mathbf{n}-\nu_2)] = 0, \\
&\Theta[\lambda_1(\mathbf{n}-\nu_3)] + \Theta[\lambda_2(\mathbf{n})] = 0.
\end{aligned} \tag{2}$$

Here, $\nu_1 = \{1,0,0\}$, $\nu_2 = \{0,1,0\}$, $-\nu_3 = \{0,0,1\}$, $\mathbf{n} = \{n_1,n_2,n_3\}$ ($\nu_1,\nu_2,\nu_3,\mathbf{n} \in Z_{N_1} \times Z_{N_2} \times Z_{N_3} \subset Z^3$), $\lambda_1(\mathbf{n})$ corresponds to the lower end of a sloping hydrogen bond and $\lambda_2(\mathbf{n})$ – to upper one. Every solution of this system conforms to set of mutually crossing contours covering the whole oxygen lattice – so-called Bernal-Fowler fibers oriented in accordance with protons' shifts.

There is no possibility to find a general solution of system (2) because unknown variables are determined in the ring which has zero divisors 2 and 3. (In this ring, $2 \times 3 = 6 = 0 (\mathrm{mod} 6)$.) Although the number of Diophantine equations in system (2) is twice that of unknowns, its solution exists. For example, $\lambda_1(n_1, n_2, n_3) = \lambda_2(n_1', n_2', n_3') = \lambda_2(n_1, n_2, n_3 + 1) + 3 (\mathrm{mod} 6)$. Quantity of solutions is rapidly growing with the numbers N_1, N_2, N_3. As an alternative to unknowns λ_i, one may choose pairs of unknowns $\sigma_i = \lambda_i (\mathrm{mod} 2)$ and $\mu_i = \lambda_i (\mathrm{mod} 3)$ which have been mentioned in (1). Their totality precisely corresponds to the number of equations (2) but, regrettably, two-moduli arithmetics appears in this case.

So it remains to take advantage of the recurrent procedure for finding solutions. For the clear presentation, each fragment (lower or upper) of the elementary cell can be imaged as a 6-pole $\Pi(\mathbf{n})$. Let us code an input $\mathbf{x} = \{x_1, x_2, x_3\}$ and outputs $\mathbf{x}' = \{x_1', x_2', x_3'\} = \{x_1'(\mathbf{n} - \nu_1), x_2'(\mathbf{n} - \nu_2), x_3'(\mathbf{n} - \nu_3)\}$ by binary integers from the ring Z_2. For shifts X, Y, Z which can be ± 1, it immediately follows

$$X = (-1)^{x_1}, \quad (-Y) = (-1)^{x_2}, \quad Z = (-1)^{x_3}$$

corresponding to the second ice rule:

$$X^2 = Y^2 = Z^2 = (-1)^{2x_i} = 1 \quad (i = 1, 2, 3).$$

Meanwhile making use of the first ice rule, one can obtain for the $\Pi(\mathbf{n})$ three-sheeted mapping

$$\Pi : Z_2^3 = Z_2 \times Z_2 \times Z_2 \to Z_2,$$

$$x_i' = \pi_i^{\mu}(x) \quad (0 = 1, 2, 3; \quad 1 \le \mu \le m(x) \le 3)$$

which is represented in Table 1.

Table 1

N	x	π_i^1	π_i^2	π_i^3	$m(x)$
0	000	000	—	—	1
1	100	100	010	001	3
2	010	100	001	—	2
3	110	110	101	011	3
4	001	100	010	001	3
5	101	110	011	—	2
6	011	110	101	011	3
7	111	111	—	—	1

Apparently, the total number of different sorts of transformation is $2^2 \times 3^4 = 324$, but when x is fixed, $m(x) \le 3$. By the way, it is easy to calculate the total amount of variants for the protons' distribution in one half-cell as the sum of all possible $m(x)$: $2 + 2 \times 2 + 4 \times 3 = 18$.

The algorithm is being built in the following manner. For a given rectangular parallelepiped P having $S_1 \times S_2 \times S_3$ layers in three orthogonal directions in the basis

$$\mathbf{b}_1 = \frac{1}{2}(\mathbf{a}_1 + \mathbf{a}_2), \quad \mathbf{b}_2 = \mathbf{a}_1 - \mathbf{a}_2, \quad \mathbf{b}_3 = \mathbf{a}_3,$$

one can fix definite mappings from Table 1 for all half-cells of P. Boundary conditions must be determined for 4 its bounds: 1 face, 2 sides and 1 lower:

$$x(s)\,|_{s_1=0} = X_1(s_2,s_3), \quad x(s)\,|_{s_2=0} = X_2^{(0)}(s_1,s_3),$$

$$x(s)\,|_{s_2=s_2-1} = X_2^{(1)}(s_1,s_3), \quad x(s)\,|_{s3=0} = X_3(s_1,s_2).$$

Because all half-cells in each layer are chess-ordered in the basis $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$, their inputs are fully independent in every row. Thereby, a complete decomposition of vertical columns is assured and wide parallelizing of output calculations is achieved for all layers. When the mapping distribution is fixed, there is possible to realize $2^{2S_3}+1$ states with various inputs x, y, z in each column. In its turn, provided inputs are given in one column, one can obtain not more than 3^{S_3} states because, in every half-cell, maximum 3 mappings can be permitted. So the upper limit of the possible states quantity in a column is $2^{2S_3+1} \times 3^{S_3}$, and respectively, for a whole parallelepiped P with dimensions $S_1 \times S_2 \times S_3$ (in half-cell units) the upper limit of the possible states number is determined as

$$M = 2^{S_1S_2(2S_3+1)} \times 3^{S_1S_2S_3}.$$

If bonds between half-cells are ignored, each of them gives 2×3^2 combinations of protons' distributions as have been mentioned above. For $S_1 \times S_2 \times S_3$ half-cells, it turns out that the upper limit of the states number in this case is

$$M_0 = 2^{S_1S_2S_3} \times 3^{2S_1S_2S_3}.$$

Comparison M_0 with M shows that, when $S_3 \gg 1$, bonding between half-cells sufficiently decreases the quantity of possible configurations for the proton subsystem. As can be seen directly, the proposed algorithm is exhausting and does not give repetitions of variants.

Now the protons' subsystem information entropy S can be estimated:

$$S \le \log_2 M = S_1S_2S_3\left[\log_2 3 + 2\left(1+\frac{1}{2S_3}\right)\right].$$

Hence, specific entropy per one half-cell has the upper limit as follows:

$$S \le \lim_{S_1\to\infty}\lim_{S_2\to\infty}\lim_{S_3\to\infty}\frac{\log_2 M}{S_1S_2S_3} = 2+\log_2 3 \simeq 3,6.$$

Without bounding between half-cells, specific entropy is found to be noticably greater:

$$S_0 = \lim_{S_1\to\infty}\lim_{S_2\to\infty}\lim_{S_3\to\infty}\frac{\log_2 M}{S_1S_2S_3} = 1+2\log_2 3 \simeq 4,2.$$

A control problem for the hydrogen subsystem consists in determining and realization of a necessary sequence in the state space of all possible protons configurations. One con consider this sequence as a one-parameter subgroup of the structural group G of a fiber bundle (X_F, P_F^t, B, F) with the orbit space $X_F = (X \times F)/G \stackrel{P_F}{\to} B$ where the base B and layer F are the state space of protons and cotangent space, respectively. From physical viewpoint, it must be required to satisfy the stability of mapping in order to secure information processing from the destruction caused by dynamic chaotization.

Controlling action on the proton subsystem from the oxygen lattice deformations is effected by variations of the potential function. As this takes place, excitation is supposed not so much to cause ionizing effects and proton transport but sufficient for displacements of protons along hydrogen bonds where they are situated. Electron shell deformations will be considered as adiabatic. Taking into account all these conditions, let us construct such a nonstationary Hamiltonian that can be decomposed by a periodically time-dependent orthogonal canonical transformation into a direct sum of Hamiltonians for bistable non-linear oscillators. Every step of transformation in the state space of the proton subsystem should correspond to a definite step of rearrangement of the potential well for protons where they are dislocated.

One of the simplest forms of bistable potential is the polynomial

$$U(q) = \frac{1}{2}q^2(q^2 - 2),$$

and a necessary Hamiltonian has the form

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2(q^2 - 2).$$

So, if energy E is fixed, the time dependence $q(t)$ can be found from the elliptic integral

$$t - t_0 = \int\limits_{q_0}^{q} \frac{dq_1}{\sqrt{2E + 2q_1^2 - q_1^4}}$$

and finally expressed in terms of the Jacobi function, but it is unnecessary to demonstrate this here.

The Hamiltonian of a direct sum of m such oscillators

$$H = \sum_{\mu=1}^{m} H_\mu = \frac{1}{2}\sum_{\mu=1}^{m}(p_\mu^2 - 2q_\mu^2 + q_\mu^4)$$

is to be subjected to an orthogonal pointwise transformation presented by the time dependent generating function

$$W(p, Q, t) = \sum_{\mu=1}^{m}\sum_{v=1}^{m} T_{\mu v}(t) p_\mu Q_v.$$

Its partial derivatives give expressions of old momemnta p_μ and coordinates q_μ in terms of new ones P_μ, Q_μ :

$$P = \frac{\partial}{\partial Q}W(p, Q, t), \quad q = \frac{\partial}{\partial p}W(p, Q, t),$$
$$p_\mu = \sum_v T_{\mu v}(t) p_v, \qquad q_\mu = \sum_v T_{\mu v}(t) Q_v.$$

A new Hamiltonian which differs from the old one by the item $\frac{\partial}{\partial t}\big|_{p=p(P)} W(p, Q, t)$ appears as

$$\widetilde{H}\,(P, Q, t) = \frac{1}{2}\sum_\mu P_\mu^2 - \sum_\mu Q_\mu^2 + \frac{1}{2}\sum_\mu \left[\sum_v T_{\mu v}(t) Q_v\right]^4 + \sum_\mu\sum_v \dot{S}_{\mu v}\,(t) P_\mu Q_v,$$

where $S_{\mu v}(t)$ are elements of the skew-symmetric matrix $\hat{S}(t) = \ln \hat{T}(t)$ and $\hat{T}(t) = \{T_{\mu v}(t)\}$, $\dot{S}_{\mu v}(t) = \frac{d}{dt} S_{\mu v}(t)$.

As soon as, in the ice I_h lattice, each proton has not more than 6 and not less than 1 protons neighboring with it, the sum over the index v in every term of the new Hamiltonian is to be calculated in the limits $1 \leq v \leq l, \quad 2 \leq l \leq 7$ (including a given proton) because another terms vanish. Number l depends on a phonon oscillations mode of the oxygen lattice.

Just above, the example of a vector fiber bundle model was demonstrated. Quite another way can be proposed on the base of a fiber bundle which has the nonabelian structure group $SU(2)$. If the well-known epimorphism $SU(2) \to SO(3)$ and trace metric $\langle \hat{H}_i, \hat{H}_j \rangle = \frac{1}{2} Tr\ \hat{H}_i \hat{H}_j$ are attracted, one can use the Pauli basis

$$\hat{h}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{h}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{h}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \left\langle \hat{h}_i, \hat{h}_j \right\rangle = \delta_{ij} \quad (i, j = \overline{1, 3})$$

in a linear space of traceless matrices $H_i \in L$. Then, vectors $\mathbf{d}_i$ representing, for instance, a protons' i-dihedron dipole moment can be imaged as vector operators $\hat{\mathbf{d}}_i \in L$. Corresponding scalar product of $\mathbf{d}_i$ over a hydrogen bond vector $\hat{\mathbf{n}}_{ij}$ may have a form

$$(\mathbf{d}_i, \mathbf{n}_{ij}) = \langle \chi_i | (\mathbf{d}_i \mathbf{n}_{ij}) | \chi_i \rangle$$

where χ_i are spinors.

The energy of dipole interaction between protons' dihedrons can be expressed as follows:

$$V = J_0 \sum_{(i,j) \in \Gamma} \left[\varepsilon(\mathbf{d}_i \mathbf{d}_j) - 3(\mathbf{d}_i \mathbf{n}_{ij})(\mathbf{d}_j \mathbf{n}_{ij}) \right].$$

The sum must be accomplished by the whole interactions graph Γ. Here, J_0 is the energy constant and $\varepsilon < 1$ is a coefficient which allows taking into account the nondirect interaction between proton' dihedrons caused by oxygen electron shells. Going to the representation of a dihedron vector by corresponding spin matrices oriented along previously introduced basis vector $\mathbf{e}_3$ (lower) and $\mathbf{e}'_3$ (upper), one can obtain the formula

$$V = J_0 \sum_{(i,j) \in \Gamma} \langle \chi_i \chi_j | (\hat{H}_{ij}) | \chi_i \chi_j \rangle,$$

where bispinors $\chi_i \chi_j$ correspond to pairs of dihedrons situated at the endpoints of each hydrogen bond. In our case, Hamiltonians $\hat{H}_{ij}$ are the next:

$$\hat{H}_{ij} = (\varepsilon - 3\cos^2 \nu_{ij})\, \hat{\sigma}_i^z \hat{\sigma}_j^z\ .$$

Fortunately, for the ice I_h structure, all the angles ν_{ij} between bonds' vectors $\mathbf{n}_{ij}$ and $\mathbf{e}_3$ (or $\mathbf{e}'_3$) are identical and $\cos^2 \nu_{ij} = \frac{1}{3}$. So Hamiltonians have the very simple expression

$$\hat{H}_{ij} = (\varepsilon - 1)\, \hat{\sigma}_i^z \hat{\sigma}_j^z\ .$$

If indirect interaction was absent, all $\hat{H}_{ij}$ were zero, but the presence of oxygen atoms makes ε much more less than unity ($0 < \varepsilon \ll 1$). Thus,

$$V = J \sum_{(i,j)\in\Gamma} \langle \chi_i \chi_j | \hat{\sigma}_i^z \hat{\sigma}_j^z | \chi_i \chi_j \rangle,$$

where $J = (\varepsilon - 1)J_0 < 0$. Finally, let us present the Hamiltonian in the explicit form

$$\hat{H} = J \sum_{(i,j)\in\Gamma} [\hat{\sigma}_1^z(\mathbf{n})\hat{\sigma}_2^z(\mathbf{n}) + \hat{\sigma}_2^z(\mathbf{n})\hat{\sigma}_3^z(\mathbf{n}) + \hat{\sigma}_3^z(\mathbf{n})\hat{\sigma}_4^z(\mathbf{n}) + \hat{\sigma}_1^z(\mathbf{n})\hat{\sigma}_4^z(\mathbf{n} - \nu_3) +$$
$$+ \hat{\sigma}_1^z(\mathbf{n})\hat{\sigma}_2^z(\mathbf{n} - \nu_1) + \hat{\sigma}_1^z(\mathbf{n} - \nu_2)\hat{\sigma}_2^z(\mathbf{n}) + \hat{\sigma}_3^z(\mathbf{n})\hat{\sigma}_4^z(\mathbf{n} - \nu_2) + \hat{\sigma}_3^z(\mathbf{n} - \nu_1)\hat{\sigma}_4^z(\mathbf{n})]$$

using notations

$$\mathbf{n} = \{n_1, n_2, n_3\}, \quad \nu_1 = \{1, 0, 0\}, \quad \nu_2 = \{0, 1, 0\}, \quad \nu_3 = \{0, 0, 1\},$$

$\hat{\sigma}_\alpha^z(\mathbf{n})$ is the spin operator of a dihedron $\mathbf{d}_\alpha(\mathbf{n})$ in the elementary cell with the number $\mathbf{n}$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 287–296.

Spinor Fields over Stochastic Loop Spaces

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Abstract

We give the construction of a line bundle over the based Brownian bridge, as well as the construction of spinor fields over the based and the free Brownian bridge.

Introduction

The Dirac operator over the free loop space is a very important object for the algebraic topology [24]; its index gives the Witten genus and it can predict the rigidity theorem of Witten: the index of some classical operator is rigid under a geometrical action of the circle over the manifold. Unfortunately, the Dirac operator over the free loop space is an hypothetical object.

In [13], we have constructed an approximation of it by considering the Brownian measure over the loop space: why is a measure important? It is to compute the adjoint of the Dirac operator, the associated Laplacian, and the Hilbert space of spinors where it acts; the choice of physicists gives a hypothetical measure over the loop space. The purpose of [13] is to replace the formal measure of physicists by a well-defined measure, that is the Brownian bridge measure. The fiber of the Dirac operator is related to the Fourier expansion. After [25, 13] the Fourier expansion has extended in an invariant by rotation way for the natural circle action over the free loop space. Unfortunately, this works only for small loops.

The problem to construct the spin bundle over the free loop space is now a well-understood problem in mathematics (see [14, 24, 26, 7, 6, 20]). In order to construct a suitable stochastic Dirac operator over the free loop space, it should be reasonable to define a Hilbert space of spinor fields over the free loop space where the operator acts. It should be nice to extend the previous work mentioned in the references above in the stochastic context. It is the subject of [17, 18] and [19]. The goal of this paper is to do a review of the results of [17, 18, 19].

In the first part, we study the problem to construct stochastic line bundles over the stochastic loop space: their transition functions are only almost surely defined. Therefore, we define the line bundle by its sections. If we consider the path space as a family of Brownian bridges, we meet the problem to glue together all the line bundles over the Brownian bridge into a line bundle over the Brownian motion. There is an obstruction which is measured in [5] for smooth loops. When the criterium of this obstruction is satisfied, the tools of the quasi-sure analysis allow one to restrict a smooth section of the line bundle over the Brownian motion into a section of the line bundle over the Brownian bridge. Moreover, if we consider the bundle associated to a given curvature (we neglect all torsion phenomena by considering the case where the loop space is simply connected) whose fiber is a circle, we cannot define it by its sections because they do not exist: we define it by its functionals. This way to define topological spaces is very useful in algebra.

In the second part, we study the case of the based Brownian bridge in order to construct spinor fields over it. This part is based upon [6] when there is no measure. We consider the case of a principal bundle $Q \to M$ over the compact manifold M. The based loop space $L_e(Q)$ is a principal bundle over the based loop space $L_x(M)$ of M with the based loop group of G as a structure group. The problem to construct spinor fields (or a string structure) is to construct a lift of $L_e(Q)$, $\tilde{L}_e(Q)$, by the basical central extension of $L_e(G)$ if G is supposed simple simply laced (see [23]). This allows us, when the first Pontryagin class of Q is equal to 0, to construct a set of transition functions with values in $\tilde{L}_{e,f}(G)$, the basical central extension of the group of finite energy based loops in G. These transition functions are almost surely defined: we can impose some rigidity by saying they belong to some Sobolev spaces. If there exists a unitary representation of $\tilde{L}_{e,f}(G)$ called $Spin_\infty$, this allows us to define the Hilbert space of sections of the associated bundle. The second part treats too the problem to construct the $\tilde{L}_{e,f}(G)$ principal bundle (and not only the associated bundle which is defined by its sections). For that, we construct a measure over $L_e(Q)$ and by using a suitable connection, we define Sobolev spaces over $L_e(Q)$. We construct a circle bundle $\tilde{L}_e(Q)$ over $L_e(Q)$ by using its functionals, called string functionals: the space of L^p string functionals is therefore defined. There is an Albeverio-Hoegh-Krohn density over $\tilde{L}_{e,f}$ associated to right and left translations by a deterministic element of $\tilde{L}_{e,2}(G)$ (the central extension of the group of loops with two derivatives), which belongs only in L^1. This shows that the stochastic gauge transform of the formal bundle $\tilde{L}_e(Q) \to L_x(M)$ operate only in $L^\infty(\tilde{L}_e(Q))$.

In the third part, we treat the free loop space case by considering the formalism of [7] related to the Chern-Simons theory. In such a case, it is possible to consider the spin representation of the free loop group when G is the finite-dimensional spinor group (see [23, 4] for instance). We construct an Hilbert space of spinor fields invariant by rotation.

1. Bundles over the Brownian bridge

Let M be a compact Riemannian manifold. Let $L_x(M)$ be the space of continuous loops over M starting from x and arriving in time 1 over x. Let $dP_{1,x}$ be the Brownian bridge measure over $L_x(M)$. Let γ_t be a loop. We consider τ_t, the parallel transport from γ_0 to γ_t. The tangent space [3, 12] of a loop γ_t is the space of sections X_t over γ_t of $T(M)$ such that

$$X_t = \tau_t H_t; \qquad H_0 = H_1 = 0 \tag{1.1}$$

with the Hilbert norm $\|X\|^2 = \int_0^1 \|H'_s\|^2 ds$. We suppose that $L_x(M)$ is simply connected in order to avoid all torsion phenomena.

Let ω be a form which is a representative of $H^3(M;Z)$. We consider the transgression

$$\tau(\omega) = \int_0^1 \omega(d\gamma_s, ., .). \tag{1.2}$$

It is the special case of a stochastic Chen form, which is closed. If we consider a smooth loop, it is Z-valued.

Let γ_i be a dense countable set of finite energy loops. Let γ_{ref} be a loop of reference. If $\gamma \in B(\gamma_i, \delta)$, the open ball of radius δ and of center γ_i for the uniform norm, we can produce a distinguished path going from γ to γ_{ref}. Between γ and γ_i, it is $s \to \exp_{\gamma_{i,s}}[t(\gamma_s - \gamma_{i,s})] = l_{i,t}(s)$ and between γ_i and γ_{ref}, it is any deterministic path. Over $B(\gamma_i, \delta)$, we say that the line bundle is trivial, and we assimilate an element α over γ to α_i over γ_{ref} by the parallel transport for the connection whose the curvature is $\tau(\omega)$: there is a choice. The consistency relation between α_i and α_j is given by the parallel transport along the path joining γ_{ref} to γ_{ref} by going from γ_{ref} to γ by l_i runned in the opposite sense and going from γ to γ_{ref} by l_j. First of all, we fulfill the small stochastic triangle γ, γ_i and γ_j by a small stochastic surface and we use for that the exponential charts as before. After we use the fact that $L_x(M)$ is supposed simply connected. We fullfill the big deterministic triangle γ_{ref}, γ_i, γ_j by a big deterministic surface. We get, if we glue the two previous surfaces, a stochastic surface $S_{i,j}(\gamma)$ constituted of the loop $l_{i,j,u,v}$. The desired holonomy should be equal to

$$\rho_{i,j}(\gamma) = \exp\left[-2i\pi \int_{S_{i,j}(\gamma)} \tau(\omega)\right].$$

Let us remark that $\int_{S_{i,j}(\gamma)} \tau(\omega)$ is well defined by using the theory of non anticipative Stratonovitch integrals. Namely, $\partial/\partial u\, l_{i,j,u,v}$ as well as the derivative with respect to v are semi-martingales. Moreover, if we choose the polygonal approximation γ^n of γ, $\rho_{i,j}(\gamma^n) \to \rho_{i,j}(\gamma)$ almost surely, we get:

Theorem 1.1. *Almost surely over $B(\gamma_i, \delta) \cap B(\gamma_j, \delta)$,*

$$\rho_{i,j}(\gamma)\rho_{j,i}(\gamma) = 1 \tag{1.3}$$

and, over $B(\gamma_i, \delta) \cap B(\gamma_j, \delta) \cap B(\gamma_k, \delta)$

$$\rho_{i,j}(\gamma)\rho_{j,k}(\gamma)\rho_{k,i}(\gamma) = 1 \tag{1.4}$$

Proof. We use the fact that $\tau(\omega)$ is Z-valued such that (1.3) and (1.4) are true surely for γ^n. It remains to pass to the limit. ♢

Moreover, $\gamma \to \rho_{i,j}(\gamma)$ belongs locally to all the Sobolev spaces.

This allows us to give the following definition:

Definition 1.2. *A measurable section ϕ of the formal bundle associated to $\tau(\omega)$ is a collection of random variables with values in C α_i over $B(\gamma_i, \delta)$ subjected to the rule*

$$\alpha_i = \rho_{i,j}\alpha_j. \tag{1.5}$$

Over $B(\gamma_i, \delta)$, we put the metric:

$$\|\alpha(\gamma)\|^2 = |\alpha_i|^2. \tag{1.6}$$

Since the transition functions are of modulus one, this metric is consistent with the change of charts. We can give the definition:

Definition 1.3. *The L^p space of sections of the formal bundle associated to $\tau(\omega)$ is the space of measurable sections ϕ endowed with the norm:*

$$\|\phi\|_{L^p} = \| \|\phi\| \|_{L^p}. \tag{1.7}$$

We can define too the circle bundle $\tilde{L}_x(M)$ (the fiber is a circle) associated to $\tau(\omega)$ by its functionals (instead of its sections).

Definition 1.4. *A measurable functional $\tilde{F}$ of $\tilde{L}_x(M)$ is a family of random variables $F_i(\gamma, u_i)$ over $B(\gamma_i, \delta) \times S^1$ submitted to the relation*

$$F_i(\gamma, u_i) = F_j(\gamma, u_j), \tag{1.8}$$

where $u_i = u_j \rho_{j,i}$ almost surely.

Over the fiber, we put:

$$\|\tilde{F}\|_{p,\gamma} = \left(\int_{S^1} | F_i(\gamma, u_i) |^p \right)^{\frac{1}{p}}. \tag{1.9}$$

Since the measure over the circle is invariant by rotation, $\|\tilde{F}\|_{p\gamma}$ is intrinsically defined and is a random functional over the basis $L_x(M)$. We can give:

Definition 1.5. *An L^p functional over $\tilde{L}_x(M)$ is a functional $\tilde{F}$ such that $\|\tilde{F}\|_{p,\gamma}$ belongs to $L^p(L_x(M))$.*

We will give a baby model due to [5] of the problem to construct a string structure. Let $P_x(M)$ be the space of continuous applications from [0,1] into M. dP_1^x is the law of the Brownian motion starting from x and $dP_{1,x,y}$ is the law of the Brownian bridge between x and y: it is a probability measure over $L_{x,y}(M)$, the space of continuous paths starting from x and arriving in y. Let $p_t(x, y)$ be the heat kernel associated to the heat semi-group. We say that $P_x(M) = \cup L_{x,y}(M)$ by using the following formula:

$$dP_1^x = p_1(x, y) dy \otimes dP_{1,x,y}. \tag{1.10}$$

By repeating the same considerations, $\tau(\omega)$ is an element of $H^2(L_{x,y}, Z)$ if we consider smooth loops. In particular, we can consider a formal line bundle $\Lambda_{x,y}$ over $L_{x,y}(M)$. The problem is to glue together all these formal line bundles $\Lambda_{x,y}$ into a formal line bundle Λ_x over $P_x(M)$. If we consider smooth paths, the obstruction is measured in [5]. It is $d\beta = -\omega$. We can perturb $\tau(\omega)$ by

$$\tilde{\tau}(\omega) = \beta(\gamma_1) + \tau(\omega). \tag{1.11}$$

such that $\tilde{\tau}(\omega) = \tau(\omega)$ over $L_{x,y}(M)$ and such that $\tilde{\tau}(\omega)$ is closed Z-valued over $P_x(M)$. So, we can construct the formal global line bundle Λ_x over $P_x(M)$ by its sections. It remains to show that a section of Λ_x restricts into a section of $\Lambda_{x,y}$. We meet the problem that a section of Λ_x is only almost surely defined. We will proceed as in the quasi-sure analysis [10, 1]: a smooth functional over the flat Wiener space restricts into a functional over a finite codimensional manifold by using integration by parts formulas. We would like to state the analogous result for a smooth section of Λ_x.

In order to speak of Sobolev spaces over $P_x(M)$, we consider the tangent space (1.1) with only the condition $H_0 = 0$. This allows us to perform integration by parts [8, 3]. We can speak of the connection one form A_i of Λ_x over $B(\gamma_i, \delta)$. We get over $P_x(M)$

$$\nabla^{\Lambda_x}\alpha_i = d\alpha_i + A_i(\gamma)\alpha_i \tag{1.12}$$

which is almost surely consistent with the change of charts. Following the convention of differential geometry, we can iterate the operation of covariant differentiation and we get the operation ∇^{k,Λ_x} which is a k Hilbert-Schmidt cotensor in the tangent space connection, if we add the trivial connection in the tangent space (the tangent space of $P_x(M)$ is trivial modulo the parallel transport τ_t).

Definition 1.6. *The space $W_{k,p}(\Lambda_x)$ is the space of sections of the formal line bundle Λ_x such that $\nabla^{k',\Lambda_x}\phi$ belongs to L^p for $k' \leq k$. The space of smooth sections $W_{\infty,\infty-}(\Lambda_x)$ is the intersection of all Sobolev spaces $W_{k,p}(\Lambda)$.*

Theorem 1.7. *A smooth section ϕ of Λ_x restricts to a section Φ_y of $\Lambda_{x,y}$.*

2. String structure over the Brownian bridge

Let us consider the finite energy based path space $P_f(G)$ and the finite energy loop group $L_f(G)$. Let us consider the two form over $P_f(G)$, which on the level of a Lie algebra satisfies to:

$$c(X_., Y_.) = \frac{1}{8\pi^2}\int_0^1 \left(\langle X_s, dY_s\rangle - \langle Y_s, dX_s\rangle\right). \tag{2.1}$$

Its restriction over $L_f(G)$ gives a central extension $\tilde{L}_f(G)$ of $L_f(G)$ if G is supposed simple simply laced [23]: in particular, $Spin_{2n}$ is simply laced.

Let us introduce the Bismut bundle over $L_x(M)$ if Q is a principal bundle over M with structure group G: q_s is a loop over γ_s such that $q_s = \tau_s^Q g_s$, where τ_s^Q is the parallel transport over Q for any connection over Q. We suppose that $g_.$ is of finite energy. Moreover, $g_1 = (\tau_1^Q)^{-1}$. Let f be the map from $L_x(M)$ to G:

$$\gamma_. \to (\tau_1^Q)^{-1}. \tag{2.2}$$

Let π be the projection from $L_e(Q)$ over $L_x(M)$ and π the projection from $P_f(G)$ over G: $g_. \to g_1$. We get a commutative diagram of bundles (see [6]):

$$\begin{array}{ccc} L_e(Q) & \to & P_f(G) \\ \downarrow & & \downarrow \\ L_x(M) & \to & G \end{array} \tag{2.3}$$

Let ω be the 3 form:

$$\omega(X, Y, Z) = \frac{1}{8\pi^2}\langle X, [Y, Z]\rangle. \tag{2.4}$$

We get [6, 17]: $\pi^*\omega = dc$. If the first Pontryagin class of Q is equal to 0, $(f^*)^*c - \pi = A8{*}\nu$ is a closed form over $L_e(Q)$: in such a case namely $f^*\omega = d\nu$, where ν is a nice iterated integral over $L_x(M)$. We can perturb $(f^*)^*c - \pi^*\nu$ by a closed iterated integral in the basis such that we get a closed Z-valued 2 form F_Q over $L_e(Q)$ (in the smooth loop context). We do the following hypothesis, for smooth loop:

Hypothesis. *$L_e(Q)$ is simply connected. $L_f(G)$ is simply connected. $L_x(M)$ is simply connected.*

The obstruction to trivialize $L_e(Q)$ is the holonomy: we can restrict the transition functions $P_f(G) \to G$ to be an element of smooth loop in G. We can introduce a connection over this bundle ∇^∞ and we can pullback this connection into a connection ∇^∞ over the bundle $L_e(Q) \to L_x(M)$. This allows us to lift the distinguished paths over $L_x(M)$ (we are now in the stochastic context) into distinguished paths over $L_e(Q)$, which will allow us to produce a system of transition functions with values in $\tilde{L}_f(Q)$, because the stochastic part of F_Q is a sum of iterated integrals. But the obstruction to trivialize $L_e(Q)$ is τ_1^Q which is almost surely defined. We cannot work over open neighborhoods in order to trivialize our lift.

Let us resume: we can find a set of subset O_i of $L_x(M)$ such that:

i) $\cup O_i = L_x(M)$ almost surely.

ii) There exists a sequence of smooth functionals G_i^n such that $G_i^n > 0$ is included into O_i and such that G_i^n tends increasingly almost surely to the indicatrix function of O_i.

iii) Over $O_i \cap O_j$, there exists a map $\tilde{\rho}_{i,j}(\gamma)$ with values in $\tilde{L}_f(G)$ such that

$$\tilde{\rho}_{i,j}(\gamma)\tilde{\rho}_{j,i}(\gamma) = \tilde{e} \tag{2.5}$$

($\tilde{e}$ is the unique element of $\tilde{L}_f(G)$) and such that, over $O_i \cup O_j \cup O_k$,

$$\tilde{\rho}_{i,j}(\gamma)\tilde{\rho}_{j,k}(\gamma)\tilde{\rho}_{k,i}(\gamma)\tilde{e} \tag{2.6}$$

iv) Moreover, $\tilde{\rho}_{i,j}(\gamma)$ is smooth in the following way: $\tilde{\rho}_{i,j}(\gamma) = (l_{i,j,.}(\gamma), \alpha_{i,j})$, where $l_{i,j,.}(\gamma)$ is a path in $L_f(G)$ starting from $e_.$ which depends smoothly on γ and $\alpha_{i,j}$ a functional in S^1 which depends smoothly on γ. (Let us recall that the central extension $\tilde{L}_f(G)$ can be seen as a set of couple of paths in $L_f(G)$ and an element of the circle, submitted to an equivalence relation as was discussed in the first part).

Let us suppose that there exists a unitary representation $Spin_\infty$ of $\tilde{L}_f(G)$.

Definition 2.1. *A measurable section of Spin is a family of random variables from O_i into $Spin_\infty$ submitted to the relation:*

$$\psi_j = \tilde{\rho}_{j,i}\psi_i. \tag{2.7}$$

The modulus $\|\psi\|$ is intrinsically defined, because the representation is unitary.

Definition 2.2. *A L^p section of Spin is a measurable section ψ such that*

$$\|\psi\|_{L^p} = \| \, \|\psi\| \, \|_{L^p} < \infty. \tag{2.8}$$

We would like to speak of a bundle whose fiber should be $\tilde{L}_f(G)$, by doing as in the first part: the problem is that there is no Haar measure over $\tilde{L}_f(G)$. We will begin to construct an S^1 bundle over $L_e(Q)$, and for that, we need to construct a measure over $L_e(Q)$. This requires to construct a measure over $L_f(G)$.

For that, we consider the stochastic differential equation:

$$dg_s = g_s(C + B_s), \qquad g_0 = e, \tag{2.9}$$

where B_s is a Brownian motion starting from 0 in the Lie algebra of G and C is an independent Gaussian variable over the Lie algebra of G with average 0 and covariance I_d. The law of g_1 has a density $q(g) > 0$. If we consider a C^2 deterministic path in G starting from e, $k_s g_s$ and $g_s k_s$ have a law which is absolutely continuous with respect to the original law, but the density is only in L^1 (unlike the traditional case of [2] for continuous loops, where the density is in all the L^p). We can get infinitesimal quasi-invariance formulas, that is integration by parts formulas: we take as tangent vector fields $X_s = g_s K_s$ or $X_s = K_s g_s$ ($K_0 = 0$), where K_s has values in the Lie algebra of G. We take as the Hilbert norm $\int_0^1 \|K"_s\|^2 ds$ (instead of $\int_0^1 \|K'_s\|^2 ds$ for the continuous case). If $K_.$ is deterministic, we get an integration by parts formula:

$$E_{P_f(G)}\left[\langle dF, X\rangle\right] = E_{P_f(G)}[F \text{ div } X], \tag{2.10}$$

where div X belongs to all the L^p.

This allows us, since $q(g) > 0$, to desintegrate the measure over the set of finite energy paths going from e to g. We get a measure dP_g, and we get a measure over $L_e(Q)$:

$$d\mu_{tot} = dP_{1,x} \otimes dP_{(\tau_1^Q)^{-1}}. \tag{2.11}$$

Let us suppose that $(\tau_1^Q)^{-1}$ belongs to a small open neighborhood G_i of G, where the bundle $P_f(G) \to G$ is trivial. Let $K_{i,.}$ be the connection one form of this bundle in this trivialization. We split the tangent space of $L_e(Q)$ into the orthonormal sum of a vertical one and a horizontal one:

— The vertical one is constituted of vector fields of the type $q_s K_s$ (K_s is a loop in the Lie algebra of G with two derivatives) with the Hilbert norm $\int_0^1 \|K"_s\|^2 ds$).

— The horizontal one is a set of vectors of the type

$$X_s^h = \tau_s H_s - K_{i,s}\left(\langle d(\tau_1^Q)^{-1}), X\rangle\right) g_s, \tag{2.12}$$

where $X_s = \tau_s H_s$ and H_s checks (1.1). The Hilbert norm is $\int_0^1 \|H'_s\|^2 ds$.

We get:

Proposition 2.3. *Let F be a cylindrical functional over $L_e(Q)$. We have the integration by parts formula:*

$$\mu_{tot}[< dF, X >] = \mu_{tot}[F \text{ div } X] \tag{2.13}$$

if X corresponds to a vertical or a horizontal vector fields which are associated to deterministic K_s or deterministic H_s. Moreover, $\operatorname{div} X$ *belongs to all the L^p.*

This integration by parts formula allows us to get Sobolev spaces over $L_e(Q)$.

We repeat the considerations which lead to the transition functions $\tilde{\rho}_{i,j}(\gamma)$ but over $L_e(Q)$. We get a system of charts $(O_i, \rho_{i,j}(q_.)$ with the difference that $O_i \subset L_e(Q)$ and $\rho_{i,j}(q_;)$ belong to S^1. It is easier to say in this context that $\rho_{i,j}(q_.)$ belong locally to all the Sobolev spaces.

Definition 2.4. *A measurable functional $\tilde{F}(\tilde{q}_.)$ associated to the formal circle bundle over $L_e(Q)$ constructed from the system of $\rho_{i,j}$ is a family of measurable functionals $F_i : O_i \times S^1 \to R$ such that almost surely in $q_.$ and u (we choose the Haar measure over the circle)*

$$F_i(q_., u_i) = F_j(q_., u_j), \tag{2.14}$$

where $u_j = u_i \rho_{i,j}(q_.)$ almost surely.

Since there is the Haar measure over the circle, we can do as in the first part in order to speak of functionals $\tilde{F}(\tilde{q}_.)$ which belong to $L^p(\tilde{\mu}_{tot})$, by integrating in the fiber.

An "element" $\tilde{q}_.$ of $\tilde{L}_e(Q)$ can be seen formally as the couple of a distinguished path in $L_e(Q)$ arriving in $q_. = \pi(\tilde{q}_.)$ and an element of the circle. An element $\tilde{g}_.$ of $\tilde{L}_2(G)$ can be seen as the couple of a distinguished path in $L_2(G)$, the group of based loops in G with two derivatives, and an element of the circle. $\tilde{g}_.$ acts over $\tilde{q}_.$ by multiplying vertically the end loop of the distinguished path associated to $\tilde{q}_.$ and multiplying the two element of the circle. Moreover, the action of $L_2(G)$ leads to quasi-invariance formulas over $P_f(G)$. This motivates the following definition:

Definition 2.5. *A stochastic gauge transform Ψ of the formal bundle $\tilde{L}_e(Q)$ is a measurable application from $L_x(M)$ into $\tilde{L}_2(G)$.*

A gauge transform induces a transformation Ψ of functionals over $\tilde{L}_2(Q)$:

$$\Psi(\tilde{F}(\tilde{q}_.) = \tilde{F}(\tilde{q}_.\Psi(q_.)). \tag{2.15}$$

This definition has a rigorous sense at the level of functionals.

Theorem 2.6. *The group of gauge transforms acts naturally by isometries over $L^\infty(\tilde{\mu}_{tot})$.*

3. The case of the free loop space

Let us suppose that we consider the free loop space of continuous applications from the circle S^1 into M with the measure

$$dP = p_1(x, x)dx \otimes dP_{1,x}. \tag{3.1}$$

The tangent space is as in (1.1), but we have to take the periodicity condition $X_0 = X_1$. We take as Hilbert structure the Hilbert structure $\int_0^1 \|H'_s\|^2 ds + \int_0^1 \|X_s\|^2 ds$ which is invariant by rotation.

Let A_Q be the connection 1-form over the principal bundle $Q \to M$: it is a one form from the tangent space of Q into the Lie algebra of Q. We associate following [7] the 2 form over $L(Q)$ to the free loop space of Q for smooth loops:

$$\omega_Q = \frac{1}{4\pi^2} \int_0^1 1/2\langle A_q, d/dt A_Q\rangle. \tag{3.2}$$

This two form gives the central extension of $L(G)$ in the fiber of $L(Q)$. We have to perturb it by a form in order to get a form which is Z-valued closed over $L(Q)$ and which gives ω_Q in the fiber.

Following [7], we perturb ω_Q into a form ω'_Q over the smooth free loop space of Q:

$$\omega'_Q = \frac{1}{4\pi^2} \int_0^1 1/2\langle A_Q, d/dt A_Q\rangle - \langle R_Q, A_Q(dq_s)\rangle, \tag{3.3}$$

where R_Q is the curvature tensor of A_Q. Let σ_Q be the 3 form over Q:

$$\sigma_Q = \frac{1}{8\pi^2}\langle A_Q, R_Q - 1/6[A_Q, A_Q]\rangle. \tag{3.4}$$

We get

$$d\sigma_Q = \pi^* p_1(Q), \tag{3.5}$$

where $p_1(Q)$ is the first Pontryiagin class of Q. If $p_1(Q) = 0$, we can choose a form over M such that $p_1(Q) = -d\nu$ and such that $\sigma_Q - \pi^*\nu$ is a Z-valued closed 3 form over Q. Let τ be the operation of transgression over $L(Q)$. Following [7], we choose as curvature term in order to define the S^1 bundle over $L(Q)$ the expression $F_Q = \omega'_Q - \tau(\pi^*\nu)$.

We repeat the considerations of the second part with the main difference that we don't have globally a commutative diagram as (2.2). This leads to some difficulties, but since we have such a diagram locally at the starting point, we can produce a system of charts $O_i, \tilde{\rho}_{i,j}(\gamma)$ with the difference that $O_i \subset L(M)$ and that $\tilde{\rho}_{i,j}(\gamma)$ takes its values in $\tilde{L}_f(G)$, the basical central extension of the free loop space of G (we consider loops with finite energy).

If $G = Spin_{2n}$, the central extension of the free loop group has got a unitary representation by using the fermionic Fock space. This allows us to repeat Definition 2.1 and Definition 2.2.

Moreover, F_q is invariant under rotation. The natural circle action lifts to the section of the spin bundle over $L(M)$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 297–306.

On a Mixed Problem for a System of Differential Equations of the Hyperbolic Type with Rotation Points

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Abstract

We propose two approaches to find asymptotic solutions to systems of linear differential equations of the hyperbolic type with slowly varying coefficients in the presence of rotation points.

In [1, 2], an asymptotic solution to a system of linear differential equations of the hyperbolic type with slowly varying coefficients is constructed for the case of constant multiplicity of the spectrum in the whole integration interval. In this paper, we propose two approaches to find asymptotic solutions to systems of such a type in the presence of rotation points.

Let us consider a differential equation of the form

$$\varepsilon^h \frac{\partial^2 u(t;x)}{\partial t^2} = A_1(t;\varepsilon)\frac{\partial^2 u}{\partial x^2} + \varepsilon A_2(t;x;\varepsilon)u(t;x) + \varepsilon A_3(t;x;\varepsilon)\frac{\partial u}{\partial t} \tag{1}$$

with the initial condititons

$$u(0;x) = \varphi_1(x), \qquad \frac{\partial u}{\partial t}(0;x) = \varphi_2(x),$$

and the boundary conditions

$$u(t;0) = u(t;l) = 0,$$

where $0 \le t \le L$, $0 \le x \le l$, $\varepsilon (0 < \varepsilon \le \varepsilon_0)$ is a small real parameter; $u(t;x)$, $\varphi_1(x)$, $\varphi_2(x)$ are n-dimensional vectors; $A_k(t;x;\varepsilon)$, $k = \overline{1,3}$ are matrices of order $n \times n$, $n \in N$.

Let the following conditions be valid:

1) coefficients of system (1) admit expansions in powers of ε

$$A_1(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_s(t), \qquad A_j(t;x;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s A_j^{(s)}(t;x), \quad j = 2,3;$$

2) matrices $A_s(t)$, $A_j^{(s)}(t;x)$ are infinitely differentiable with respect to $t \in [0;L]$ and continuous in $x \in [0;l]$ together with derivatives up to the second order inclusively; functions $\varphi_1(x)$, $\varphi_2(x)$ are twice continuously differentiable;

3) in the interval $[0;L]$ for arbitrary $r,s=0,1,\ldots$, the series

$$\sum_{k=1}^{\infty}\left\|\frac{d^rA_{mk}^{(s)}(t)}{dt^r}\right\|^2, \quad \sum_{k=1}^{\infty}\left\|\frac{d^rC_{mk}^{(s)}(t)}{dt^r}\right\|^2,$$

converge uniformly, where

$$A_{mk}^{(s)}(t)=\frac{2}{l}\int_0^l A_2^{(s)}(t;x)\sin\omega_k x\sin\omega_m x dx;$$
$$C_{mk}^{(s)}(t)=\frac{2}{l}\int_0^l A_3^{(s)}(t;x)\sin\omega_k x\sin\omega_m x dx.$$

We shall look for a solution of problem (1) in the form

$$u(t;x;\varepsilon)=\sum_{k=1}^{\infty}v_k(x)z_k(t;\varepsilon), \tag{1a}$$

where $v_k(x)$ is the orthonormal system

$$v_k(x)=\sqrt{\frac{2}{l}}\sin\omega_k x, \quad \omega_k=\frac{k\pi}{l}, \quad k=1,2\ldots,$$

and $z_k(t;\varepsilon)$ are n-dimensional vectors that are defined from the denumerable system of differential equations

$$\begin{aligned}\varepsilon^h\frac{d^2z_k(t;\varepsilon)}{dt^2}&=-\omega_k^2\sum_{s=0}^{\infty}\varepsilon^sA_1^{(s)}(t)z_k(t;\varepsilon)+\\&+\varepsilon\sum_{m=1}^{\infty}\left(\sum_{s=0}^{\infty}\varepsilon^sA_{km}^{(s)}(t)z_m(t;\varepsilon)+\sum_{s=0}^{\infty}\varepsilon^sC_{km}^{(s)}(t)\frac{dz_m(t;\varepsilon)}{dt}\right),\end{aligned} \tag{2}$$

with the initial conditions

$$z_k(0;\varepsilon)=\varphi_1, \qquad \frac{dz_k(0;\varepsilon)}{dt}=\varphi_2,$$

where

$$\varphi_1=\sqrt{\frac{2}{l}}\int_0^l\varphi_1(x)\sin\omega_k x dx, \quad \varphi_2=\sqrt{\frac{2}{l}}\int_0^l\varphi_2(x)\sin\omega_k x dx.$$

Putting in (2) $z_k(t;\varepsilon)=q_{1k}(t\ \varepsilon)$, $\dfrac{dz_k(t;\varepsilon)}{dt}=q_{2k}(t;\varepsilon)$, we get the following system of the first order

$$\varepsilon^h\frac{dq_k(t;\varepsilon)}{dt}=H_k(t;\varepsilon)q_k(t;\varepsilon)+\varepsilon\sum_{m=1}^{\infty}H_{km}(t;\varepsilon)q_k(t;\varepsilon), \tag{3}$$

with the initial condition $q_k(0;\varepsilon) = x_{k0}$, where $q_k(t;\varepsilon)$, x_{k0} are $2n$-dimensional vectors, $H_{km}(t;\varepsilon)$, $H_k(t;\varepsilon)$ are square $(2n \times 2n)$-matrices of the form

$$q_k(t;\varepsilon) = \left\| \begin{matrix} q_{1k}(t;\varepsilon) \\ q_{2k}(t;\varepsilon) \end{matrix} \right\|,$$

$$H_{km}(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_{km}^{(s)}(t) = \left\| \begin{matrix} 0 & 0 \\ \sum\limits_{s=0}^{\infty} \varepsilon^s A_{km}^{(s)}(t) & \sum\limits_{s=0}^{\infty} \varepsilon^s C_{km}^{(s)}(t) \end{matrix} \right\|,$$

$$H_k(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s H_k^{(s)}(t) = \left\| \begin{matrix} 0 & E \\ -\sum\limits_{s=0}^{\infty} \omega_k^2 \varepsilon^s A_s(t) & 0 \end{matrix} \right\|.$$

Let roots of the characteristic equation

$$\det \left\| H_k^{(0)}(t) - \lambda_k^{(i)}(t) E \right\| = 0, \quad k = 1, 2 \ldots, \quad i = \overline{1, 2n},$$

for the previous system coincide at the point $t = 0$ and be different for $t \in (0; L]$ (i.e., $t = 0$ is a rotation point of system (3)). The following theorem tells us about the form of a formal solution of system (3).

Theorem 1. *If conditions 1)–3) are valid and roots of the equation*

$$det \left\| H_k^{(0)}(t) + \varepsilon H_k^{(1)}(t) - \lambda_k(t;\varepsilon) E \right\| = 0$$

are simple $\forall\, t \in [0; L]$*, then system (3) has the formal matrix solution*

$$Q_k(t;\varepsilon) = U_k(t;\varepsilon) \exp\left(\frac{1}{\varepsilon^h} \int_0^t \Lambda_k(t;\varepsilon) dt \right), \tag{4}$$

where $U_k(t;\varepsilon)$, $\Lambda_k(t;\varepsilon)$ *are square matrices of order* $2n$*, that are presented as formal series*

$$U_k(t;\varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r U_k^{(r)}(t), \qquad \Lambda_k(t;\varepsilon) = \sum_{r=0}^{\infty} \Lambda_k^{(r)}(t). \tag{5}$$

Proof. Hawing substituted (4), (5) into (3), we get the following system of matrix equations

$$H_k^{(1)}(t;\varepsilon) U_k^{(0)}(t;\varepsilon) - U_k^{(0)}(t;\varepsilon) \Lambda_k^{(0)}(t;\varepsilon) = 0, \tag{6}$$

$$H_k^{(1)}(t;\varepsilon) U_k^{(s)}(t;\varepsilon) - U_k^{(s)}(t;\varepsilon) \Lambda_k^{(0)}(t;\varepsilon) = -U_k^{(0)}(t;\varepsilon) \Lambda_k^{(s)}(t;\varepsilon) + B_k^{(s)}(t;\varepsilon), \tag{7}$$

where

$$B_k^{(s)}(t;\varepsilon) = \sum_{r=1}^{s-1} U_k^{(r)}(t;\varepsilon) \Lambda_k^{(s-r)}(t;\varepsilon) + \frac{\partial U_k^{(s-h)}(t;\varepsilon)}{\partial t} - \\ - \sum_{r=2}^{s} H_k^{(r)}(t;\varepsilon) U_k^{(s-r)}(t;\varepsilon) - \sum_{r=1}^{s-1} H_{km}^{(r)}(t;\varepsilon) U_k^{(s-r)}(t;\varepsilon),$$

$$H_k^{(1)}(t;\varepsilon) = H_k^{(0)}(t) + \varepsilon H_k^{(1)}(t).$$

It follows from the conditions of the theorem that there exists a nonsingular matrix $V_k(t;\varepsilon)$ such that

$$H_k^{(1)}(t;\varepsilon)V_k(t;\varepsilon) = V_k(t;\varepsilon)\Lambda_k(t;\varepsilon),$$

where

$$\Lambda_k(t;\varepsilon) = \operatorname{diag}\left\{\lambda_{1k}(t;\varepsilon),\ldots,\lambda_{\nu k}(t;\varepsilon)\right\}, \quad \nu = 2n.$$

Multiply (6) and (7) from the left by the matrix $V_k^{-1}(t;\varepsilon)$ and introduce the notations

$$P_k^{(s)}(t;\varepsilon) = V_k^{-1}(t;\varepsilon)U_k^{(s)}(t;\varepsilon), \quad F_k^{(s)}(t;\varepsilon) = V_k^{-1}(t;\varepsilon)B_k^{(s)}(t;\varepsilon).$$

Finally, we obtain the system

$$\begin{aligned}&\Lambda_k(t;\varepsilon)P_k^{(0)}(t;\varepsilon) - P_k^{(0)}(t;\varepsilon)\Lambda_k^{(0)}(t;\varepsilon) = 0,\\ &\Lambda_k(t;\varepsilon)P_k^{(s)}(t;\varepsilon) - P_k^{(s)}(t;\varepsilon)\Lambda_k^{(0)}(t;\varepsilon) = P_k^{(0)}(t;\varepsilon)\Lambda_k^{(s)}(t;\varepsilon) + F_k^{(s)}(t;\varepsilon).\end{aligned} \tag{8}$$

Let us put here $P_k^{(0)}(t;\varepsilon) = E$. Then $\Lambda_k^{(0)}(t;\varepsilon) = \Lambda_k(t;\varepsilon)$. From (8), we determine $\Lambda_k^{(s)}(t;\varepsilon) = -F_k^{(s0)}(t;\varepsilon)$, where $F_k^{(s0)}(t;\varepsilon)$ is a diagonal matrix that consists of diagonal elements of the matrix $F_k^{(s)}(t;\varepsilon)$. Elements $P_k^{(s)}(t;\varepsilon)$ that are not situated on the main diagonal, are determined by formulas

$$\left\{P_k^{(s)}(t;\varepsilon)\right\}_{ij} = \frac{\{F_k^{(s)}(t;\varepsilon)\}_{ij}}{\lambda_{ik}(t;\varepsilon) - \lambda_{jk}(t;\varepsilon)}, \quad i \neq j, \ i,j = \overline{1,2n},$$

and diagonal elements $\left\{P_k^{(s)}(t;\varepsilon)\right\}_{ii} = 0$. The theorem is proved.

In investigating formal solutions, it has been shown that the following asymptotic equalities ave valid for $t \in [0; L\varepsilon]$:

$$P_k^{(s)}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^{\alpha_1}}\right), \quad \Lambda_k^{(s)}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^{\alpha_2}}\right),$$

where α_1, α_2 are positive numbers, and $t \in (L\varepsilon; L]$, then $P_k^{(s)}(t;\varepsilon)$ and $\Lambda_k^{(s)}(t;\varepsilon)$ are bounded for $\varepsilon \to 0$. Let us consider the character of formal solutions in the sense [3]. Let us write down the p-th approximation

$$q_{kp}(t;\varepsilon) = Q_{kp}(t;\varepsilon)a_k(\varepsilon),$$

where

$$Q_{kp}(t;\varepsilon) = U_{kp}(t;\varepsilon)\exp\left(\frac{1}{\varepsilon^h}\int_0^t \Lambda_{kp}(t;\varepsilon)dt\right),$$

$$U_{kp}(t;\varepsilon) = \sum_{m=1}^{p} \varepsilon^m U_k^{(m)}(t;\varepsilon), \quad \Lambda_{kp}(t;\varepsilon) = \sum_{m=1}^{p} \varepsilon^r \Lambda_k^{(m)}(t;\varepsilon),$$

$a_k(\varepsilon)$ is an arbitrary constant vector. Having substituted the p-th approximation of solutions (4) in the differential operator

$$Mq_k \equiv \varepsilon^h \frac{dq_k}{dt} - H_k(t;\varepsilon)q_k(t;\varepsilon) + \varepsilon \sum_{m=1}^{\infty} H_{km}(t;\varepsilon)q_k(t;\varepsilon),$$

and taking into account (6), (7), we obtain

$$Mq_{kp}(t;\varepsilon) = O\left(\varepsilon^{p+1}\right) Y_{kp}(t;\varepsilon) \exp\left(\frac{1}{\varepsilon^h}\int_0^t \Lambda_{kp}(t;\varepsilon)dt\right),$$

where

$$Y_{kp}(t;\varepsilon) = O\left(\frac{1}{\varepsilon^\beta}\right), \quad \beta > 0 \quad \text{for } t \in [0; L\varepsilon],$$
$$Y_{kp}(t;\varepsilon) = O(1), \quad \text{for } t \in [L\varepsilon; L],$$

Let, in addition, the following condititons are valid:
4) $\operatorname{Re}\lambda_{ik}(t;\varepsilon) < 0 \;\forall\, t \in [0; L]$;
5) $\operatorname{Re}\Lambda_k^{(p)}(t;\varepsilon) < 0$; for $h = 1$, $\forall\, t \in [0; L\varepsilon]$.

Then there exists ε_{k1} $(0 < \varepsilon_{k1} < \varepsilon_0)$ such that, for all $t \in [0; L]$, the asymptotic equality $Mq_{kp}(t;\varepsilon) = O(\varepsilon^p)$ is fulfilled. The following theorem is true.

Theorem 2. *If the conditions of Theorem 1 and conditions 4), 5) are fulfilled, and for $t = 0, q_{kp}(0;\varepsilon) = q_k(0;\varepsilon)$, where $q_k(t;\varepsilon)$ are exact solutions of system (3), then, for every $L_k > 0$, there exist constants $C_k > 0$ not depending on ε and such that, for all $t \in [0; L]$ and $\varepsilon \in (0; \varepsilon_{k1}]$, the following inequalities*

$$\|q_{kp} - q_k\| < C_k \varepsilon^{p+1-h-\frac{1}{2n}}$$

are fulfilled.

Proof. Vector functions $y_k(t;\varepsilon) = q_k(t;\varepsilon) - q_{kp}(t;\varepsilon)$ are solutions to the equations

$$\varepsilon^h \frac{dy_k}{dt} = H_k(t;\varepsilon)y_k + O\left(\varepsilon^{p+1}\right) + \sum_{m=1}^{\infty} H_{km}y_k.$$

With the help of the transformation

$$y_k(t;\varepsilon) = V_k(t;\varepsilon)z_k(t;\varepsilon),$$

we reduce the latter system to the form

$$\varepsilon^h \frac{dz_k}{dt} = \left(\Lambda_k(t;\varepsilon) + \varepsilon B_{1k}(t;\varepsilon)\right) z(t;\varepsilon) + O\left(\varepsilon^{p+1-\frac{1}{2n}}\right). \tag{9}$$

Let us replace system (9) by the equivalent system of integral equations

$$z_k(t;\varepsilon) = \int_0^t \exp\left(\frac{1}{\varepsilon^h}\int_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\left(B_k(t;\varepsilon)z_k(t;\varepsilon) + O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right).$$

Let us bound $\|z_k(t;\varepsilon)\|$:

$$\|z_k(t;\varepsilon)\| \leq \int_0^t \left\|\exp\left(\frac{1}{\varepsilon^h}\int_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\right\| \left(\|B_k(t;\varepsilon)\|\,\|z_k(t;\varepsilon)\| + \left\|O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right\|\right). \quad (10)$$

Since

$$\left\|\exp\left(\frac{1}{\varepsilon^h}\int_{t_1}^t \Lambda_k(s;\varepsilon)ds\right)\right\| \leq 1, \quad \forall\, t \in [0;L],$$

$$\|B_k(t;\varepsilon)\| \leq \frac{C_{1k}}{\varepsilon^\alpha}, \quad \alpha > 0,\ \forall\, t \in [0;L\varepsilon];$$

$$\left\|O\left(\varepsilon^{p+1-h-\frac{1}{2n}}\right)\right\| \leq C_{2k}\varepsilon^{p+1-h-\frac{1}{2n}},$$

we have

$$\|z_k(t;\varepsilon)\| \leq C_{3k}\int_0^t \|z_k(t_1;\varepsilon)\|dt_1 + C_{2k}L\varepsilon^{p+1-h-\frac{1}{2n}},$$

where $C_{3k} = C_{1k}/\varepsilon^\alpha$.

Using the Gronwall-Bellman lemma, we get the inequality

$$\|z_k(t;\varepsilon)\| \leq C_k\varepsilon^{p+1-h-\frac{1}{2n}}.$$

Then

$$\|y_k(t;\varepsilon)\| = \|q_k - q_{kp}\| \leq \|V_k(t;\varepsilon)\| \cdot \|z_k\| \leq C_k\varepsilon^{p+1-h-\frac{1}{2n}}.$$

The theorem is proved.

The other approach to constructing an asymptotic solution is based on the "joining" of solutions in a neighbourhood of a rotation point with solutions that are constructed outside this neighbourhood. For this purpose, we suppose that the following conditions are fulfilled:

4) the equation $\det\|H_k^{(0)}(0) - \lambda_k(t)E\| = 0$ has a multiple root with an elementary divisor;

5) an matrix element

$$\left\{T_k^{-1}\left(\frac{dH_k^{(0)}(t)}{dt}\right)_{t=0}\frac{t}{\varepsilon} + P_k^{(1)}(0)T_k\right\}_{n1}$$

differs from zero for all $t \in [0;L\varepsilon]$, where T_k is a transformation matrix of the matrix $H_k^{(0)}(0)$;

6) matrices $H_k^{(r)}(t)$ and $H_{km}^{(r)}(t)$ are expandable in the interval $t \in [0;L\varepsilon]$ into convergent Taylor series

$$\begin{aligned} H_k^{(r)}(t) &= \sum_{s=0}^\infty \frac{1}{s!}\frac{d^sH_k^{(r)}(t)}{dt^s}\Big|_{t=0}t^s, \quad r = 0,1,\ldots \\ H_{km}^{(r)}(t) &= \sum_{s=0}^\infty \frac{1}{s!}\frac{d^sH_{km}^{(r)}(t)}{dt^s}t^s. \end{aligned} \quad (11)$$

To construct an expansion of a solution in the interval $[0; L\varepsilon]$, let us introduce a new variable $t_1 = \dfrac{t}{\varepsilon}$. Let us pass to this variable in system (3). Having grouped together coefficients of the same powers of ε on the right-hand side, we get

$$\varepsilon^{h-1}\frac{dq_k}{dt_1} = F_k(t_1;\varepsilon)q_k(t;\varepsilon) + \varepsilon\sum_{m=1}^{\infty} F_{km}(t_1;\varepsilon)q_k(t;\varepsilon), \tag{12}$$

where

$$F_k(t_1;\varepsilon) = \sum_{r=0}^{\infty} F_k(t_1)\varepsilon^r, \quad F_{km}(t_1;\varepsilon) = \sum_{r=0}^{\infty} F_{km}(t_1)\varepsilon^r,$$
$$F_k(t_1) = \sum_{s=0}^{\infty}\frac{1}{s!}\frac{d^s H_k^{(0;r-s)}(t)}{dt^s}t_1^s, \quad F_{km}(t_1) = \sum_{s=0}^{r}\frac{1}{s!}\frac{d^s H_{km}^{(0;r-s)}}{dt^s}t_1^s.$$

Roots of the characteristic equation for system (12) satisfy condition 4), therefore according to [2], we can look for a solution of equation (12) for $t \in [0; L\varepsilon]$ in the form

$$x_k^{(i)}(t;\varepsilon) = u_k^{(i)}\left(\frac{t}{\varepsilon};\mu\right)\exp\left(\frac{1}{\varepsilon^{h-1}}\int_0^{\frac{t}{\varepsilon}}\lambda_k^{(i)}(t;\mu)dt\right), \tag{13}$$

where a $2n$-dimensional vector $u_k^{(i)}(t_1;\mu)$ and the function $\lambda_k^{(i)}(t_1;\mu)$ admit expansions

$$u_k^{(i)}(t_1;\mu) = \sum_{r=0}^{\infty}\mu^r u_{kr}^{(i)}(t_1), \quad \lambda_k^{(i)}(t_1;\mu) = \sum_{r=0}^{\infty}\mu^r\lambda_{kr}^{(i)}(t_1), \quad \mu = \sqrt[2n]{\varepsilon}.$$

In the interval $[L\varepsilon; L]$, roots $\lambda_{ik}(t)$, $i = \overline{1, 2n}$, of the characteristic equation for system (3) are simple. Then, in this interval $2n$, independent formal solutions to system (3) are constructed in the form

$$y_k^{(i)}(t;\varepsilon) = v_k^{(i)}(t;\varepsilon)\exp\left(\frac{1}{\varepsilon^h}\int_0^t \xi_k^{(i)}(t;\varepsilon)dt\right), \tag{14}$$

where $v_k^{(i)}(t;\varepsilon)$ is an n-dimensional vector and $\xi_k^{(i)}(t;\varepsilon)$ is a scalar function which admit the expansions

$$v_k^{(i)}(t;\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r v_{kr}^{(i)}(t), \quad \xi_k^{(i)}(t;\varepsilon) = \sum_{r=0}^{\infty}\varepsilon^r\xi_{kr}^{(i)}(t).$$

The functions $u_{kr}^{(i)}(t_1)$, $\lambda_{kr}^{(i)}(t_1)$, $v_{kr}^{(i)}(t)$, $\xi_{kr}^{(i)}(t)$ are determined by the method from [2].

Denote by $x_{kp}^{(i)}(t;\varepsilon)$, $y_{kp}^{(i)}(t;\varepsilon)$ p-th approximations of solutions (13), (14), that are formed by cutting off the corresponding expansions at the p-th place. The p-th approximation of a general solution for $t \in [0; L\varepsilon]$ is of the form

$$\overline{x}_{kp}(t;\varepsilon) = \sum_{i=1}^{2n} x_{kp}^{(i)}(t;\varepsilon)a_{ki}(\varepsilon),$$

and, in the interval $[L\varepsilon; L]$:

$$\overline{y}_{kp}(t;\varepsilon) = \sum_{i=1}^{2n} y_{kp}^{(i)}(t;\varepsilon) b_{ki}(\varepsilon),$$

where $a_{ki}(\varepsilon)$, $b_{ki}(\varepsilon)$ are arbitrary numbers. We choose numbers $a_{ki}(\varepsilon)$ from the initial condition for system (3), that is equivalent to the relation $\overline{x}_{kp}(0;\varepsilon) = x_{0k}$. Let us "join" the constructed p-th approximations $\overline{x}_{kp}(t;\varepsilon)$, $\overline{y}_{kp}(t;\varepsilon)$ at the point $t = L\varepsilon$. We can do this by choosing numbers $b_{ki}(\varepsilon)$ in $\overline{y}_{kp}(L\varepsilon;\varepsilon)$ so that the equality

$$\overline{x}_{kp}(L\varepsilon;\varepsilon) = \overline{y}_{kp}(L\varepsilon;\varepsilon). \tag{15}$$

is fulfilled. Therefore, Theorem 2 is proved.

Theorem 3. *If conditions 1)–5) and relation (15) are fulfilled, then the Cauchy problem for system (3) has the p-th approximation of a solution of the form*

$$q_{kp}(t;\varepsilon) = \begin{cases} \overline{x}_{kp}(t;\varepsilon) & \text{for } 0 \le t \le L\varepsilon; \\ \overline{y}_{kp}(t;\varepsilon) & \text{for } L\varepsilon \le t \le L. \end{cases}$$

So, the theorem on the asymptotic character of formal solutions is proved.

Theorem 4. *If the conditions of Theorem 3 are fulfilled, then the following asymptotic bounds are valid:*

$$\|q_{kp}(t;\varepsilon) - q_k(t;\varepsilon)\| \le C \cdot \mu^{p+3-2n-h} \sup_{t\in[0;L\varepsilon]} \exp\left(\varepsilon^{1-h} \int_0^{\frac{t}{\varepsilon}} \sum_{m=0}^{2n(h-1)} \mu^k \, Re\, \lambda_{km}^{(i)}(t) dt\right)$$

for $t \in [0; L\varepsilon]$,

$$\|q_{kp}(t;\varepsilon) - q_k(t;\varepsilon)\| \le C \cdot \varepsilon^{p+1-h} \sup_{t\in[L\varepsilon;L]} \exp\left(\varepsilon^{-h} \int_{L\varepsilon}^{t} \sum_{m=0}^{h-1} \varepsilon^k \xi_{km}^{(i)}(t) dt\right)$$

for $t \in [L\varepsilon; L]$.

So, we get asymptotics of equation (1) for the case, when the rotation point is some inner point $t = L$ of the interval $[0; L]$ and also for two rotation points.

Now let us consider an inhomogeneous system of the hyperbolic type

$$\varepsilon^h \frac{\partial^2 u(t;x)}{\partial t^2} = A_1(t;\varepsilon) \frac{\partial^2 u}{\partial x^2} + g(t;x,\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right) \tag{16}$$

where

$$g(t;x;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s g_s(t;x).$$

With the help of transformation (1a), system (16) takes the form

$$\varepsilon^h \frac{dq_k}{dt} = H_k(t;\varepsilon) q_k(t;\varepsilon) + p_k(t;\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{17}$$

where

$$p_k(t;\varepsilon) = \sum_{s=0}^{\infty} \varepsilon^s p_k^{(s)}(t) = \left\| \begin{matrix} 0 \\ \sum_{s=0}^{\infty} \varepsilon^s f_k^{(s)}(t) \end{matrix} \right\|,$$

$$f_k^{(s)}(t) = \frac{2}{l} \int_0^l g_s(t;x) \sin \omega_k x dx.$$

Let one of the following cases hold: 1) "nonresonance", when the function $ik(t)$ $\left(k(t) = \frac{d\theta(t)}{dt}\right)$ is not equal to any root of the characteristic equation for all $t \in [0;L]$; 2) "resonance" when the function $ik(t)$ is equal identically to one of roots of the characteristic equation, for example, $ik(t) = \lambda_k^{(1)}(t)$. Then, in the "nonresonance" case, the following theorem is valid.

Theorem 5. *If the conditions of Theorem 1 are fulfilled, then, in the "nonresonance" case, system (17) has a partial formal solution of the form*

$$q_k(t;\varepsilon) = \sum_{m=0}^{\infty} \overline{q}_k(t;\varepsilon) \exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{18}$$

where $\overline{q}_k(t;\varepsilon)$ is an n-dimensional vector that admits the expansion

$$\overline{q}_k(t;\varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m q_k^{(m)}(t). \tag{19}$$

Proof. Having substituted (19), (18) in (17) and equated coefficients of the same powers of ε, we get

$$\begin{gathered} \left(H_k^{(0)}(t) - ik(t)\right) q_k^{(0)}(t) = -p_k^{(0)}(t), \\ \left(H_k^{(0)}(t) - ik(t)\right) q_k^{(s)}(t) = \frac{dq_k^{(s-h)}}{dt} - p_k^{(s)}(t) - \sum_{m=1}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t),\ s = 1, 2, \dots, \end{gathered} \tag{20}$$

Let us prove that system (20) has a solution. Since $\forall\, t \in [0;L]\ ik(t) \neq \lambda_k^{(j)}(t)$, we have

$$\det \|H_k^{(0)}(t) - ik(t)E\| \neq 0, \quad j = \overline{1, 2n}.$$

For this reason,

$$q_k^{(0)}(t) = -\left(H_k^{(0)}(t) - ik(t)E\right)^{-1} P_k^{(0)}(t),$$

$$q_k^{(s)}(T) = \left(H_k^{(0)}(t) - ik(t)E\right)^{-1} \left(\frac{dq_k^{(s-h)}}{dt} - p_k^{(s)}(t) - \sum_{m=1}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t)\right).$$

Theorem 5 is proved.

So, in the case of "nonresonance", the presence of a rotation point doesn't influence the form of a formal solution. In the "resonance" case, the following theoren is true.

Theorem 6. *If the conditions of Theorem 1 are fulfilled, then, in the case of "resonance", system (16) has a partial formal solution of the form*

$$q_k(t;\varepsilon) = \overline{q}_k(t;\varepsilon)\exp\left(\frac{i\theta(t)}{\varepsilon^h}\right), \tag{21}$$

where $\overline{q}_k(t;\varepsilon)$ are $2n$-dimensional vectors presented by the formal series

$$\overline{q}_k(t;\varepsilon) = \sum_{m=0}^{\infty}\varepsilon^m q_k^{(m)}(t;\varepsilon). \tag{22}$$

Proof. Substitute (21), (22) in (17) and determine vectors $q_k^{(m)}(t;\varepsilon), m = 0,1\ldots$, from the identity obtained with the help of equalities

$$\begin{aligned}\left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right) q_k^{(0)}(t;\varepsilon) &= -P_k^{(0)}(t),\\ \left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right) q_k^{(s)}(t;\varepsilon) &= h_k^{(s)}(t;\varepsilon),\end{aligned} \tag{23}$$

where

$$h_k^{(s)}(t;\varepsilon) = -p_k^{(s)}(t) + \frac{\partial p_k^{(s-h)}(t)}{\partial t} - \sum_{m=2}^{s} H_k^{(m)}(t) q_k^{(s-m)}(t;\varepsilon).$$

Prove that the system of equations (23) has a solution. Since $ik(t)$ coincides with a root $\lambda_k^{(1)}(t)$, but $\lambda_{ik}(t;\varepsilon) \neq ik(t)$, $i = \overline{1,2n}$, we get $\det \|H_k^{(1)}(t;\varepsilon) - ik(t)E\| \neq 0$. Therefore, from (23) we obtain

$$\begin{aligned}q_k^{(0)}(t;\varepsilon) &= -\left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right)^{-1} p_k^{(0)}(t),\\ q_k^{(s)}(t) &= \left(H_k^{(1)}(t;\varepsilon) - ik(t)E\right)^{-1} h_k^{(s)}(t;\varepsilon), \quad s = 1,2,\ldots .\end{aligned}$$

Theorem 6 is proved.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 307–312.

Conformal Invariance of the Maxwell-Minkowski Equations

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Abstract

We study the symmetry of Maxwell's equations for external moving media together with the additional Minkowski constitutive equations (or Maxwell-Minkowski equations). We have established the sufficient condition for a solution found with the help of conditional symmetry operators to be an invariant solution of the considered equation in the classical Lie sense.

In the present paper, we study the symmetry properties of Maxwell's equations in an external moving medium

$$\begin{aligned} &\frac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H} - \vec{j}, && \frac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\vec{E}, \\ &\mathrm{div}\vec{D} = \rho, && \mathrm{div}\vec{B} = 0, \end{aligned} \tag{1}$$

together with the additional Minkowski constitutive equations

$$\begin{aligned} &\vec{D} + \vec{u} \times \vec{H} = \varepsilon(\vec{E} + \vec{u} \times \vec{B}), \\ &\vec{B} + \vec{E} \times \vec{u} = \mu(\vec{H} + \vec{D} \times \vec{u}), \end{aligned} \tag{2}$$

where $\vec{u}$ is the velocity of the medium, ε is the permittivity and μ is the permeance of the stationary medium, ρ and $\vec{j}$ are the charge and current densities.

In [1], we have established the infinite symmetry of (1) for $\rho = 0$, $\vec{j} = 0$. In this case, the Maxwell equations admit an infinite-dimensional algebra whose elements have the form

$$X = \xi^{\mu}(x)\frac{\partial}{\partial x_{\mu}} + \eta_{E_a}\frac{\partial}{\partial E_a} + \eta_{B_a}\frac{\partial}{\partial B_a} + \eta_{D_a}\frac{\partial}{\partial D_a} + \eta_{H_a}\frac{\partial}{\partial H_a}, \tag{3}$$

where

$$\begin{aligned}
&\eta^{E_1} = \xi^3_0 B_2 - \xi^2_0 B_3 - (\xi^1_1 + \xi^0_0)E_1 - \xi^2_1 E_2 - \xi^3_1 E_3, \\
&\eta^{E_2} = -\xi^3_0 B_1 + \xi^1_0 B_3 - (\xi^2_2 + \xi^0_0)E_2 - \xi^1_2 E_1 - \xi^3_2 E_3, \\
&\eta^{E_3} = \xi^2_0 B_1 - \xi^1_0 B_2 - (\xi^3_3 + \xi^0_0)E_3 - \xi^2_3 E_2 - \xi^1_3 E_1, \\
&\eta^{B_1} = \xi^1_2 B_2 - \xi^1_3 B_3 - (\xi^2_2 + \xi^3_3)B_1 - \xi^0_3 E_2 + \xi^0_2 E_3, \\
&\eta^{B_2} = \xi^0_3 E_1 - \xi^0_1 E_3 - (\xi^1_1 + \xi^3_3)B_2 - \xi^2_1 B_1 + \xi^2_3 E_3, \\
&\eta^{B_3} = \xi^3_1 B_1 + \xi^3_2 B_2 - (\xi^1_1 + \xi^2_2)B_3 - \xi^0_2 E_1 + \xi^0_1 E_2, \\
&\eta^{D_1} = \xi^1_2 D_2 + \xi^1_3 D_3 - (\xi^1_1 + \xi^0_0)D_1 + \xi^0_3 H_2 - \xi^0_2 H_3, \\
&\eta^{D_2} = -\xi^0_3 H_1 + \xi^0_1 H_3 - (\xi^2_2 + \xi^0_0)D_2 - \xi^2_1 D_1 + \xi^2_3 D_3,
\end{aligned}$$

$$\eta^{D_3} = \xi_1^3 D_1 + \xi_2^3 D_2 - (\xi_3^3 + \xi_0^0) D_3 - \xi_2^0 H_1 - \xi_1^0 H_2,$$
$$\eta^{H_1} = -\xi_0^3 D_2 + \xi_0^2 D_3 - (\xi_1^1 + \xi_0^0) H_1 - \xi_1^2 H_2 - \xi_1^3 H_3,$$
$$\eta^{H_2} = \xi_0^3 D_1 - \xi_0^1 D_3 - (\xi_2^2 + \xi_0^0) H_2 - \xi_2^1 H_1 - \xi_2^3 H_3,$$
$$\eta^{H_3} = -\xi_0^2 D_1 + \xi_0^1 D_2 - (\xi_3^3 + \xi_0^0) H_3 - \xi_2^3 H_2 - \xi_3^1 H_1,$$

and $\xi^\mu(x)$ are arbitrary smooth functions of $x = (x_0, x_1, x_2, x_3)$, $\xi^\mu_\nu \equiv \dfrac{\partial \xi^\mu}{\partial x_\nu}$, $\mu, \nu = \overline{0,3}$, $a = \overline{1,3}$. We prove that the Maxwell equations (1) with charges and currents ($\rho \neq 0$, $\vec{j} \neq 0$) are invariant with respect to an infinite-parameter group provided that ρ and $\vec{j}$ are transformed in appropriate way.

Theorem 1. *The system of equations (1) is invariant with respect to an infinite-dimensional Lie algebra whose elements are given by*

$$Q = X + \eta^{j^a} \frac{\partial}{\partial j^a} + \eta^\rho \frac{\partial}{\partial \rho}, \tag{4}$$

where

$$\eta^{j^a} = -dj^a + \xi^a_b j^b + \xi^a_0 \rho, \quad \eta^\rho = -d\rho + \xi^0_0 \rho + \xi^0_b j^b, \quad d = -(\xi^0_0 + \xi^1_1 + \xi^2_2 + \xi^3_3) \tag{5}$$

and the summation from 1 *to* 3 *is understood over the index* b.

Proof. The proof of Theorem 1 requires long cumbersome calculations which are omitted here. We use in principle the standard Lie scheme which is reduced to realization of the following algorithm:
Step 1. The prolongating operator (4) is constructed by using the Lie formulae (see, e.g., [4]).
Step 2. Using the invariance condition

$$\underset{1}{Q} L\Psi \Big|_{L\Psi=0} = 0, \tag{6}$$

where $\underset{1}{Q}$ is the first prolongation of operator (4) and $L\Psi = 0$ is the system of equations (1), we obtain the determining equations for the functions ξ^μ, η^{j^a}, and η^ρ.
Step 3. Solving the corresponding determining equations, we obtain the condition of Theorem 1.

From the invariance condition (6) for the equation $\dfrac{\partial \vec{D}}{\partial t} = \mathrm{rot}\vec{H} - \vec{j}$, we obtain $\eta^{j^a} = -dj^a + \xi^a_b j^b + \xi^a_0 \rho$. By applying criterium (6) to the equation $\mathrm{div}\, \vec{D} = \rho$, we get $\eta^\rho = -d\rho + \xi^0_0 \rho + \xi^0_b j^b$. As follows from [1] the invariance condition for the equations $\dfrac{\partial \vec{B}}{\partial t} = -\mathrm{rot}\, \vec{E}$ and $\mathrm{div}\, \vec{B} = 0$ gives no restriction on η^{j^a} and η^ρ. Theorem 1 is proved.

The invariance algebra (4), (5) of the Maxwell equations contains the Galilei algebra $AG(1,3)$, Poincaré algebra $AP(1,3)$, and conformal algebra $AC(1,3)$ as subalgebras.

It is well known that the Maxwell equations in vacuum are invariant with respect to the conformal group [2]. In [1], we showed that there exists the class of conformally invariant constitutive equations of the type

$$\vec{D} = M(I)\vec{E} + N(I)\vec{B}, \quad \vec{H} = M(I)\vec{B} - N(I)\vec{E}, \quad I = \frac{\vec{E}^2 - \vec{B}^2}{\vec{B}\vec{E}}, \tag{7}$$

where M, N are smooth functions of I. System (1), (7) admits the conformal algebra $AC(1,3)$.

It is surprisingly but true that the symmetry of equations for electromagnetic fields in a moving medium was not investigated at all. The following statement gives information on a local symmetry of system (1), (2) which may be naturally called Maxwell's equations with the supplementary Minkowski conditions (or Maxwell-Minkowski equations).

Theorem 2. *System (1), (2) is invariant with respect to the conformal algebra AC(1,3) whose basis operators have the form*

$$
\begin{aligned}
&P_0 = \partial_0 = \frac{\partial}{\partial t}, \qquad P_a = -\partial_a, \qquad \partial_a = \frac{\partial}{\partial_{x_a}}, \quad a, b = \overline{1,3}\\
&J_{ab} = x_a\partial_b - x_b\partial_a + \tilde{S}_{ab} + V_{ab} + R_{ab},\\
&J_{0a} = x_0\partial_a + x_a\partial_0 + \tilde{S}_{0a} + V_{0a} + R_{0a},\\
&D = t\partial_t + x_k\partial_{x_k} - 2\left(E_k\partial_{E_k} + B_k\partial_{B_k} + D_k\partial_{D_k} + H_k\partial_{H_k}\right) - 3\left(j_k\partial_{j_k} + \rho\partial_\rho\right),\\
&K_\mu = 2x_\mu D - x^2 P_\mu + 2x^\nu(\tilde{S}_{\mu\nu} + V_{\mu\nu} + R_{\mu\nu}), \quad \mu, \nu = \overline{0,3},
\end{aligned}
\tag{8}
$$

where $\tilde{S}_{ab}, \tilde{S}_{0a}$ *are given by (6), and* $V_{ab}, V_{0a}, R_{ab}, R_{0a}$ *have the form*

$$
\begin{aligned}
&V_{ab} = u_a\partial_{u_b} - u_b\partial_{u_a}, \quad V_{0a} = \partial_{u_a} - u_a\left(u_b\partial_{u_b}\right),\\
&R_{ab} = j_a\partial_{j_b} - j_b\partial_{j_a}, \quad R_{0a} = j_a\partial_\rho + \rho\partial_{j_a}.
\end{aligned}
\tag{9}
$$

To prove the theorem, we use in principle the standard Lie scheme and therefore it is given without proof.

As follows from the theorem, vectors $\vec{D}, \vec{B}, \vec{E}, \vec{H}$ are transformed according to a standard linear representation of the Lorentz group, and the velocity of a moving medium and the density of charge are nonlinearity transformed

$$
u_a \to u'_a = \frac{u_a + \theta_a}{1 + \vec{u}\vec{\theta}}, \qquad \rho \to \rho' = \frac{\rho(1 - \vec{\theta}\vec{u})}{\sqrt{1 - \vec{\theta}^2}}.
$$

Components of the velocity vector $\vec{u}$ are transformed in the following way:

$$
u_k \to u'_k = \frac{u_k\sigma - 2b_0x_k - 2b_0^2x_k(x_0 - \vec{x}\vec{u})}{1 + 2b_0(x_0 - \vec{x}\vec{u}) + b_0^2(x_0^2 + \vec{x}^2 - 2x_0\vec{x}\vec{u})},
\tag{10}
$$

where $\sigma = 1 + 2b_0x_0 + b_0^2x^2$, $x^2 = x_0^2 - \vec{x}^2$, b_0 is a group parameter under the transformations generated by K_μ.

Operators K_a generate the following transformations for the velocity vector:

$$
u_a \to u'_a = \frac{u_a\sigma + 2(x_0 - \vec{x}\vec{u})(b_a - b_a^2x_a) - 2b_au_a(x_a + b_ax^2)}{\sigma + 2b_a^2x_0(x_0 - \vec{x}\vec{u}) - 2b_au_ax_0},
\tag{11}
$$

$$
u_c \to u'_c = \frac{u_c\sigma + 2(x_0 - \vec{x}\vec{u})b_a^2x_c - 2b_au_ax_c}{\sigma + 2b_a^2x_0(x_0 - \vec{x}\vec{u}) - 2b_au_ax_0}, \qquad c \neq a,
\tag{12}
$$

where $\sigma = 1 - 2b_ax_a - b_a^2x^2$, b_a are group parameters and there is no summation over a.

If the permittivity ε and permeance μ are functions of the ratio of invariants of electromagnetic field, i.e.,

$$\varepsilon = \varepsilon\left(\frac{\vec{B}^2 - \vec{E}^2}{\vec{B}\vec{E}}\right), \qquad \mu = \mu\left(\frac{\vec{B}^2 - \vec{E}^2}{\vec{B}\vec{E}}\right),$$

then system (1), (2) is invariant with respect to the conformal algebra AC(1,3).

Thus, the system of Maxwell's equations (1), (2) in a moving external medium is invariant with respect to the conformal group C(1,3). Here, the velocity is changed nonlinearly under the transformations generated by K_μ according to formulae (10), (11), (12).

In the all above-given equations, the fields $\vec{D}$, $\vec{B}$, $\vec{E}$, $\vec{H}$ are transformed in a linear way.

Here, we give one more system of nonlinear equations for which a nonlinear representation of the Poincaré algebra AP(1,3) is realized on the set of its solutions. The system has the form

$$\frac{\partial \Sigma_k}{\partial x_0} + \Sigma_l \frac{\partial \Sigma_k}{\partial x_l} = 0, \qquad k, l = 1, 2, 3, \tag{13}$$

where $\Sigma_k = E_k + iH_k$. The complex system (12) is equivalent to the real system of equations for $\vec{E}$ and $\vec{H}$:

$$\begin{aligned} &\frac{\partial E_k}{\partial x_0} + E_l \frac{\partial E_k}{\partial x_l} - H_l \frac{\partial H_k}{\partial x_l} = 0, \\ &\frac{\partial H_k}{\partial x_0} + H_l \frac{\partial E_k}{\partial x_l} + E_l \frac{\partial H_k}{\partial x_l} = 0. \end{aligned} \tag{14}$$

Having used the Lie algorithm [4], we have proved the theorem.

Theorem 3. *The system of equations (14) is invariant with respect to the 24-dimensional Lie algebra with basis operators*

$$\begin{aligned}
&P_\mu = \frac{\partial}{\partial_{x_\mu}} = \partial_\mu, \qquad \mu = \overline{0,3} \\
&J^{(1)}_{kl} = x_k\partial_l - x_l\partial_k + E_k\partial_{E_l} - E_l\partial_{E_k} + H_k\partial_{H_l} - H_l\partial_{H_k}, \\
&J^{(2)}_{kl} = x_k\partial_l + x_l\partial_k + E_k\partial_{E_l} + E_l\partial_{E_k} + H_k\partial_{H_l} + H_l\partial_{H_k}, \\
&G^{(1)}_a = x_0\partial_a + \partial_{E_a}, \\
&G^{(2)}_a = x_a\partial_0 - (E_aE_k - H_aH_k)\partial_{E_a} - (E_aH_k + H_aE_k)\partial_{H_k}, \\
&D_0 = x_0\partial_0 - E_k\partial_{E_k} - H_k\partial_{H_k}, \\
&D_a = x_a\partial_a + E_a\partial_{E_a} + H_a\partial_{H_a} \qquad \textit{(there is no summation by k)}, \\
&K_0 = x_0^2\partial_0 + x_0x_k\partial_k + (x_k - x_0E_k)\partial_{E_k} - x_0H_k\partial_{H_k}, \\
&K_a = x_0x_a\partial_0 + x_ax_k\partial_k + [x_kE_a - x_0(E_aE_k - H_aH_k)]\partial_{E_k} + \\
&\qquad\qquad [x_kH_a - x_0(H_aE_k + E_aH_k)]\partial_{H_k}.
\end{aligned} \tag{15}$$

The invariance algebra of (14) given by (15) contains the Poincaré algebra AP(1,3), the conformal algebra AC(1,3), and the Galilei algebra AG(1,3) as subalgebras.

The operators $J_{0k} = G_k^{(1)} + G_k^{(2)}$ generate the standard transformations for x:

$$\begin{aligned} &x_0 \to x_0' = x_0 \text{ ch } \theta_k + x_0 \text{ sh } \theta_k, \\ &x_k \to x_k' = x_k \text{ ch } \theta_k + x_0 \text{ sh } \theta_k, \\ &x_l \to x_l' = x_l, \qquad \text{if } l \neq k, \end{aligned} \tag{16}$$

and nonlinear transformations for $\vec{E}, \vec{H}$:

$$\begin{aligned} &E_k + iH_k \to E_k' + iH_k' = \frac{(E_k + iH_k) \text{ ch } \theta_k + \text{ sh } \theta_k}{(E_k + iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \\ &E_k - iH_k \to E_k' - iH_k' = \frac{(E_k - iH_k) \text{ ch } \theta_k + \text{ sh } \theta_k}{(E_k - iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \end{aligned} \tag{17}$$

$$\begin{aligned} &E_l + iH_l \to E_l' + iH_l' = \frac{E_l + iH_l}{(E_k + iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \quad l \neq k, \\ &E_l - iH_l \to E_l' - iH_l' = \frac{E_l - iH_l}{(E_k - iH_k) \text{ sh } \theta_k + \text{ ch } \theta_k}, \quad l \neq k. \end{aligned} \tag{18}$$

There is no summation over k in formulas (16), (17), (18).

Conformal invariance can be used to construct exact solutions of Maxwell's equations. In conclusion, we give the theorem determining the relationship betwen invariant and conditionally invariant solutions of differential equations.

Let consider a nonlinear partial differential equation

$$Lu = 0. \tag{19}$$

Suppose that (19) is Q-conditionally-invariant under the k-dimensional algebra AQ_k [4, 5, 6] with basis elements $\langle Q_1, Q_2, \ldots, Q_k \rangle$, where

$$Q_i = \xi_i^a \partial_{x_a} + \eta_i \partial_u,$$

and the ansatz corresponding to this algebra reduces equation (19) to an ordinary differential equation. A general solution of the reduced equation is called the general conditionally invariant solution of (19) with respect to AQ_k. Then the following theorem has been proved.

Theorem 4. *Let (19) is invariant (in the Lie sense) with respect to the m-dimensional Lie algebra AG_m and Q-conditionally invariant under the k-dimensional Lie algebra AQ_k. Suppose that a general conditionally invariant solution of (19) depends on t constants c_1, c_2, ..., c_t.*

If the system

$$\xi_i^a \frac{\partial u}{\partial x_a} = \eta_i(x, u), \quad i = \overline{1, t}, \tag{20}$$

is invariant with respect to a p-dimensional subalgebra of AG_m and $p \geq t + 1$, then the conditionally invariant solution of (19) with respect to AQ_k is an invariant solution of this equation in the classical Lie sense.

Thus, we obtain the sufficient condition for the solution found with the help of conditional symmetry operators to be an invariant solution in the classical sense. It is obvious that this theorem can be generalized and applicable to construction of exact solutions of partial differential equations by using the method of differential constraints [7], Lie-Bäcklund symmetry method [8], and the approach suggested in [9].

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 313–320.

Symmetry Reduction and Exact Solutions of the $SU(2)$ Yang-Mills Equations

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Abstract

We present a detailed account of symmetry properties of $SU(2)$ Yang-Mills equations. Using a subgroup structure of the conformal group $C(1,3)$, we have constructed $C(1,3)$-inequivalent ansatzes for the Yang-Mills field which are invariant under three-dimensional subgroups of the conformal group. With the aid of these ansatzes, reduction of Yang-Mills equations to systems of ordinary differential equations is carried out and wide families of their exact solutions are constructed.

Classical ideas and methods developed by Sophus Lie provide us with a powerful tool for constructing exact solutions of partial differential equations (see, e.g., [1–3]). In the present paper, we apply the above methods to obtain new explicit solutions of the $SU(2)$ Yang-Mills equations (YME). YME is the following nonlinear system of twelve second-order partial differential equations:

$$\begin{aligned}&\partial_\nu\partial^\nu \mathbf{A}_\mu - \partial^\mu\partial_\nu \mathbf{A}_\nu + e[(\partial_\nu \mathbf{A}_\nu)\times \mathbf{A}_\mu - \\ &\quad -2(\partial_\nu \mathbf{A}_\mu)\times \mathbf{A}_\nu + (\partial^\mu \mathbf{A}_\nu)\times \mathbf{A}^\nu] + e^2 \mathbf{A}_\nu \times (\mathbf{A}^\nu \times \mathbf{A}_\mu) = 0.\end{aligned} \tag{1}$$

Here $\partial_\nu = \dfrac{\partial}{\partial x_\nu}$, $\mu,\nu = 0,1,2,3$; $e = \mathrm{const}$, $\mathbf{A}_\mu = \mathbf{A}_\mu(x) = \mathbf{A}_\mu(x_0,x_1,x_2,x_3)$ are three-component vector-potentials of the Yang-Mills field. Hereafter, the summation over the repeated indices μ,ν from 0 to 3 is supposed. Raising and lowering the vector indices are performed with the aid of the metric tensor $g_{\mu\nu}$, i.e., $\partial^\mu = g_{\mu\nu}\partial_\nu$ ($g_{\mu\nu} = 1$ if $\mu = \nu = 0$, $g_{\mu\nu} = -1$ if $\mu = \nu = 1,2,3$ and $g_{\mu\nu} = 0$ if $\mu \neq \nu$).

It should be noted that there are several reviews devoted to classical solutions of YME in the Euclidean space R_4. They have been obtained with the help of *ad hoc* substitutions suggested by Wu and Yang, Rosen, 't Hooft, Carrigan and Fairlie, Wilczek, Witten (for more detail, see review [4] and references cited therein). However, symmetry properties of YME were not used explicity. It is known [5] that YME (1) are invariant under the group $C(1,3)\otimes SU(2)$, where $C(1,3)$ is the 15-parameter conformal group and $SU(2)$ is the infinite-parameter special unitary group. Symmetry properties of YME have been used for obtaining some new exact solutions of equations (1) by W. Fushchych and W. Shtelen in [6].

The present talk is based mainly on the investigations by the author together with W. Fushchych and R. Zhdanov [7–12].

1. Linear form of ansatzes

The symmetry group of YME (1) contains as a subgroup the conformal group $C(1,3)$ having the following generators:

$$\begin{aligned} &P_\mu = \partial_\mu, \\ &J_{\mu\nu} = x^\mu \partial_\nu - x^\nu \partial_\mu + A^{a\mu}\partial_{A^a_\nu} - A^{a\nu}\partial_{A^a_\mu}, \\ &D = x_\mu \partial_\mu - A^a_\mu \partial_{A^a_\mu}, \\ &K_\mu = 2x^\mu D - (x_\nu x^\nu)\partial_\mu + 2A^{a\mu}x_\nu \partial_{A^a_\nu} - 2A^a_\nu x^\nu \partial_{A^a_\mu}. \end{aligned} \tag{2}$$

Here $\partial_{A^a_\mu} = \dfrac{\partial}{\partial A^a_\mu}$, $a = 1,2,3$.

Using the fact that operators (2) realize a linear representation of the conformal algebra, we suggest a direct method for construction of the invariant ansätze enabling us to avoid a cumbersome procedure of finding a basis of functional invariants of subalgebras of the algebra $AC(1,3)$.

Let $L = \langle X_1, \ldots, X_s\rangle$ be a Lie algebra, where

$$X_a = \xi_{a\mu}(x)\partial_\mu + \rho_{amk}(x)u_k\partial_{u_m}. \tag{3}$$

Here $\xi_{a\mu}(x)$, $\rho_{amk}(x)$ are smooth functions in the Minkowski space $R_{1,3}$, $\mu = 0,1,2,3$, $m,k = 1,2,\ldots,n$. Let also $\mathrm{rank}L = 3$, i.e.,

$$\mathrm{rank}||\xi_{a\mu}(x)|| = \mathrm{rank}||\xi_{a\mu}(x), \rho_{amk}(x)|| = 3 \tag{4}$$

at an arbitrary point $x \in R_{1,3}$.

Lemma [3]. *Assume that conditions (3), (4) hold. Then, a set of functionally independent first integrals of the system of partial differential equations*

$$X_a F(x,u) = 0, \qquad u = (u_1, \ldots, u_n)$$

can be chosen as follows

$$\omega = \omega(x), \qquad \omega_i = h_{ik}(x)u_k, \quad i,k = 1,\ldots,n$$

and, in addition,

$$\det||h_{ik}(x)||^{n}_{i=1}{}^{n}_{k=1} \neq 0.$$

Consequently, we can represent L-invariant ansatzes in the form

$$u_i = h_{ik}(x)v_k(\omega)$$

or

$$\mathbf{u} = \Lambda(x)\mathbf{v}(\omega), \tag{5}$$

where

$$\mathbf{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

$\Lambda(x))$ is a nonsingular matrix in the space $R_{1,3}$.

Let

$$S_{01} = \begin{pmatrix} 0 & -I & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad S_{02} = \begin{pmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{03} = \begin{pmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \qquad S_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S_{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}, \qquad S_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \end{pmatrix},$$

where 0 is zero and I is the unit 3×3 matrices, and E is the unit 12×12 matrix. Now we can represent generators (2) in the form

$$\begin{aligned}
&P_\mu = \partial_{x_\mu}, \\
&J_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu} - (S_{\mu\nu} A \cdot \partial_{\mathbf{A}}), \\
&D = x_\mu \partial_{x_\mu} - k(E\ A \cdot \partial_{\mathbf{A}}), \\
&K_0 = 2x_0 D - (x_\nu x^\nu)\partial_{x_0} - 2x_a (S_{0a}\ \mathbf{A} \cdot \partial_{\mathbf{A}}), \\
&K_1 = -2x_1 D - (x_\nu x^\nu)\partial_{x_1} + 2x_0(S_{01} A \cdot \partial_{\mathbf{A}}) - 2x_2(S_{12} A \cdot \partial_{\mathbf{A}}) - 2x_3(S_{13} A \cdot \partial_{\mathbf{A}}), \\
&K_2 = -2x_2 D - (x_\nu x^\nu)\partial_{x_2} + 2x_0(S_{02} A \cdot \partial_{\mathbf{A}}) + 2x_1(S_{12} A \cdot \partial_{\mathbf{A}}) - 2x_3(S_{23} A \cdot \partial_{\mathbf{A}}), \\
&K_3 = -2x_3 D - (x_\nu x^\nu)\partial_{x_3} + 2x_0(S_{03} A \cdot \partial_{\mathbf{A}}) + 2x_1(S_{13} A \cdot \partial_{\mathbf{A}}) + 2x_2(S_{23} A \cdot \partial_{\mathbf{A}}).
\end{aligned} \tag{6}$$

Here, the symbol $(* \cdot *)$ denotes a scalar product,

$$A = \begin{pmatrix} A_0^1 \\ A_0^2 \\ A_0^3 \\ \vdots \\ A_3^2 \\ A_3^3 \end{pmatrix}, \qquad \partial_A = (\partial_{A_0^1}, \partial_{A_0^2}, \ldots, \partial_{A_3^3}).$$

Let L be a subalgebra of the conformal algebra $AC(1,3)$ with the basis elements (2) and $\operatorname{rank} L = 3$. According to Lemma, it has twelve invariants

$$f_{ma}(x) A_a, \quad a, m = 1, \ldots, 12,$$

which are functionally independent. They can be considered as components of the vector

$$F \cdot A,$$

where $F = ||f_{mn}(x)||$, $m, n = 1, \ldots, 12$. Furthemore, we suppose that the matrix F is non-singular in some domain of $R_{1,3}$. Providing the rank $L = 3$, there is one additional invariant

ω independent of components of A. According to [1], the ansatz $FA = B(\omega)$ reduces system (1) to a system of ordinary differential equations which contains the independent variable ω, dependent variables $B_0^1, B_0^2, \ldots, B_3^3$, and their first and second derivatives. This ansatz can be written in the form (5):

$$A = Q(x)B(x), \qquad Q(x) = F^{-1}(x), \tag{7}$$

where a function ω and a matrix F satisfy the equations

$$X_a\omega = 0, \qquad a = 1, 2, 3,$$
$$X_aF = 0, \qquad a = 1, 2, 3,$$

or

$$\begin{aligned} &\xi_{a\mu}(x)\frac{\partial\,\omega}{\partial\,x_\mu} = 0, \\ &\xi_{a\mu}(x)\frac{\partial\,F}{\partial\,x_\mu} + F\Gamma_a(x) = 0, \qquad a = 1, 2, 3, \quad \mu = 0, 1, 2, 3, \end{aligned} \tag{8}$$

where $\Gamma_a(x)$ are certain 12×12 matrices.

It is not difficult to make that matrices Γ_a have the form (6):

$$\begin{aligned} P_\mu \quad &: \quad \Gamma_\mu = 0; \\ J_{\mu\nu} \quad &: \quad \Gamma_{\mu\nu} = -S_{\mu\nu}; \\ D \quad &: \quad \Gamma = -E; \\ K_0 \quad &: \quad \tilde{\Gamma}_0 = -2x_0E - 2x_aS_{0a} \quad (a = 1, 2, 3); \\ K_1 \quad &: \quad \tilde{\Gamma}_1 = 2x_1E + 2x_0S_{01} - 2x_2S_{12} - 2x_3S_{13}; \\ K_2 \quad &: \quad \tilde{\Gamma}_2 = 2x_2E + 2x_0S_{02} + 2x_1S_{12} - 2x_3S_{23}; \\ K_3 \quad &: \quad \tilde{\Gamma}_3 = 2x_3E + 2x_0S_{03} + 2x_1S_{13} + 2x_2S_{23}. \end{aligned}$$

It is natural to look for a matrix F in the form

$$\begin{aligned} F(x) = {} &\exp\{(-\ln\theta)E\}\exp(\theta_0S_{03})\exp(-\theta_3S_{12})\exp(-2\theta_1H_1)\times \\ &\exp(-2\theta_2H_2)\exp(-2\theta_4\tilde{H}_1)\exp(-2\theta_5\tilde{H}_2), \end{aligned} \tag{9}$$

where $\theta = \theta(x)$, $\theta_0 = \theta_0(x)$, $\theta_m = \theta_m(x)$ $(m = 1, 2, \ldots, 5)$ are arbitrary smooth functions, $H_a = S_{0a} - S_{a3}$, $\tilde{H}_a = S_{0a} + S_{a3}$ $(a = 1, 2)$.

Generators X_a $(a = 1, 2, 3)$ of a subalgera L can be writen in the next general form:

$$X_a = \xi_{a\mu}(x)\partial_{x_\mu} + Q_aA\partial_A,$$

where

$$Q_a = f_aE + f_{0a}S_{03} + f_{1a}H_1 + f_{2a}H_2 + f_{3a}S_{12} + f_{4a}\tilde{H}_1 + f_{5a}\tilde{H}_2,$$

and $F_a = f_a(x)$, $f_{0a} = f_{0a}(x)$, $f_{ma} = f_{ma}(x)$ $(m = 1, \ldots, 5)$ are certain functions. Consequently, the determining system for the matrix F (9) reduces to the system for finding functions θ, θ_0, θ_m $(m = 1, \ldots, 5)$:

$$\begin{aligned}
&\xi_{a\mu}\frac{\partial\theta}{\partial x_\mu} = f_a\theta,\\
&\xi_{a\mu}\frac{\partial\theta_0}{\partial x_\mu} = 4(\theta_4 f_{1a} + \theta_5 f_{2a}) - f_{0a},\\
&\xi_{a\mu}\frac{\partial\theta_3}{\partial x_\mu} = 4(\theta_4 f_{2a} - \theta_5 f_{1a}) + f_{3a},\\
&\xi_{a\mu}\frac{\partial\theta_1}{\partial x_\mu} = 4(\theta_1\theta_4 + \theta_2\theta_5)f_{1a} + 4(\theta_1\theta_5 - \theta_2\theta_4)f_{2a} - \theta_1 f_{0a} - \theta_2 f_{3a} + \frac{1}{2}f_{1a},\\
&\xi_{a\mu}\frac{\partial\theta_2}{\partial x_\mu} = 4(\theta_2\theta_4 - \theta_1\theta_5)f_{1a} + 4(\theta_2\theta_5 + \theta_1\theta_4)f_{2a} - \theta_2 f_{0a} + \theta_1 f_{3a} + \frac{1}{2}f_{2a},\\
&\xi_{a\mu}\frac{\partial\theta_4}{\partial x_\mu} = \theta_4 f_{0a} - 2(\theta_4^2 - \theta_5^2)f_{1a} - 4\theta_4\theta_5 f_{2a} - \theta_5 f_{3a} + \frac{1}{2}f_{4a},\\
&\xi_{a\mu}\frac{\partial\theta_5}{\partial x_\mu} = \theta_5 f_{0a} - 4\theta_4\theta_5 f_{1a} + 2(\theta_4^2 - \theta_5^2)f_{2a} + \theta_4 f_{3a} + \frac{1}{2}f_{5a}.
\end{aligned} \tag{10}$$

Here, $\mu = 0, 1, 2, 3$, $a = 1, 2, 3$.

2. Reduction and exact solutions of YME

Substituting (7), (9) into YME we get a system of ordinary differential equations. However, owing to an asymmetric form of the ansatzes, we have to repeat this procedure 22 times (if we consider Poincaré-invariant ansatzes). For the sake of unification of the reduction procedure, we use the solution generation routine by transformations from the Lorentz group (see, for example, [8]). Then ansatz (7), (8) is represented in a unified way for all the subalgebras. In particular, $P(1,3)$-invariant ansatzes have the following form:

$$\mathbf{A}_\mu(x) = a_{\mu\nu}(x)\mathbf{B}^\nu(\omega), \tag{11}$$

where

$$\begin{aligned}
a_{\mu\nu}(x) = {}&(a_\mu a_\nu - d_\mu d_\nu)\cosh\theta_0 + (d_\mu a_\nu - d_\nu a_\mu)\sinh\theta_0 +\\
&+2(a_\mu + d_\mu)[(\theta_1\cos\theta_3 + \theta_2\sin\theta_3)b_\nu + (\theta_2\cos\theta_3 - \theta_1\sin\theta_3)c_\nu +\\
&+(\theta_1^2 + \theta_2^2)e^{-\theta_0}(a_\nu + d_\nu)] + (b_\mu c_\nu - b_\nu c_\mu)\sin\theta_3 -\\
&-(c_\mu c_\nu + b_\mu b_\nu)\cos\theta_3 - 2e^{-\theta_0}(\theta_1 b_\mu + \theta_2 c_\mu)(a_\nu + d_\nu).
\end{aligned} \tag{12}$$

Here, $\mu, \nu = 0, 1, 2, 3$; $x = (x_0, \mathbf{x})$, $a_\mu, b_\mu, c_\mu, d_\mu$ are arbitrary parameters satisfying the equalities

$$\begin{aligned}
&a_\mu a^\mu = -b_\mu b^\mu = -c_\mu c^\mu = -d_\mu d^\mu = 1,\\
&a_\mu b^\mu = a_\mu c^\mu = a_\mu d^\mu = b_\mu c^\mu = b_\mu d^\mu = c_\mu d^\mu = 0.
\end{aligned}$$

Theorem. *Ansatzes (11), (12) reduce YME (1) to the system*

$$k_{\mu\gamma}\ddot{\mathbf{B}}^\gamma + l_{\mu\gamma}\dot{\mathbf{B}}^\gamma + m_{\mu\gamma}\mathbf{B}^\gamma + eg_{\mu\nu\gamma}\dot{\mathbf{B}}^\nu\times\mathbf{B}^\gamma + eh_{\mu\nu\gamma}\mathbf{B}^\nu\times\mathbf{B}^\gamma + e^2\mathbf{B}_\gamma\times(\mathbf{B}^\gamma\times\mathbf{B}_\mu) = 0. \tag{13}$$

Coefficients of the reduced equations are given by the following formulae:

$$\begin{aligned} & k_{\mu\gamma} = g_{\mu\gamma}F_1 - G_\mu G_\gamma, \quad l_{\mu\gamma} = g_{\mu\gamma}F_2 + 2S_{\mu\gamma} - G_\mu H_\gamma - G_\mu \dot{G}_\gamma, \\ & m_{\mu\gamma} = R_{\mu\gamma} - G_\mu \dot{H}_\gamma, \quad g_{\mu\nu\gamma} = g_{\mu\gamma}G_\nu + g_{\nu\gamma}G_\mu - 2g_{\mu\nu}G_\gamma, \\ & h_{\mu\nu\gamma} = \frac{1}{2}(g_{\mu\gamma}H_\nu - g_{\mu\nu}H_\gamma) - T_{\mu\nu\gamma}, \end{aligned} \tag{14}$$

where F_1, F_2, G_μ, H_μ, $S_{\mu\nu}$, $R_{\mu\nu}$, $T_{\mu\nu\gamma}$ *are functions of* ω *determined by the relations*

$$F_1 = \frac{\partial\omega}{\partial x_\mu}\frac{\partial\omega}{\partial x^\mu}, \qquad F_2 = \Box\omega, \qquad G_\mu = a_{\gamma\mu}\frac{\partial\omega}{\partial x_\gamma},$$

$$H_\mu = \frac{\partial a_{\gamma\mu}}{\partial x_\gamma}, \qquad S_{\mu\nu} = a_\mu^\gamma \frac{\partial a_{\gamma\nu}}{\partial x_\delta}\frac{\partial\omega}{\partial x^\delta}, \qquad R_{\mu\nu} = a_\mu^\gamma \Box a_{\gamma\nu},$$

$$T_{\mu\nu\gamma} = a_\mu^\delta \frac{\partial a_{\delta\nu}}{\partial x_\sigma}a_{\sigma\gamma} + a_\nu^\delta \frac{\partial a_{\delta\gamma}}{\partial x_\sigma}a_{\sigma\mu} + a_\gamma^\delta \frac{\partial a_{\delta\mu}}{\partial x_\sigma}a_{\sigma\nu}.$$

A subalgebraic structure of subalgebras of the conformal algebra $AC(1,3)$ is well known (see, for example, [13]). Here we restrict our considerations to the case of the subalgebra $\langle G_a = J_{0a} - J_{03},\ J_{03}, a = 1,2\rangle$ of the algebra $AP(1,3)$. In this case, $\theta = 1$, $\theta_4 = \theta_5 = 0$ and functions f_a, f_{0a}, f_{ma} $(a, m = 1,2,3)$ have following values:

$$\begin{aligned} G_1 \quad &: \quad f_1 = f_{01} = f_{21} = f_{31} = 0, \quad f_{11} = -1; \\ G_2 \quad &: \quad f_2 = f_{02} = f_{12} = f_{32} = 0, \quad f_{22} = -1; \\ J_{03} \quad &: \quad f_3 = f_{13} = f_{23} = f_{33} = 0, \quad f_{03} = -1; \end{aligned}$$

Consequently, system (10) has the form:

$$\begin{aligned} & G_1^{(1)}\theta_0 = G_1^{(1)}\theta_2 = G_1^{(1)}\theta_3 = 0, \qquad G_1^{(1)}\theta_1 = -\frac{1}{2}; \\ & G_2^{(1)}\theta_0 = G_2^{(1)}\theta_1 = G_2^{(1)}\theta_3 = 0, \qquad G_2^{(1)}\theta_2 = -\frac{1}{2}; \\ & J_{03}^{(1)}\theta_0 = 1, \qquad J_{01}^{(1)}\theta_3 = 0, \qquad J_{03}^{(1)}\theta_a = \theta_a \quad (a = 1,2). \end{aligned}$$

Here,

$$\begin{aligned} & G_a^{(1)} = (x_0 - x_3)\partial_{x_a} + x_a(\partial_{x_0} + \partial_{x_3}) \qquad (a = 1,2), \\ & J_{03}^{(1)} = x_0\partial_{x_3} + x_3\partial_{x_0}. \end{aligned}$$

In particular, the system for the function θ_0 reads

$$(x_0 - x_3)\frac{\partial\theta_0}{\partial x_a} + x_a\left(\frac{\partial\theta_0}{\partial x_0} + \frac{\partial\theta_0}{\partial x_3}\right) = 0 \quad (a = 1,2), \qquad x_0\frac{\partial\theta_0}{\partial x_3} + x_3\frac{\partial\theta_0}{\partial x_0} = 1,$$

and the function $\theta_0 = -\ln|x_0 - x_3|$ is its particular solution. In a similar way, we find that $\theta_1 = -\frac{1}{2}x_1(x_0 - x_3)^{-1}$, $\theta_2 = -\frac{1}{2}x_2(x_0 - x_3)^{-1}$, $\theta_3 = 0$. The function w is a solution of the system

$$G_a^{(1)}w = J_{03}^{(1)}w = 0 \qquad (a = 1,2),$$

and is equal to $x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Finally, we arrive at ansatz (11), (12), where

$$\theta_0 = -\ln|kx|, \quad \theta_1 = \frac{1}{2}bx(kx)^{-1}, \quad \theta_2 = \frac{1}{2}cx(kx)^{-1},$$
$$\theta_3 = 0, \quad w = (ax)^2 - (bx)^2 - (cx)^2 - (dx)^2.$$

Here, $ax = a_\mu x^\mu$, $bx = b_\mu x^\mu$, $cx = x_\mu x^\mu$, $dx = d_\mu x^\mu$, $kx = k_\mu x^\mu$, $k_\mu = a_\mu + d_\mu$, $\mu = 0, 1, 2, 3$. According to the theorem, the reduced system (13) has the following coefficients (14):

$$\begin{aligned} k_{\mu\gamma} &= 4wg_{\mu\gamma} - (a_\mu - d_\mu + k_\mu w)(a_\gamma - d_\gamma + k_\gamma w), \\ l_{\mu\gamma} &= 4[2g_{\mu\gamma} - a_\mu a_\gamma + d_\mu d_\gamma - wk_\mu k_\gamma], \\ m_{\mu\gamma} &= -2k_\mu k_\gamma, \\ g_{\mu\nu\gamma} &= \epsilon(g_{\mu\gamma}(a_\nu - d_\nu + k_\nu w) + g_{\nu\gamma}(a_\mu - d_\mu + k_\mu w) - 2g_{\mu\nu}(a_\gamma - d_\gamma + k_\gamma w), \\ h_{\mu\nu\gamma} &= \frac{3}{2}\epsilon(g_{\mu\gamma}k_\nu - g_{\mu\nu}k_\gamma), \end{aligned} \tag{15}$$

where $\epsilon = 1$ for $kx > 0$ and $\epsilon = -1$ for $kx < 0$, $\mu, \nu, \gamma = 0, 1, 2, 3$. We did not succeed in finding general solutions of system (13), (15). Nevertheless, we obtain a particular solution of these equations. The idea of our approach to integration of this system is rather simple and quite natural. It is a reduction of this system by the number of components with the aid of an *ad hoc* substitution. Let

$$\mathbf{B}_\mu = b_\mu \mathbf{e}_1 f(w) + c_\mu \mathbf{e}_2 g(w), \tag{16}$$

where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2(0, 1, 0)$, f and g are arbitrary smooth functions. Then the corresponding equations have the form

$$4w\ddot{f} + 8\dot{f} + e^2 g^2 f = 0, \qquad 4w\ddot{g} + 8\dot{g} + e^2 f^2 g = 0. \tag{17}$$

System (17) with the substitution $f = g = u(w)$ reduces to

$$w\ddot{u} + 2\dot{u} + \frac{e^2}{4}u^3 = 0. \tag{18}$$

The ordinary differential equation (18) is the Emden-Fowler equation and the function $u = e^{-1}w^{-\frac{1}{2}}$ is its particular solution.

Substituting the result obtained into formula (16) and then into ansatz (11), (12) we get a non-Abelian exact solution of YME (1):

$$\mathbf{A}_\mu = \{\mathbf{e}_1(b_\mu - k_\mu bx(kx)^{-1}) + \mathbf{e}_2(c_\mu - k_\mu cx(kx)^{-1})\}\times$$
$$\times e^{-1}\{(ax)^2 - (bx)^2 - (cx)^2 - (dx)^2)\}^{-\frac{1}{2}}.$$

Analogously, we consider the rest of subalgebras of the conformal algebra.

For example, for the subalgebra $\langle J_{12}, P_0, P_3\rangle$, we get the following non-Abelian solutions of YME (1):

$$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu Z_0\left[\frac{i}{2}e\lambda((bx)^2 + (cx)^2)\right] + \mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda,$$
$$\mathbf{A}_\mu = \mathbf{e}_1 k_\mu\left[\lambda_1((bx)^2 + (cx)^2)^{\frac{e\lambda}{2}} + \lambda_2((bx)^2 + (cx)^2)^{-\frac{e\lambda}{2}}\right]+$$
$$+\mathbf{e}_2(b_\mu cx - c_\mu bx)\lambda((bx)^2 + (cx)^2)^{-1}.$$

Here $Z_0(w)$ is the Bessel function, $\lambda_1, \lambda_2, \lambda_2 = \text{const}$.

Acknowledgements. The author gratefully acknowledges fruitful collaboration and stimulating discussions with Renat Zhdanov.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 321–327.

Higher Symmetries of the Wave Equation with Scalar and Vector Potentials

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Abstract

Higher order symmetry operators for the wave equation with scalar and vector potentials are investigated. All scalar potentials which admit second order symmetry operators are found explicitly.

Symmetries of partial differential equations are used for separation of variables [1], description of conservation laws [2], construction of exact solutions [3], etc. Before can be applied, symmetries have to be found, therefore search for symmetries attracts attention of many investigators.

We say a linear differential operator of order n is a symmetry operator (or simply a symmetry) of a partial differential equation if it transforms any solution of this equation into a solution. If $n = 1$, then the symmetry is nothing but a generator of a Lie group being a symmetry group of the equation considered. If $n > 1$, the related symmetries are referred as non-Lie or higher symmetries.

The symmetry aspects of the Schrödinger equation have been investigated by many authors (see [1] and references cited in), the higher symmetries of this equation where investigated in [4–9]. In contrast, the higher symmetries of the relativistic wave equation have not been studied well yet.

In this paper, we investigate the higher symmetries of the wave equation with an arbitrary scalar potential

$$L\psi \equiv (\partial_\mu \partial^\mu - V)\psi = 0 \qquad (1)$$

where $\mu = 0, 1, \ldots, m$. We deduce equations describing both potentials and coefficients of the corresponding symmetry operators of order n and present the complete list of potentials admitting symmetries for the case $m = 1$, $n = 2$, which completes the results of papers [10, 11]. In addition, we find all the possible scalar potentials V and vector potentials A_μ such that the equation

$$\widehat{L}\psi \equiv (D_\mu D^\mu - V)\psi = 0, \qquad D_\mu = \partial_\mu - eA_\mu, \quad \mu = 0, 1 \qquad (2)$$

admits any Lie symmetry. Moreover, we consider the case of time-dependent potential V and present a constructive test in order to answer the question if the corresponding wave equation admits any Lie symmetry.

Let us represent a differential operator of arbitrary order n in the form

$$Q_n = \sum_{j=0}^{n} \Big[[\cdots [K^{a_1 a_1 \dots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \cdots \partial_{a_j} \Big]_+, \tag{3}$$

where $K^{a_1 a_1 \dots a_j}$ are arbitrary functions of $x = (x_0, x_1, \dots, x_m)$, $[A, B]_+ = AB + BA$. We say Q_n is a symmetry operator of equation (1) if it satisfies the invariance condition [13]

$$[Q_n, L] = \alpha_{n-1} L, \tag{4}$$

where α_{n-1} is a differential operator of order $n-1$ which we represent in the form

$$\alpha_n = \sum_{j=0}^{n-1} \Big[[\cdots [\alpha^{a_1 a_2 \dots a_j}, \partial_{a_1}]_+, \partial_{a_2}]_+, \cdots \partial_{a_j} \Big]_+. \tag{5}$$

To find the determining equations for coefficients of a symmetry operator of arbitrary order, it is sufficient to equate coefficients of the same differentials in (4). We start with the case $m = 1$ in order to verify the old results [10]. For the first order symmetries

$$Q = [K^a, \partial^a]_+ + K \tag{6}$$

we obtain the following determining equations

$$\partial^{(a} K^{b)} = \frac{1}{2} g^{ab} \alpha, \tag{7}$$

$$\partial^a K = -\frac{1}{2} \partial^a \alpha, \tag{8}$$

$$2K^a \partial^a V = -\alpha V - \frac{1}{4} (\partial^b \partial_b \alpha). \tag{9}$$

Formula (7) defines the equation for a conformal Killing vector whose general form is

$$K^0 = \varphi(x - t) + f(x + t) + c, \qquad K^1 = \varphi(x - t) - f(x + t), \tag{10}$$

where φ and f are arbitrary functions, c is an arbitrary constant. Moreover, in accordance with (7), (8), (10), $\alpha = 4(\varphi' - f')$, $K = \alpha/2 + C$, and the remaining condition (9) takes the form

$$\partial^0 (K^0 V) + \partial^1 (K^1 V) = 0. \tag{11}$$

Using the fact that K^1 satisfies the wave equation and changing $V = 1/U^2$, we come to the following differential consequence of (11):

$$U'' = \omega^2 U + C, \qquad \omega = \text{const}, \qquad C = 0, \tag{12}$$

which is clearly simply the compatibility condition for system (6)–(9).

Nonequivalent solutions of (12) have the form

$$
\begin{gathered}
V = C, \qquad V = \frac{C}{x^2}, \qquad V = C\exp(-2\omega x), \\
V = \frac{C}{\cos^2(\omega x)}, \qquad V = \frac{C}{\cosh^2(\omega x)}, \qquad V = \frac{C}{\sinh^2(\omega x)},
\end{gathered} \tag{13}
$$

where C and ω are arbitrary real constants. These potentials are defined up to equivalence transformations, $x \to ax + b$, where a, b are constants.

In comparison with the list of symmetry adopting potentials present in [10], formula (13) includes one additional potential (the last one) which was not found in [10].

We see the list of potentials admitting Lie symmetries is very restricted and is exhausted by the list of potentials enumerated in (13). The corresponding symmetry operators are easily calculated using relations (6)–(11).

We notice that the determining equations (6)-(9) are valid in the case of time-dependent potentials also. Moreover, we again come to equation (11) where K^0 and K^1 are functions defined in (10). These functions can be excluded by a consequent differentiation of (11). As a result, we obtain the following relation for V:

$$
\begin{gathered}
\Box\left\{\ln[\ln(\Box \ln V)_\eta - (\ln V)_\eta]\right\} = \Box\left\{\ln[\ln(\Box \ln V)_\zeta - (\ln V)_\zeta]\right\}, \\
\Box = \partial_t^2 - \partial_x^2, \qquad (\cdot)_\eta = \partial_\eta(\cdot), \qquad \eta = x + t, \qquad \zeta = x - t,
\end{gathered} \tag{14}
$$

which is a necessary and sufficient condition for the corresponding equation (1) to admit a Lie symmetry. It is the case, e.g., if the potential V satisfies the wave equation or has the form $V = f(\eta)\varphi(\zeta)$.

For the classification of the potentials admitting invariance algebras of dimension $k > 1$, refer to [12].

The second order symmetries are searched in the form (see (3))

$$
Q = [[K^{ab}, \partial^a,]_+, \partial^b]_+ + [K^a, \partial^a]_+ + K, \qquad \widehat{\alpha} = [\alpha^a, \partial^a]_+ + \alpha \tag{15}
$$

which leads to the following determining equations:

$$
\begin{gathered}
\partial^{(a} K^{bc)} = -\frac{1}{4}\alpha^{(a} g^{bc)}, \qquad \partial^{(a} K^{b)} = \frac{1}{2}(\partial^{(a}\alpha^{b)} - g^{ab}\alpha), \\
\partial^a K = \frac{1}{2}\partial^a\alpha + \alpha^a V - 4K^{ab}\partial^b V, \qquad K^a\partial^a V = \frac{1}{2}(\alpha^a\partial^a V + \alpha V).
\end{gathered} \tag{16}
$$

These equations can be solved by analogy with (6)–(12). Omitting straightforward but cumbersome calculations, we notice that the compatibility condition for system (15) again has the form (12) where C is an arbitrary constant. Moreover, $V = U/2(U')$. The corresponding general solution for the potential is given by formulas (13) and (17):

$$
\begin{gathered}
V = Cx, \qquad V = C_1 + \frac{C_2}{x}, \qquad V = C\exp(\omega x)x + C\exp(2\omega x), \\
V = C_1\frac{\cos(\omega x) + C_2}{\sin^2(\omega x)}, \qquad V = C_1\frac{\sinh(\omega x) + C_2}{\cosh^2(\omega x)}, \qquad V = C_1\frac{\cosh(\omega x) + C_2}{\sinh^2(\omega x)}.
\end{gathered} \tag{17}
$$

The list of solutions (17) includes all the potentials found in [11] and one additional potential given by the last term. We notice that potentials (13), (17) make it possible to separate variables in equation (1) [14]. In other words, the potentials admitting a separation of variables are exactly the same as the potentials admitting second order symmetries.

Let us consider equation (2). Without loss of generality, we set $A_1 = 0$, bearing in mind ambiguities arising due to gauge transformations. Then the corresponding determining equations for the first-order symmetries reduce to the form

$$\begin{aligned} &\partial^{(a} K^{b)} = -\frac{1}{2} g^{ab}\alpha, \quad \partial^0 K - 2iA_1\dot{K}^0 + 2iA_1' K^1 = \frac{1}{2}\dot{\alpha} + i\alpha A_1, \\ &K' + 2i\dot{K}^1\ A_1 = \frac{1}{2}\alpha', \quad 2iA_0\dot{K} - 2VF^1 = -\alpha(V + A_1^2) + i\dot{\alpha}(V + A_1^2) + \frac{1}{4}\Box\alpha. \end{aligned} \tag{18}$$

These equations are valid for time-dependent as well as time-independent potentials. Restricting ourselves to the last case, we find the compatibility condition for this system can be represented in the form (12), moreover,

$$k_1 A_1' = k_2 V = \frac{1}{U^2} \tag{19}$$

where k_1, k_2 are arbitrary constants. Using (13), we find the admissible vector potentials in the form

$$A_1 = \widetilde{C}x, \tag{20a}$$

$$A_1 = \frac{\widetilde{C}}{x}, \tag{20b}$$

$$A_1 = \widetilde{C}\exp(-2\omega x), \tag{20c}$$

$$A_1 = \widetilde{C}\tan(\omega x), \tag{20d}$$

$$A_1 = \widetilde{C}\tanh(\omega x), \tag{20e}$$

$$A_1 = \widetilde{C}\coth(\omega x). \tag{20f}$$

The list (20) exhausts all time independent vector potentials which admit Lie symmetries. Let us present explicitly the corresponding symmetry operators:

$$Q_1 = \partial_t, \qquad Q_2 = \partial_x - i\widetilde{C}t, \qquad Q_3 = t\partial_x - x\partial_t - i\widetilde{C}(x^2 + t^2); \tag{21a}$$

$$Q = \partial_t, \qquad Q_2 = t\partial_t + x\partial_x, \qquad Q_3 = (t^2 + x^2)\partial_t + 2tx\partial_x - 2i\widetilde{C}x; \tag{21b}$$

$$\begin{aligned} &Q_1 = \partial_t, \qquad Q_2 = \exp[\omega(x+t)](\partial_t + \partial_x) + i\widetilde{C}\exp[\omega(x-t)], \\ &Q_3 = \exp[\omega(x-t)](\partial_t - \partial_x)] + i\widetilde{C}\exp[-\omega(x+t)]; \end{aligned} \tag{21c}$$

$$\begin{aligned} &Q_1 = \partial_t, \\ &Q_2 = \sin(\omega t)\sin(\omega t)\partial_t - \cos(\omega t)\cos(\omega x)\partial_x + \widetilde{C}\sin(\omega t)\cos(\omega x); \\ &Q_3 = \cos(\omega t)\sin(\omega x)\partial_t + \sin(\omega t)\cos(\omega x)\partial_x - \widetilde{C}\cos(\omega t)\cos(\omega x); \end{aligned} \tag{21d}$$

$$\begin{aligned} &Q_1 = \partial_t, \\ &Q_2 = \sinh(\omega t)\sinh(\omega x)\partial_t + \cosh(\omega t)\cosh(\omega x)\partial_x - i\widetilde{C}\sinh(\omega t)\cosh(\omega x), \\ &Q_3 = \cosh(\omega t)\sinh(\omega x)\partial_t + \cosh(\omega t)\sinh(\omega t)\partial_x - i\widetilde{C}\cosh(\omega t)\cosh(\omega x); \end{aligned} \tag{21e}$$

$$
\begin{aligned}
&Q_1 = \partial_t,\\
&Q_2 = \cosh(\omega t)\cosh(\omega x)\partial_t + \sinh(\omega t)\sinh(\omega x)\partial_x - i\widetilde{C}\cosh(\omega t)\sinh(\omega x),\\
&Q_3 = \sinh(\omega t)\cosh(\omega x)\partial_t + \cosh(\omega t)\sinh(\omega x)\partial_x - i\widetilde{C}\sinh(\omega t)\sinh(\omega x).
\end{aligned}
\tag{21f}
$$

Operators (21a) and (21b) form the Lie algebra isomorphic to $AO(1,2)$ while the algebras of operators (21c)–(21f) are isomorphic to $AE(2)$. The corresponding potentials V have to satisfy relation (20). We notice that formulae (21) define symmetry operators for equation (1) with potentials (14) also if we set $\ddot{C} = 0$. These results are in accordance with the general classification scheme present in [12].

In conclusion, we return to equation (1) and present the determining equations for the symmetry operators (3) of arbitrary order n in a space of any dimension $m+1$:

$$
\begin{aligned}
&\partial^{(a_{n+1}} K^{a_1 a_2 \ldots a_n)} = \frac{1}{4} g^{(a_n a_{n+1}} \alpha^{a_1 a_2 \ldots a_{n-1})},\\
&\partial^{(a_n} K^{a_1 a_2 \ldots a_{n-1})} = \frac{1}{4} g^{(a_n a_{n-1}} \alpha^{a_1 a_2 \ldots a_{n-2})} - \frac{1}{2}\partial^{(a_n} \alpha^{a_1 a_2 \ldots a_{n-1})},\\
&\partial^{(a_{m-n+1}} K^{a_1 a_2 \ldots a_{n-m})} = -\frac{1}{4}\partial^b \partial_b \alpha^{a_1 a_2 \ldots n-m+1} +\\
&\quad + \sum_{k=0}^{\left[\frac{n-m}{2}\right]} (-1)^k \frac{2(n-m+2+2k)!}{(2k+1)!(n-m+1)!} U^{a_1 a_2 \ldots a_{n-m+1}} + W^{a_1 a_2 \ldots a_{n-m+1}},\\
&\sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-1)^{p+1} K^{b_1 b_2 \ldots b_{2p+1}} \partial_1^b \partial_2^b \ldots \partial_{2p+1}^b V + \sum_{k=0}^{n-1} \alpha^{b_1 b_2 \ldots b_k} \partial_1^b \partial_2^b \ldots \partial_k^b V = 0.
\end{aligned}
\tag{22}
$$

Here,

$$
\begin{aligned}
&U^{a_1 a_2 \ldots a_{n-m+1}} = K^{a_1 a_2 \ldots a_{n-m+1} b_1 b_2 \ldots b_{2k+1}} \partial^{b_1} \partial^{b_2} \ldots \partial^{b_{2k+1}} V,\\
&W^{a_1 a_2 \ldots a_{n-1}} = \alpha^{a_1 a_2 \ldots a_{n-1}} V,\\
&W^{a_1 a_2 \ldots a_{n-2}} = -(n-1)\alpha^{a_1 a_2 \ldots a_{n-2} b} \partial^b V - \alpha^{a_1 a_2 \ldots a_{n-2}} V,\\
&W^{a_1 a_2 \ldots a_{n-2q-1}} = -\sum_{k=0}^{q-1} (-1)^k \frac{(n-2k-2q)!}{(2k+1)!(n-2q-1)!} \times\\
&\quad \times \alpha^{a_1 a_2 \ldots a_{n-2q-1} b_1 b_2 \ldots b_{2k+1}} \partial^{b_1} \partial^{b_2} \ldots \partial^{b_{2k+1}} V - \alpha^{a_1 a_2 \ldots a_{n-2p-1}} V -\\
&\quad - \frac{1}{2} \sum_{k=0}^{q-1} (-1)^{k+q} \frac{(n-2k-1)!}{(n-2p-1)!(2p-2k-1)!(p-k)} \times\\
&\quad \times \alpha^{a_1 a_a \ldots a_{n-2k-1} b_1 b_2 \ldots b_{2p-2k}} \partial^{b_1} \partial^{b_2} \ldots \partial^{b_{2p-2k}} V,\\
&q = 1, 2, \ldots \left[\frac{n-2}{2}\right],\\
&W^{a_1 a_2 \ldots a_{n-2q}} = -\sum_{k=0}^{q-1} (-1)^k \frac{(n-2q+2k+1)!}{(2k+1)!(n-2q)!} \times
\end{aligned}
$$

$$\times\alpha^{a_1a_2\ldots a_{n-2q+2k+1}b_1b_2\ldots b_{2k+1}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2k+1}}V-\alpha^{a_1a_2\ldots a_{n-2q}}V-$$

$$-\frac{1}{2}\sum_{k=0}^{q-2}(-1)^{k+q}\frac{(n-2k-2)!}{(n-2q)!(2q-2k-3)!(q-k-1)}\times$$

$$\times\alpha^{a_1a_2\ldots a_{n-2k}b_1b_2\ldots b_{2q-2k-2}}\partial^{b_1}\partial^{b_2}\ldots\partial^{b_{2q-2k-2}}V,$$

$$q=2,3,\ldots,\left[\frac{n-1}{2}\right].$$

Equations (22) define the potentials admitting non-trivial symmetries of order n, and the coefficients $K^{\cdots}$ of the corresponding symmetry operators, as well.

For $V=0$, equations (22) reduce to the following form

$$\partial^{(a_{j+1}}K^{a_1a_2\ldots a_j)}=\frac{2}{m+2j-1}\partial^bK^{b(a_1a_2\ldots a_{j-1}}g^{a_ja_{j+1})}=0, \tag{23}$$

$$\alpha^{a_1a_2\ldots a_{j-1}}=\frac{2}{m+2j-1}\partial^bK^{ba_1a_2\ldots a_{j-1}}, \tag{24}$$

where $K^{\cdots}$ and $\alpha^{\cdots}$ are symmetric and traceless tensors.

Thus, in the above case, the system of determining equations is decomposed into the uncoupled subsystems (24), which are the equations for a conformal Killing tensor that can be integrated for any m [8]. The corresponding symmetry operators reduce to polynomials in generators of the conformal group, moreover, the number of linearly independent symmetries for $m=2$ and $m=3$ [8] is given by the formulas:

$$N_2=\frac{1}{3}(n+1)(2n+1)(2n+3),\qquad N_3=\frac{1}{12}(n+1)^2(n+2)^2(2n+3). \tag{25}$$

Formulae (25) take into account n-order symmetries but no symmetries of order $j<n$.

Acknowledgments

One of the author (AN) is grateful to the DFFD of Ukraine (project 1.4/356) for the financial support.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 328–330.

Higher Order Symmetry Operators for the Schödinger Equation

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Abstract

Potential $V(x,t)$ classes, depending on both independent variables x and t, which allow a symmetry of the Schrödinger equation in the differential operators class of the third order are found.

One-dimensional non-relativistic systems can be considered with the help of the Schrödinger equation

$$L\Psi(x,t) = \left[p_0 - \frac{p^2}{2m} - V(x,t)\right]\Psi(x,t) = 0, \tag{1}$$

where $p_0 = i\dfrac{\partial}{\partial t}$, $p = -i\dfrac{\partial}{\partial x}$, $V(x,t)$ is a potential.

For equation (1), its symmetric analysis plays an important role. The Schrödinger equation symmetry is also surveyed in works [1–7].

The operator of a symmetry Q of equation (1) is an operator, which complies with the condition

$$[L,Q] = 0, \tag{2}$$

where

$$[L,Q] = LQ - QL. \tag{3}$$

Let us find classes of potentials $V(x,t)$, which possess symmetry with respect to a differential operator Q_3 of the third order

$$Q_3 = a_3p^3 + a_2p^2 + a_1p + a_0, \tag{4}$$

where a_3, a_2, a_1, a_0 are unknown functions depending on variables x and t. Substituting operator (4) to equation (2) and, after corresponding changes, equating coefficients of the corresponding operators of differentiation, we obtain the system of differential equations

$$\begin{aligned}
&a_3' = 0,\\
&\dot a_3 + \frac{1}{2m}a_2' = 0,\\
&\dot a_2 + \frac{1}{2m}a_1' - 6a_3V' = 0,\\
&\dot a_1 + \frac{1}{2m}a_0' - 4a_2V' = 0,\\
&\dot a_0 - 2a_1V' + 2a_3V''' = 0.
\end{aligned} \tag{5}$$

Integrating system (5), we obtain

$$\begin{aligned}
&a_3 = a_3(t),\\
&a_2 = -2m\dot{a}_3 x + a_2^0(t),\\
&a_1 = 2m^2\ddot{a}_3 x^2 + 12ma_3V - 2m\dot{a}_2^0 x + a_1^0(t),\\
&a_0' = -4m^3\dddot{a}_3 x^2 - 16m^2\dot{a}_3 xV' - 24m^2\dot{a}_3 V - 24m^2 a_3\dot{V} +\\
&\qquad 8ma_2^0V' + 4m^2\ddot{a}_2^0 x - 2m\dot{a}_1^0,\\
&\dot{a}_0 - 2a_1V' + 2a_3V''' = 0,
\end{aligned} \tag{6}$$

where $a_3(t)$, $a_2^0(t)$, $a_1^0(t)$ are unrestricted functions depending on t.

Some classes of potentials $V(x)$ which comply with system (6) are found in [6, 8]:

$$V(x) = \frac{2c^2}{m\cos^2 cx}, \qquad V(x) = \frac{2}{m}c^2\tan^2 cx, \qquad V(x) = \frac{2c^2}{m}(\tanh^2 cx - 1),$$
$$V(x) = \frac{2c^2}{m}(\coth^2 cx - 1), \qquad V(x) = \frac{1}{m}\left(\frac{c^2}{\sinh^2 cx} \pm \frac{c^2\cosh cx}{\sinh^2 cx}\right),$$

where c is some unrestricted constant.

We succeeded to indentify other kinds of potentiales $V(x,t)$ (depending on variables x and t) with the symmetry under the class of differential operators Q_3.

If we set $a_3 = \text{const}$, $a_2 = a_1^0 = a_2^0 = 0$ in (6), then we obtain the system

$$\begin{aligned}
&a_1 = 12ma_3V,\\
&a_0' = -24m^2a_3\dot{V},\\
&\dot{a}_0 - 2a_1V' + 2a_3V''' = 0.
\end{aligned} \tag{7}$$

After some changes, we obtain the equation in partial derivatives for finding $V(x,t)$:

$$12m^2\ddot{V} + 12mVV'' + 12m(V')^2 - V'''' = 0. \tag{8}$$

Equation (8) can be written as

$$12m^2\ddot{V} = (V'' - 6mV^2)''. \tag{9}$$

A solution of the given equation is the function

$$V(x) = \left(-\frac{c_1^2}{2m}t^2 + c_2t + c_3\right)(c_1x + c_4), \tag{10}$$

where c_1, c_2, c_3, c_4 are unrestricted constants.

If we make the substitution $V(x,t) = \dfrac{U(x,t)}{m}$ in equation (9), then we will obtain the equation

$$12m^2\ddot{U} = (U'' - 6U^2)''. \tag{11}$$

We know [9], that a solution of the equation $y'' - 6y^2 = 0$ ($y = y(x)$) is the function $y(x) = \wp(x + c_0)$, where $\wp$ is the Weierstrass function with invariants $g_2 = 0$ and $g_3 = c_1$, and c_1 is an unrestricted constant. Using this fact, we can write a solution of equation (11) as

$$U(x,t) = (\alpha t + \beta)\wp(x + c_0),$$

and a solution of equation (9)

$$V(x,t) = \frac{1}{m}(\alpha t + \beta)\wp(x + c_0), \qquad (12)$$

where α, β, c_0 are unrestricted constants.

Thus, the found operators $V(x,t)$ (10), (12) exhibit the symmetry of the Schrödinger equation (1) in the class of differential operators of the third order Q_3.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 331–335.

On Parasupersymmetries in a Relativistic Coulomb Problem for the Modified Stueckelberg Equation

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Abstract

We consider a Coulomb problem for the modified Stueckelberg equation. For some specific values of parameters, we establish the presence of parasupersymmetry for spin-1 states in this problem and give the explicit form of corresponding parasupercharges.

Introduction

In spite of the striking progress of quantum field theory (QFT) during last three decades, the problem of description of bound states in QFT context isn't yet completely solved. This state of things stimulates the use of various approximate methods in order to obtain physically important results.

In this paper, we work within the frame of the so-called one-particle approximation (OPA) which consists in neglecting creation and annihilation of particles and the quantum nature of external fields which the only particle we consider interacts with. This approximation is valid for the case where the energy of the interaction between the particle and the field is much less than the mass of the particle. In OPA, the operator of the quantized field corresponding to our particle is replaced by the "classical" (i.e., non-secondly quantized) quantity which is in essence nothing but the matrix element of that operator between vacuum and one-particle state (cf. [1] for the case of spin 1/2 particles). This quantity may be interpreted as a wave function of the particle and satisfies a *linear* equation, in which external field also appears as a classical quantity.

But for the case of massive charged spin-1 particles interacting with external electromagnetic field, even such relatively simple equations lead to numerous difficulties and inconsistencies [2, 3]. Only in 1995, Beckers, Debergh and Nikitin [2] overcame most of them and suggested a new equation describing spin-1 particles. They exactly solved it and pointed out its parasupersymmetric properties for the case of external constant homogeneous magnetic field [2]. (To find out more about parasupersymmetry, refer to [4] and references therein.)

The evident next step is to study this model for another physically interesting case of external Coulomb field (and in particular to check the possibility of existence of the parasupersymmetry in this case too).

In order to overcome some difficulties arising in the process of such a study, we consider in [3] and here a "toy" model corresponding to a particle with two possible spin states:

spin-1 and spin-0. Beckers, Debergh and Nikitin in [2] suggested a new equation for this case too, but our one is slightly more general (refer to Section 1).

The plan of the paper is as follows. In Section 1, we write down the equation describing our model (we call it modified Stueckelberg equation). In Section 2, we briefly recall the results from [3] concerning the exact solution of this model for the case of the Coulomb field of attraction. Finally, in Section 3, we point out the existence of the parasupersymmetry in the Coulomb field for spin-1 states at particular values of some parameters of our model.

1 Modified Stueckelberg equation

We consider the so-called modified Stueckelberg equation written in the second-order formalism [3]:

$$(D_\mu D^\mu + m_{eff}^2)B^\nu + iegF^\mu_\rho B^\rho = 0, \tag{1}$$

where

$$m_{eff}^2 = M^2 + k_2 \mid e^2 F_{\mu\nu}F^{\mu\nu} \mid^{1/2}, \quad D_\mu = \partial_\mu + ieA_\mu, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

We use here the $\hbar = c = 1$ units system and following notations: small Greek letters denote indices which refer to the Minkowski 4-space and run from 0 to 3 (unless otherwise stated); we use the following metrics of the Minkowski space: $g_{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$; the four-vector is written as $N^\mu = (N^0, \mathbf{N})$, where the bold letter denotes its three-vector part; coordinates and derivatives are $x^\mu = (t, \mathbf{r})$, $\partial_\mu = \partial/\partial x^\mu$; A^μ are potentials of external electromagnetic field; e, g and M are, respectively, charge, gyromagnetic ratio and mass of the particle described by (1). Its wave function is given by the four-vector B^μ. In the free ($e = 0$) case [2] this particle has two possible spin states: spin-0 and spin-1 ones with the same mass M.

Equation (1) generalizes the modification of the Stueckelberg equation from [2] for the case of an arbitrary gyromagnetic ratio g (authors of [2] consider only the $g = 2$ case). We also put the module sign at expression (2) for m_{eff}^2 (in spite of [2]) in order to avoid complex energy eigenvalues in the Coulomb problem (else μ_i (9) and, hence, energy eigenvalues of the discrete spectrum E^{inj} will be complex).

2 Coulomb field

The 4-potential, corresponding to the Coulomb field of attraction, is:

$$\mathbf{A} = 0, \quad A^0 = -Ze/r, \quad Z > 0. \tag{3}$$

Since it is static and spherically symmetric, energy E and total momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$ ($\mathbf{L}$ is an angular momentum and $\mathbf{S}$ is a spin) are integrals of motion. Let us decompose the wavefunctions of stationary states with fixed energy E in the basis of common eigenfunctions of $\mathbf{J}^2$ and $\mathbf{J}_z$ with eigenvalues $j(j+1)$ and m, respectively, $j = 0, 1, 2, \ldots$, for a given j, $m = -j, -j+1, \ldots, j$. The corresponding eigenmodes are:

$$\begin{aligned} B^0_{Ejm} &= iF_{Ej}(r)Y_{jm}\exp(-iEt), \\ \mathbf{B}_{Ejm} &= \exp(-iEt)\sum_{\sigma=-1,0,1} B^{(\sigma)}_{Ej}(r)\mathbf{Y}^{(\sigma)}_{jm}, \end{aligned} \tag{4}$$

where $\mathbf{Y}_{jm}^{(\sigma)}$ are spherical vectors (see [5] for their explicit form) and Y_{jm} are usual spherical functions. It may be shown that F_{Ej} and B_{Ej}^{σ}, $(\sigma = -1, 0, 1)$ in fact don't depend on m [5]. For the sake of brevity from now on, we often suppress indices Ej at F_{Ej} and $B_{Ej}^{(\sigma)}$ $(\sigma = -1, 0, 1)$.

The substitution of (3) and (4) into (1) yields the following equations for F and $B^{(\sigma)}$:

$$\mathrm{T}V = (2/r^2)\mathrm{P}V, \qquad \mathrm{T}B^{(0)} = 0, \tag{5}$$

where

$$V = (FB^{(-1)}B^{(1)})^{\dagger}, \qquad \mathrm{P} = \begin{pmatrix} 0 & -b & 0 \\ b & 1 & -a \\ 0 & -a & 0 \end{pmatrix}$$

($\dagger$ denotes matrix transposition, $a = \sqrt{j(j+1)}$, $\beta = Ze^2$, $b = \beta g/2$);

$$\mathrm{T} = (E + \beta/r)^2 + d^2/dr^2 + (2/r)d/dr - j(j+1)/r^2 - M^2 - k_2\beta/r^2. \tag{6}$$

Notice that for $j = 0$ $B_{Ej}^{(0)} = B_{Ej}^{(1)} \equiv 0$ [5].

By the appropriate replacement of the basis (introducing new unknown functions $K^{(i)}$, $i = 1, 2, 3$, being linear combinations of $B^{(\sigma)}$, $\sigma = -1, 1$ and $F(r)$), equations (5) for radial functions may be reduced to the following form [3] (for convenience, we denote $K^{(0)} \equiv B^{(0)}$):

$$\mathrm{T}K^{(i)} = (2\lambda_i/r^2)K^{(i)} \qquad i = 0, 1, 2, 3, \tag{7}$$

where

$$\lambda_l = 1/2 + (-1)^{l-1}\sqrt{(j+1/2)^2 - (\beta g/2)^2}, \qquad l = 1, 2, \quad \lambda_0 = \lambda_3 = 0. \tag{8}$$

Energy eigenvalues of the discrete spectrum for (7) are [3]:

$$E^{inj} = M/\sqrt{1 + \beta^2/(n + \mu_i + 1)^2}, \tag{9}$$

where $n = 0, 1, 2, \ldots$; $j = 0, 1, 2 \ldots$; $i = 0, 1, 2, 3$ and

$$\mu_i = -1/2 + \sqrt{(j+1/2)^2 - \beta^2 + 2\lambda_i + k_2\beta} \tag{10}$$

(index i corresponds to the following eigenmode: $K^{(i)} \neq 0$, other functions $K^{(l)} = 0$; since for $j = 0$ $K^{(0)} = K^{(3)} \equiv 0$, in this case, we have only two branches corresponding to $i = 1, 2$). Branches of the spectrum for $i = 0$ and $i = 3$ are completely identical, i.e., we meet here a twofold degeneracy.

The discrete spectrum eigenfunctions are [3]

$$K^{(i)nj} = c^{inj}x^{\mu_i}\exp(-x/2)L_n^{\mu_i}(x), \tag{11}$$

where c^{inj} are normalization constants, $x = 2r\sqrt{M^2 - E^2}$, L_n^{α} are Laguerre polynomials, $n = 0, 1, 2, \ldots$; $j = 0, 1, 2 \ldots$; $i = 0, 1, 2, 3$.

3 Parasupersymmetry

We consider the case $k_2 = 0, g = 2, j > 0$ where [3]

$$\mu_1 = \lambda_1, \quad \mu_0 = \lambda_1 - 1, \quad \mu_2 = \lambda_1 - 2$$

and hence the above energy eigenvalues (9) possess the threefold extra degeneracy:

$$E^{1,n+1,j} = E^{0,n,j} = E^{3,n,j} = E^{2,n-1,j}, \quad n > 1. \tag{12}$$

Moreover, we restrict ourselves by considering only spin-1 states, setting the condition (compatible with (1) for the case of Coulomb field if $k_2 = 0$, $g = 2$ for the states with $j > 0$ [3])

$$\begin{aligned} D_\mu B^\mu_{Ejm} = \Big\{ E K^{(3)}_{Ej} + \sum_{i=1}^{2} \Big[dK^{(i)}_{Ej}/dr + \\ +(1+\lambda_i) K^{(i)}_{Ej}/r - E\beta K^{(i)}_{Ej}/\lambda_i \Big] \Big\} \exp(-iEt) Y_{jm} = 0. \end{aligned} \tag{13}$$

We use here the term "spin-1 states" by the analogy with the free ($e = 0$) case (cf. [1]), i.e., we call spin-1 states the states satisfying (13). In virtue of (13) the component $K^{(3)}$ is expressed via the remaining ones and isn't more independent [3]. This implies that we must deal with only three branches of the discrete spectrum E^{inj}, $i = 0, 1, 2$.

Equations (7) which remain after the exclusion of $K^{(3)}$ may be rewritten in the following form:

$$\mathrm{H}\psi = \varepsilon\psi, \tag{14}$$

where

$$\begin{aligned} &\psi = (K^{(1)} K^{(0)} K^{(2)})^\dagger, \quad \mathrm{H} = 2\mathrm{diag}\,[\mathrm{H}_1, \mathrm{H}_0, \mathrm{H}_2]; \\ &\mathrm{H}_i = -d^2/dr^2 - (2/r)d/dr - \lambda_i(\lambda_i+1)/r^2 - 2\beta E/r + \\ &\qquad +(1/2)\beta E(1/\lambda_1 + 1/\lambda_2) \qquad \text{for} \quad i = 1, 2, \\ &\mathrm{H}_0 = -d^2/dr^2 - (2/r)d/dr - \lambda_1(\lambda_1 - 1)/r^2 - 2\beta E/r + (1/2)\beta E(1/\lambda_1 + 1/\lambda_2), \\ &\varepsilon = \beta E(1/\lambda_1 + 1/\lambda_2) - 2(M^2 - E^2). \end{aligned} \tag{15}$$

Let us introduce parasupercharges Q^+ and Q^- of the form:

$$\mathrm{Q}^+ = \begin{pmatrix} 0 & \mathrm{S}_1 & 0 \\ 0 & 0 & \mathrm{R}_2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{Q}^- = \begin{pmatrix} 0 & 0 & 0 \\ \mathrm{R}_1 & 0 & 0 \\ 0 & \mathrm{S}_2 & 0 \end{pmatrix},$$

where

$$\mathrm{R}_i = d/dr + (1+\lambda_i)/r - E\beta/\lambda_i, \qquad \mathrm{S}_i = -d/dr + (\lambda_i - 1)/r - E\beta/\lambda_i. \tag{16}$$

Now it is straightforward to check that Q^+, Q^-, $\mathrm{Q}_1 = (\mathrm{Q}^+ + \mathrm{Q}^-)/2$, $\mathrm{Q}_2 = (i/2)(\mathrm{Q}^+ - \mathrm{Q}^-)$ and H satisfy the commutation relations of the so-called $p = 2$ parasupersymmetric quantum mechanics of Rubakov and Spiridonov (see, e.g., [2, 4]):

$$\begin{aligned} &(\mathrm{Q}^\pm)^3 = 0, \qquad \mathrm{Q}_i^3 = \mathrm{H}\mathrm{Q}_i, \qquad [\mathrm{Q}^\pm, \mathrm{H}] = 0, \\ &\{\mathrm{Q}_i^2, \mathrm{Q}_{3-i}\} + \mathrm{Q}_i \mathrm{Q}_{3-i} \mathrm{Q}_i = \mathrm{H}\mathrm{Q}_{3-i}, \qquad i = 1, 2, \end{aligned} \tag{17}$$

where $[\mathrm{A}, \mathrm{B}] = \mathrm{AB} - \mathrm{BA}$, $\{\mathrm{A}, \mathrm{B}\} = \mathrm{AB} + \mathrm{BA}$.

We see that the parasupercharges Q^+, Q^- commute with H and hence are (non-Lie) symmetries of our Coulomb problem. Their existence is just the reason of the above-mentioned threefold extra degeneracy.

4 Conclusions and discussion

Thus, in this paper, we the explained the threefold extra degeneracy of discrete spectrum levels in a Coulomb problem for the modified Stueckelberg equation for $k_2 = 0$, $g = 2$, $j > 0$, $n > 1$ by the presence of parasupersymmetry. Our results present a natural generalization of supersymmetry in a Coulomb problem for the Dirac equation [4].

It is also worth noticing that, in the case of Coulomb field for $k_2 = 0$, $g = 2$, the equations of our model for the spin-1 states with $j > 1$ coincide with the equations for the same states of the Corben-Schwinger model [6] for $g = 2$. Hence, their model also is parasupersymmetric and possesses non-Lie symmetries – parasupercharges Q^+, Q^-.

Finally, we would like to stress that, to the best of author's knowledge, our model give one of the first examples of parasupersymmetry in a relativistic system with non-oscillator-like interaction.

I am sincerely grateful to Prof. A.G. Nikitin for statement of the problem and stimulating discussions.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 336–342.

Discrete Integrable Systems and the Moyal Symmetry

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Abstract

The Hirota bilinear difference equation plays the central role in the study of integrable nonlinear systems. A direct correspondence of the large symmetry characterizing this classical system to the Moyal algebra, a quantum deformation of the Poisson bracket algebra, is shown.

1 Introduction

In the study of nonlinear systems, completely integrable systems play special roles. They are not independent at all, but are strongly correlated with each other owing to the large symmetries shared among themselves. We like to know how far we can extend such systems without loosing integrability. The question could be answered if we know how much we can deform the symmetries characterizing the integrable systems.

In this note, we would like to discuss the Moyal bracket algebra [1], a quantum deformation of the Poisson bracket algebra, as a scheme which should describe a large class of integrable systems. Here, however, we focus our attention to the Hirota bilinear difference equation (HBDE) [2], a difference analogue of the two-dimensional Toda lattice.

This paper is organized as follows. We first explain HBDE and what is the Moyal quantum algebra in the following two sections. Using the Miwa transformation [3] of soliton variables, the correspondence of the shift operator appeared in HBDE to the Moyal quantum operator is shown in sections 4 and 5. The large symmetry possessed by the universal Grassmannian of the KP hierarchy [4, 5] is explained in section 6 within our framework, and the connection of their generators to the Moyal quantum operators is shown in the last section.

2 Hirota bilinear difference equation

The Hirota bilinear difference eqution (HBDE) is a simple single equation which is given by [2]

$$\begin{aligned}&\alpha f(k_1+1,k_2,k_3)f(k_1,k_2+1,k_3+1)+\beta f(k_1,k_2+1,k_2)f(k_1+1,k_2,k_3+1)\\&\qquad+\gamma f(k_1,k_2,k_3+1)f(k_1+1,k_2+1,k_3)=0,\end{aligned}\tag{1}$$

This paper is dedicated to the memory of Professor Wilhelm Fushchych where $k_j \in \boldsymbol{Z}$ are discrete variables and $\alpha,\ \beta,\ \gamma \in \boldsymbol{C}$ are parameters subject to the constraint $\alpha + \beta + \gamma = 0$.

The contents of this equation is, however, very large. In fact, this single difference equation

1. is equivalent to the soliton equations of the KP-hierarchy [2, 4, 5],
2. characterizes algebraic curves (Fay's trisecant formula) [4],
3. is a consistency relation for the Laplace maps on a discrete surface [6],
4. is satisfied by string correlation functions in the particle physics [7], and
5. by transfer matrices of certain solvable lattice models [8, 9, 10, 11, 12],

etc..

We note here that HBDE (1) is a collection of infinitely many 'classical' soliton equations [2, 3]. For the purpose of describing the symmetric nature of this equation, it will be convenient to rewrite it as

$$\begin{aligned}\Big(\alpha \exp\Big[\partial_{k_1} + \partial_{k_2'} + \partial_{k_3'}\Big] + \beta \exp\Big[\partial_{k_1'} + \partial_{k_2} + \partial_{k_3'}\Big] + \gamma \exp\Big[\partial_{k_1'} + \partial_{k_2'} + \partial_{k_3}\Big]\Big) \\ \times\ f(k_1, k_2, k_3) f(k_1', k_2', k_3')|_{k_j'=k_j}\ .\end{aligned} \tag{2}$$

As we will see later, it is this shift operator $\exp \partial_{k_j}$ that generates the symmetry of a system associated with the Moyal 'quantum' algebra. Classical soliton equations are obtained by expanding the shift operators in (2) into power series of derivatives [4, 5].

The space of solutions to this equation, hence to the KP hierarchy, is called the Universal Grassmannian [4]. Every point on this space corresponds to a solution of HBDE, which can be given explicitly. Starting from one solution, we can obtain other solutions via a sequence of Bäcklund transformations. The transformations generate a large symmetry which characterizes this particular integrable system.

3 Moyal quantum algebra

Let us explain what is the Moyal algebra[1]. We will show, in other sections, its relation to the integrable systems characterized by HBDE.

The Moyal bracket is a quantum deformation of the Poisson bracket and is given by [1]

$$i\{f, g\}_M := \frac{1}{\lambda} \sin\left\{\lambda\left(\boldsymbol{\partial}_x \boldsymbol{\partial}_{p'} - \boldsymbol{\partial}_{x'} \boldsymbol{\partial}_p\right)\right\} f(\boldsymbol{x}, \boldsymbol{p}) g(\boldsymbol{x'}, \boldsymbol{p'})\Big|_{\boldsymbol{x'}=\boldsymbol{x}, \boldsymbol{p'}=\boldsymbol{p}} \tag{3}$$

where $\boldsymbol{x}$ and $\boldsymbol{p}$ are the coordinates and momenta in $\boldsymbol{R}^N$ with N being the number of degres of freedom. It turns to the Poisson bracket in the small λ limit. In this particular limit, the symmetry is well described by the language of differential geometry. Therefore, our question is whether there exists a proper language which can describe concepts, in the case of finite values of λ, corresponding to the terms such as vector fields, differential forms, and Lie derivatives. The answer is "yes" [13]. In order to show that, we first define a difference operator by

$$\boldsymbol{\nabla}_{\boldsymbol{a}_x, \boldsymbol{a}_p} := \frac{1}{\lambda} \sinh\left[\lambda(\boldsymbol{a}\boldsymbol{\partial})\right], \qquad (\boldsymbol{a}\boldsymbol{\partial}) := \boldsymbol{a}_x \boldsymbol{\partial}_x + \boldsymbol{a}_p \boldsymbol{\partial}_p \tag{4}$$

For a given $\boldsymbol{C}^\infty$ function $f(\boldsymbol{x},\boldsymbol{p})$, the Hamiltonian vector field is defined as [13]

$$\boldsymbol{X}_f^D := \left(\frac{\lambda}{2\pi}\right)^{2N} \int d\boldsymbol{a}_x d\boldsymbol{a}_p \int d\boldsymbol{b}_x d\boldsymbol{b}_p \, e^{-i\lambda(\boldsymbol{a_x}\boldsymbol{b}_p-\boldsymbol{a}_p\boldsymbol{b}_x)} f(\boldsymbol{x}+\lambda\boldsymbol{b}_x,\boldsymbol{p}+\lambda\boldsymbol{b}_p)\boldsymbol{\nabla}_{\boldsymbol{a}_x,\boldsymbol{a}_p}. \quad (5)$$

The operation of the Hamiltonian vector field to a function g on the phase space yields the Moyal bracket:

$$\boldsymbol{X}_f^D g(\boldsymbol{x},\boldsymbol{p}) = i\{f,g\}_M. \quad (6)$$

The above definition of the Hamiltonian vector fields enables us to extend the concept of Lie derivative in the differential geometry to a discrete phase space [13]. In fact, we can show directly the following relation between two vector fields:

$$\left[\boldsymbol{X}_f^D, \boldsymbol{X}_g^D\right] = \boldsymbol{X}_{-i\{f,g\}_M}^D. \quad (7)$$

This exhibits a very large symmetry generated by Hamiltonian vector fields, which should be shared commonly by the physical systems formulated in our prescription. This algebra itself was derived and discussed [14, 15, 16] in various contexts including some geometrical arguments.

For the vector field $\boldsymbol{X}_f^D$ to be associated with a quantum operator, we must introduce a difference one form whose pairing with the vector field yields the expectation value of f. It is shown in [13] that such a pairing can be defined properly by considering a quantum state characterized by the Wigner distribution function [17].

4 Miwa transformation

The Miwa transformation [3] of variables enables us to interpret the shift operators in HBDE (2) as the Hamiltonian vector field discussed above. It is defined by

$$t_n := \frac{1}{n}\sum_j k_j z_j^n, \qquad n = 1,2,3,\ldots. \quad (8)$$

In this expression, k_j ($j \in \boldsymbol{N}$) are integers among which k_1, k_2, k_3 belong to (1). t_n's are new variables which describe soliton coordinates of the KP hierarchy. For instance, $t_1 = t$ and $t_3 = x$ are the time and space variables, respectively, of the KdV equation. z_j's are complex parameters which are defined on the Riemann surface.

In the language of string models, the soliton variables t_n's correspond to the oscillation parts of open strings [7]. The center of momenta x_0 and the total momentum p_0 should be also included as dynamical variables, which are related to k_j's by

$$p_0 = \sum_j k_j, \qquad x_0 = i\sum_j k_j \ln \bar{z}_j. \quad (9)$$

In addition to (8), we also define

$$\bar{t}_n := \frac{1}{n}\sum_j k_j \bar{z}_j^{-n}, \qquad n = 1,2,3,\ldots, \quad (10)$$

so that the physical space of oscillations is doubled. The phase space of oscillations are described by t_n's and $\bar{t}_n$'s as

$$\begin{aligned} x_n &= \sqrt{\frac{n}{2}}(t_n + \bar{t}_n) = \sqrt{\frac{1}{2n}}\sum_j k_j \left(z_j^n + \bar{z}_j^{-n}\right), \qquad n = 1,2,3,\ldots, \\ p_n &= \frac{1}{i}\sqrt{\frac{n}{2}}(t_n - \bar{t}_n) = \frac{1}{i}\sqrt{\frac{1}{2n}}\sum_j k_j \left(z_j^n - \bar{z}_j^{-n}\right), \qquad n = 1,2,3,\ldots. \end{aligned} \tag{11}$$

In terms of these new variables, the shift operators appearing in HBDE (2) become

$$\exp \partial_{k_j} = \exp\left[\lambda(\boldsymbol{a}_j\boldsymbol{\partial})\right]. \tag{12}$$

Here $(\boldsymbol{a}_j\boldsymbol{\partial})$ is the sum of an infinite number of components in the notation of (4). The values of the components of $\boldsymbol{a}_{j,x}$ and $\boldsymbol{a}_{j,p}$ are specialized to

$$\begin{aligned} a_{j,x_0} &= \frac{i}{\lambda}\ln \bar{z}_j, \qquad a_{j,p_0} = \frac{1}{\lambda}, \\ a_{j,x_n} &= \frac{z_j^n + \bar{z}_j^{-n}}{\lambda\sqrt{2n}}, \quad a_{j,p_n} = \frac{z_j^n - \bar{z}_j^{-n}}{i\lambda\sqrt{2n}}, \qquad n = 1,2,3,\ldots. \end{aligned} \tag{13}$$

5 Gauge covariant shift operator

We now gauge the shift operator to obtain a gauge covariant shift operator:

$$e^{\partial_{k_j}} \quad \to \quad \boldsymbol{X}^S_{u_j} := U(\boldsymbol{k})e^{\partial_{k_j}}U^{-1}(\boldsymbol{k}). \tag{14}$$

Here $U(\boldsymbol{k})$ is a function of $\boldsymbol{k} = \{k_1, k_2, \cdots\}$. Using the Miwa transformation, we can write it in terms of the soliton variables as

$$\boldsymbol{X}^S_{u_j} = u_j(\boldsymbol{x},\boldsymbol{p})e^{\lambda(\boldsymbol{a}_j\boldsymbol{\partial})}, \qquad u_j(\boldsymbol{x},\boldsymbol{p}) := U(\boldsymbol{k})U^{-1}(\boldsymbol{k}^{(j)}), \tag{15}$$

where $\boldsymbol{k}^{(j)}$ denotes the set of k's but k_j is replaced by $k_j + 1$.

The covariant shift operators also form a closed algebra as follows:

$$\left[\boldsymbol{X}^S_{u_j},\ \boldsymbol{X}^S_{v_l}\right] = \boldsymbol{X}^S_{\{u_j,v_l\}_S} \tag{16}$$

where

$$\{u_j, v_l\}_S := \left(e^{\lambda(\boldsymbol{a}_l\boldsymbol{\partial}')} - e^{\lambda(\boldsymbol{a}_j\boldsymbol{\partial})}\right) u_j(\boldsymbol{x},\boldsymbol{p})v_l(\boldsymbol{x}',\boldsymbol{p}')\Big|_{\boldsymbol{x}'=\boldsymbol{x},\boldsymbol{p}'=\boldsymbol{p}}. \tag{17}$$

In this formula, $\boldsymbol{\partial}$ and $\boldsymbol{\partial}'$ act on u_j and v_l selectively.

We now want to establish the correspondence of gauge covariant shift operators to the Hamiltonian vector field of the Moyal quantum algebra discussed before. For this purpose, we denote by $\boldsymbol{X}^D_{u_j}$ the antisymmetric part of $-\boldsymbol{X}^S_{u_j}$,

$$\boldsymbol{X}^D_{u_j} = \frac{1}{2\lambda}\left(\overline{\boldsymbol{X}}^S_{u_j} - \boldsymbol{X}^S_{u_j}\right), \tag{18}$$

Here $\overline{\boldsymbol{X}}_{u_j}^S$ is obtained from $\boldsymbol{X}_{u_j}^S$ by reversing the direction of shift in (15). Then it is not difficult to convince ourselves that $\boldsymbol{X}_{u_j}^D$ is a Hamiltonian vector field of (5) with f being specialized to u_j, and u_j itself is given by $u_j(\boldsymbol{x},\boldsymbol{p}) = \exp\left[i\lambda\left(\boldsymbol{a}_{j,p}\boldsymbol{x} - \boldsymbol{a}_{j,x}\boldsymbol{p}\right)\right]$. We thus obtain an expression for the shift operator

$$\boldsymbol{X}_{u_j}^S = \exp\left[i\lambda\left(\boldsymbol{a}_{j,p}\boldsymbol{x} - \boldsymbol{a}_{j,x}\boldsymbol{p}\right)\right]\exp\left[\lambda\left(\boldsymbol{a}_{j,x}\boldsymbol{\partial}_x + \boldsymbol{a}_{j,p}\boldsymbol{\partial}_p\right)\right], \tag{19}$$

from which we can form a Hamiltonian vector field of the Moyal algebra. We notice that this operator $\boldsymbol{X}_{u_j}$ is nothing but the vertex operator for the interaction of closed strings [7].

Does there exist U associated to this $u_j(\boldsymbol{x},\boldsymbol{p})$? We can see that

$$U(\boldsymbol{k}) := \prod_{j,l}\left(\frac{\bar{z}_j - z_l}{\bar{z}_l - z_j}\right)^{-k_l k_j} \tag{20}$$

satisfies this requirement if we substitute it to (15) and use the Miwa transformations.

6 Symmetry of HBDE

One of the symmetries which characterize the solution space of HBDE is generated by the Bäcklund transformation [4, 5]. The generators of this symmetry are given by

$$\boldsymbol{B}(z_j, z_l) := 4\pi V(z_j)\bar{V}(z_l) \tag{21}$$

where $V(z_j)$ is obtained from $\boldsymbol{X}_{u_j}^S$ of (19) simply by ignoring the $\bar{t}_n$ and x_0 dependence. In terms of the soliton variables, we can express it in the form

$$\begin{aligned} V(z_j) &:= \exp\left[p_0 \ln \bar{z}_j - \sum_{n=1}^{\infty} \bar{z}_j^{-n} t_n\right] \exp\left[\partial_{p_0} + \sum_{n=1}^{\infty} \frac{1}{n} z_j^n \partial_{t_n}\right], \\ \bar{V}(z_j) &:= \exp\left[-p_0 \ln \bar{z}_j + \sum_{n=1}^{\infty} \bar{z}_j^{-n} t_n\right] \exp\left[-\partial_{p_0} - \sum_{n=1}^{\infty} \frac{1}{n} z_j^n \partial_{t_n}\right]. \end{aligned} \tag{22}$$

These vertex operators must be thought being local field operators which might behave singularly when their coordinates z get close with each other. For instance, after simple calculation we find

$$\begin{aligned} V(z_j)\bar{V}(z_l) &= \frac{1}{\bar{z}_l - z_j}\exp\left[p_0 \ln\left(\frac{\bar{z}_j}{\bar{z}_l}\right) - \sum_{n=1}^{\infty}\left(\bar{z}_j^{-n} - \bar{z}_l^{-n}\right) t_n\right]\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(z_j^n - z_l^{-n}\right)\partial_{t_n}\right], \\ \bar{V}(z_l)V(z_j) &= \frac{1}{\bar{z}_j - z_l}\exp\left[p_0 \ln\left(\frac{\bar{z}_j}{\bar{z}_l}\right) - \sum_{n=1}^{\infty}\left(\bar{z}_j^{-n} - \bar{z}_l^{-n}\right) t_n\right]\exp\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(z_j^n - z_l^{-n}\right)\partial_{t_n}\right]. \end{aligned} \tag{23}$$

If we are interested in the behaviour of these quantities on the real axis, the summation of $V(z_j)\bar{V}(z_l)$ and $\bar{V}(z_l)V(z_j)$ is zero. If we are interested in the behaviour, not on the real axis but near the real axis, we must be more careful. Let us write $\bar{z}_m - z_m = -2i\epsilon$, $\forall m$, and take the limit of $\epsilon \to 0$. We find

$$V(z_j)\bar{V}(z_l) + \bar{V}(z_l)V(z_j) = 4\pi i\delta(z_l - z_j). \tag{24}$$

This is the result known as bosonization. Using this result, we can show that $\boldsymbol{B}$'s form the algebra $gl(\infty)$, which characterizes the symmetry of the universal Grassmannian [4, 5],

$$[\boldsymbol{B}(z_j, z_k),\ \boldsymbol{B}(z_l, z_m)] = i\boldsymbol{B}(z_j, z_m)\delta(z_k - z_l) - i\boldsymbol{B}(z_k, z_l)\delta(z_j - z_m). \tag{25}$$

This symmetry includes the conformal symmetry of the theory [18]. In fact, the Virasoro generators are included in (21) in the limit of $z_j \to z_l$. $W_{1+\infty}$ symmetry of the KP-hierarchy [19, 20, 21] can be also described in our scheme.

7 Quantum deformation

We now look at the vertex operator $V(z_j)$ from other view point. It must be also represented as a gauge covariant shift operator as $\boldsymbol{X}^S_{u_j}$ in (14). Indeed, we can rewrite it in the form of (14), i.e., $U \exp \partial_{k_j} U^{-1}$, by choosing $U(\boldsymbol{k})$ as

$$U(\boldsymbol{k}) = \prod_{j \neq l} (\bar{z}_l - z_j)^{k_j k_l}. \tag{26}$$

It then follows, from their construction, that $V(z_j)$ and $\bar{V}(z_l)$ must commute with each other; $[V(z_j),\ \bar{V}(z_l)] = 0,\quad {}^\forall j, l$. By the same reason, we should have $\left[\boldsymbol{X}^S_{u_j},\ \boldsymbol{X}^S_{u_l}\right] = 0$, ${}^\forall j, l$, as long as $\boldsymbol{X}^S_{u_j}$ and $\boldsymbol{X}^S_{u_l}$ are induced by the same function $U(\boldsymbol{k})$. Apparently, this is not compatible with the previous result, e.g., (24). How can we get rid of this contradiction?

Instead of trying to avoid this problem, we like to interpret it as a transition from a classical view to a quantum view. To do this, we recall that the shift operators $\exp \partial_{k_j}$ define HBDE, a classical soliton equation. The change of variables, via Miwa transformations, introduces infinitely many new variables and the shift operator is represented in terms of quantized local fields. If we expand $\exp \partial_{k_j}$ in (1) into powers of ∂_{t_n}'s, we obtain infinitely many classical soliton equations which belong to the KP hierarchy [4, 5]. But their infinite collection may not be classical anymore.

This transition is, however, not sufficient to claim that $V(z_j)$, expressed in the form of (22), is an operator of quantum mechanics. A quantum mechanical operator must be described in terms of a Moyal Hamiltonian vector field of (5). From this point of view, the operator $\boldsymbol{X}^D_{u_j}$ of (18) could be a candidate of a quantum mechanical object.

Before closing this note, let us see the algebra corresponding to (24), but $V(z_j)$ is replaced by $\boldsymbol{X}^S_{u_j}$ of (19). It is more convenient to introduce the notations $\boldsymbol{V}(z_j) := \boldsymbol{X}^S_{u_j}$ and $\bar{\boldsymbol{V}}(z_j)$ being defined by substituting $-\lambda$ in the place of λ on the right hand side of (19). After some calculations similar to (23), we find

$$\boldsymbol{V}(z_j)\bar{\boldsymbol{V}}(z_l) = \frac{\bar{z}_j - z_l}{\bar{z}_l - z_j} \exp\left[i\lambda\left\{(\boldsymbol{a}_{j,p} - \boldsymbol{a}_{l,p})\boldsymbol{x} - (\boldsymbol{a}_{j,x} - \boldsymbol{a}_{l,x})\boldsymbol{p}\right\}\right] \exp\left[\lambda\left((\boldsymbol{a}_j - \boldsymbol{a}_l)\boldsymbol{\partial}\right)\right],$$

$$\bar{\boldsymbol{V}}(z_l)\boldsymbol{V}(z_j) = \frac{\bar{z}_l - z_j}{\bar{z}_j - z_l} \exp\left[i\lambda\left\{(\boldsymbol{a}_{j,p} - \boldsymbol{a}_{l,p})\boldsymbol{x} - (\boldsymbol{a}_{j,x} - \boldsymbol{a}_{l,x})\boldsymbol{p}\right\}\right] \exp\left[\lambda\left((\boldsymbol{a}_j - \boldsymbol{a}_l)\boldsymbol{\partial}\right)\right],$$

from which we obtain, in the limit of small $\bar{z}_m - z_m = -2i\epsilon$,

$$\boldsymbol{V}(z_j)\bar{\boldsymbol{V}}(z_l) - \bar{\boldsymbol{V}}(z_l)\boldsymbol{V}(z_j) = \lim_{\epsilon \to 0} \frac{8i\epsilon(z_j - z_l)}{(z_j - z_l)^2 + 4\epsilon^2}. \tag{27}$$

This does not vanish in the limit of $\epsilon \to 0$ *iff* $z_j - z_l$ is the same order of ϵ.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 343–352.

An Orbit Structure for Integrable Equations of Homogeneous and Principal Hierarchies

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Abstract

A construction of integrable Hamiltonian systems associated with different graded realizations of untwisted loop algebras is proposed. These systems have the form of Euler-Arnold equations on orbits of certain subalgebras of loop algebras and coincide with hierarchies of higher stationary equations for some nonlinear partial differential equations integrable by the inverse scattering method. We apply the general scheme for the loop algebras sl(3) $\otimes\mathcal{P}\left(\lambda,\lambda^{-1}\right)$. The corresponding equations on the orbit are interpreted as a two-(or three-)component nonlinear Schrödinger-type equation, an SU(3) – Heisenberg magnet type equation for the homogeneous realization, and the Boussinesq's equation for the principal realization.

We dedicate this work to the memory of W.I. Fushchych.

1. Introduction

In this paper, we deal with hierarchies of nonlinear evolutionary equations which are integrable Hamiltonian systems in the phase space of functions of one variable "x" and can be written in the form of a "zero-curvature" equation

$$\frac{\partial\Lambda}{\partial\tau_n}+\frac{\partial A_n}{\partial x}+[\Lambda,A_n]=0,\qquad n=1,2,\ldots. \tag{1}$$

Here, Λ and A_n are elements of $\tilde{\tilde{g}}$, an algebra of Laurent polynomials with coefficients belonging to the central extention of a current algebra on the circle.

Let $\widetilde{g}$ denote an algebra of Laurent polynomials with coefficients in a semisimple Lie algebra g, with σ being an internal finite-order automorphism of g. The automorphism σ lifted up to $\widetilde{g}$ fixes a gradation. The gradation in $\widetilde{g}$ and the symmetry of the elements Λ and A_n with respect to σ determine the hierarchy type.

It is known that classes of equivalent finite-order automorphisms coincide with conjugate classes of the Weyl group of a semisimple Lie algebra g [1, 2]. In the case of the identity automorphism, one has the homogeneous gradation and, accordingly, the homogeneous hierarchy of equations (1) involving the many-component nonlinear Schrödinger equation, the Heisenberg magnet type equation as well as many other equations. The maximal order automorphism is related to the principal gradation, with corresponding equations such as KdV type equations, the Boussinesq's equation and others.

The dressing procedure is used in the standard approach [3, 4] to find the matrices A_n for hierarchy (1) . Given the operator $L = \frac{\partial}{\partial x} - \Lambda$, the dressing operator K is determined by the equation:

$$L = K^{-1}\left(\frac{\partial}{\partial x} - \Lambda_0\right) K,$$

where Λ_0 is a constant σ-invariant element of Heisenberg subalgebras in $\widetilde{g}$.

In the approach developed earlier by one of the authors [5, 6], the objects A_n were interpreted as points of a coadjoint orbit of the subalgebra of Laurent series with positive or negative powers only and it may be obtained by restriction onto a coadjoint orbit of the general position element. The stationary (finite-zone) equations $\partial_x A_n = [\Lambda, A_n]$ form a hierarchy of finite-dimensional Hamiltonian systems on the orbit, with the standard set of commuting integrals. The number of these integrals suffices for integrability in the case of a homogeneous hierarchy. A Hamiltonian reduction of orbits is needed in other cases (the principal and intermediate hierarchies). As to the principal hierarchy, this reduction is the Drinfeld-Sokolov reduction restricted onto a finite-dimensional phase-space. From the point of view of the orbit approach, the dressing procedure is nothing but a restriction onto orbits as algebraic manifolds in the case of the homogeneous gradation or, for other gradations, onto a reduced phase space. In this paper, we develop this approach and expand it for higher range algebras as well as for nonhomogeneous gradations.

2. Constructing integrable Hamiltonian equations on coadjoint orbits

1. Let g be a semisimple finite-dimensional Lie algebra of rank r and $\mathcal{P}(\lambda, \lambda^{-1})$ the associative algebra of Laurent polynomials in a complex parameter λ . Let us consider the loop algebra $\widetilde{g} = g \otimes \mathcal{P}(\lambda, \lambda^{-1})$ with the commutator [7]:

$$\left[\sum A_i\lambda^i, \sum B_j\lambda^j\right] = \sum [A_i, B_j]\,\lambda^{i+j}.$$

Define the family of Ad-invariant nondegenerate forms on $\widetilde{g}$:$\langle A, B\rangle_k = \sum_{i+j=k}(A_i, B_j)$, $k \in \mathbf{Z}$, where (,) denotes the Killing form in g. Decompose $\widetilde{g}$ in the sum of two subspaces, $\widetilde{g} = \widetilde{g}_- + \widetilde{g}_+$, where

$$\widetilde{g}_+ = \left\{\sum_{i\geq 0} A_i\lambda^i\right\}, \qquad \widetilde{g}_- = \left\{\sum_{i<0} A_i\lambda^i\right\}.$$

Then $\widetilde{g}_+$ and $\widetilde{g}_-$ are subalgebras of $\widetilde{g}$ and form the dual pair relative to $\langle,\rangle_{-1}$ with the coadjoint action

$$ad^*_A\mu = \mathcal{P}_+[\mu, A], \qquad A \in \widetilde{g}_-, \qquad \mu \in \widetilde{g}_-{}^* \simeq \widetilde{g}_+, \tag{2}$$

where $\mathcal{P}_+$ denotes the projector onto $\widetilde{g}_+$. Let $\{X_i\}_1^{\dim g}$ be a basis in g. Keeping in mind our further purposes, it is more convenient to fix a dual basis $\{X_i\}_1^{\dim g}$ determined by $(X_i^*, X_j) = \delta_{ij}$. The finite-dimensional subspaces

$$M^{N+1} = \left\{\mu \in \widetilde{g}_-^* : \quad \mu = \sum_{\ell=0}^{N+1}\sum_{i=1}^{\dim g} \mu_i^\ell X_i^* \lambda^\ell\right\} \subset \widetilde{g}_+, \quad N = 0, 1, 2, \ldots < \infty,$$

where $\mu_i^\ell = \langle \mu, X_i^{-\ell-1}\rangle_{-1}$ are the coordinates on M^{N+1}, are invariant under the coadjoint action of $\widetilde{g}_-$. The coadjoint action of $\widetilde{g}_+$ is defined on M^{N+1} also. In this case, $\mu(A) = \langle \mu, A\rangle_{N+1}$, $A \in \widetilde{g}_+$, and we identify $\widetilde{g}_-^*$ with the subspace $\widetilde{g}_- \otimes M^{N+1}$. The coordinates on M^{N+1} can be written down as $\mu_i^\ell = \langle \mu, X_i^{-\ell+N+1}\rangle_{N+1}$, and M^{N+1} is invariant under the action of $\widetilde{g}_+$.

The coadjoint actions induce the family of Lie-Poisson brackets on M^{N+1}:

$$\{f_1, f_2\}_n = \sum_{\ell=0}^{N+1}\sum_{i=1}^{\dim g} W_{ij}^{\ell s}(n)\frac{\partial f_1}{\partial \mu_i^\ell}\frac{\partial f_2}{\partial \mu_j^s}, \quad \forall f_1, f_2 \in \mathbf{C}^\infty(M^{N+1}), \tag{3}$$

where

$$W_{ij}^{\ell s}(n) = \langle \mu, [X_i^{-\ell+n}, X_j^{-s+n}]\rangle_n, \qquad n \in \mathbf{Z}. \tag{4}$$

Symplectic leaves of the Lie-Poisson structures $W(n)$ are called generic orbits of the corresponding loop subalgebras acting on M^{N+1}. Two following cases are important: $\mathcal{O}_-^{gen}$ denotes the generic orbit for $n = -1$, and the generic orbit for $n = N+1$ will be denoted $\mathcal{O}_+^{gen}$.

Let H^ν, $\nu = 2, 3, \ldots, r+1$, be Casimir functions in the enveloping algebra of g. They are polynomials in the variables $\mu_k = (\mu, X_k)$ on the dual g^* of g. The substitution $\mu_k \mapsto \mu_k(\lambda) = \sum_{\ell=0}^{N+1} \mu_k^\ell \lambda^\ell$ provides the continuation of H^ν to $\mathbf{C}^\infty(M^{N+1})$:

$$H^\nu = \sum_{\alpha=0}^{\nu(N+1)} h_\alpha^\nu \lambda^\alpha, \qquad h_\alpha^\nu \in \mathbf{C}^\infty(M^{N+1}). \tag{5}$$

Theorem 1. *1. The functions $\{h_\alpha^\nu\}$ constitute an involutive collection in $\mathbf{C}^\infty(M^{N+1})$, relative to the Lie-Poisson brackets $W(-1)$ and $W(N+1)$.*
2. The functions $\{h_\alpha^\nu\}$, $\alpha \geq (\nu-1)(N+1)$, annihilate the Lie-Poisson brackets $W(-1)$.
3. The functions $\{h_\alpha^\nu\}$, $\alpha = 0, 1, \ldots, N+1$, annihilate the Lie-Poisson brackets $W(N+1)$.

Proof. Let $\widetilde{X}_i^{-\ell+n}(n)$ be the tangent vector field corresponding to the basis element $X_i^{-\ell+n}$ and the coadjoint action (2) of $\widetilde{g}_-$ (for $n=-1$) or $\widetilde{g}_+$ (for $n = N+1$). Then one can show that

$$\widetilde{X}_i^{-\ell+n}(n) = \sum_{k=1}^{\dim g}\sum_{r=0}^{N+1} W_{ik}^{\ell r}(n)\frac{\partial}{\partial \mu_k^r}. \tag{6}$$

Note that

$$\frac{\partial}{\partial \mu_k^r} = \frac{\partial \mu_k(\lambda)}{\partial \mu_k^r}\frac{\partial}{\partial \mu_k(\lambda)} = \lambda^r \frac{\partial}{\partial \mu_k(\lambda)}.$$

By (6) and (4),

$$\sum_\ell \left(\lambda^{\ell+1}\widetilde{X}_i^{-\ell-1}(-1) + \lambda^{\ell-N-1}\widetilde{X}_i^{-\ell+N+1}(N+1)\right) = \sum_{j,k} C_{ik}^j \mu_j(\lambda)\frac{\partial}{\partial \mu_k(\lambda)},$$

where C_{ik^j} denotes the structure constants of g. The ad^*-invariance of H^ν means that

$$\sum_{j,k} C_{ik}^{j} \mu_j(\lambda) \frac{\partial}{\partial \mu_k(\lambda)} H^\nu = 0,$$

or, by the previous formula,

$$\sum_{\ell} \left(\lambda^{\ell+1} \widetilde{X}_i^{-\ell-1}(-1) + \lambda^{\ell-N-1} \widetilde{X}_i^{-\ell+N+1}(N+1) \right) H^\nu = 0.$$

Substitute (5) herein and equate the coefficients at the same powers of λ. Then the following relations arise:

$$\widetilde{X}_i^{-\ell-1}(-1) h_\alpha^\nu = 0, \qquad \alpha \geq (\nu-1)(N+1); \tag{7}$$

$$\widetilde{X}_i^{-\ell+N+1}(N+1) h_\alpha^\nu = 0, \qquad 0 \leq \alpha \leq N+1; \tag{8}$$

$$\widetilde{X}_i^{-\ell-1}(-1) h_\alpha^\nu + \widetilde{X}_i^{-\ell+N+1}(N+1) h_\alpha^\nu = 0, \qquad N+1 < \alpha < (\nu-1)(N+1). \tag{9}$$

The second and third assertions of the theorem follow immediately from (7) and (8), respectively.

The first assertion is clear if $\nu \neq \mu$. Let $\nu = \mu$. Then (9) leads to the sequence of equations:

$$\{h_\alpha^\nu, h_\beta^\nu\}_{-1} = \{h_{\alpha-N-2}^\nu, h_{\beta+N+2}^\nu\}_{-1} = \cdots = \{h_{\alpha-m(N+2)}^\nu, h_{\beta+m(N+2)}^\nu\}_{-1},$$

where m is a natural number. For any pair of nonnegative integer numbers α and β that are less than $(\nu-1)(N+1)$, there exists a number m such that one of the following inequalities holds: $\alpha - m(N+2) < 0$, or $\beta + m(N+2) \geq (\nu-1)(N+1)$. The first one implies that $h_{\alpha-m(N+2)}^\nu \equiv 0$, and the second that $h_{\beta-m(N+2)}^\nu$ annihilates the Poisson structure $W(-1)$. In the both cases, the vanishing of the brackets $\{h_\alpha^\nu, h_\beta^\nu\}_{-1}$ is guaranteed. The involutivity of the functions h_α^ν and h_β^ν with respect to $\{,\}_{N+1}$ are proved in the same way.

Remark. This proof is a realization of the "Adler scheme" [7, 8] and carried out in a gradation invariant way. A.G.Reyman and M.A.Semenov-Tian-Shansky applied the R-matrix technique to prove an analogue of Theorem 1 for the homogeneous gradation [10, 11]. By Theorem 1, the generic orbit $\mathcal{O}_-^{gen}$ is a real algebraic manifold embedded into M^{N+1} by the constraints h_α^ν, $\alpha \geq (\nu-1)(N+1)$. Fixing the functions h_α^ν, $\alpha = 0, 1, \ldots, N+1$, determines the real algebraic manifold being the generic orbit $\mathcal{O}_+^{gen}$. Set Hamiltonian flows of the form

$$\frac{d\mu}{d\tau_\alpha^\nu} = \{\mu, h_\alpha^\nu\}_n = ad^*_{dh_\alpha^\nu}\mu = [\mu, dh_\alpha^\nu], \tag{10}$$

on $\mathcal{O}_-^{gen}$ (resp., $\mathcal{O}_+^{gen}$), where $\alpha < (\nu-1)(N+1)$ (resp., $\alpha > N+1$) and dh_α^ν is the differential of the Hamiltonian h_α^ν (τ_α^ν is the corresponding trajectory parameter).

In case $g \simeq sl(n)$, one checks easily that:

1. The dimensions of the generic orbits $\mathcal{O}_-^{gen}$ and $\mathcal{O}_+^{gen}$ are equal to (N+1)(n-1)n.
2. The number of nonannihilators on the generic orbits is equal to (N+1)(n-1)n/2.

3a. The functions h_α^ν, $(\nu-1)(N+1) \le \alpha < \nu(N+1)$, are functionally independent almost everywhere on M^N.

3b. The functions h_α^ν, $0 \le \alpha < (\nu-1)(N+1)$, are functionally independent almost everywhere on $\mathcal{O}_-^{gen}$.

4a. The functions h_α^ν, $0 < \alpha \le N+1$, are functionally independent on M^{N^ℓ}.

4b. The functions h_α^ν, $\alpha > N+1$, are functionally independent on $\mathcal{O}_+^{gen}$.

A consequence of Theorem 1 and propositions 1–4 is that the Hamiltonian flows (10) are integrable in the Liouville sense.

3. Integrable Hamiltonian systems on orbits of the algebra $su(3) \otimes \mathcal{P}(\lambda^{-1})$

Let $\widetilde{g} = sl(3,\mathbf{C}) \otimes \mathcal{P}(\lambda,\lambda^{-1})$ be an algebra of polynomial loops with values in the Lie algebra $sl(3,\mathcal{C})$ and let $\widetilde{g} = \widetilde{g}_+ + \widetilde{g}_-$ be the decomposition into a direct sum of two subspaces, as was described in the previuos section. Let $\{H_1, H_2, E_{\pm\alpha i} \equiv E_i, i = 1,2,3\}$ be the standard root basis in $sl(3,\mathbf{C})$. If A_i is a basis element in g, then $A_i^l = \lambda^l A_i, l \in \mathbf{Z}$ is a basis element in $\widetilde{g}$. Fix the following finite-dimensional subspace in $\widetilde{g}_+$:

$$M^{N+1} = \left\{ \mu \in \widetilde{g}_-^* |_\mu = \sum_{\ell=0}^{N+1} \left[\alpha_1^\ell H_1^\ell + \alpha_2^\ell H_2^\ell + \sum_{i=1}^{3} \left(E_{+i}^\ell \beta_i^\ell + E_{-i}^\ell \gamma_i^\ell \right) \right] \right\}.$$

M^{N+1} is manifestly invariant under the coadjoint action of $\widetilde{g}_-$. Define two Lie-Poisson brackets on M^{N+1}, putting $n = -1$ and $n = N+1$ in (3). We indentify the coadjoint orbits of $\widetilde{g}_-$ and $\widetilde{g}_+$ in the subspace M^{N+1} with the symplectic leaves of the corresponding Lie-Poisson brackets.

It follows from the explicit form of the first brackets $\{f_1, f_2\}_{-1}$ that the variables α_1^{N+1}, α_2^{N+1}, β_i^{N+1}, γ_i^{N+1}, are its annihilators. Thus, we may set

$$\beta_i^{N+1} = \gamma_i^{N+1} = 0, \qquad \alpha_{1,2}^{N+1} = \text{const} \neq 0. \tag{11}$$

This means that symplectic leaves are fully embedded into the subspace $M^N \subset M^{N+1}$, which location is fixed by the constants α_1^{N+1} and α_2^{N+1}.

The ad^*-invariant functions $I_2(\mu(\lambda)) = \dfrac{1}{2} Tr \mu^2(\lambda)$ and $I_3(\mu(\lambda)) = \dfrac{1}{3} Tr \mu^3(\lambda)$ are polynomials in λ of order $2(N+1)$ and $3(N+1)$, respectively:

$$I_2(\mu(\lambda)) = h_0 + \lambda h_1 + \cdots + \lambda^{2(N+1)} h_{2(N+1)},$$
$$I_3(\mu(\lambda)) = f_0 + \lambda f_1 + \cdots + \lambda^{3(N+1)} f_{3(N+1)}.$$

The coefficient functions h_α and f_β can be found easily. By Theorem 1, they are in involution with respect to the both Lie-Poisson brackets. The same holds for the restrictions of h_α and f_β onto the subspace M^N. Under this restriction, the functions $h_{2(N+1)}$ and $f_{3(N+1)}$ become fixed constants expressed via $\alpha_{1,2}^{N+1}$ and, hence, should be left out of the consideration.

The functions $h_{N+1}, h_{N+2}, \ldots, h_{2N+1}$ and $f_{2N+2}, f_{2N+3}, \ldots, h_{3N+2}$ ($2(N+1)$ functions) are Casimir functions with respect to the first brackets. By fixing their values, one obtains

$2(N+1)$ algebraic equations which determine a coadjoint orbit of the algebra $\widetilde{g}_-$ in the subspace M^N. The dimension of such a generic orbit is equal to $6(N+1)$. The $3(N+1)$ remaining functions give rise to nontrivial commuting flows on the orbit and provide the integrability in the sense of the Liouville theorem.

Choose the function h_{N-1} as the Hamiltonian and write down the corresponding Hamiltonian equations in terms of the coordinates $\{\mu_a^\ell\} \equiv \{\alpha_1^\ell, \alpha_2^\ell, \beta_1^\ell, \beta_2^\ell, \beta_3^\ell, \gamma_1^\ell, \gamma_2^\ell, \gamma_3^\ell\}$, $a = \overline{1,8}$,

$$\frac{d\mu_a^\ell}{d\tau_{N-1}} = \left\{\mu_a^\ell, h_{N-1}\right\}_{-1}. \tag{12}$$

The algebraic equations $h_\alpha = c_\alpha$, $\alpha = \overline{N+1, 2N+1}$, and $f_\beta = c_\beta$, $\beta = \overline{2N+2, 3N+2}$, can be solved for the variables α_1^ℓ and α_2^ℓ, $\ell = N, N-1, \ldots, 0$, expressing them via β_i^ℓ, γ_i^ℓ, $i = 1, 2, 3$. This implies that the coadjoint orbit of $\widetilde{g}_-$ is diffeomorphic to a flat space, with the variables β_i^ℓ, γ_i^ℓ being its global coordinates.

The structure of equations (12) allows us to express the variables $\beta_i^{\ell-1}$, $\gamma_i^{\ell-1}$ through β_i^ℓ, γ_i^ℓ and their first derivatives and, therefore, to reduce the system of $6(N+1)$ first order equations to six equations of order $N+1$ for the variables β_i^N, γ_i^N. These equations are interpreted as the higher stationary equations for a many-component nonlinear Shrodinger equation.

In order to ground this interpretation, reduce system (12) onto the orbit of the algebra $su(3) \otimes \mathcal{P}(\lambda^{-\infty})$. To this end, set $\alpha_{1,2}^\ell = \sqrt{-1} a_{1,2}^\ell$, $a_{1,2}^\ell$ being real numbers, and $\gamma_i^\ell = -\beta_i^{\ell *}$. Consider the Hamiltonian flow generated by h_{N-2}, $N > 1$, on the phase space reduced in the above sense:

$$\frac{d\beta_i^\ell}{dt} = \left\{\beta_i^\ell, h_{N-2}\right\}_{-1}. \tag{13}$$

Let the variables β_i^ℓ lie on trajectories of (12). This amounts to the following. First, β_i^ℓ depend on a "parameter" $\tau_{N-1} \equiv x$ and, second, they are expressed through β_i^N and their derivatives. Then the system (13) takes on the form:

$$\begin{aligned}
i\frac{d\beta_1}{dt} &= -\frac{\partial^2\beta_1}{\partial x^2} - \left(2|\beta_1|^2 - |\beta_2|^2 + \frac{1}{2}|\beta_3|^2\right)\beta_1 - \beta_3\frac{\partial\beta_3^*}{\partial x} - \frac{1}{2}\beta_2^*\frac{\partial\beta_3}{\partial x},\\
i\frac{\partial\beta_2}{\partial t} &= -\frac{\partial^2\beta_2}{\partial x^2} - \left(2|\beta_2|^2 - |\beta_1|^2 + \frac{1}{2}|\beta_3|^2\right)\beta_2 - \beta_3\frac{\partial\beta_1^*}{\partial x} - \frac{1}{2}\beta_1^*\frac{\partial\beta_3}{\partial x},\\
i\frac{\partial\beta_3}{\partial t} &= -\frac{\partial^2\beta_3}{\partial x^2} - \left(|\beta_1|^2 + |\beta_2|^2 + |\beta_3|^2\right)\beta_3 + \beta_2\frac{\partial\beta_1}{\partial x} - \beta_1\frac{\partial\beta_2}{\partial x},
\end{aligned} \tag{14}$$

where we denote $\beta_i^N \equiv \beta_i$. Equations (12) are higher stationary equations for the evolutionary system (14).

Let us focus on the degenerate orbit passing through the point $\{\mu_a^\ell\} \in M^N$ for which $\alpha_2^{N+1} = 2\alpha_1^{N+1}$, and consider the restriction of (12) onto this orbit. In the Lie algebra $g \simeq su(3)$, the point with the coordinates $\alpha_2 = 2\alpha_1$ remains invariant under the coadjoint action of the subgroup $SU(2) \times U(1)$. The corresponding orbit going through this point is the quotient $SU(3)/SU(2) \times U(1) \simeq \mathbf{CP}^2$.

The analogue of this in the space M^N is a $4(N+1)$-dimensional complex manifold with the structure of the vector fibre over $\mathbf{CP}^2$. The local coordinates on this manifold are $\beta_2^0, \beta_3^0, \beta_2^1, \beta_3^1, \ldots, \beta_2^N, \beta_3^N$. On the degenerate orbit, equation (14) takes the form:

$$\begin{aligned} i\frac{\partial \beta_2^N}{\partial t} &= -\frac{\partial^2 \beta_2^N}{\partial x^2} + 2\left(|\beta_2^N|^2 + |\beta_3^N|^2\right)\beta_2^N - 3ia_1^N \frac{\partial \beta_2^N}{\partial x}, \\ i\frac{\partial \beta_3^N}{\partial t} &= -\frac{\partial^2 \beta_3^N}{\partial x^2} + 2\left(|\beta_2^N|^2 + |\beta_3^N|^2\right)\beta_3^N - 3ia_1^N \frac{\partial \beta_3^N}{\partial x}. \end{aligned} \tag{15}$$

This system coincides with the two-component nonlinear Schrödinger equation.

4. Systems on orbits of the algebra $\widetilde{g}_+ \simeq su(3) \otimes \mathcal{P}(\lambda)$. The $SU(3)$-Heisenberg magnet equation

As is mentioned in the second section, there is the coadjoint action of the subalgebra $\widetilde{g}_+$ defined on the space M^{N+1}, apart from that of $\widetilde{g}_-$. The corresponding Lie-Poisson brackets have the form:

$$\{f_1, f_2\}_{N+1} = \sum \langle \mu, \left[X_i^{-\ell+N+1}, X_j^{-s+N+1}\right]\rangle_{N+1} \frac{\partial f_1}{\partial \mu_i^\ell}\frac{\partial f_2}{\partial \mu_j^s}. \tag{16}$$

In order to fix the relative orbit (symplectic leaf of the brackets (16)) $\mathcal{O}_2^N$, one equalizes $2(N+2)$ functions $h_0, h_1, \ldots, h_{N+1}$, $f_0, f_1, \ldots, f_{N+1}$, to constans.

The functions $h_{N+2}, \ldots, h_{2N+1}, f_{N+2}, \ldots, f_{3N+2}$ are independent almost everywhere on $\mathcal{O}_2^N \cap M^N$ and form an involutive collection relative to the Lie-Poisson brackets, in accordance with Theorem 1. This implies the complete integrability of the Hamiltonian systems generated by $h_{N+2}, \ldots, h_{2N+1}$, $f_{N+2}, \ldots, f_{3N+2}$. We pay special attention to the geometry of the phase space $\mathcal{O}_2^N$ and the systems generated by h_{N+2} and h_{N+3}.

To choose the dynamical variables $\mu_a = (\mu, X_a)$ conveniently, fix the following orthonormalized basis in the Lie algebra $su(3) : \{X_a,\ a = 1, 2, \ldots, 8\}$, where X_a are anti-Hermitian matrices, with the orthonormalization performed according to the scalar product $(A, B) = -2TrAB$. The commutation relations $[X_a, X_b] = C_{abc}X_c$, are fulfilled by matrices X_a, where the structure constants C_{abc} are real and antisymmetric. The polynomial $I_2 = \sum_{a=1}^{8} \mu_a^2$ and $I_3 = \sum_{a,b,c} d_{abc}\mu_a\mu_b\mu_c$ are independent ad^*-invariant Casimir functions on $su^*(3)$, where $d_{abc} = -2Tr(X_aX_bX_c + X_bX_aX_c)$. They are continued to functions on M^{N+1} by the substitution $\mu_a \longrightarrow \mu_a(\lambda) = \mu_a^0 + \lambda\mu_a^1 + \ldots + \lambda^{N+1}\mu_a^{N+1}$. Some of the coefficient functions h_α, f_β (see the previous subsection) have the form: $h_0 = \mu_a^0\mu_a^0$, $h_1 = 2\mu_a^1\mu_a^0, \ldots, f_0 = d_{abs}\mu_a^0\mu_b^0\mu_c^0$, $f_1 = 3d_{abc}\mu_a^1\mu_b^0\mu_c^0, \ldots$, where the summation over repeated indices is assumed.

The orbit $\mathcal{O}_+^N$ is fixed by the set of algebraic equations

$$h_\alpha = R_\alpha^2, \qquad f_\beta = C_\beta^3, \qquad \alpha, \beta = \overline{0, N+1}. \tag{17}$$

In particular, the equations $h_0 = \mu_a^0\mu_a^0 = R_0^2$, $f_0 = d_{abc}\mu_a^0\mu_b^0\mu_c^0 = C_0^3$ determine the generic coadjoint orbit in $su^*(3)$, which is diffeomorphic to the factor-space $SU(3)/U(1) \times U(1)$. The whole orbit $\mathcal{O}_+^N$ has the structure of the vector bundle over $SU(3)/U(1) \times U(1)$ (see eq.(17)).

Write down the Hamiltonian equations generated by h_{N+r} on the orbit $\mathcal{O}_+^N$:

$$\frac{d\mu_a^\ell}{dx} = \{\mu_a^\ell, h_{N+2}\}_{N+1} = f_{abc}\mu_c^0\mu_b^{\ell+1}, \tag{18}$$

that coincides with the higher stationary equations for the evolutionary system considered earlier in [14]. The latter can be obtained when reducing the Hamiltonian equations

$$\frac{d\mu_a^0}{dt} = \{\mu_a^0, h_{N+3}\}_{N+1} = f_{abc}\mu_c^0\mu_b^{\ell+2} \tag{19}$$

onto trajectories of system (18).

Degenerate orbits. Let the initial point $\mu_0 \in su^*(3)$ have the coordinates $(\mu_0, x_a) = 0$, $a = 1, 2, \ldots, 7$, $(\mu_0, x_8) = R_0$. The stationary subgroup of this point is isomorphic to $SU(2) \times U(1)$. Then the orbit passing through μ_0 is diffeomorphic to the factor-space $SU(3)/SU(2) \times U(1) \simeq \mathbf{CP}^2$ which is a two-dimensional complex projective space. There is a degenerate four-dimensional orbit that can be fixed by the system of quadrics in the space $su^*(3)$:

$$d_{abc}\mu_b\mu_c + \frac{R_0}{\sqrt{3}}\mu_a = 0,$$

amongst which there are only four independent. The whole degenerate orbit has the structure of a vector bundle over $\mathbf{CP}^2$.

The variables μ_a^ℓ, $\ell = 1, 2, \ldots, N$, are expressed via μ_a^0 and the derivatives $\mu_{a_x}^0$, $\mu_{a_{xx}}^0, \ldots$, on trajectories of system (18). In view of this, equation (19) reduced on the degenerate orbit renders the form:

$$\frac{\partial\mu_a^0}{\partial t} = \frac{4}{3I_2} f_{abc}\mu_b^0\mu_{c-xx}^0 \tag{20}$$

which represents an $SU(3)$-generalization of the dynamical equation for the continuous Heisenberg magnet [14].

5. Higher stationary equations of the principal hierarchy. The Boussinesq's equation

Let σ be a third order automorphism of the algebra $g = sl(3, \mathbf{C})$, and g_1, g_ω, g_{ω^2} be the eigenspaces of σ with the eigenvalues 1, ω, ω^2, respectively. Here, $\omega = \exp\dfrac{2\pi i}{3}$. Let $\widetilde{g}$ be a loop algebra with the principal gradation. This means that the following decomposition is given:

$$\widetilde{g} = \sum_l \oplus g_1^l \oplus g_\omega^l \oplus g_{\omega^2}^l,$$

where $g^l_{1,\ \omega,\ \omega^2} = \mathbf{C}X^l_{1,\ \omega,\ \omega^2}$ are the 1-dim. eigenspaces of the gradiation operator $d' = 3\lambda\dfrac{d}{d\lambda} + ad\ H$, $H = \mathrm{diag}(1, 0, -1)$. It is not difficult to see that:

$$X_\omega^{3k} = \lambda^k \mathrm{diag}(\omega^2, \omega, 1), \qquad X_{\omega^2}^{3k} = \lambda^k \mathrm{diag}(\omega, \omega^2, 1)$$

$$X_1^{1+3k} = \lambda^k \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda & 0 & 0 \end{bmatrix}, \qquad X_\omega^{1+3k} = \lambda^k \begin{bmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ \lambda & 0 & 0 \end{bmatrix},$$

$$X_{\omega^2}^{1+3k} = \lambda^k \begin{bmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ \lambda & 0 & 0 \end{bmatrix}, \qquad X_1^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$X_\omega^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ \omega^2 & 0 & 0 \\ 0 & \omega & 0 \end{bmatrix}, \qquad X_{\omega^2}^{2+3k} = \lambda^{k+1} \begin{bmatrix} 0 & 0 & \lambda^{-1} \\ \omega & 0 & 0 \\ 0 & \omega^2 & 0 \end{bmatrix}.$$

Elements X_1^{1+3k}, X_1^{2+3k} generate a maximaly commutative subalgebra in $\widetilde{g}$, which turns into Heisenberg subalgebra after central extension.

Let's consider coadjoint action of the subalgebra$\widetilde{g}_-(\sigma) = \mathrm{span}_{\mathbf{C}}\{X_1^{-1},\ X_1^{-2},\ X_\omega^{-2},\ X_{\omega^2}^{-2},$ $X_\omega^{-3},\ X_{\omega^2}^{-3},\ \ldots\}$ in the dual space $\widetilde{g}_-^*(\sigma) \sim \widetilde{g}_+(\sigma)$. The subspace

$$M^N = \{\mu \in \widetilde{g}_+(\sigma); \qquad \mu = \sum_{k=-2}^{3N-1} (\mu_1^k X_\omega^{*k} + \mu_\omega^k X_\omega^{*k} + \mu_{\omega^2}^k X_{\omega^2}^{*k})\},$$

is invariant with respect to this action. We define the Lie-Poisson bracket on this subspace according to formulas (3) and (4) and putting $n = -1$. The symplectic leaf (orbit) of this bracket is fixed by equations:

$$\mu_\omega^{3N-1} = \mu_{\omega^2}^{3N-1} = 0;$$

$$\mu_1^{3N-1} \neq 0,\ h_\alpha^2 = c_\alpha^2,\ \alpha = N, \ldots, 2N-2;\ h_\beta^3 = c_\beta^3,\ \beta = 2N-1, \ldots, 3N-2$$

The next step is reduction of the orbit over the Hamiltonian flow of h_{N-1}^2. On the reduced orbit, $\omega\mu_\omega^{3(N-1)} + \omega^2\mu_{\omega^2}^{3(N-1)} = \mu_\omega^{3(N-1)} + \mu_{\omega^2}^{3(N-1)} = \dfrac{1}{3}u$ and we can define the integrable system with the Hamiltonian h_{2N-2}^3 on it. It's integrability is provided by integrals: $h_0^2,\ h_1^2, \ldots,\ h_{N-2}^2;\ h_0^3,\ h_1^3, \ldots,\ h_{2N-2}^3$. This system could be written as higher-order differential equation for the function $u(x)$ and coincides with the higher stationary Boussinesq's equation.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 353–358.

Three-Gap Elliptic Solutions of the KdV Equation

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1 Introduction

The theory of finite-gap integration of the KdV equation is interesting both in applied problems of mathematical physics and in connection with the development of general methods for the solving of integrable nonlinear partial differential equations. The finite-gap solutions are the coefficient functions of auxiliary linear differential equations which have the Baker-Akhiezer eigenfunctions associated with determined complex algebraic curves and the respective Riemann surfaces (see [1, 2, 3]). With the aid of the Baker-Akhiezer functions, these solutions are expressed via theta functions in an implicit form. As parameters, ones contain characteristics of the Riemann surfaces, the computation of which is a special algebraic geometry problem (see [3, 4]).

In this paper, a simple method is proposed for calculating the three-gap elliptic solutions of the KdV equations in the explicit form. One is based on the usage of a system of trace formulae and auxiliary time evolution equations in the representation of the elliptic Weierstrass function ($\wp$-representation) [5]. It is shown that the initial three-gap elliptic solutions (at $t = 0$) are the linear combinations of $\wp$-functions with shifted arguments which are determined by the trace formulae. In view of the evolution equations, the three-gap elliptic solutions of the KdV equation are a double sum of $\wp$-functions with the time dependent shifts of poles. The number of terms in this sum is determined by the condition of a coincidence of the general expression with the initial conditions at $t \to 0$. It is shown that the time evolution of the finite-gap elliptic solution is determined through X_i-functions which are determined by the trace formulae and which are roots of the algebraic equations of corresponding orders.

In distinct to the known methods (see [6, 7]), our method is characterized by a simple and general algorithm which is valid for the computation of finite-gap elliptic solutions of the KdV equation in cases of arbitrary finite-gap spectra of the auxiliary linear differential equations.

2 The finite-gap equations

The finite-gap solutions of integrable nonlinear equations, in particular the KdV equation, are solutions of the spectral problem for auxiliary linear differential equations. In so doing, the first motion integral of these equations must be the polynomial in their eigenvalues E.

In the case of the KdV equation, the finite-gap solutions ($U(x,t)$) are solutions of the spectral problem of the Schrödinger equation $[-\partial_x^2 + U(x,t)]\Psi = E\Psi$ with the eigenfunctions (Ψ-functions) which satisfy the condition

$$\sqrt{P(E)} = \Psi_- \partial_x \Psi_+ - \Psi_+ \partial_x \Psi_- . \tag{1}$$

The right-hand side in (1) is the motion integral which follows from the known (see [8]) Ostrogradskii-Liouville formula $W(x) = W(x_0) \times \exp(\int\limits_{x_0}^{x} dt a_1(t))$ for the Wronski determinant $W(x)$ of the fundamental solutions Ψ_-, Ψ_+ (a_1 is the coefficient at the $(n-1)$th derivative in a linear differential equation of nth order). The dependence of the Ψ-function on the time t is described by the auxiliary linear equation $\partial_t \Psi = A\Psi$, where $A = 4\partial_x^3 - 3[U, \partial_x + U, \partial_x]$ is the time evolution operator (see [1]). From equation (1) in accordance with the asymptotic relation $\Psi \to \exp i\sqrt{E}x$, $E \to \infty$, the finite-gap Ψ-function has the form

$$\Psi = \sqrt{\chi_R(x,t,E)} \exp i \int\limits_{x_0}^{x} dx \chi_R(x,t,E), \tag{2}$$

where

$$\chi_R(x,t,E) = \Psi_- \Psi_+ = \frac{\sqrt{P(E)}}{\prod\limits_{i=1}^{g} (E - \mu_i(x,t))}$$

is equivalent to

$$\begin{aligned} \chi_R &= \sqrt{E} \left(1 + \sum_{n=1}^{\infty} A_n E^{-n} \right), \\ A_n &= \frac{1}{n!} \partial_z^n \left(\frac{\sqrt{\sum\limits_{n=0}^{2g+1} a_n z^n}}{\sum\limits_{n=0}^{g} b_n z^n} \right) \Big|_{z=0}. \end{aligned} \tag{3}$$

Here a_i are symmetrized products of the spectrum boundaries E_j of ith order and b_i are coefficient functions of x changing in the spectrum gaps.

On the other hand, with the aid of substitution (2) into the auxiliary Schrödinger equation, we obtain

$$\begin{aligned} \chi_R &= \sqrt{E} \left(1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1} E^{-(n+1)} \right), \\ \chi_{n+1} &= \partial_x \chi_n + \sum_{k=1}^{n-1} \chi_k \chi_{n-k},\ \chi_1 = -U(x), \end{aligned} \tag{4}$$

where the second recurrent equation determines the coefficient functions χ_n in the form of polynomials in U-functions and their derivatives.

The equalizing of the coefficient at similar powers of E of expressions (3) and (4) leads to the trace formulae

$$A_{n+1} = \frac{(-1)^n}{2^{2n+1}} \chi_{2n+1}, \tag{5}$$

which presents the system of equations which describe the finite-gap elliptic function $U(x,t)$.

The finite-gap elliptic solutions of the KdV equation admit the $\wp$-representation. Using this representation in the trace formulae (5) at the initial time $t = 0$ and comparing the Laurent expansion in $\wp$ of their left-hand and right-hand sides, we obtain the general expression

$$U(z) = \alpha_0 \wp(z) + \sum_i (\alpha_i \wp(z+\omega_i) + \beta_i(\wp(z+\varphi_i) + \wp(z-\varphi_i)) + C, \tag{6}$$

($\wp(z) \equiv \wp(z|\omega,\omega')$, $\omega_i = (\delta_{i,1} - \delta_{i,2})\omega + (\delta_{i,3} - \delta_{i,2})\omega'$) describing even initial finite-gap elliptic solutions of the KdV equation (see [6]). Here α, β and φ_i are unknown parameters, ω, ω' or ω, $\tau = \omega'/\omega$ are independent parameters. The constant C is determined from the condition of vanishing a constant in the Laurent expansion in $\wp$ of function (6). This correspond to a vanishing shift of the spectrum the Schrödinger equation. Then the mentioned unknown parameters are determined by substitution (6) in the trace formulae and comparison of the Laurent expansion in $\wp$ of their left- and right-hand sides.

3 Initial three-gap elliptic solutions

In the case of the three-gap spectrum, the unknown parameters of expression (6) and spectrum parameters a_i are described by the system of five trace formulae (5). Index n in these formulae receives the values $n = (0,1,2,3)$ and $b_i = 0$, $i \geq 3$. These four trace formulae are reduced to the equation

$$\begin{aligned} &-16a_2^2 + 64a_4 + 32a_3U(x) + 24a_2U(x)^2 + 35 * U(x)^4 - \\ &\quad 70 * U(x)U'(x)^2 - 8 * a_2U''(x) - 70 * U(x)^2U''(x) + \\ &\quad 21U(x)''^2 + 28U'(x)U'''(x) + 14U(x)U^{(4)}(x) - U^{(6)}(x) = 0, \end{aligned} \tag{7}$$

determining the unknown parameters of initial three-gap elliptic solutions of the KdV equation. Under the condition of vanishing of coefficients of the Laurent expansion in $\wp$ of the left-hand side (7), we obtain a closed system of algebraic equations for the mentioned parameters α, α_i, β_i and φ_i. The general relation $\alpha \neq 0$ and *a*) $\alpha_i = \beta_i = 0$; *b*) $\alpha_i \neq 0$, $\beta_i = 0$, *c*) $\alpha_i = 0$, $\beta_i \neq 0$, following from these equations determine three kinds of initial three-gap elliptic solutions. In the case *a*), the substitution of expression (6) into equation (7) under the condition of vanishing the coefficients of its Laurent expansion in $\wp$ gives $\alpha = 12$. In so doing, from (6) we obtain the expression

$$U(z) = 12\wp(z) \tag{8}$$

for the well-known [5] three-gap Lamé potential.

In the case *b*), the parameters α_i have the form $\alpha_i = \sum_{j=1}^{m} \delta_{i,j} \times \text{const}$, $j = (1,2,3)$. In so doing, the substitution of expression (6) in equation (7) and nuliifying coefficients of the Laurent expansion of its left-hand side give a simple algebraic equation for the unknown parameter. Taking into account values of these parameters in (6), we obtain the expressions

$$U(z) = 12\wp(z) + 2\wp(z+\omega_i) - 2e_i, \qquad e_i = \wp(\omega_i) \tag{9}$$

$$U(z) = 12\wp(z) + 2(\wp(z+\omega_i) + \wp(z+\omega_j)) - 2(e_i + e_j) \tag{10}$$

($e_i = \wp(\omega_i)$) which describe the initial three-gap elliptic solutions of the KdV equation which are similar to the well-known [9] two-gap Treibich-Verdier potentials.

In the case of relations c), the parameters β_i have the form $\alpha_i = \sum_{j=1}^{m} \delta_{i,j}\text{const}$, $j = (1,2,3)$. Then the substitution of expression (6) into equation (7) and nullifying of coefficients of the Laurent expansion in $\wp$ of its left-hand side lead to $\alpha = 12$, $\beta = 2$. In so doing, the condition of vanishing the coefficient of the fifth order pole at the point $\wp = h$, $h = \wp(\varphi)$ determines the equation

$$h^6 + \frac{101}{196}g_2h^4 + \frac{29}{49}g_3h^3 - \frac{43}{784}g_2^2h^2 - \frac{23}{196}g_2g_3h - \left(\frac{1}{3136}g_2^3 + \frac{5}{98}g_3^2\right) = 0$$

for $h = h(\varphi)$. Six values of h determine six values of the parameter $\varphi = \wp^{-1}(h)$. The substitution of the obtained parameters into (6) leads to the expression

$$U(z) = 12\wp(z) + 2(\wp(z+\varphi) + \wp(z-\varphi)) - 4\wp(\varphi) \tag{11}$$

describing the initial three-gap elliptic solutions of the KdV equations similar to the two-gap potential in [6].

4 Dynamics of three-gap elliptic solutions

The time dependent three-gap elliptic solutions of the KdV equation are solutions of the system involving both the trace formulae (5) and the above-mentioned auxiliary evolution equation. Substitution (2) into the last equation and separation of the real and imaginary parts transform one to the form

$$\partial_t\chi_R(x,t,E) = \partial_x\{(\lambda\chi_R(x,t,E)\}, \quad \lambda = -2(U(x,t,E) + E)$$

Relation (3) reduces the last equality to

$$\partial_t b_n - 2\{b_n\partial_x U - U\partial_x b_n + 2\partial_x b_{n+1}\} = 0, \quad n = (1,\ldots,g) \tag{12}$$

(g means the number of spectral gaps), where b_n-functions in view of the trace formulae (5) are polynomials in U-functions and their derivatives. Substitution of the U-function in the $\wp$-representation into equation (12) and a comparison of the Laurent expansions of their left-hand and right-hand sides give simple algebraic equations determining the general form and the time dependence of the U-function. The general finite-gap elliptic solutions of the KdV equation have the form

$$U(z,t) = 2\sum_{i=1}^{N}\wp(z-\varphi_i(t)) + C, \tag{13}$$

where the number N and the constant C are determined by the condition of coincidence of (13) with the initial finite-gap elliptic solutions at $t \to 0$. In so doing, the dynamic equations (12) are reduced to the equations

$$\begin{aligned} &\partial_t \varphi_i(t) = -12 X_i(t) + C,\ X_i(\varphi_i(t)) = \sum_{\substack{i \neq j \\ i=1}}^{N-1} \wp(\varphi_i(t) - \varphi_j(t)),\ (g \geq 2), \\ &\sum_{n=1}^{N} \partial_t \wp(z - \varphi_i(t)) = 0, \quad (g = 1) \end{aligned} \tag{14}$$

describing the time evolution of poles φ_i. Here, the N functions $X_i(t) = X_i(h_i(\varphi_i(t)))$ are determined by the trace formula (5) with the index $n = N - 1$. A compariszon of the Laurent expansions in $\wp$ of the left-hand and right-hand sides of the last equation leads to the algebraic equation of Nth order for the unknown functions $X_i(h_i)$. Then (14) can be transformed to the equality

$$\int\limits_{\varphi_{0i}}^{\varphi_i} \frac{d\varphi_i}{X_i(h_i(\varphi_i))} = -12t, \tag{15}$$

describing the dynamics of poles in expression (13). Here, initial values φ_{0i} are determined from the initial conditions describing by expressions (8)-(11) in the three-gap case. Thus, the problem of time evolution of three-gap elliptic solutions of the KdV equation is reduced to computation of the functions $X_i(h_i(\varphi_i))$.

In the three-gap case, the condition of coincidence of the general expression (13) with (8)–(11) at $t \to 0$ leads to $N = (6, 7, 8)$.

The values $N = 6$ and $N = 7$ determine two three-gap elliptic solutions of the KdV equation with the initial conditions (8) and (9), respectively. The substitution of expression (13) with $N = 6$ and $N = 7$ in the trace formulae with the index $n = 5$ and $n = 6$, respectively, and a usage of the Laurent expansion in $\wp$ lead to the algebraic equations

$$\sum_{i=0}^{6} c_i^{(6)}(h) X^i = 0, \qquad \text{and} \qquad \sum_{i=0}^{7} c_i^{(7)}(h) X^i = 0.$$

of the 6th and 7th orders, respectively. Here, the coefficients $c_i^{(6)}(h)$ and $c_i^{(7)}(h)$ are rational functions of $h(\varphi)$ and $h'(\varphi)$; $c_6^{(6)}(h) = c_7^{(7)}(h) = 1$. The roots of these equations coincide with the functions $X_i = \sum_{j=1}^{6} \wp(\varphi_i - \varphi_j)$ and $X_i = \sum_{j=1}^{7} \wp(\varphi_i - \varphi_j)$ which enter into the dynamic equation (15).

The substitution $N = 8$ into (13) gives the three-gap elliptic solution of the KdV equation with the possible initial conditions (10) and (11). In so doing, X_i-functions of the dynamic equation (15) are determined by the trace formula (5) with the index $n = 7$. In this case, a comparison of the Laurent expansions in $\wp$ of the left-hand and right-hand sides of the last equation leads to the algebraic equation of the 8th order

$$\sum_{i=0}^{8} c_i^{(8)}(h)^8 X^i = 0, \qquad c_8^{(8)}(h) = 1.$$

The solutions of this equation coincide with the functions $X_i = \sum_{j=1}^{8} \wp(\varphi_i - \varphi_j)$ entering into the dynamic equation (15).

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 359–365.

On a Construction Leading to Magri-Morosi-Gel'fand-Dorfman's Bi-Hamiltonian Systems

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Abstract

We present a method of generating Magri-Morosi-Gel'fand-Dorfman's (MMGD) bi-Hamiltonian structure leading to complete integrability of the associated Hamiltonian system and discuss its applicability to study finite-dimensional Hamiltonian systems from the bi-Hamiltonian point of view. The approach is applied to the finite-dimensional, non-periodic Toda lattice.

1 Introduction

The generalization due to Lichnerowicz [1] of symplectic manifolds to Poisson ones gives a possibility to reconsider the basic notions of the Hamiltonian formalism accordingly. As a fundamental tool in this undertaking, we use the Schouten bracket [2] of two contravariant objects $Q^{i_1,\dots,i_q}$ and $R^{j_1,\dots,j_r}$, given on a local coordinate chart of a manifold M by

$$
\begin{aligned}
[Q,R]^{i_1,\dots,i_{q+r-1}} = {} & \left(\sum_{k=1}^{q} Q^{(i_1,\dots,i_{k-1}|\mu|i_k,\dots,i_{q-1}}\right)\partial_\mu R^{i_p,\dots,i_{q+r-1})} + \\
& \left(\sum_{k=1}^{q} (-1)^k Q^{\{i_1,\dots,i_{k-1}|\mu|i_k,\dots,i_{q-1}}\right)\partial_\mu R^{i_q,\dots,i_{q+r-1}\}} - \\
& \left(\sum_{l=1}^{r} R^{(i_1,\dots,i_{l-1}|\mu|i_l,\dots,i_{r-1}}\right)\partial_\mu Q^{i_r,\dots,i_{q+r-1})} - \\
& \left(\sum_{l=1}^{r} (-1)^{qr+q+r+l} R^{\{i_1,\dots,i_{l-1}|\mu|i_l\dots i_{r-1}}\right)\partial_\mu Q^{i_r,\dots,i_{q+r-1}\}}.
\end{aligned}
\tag{1}
$$

Here the brackets (,) and { , } denote the symmetric and skew-symmetric parts of these contravariant quantities respectively. We note that throughout this paper the bracket [,] is that of Schouten (1). As is follows from formula (1), even the usual commutator of two vector fields $[X,Y]$, $X,Y \in TM$, is a sub-case of the general formula (1).

Consider a Poisson manifold (M,P), i.e., a differential manifold M equipped with a Poisson bi-vector P, that is a skew-symmetric, 2-contravariant tensor satisfying the vanishing condition: $[P, P] = 0$. Then a general Hamiltonian vector field X_H defined on (M, P) takes the following form on a local coordinate chart $x^1, \ldots, x^{2n}$ of M:

$$x^i(t) = X^i(x) = P^{i\alpha}\frac{\partial H(x)}{\partial x^\alpha}.$$

Here H is the Hamiltonian function (total energy). We use the Einstein summation convention. Comparing this formula with (1), we come to the conclusion that it can be rewritten in the following coordinate-free form, employing the Schouten bracket:

$$X_H = [P, H] \tag{2}$$

Note that the dimension of M may be arbitrary: not necessarily even. An example of an odd-dimensional Hamiltonian system defined by (2) is the Volterra model (see, for example [3]). An alternative way of defining the basic notions of the Hamiltonian formalism is by employing the Poisson calculus (see, for example, [4, 5]).

Integrability of the Hamiltonian systems (2) defined on an even-dimensional manifold is the subject to the classical Arnol'd-Liouville's theorem [7, 6]. We study the bi-Hamiltonian approach to the Arnol'd-Liouville's integrability originated in works by Magri [8], Gel'fand & Dorfman [9] and Magri & Morosi [10]. It concerns the following systems. Given a Hamiltonian system (2), assume it admits two distinct bi-Hamiltonian representations, i.e.,

$$X_{H_1,H_2} = [P_1, H_1] = [P_2, H_2], \tag{3}$$

provided that P_1 and P_2 are compatible: $[P_1, P_2] = 0$. We call systems of the form (3), that is the quadruples $(M, P_1, P_2, X_{H_1,H_2})$ defined by compatible Poisson bi-vectors — *Magri-Morosi-Gel'fand-Dorfman's (MMGD) bi-Hamiltonian systems.* To distinguish them from the bi-Hamiltonian systems with incompatible Poisson bi-vectors, see Olver [11], Olver and Nutku [12] and Bogoyavlenskij [13]. The bi-Hamiltonian approach to integrability of the MMGD systems is the subject of the theorem that follows.

Theorem 1 (Magri-Morosi-Gel'fand-Dorfman) *Given an MMGD bi-Hamiltonian system: $(M^{2n}, P_1, P_2, X_{H_1,H_2})$. Then, if the linear operator $A := P_1P_2^{-1}$ (assuming P_2 is non-degenerate) has functionally independent eigenvalues of minimal degeneracy, i.e., — exactly n eigenvalues, the dynamical system determined by the vector field X_{H_1,H_2}*

$$\dot{x}(t) = X_{H_1,H_2}(x)$$

is completely integrable in Arnol'd-Liouville's sense.

Remark 1. This approach to integrability gives a constructive way to derive the set of n first integrals for the related MMGD bi-Hamiltonian system:

$$I_k := \frac{1}{k}Tr(A^k), \qquad k = 1, 2, \ldots \tag{4}$$

The result of Theorem 1 suggests to pose the following problem: *Given a Hamiltonian system (2). How to transform such a system into the MMGD bi-Hamiltonian form?* This representation definitely will allow us to study integrability of the related Hamiltonian vector field.

2 A constructive approach to the bi-Hamiltonian formalism

We note that the problem posed below can be easily solved in some instances, for example, if we study a system with two degrees of freedom. Then the bi-Hamiltonian construction may be defined by two Poisson bi-vectors P_1, P_2 with constants coefficients, which are easy to work with (see [14]).

In general, the problem is far from being simple. The compatibility imposes a complex condition on P_1 and P_2, and so it would be desirable to circumvent this difficulty.

To solve the problem, we introduce the following geometrical object.

Definition 1 *Given a Hamiltonian system (M^{2n}, P, X_H). A vector field $Y_P \in TM^{2n}$ is called a master locally Hamiltonian (MLH) vector field of the system if it is not locally Hamiltonian with respect to the Poisson bi-vector (i.e., $L_{Y_P}(P) \neq 0$), while the commutator $[Y_P, X_H]$ is:*

$$L_{[Y_P, X_H]}(P) = 0.$$

We note that the notion of an MLH vector field is in a way reminiscent of the notion of *master symmetry* (MS) introduced by Fuchssteiner [15]:

Definition 2 *A vector field $Z \in TM^{2n}$ is called the master symmetry (MS) of a vector field $X \in TM^{2n}$ if it satisfies the condition*

$$[[Z, X], X] = 0,$$

provided that $[Z, X] \neq 0$.

For a complete classification of master symmetries related to integrable Hamiltonian systems, see [16].

Theorem 2 *Let (M, P, X_H) be a Hamiltonian system defined on a non-degenerate Poisson (symplectic) manifold (M, P). Suppose, in addition, that there exists an MLH vector field $Y_P \in TM$ for X_H. Then, if X_H is a Hamiltonian vector field with respect to $\tilde{P} = L_{Y_P}(P) : X_H = [\tilde{P}, \tilde{H}]$ (it implies that $\tilde{P}$ is a Poisson bi-vector and there is a second Hamiltonian $\tilde{H}$), X_H is an MMGD bi-Hamiltonian vector field:*

$$X_{H,\tilde{H}} = [P, H] = [\tilde{P}, \tilde{H}].$$

We note that the proof of this theorem employs some basic properties of the Schouten bracket [5].

Using this theorem, we can circumvent difficulties connected with finding a second Poisson bi-vector for a given Hamiltonian system. It is definitely easier to find a suitable MLH vector field (non-Hamiltonian symmetry) generating a Poisson bi-vector compatible with the initial one, rather than a second Poisson bi-vector itself, for it is easier in general to find a "vector" (i.e., MLH vector field or non-Hamiltonian symmetry), than a "matrix" (i.e., Poisson bi-vector). Besides, in this case the pair of Poisson bi-vectors so constructed is automatically compatible; we do not need to verify the compatibility condition, which, as was already mentioned, is in general a very complicated problem (for instance, see [17]).

3 Application

As an example illustrating this approach, consider the non-periodic, finite-dimensional Toda lattice, i.e., the system of equations that describes the dynamics of a one-dimensional lattice of particles with exponential interaction of nearest neighbors. In terms of the canonical coordinates q^i and momenta p_i, $(i = 1, 2, \ldots, n)$ it is given by (with the understanding that $e^{q^0-q^1} = e^{q^n-q^{n+1}} = 0$)

$$\begin{aligned} \mathrm{d}q^i/\mathrm{d}t &= p_i, \\ \mathrm{d}p_i/\mathrm{d}t &= e^{q^{i-1}-q^i} - e^{q^i-q^{i+1}}, \end{aligned} \tag{5}$$

where $q^i(t)$ can be interpreted as the coordinate of the i-th particle in the lattice. This system takes the Hamiltonian form (2) with the Hamiltonian function H_0 defined by

$$H_0 := \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q^i-q^{i+1}},$$

while the corresponding Poisson bi-vector P_0 is canonical. We obviously deal with the same Poisson manifold: $(\mathbb{R}^{2n}, P_0)$.

Consider the following vector field:

$$Y_P = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} p_j \frac{\partial}{\partial q^i} + \left(-\sum_{i=1}^{n-1} e^{q^i-q^{i+1}} + \frac{1}{2}\sum_{i=1}^{n} p_i^2\right)\frac{\partial}{\partial p_i}. \tag{6}$$

Direct verification shows that the vector field (6) is an MLH vector field for system (5). The result of its action on the canonical Poisson bi-vector through the Lie derivative is the following tensor: $P_1 = L_{Y_P}(P_0) = [Y_P, P_0] = \sigma_0 Y_P$, where

$$P_1 = \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} \frac{\partial}{\partial p_{i+1}} \wedge \frac{\partial}{\partial p_i} + \sum_{i=1}^{n} p_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} + \frac{1}{2}\sum_{i<j} \frac{\partial}{\partial q^j} \wedge \frac{\partial}{\partial q_i}.$$

We first observe that the Hamiltonian vector field X_H of (5) can be expressed by means of this tensor and the function $H_1 = \sum_{i=1}^{n} p_i$, which is its first integral:

$$X_H = P_1^{i\alpha} H_{1,\alpha}.$$

The last expression is not quite yet the Hamiltonian representation (2), since at this point it is unclear whether the tensor P_1 is a Poisson bi-vector or not. The direct checking of the condition $[P, P] = 0$ on the tensor P_1 would involve a lot of computational stamina, and so we observe further that P_1 can be expressed as follows:

$$P_1 = P_0 \omega_1 P_0, \tag{7}$$

where P_0 is the canonical Poisson bi-vector, while ω_1:

$$\omega_1 := \sum_{i=1}^{n-1} e^{q^i-q^{i+1}} \mathrm{d}q^i \wedge \mathrm{d}q^{i+1} + \sum_{i=1}^{n} p_i \mathrm{d}q^i \wedge \mathrm{d}p_i + \frac{1}{2}\sum_{i<j}^{n} \mathrm{d}p_i \wedge \mathrm{d}p_j$$

is the second symplectic structure for system (5) found in [17] and proved to be compatible with the canonical symplectic form $\omega_0 := P_0^{-1}$. Relation (7) is equivalent to the vanishing: $[P_1, P_1] = 0$, which means that P_1 is a Poisson bi-vector. This fact follows from the following formula, taking into account the compatibility in terms of the corresponding Nijenhuis tensor of P_0 and ω_1:

$$\begin{aligned}&[P_1, P_1](\alpha, \beta) = \\ &\quad A[P_0, P_0](A^t\alpha, \beta) - A[P_0, P_0](A^t\beta, \alpha) + N_A(P_0\alpha, \omega_1^{-1}\beta) - AP_0\mathrm{d}\omega_1(P_0\alpha, P_0\beta),\end{aligned} \tag{8}$$

where $\alpha, \beta \in T^*(M), A := P_0\omega_1(A^t := \omega_1 P_0)$ and N_A is the corresponding Nijenhuis tensor. Clearly, $[P_1, P_1 = 0]$ holds in view of compatibility of ω_0 and ω_1 [17]. Note that for the first time a formula analogous to (8) for a pre-symplectic form appeared in [10].

Now applying Theorem 2, we draw the conclusion: P_0 and P_1 constitute a compatible Poisson pair, the non-periodic, finite-dimensional, Toda lattice is an MMGD bi-Hamiltonian system defined by the Poisson bi-vectors P_0, P_1 and the corresponding Hamiltonians H_0, H_1. Now one can construct the set of first integrals $I_1, ...I_n$(mutually in involution, according to Theorem 1) by employing the formula $I_i := \frac{1}{i}\mathrm{Tr}(\tilde{A}^i)$, where $\tilde{A} := P_1 P_0^{-1}$. We present here a few first integrals obtained by using this method.

$$\frac{1}{2}I_1 = \frac{1}{2}\mathrm{Tr}(A) = \sum_{i=1}^{n} p_i = H_1,$$

$$\frac{1}{2}I_2 = \frac{1}{4}\mathrm{Tr}(A^2) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}} = H_0,$$

$$\frac{1}{2}I_3 = \frac{1}{4}\mathrm{Tr}(A^3) = \frac{1}{3}\sum_{i=1}^{n} p_i^3 + \sum_{i=1}^{n-1}(p_i + p_{i+1})e^{q_i - q_{i+1}},$$

$$\frac{1}{2}I_4 = \frac{1}{8}\mathrm{Tr}(A^4) = \frac{1}{4}\sum_{i=1}^{n} p_i^4 + \sum_{i=1}^{n-1}\Big((p_i^2 + p_{i+1}^2 + p_i p_{i+1})e^{q_i - q_{i+1}} + \frac{1}{2}e^{2(q_i - q_{i+1})} + e^{q_i - q_{i+2}}\Big),$$

where $A = P_1 P_0^{-1}$.

4 Concluding remarks

We have presented a method of transforming a Hamiltonian system into a bi-Hamiltonian system in the MMGD sense, which may (under some extra assumptions) lead to its complete integrability according to the Magri-Morosi-Gel'fand-Dorfman scheme (see Theorem 1). It has a certain attractiveness: the second Poisson bi-vector generated from the initial one is compatible with the latter. Thus, the condition of compatibility leading to complete integrability in Arnol'd-Liouville's sense of the corresponding Hamiltonian system in this case is assured. Besides, the concepts of Poisson calculus have proved to form a quite natural setting for the Hamiltonian formalism. This method was also employed in [18] to study the classical Kepler problem from the bi-Hamiltonian point of view. A similar problem was studied by Kosmann-Schwarzbach [19]: there was presented a method of transforming a Hamiltonian system into the bi-Hamiltonian form based on the existence of an appropriate Lax representation.

The bi-Hamiltonian formalism emerged in the theory of soliton equations [8], and so it is natural to apply this method to the soliton systems as well. The work in this direction is under way.

Acknowledgements

The author acknowledges with gratitude the stimulating discussions on the subject of this work with Oleg Bogoyavlenskij, Jerrold Marsden and Tudor Ratiu.

Many thanks to the organizers for the invitation to participate in the Wilhelm Fushchych Memorial Conference on *Symmetry in Nonlinear Mathematical Physics*, where this work has been presented.

The research was supported in part by the National Science and Engineering Research Council of Canada.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 366–371.

On Some Integrable System of Hyperbolic Type

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Abstract

The example of an integrable hyperbolic system with exponential nonlinearities, which somewhat differs from known integrable systems of the Toda type as well as its local conservation laws and reductions is presented. A wide class of exact solutions in some particular case of the system is found.

1 Introduction

A large class of hyperbolic systems with exponential nonlinearities of the type

$$u^i_{xt} = \sum_{j=1}^{m} a^i_j \exp\left(\sum_{k=1}^{n} b^j_k u^k\right), \quad i = 1, \ldots, n, \tag{1}$$

as is known, can be written as the zero-curvature condition (see, for example, [1])

$$P_t - Q_x + [P, Q] = 0, \tag{2}$$

where $P = P(x, t, \zeta)$ and $Q = Q(x, t, \zeta)$ are two matrix-functions having a simple pole at $\zeta = \infty$ and $\zeta = 0$, respectevely. Representation (2) is one of the distinguished features of partial differential equations in one spatial and one temporal dimensions, which have the infinite sequences of symmetries and local conservation laws.

The aim of this work is to present an example of hyperbolic systems, which has representation (2) and, in the same time, somewhat differs from systems of the type (1). Also we present possible reductions of this system, which can be interesting from the physical point of view.

2 Auxiliary linear problem

To begin with, we consider a linear problem in the form of the first-order system

$$\Psi_x = P(x, \zeta)\Psi, \tag{3}$$

where $\Psi = (\Psi_1, \Psi_2, \Psi_3)^T$ is the column-vector depending upon the variable $x \in \mathbf{R}^1$ and spectral parameter $\zeta \in \mathbf{C}^1$. Matrix P is written down in its explicit form as follows:

$$P(x, \zeta) = \begin{pmatrix} -i\zeta + r^1 & 1 & r^3 \\ i\zeta r^2 & -2r^1 & -i\zeta r^2 \\ r^3 & 1 & i\zeta + r^1 \end{pmatrix}. \tag{4}$$

Thus, the dependence of matrix elements of $P(x,\zeta)$ on the spatial variable $x \in \mathbf{R}^1$ is defined by the collection of complex-valued functions $\{r^i = r^i(x),\ i = 1,2,3\}$, which are assumed to be smooth everywhere in some domain.

Let us give some remarks about notations. In what follows, we shall omit notations to indicate the evolution parameters dependence. Any vector-function $(r^1, r^2, r^3)^T$ will be denoted by r. We denote a ring of differential functions of r by A_r and a ring of matrix differential operators with coefficients from A_r by $A_r[\partial_x]$.

The linear system (3) is intimately linked with other linear problem. Let us consider the linear equation

$$Ly = (i\zeta)^2 My, \tag{5}$$

where $L = \partial_x^3 + q^1(x)\partial_x + q^2(x)$ and $M = \partial_x + q^3(x)$ are two linear differential operators. Eq.(5) can be used as an auxiliary linear problem for bi-Hamiltonian evolution equations [6]

$$q_{\tau_n} = \mathcal{E}\mathrm{grad}_q H_n = \mathcal{D}\mathrm{grad}_q H_{n+2} \tag{6}$$

with the sequence of Hamiltonians $H_n = \int\limits_{-\infty}^{+\infty} h_n dx$. The sequence of Hamiltonian densities h_n, in one's turn, can be calculated as logarithmic derivative coefficients

$$\psi^{-1}\psi_x = -i\zeta + \sum_{k=0}^{\infty} \frac{h_k[q]}{(i\zeta)^k}$$

of the formal solution of Eq.(5)

$$\psi(x,\zeta) = \mathrm{e}^{-i\zeta x} \sum_{j=0}^{\infty} \frac{\psi_j(x)}{(i\zeta)^j}.$$

Several first Hamiltonian densities $h_n \in A_q$ read as

$$h_0 = \frac{1}{2}q^3, \quad h_1 = \frac{1}{2}q^1 + \frac{3}{8}(q^3)^2, \quad h_2 = -\frac{1}{2}q^2 + \frac{1}{2}q^1 q^3 + \frac{1}{2}(q^3)^3,$$

$$h_3 = \frac{1}{8}(q^1)^2 + \frac{105}{128}(q^3)^4 + \frac{15}{16}q^1(q^3)^2 - \frac{3}{4}q^2 q^3 - \frac{3}{8}q^1 q^3_x - \frac{15}{32}(q^3_x)^2, \text{ etc.}$$

Now we look for the Hamiltonian Miura map linking with the 'second' Hamiltonian structure $\mathcal{E} \in A_q[\partial_x]$, which is explicitly given by

$$\mathcal{E} = \begin{pmatrix} 4\partial_x^3 + 4q^1\partial_x + 2q^1_x & -2\partial_x^4 - 2q^1\partial_x^2 + (6q^2 - 4q^1_x)\partial_x + (4q^2_x - 2q^1_{xx}) & 0 \\ 2\partial_x^4 + 2q^1\partial_x^2 + 6q^2\partial_x + 2q^2_x & -\frac{4}{3}\partial_x^5 - \frac{8}{3}q^1\partial_x^3 - 4q^1_x\partial_x^2 + (4q^2_x - 4q^1_{xx} - \frac{4}{3}(q^1)^2)\partial_x + (2q^2_{xx} - \frac{4}{3}q^1_{xxx} - \frac{4}{3}q^1 q^1_x) & 0 \\ 0 & 0 & \frac{4}{3}\partial_x \end{pmatrix}. \tag{7}$$

By definition [2], the noninvertible differential relationship $q = F[r]$ is a Miura map for a certain Hamiltonian operator $\tilde{\mathcal{E}} \in A_r[\partial_x]$ if one generates the transformation $A_r[\partial] \to A_q[\partial]$ by virtue of the relation

$$\mathcal{E}|_{q=F[r]} = F'[r]\tilde{\mathcal{E}}(F'[r])^\dagger, \tag{8}$$

where $F'[r] \in A_r[\partial_x]$ is a Fréchet derivative. To get one of the posible solutions of (8), we try out the factorization approach. If we require, for example, $L = \left(\partial_x + 2r^1\right) \times \left(\partial_x - r^1 - r^3\right)\left(\partial_x - r^1 + r^3\right)$, while $q^3 = 2(r^1 - r^2)$, the linear equation (5) may be rewritten equivalently as (3).

To write more suitable variables in a more symmetric form, the modifications of (6) and the hyperbolic system associated with (1) can be introduced as follows: $r^1 = \frac{1}{12}v^1 + \frac{1}{6}v^2 + \frac{1}{6}v^3$, $r^2 = \frac{1}{4}v^1$, $r^3 = -\frac{1}{4}v^1 + \frac{1}{2}v^2$. Then, as can be checked by direct calculation, the operator

$$\tilde{\mathcal{E}} = \begin{pmatrix} 0 & 0 & -8 \\ 0 & -4 & 0 \\ -8 & 0 & 0 \end{pmatrix} \partial_x \tag{9}$$

solves relation (8) with the corresponding ansatz $q = F[v]$. It is obvious that this operator is Hamiltonian [3]. Also, we can calculate Hamiltonian densities for the modified evolution equations

$$\tilde{h}_n = h_n|_{q=F[v]} \in A_v \ (\mathrm{mod}\ \mathrm{Im}\,\partial_x)$$

to obtain

$$\tilde{h}_0 = \frac{1}{6}(-v^1 + v^2 + v^3),\ \ \tilde{h}_1 = -\frac{1}{8}\left(v^1v^3 + (v^2)^2\right),\ \ \tilde{h}_2 = \frac{1}{16}v^1v^3\left(v^1 - 2v^2 - v^3\right) -$$

$$-\frac{1}{16}\left(v^1v^3_x - v^3v^1_x\right),\ \ \tilde{h}_3 = \frac{1}{32}\left(4v^1_xv^3_x + (v^2_x)^2\right) + \frac{3}{16}v^1v^3\left(v^1_x + v^3_x\right) + \frac{3}{32}v^2\left(v^1v^3_x - v^3v^1_x\right) +$$

$$+\frac{1}{128}\left(9(v^1v^3)^2 + (v^2)^4\right) - \frac{1}{32}v^1v^3\left((v^1)^2 + (v^3)^2\right) + \frac{3}{32}v^1v^2v^3\left(v^1 - \frac{1}{2}v^2 - v^3\right), \text{ etc.} \tag{10}$$

From the form of the Hamiltonian structure $\mathcal{E}$ given by (9), it is evident that every system of the modified hierarchy $v_{\tau_n} = \tilde{\mathcal{E}}\mathrm{grad}_v\tilde{H}_n$ can be written in potential form. After introducing potentials $v_i = u_{ix}$, the first nontrivial system in these variables is explicitly given by

$$\begin{cases} u^1_{\tau_2} = -u^1_{xx} - \dfrac{1}{2}(u^1_x)^2 + u^1_xu^2_x + u^1_xu^3_x, \\ u^2_{\tau_2} = \dfrac{1}{2}u^1_xu^3_x, \\ u^3_{\tau_2} = u^3_{xx} + \dfrac{1}{2}(u^3_x)^2 + u^2_xu^3_x - u^1_xu^3_x. \end{cases} \tag{11}$$

Now we observe the following fact. Let us write Eqs.(11) as one evolution equation

$$u_{\tau_2} = K(u_{xx}) + u_x \circ u_x \tag{12}$$

on the element $u = \sum\limits_{i=1}^{3} u^ie_i$ of some commutative algebra $\mathcal{A}$, where $K : \mathcal{A} \to \mathcal{A}$ is the endomorphism of $\mathcal{A}$ and $\circ$ denote multiplication in this algebra defining by Eqs. (11). Then we can state that the algebra $\mathcal{A}$ is Jordan [4] (compare with [5]).

3 Hyperbolic integrable system

The following question will be of interest: what systems of partial differential equations can be written as the zero-curvature condition

$$P_t - Q_x + [P, Q] = 0 \tag{13}$$

if $Q(x,\zeta) = Q_{-1}(x)(i\zeta)^{-1} + Q_0(x) \in \mathrm{sl}(3, \mathbf{C})$ and matrix elements of Q_{-1} and Q_0 are differential functions of variables u^i. In such circumstances, it is easy to get

$$Q_{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} c_1 \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \frac{c_3}{4} \exp\left(-\frac{1}{2}u^1 + u^2\right),$$

$$Q_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \frac{c_1}{3} \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \frac{c_2}{4} \exp\left(-u^2 - \frac{1}{2}u^3\right),$$

while the system of equations, in general form, having representation (13) reads

$$\begin{cases} u^1_{xt} = c_1 u^1_x \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right), \\ u^2_{xt} = c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right) - c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right), \\ u^3_{xt} = -c_1 u^3_x \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right) + c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right). \end{cases} \tag{14}$$

Here $c_1,\ c_2,\ c_3$, in general case, are arbitrary complex constants. By direct calculation, it can be checked that Eqs.(11) present the one-parameter Lie-Bäcklund group for the hyperbolic system (14). Using an explicit form for the Hamiltonian densities of polynomial flows, we can calculate any densities-fluxes of the conservation laws $\partial_t q_k + \partial_x p_k = 0,\ q_k,\ p_k \in A_u,\ k = 0, 1, ...$ of system (11). We have

$$q_0 = -u^1_x + u^2_x + u^3_x, \ \ p_0 = 2c_1 \exp\left(\frac{1}{2}u^1 + \frac{1}{2}u^3\right),$$

$$q_1 = u^1_x u^3_x + (u^2_x)^2, \ \ p_1 = 2c_2 \exp\left(-u^2 - \frac{1}{2}u^3\right) + 2c_3 \exp\left(-\frac{1}{2}u^1 + u^2\right), \ \text{ etc.}$$

Consider different particular cases of system (14) and possible reductions. Choose, for example, $c_2 = c_3 = -\frac{1}{2}$ and $c_1 = ic$, where c is a real number. Putting in (14) $u^2 = in$, where $n = n(x,t)$ is a real-valued function and $u^1 = u^{3*} = \varphi$, where $\varphi = \varphi(x,t)$ is a complex-valued function, Eqs.(14) becomes

$$\begin{cases} n_{xt} = \exp\left(-\frac{1}{2}\mathrm{Re}\,\varphi\right) \sin\left(n - \frac{1}{2}\mathrm{Im}\,\varphi\right), \\ \varphi_{xt} = ic\varphi_x \exp\left(\mathrm{Re}\,\varphi\right) + \frac{1}{2}\exp\left(-in - \frac{1}{2}\varphi^*\right), \end{cases} \tag{15}$$

For the case $c_1 = 0$, Eqs.(14) reduce to the system of Toda type [7]. Putting $u^2 = u^1 - u^3$, $c_2 = c_3 = -2$ in (14), we get

$$\begin{cases} v^1_{xt} = \exp\left(2v^1 - v^2\right), \\ v^2_{xt} = \exp\left(-v^1 + 2v^2\right), \end{cases} \tag{16}$$

where $v^1 = -\frac{1}{2}u^1$ and $v^2 = -\frac{1}{2}u^3$.

Finally, let us consider the case $c_2 = c_3 = 0$. Without loss of generality, we can put $c_1 = -1$. In this particular case, we have

$$\begin{cases} u^1_{xt} = -u^1_x \exp\left(\dfrac{1}{2}u^1 + \dfrac{1}{2}u^3\right), \\ u^2_{xt} = 0, \\ u^3_{xt} = u^3_x \exp\left(\dfrac{1}{2}u^1 + \dfrac{1}{2}u^3\right). \end{cases} \tag{17}$$

For this case, a large class of exact solutions can be found in the form

$$u^1 = F^1(\xi) + \ln(\xi_t) + g_3(t), \;\; u^2 = \eta,$$

$$u^3 = F^2(\xi) + \ln(\xi_t) - g_3(t),$$

where $\xi(x,t) = f_1(x) + g_1(t)$, $\eta(x,t) = f_2(x) + g_2(t)$ and $f_i(x)$, $i = 1, 2$, $g_i(t)$, $i = 1, 2, 3$ are arbitrary smooth functions of variables $x \in \mathbf{R}^1$ and $t \in \mathbf{R}^1$, respectively. Putting this ansatz in (17), we lead to the system of ordinary differential equations

$$\begin{cases} F^{1\prime\prime} = -F^{1\prime} \exp\left(\dfrac{1}{2}F^1 + \dfrac{1}{2}F^2\right), \\ F^{2\prime\prime} = F^{2\prime} \exp\left(\dfrac{1}{2}F^1 + \dfrac{1}{2}F^2\right). \end{cases} \tag{18}$$

System (18) has a solution in the form

$$F^1(\xi) = \ln\left\{\frac{2k_1k_3}{\left(\cos(k_1\xi + k_2) + k_3\sin(k_1\xi + k_2)\right)^2}\right\} + \frac{k_1}{k_3}\left(k_3^2 - 1\right)\xi + k_4,$$

$$F^2(\xi) = \ln\left\{\frac{2k_1k_3}{\left(\cos(k_1\xi + k_2) - k_3\sin(k_1\xi + k_2)\right)^2}\right\} - \frac{k_1}{k_3}\left(k_3^2 - 1\right)\xi - k_4, \;\; k_1, \;\; k_3 \neq 0.$$

As $k_1\zeta + k_2$ also can be written in the form $f(x) + g(t)$, then, without loss of generality, we can put $k_1 = 1$, $k_2 = 0$. So, for system (17) we have a solution in the form

$$u^1 = \ln\left\{\frac{2p\xi_t}{(\cos\xi + p\sin\xi)^2}\right\} + \left(p - p^{-1}\right)\xi + g_3(t), \;\; u^2 = \eta,$$

$$u^3 = \ln\left\{\frac{2p\xi_t}{(\cos\xi - p\sin\xi)^2}\right\} + \left(p^{-1} - p\right)\xi - g_3(t),$$

where $p = k_3 \neq 0$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 372–383.

Representations of a Cubic Deformation of *su(2)* and Parasupersymmetric Commutation Relations

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Abstract

Application of the twisted generalized Weyl construction to description of irreducible representations of the algebra generated by two idempotents and a family of graded-commuting selfadjoint unitary elements which are connected by relations of commutation and anticommutation is presented. It's also discussed $*$-representations and the theory of cubic deformation $A^+_{pq}(3,1)$ of the enveloping algebra $su(2)$.

1 Introduction.

In the last few years, quantum groups, different q-deformations of the universal algebra of Lie algebras, their $\mathbb{Z}_2$-graded analogs, superalgebras and quantum superalgebras have attracted more interest and play an important role in various branches of modern physics. For applications, in particular in particle physics, knot theory, supersymmetric models and others, it is desirable to have a well-developed representation theory.

The purpose of this paper is to study $*$-representations of some nonlinear deformation of the enveloping algebra $su(2)$ and the algebra generated by two idempotents, and a family of graded-commuting selfadjoint elements which are connected by relations of commutation and anticommutation. Namely, in Section 2, we study representations of a cubic deformation of $su(2)$ such as Witten's deformation $A^+_{pq}(3,1)$ [4]. This algebra and their $*$-representations have recently been studied in connection with some physically interesting applications (see [2, 3, 4] and references therein). There are some other non-linear generalizations of $su(2)$ intensively studied in the literature and worth to mention here: in particular, the quantum algebra $su_q(2)$ [23], Witten's first deformation [27, 19], the Higgs algebras [7, 1], the Fairlie q-deformation of $so(3)$ [6, 22], nonlinear $sl(2)$ algebras [1] and others. Our purpose is to describe all irreducible representations of the algebra $A^+_{pq}(3,1)$ by bounded and unbounded operators. The method of solving this problem is based on the study of some dynamical system. The important point to note here that this method allows us to give a complete classification of representations, up to a unitary equivalence, in the class of "integrable" representations and it can be applied, in particular, to the study of $*$-representations of the Higgs algebra [7] and other nonlinear $sl(2)$ algebras [1].

Partially supported by the Fundamental Research Foundation of Ukraine grant no. 1.4/62.

Using some generalization of this method, in Section 3, we will study collections of unitary graded-commuting selfadjoint operators $(\gamma_k)_{k=1}^n$ which commute or anticommute with a pair of unitary selfadjoint operators u, v. Note that these operators determine representations of the $*$-algebra which can be considered as an algebra obtained by the twisted generalized Weyl construction investigated in [12]. The study of their representations can be reduced to the study of a dynamical system on the set of irreducible representations of some noncommutative subalgebra. Such methods of studying representations by using dynamical systems go back to the classical papers [5, 11, 17, 24]. In particular, they have been developed and extended to the families of operators satisfying relations of some special type (see [13, 16, 25, 26]), and then applied for the study of many objects important in mathematical physics ([14, 15] and references therein).

The paper is organized as follows: in Section 2, we describe briefly the method of dynamical systems and study $*$-representations of the algebra $A_{pq}^+(3,1)$. In Section 3, we review some facts on twisted generalized Weyl constructions. Then, using these results and results on representations of commutative Lie superalgebras ([21]), we will give a complete classification of irreducible representations of the algebra generated by unitary selfadjoint elements u, v and the collection of unitary graded-commuting selfadjoint elements $(\gamma_k)_{k=1}^n$ which are connected with each other by the relations of commutation or anticommutation. Families of unitary selfadjoint operators and families of idempotents in the algebra of bounded operators in a Hilbert space were studied, in particular, in [9, 10].

2 Representations of a cubic deformation of $su(2)$

2.1 Representations of commutation relations and dynamical systems

Consider the operator relation

$$A_k B = B F_k(\mathbb{A}), \qquad (k = 1, \dots, n), \tag{1}$$

where $\mathbb{A} = (A_k)_{k=1}^n$ is a family of selfadjoint, generally speaking, unbounded operators in a complex separable Hilbert space H, $F_k(\cdot)$ is a real measurable function on $\mathbb{R}^n$. The method of study of operators satisfying (1) was developed in [13, 25, 26]. It has the origin in the theory of imprimitivity and induced representations of groups ([11]), on the other hand, in the theory of C^*-products and their representations ([5, 17, 24]). At the same time, this method has some new aspects which allow us to study many objects appearing in mathematical physics (see [15]).

We allow the operators $\mathbb{A}$, B to be unbounded. Thus, we have to make precise the sense in which relations (1) hold.

We call the operators $\mathbb{A} = (A_k)_{k=1}^n$ *and* B *a representation of (1) if*

$$E_{\mathbb{A}}(\Delta) U = U E_{\mathbb{A}}(\mathbb{F}^{-1}(\Delta)), \quad [E_{|B|}(\Delta'), E_{\mathbb{A}}(\Delta)] = 0, \quad \Delta \in \mathfrak{B}(\mathbb{R}^n),\ \Delta' \in \mathfrak{B}(\mathbb{R}), \tag{2}$$

where $\mathbb{F}(\cdot) = (F_1(\cdot), \dots, F_n(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$, $E_{\mathbb{A}}(\cdot)$ *is a joint resolution of the identity for the commuting family* $\mathbb{A}$, $B = U|B|$ *is a polar decomposition of the closed operator* B.

For bounded A_k, B, this definition and the usual pointwise definition are equivalent. We will say that the representation $(\mathbb{A}, B)$ is irreducible if any bounded operator such that $CX \subseteq XC$, $C^*X \subseteq XC^*$, where X is one of the operators A_k, B, B^*, is a multiple of the identity operator.

The important role in the study of the commutation relations (1) is played by the dynamical system (d.s.) on $\mathbb{R}^n$ generated by $\mathbb{F}$. We will assume that $\mathbb{F}$ is bijective. It follows from (2) that for any $\mathbb{F}$-invariant set Δ, the operator $E_{\mathbb{A}}(\Delta)$ is a projection on an invariant subspace. If the dynamical system is *simple*, i.e., there exists a measurable set intersecting every orbit of the dynamical system exactly at one point, then any irreducible representation arises from an orbit of d.s., i.e., the spectral measure of the family $\mathbb{A}$ is concentrated on an orbit. We restrict ourselves by considering only this case.

If no conditions are imposed on the operator B, then the problem of unitary classification of all families $(\mathbb{A}, B)$ is a very difficult problem. It contains as a subproblem the problem of unitary classification of pairs of selfadjoint operators without any relations (see [18, 10]). We will assume that the operators B, B^* are additionally connected by the relation

$$B^*B = \varphi(\mathbb{A}, BB^*), \tag{3}$$

where $\varphi(\cdot) : \mathbb{R}^{n+1} \to \mathbb{R}$ is a continuous function. If the operators $\mathbb{A}$, B, B^* are bounded, (3) is equivalent to the following equality

$$|B|^2 U = UF_{n+1}(\mathbb{A}, |B|^2) \tag{4}$$

with $F_{n+1}(x_1, \ldots, x_{n+1}) = \varphi(F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n), x_{n+1})$. Relation (4) is of the form (1), hence we can give a definition of unbounded representations of relations (1) and (3). Note that the assumed condition implies the following relations for the operator U: operators $U^l(U^*)^l$, $(U^*)^l U^l$, $l = 1, 2, \ldots$ form a commuting family (i.e., the operator U is centered).

The complete classification of all irreducible families $(\mathbb{A}, B, B^*)$ satisfying (1), (3) was given in [25]. Moreover, there was proved the structure theorem which defines the form of any such operators as a direct sum or a direct integral of irreducible ones. Let

$$\mathcal{F}(x_1, \ldots, x_{n+1}) = (F_1(x_1, \ldots, x_n), \ldots, F_n(x_1, \ldots, x_n), F_{n+1}(x_1, \ldots, x_{n+1})$$

be bijective. By [25], if the dynamical system is simple, any irreducible representation of relations (1), (3) arises from certain subsets of an orbit of the dynamical system on $\mathbb{R}^{n+1}$ generated by $\mathcal{F}$. Namely, any irreducible representation is unitarily equivalent to one of the following:

$$A_k e_{\mathbf{x}} = x_k e_{\mathbf{x}}, \ Be_{\mathbf{x}} = \sqrt{x_{n+1}}\, u(\mathbf{x}) e_{\mathcal{F}(\mathbf{x})}, \ \mathbf{x} = (x_1, \ldots, x_{n+1}) \in \Omega_0,$$

where Ω_0 is a connected subset of some orbit of the dinamcal system on $\mathbb{R}^{n+1}$ generated by $\mathcal{F}$ [16].

2.2 Representations of the cubic deformation $A^+_{pq}(3,1)$ of $su(2)$

The $*$-algebra $A^+_{pq}(3,1)$ was introduced by Delbeq and Quesne ([4]) as a two-parameter nonlinear cubic deformation of $su(2)$. It is generated by generators J_0, J_+, J_- satisfying the relations:

$$\begin{aligned} &[J_0, J_+] = (1 + (1-q)J_0)J_+, \qquad [J_0, J_-] = -J_-(1 + (1-q)J_0), \\ &[J_+, J_-] = 2J_0(1 + (1-q)J_0)(1 - (1-p)J_0), \end{aligned} \tag{5}$$

with involution defined as follows: $J_0^* = J_0$, $J_+^* = J_-$. We will assume that $0 < p < 1$, $0 < q < 1$.

This algebra has a Casimir operator

$$C = J_- J_+ + \frac{2(q-1)}{(q+1)(q^3-1)} J_0(J_0+1)(1+(p+q)q - (1-p)(1+q)J_0).$$

Representations of the $*$-algebra $A^+_{pp}(3,1)$ were classified in [4]. In this paper we allow parameters p, q to be different.

It is clear that bounded representations of the $*$-algebra are defined by their value on the generators, i.e., operators $J_0^* = J_0$, $J_+ \in L(H)$ satisfying (5). An unbounded representation of $A^+_{pq}(3,1)$ is defined to be formed by unbounded operators $J_0^* = J_0$, J_+ satisfying (1), (3) in the sense of the definition given in the previous subsection. It follows from the definition that the spectral projections of the Casimir operator C commute with the generators of the algebra, i.e., $E_C(\Delta)A \subseteq AE_C(\Delta)$, where A is one of the operators J_0, J_+, J_+^* and $\Delta \in \mathfrak{B}(\mathbb{R})$. Thus, given an irreducible triple (J_0, J_+, J_-), the operator C is a multiple of the identity operator: $C = \mu I$, where $\mu \in \mathbb{R}$. Moreover,

$$J_0 J_+ = q^{-1} J_+ (J_0+1), \quad J_+^* J_+ = g(J_0, \mu), \tag{6}$$

where $g(x,\mu) = \mu - \dfrac{2(q-1)}{(q+1)(q^3-1)} x(x+1)(1+(p+q)q-(1-p)(1+q)x)$.

Conversely, any irreducible representation of the algebra generated by $J_0 = J_0^*$, J_+, J_+^* and relations (6) which is defined on a Hilbert space H, $\dim H \geq 2$, is an irreducible representation of $A^+_{pq}(3,1)$ (see [20], Lemma 1). In what follows, we will study representations of relations (6).

To (6) there corresponds the dynamical system generated by $\mathbb{F}(x,y) = (q^{-1}(x+1), g(q^{-1}(x+1),\mu)) : \mathbb{R}^2 \to \mathbb{R}^2$. It has the measurable section $\tau = ([q^{-1}(\delta_1+1), \delta_1) \cup \left\{\dfrac{1}{q-1}\right\} \cup (\delta_2, q^{-1}(\delta_2+1)]) \times \mathbb{R}$, where $\delta_1 < \dfrac{1}{q-1} < \delta_2$. Thus, any irreducible representation arises from an orbit of the dynamical system and can be described by the formulae given above. It is easy to show that any orbit of the dynamical system is of the form

$$\Omega_x = \left\{ (q^{-n}\left(x - \frac{1}{q-1}\right) + \frac{1}{q-1}, g\left(q^{-n}\left(x - \frac{1}{q-1}\right) + \frac{1}{q-1}, \mu\right) \mid n \in \mathbb{Z} \right\}.$$

In the sequel we will denote the point of the orbit by $(f_1(x,\mu,n), f_2(x,\mu,n))$. It depends on a behavior of the function $g(x,\mu)$ what kind of representations relations (6) will have. Since the calculations are rather lengthy, we leave them out and state the final result. For the deeper discussion, we refer the reader to [20]. First, let us introduce some notations.

Denote by $x_1(\mu) \leq x_2(\mu) \leq x_3(\mu)$ the real roots of the equation $g(x,\mu) = 0$. Let $a = 1+(p+q)q$, $b = (1-p)(1+q)$, $\gamma(p,q) = ab^{-1}$. Then

$$\varepsilon_1(p,q) = \frac{a-b+\sqrt{a^2+b^2+ab}}{3b}, \quad \varepsilon_2(p,q) = \frac{a-b-\sqrt{a^2+b^2+ab}}{3b}$$

are the extreme points of $g(x,\mu)$. Write $y_1(p,q) = \min g(x,0) \equiv g(\varepsilon_1(p,q),0)$, $y_2(p,q) = \max g(x,0) \equiv g(\varepsilon_2(p,q),0)$ and $\psi(\mu,p,q) = \dfrac{x_3(\mu)-(q-1)^{-1}}{x_2(\mu)-(q-1)^{-1}}$. Then

$$\max_{\mu \in [-y_2(p,q), -y_1(p,q)]} \psi(\mu,p,q) = \frac{ab^{-1} - 1 - 2\varepsilon_2(p,q) - (q-1)^{-1}}{\varepsilon_2(p,q) - (q-1)^{-1}},$$

which will be denoted by $\psi(p,q)$.

Below we give a list of irreducible representations of the $*$-algebra $A^+_{pq}(3,1)$:

1. one-dimensional representations: $J_0 = (q-1)^{-1}$, $J_+ = \lambda$, $\lambda \in \mathbb{C}$;

2. finite-dimensional representations:

a) for any p, q such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) \le (q-1)^{-1}$, for any $n \ge 2$, there exists the representation of the dimension $n+1$ with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k),\ f_2(x_1(\mu),\mu,k) \mid k = 0,\dots,n\}$, where μ is uniquely defined from the equation $\kappa(\mu,p,q) = q^{-n}$; where $\kappa = \dfrac{x_2 - (q-1)^{-1}}{x_1 - (q-1)^{-1}}$.

b) for any p, q such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) > (q-1)^{-1}$, there exist the representations of any dimension $n \le \log_{q^{-1}}(\gamma(p,q) - 1 - 2\varepsilon_1(p,q)) + 1$ with $\Omega_0 = \{(f_1(x_1(-y_1(p,q)), -y_1(p,q),k),\ f_2(x_1(-y_1(p,q)),\ -y_1(p,q),k) \mid k = 0,\dots,n\}$;

c) for any $n \in \mathbb{N}$ and $\mu \in \mathbb{R}$ such that $x_1(\mu) > (q-1)^{-1}$ and $q^n(x_3(\mu) - (q-1)^{-1}) = x_1(\mu) - (q-1)^{-1}$ and $x_3(\mu) < q^{-1}(x_2(\mu)+1)$, there exists the representation of the dimension $n+1$ with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k = 0,\dots,n\}$;

3. representations with higher weight:

a) for any p, q such that $\gamma(p,q) - 1 - 2\varepsilon_1(p,q) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(-y_1(p,q)), -y_1(p,q),k), f_2(x_2(-y_1(p,q)),\ -y_1(p,q),k) \mid k \le 0\}$;

b) for any μ such that $x_1(\mu) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(\mu),\mu,k), f_2(x_2(\mu),\mu,k) \mid k \le 0\}$;

c) for any p, q such that $\psi(p,q) \le q^{-1}$ and $\mu \in (-y_2(p,q), -y_1(p,q))$, there exists the representations with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \le 0\}$;

d) for any p, q such that $\psi(p,q) > q^{-1}$ and any $\mu \in (\mu_0, -y_1(p,q))$ such that $x_1(\mu) \le (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \le 0\}$, where μ_0 is uniquely defined by the condition $\psi(\mu_0,p,q) = q^{-1}$;

4. representations with a lower weight:

a) for any p, q and $\mu \in (-\infty, -y_1(p,q))$, there exists the representation with $\Omega_0 = \{(f_1(x_3(\mu),\mu,k), f_2(x_3(\mu),\mu,k) \mid k \ge 0\}$;

b) for any p, q and $\mu \in (-y_1(p,q), +\infty)$ and $x_1(\mu) > (q-1)^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \ge 0\}$;

c) for any p, q such that $\psi(p,q) > q^{-1}$ and $\mu \in (-y_2(p,q), \mu_0)$, there exists the representation with $\Omega_0 = \{(f_1(x_2(\mu),\mu,k), f_2(x_2(\mu),\mu,k) \mid k \ge 0\}$, where μ_0 is uniquely defined by the condition $\psi(\mu_0,p,q) = q^{-1}$;

d) for any p, q such that $\psi(p,q) < q^{-1}$, there exists the representation with $\Omega_0 = \{(f_1(x_2(-y_2(p,q)), -y_2(p,q),k), f_2(x_2(-y_2(p,q)),\ -y_2(p,q),k) \mid k \ge 0\}$;

e) for any p, q and $\mu \in (-y_2(p,q), \mu_0)$ such that there exists $n \in \mathbb{N} \cup \{0\}$ satisfying the condition $q^{-n} < \dfrac{x_2(\mu) - (q-1)^{-1}}{x_1(\mu) - (q-1)^{-1}}$, $q^{-n-1} > \dfrac{x_3(\mu) - (q-1)^{-1}}{x_1(\mu) - (q-1)^{-1}}$, we have the representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \ge 0\}$, where μ_0 is defined as follows: $x_1(\mu_0) = (q-1)^{-1}$;

f) for any p, q and μ such that $x_1(\mu) < (q-1)^{-1}$, there is representation with $\Omega_0 = \{(f_1(x_1(\mu),\mu,k), f_2(x_1(\mu),\mu,k) \mid k \le 0\}$

5. nondegenerate representations:

a) for any $\lambda \in \left[\dfrac{(2q\varepsilon_1(p,q) - 1)}{q+1}, \dfrac{(2\varepsilon_1(p,q) - 1)}{q+1}\right) \setminus \{\varepsilon_1(p,q)\}$, if $\gamma - 1 - 2\varepsilon_1 < (q-1)^{-1}$, then there exists the representation with $\Omega_0 = \{(f_1(\lambda, -y_1(p,q),k), f_2(\lambda, -y_1(p,q),k) \mid k \in \mathbb{Z}\}$;

b) for any $\lambda \in \left[\frac{(2q\varepsilon_1(p,q)-1)}{q+1}, \frac{(2\varepsilon_1(p,q)-1)}{q+1}\right) \setminus \{\varepsilon_1(p,q)\}$ and $\mu \in (\mu_\lambda, -y_1(p,q))$ such that $x_1(\mu) \leq (q-1)^{-1}$ there exists the representation with $\Omega_0 = \{(f_1(\lambda,\mu,k), f_2(\lambda,\mu,k) \mid k \in \mathbb{Z}\}$, where μ_λ is defined by the condition: $\mathrm{dist}(\{x \mid g(x,\mu_\lambda) = 0\}, \varepsilon_1(p,q)) = \mathrm{dist}(\{q^{-n}(\lambda-(q-1)^{-1}) + (q-1)^{-1} \mid n \in \mathbb{Z}\}, \varepsilon_1(p,q))$ (here $\mathrm{dist}(M,x)$ is the distance between the subset $M \subseteq \mathbb{R}$ and $x \in \mathbb{R}$).

3 On the structure of families of unitary selfadjoint operators

3.1 Twisted generalized Weyl construction.

Let R be a unital $*$-algebra, $t = t^*$ a central element and σ an automorphism such that $\sigma(r^*) = (\sigma(r))^*$. Define the $*$-algebra $\mathfrak{A}^1_R$ as the R-algebra generated by two elements X, X^* subjected to the following relations:

- $Xr = \sigma(r)X$ and $rX^* = X^*\sigma(r)$ for any $r \in R$,
- $X^*X = t$ and $XX^* = \sigma(t)$.

We will say that the $*$-algebra $\mathfrak{A}^1_R$ is obtained from R, σ, t by the twisted generalized Weyl construction. Such $*$-algebras were introduced in [12] and their Hilbert space representations were studied up to a unitary equivalence.

In this subsection, we set up a notation and give a brief exposition of results from [12] which will be needed below.

Let H be a complex separable Hilbert space, $L(H)$ denotes the set of all bounded operators on H, $\mathcal{M}' = \{c \in L(H) \mid [c,a] = 0, a \in \mathcal{M}\}$ is the commutator of the operator algebra $\mathcal{M}$.

Assume that R is an algebra of type I, i.e., the W^*-algebra $\{\pi(r), r \in R\}''$ is of type I for any representation π of R, and, given a representation π of R, the automorphism σ can be extended to the corresponding von Neumann algebra. Let $\widehat{R}$ be the set of equivalence classes of irreducible representations of R. The automorphism σ generates the dynamical system on the set $\widehat{R}$. Indeed, if π is an irreducible representation of R, then so is $\pi(\sigma)$. Denote by Ω_π the orbit of the dynamical system, i.e., $\Omega_\pi = \{\pi(\sigma^k), k \in \mathbb{Z}\}$.

The next assumption will be needed throughout the section. Suppose that it is possible to choose the subset $\tau \subset \widehat{R}$ which meets each orbit just once in such a way that τ is a Borel subset. In this case, we will say that the dynamical system $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ is *simple*. Then any irreducible representation $w : \mathfrak{A}^1_R \to L(H)$ is concentrated on an orbit of the dynamical system, i.e., $H = \oplus_{\Omega_0 \subset \Omega_\pi} H_{\pi_k}$, where H_{π_k} is invariant with respect to $w(r)$ for any $r \in R$, and $w|_{H_{\pi_k}}$, as a representation of R, is unitarily equivalent to $\pi(\sigma^k) \otimes I$ (here I is the identity operator of dimension $n(k) \leq \infty$). We will call Ω_0 the support of w and denote by supp$w$. Without loss of generality, we can assume that $\pi \in \Omega_0$. Then $\Omega_0 = \{\pi(\sigma^k), k \in \mathbb{Z} \mid \pi(\sigma^l(t) > 0$ for any $0 \leq l \leq k$ if $k \geq 0$ or $k < l < 0$ if $k < 0\}$. Denote by $\tilde{K}$ the subgroup of $\mathbb{Z}$ consisting of $k \in \mathbb{Z}$ such that $\pi(\sigma^k)$ is unitarily equivalent to π and $\pi(\sigma^l) \in \Omega_0$ for any $0 < l < k$ if $k > 0$ or $0 > l > k$ if $k < 0$.

Theorem 1 *Any irreducible representation w of the $*$-algebra $\mathfrak{A}^1_r$ such that supp$w = \Omega_0$ coincides, up to a unitary equivalence, with one of the following:*

1. *If $\tilde{K} = \emptyset$, then $H = \oplus_{\Omega_0} H_{\pi_k}$*

$$w(r)|_{H_{\pi_k}} = \pi(\sigma^k(r)), \quad X : H_{\pi_k} \to H_{\pi_{k+1}}, \quad X|_{H_{\pi_k}} = \begin{cases} \pi(\sigma^k(t)), & \pi(\sigma_{k+1}) \in \Omega_0, \\ 0, & \pi(\sigma_{k+1}) \notin \Omega_0. \end{cases}$$

2. *If $\tilde{K} \neq \emptyset$ and $n \in \mathbb{N}$ is the smallest number such that $\pi(\sigma^n)$ and π are unitarily equivalent, then $H = \oplus_{k=0}^{n-1} H_{\pi_k}$*

$$w(r)|_{H_{\pi_k}} = \pi(\sigma^k(r)), \quad X : H_{\pi_k} \to H_{\pi_{k+1}}, \quad X|_{H_{\pi_k}} = \begin{cases} \pi(\sigma^k(t)), \ k \neq n-1, \\ e^{i\varphi} W \pi(\sigma^{n-1}(t)), \ k = n-1, \end{cases}$$

where $W^{-1}\pi W = \pi(\sigma^n)$, $\varphi \in [0, 2\pi)$.

This technique can be applied to the study of many objects important in mathematical physics such as $Q_{ij} - CCR$ ([8]), $su_q(3)$ and others. Next section is devoted to the study of one of them.

3.2 Representations of $*$-algebras generated by unitary selfadjoint generators

The purpose of this subsection is to describe representations of the $*$-algebra $\mathfrak{A}$ generated by selfadjoint unitary generators u, v and j_k ($k = 1, \ldots, n$), and the relations

$$j_i j_k = (-1)^{g(i,k)} j_k j_i, \tag{7}$$

$$u j_k = (-1)^{h(k)} j_k u, \quad v j_k = (-1)^{w(k)} j_k v, \tag{8}$$

here $g(i,k) = g(k,i) \in \{0,1\}$, $g(i,i) = 0$, $h(k)$, $w(k) \in \{0,1\}$ for any $i, k = 1, \ldots, n$. Any family of elements $(j_k)_{k=1}^n$ satisfying (7) is said to be a graded-commuting family.

Any representations of $\mathfrak{A}$ is determined by representation operators corresponding to the generators. Thus, instead of representations of the $*$-algebra $\mathfrak{A}$, we will study collections of unitary selfadjoint operators u, v, j_k ($k = 1, \ldots, n$) on a complex separable Hilbert space H satisfying (7), (8).

We start with the study of graded-commuting selfadjoint operators $\mathcal{J} = (j_k)_{k=1}^n$ with the condition $j_k^2 = I$ ($k = 1, \ldots, n$). For these operators, the structure question was solved in [21]. We will present only main results and constructions from [21].

To collection of selfadjoint unitary operators $(j_k)_{k=1}^n$, there corresponds a simple graph $\Gamma = (S, R)$ (without loops and multiple edges). Here S is element subsets of S corresponding to the edges. The vertices a_k and a_m are connected with an edge if $\{j_k, j_m\} \equiv j_k j_m + j_m j_k = 0$ and there is no edge if the operators commute. In what follows, we will regard such a collection as selfadjoint and unitary representations of the graph Γ.

Denote by σ_x, σ_y, σ_z and σ_0 the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Consider the following construction of irreducible representations of the graph Γ, which are defined inductively.

1. If $\Gamma = \begin{pmatrix} a_1 & a_2 & & a_n \\ \cdot & \cdot & \dots & \cdot \end{pmatrix}$ then collections of unitary commuting selfadjoint operators form representations of the graph. Set $j_k = (-1)^{i_k}$, where $i_k \in \{0,1\}$, $k = 1, \dots, n$, and $m(\Gamma) = 0$. We get 2^n unitarily inequivalent representations.

2. Suppose now that at least two of vertices (without loss of generality, we can assume that these are a_1, a_2) are connected by an edge. In the space $H = \mathbb{C}^2 \otimes H_1$, consider $j_1 = \sigma_z \otimes I$, $\quad j_2 = \sigma_x \otimes I$, where I is the identity operator on a Hilbert space H_1.

If j_k commutes with j_1 and j_2, set $j_k = \sigma_0 \otimes B_k$;

if j_k commutes with j_1 and anticommutes with j_2, set $j_k = \sigma_z \otimes B_k$;

if j_k anticommutes with j_1 and commutes with j_2, set $j_k = \sigma_x \otimes B_k$;

if j_k anticommutes with j_1 and j_2, set $j_k = \sigma_y \otimes B_k$, where $(B_k)_{k=3}^n$ is the representation of the derivative graph $\Gamma_1 = (S_1, R_1)$ which is defined in the following way:

a) the graph Γ_1 contains the vertices $(b_k)_{k=3}^n$;

b) if, in the graph Γ, the vertex a_k is contained in the star of the vertex a_1 (i.e., it is connected with a_1 by an edge) and a_m is contained in the star of a_2 but at least one of them is not in both of the stars (i.e., is not connected with both vertices), then the edge $(b_k, b_m) \in R_1$ if $(a_k, a_m) \notin R$ and conversely, $(b_k, b_m) \notin R_1$ if $(a_k, a_m) \in R$;

c) in all other cases $(b_k, b_m) \in R_1$ if $(a_k, a_m) \in R$ and $(b_k, b_m) \notin R_1$ if $(a_k, a_m) \notin R$.

Proceeding in such a manner at the end we find that either there exists $m(\Gamma) \in \mathbb{N}$ such that all vertices of the graph $\Gamma_{m(\Gamma)}$ are isolated and we get $2^{n-2m(\Gamma)}$ representations of the dimension $2^{m(\Gamma)}$, or $m(\Gamma) \equiv n \in 2\mathbb{N}$ and $\Gamma_{m(\Gamma)/2} = \emptyset$. In the last case, we get the unique representations of the dimension 2^n.

Theorem 2 *A simple graph Γ with n vertices has $2^{r(\Gamma)}$ $(0 \le r(\Gamma) \le n)$ unitarily inequivalent irreducible representations of the same dimension $2^{m(\Gamma)}$ with $r(\Gamma) = n - 2m(\Gamma)$. Any of them is unitarily equivalent to one defined by the construction above.*

It follows from Theorem 2 that any irreducible unitary selfadjoint representation of the graph Γ is realized, up to a permutation and a unitary equivalence, on $H = \mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2$ by the formulae $j_k = i_k \sigma_{k1} \otimes \dots \otimes \sigma_{km(\Gamma)}$ $\quad (k = 1, \dots, n)$, where σ_{km} is the Pauli matrix contained as the m-th factor in j_k, $i_1 = \dots = i_{2m(\Gamma)} = 1$, $i_k \in \{0,1\}$ for $k > 2m(\Gamma)$.

To distinguish families of unitary selfadjoint operators $(j_k)_{k=1}^n$ satisfying (7), we will say that $(j_k)_{k=1}^n$ is a family of unitary graded-commuting selfadjoint operators corresponding to $g : M \times M \to \{0,1\}$, where $M = \{1, 2, \dots, n\}$.

2. Next we describe collections $(u,\, v,\, j_1, \dots, j_n)$ of selfadjoint unitary operators which satisfy the relations:

$$j_i j_k = (-1)^{g(i,k)} j_k j_i, \tag{9}$$

$$u j_k = (-1)^{h(k)} j_k u, \quad v j_k = (-1)^{h(k)} j_k v, \tag{10}$$

where $g(i,k) = g(k,i) \in \{0,1\}$, $g(i,i) = 0$, $h(k) \in \{0,1\}$ for any $i, k = 1, \dots, n$.

They determine representations of the $*$-algebra $\mathcal{A}$ generated by $u = u^*$, $v = v^*$, $j_k^* = j_k$ $(k = 1, \dots, n)$ satisfying (9), (10) and the condition $u^2 = v^2 = j_k^2 = 1$. Denote by $\mathcal{A}_0$ the subalgebra of $\mathcal{A}$ generated by j_k, $k = 1, \dots, n$. The $*$-algebra $\mathcal{A}$ can be treated as a $*$-algebra obtained by the twisted generalized Weyl construction. Indeed, let $X = u + iv$,

$X^* = u - iv$. It is easy to check that relations (10) and $u^2 = v^2 = 1$ are equivalent to the following ones:

$$\begin{aligned} &XX^* + X^*X = 4, \quad X^2 + (X^*)^2 = 0, \\ &Xj_k = (-1)^{h(k)} j_k X, \quad X^* j_k = (-1)^{h(k)} j_k X^* \end{aligned} \tag{11}$$

Consider the unital $*$-algebra $\mathcal{A}_0 \oplus \mathbb{C}X^*X$ as the ground $*$-algebra R with the central element $t = X^*X$ and the automorphism σ which is defined in the following way: $\sigma(X^*X) = 4 - X^*X$, $\sigma(j_k) = (-1)^{h(k)} j_k$, $k = 1, \dots, n$. Then $\mathcal{A}$ is $*$-isomorphic to $\mathfrak{A}^1_R$.

Theorem 3 *Any irreducible representation of $\mathcal{A}$ is unitarily equivalent to one of the following:*

1. $H = H_0 \otimes \mathbb{C}^2$

$$u = I \otimes \sigma_x \ v = I \otimes \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix}, \ j_k = \begin{cases} -j'_k \otimes \sigma_y, & h(k) = 1, \\ j'_k \otimes \sigma_0, & h(k) = 0, \end{cases} \tag{12}$$

where $\varphi \in (-\pi, \pi)$, $(j'_k)_{k=1}^n$ is an irreducible family of unitary graded-commuting selfadjoint operators on H_0 corresponding to g.

2. $H = H_0$

$$j_k = j'_k, \ k = 1, \dots, n, \quad u = j'_{n+1}, \quad v = j'_{n+2}, \tag{13}$$

where $(j'_k)_{k=1}^{n+2}$ is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to g' defined as follows:

$$g'(k,i) = \begin{cases} g(k,i), & k,i \le n, \\ h(k), & k \le n < i, \\ 0, & \textit{otherwise}. \end{cases}$$

Proof. To the $*$-algebra $\mathcal{A}$, there corresponds the dynamical system $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ with $\sigma^2 = 1$. Any representation $\pi \in \widehat{R}$ is defined by the collection $(j'_1, \dots, j'_n, X^*X = \lambda I)$, where $(j'_k)_{k=1}^n$ is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to g, and $\lambda \in [0,4]$. Any irreducible representation arises from an orbit of the dynamical system.

If $\lambda \neq 0, 2, 4$, then the representations π, $\pi(\sigma)$ are not unitarily equivalent and and $\pi(t)$, $\pi(\sigma(t)) > 0$, hence, by Theorem 1, the corresponding irreducible representation of the $*$-algebra $\mathcal{A}$ is of the form

$$j_k = j'_k \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{h(k)} \end{pmatrix} \quad X = I \otimes \begin{pmatrix} 0 & e^{i\psi}\sqrt{4-\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix},$$

$\psi \in [0, 2\pi)$, $\lambda \in (0,2)$. Moreover, since $X^2 + (X^*)^2 = 0$, $e^{i\psi} = \pm i$. Thus,

$$u = I \otimes \begin{pmatrix} 0 & e^{i\delta} \\ e^{-i\delta} & 0 \end{pmatrix}, \ v = I \otimes \begin{pmatrix} 0 & ie^{-i\delta} \\ -ie^{i\delta} & 0 \end{pmatrix}, \tag{14}$$

where $\lambda = 4\cos^2\delta$, $\delta \in (-\pi/4, \pi/4)$.

If $\lambda = 4$, then $\pi(\sigma(t)) = 0$, $\pi(t) = 4$ and the corresponding irreducible representation is of the form

$$j_k = j'_k \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{h(k)} \end{pmatrix}, \quad X = I \otimes \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix},$$

and $u = I \otimes \sigma_x$, $v = I \otimes (-\sigma_y)$.

If $\lambda = 0$, then $\pi(t) = 0$, $\pi(\sigma(t)) = 4$ and the corresponding irreducible representation is unitarily equivalent to one given in the previous case.

It is easy to check that these representations are unitarily equivalent to that given by (12) with the unitary operator

$$W = \frac{1}{\sqrt{2}} I \otimes \begin{pmatrix} 1 & i \\ e^{-i\delta} & -ie^{-i\delta} \end{pmatrix} \quad \text{and} \quad \varphi = 2\delta.$$

If $\lambda = 2$, then $\pi(t) = \pi(\sigma(t))$. Hence, for the corresponding irreducible representation of $\mathcal{A}$, we have $XX^* = X^*X = 2$, which is equivalent to the following $[u, v] = 0$, $u^2 = v^2 = I$. The corresponding irreducible family $(u, v, j_1, \ldots, j_n)$ is defined by (13). Moreover, any collection of the form (13) determines a representation of the $*$-algebra $\mathcal{A}$. ■

3. Now consider representations of the $*$-algebra generated by u, v, j_k, $k = 1, \ldots, n$, and relations (7), (8). Suppose that there exists $k \le n$ such that $h(k) \ne w(k)$. Without loss of generality, we can assume that $h(k) = w(k)$ for some $k \le s < n$, $h(k) \ne w(k)$ if $s < k \le n$ and $h(n) = 0$, $w(n) = 1$. Consider the selfadjoint unitary generators

$$\tilde{j}_k = \begin{cases} j_k, & k \le s \text{ or } k = n, \\ i^{g(n,k)} j_k j_n, & s < k < n. \end{cases}$$

One can check that relations (7), (8) are equivalent to the following

$$\tilde{j}_k \tilde{j}_l = (-1)^{\tilde{g}(k,l)} \tilde{j}_l \tilde{j}_k \tag{15}$$

$$\tilde{j}_k u = (-1)^{\tilde{h}(k)} u \tilde{j}_k, \ \tilde{j}_k v = (-1)^{\tilde{h}(k)} v \tilde{j}_k, \ k \ne n, \tag{16}$$

$$\tilde{j}_n u = u \tilde{j}_n, \ \tilde{j}_n v = -v \tilde{j}_n, \tag{17}$$

where $\tilde{g}(k,l) = \begin{cases} g(k,l), & k < l \le s \text{ or } l = n, \\ (g(k,l) + g(k,n))(\text{mod}2), & k \le s < l < n, \\ (g(l,n) + g(k,l) + g(k,n))(\text{mod}2), & s < k < l < n, \end{cases}$

$$\tilde{g}(k,l) = \tilde{g}(l,k), \ \tilde{g}(l,l) = 0, \ \tilde{h}(k) = \begin{cases} h(k), & k \le s, \\ (h(n) + h(k))(\text{mod}2), & s < k < n. \end{cases}$$

The $*$-algebra $\mathcal{A}'$ generated by selfadjoint unitary elements (u, v, $(\tilde{j}_k)_{k=1}^n$) satisfying (15)–(17) is $*$-isomorphic to $\mathcal{A}$ and can be considered as a $*$-algebra obtained by the twisted generalized Weyl construction from the ground ring R generated by (u, v, $\tilde{j}_1, \ldots, \tilde{j}_{n-1}$) satisfying (15)–(16), the central element $t = 1$ and the automorphism σ defined in the following way:

$$\sigma(u) = u, \ \sigma(v) = -v \ \sigma(\tilde{j}_k) = (-1)^{\tilde{g}(k,n)} \tilde{j}_k.$$

We can use the technique above to describe representations of the algebra.

Theorem 4 *Any irreducible representation of the $*$-algebra coincides, up to a unitary equivalence, with the following ones:* 1. $H = H_0 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$

$$u = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes I, \quad v = I \otimes \begin{pmatrix} \sin\varphi & \cos\varphi \\ \cos\varphi & -\sin\varphi \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\tilde{j}_k = \begin{cases} -\tilde{j}'_k \otimes \sigma_y \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\tilde{g}(n.k)} \end{pmatrix}, & \tilde{h}(k) = 1, k \neq n, \\ \tilde{j}'_k \otimes \sigma_0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & (-1)^{\tilde{g}(n.k)} \end{pmatrix}, & \tilde{h}(k) = 0, k \neq n, \end{cases} \qquad j_n = I \otimes I \otimes \sigma_x, \tag{18}$$

where $\varphi \in (-\pi, \pi)$, $(\tilde{j}'_k)_{k=1}^n$ *is an irreducible family of unitary graded-commuting selfadjoint operators on* H_0 *corresponding to* $\tilde{g}$.

2. $H = H_0$

$$\tilde{j}_k = \tilde{j}'_k, \ k = 1, \ldots, n, \ u = \tilde{j}'_{n+1}, \quad v = \tilde{j}'_{n+2}, \tag{19}$$

where $(\tilde{j}'_k)_{k=1}^{n+2}$ *is an irreducible family of unitary graded-commuting selfadjoint operators corresponding to* $\tilde{g}'$ *defined as follows:*

$$\tilde{g}'(k,i) = \begin{cases} \tilde{g}(k,i), & k < i \leq n, \\ \tilde{h}(k), & k < n < i, \\ 0, & (k,i) = (n,n+1), (n+1,n+2), \\ 1, & (k,i) = (n,n+2). \end{cases}$$

Proof. By Theorem 3, given an irreducible representations π of the $*$-algebra R, we have either $\pi([u,v]) = 0$ or, up to a unitary equivalence, $\pi(u) = I \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\pi(v) = I \otimes \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$. It is easy to show that the pairs $(\pi(u), \pi(v))$ and $(\pi(\sigma(u)), \pi(\sigma(v)))$ are not unitarily equivalent. Hence, by Theorem 1, if $\pi([u,v]) \neq 0$, then the irreducible representation corresponding to the orbit $\widehat{R} \ni \pi \to \pi(\sigma) \in \widehat{R}$ is unitarily equivalent to the representation defined by (18). If $\pi([u,v]) = 0$, the corresponding irreducible representation is defined by the family of graded-commuting unitary selfadjoint operators described by (19). ∎

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 384–392.

q-Deformed Inhomogeneous Algebras $U_q(iso_n)$ and Their Representations

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Abstract

Proceeding from the nonstandard q-deformed algebras $U'_q(so_{n+1})$ and their finite-dimensional representations in a q-analog of the Gel'fand-Tsetlin basis, we obtain, by means of the contraction procedure, the corresponding q-deformed inhomogeneous algebras $U'_q(iso_n)$ in a uniform fashion for all $n \geq 2$ as well as their infinite-dimensional representations.

1. Introduction

Lie algebras of inhomogeneous orthogonal or pseudoorthogonal Lie groups are important for various problems of theoretical and mathematical physics. Recently, certain efforts were devoted to the problem of constructing quantum, or q-deformed, analogs of inhomogeneous (Euclidean) algebras [1–3]. Practically, all these works exploit as starting point the standard deformations $U_q(B_r)$, $U_q(D_r)$, given by Jimbo and Drinfeld [4], of Lie algebras of the orthogonal groups $SO(2r+1)$ and $SO(2r)$. In addition to the fact that most of papers [1–3] concern Euclidean algebras *of low dimension* $2, 3, 4$, their q-analogs may be examined from the viewpoint of (non-)possessing the following two characteristic features:

(i) after deformation, a rotation subalgebra remains closed;

(ii) both rotation and translation subalgebras in the q-analog are nontrivially deformed.

Examination shows that a rotation subalgebra may become nonclosed within a specific approach (cf. Celeghini et al. in [2]) to q-deformation; moreover, in most of the examples of q-deformed Euclidean algebras [1–3], either the whole (rotation or translation) subalgebra remains nondeformed (i.e., coincides with classical one) or at least some from the set of translations are still commuting.

The purpose of this contribution is to describe a certain nonstandard version of the q-deformed inhomogeneous algebras $U_q(iso_n)$ (i.e., q-Euclidean algebras) as well as their representations, obtained by a simple contraction procedure from the nonstandard q-deformed algebras $U'_q(so_{n+1})$, and their representations which were proposed and studied in [5]. As will be seen, our q-analogs are obtained in a uniform fashion for all values $n \geq 2$, and the same concerns their representations. Other viable features are: the homogeneous (rotation) subalgebra remains closed and becomes completely deformed; moreover, the translation generators are all mutually noncommuting (in fact, they q-commute).

2. q-Deformed inhomogeneous algebras $U'_q(iso_n)$

The well-known connection of the inhomogeneous (Euclidean) algebras $iso(n)$ to the Lie algebras $so(n+1)$ of orthogonal Lie groups performed by means of the procedure of contraction [6] is applied here to the non-standard q-deformed algebras $U'_q(so_{n+1})$ (studied in [5]) in order to obtain the corresponding version $U'_q(iso_n)$ of q-deformed inhomogeneous algebras.

A. Nonstandard q-deformed algebras $U'_q(so_{n+1})$

We consider a q-deformation of the orthogonal Lie algebras $so(n, C)$ that essentially differs from the standard quantum algebras $U_q(B_r)$, $U_q(D_r)$ given by Jimbo and Drinfeld [4]. It is known that, in order to describe explicitly finite-dimensional irreducible representations (irreps) of the q-deformed algebras $U_q(so(n, C))$ and their compact real forms, one needs a q-analog of the Gelfand-Tsetlin (GT) basis and the GT action formulas that require the existence of canonical embeddings q-analogous to the chain $so(n, C) \supset so(n-1, C) \supset \cdots \supset so(3, C)$. Evidently, such embeddings do not hold for the standard quantum algebras $U_q(B_r)$ and $U_q(D_r)$. Another feature is the restricted set of possible noncompact real forms admitted by Drinfeld-Jimbo's q-algebras, which exclude the Lorentz signature in multidimensional cases. On the contrary, the nonstandard q-deformation $U'_q(so(n, C))$ does admit [5] all noncompact real forms that exist in the classical case. Moreover, validity of the chain of embeddings $U_q(so(n,1)) \supset U_q(so(n)) \supset U_q(so(n-1)) \supset \cdots \supset U_q(so(3))$ allows us to construct and analyze infinite-dimensional representations of the q-Lorentz algebras $U'_q(so_{n,1})$.

According to [5], the nonstandard q-deformation $U'_q(so(n, C))$ of the Lie algebra$so(n, C)$ is given as a complex associative algebra with $n-1$ generating elements $I_{21}, I_{32}, \ldots, I_{n,n-1}$ obeying the defining relations (denote $q + q^{-1} \equiv [2]_q$)

$$I^2_{j,j-1} I_{j-1,j-2} + I_{j-1,j-2} I^2_{j,j-1} - [2]_q \; I_{j,j-1} I_{j-1,j-2} I_{j,j-1} = -I_{j-1,j-2}, \tag{1}$$

$$I^2_{j-1,j-2} I_{j,j-1} + I_{j,j-1} I^2_{j-1,j-2} - [2]_q \; I_{j-1,j-2} I_{j,j-1} I_{j-1,j-2} = -I_{j,j-1}, \tag{2}$$

$$[I_{i,i-1}, I_{j,j-1}] = 0 \qquad \text{if} \quad |\, i-j \,| > 1. \tag{3}$$

The compact and noncompact (of the Lorentz signature) real forms $U_q(\mathrm{so}_n)$and $U_q(\mathrm{so}_{n-1,1})$ are singled out from the complex q-deformed algebra $U_q(\mathrm{so}(n, C))$ by means of appropriate $*$-structures [5] which read in the compact case:

$$I^*_{j,j-1} = -I_{j,j-1}, \qquad j = 2, \ldots, n. \tag{4}$$

Besides the definition in terms of trilinear relations, *one can also give a 'bilinear' presentation* (useful for comparison to other approaches). To this end, one introduces the generators (here, $k > l+1, \;\; 1 \le k, l \le n$)

$$I^{\pm}_{k,l} \equiv [I_{l+1,l}, I^{\pm}_{k,l+1}]_{q^{\pm 1}} \equiv q^{\pm 1/2} I_{l+1,l} I^{\pm}_{k,l+1} - q^{\mp 1/2} I^{\pm}_{k,l+1} I_{l+1,l}$$

together with $I_{k+1,k} \equiv I^+_{k+1,k} \equiv I^-_{k+1,k}$. Then (1)–(3) imply

$$\begin{aligned} &[I^+_{lm}, I^+_{kl}]_q = I^+_{km}, \quad [I^+_{kl}, I^+_{km}]_q = I^+_{lm}, \quad [I^+_{km}, I^+_{lm}]_q = I^+_{kl} \quad \text{if} \;\; k > l > m, \\ &[I^+_{kl}, I^+_{mn}] = 0 \qquad \text{if} \;\; k > l > m > n \qquad \text{or if} \;\; k > m > n > l; \\ &[I^+_{kl}, I^+_{mn}] = (q - q^{-1})(I^+_{ln} I^+_{km} - I^+_{kn} I^+_{ml}) \qquad \text{if} \;\; k > m > l > n. \end{aligned} \tag{5}$$

An analogous set of relations is obtained for I^-_{kl} combined with $q \to q^{-1}$ (denote this alternative set by (5′)). When $q \to 1$, i.e., in the 'classical' limit, both relations (5) and (5′) go over into those of $so(n+1)$.

B. Contraction of $U_q(so_{n+1})$ into $U_q(iso_n)$ in terms of trilinear relations

To obtain the deformed algebra $U_q(iso_n)$, we apply the contraction procedure in its usual form [6] first to the q-deformed algebra $U_q(so_{n+1})$ given in terms of the trilinear relations (1)–(3): replacing $I_{n+1,n} \to \rho\, P_n$, with the trivial replacement $I_{k+1,k} \to \tilde I_{k+1,k}$ for $1 \le k \le n-1$, and sending $\rho \to \infty$, we arrive at the relations [7]

$$\tilde I^2_{n,n-1} P_n + P_n \tilde I^2_{n,n-1} - [2]_q\ \tilde I_{n,n-1} P_n \tilde I_{n,n-1} = -P_n, \tag{6}$$

$$P_n^2 \tilde I_{n,n-1} + \tilde I_{n,n-1} P_n^2 - [2]_q\ P_n \tilde I_{n,n-1} P_n = 0, \tag{7}$$

$$[\tilde I_{k,k-1}, P_n] = 0 \qquad \text{if} \quad k < n, \tag{8}$$

which together with the rest of relations (that remain intact and form the subalgebra $U'_q(so_n)$) define the q-deformed inhomogeneous algebra $U'_q(iso_n)$. Of course, this real form of the complex inhomogeneous algebra $U'_q(iso(n, C)$ requires that the involution

$$I^*_{j,j-1} = -I_{j,j-1}, \quad j = 2, \ldots, n, \qquad P^*_n = -P_n, \tag{9}$$

be imposed (compare with (4)).

Observe that in the formulation just given, we have only a single ('senior') component of translation generators. The whole set of translations emerges when one uses the 'bilinear' approach discussed right below.

C. Contraction into $U'_q(iso_n)$ of the bilinear version of $U'_q(so_{n+1})$

Now let us contract relations (5), (5′). Set $I^\pm_{n+1,k} = \rho\, P^\pm_k$ for $1 \le k \le n$ as well as $I^\pm_{kl} = \tilde I^\pm_{kl}$ for $1 \le l < k \le n$, and then send $\rho \to \infty$. As a result, we get the equality

$$[P^\pm_l, P^\pm_m]_{q^{\pm 1}} = 0, \qquad 1 \le m < l \le n, \tag{10}$$

as well as the rest of relations that remain unchanged (formally, i.e., modulo replacement $I^\pm_{kl} \to \tilde I^\pm_{kl}$ and $I^\pm_{n+1,k} \to P^\pm_k$): those which form the subalgebra $U'_q(so_n)$ and those which characterize the transformation property of $P^\pm_k$ with respect to $U'_q(so_n)$.

If $q \to 1$, the set of relations defining $U'_q(iso_n)$ turns into commutation relations of the 'classical' algebra $iso(n)$. In what follows, we shall omit tildas over I_{kl} and the prime in the notation of q-deformed algebras.

Remark 1. The algebra $U_q(iso_n)$ contains the subalgebra $U_q(so_n)$ in canonical way, i.e., similarly to the embedding of nondeformed algebras: $so(n) \subset iso(n)$.

Remark 2. The generators of translations in $U_q(iso_n)$ are noncommuting: as seen from (10), they q-commute.

Remark 3. It can be proved that the element

$$C_2(U_q(iso_n)) \equiv \sum_{k=1}^{n} ([2]/2)^{k-1}\ (1/2)\{P^+_k, P^-_k\}$$

is central for the inhomogeneous q-algebra $U_q(iso_n)$. When $q \to 1$, it reduces to the Casimir C_2 of the classical $iso(n)$.

Let us quote examples of $U_q(iso_n)$ for small values $n = 2, 3$.

n = 2

$$[I_{21}, P_2]_q = P_1^+ \qquad [P_1^+, I_{21}]_q = P_2 \qquad [P_2, P_1^+]_q = 0 \tag{11}$$

n = 3

$$\begin{aligned} &[I_{21}, I_{32}]_q = I_{31}^+ && [I_{21}, P_3] = 0 \\ &[I_{32}, I_{31}^+]_q = I_{21} && [I_{32}, P_1^+] = 0 \\ &[I_{31}^+, I_{21}]_q = I_{32} && [I_{31}^+, P_2^+] = (q - q^{-1})(P_1^+ I_{32} - P_3 I_{21}) \end{aligned} \tag{12}$$

$$\begin{aligned} &[I_{32}, P_3]_q = P_2^+ && [I_{21}, P_2^+]_q = P_1^+ && [I_{31}^+, P_3]_q = P_1^+ \\ &[P_2^+, I_{32}]_q = P_3 && [P_1^+, I_{21}]_q = P_2^+ && [P_1^+, I_{31}^+]_q = P_3 \\ &[P_3, P_2^+]_q = 0 && [P_2^+, P_1^+]_q = 0 && [P_3, P_1^+]_q = 0 \end{aligned} \tag{13}$$

Remark 4. The Euclidean q-algebra $U_q(iso_3)$ contains the homogeneous subalgebra $U_q(so_3)$, given by the left column in (12), isomorphic to the (cyclically symmetric, Cartesian) q-deformed Fairlie-Odesskii algebra [8]. Besides, three columns in (13) represent three distinct *inhomogeneous subalgebras* of $U_q(iso_3)$, each isomorphic to $U_q(iso_2)$, conf. (11). This feature extends to higher n: the algebra $U_q(iso_n)$ contains n distinct subalgebras isomorphic to $U_q(iso_{n-1})$.

3. Representations of inhomogeneous algebras $U_q(iso_n)$

We proceed with finite-dimensional representations of the algebras $U_q(so_{n+1})$. These representations denoted by $T_{m_{n+1}}$ are given by 'highest weights' m_{n+1} consisting of $[\frac{n+1}{2}]$ components $m_{1,n}, m_{2,n}, \ldots, m_{[\frac{n+1}{2}],n+1}$ (here $[r]$ means the integer part of r) which are all integers or all half-integers satisfying the dominance conditions

$$m_{1,2k+1} \geq m_{2,2k+1} \geq \ldots \geq m_{k,2k+1} \geq 0 \qquad \text{if} \qquad n = 2k, \tag{14}$$

$$m_{1,2k} \geq m_{2,2k} \geq \ldots \geq m_{k-1,2k} \geq |m_{k,2k}| \qquad \text{if} \qquad n = 2k-1. \tag{15}$$

When restricted to subalgebra $U_q(so_n)$, the representation $T_{m_{n+1}}$ contains with multiplicity 1 those and only those irreps T_{m_n} for which the inequalities ('branching rules') similar to the nondeformed case [9] are satisfied:

$$m_{1,2k+1} \geq m_{1,2k} \geq m_{2,2k+1} \geq m_{2,2k} \geq \ldots \geq m_{k,2k+1} \geq m_{k,2k} \geq -m_{k,2k+1}, \tag{16}$$

$$m_{1,2k} \geq m_{1,2k-1} \geq m_{2,2k} \geq m_{2,2k-1} \geq \ldots \geq m_{k-1,2k-1} \geq | m_{k,2k} | \,. \tag{17}$$

For a basis in the representation space, we take (q-analogue of) the GT basis [9]. Its elements are labelled by the GT schemes

$$\{\xi_{n+1}\} \equiv \{m_{n+1}, m_n, \ldots, m_2\} \equiv \{m_{n+1}, \xi_n\} \equiv \{m_{n+1}, m_n, \xi_{n-1}\} \tag{18}$$

and denoted as $|\{\xi_{n+1}\}\rangle$ or simply $|\xi_{n+1}\rangle$.

We use the notation $[x] \equiv \dfrac{q^x - q^{-x}}{q - q^{-1}}$ for the q-number corresponding to a real number x. In what follows, q is not a root of unity.

The infinitesimal generator $I_{2k+1,2k}$ in the representation $T_{m_{2k+1}}$ of $U_q(so_{2k+1})$ acts upon the basis elements (18) according to (here $\beta \equiv \xi_{n-1}$)

$$
\begin{aligned}
T_{m_{2k+1}}(I_{2k+1,2k})|m_{2k+1}, m_{2k}, \beta\rangle &= \sum_{j=1}^{k} A_{2k}^{j}(m_{2k})|m_{2k+1}, m_{2k}^{+j}, \beta\rangle \\
&- \sum_{j=1}^{k} A_{2k}^{j}(m_{2k}^{-j})|m_{2k+1}, m_{2k}^{-j}, \beta\rangle
\end{aligned}
\tag{19}
$$

and the generator $I_{2k,2k-1}$ in the representation $T_{m_{2k}}$ of $U_q(so_{2k})$ acts as

$$
\begin{aligned}
T_{m_{2k}}(I_{2k,2k-1})|m_{2k}, m_{2k-1}, \beta\rangle &= \sum_{j=1}^{k-1} B_{2k-1}^{j}(m_{2k-1})|m_{2k}, m_{2k-1}^{+j}, \beta\rangle \\
- \sum_{j=1}^{k-1} B_{2k-1}^{j}(m_{2k-1}^{-j})|m_{2k}, m_{2k-1}^{-j}, \beta\rangle &+ \ i\ C_{2k-1}(m_{2k-1})|m_{2k}, m_{2k-1}, \beta\rangle.
\end{aligned}
\tag{20}
$$

In these formulas, $m_n^{\pm j}$ means that the j-th component $m_{j,n}$ of the highest weight m_n is to be replaced by $m_{j,n} \pm 1$; matrix elements A_{2k}^{j}, B_{2k-1}^{j}, C_{2k-1} are given in terms of 'l-coordinates' $l_{j,2k+1} = m_{j,2k+1} + k - j + 1$, $\quad l_{j,2k} = m_{j,2k} + k - j$ by the expressions

$$
\begin{aligned}
A_{2k}^{j}(\xi) = \left(\frac{[l_{j,2k}][l_{j,2k}+1]}{[2l_{j,2k}][2l_{j,2k}+2]}\right)^{\frac{1}{2}} &\left| \frac{\prod_{i=1}^{k}[l_{i,2k+1}+l_{j,2k}][l_{i,2k+1}-l_{j,2k}-1]}{\prod_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}][l_{i,2k}-l_{j,2k}]} \right. \\
&\times \left. \frac{\prod_{i=1}^{k-1}[l_{i,2k-1}+l_{j,2k}][l_{i,2k-1}-l_{j,2k}-1]}{\prod_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}+1][l_{i,2k}-l_{j,2k}-1]} \right|^{\frac{1}{2}},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
B_{2k-1}^{j}(\xi) = &\left| \frac{\prod_{i=1}^{k}[l_{i,2k}+l_{j,2k-1}][l_{i,2k}-l_{j,2k-1}]}{[2l_{j,2k-1}+1][2l_{j,2k-1}-1]\prod_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}][l_{i,2k-1}-l_{j,2k-1}]} \right. \\
&\times \left. \frac{\prod_{i=1}^{k-1}[l_{i,2k-2}+l_{j,2k-1}][l_{i,2k-2}-l_{j,2k-1}]}{[l_{j,2k-1}]^2\prod_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}-1][l_{i,2k-1}-l_{j,2k-1}-1]} \right|^{\frac{1}{2}},
\end{aligned}
\tag{22}
$$

$$
C_{2k-1}(\xi) = \frac{\prod_{s=1}^{k}[l_{s,2k}]\prod_{s=1}^{k-1}[l_{s,2k-2}]}{\prod_{s=1}^{k-1}[l_{s,2k-1}][l_{s,2k-1}-1]}.
\tag{23}
$$

The detailed proof that the representation operators defined by (19)–(23) satisfy the basic relations (1)–(3) of the algebra $U_q(\mathrm{so}_n)$ for $n = 2k+1$ and $n = 2k$ is given in [10]. It can be verified that the $*$-condition $T(I_{j,j-1})^* = -T(I_{j,j-1})$, $j = 2, \dots, n+1$ (compare with (4)), for representation operators given in (19)-(23) is fulfilled if $q \in \mathbf{R}$ or $q = \exp ih$, $h \in \mathbf{R}$. Therefore, the action formulas for $T_{m_{n+1}}(I_{n+1,n})$ together with similar formulas for the operators $T_{m_{n+1}}(I_{i,i-1})$, $i < n+1$, give irreducible infinitesimally unitary (or $*$-) representations of the algebra $U_q(\mathrm{so}_{n+1})$.

Representations of $U_q(iso_n)$

Representations of inhomogeneous algebras $U_q(iso_n)$ are obtained from the representations of $U_q(so_{n+1})$ given above in a manner similar to that followed by Chakrabarti [6] for obtaining irreps of Euclidean algebras $iso(n)$ from finite-dimensional irreps of rotation algebras $so(n+1)$.

The representations of $U_q(iso_n)$ are characterized by a complex number a and the set $\tilde{m}_{n+1} \equiv \{m_{2,n+1}, m_{3,n+1}, \dots, m_{[\frac{n+1}{2}],n+1}\}$ of numbers which are all integers or all half-integers. Due to validity of the chain of inclusions

$$U_q(\mathrm{iso}_n) \supset U_q(\mathrm{so}_n) \supset \dots \supset U_q(\mathrm{so}_4) \supset U_q(\mathrm{so}_3), \tag{24}$$

the representation space $\mathcal{V}_{a,\tilde{m}_{n+1}}$ for a representation of $U_q(iso_n)$ is taken as a direct sum of the representation spaces of the q-rotation subalgebra $U_q(so_n)$ given by m_n, whose components $m_{2,n}, \dots, m_{[\frac{n}{2}],n}$ satisfy the inequalities (16)-(17) and the first component $m_{1,n}$ is bounded only from below, $\infty \geq m_{1,n} \geq m_{2,n+1}$. In this way, one is led to infinite-dimensional representations of the inhomogeneous algebras $U_q(iso_n)$.

The representation operators $T_{a,\tilde{m}_{2k+1}}(I_{j,j-1})$ that correspond to the generators $I_{j,j-1}$ of the compact subalgebra $U_q(so_n)$ act according to formulas coinciding with (19)–(23).

The representation operator $T_{a,\tilde{m}_{2k+1}}(P_{2k})$ which corresponds to the translation generator P_{2k} of the algebra $U_q(iso_{2k})$ acts upon basis elements (12) according to (here $\beta \equiv \xi_{n-1}$)

$$\begin{aligned} T_{a,\tilde{m}_{2k+1}}(P_{2k})|\tilde{m}_{2k+1}, m_{2k}, \beta\rangle = &\sum_{j=1}^{k} \mathcal{A}^j_{2k}(m_{2k})|\tilde{m}_{2k+1}, m^{+j}_{2k}, \beta\rangle \\ &- \sum_{j=1}^{k} \mathcal{A}^j_{2k}(m^{-j}_{2k})|\tilde{m}_{2k+1}, m^{-j}_{2k}, \beta\rangle \end{aligned} \tag{25}$$

and the representation operator $T_{a,\tilde{m}_{2k}}(P_{2k-1})$ which corresponds to the translation generator P_{2k-1} of the algebra $U_q(iso_{2k-1})$ acts as

$$\begin{aligned} &T_{a,\tilde{m}_{2k}}(P_{2k-1})|\tilde{m}_{2k}, m_{2k-1}, \beta\rangle = \sum_{j=1}^{k-1} \mathcal{B}^j_{2k-1}(m_{2k-1})|\tilde{m}_{2k}, m^{+j}_{2k-1}, \beta\rangle \\ &- \sum_{j=1}^{k-1} \mathcal{B}^j_{2k-1}(m^{-j}_{2k-1})|\tilde{m}_{2k}, m^{-j}_{2k-1}, \beta\rangle + \; i\, \mathcal{C}_{2k-1}(m_{2k-1})|\tilde{m}_{2k}, m_{2k-1}, \beta\rangle \end{aligned} \tag{26}$$

where

$$\mathcal{A}^j_{2k}(\xi) = a\left(\frac{[l_{j,2k}][l_{j,2k}+1]}{[2l_{j,2k}][2l_{j,2k}+2]}\right)^{\frac{1}{2}}\left|\frac{\prod_{i=2}^{k}[l_{i,2k+1}+l_{j,2k}][l_{i,2k+1}-l_{j,2k}-1]}{\prod_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}][l_{i,2k}-l_{j,2k}]}\right.$$
$$\left.\times\frac{\prod_{i=1}^{k-1}[l_{i,2k-1}+l_{j,2k}][l_{i,2k-1}-l_{j,2k}-1]}{\prod_{i\neq j}^{k}[l_{i,2k}+l_{j,2k}+1][l_{i,2k}-l_{j,2k}-1]}\right|^{\frac{1}{2}}, \tag{27}$$

$$\mathcal{B}^j_{2k-1}(\xi) = a\left|\frac{\prod_{i=2}^{k}[l_{i,2k}+l_{j,2k-1}][l_{i,2k}-l_{j,2k-1}]}{[2l_{j,2k-1}+1][2l_{j,2k-1}-1]\prod_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}][l_{i,2k-1}-l_{j,2k-1}]}\right.$$
$$\left.\times\frac{\prod_{i=1}^{k-1}[l_{i,2k-2}+l_{j,2k-1}][l_{i,2k-2}-l_{j,2k-1}]}{[l_{j,2k-1}]^2\prod_{i\neq j}^{k-1}[l_{i,2k-1}+l_{j,2k-1}-1][l_{i,2k-1}-l_{j,2k-1}-1]}\right|^{\frac{1}{2}}, \tag{28}$$

$$\mathcal{C}_{2k-1}(\xi) = a\frac{\prod_{s=2}^{k}[l_{s,2k}]\prod_{s=1}^{k-1}[l_{s,2k-2}]}{\prod_{s=1}^{k-1}[l_{s,2k-1}][l_{s,2k-1}-1]}. \tag{29}$$

The representation operators given by formulas (25)–(29) (together with formulas (19)–(23) for the subalgebra $U_q(\mathrm{so}_n)$) can be proved to satisfy the defining relations (6)–(8), in complete analogy to the proof [10] in the case of homogeneous algebra $U_q(\mathrm{so}_n)$. Moreover, it can be verified that these representations are $*$-representations (satisfy $*$-relations (9)), for q real or the pure phase if a is real in formulas (25)–(29).

4. Class 1 representations of the inhomogeneous algebra $U_q(iso_n)$

We use the term class 1 (or C1)representation for those representations of either $U_q(so_{n+1})$ or $U_q(iso_n)$ which contain the trivial (identical) representation of the maximal compact subalgebra $U_q(\mathrm{so}_n)$. Note that among representations of $U_q(\mathrm{so}_3)$ all irreps T_l are given by an integer l and only these are C1 representations with respect to the Abelian subalgebra generated by I_{21}.

The particular case T_a (C1 representations of $U_q(iso_n)$ characterized by a single complex number a) was considered in [11]. These special representations are obtainable from our general formulas (19)–(23), (25)–(29) if we set $m_{2,n+1} = m_{3,n+1} = \cdots = m_{[\frac{n+1}{2}],n+1} = 0$.

The carrier space V_a of T_a is composed of carrier spaces $V_{\mathbf{m}_n}$ of irreps $T_{\mathbf{m}_n}$ of the subalgebra $U_q(\mathrm{so}_n)$ with the signatures $(m_{1,n}, 0, \ldots, 0)$, $\infty \geq m_{1,n} \geq 0$ (which in turn are

C1 irreps w.r.t. $U_q(\mathrm{so}_{n-1})$). Accordingly, the basis in V_a is composed as the union of G-T bases of such subspaces $V_{\mathbf{m}_n}$. We denote basis elements in the representation space $V_{\mathbf{m}_n}$ by $|m_n, m_{n-1}, \cdots, m_3, m_2\rangle$.

The operator $T_a(I_{21})$ and operators $T_a(I_{i,i-1})$, $3 \le i \le n$, representing generators of subalgebra $U_q(iso_n)$ act in this basis by the formulas

$$
\begin{aligned}
&T_a(I_{21})|m_n, m_{n-1}, \ldots, m_2\rangle = i[m_2]|m_n, m_{n-1}, \ldots, m_2\rangle, \\
&T_a(I_{i,i-1})|m_n, m_{n-1}, \ldots, m_2\rangle = \\
&\quad = \left([m_i^{(+)} + i - 2][m_i^{(-)}]\right)^{1/2} \; R(m_{i-1})|m_n, \ldots, m_{i-1} + 1, \ldots, m_2\rangle \\
&\quad - \left([m_i^{(+)} + i - 3][m_i^{(-)} + 1]\right)^{1/2} R(m_{i-1} - 1)|m_n, \ldots, m_{i-1} - 1, \ldots, m_2\rangle,
\end{aligned}
\tag{30}
$$

where $m_i^{(\pm)} \equiv m_i \pm m_{i-1}$ and

$$
R(m_i) = \left(\frac{[m_i^{(+)} + i - 2][m_i^{(-)} + 1]}{[\,2\ m_i + i - 2][\,2\ m_i + i]} \right)^{1/2} .
$$

The operator $T_a(P_n)$ of the representation T_a of $U_q(iso_n)$, $n \ge 2$, corresponding to the translation P_n is given by the formula

$$
\begin{aligned}
T_a(P_n)|m_n, m_{n-1}, \cdots, m_2\rangle &= a\ R(m_n)|m_n + 1, m_{n-1}, m_{n-2}, \cdots m_2\rangle \\
&\quad -a\ R(m_n - 1)|m_n - 1, m_{n-1}, m_{n-2}, \cdots, m_2\rangle
\end{aligned}
\tag{31}
$$

In summary, we have presented the nonstandard q-deformed inhomogeneous algebras $U_q(iso_n)$ defined in a uniform manner for all $n \ge 2$, for which both 'trilinear' and 'bilinear' presentations were given. All the infinite-dimensional representations of $U_q(iso_n)$ that directly correspond to well-known irreducible representations of the classical limit iso_n are obtained and illustrated with the particular case of class 1 irreps. It is an interesting task to analyze $U_q(iso_n)$ representations for cases where discrete components characterizing representations are not all integers or all half-integers as well as the cases of q being roots of 1.

Acknowledgements.

The authors are thankful to Prof. A.U. Klimyk for valuable discussions. This research was supported in part by Award No. UP1-309 of the U.S. Civilian Research & Development Foundation for the Independent States of the Former Soviet Union (CRDF), and by DFFD Grant 1.4/206.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 393–396.

Representation of Real Forms of Witten's First Deformation

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Abstract

Special star structures on Witten's first deformation are found. Description of all irreducible representations in the category of Hilbert spaces of the found $*$-algebras in both bounded and unbounded operators is obtained.

1 Introduction

Studying Jone's polynomials in node theories, their generalization and their connections with "vertex models" in two-dimensional statistical mechanics, Witten presents Hoph algebra deformations of the universal enveloping algebra $su(2)$. There is a family of associative algebras depending on a real parameter p. These algebras are given by the generators E_0, E_+, E_- and relations [2]:

$$pE_0E_+ - \frac{1}{p}E_+E_0 = E_+, \tag{1}$$

$$[E_+, E_-] = E_0 - \left(p - \frac{1}{p}\right) E_0^2, \tag{2}$$

$$pE_-E_0 - \frac{1}{p}E_0E_- = E_-. \tag{3}$$

In Section 1, we write down all (but from a certain class) star algebra structures on Witten's first deformation. In Section 3, we give description of all irreducible representations in bounded or unbounded operators of the found $*$-algebras in the category of Hilbert's spaces. We widely use the "dynamical relations" method developed in [3].

2 Real forms

Witten's first deformation is a family of associative algebras A_p given by generators and relations (1)–(3), with the parameter p in $(0, 1)$. Extreme cases are $p = 1$, which is $su(2)$, and $p = 0$, with relations $E_+E_0 = E_0E_+ = E_0^2 = 0$. We consider stars on A_p, which are obtained from involutions on the free algebra with the invariant lineal subspace generated by relations (1)–(3).

Stars are equivalent if the corresponding real forms are isomorphic.

Lemma 1 *There are only two unequivalent stars on Witten's first deformation:*

$$E_0^* = E_0, \qquad E_+^* = E_-, \tag{4}$$

$$E_0^* = E_0, \qquad E_+^* = -E_-. \tag{5}$$

3 Dynamical relations.

Relations (1)–(3) with star (4) or (5) form a dynamic relation system [3]. The corresponding dynamic system on $\mathbb{R}^2$:

$$F(x,y) = \Big(p^{-1}(1+p^{-1}x), g(gy - x + (p-p^{-1})x^2)\Big),$$

where $g = 1$ ($g = -1$) for the first (second) real form.

It's a quite difficult task to find positive orbits of a two-dimensional nonlinear dynamic system, to avoid these difficulties, we use Casimir element:

$$C_p = p^{-1}E_+E_- + pE_-E_+ + E_0^2.$$

For any irreducible representation T: $T(C_p) = \mu I$, where μ is complex.

We will be working with the following system:

$$E_0E_+ = E_+f(E_0), \tag{6}$$

$$E_+^*E_+ = G_\nu(E_0), \tag{7}$$

where $\nu = \mu p$,

$$f(X) = p^{-1}(1+p^{-1}x),$$

$$G_\nu(y) = \frac{g}{1+p^2}(-y - p^{-1}y^2 + \nu I).$$

Lemma 2 *For any irreducible representation T of the real form A_p, there is a unique ν ($\nu \geq 0$ for the first real form) such that T is the representation of (6), (7).*

For an arbitrary ν ($\nu \geq 0$ for the first real form), every irreducible representation T of (6), (7) with $dim T > 1$ is a representation of A_p.

The dynamic system corresponding to relations (6), (7) is actually one-dimensional and linear depending on one real parameter.

Below we compile some basic facts from [3]. Every irreducible representation of a dynamic system is determined by the subset $\mathbb{R}^2 \supset \Delta$:

$$\Delta = \{(\lambda_k, \mu_k), j < k < J\},$$

where $\lambda_{k+1} = f(\lambda_k)$, $\mu_{k+1} = G_\nu(\lambda_k)$, $\mu_k \geq 0$, $\mu_k = 0$ for extreme k; j, J are integer or infinite; $l_2(\Delta)$ is a Hilbert space with the orthonormed base: $\{e_{(\lambda_k,\mu_k)} : (\lambda_k,\mu_k) \in \Delta\}$,

$$T(E_0)e_{(\lambda_k,\mu_k)} = \lambda_k e_{(\lambda_k,\mu_k)},$$

$$T(E_+)e_{(\lambda_k,\mu_k)} = \mu_{k+1}^{1/2} e_{(\lambda_{k+1},\mu_{k+1})}, \qquad j < k+1 < J,$$

$$T(E_-)e_{(\lambda_k,\mu_k)} = \mu_k^{1/2} e_{(\lambda_{k-1},\mu_{k-1})}, \qquad j < k-1,$$

$$T(E_+)e_{(\lambda_{J-1},\mu_{J-1})} = 0, \qquad T(E_-)e_{(\lambda_{j+1},\mu_{j+1})} = 0.$$

4 Classification of representations

Theorem 1 *Every irreducible representation of the first real form is bounded*

1. For every nonnegative integer m*, there is a representation of dimension* $m+1$*, with* $\nu = \frac{p}{4}\left((\frac{(1-p^{2m})(1+p^2)}{(1+p^{2m})(1-p^2)})^2 - 1 \right)$*,* $\Delta_\nu = \{f(k, x_1), -1 < k \leq m+1\}$*;*

2. There is a family of one-dimensional representations $E_0 = \frac{p}{(p^2-1)}$*;* $E_+ = \lambda$*;* $E_- = \bar{\lambda}$*;* λ *is complex,* $\nu = \frac{p^3}{(1-p^2)^2}$*;*

3. For every $\nu \in \left[\frac{p^3}{(1-p^2)^2}; +\infty\right)$*, there is a representation with the upper weight* $\Delta_\nu = \{f(k, x_2),\ k < 1\}$*;*

4. For every nu $\in \left[\frac{p^3}{(1-p^2)^2}; +\infty\right)$*, there is a representation with the lower weight:* $\Delta_\nu = \{f(k, x_1),\ k < 1\}$*.*

In the theorem, we have used notation

$$f(k, x) = \frac{1}{p^{2k}} \left(x + \frac{p^{2k}-1}{p^2-1} p \right); \qquad g_\nu(x) = -x - p^{-1}x^2 + \nu,$$

were $x_1 < x_2$ are roots of the equation $g_\nu(x) = 0$.

Theorem 2 *There are bounded and unbounded irreducible representations of the second real form in an infinite-dimensional Hilbert space, except for a one-dimensional representation.*

1. *All bounded irreducible representations have the upper weight*

$$\nu \in \left[-\frac{p}{4}; \frac{p^3}{(1-p^2)^2}\right), \qquad \Delta_\nu = \{f(k, x_1), k < 1\};$$

2. *Unbounded irreducible representations:*

   (a) *There are two families with the upper weight*
   first family:

$$\nu \in \left(-\frac{p}{4}\ ;\ 0\right), \qquad \Delta_\nu = \{f(k, x_2), k < 1\}$$

   second family:

$$\nu \in \left(\frac{p^3}{(1-p^2)^2}; +\infty\right), \qquad \Delta_\nu = \{f(k, x_1), k > -1\}$$

   (b) *There are two families with the lower weight*
   first family:

$$\nu \in \left[\frac{p}{4}\ ; 0\right), \qquad \Delta_\nu = \{f(k, x_2), k > -1\}$$

   second family:

$$\nu \in \left(\frac{p^3}{(1-p^2)^2}; +\infty\right), \qquad \Delta_\nu = \{f(k, x_1), k > -1\}$$

(c) There are two families without the upper and lower weights:
first family is numerated by the set

$$\tau = \{(\lambda,\nu) : \lambda \in \left(-p\ ; -\frac{p}{2}\right) \cup \left(-\frac{p}{2}\ ; 0\right], \nu \in (-\infty\ ; \lambda(\lambda+p)p^{-1})\}$$

$$\Delta_{(\lambda,\nu)} = \{f(k,\lambda), k \in \mathbb{Z}\}$$

second family is numerated by the set

$$\epsilon = [-p^{-2}\ ; -p-1) \times (-\infty\ ; p^3(1-p^2)^{-1}), \quad \Delta_{(\lambda,\nu)} = \{f(k,\lambda), k \in \mathbb{Z}\}.$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 397–402.

Deformed Oscillators with Interaction

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Abstract

Deformation of the Heisenberg-Weyl algebra W^s of creation-annihilation operators is studied and the problem of eigenvalues of the Hamiltonian for s deformed oscillators with interaction is solved within this algebra. At first, types of deformation are found for which solutions could be presented analytically and a simple $q-$deformation is considered by the means of the perturbation theory. Cases of reducing Hamiltonians which do not preserve the total particle number to that studied here are indicated.

Investigations of new types of symmetries in different areas of mathematical physics by using the Inverse Scattering Method (ISM) and nonlinear differential equations, led to the appearance of notions "Quantum Group", "Quantum Algebra" [1]. Nowadays, the deformed quantum statistics, parastatistics and corresponding algebras of creation-annihilation operators are widely studied.

The common Heisenberg-Weyl algebra W^s consists of generators $\widehat{a}_i$, $\widehat{a}_i^+$, $\widehat{n}_i$, $i = 1, \ldots, s$, satisfying commutative relations:

$$\begin{aligned} &[\widehat{a}_i, \widehat{a}_j^+] = \delta_{ij}, \qquad [\widehat{n}_i, \widehat{a}_j^\pm] = \pm \widehat{a}_j^\pm \delta_{ij}, \\ &[\widehat{a}_i, \widehat{a}_j] = [\widehat{a}_i^+, \widehat{a}_j^+] = 0, \qquad \widehat{n}_i^+ = \widehat{n}_i. \end{aligned} \tag{1}$$

In this paper, the deformed algebra $A_d(s)$ with generators $\widehat{a}_i$, $\widehat{a}_i^+$, $\widehat{n}_i$, $i = 1, \ldots, s$, satisfying:

$$\begin{aligned} &[\widehat{a}_i, \widehat{a}_j^+] = f_i(\widehat{n}_1, \ldots, \widehat{n}_s)\delta_{ij}, \\ &\text{the rest relations are the same as in (1)} \end{aligned} \tag{2}$$

is considered. If we assume $f_i(\widehat{n}_1, \ldots, \widehat{n}_s) = 1$, we return to the algebra W^s (1). One-dimensional deformed oscillators were studied in different papers [2, 3, 10].

Here, we study the problem of eigenvalues of the Hamiltonian

$$\widehat{H} = \sum_{i,j=1}^{s} w_{ij} \widehat{a}_i^+ \widehat{a}_j \tag{3}$$

which gives us s deformed oscillators with interaction.

Algebraically solvable Case

By means of the linear transformation

$$\widehat{b}_i = \sum_k \alpha_{ik} \widehat{a}_k, \tag{4}$$

where $||\alpha_{ik}||$ is unitary matrix with

$$\sum_k \alpha_{ik} \alpha^*_{jk} = \delta_{ij}, \tag{5}$$

one can receive a new set of generators $b_i, b_i^+, N_i, \quad i = 1, \ldots, s$, which determine a new algebra $B_d(s)$.

Now we can rewrite (3) using the generators of the algebra $B_d(s)$:

$$\widehat{H} = \sum_{ij} w'_{ij} \widehat{b}_i^+ \widehat{b}_j, \tag{6}$$

where $w'_{ij} = \sum_{kl} w_{kl} \alpha_{ik} \alpha^*_{jl}$.

Choosing the appropriate matrix $||\alpha_{ik}||$, we can diagonalize $||w'_{ij}||$ and obtain

$$\widehat{H} = \sum_i w'_i \widehat{b}_i^+ \widehat{b}_i. \tag{7}$$

This Hamiltonian gives us a set of noninteracting deformed oscillators. To find eigenvalues of (7), one should know the commutative relations on $B_d(s)$:

$$[\widehat{b}_i, \widehat{b}_j^+] = F_{ij}(\widehat{n}_1, \ldots, \widehat{n}_s),$$

the rest are the same as in (1) (8)

(considering the substitution $\widehat{a}_i \to \widehat{b}_i,\ \widehat{n}_i \to \widehat{N}_i$),

where

$$F_{ij}(\widehat{n}_1, \ldots, \widehat{n}_s) = \sum \alpha_{ik} \alpha^*_{jk} f_i(\widehat{n}_1, \ldots, \widehat{n}_s) \tag{9}$$

are the functions of "old" generators $\widehat{n}_1, \ldots, \widehat{n}_s$, but we should express the right side of the first equation (8) in the terms of "new" generators $\widehat{N}_1, \ldots, \widehat{N}_s$

The operators $\widehat{N}_i$ cannot be represented as functions of operators $\widehat{n}_i$ generally. Therefore, in the case of the arbitrary algebra $A_d(s)$, the transition from $\widehat{n}_i$ to $\widehat{N}_i$ on the right side of the first equation (8) cannot be fulfilled.

One can introduce operators

$$\widehat{n} = \widehat{n}_1 + \cdots + \widehat{n}_s, \qquad \widehat{N} = \widehat{N}_1 + \cdots + \widehat{N}_s, \tag{10}$$

of the total particle number for the old ($A_d(s)$) and new ($B_d(s)$) algebras, respectively.

It can be easily proved that

$$\widehat{n} = \widehat{N}. \tag{11}$$

So, if we have deformed the algebra $A_d(s)$ with functions f_i satisfying

$$f_i(\widehat{n}_1, \ldots, \widehat{n}_s) = f_i(\widehat{n}_1 + \cdots + \widehat{n}_s) = f_i(\widehat{n}), \tag{12}$$

then the first relation (8) will read due to (11) as

$$[\widehat{b}_i, \widehat{b}_j^+] = F_{ij}(\widehat{N}), \tag{13}$$

where

$$F_{ij}(\widehat{N}) = \sum_k \alpha_{ik}\alpha_{jk}^* f_i(\widehat{N}). \tag{14}$$

Eigenvalues of (7) can be immediately found now:

$$E_{N_1 \ldots N_S} = \sum_{i=1}^{s} w_i' \beta_i(N, N_i) \tag{15}$$

with the corresponding eigenfunctions $|N_1, \ldots, N_s >$, where

$$\beta_i(N, N_i) = \sum_{j=1}^{N_i} F_{ii}(N - j), \qquad N = N_1 + \cdots + N_s.$$

These deformed algebras $A_d(s)$ (2), (12), for which the problem of eigenvalues is solvable, don't factorize, i.e., cannot be represented as sets of s independent deformed algebras.

q-Deformation

Let us consider now the deformed algebra $A_q(2)$ consisting of 2 one-dimensional q-deformed algebras with generators $\widehat{a}$, $\widehat{b}$ ($\widehat{a} = \widehat{a}_1$, $\widehat{b} = \widehat{a}_2$) and standard particle number operators $\widehat{n}_a$, $\widehat{n}_b$ satisfying:

$$\widehat{a}\widehat{a}^+ - q\widehat{a}^+\widehat{a} = 1, \qquad \widehat{b}\widehat{b}^+ - q\widehat{b}^+\widehat{b} = 1. \tag{16}$$

After introducing new operators

$$\widehat{S}^+ = \widehat{a}^+ b, \qquad \widehat{S}^- = \widehat{b}^+ a, \tag{17}$$

Hamiltonian (3) will read:

$$\widehat{H} = w_a\widehat{n}_a + w_b\widehat{n}_b + v(\widehat{S}^+ + \widehat{S}^-). \tag{18}$$

The set of generators $\{\widehat{n}_a, \widehat{b}_a, \widehat{S}^+, \widehat{S}^-\}$ satisfies:

$$[\widehat{n}_a, \widehat{S}^\pm] = \pm\widehat{S}^\pm, \quad [\widehat{n}_b, \widehat{S}^\pm] = \mp\widehat{S}^\pm, \quad [\widehat{n}_a, \widehat{n}_b] = 0, \quad [\widehat{S}^+, \widehat{S}^-] = [\widehat{n}_a] - [\widehat{n}_b], \tag{19}$$

where $[x]$ means the function $[x] = \dfrac{q^x - 1}{q - 1}$.

The Casimir operator of the algebra $\{\widehat{n}_a, \widehat{b}_a, \widehat{S}^+, \widehat{S}^-\}$ is

$$\widehat{K} = \widehat{n} = \widehat{n}_a + \widehat{n}_b. \tag{20}$$

Consider the problem of eigenvalues of the reduced Hamiltonian (18) on invariant subspaces $K_n = L(|i, n-i> \;\; | \; i = 1, \ldots, n)$. One can obtain basis vectors in K_n by using the vector $|0, n>$:

$$|i, n-i> = \sqrt{\frac{[n-1]!}{[n]![i]!}}(S^+)^i|0, n>, \tag{21}$$

where $[n]! = [n][n-1]\cdots[1]$.

Now let us take the representation where operators $\{\widehat{S}^+|_{K_n}, \widehat{S}^-|_{K_n}, \widehat{n}_a|_{K_n}, \widehat{n}_b|_{K_n}\}$ reduced on the subspaces K_n are given by:

$$\begin{aligned}
&\widehat{S}^+|_{K_n} = t,\\
&\widehat{S}^+|_{K_n} = \frac{1}{t}\left[t\frac{t}{dt}\right]\left[n+1-t\frac{t}{dt}\right],\\
&\widehat{n}_a|_{K_n} = t\frac{t}{dt},\\
&\widehat{n}_b|_{K_n} = n - t\frac{t}{dt}.
\end{aligned} \tag{22}$$

Vectors $\psi \in \widehat{K}_n$ read:

$$\psi = \sum_{i=1}^{n} C_i t^i. \tag{23}$$

In view of (21), one can write $\psi = \left(\sum_{i=1}^{n} C_i S^{+i}\right)\Big|0, n>$ and, taking into account the first equation (22), we receive (23). Now we obtain the equation

$$\widehat{T}_n\psi(t) = E\psi(t), \tag{24}$$

where

$$\widehat{T}_n = v \cdot \left(t + \frac{1}{t}\left[t\frac{t}{dt}\right]\left[n+1-\frac{t}{dt}\right]\right) + w_1 \cdot \left[t\frac{t}{dt}\right] + w_2 \cdot \left[n - t\frac{t}{dt}\right], \tag{25}$$

on the condition that $\psi(t)$ is an analytic function. In the nondeformed case ($q = 1$), equation (24) can be transformed into a degenerated hypergeometric equation and easily solved:

$$E_{nm} = n\frac{w_1 + w_2}{2} + (2m - n)\sqrt{\left(\frac{w_1 - w_2}{2}\right)^2 + v^2}. \tag{26}$$

In the deformed case, equation (24) was solved by the means of perturbation theory, using the potential of interaction v between oscillators as a small parameter. Energy eigenvalues were found up to the 5-th order of perturbation theory:

$$\begin{aligned}
&E_0^{(k)} = w_1 \cdot [k] + w_2 \cdot [n-k], \qquad E_2^{(k)} = \frac{B(k-1)}{\Gamma(k-1)} + \frac{B(k)}{\Gamma(k+1)},\\
&E_4^{(k)} = \frac{B(k-1)}{\Gamma(k-1)^2}\left\{\frac{B(k-2)}{\Gamma(k-2)} - \frac{B(k-1)}{\Gamma(k-1)} - \frac{B(k)}{\Gamma(k+1)}\right\} +\\
&\qquad + \frac{B(k)}{\Gamma(k-1)^2}\left\{\frac{B(k+1)}{\Gamma(k+2)} - \frac{B(k-1)}{\Gamma(k-1)} - \frac{B(k)}{\Gamma(k+1)}\right\},
\end{aligned} \tag{27}$$

where $B(p) = [p+1][n-p]$, $\Gamma(p) = w\left([k]-[p]\right) + w\left([n-k]+[n-p]\right).$

All odd approximations are equal to zero:

$$E_1 = E_3 = \cdots = E_{2p+1} = 0. \tag{28}$$

Generalized Weyl Shift

A Weyl shift of the nondeformed algebra W^s reads:

$$\widehat{b}_i = \widehat{a}_i + \alpha_i. \tag{29}$$

For the deformed algebra $A_q(s)$, it can be assumed in the form proposed in [4]:

$$\widehat{b}_i = F_i(\widehat{n}_i)\widehat{a}_i - f_i(\widehat{n}_i), \qquad f_i = w_i q^{\widehat{n}_i}, \qquad F_i = \sqrt{1-|w_i|^2(1-q)q^{\widehat{n}_i}}. \tag{30}$$

Hamiltonian (3) preserves the total particle number $n = n_1 + \cdots + n_s$, but if we make the generalized Weyl shift (30), it will not. There will be some terms in a shifted Hamiltonian, which will increase or decrease the particle number by 1.

Our aim is to find "shifted" Hamiltonians (which violate the particle number at 1 unit) which can be reduced to "neutral" Hamiltonians (preserving the particle number n). The general "neutral" Hamiltonian which covers case (3) reads:

$$\widehat{H} = \sum_{ij} \phi_{ij}(\widehat{b}_i^+\widehat{b}_i, \widehat{b}_j^+\widehat{b}_j, \widehat{b}_i^+\widehat{b}_j, \widehat{b}_j^+\widehat{b}_i). \tag{31}$$

After shift (30), arguments of (31) will read as

$$\begin{aligned} \widehat{b}_i^+\widehat{b}_j &= \left(\widehat{a}_i^+F_i^+ - f_i^+\right)\left(F_j\widehat{a}_j - f_j\right) = \\ &= \widehat{a}_i^+\widehat{\sigma}_{ij}(\widehat{n}_i,\widehat{n}_j)\widehat{a}_j + \widehat{a}_i^+\widehat{\pi}_{ij}(\widehat{n}_i,\widehat{n}_j) + \widehat{\pi}^+ij(\widehat{n}_i,\widehat{n}_j)\widehat{a}_j + \widehat{\omega}ij(\widehat{n}_i,\widehat{n}_j), \end{aligned} \tag{32}$$

where $\widehat{\sigma}^{ij}, \widehat{\pi}^{ij}, \widehat{\pi}^{ij}$ are some "neutral" functions. Developing (31) into the series in new shifted arguments (right side of (32)), we will receive some "shifted" Hamiltonian $\widehat{H}'$. If we demand that $\widehat{H}'$ violate n not more than by 1 unit, we should consider ϕ_{ij} to be a linear function of its arguments, because these arguments (32) do violate n by one unit, and higher powers of (32) will violate n more than by one unit. ϕ_{ij} reads:

$$\phi_{ij}(x,y,z,t) = \alpha_{ij}x + \beta_{ij}y + \gamma_{ij}z + \zeta_{ij}t, \tag{33}$$

and we receive Hamiltonian (3):

$$\widehat{H} = \sum_{i,j=1}^{s} w_{ij}\widehat{b}_i^+\widehat{b}_j, \tag{34}$$

where

$$w_{ij} = \gamma_{ij} + \zeta_{ij} + \delta_{ij}\sum_k(\alpha_{ik} + \beta_{ik}).$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 403–408.

On Duality for a Braided Cross Product

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Abstract

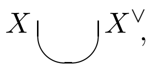
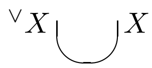
In this note, we generalize a result of [4] (see also [9]) and set the isomorphism between the iterated cross product algebra $H^\vee\#(H\#A)$ and braided analog of an A-valued matrix algebra $H^\vee\otimes A\otimes H$ for a Hopf algebra H in the braided category $\mathcal{C}$ and for an H-module algebra A. As a preliminary step, we prove the equivalence between categories of modules over both algebras and category whose objects are Hopf H-modules and A-modules satisfying certain compatibility conditions.

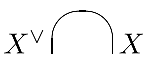
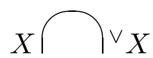
Introduction and preliminaries

A purpose of this note is to generalize the result of [4] (see also [9]) about the isomorphism between the iterated cross product algebra $H^*\#(H\#A)$ and A-valued matrix algebra $\mathrm{M}(H)\otimes A$ (for an H-module algebra A) to the fully braided case.

Throughout this paper, the symbol $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ denotes a strict monoidal category with braiding Ψ. For convenience of the reader, we recall the necessary facts about braided monoidal categories and Hopf algebras in them.

For object $X \in \mathcal{C}$, we say that $X^\vee$ and ${}^\vee X \in \mathcal{C}$ are dual objects if evaluation and coevaluation morphisms

$\mathrm{ev} : X \otimes X^\vee \to \mathbb{1} =$ , $\quad \mathrm{ev} : {}^\vee X \otimes X \to \mathbb{1} =$ ,

$\mathrm{coev} : \mathbb{1} \to X^\vee \otimes X =$ , $\quad \mathrm{coev}: \mathbb{1} \to X \otimes {}^\vee X =$ 

can be chosen so that the compositions

$$X = X \otimes \mathbb{1} \xrightarrow{1\otimes\mathrm{coev}} X \otimes (X^\vee \otimes X) = (X \otimes X^\vee) \otimes X \xrightarrow{\mathrm{ev}\otimes 1} \mathbb{1} \otimes X = X,$$
$$X = \mathbb{1} \otimes X \xrightarrow{\mathrm{coev}\otimes 1} (X \otimes {}^\vee X) \otimes X = X \otimes ({}^\vee X \otimes X) \xrightarrow{1\otimes\mathrm{ev}} X \otimes \mathbb{1} = X,$$
$$X^\vee = \mathbb{1} \otimes X^\vee \xrightarrow{\mathrm{coev}\otimes 1} (X^\vee \otimes X) \otimes X^\vee = X^\vee \otimes (X \otimes X^\vee) \xrightarrow{1\otimes\mathrm{ev}} X^\vee \otimes \mathbb{1} = X^\vee,$$
$${}^\vee X = {}^\vee X \otimes \mathbb{1} \xrightarrow{1\otimes\mathrm{coev}} {}^\vee X \otimes (X \otimes {}^\vee X) = ({}^\vee X \otimes X) \otimes {}^\vee X \xrightarrow{\mathrm{ev}\otimes 1} \mathbb{1} \otimes {}^\vee X = {}^\vee X$$

are all identity morphisms.

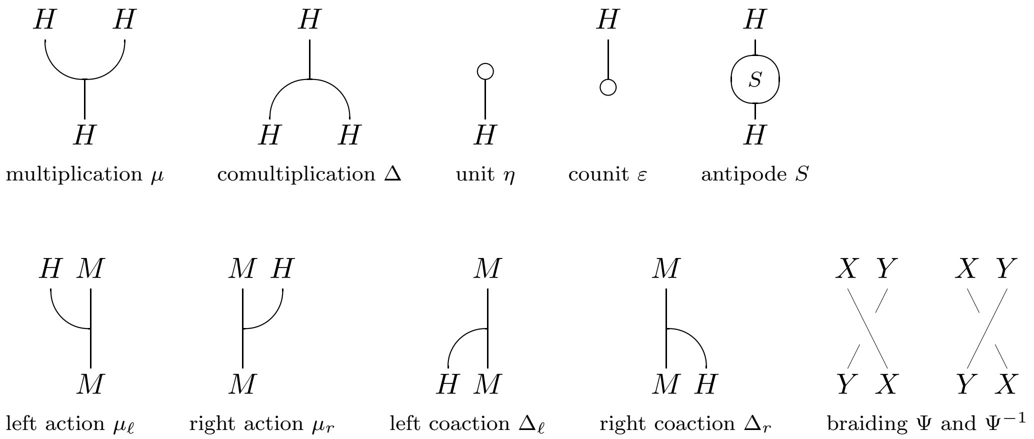


Figure 1: Graphical notations

Recall that a Hopf algebra $H \in \mathcal{C}$ [7] is an object $H \in \operatorname{Obj}\mathcal{C}$ together with an associative multiplication $m : H \otimes H \to H$ and an associative comultiplication $\Delta : H \to H \otimes H$, obeying the bialgebra axiom

$$\begin{aligned}&\big(H \otimes H \xrightarrow{m} H \xrightarrow{\Delta} H \otimes H\big)\\&\quad = \big(H \otimes H \xrightarrow{\Delta\otimes\Delta} H \otimes H \otimes H \otimes H \xrightarrow{H\otimes\Psi\otimes H} H \otimes H \otimes H \otimes H \xrightarrow{m\otimes m} H \otimes H\big),\end{aligned}$$

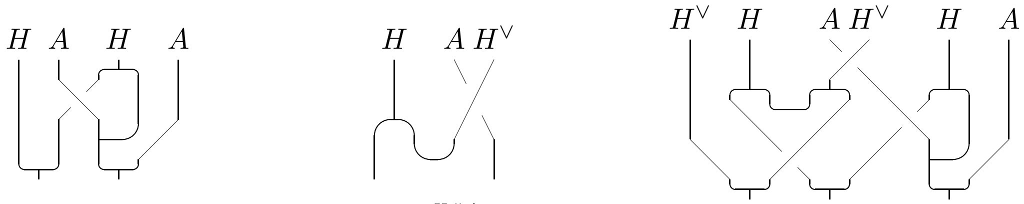
which possesses the unit $\eta : \mathbb{1} \to H$, the counit $\varepsilon : H \to \mathbb{1}$, the antipode $S : H \to H$, and the inverse antipode $S^{-1} : H \to H$ (definitions are the same as in the classical case).

A left (resp., right) module over an algebra H is an object $M \in \mathcal{C}$ equipped with an associative action $\mu_\ell : H \otimes M \to M$ (resp., $\mu_r : M \otimes H \to M$). The category of left (resp., right) H-modules will be denoted by ${}_H\mathcal{C}$ (resp., $\mathcal{C}_H$). A left (resp., right) comodule over a coalgebra H is an object $M \in \mathcal{C}$ equipped with an associative coaction $\Delta_\ell : M \to H \otimes M$ (resp., $\Delta_r : M \to M \otimes H$). The category of left (resp., right) H-comodules will be denoted by ${}^H\mathcal{C}$ (resp., $\mathcal{C}^H$).

If $(\mathcal{C}, \otimes, \mathbb{1}, \Psi)$ is a braided monoidal category, then $\overline{\mathcal{C}} = (\mathcal{C}, \otimes, \mathbb{1}, \overline{\Psi})$ denotes the same monoidal category with the mirror-reversed braiding $\overline{\Psi}_{X,Y} := {\Psi_{Y,X}}^{-1}$. For a Hopf algebra H in $\mathcal{C}$, we denote by H^{op} (resp., H_{op}) the same coalgebra (resp., algebra) with opposite multiplication μ^{op} (resp., opposite comultiplication Δ^{op}) defined through

$$\mu^{\mathrm{op}} := \mu \circ {\Psi_{H\,H}}^{-1} \qquad \big(\text{resp., } \Delta^{\mathrm{op}} := {\Psi_{H\,H}}^{-1} \circ \Delta\big)\ . \tag{1}$$

It is easy to see that H^{op} and H_{op} are Hopf algebras in $\overline{\mathcal{C}}$ with antipode S^{-1}. We will always consider H^{op} and H_{op} as objects of the category $\overline{\mathcal{C}}$. In what follows, we often use a graphical notation for morphisms in monoidal categories [1, 5, 6, 8]. The graphics and notation for (co-)multiplication, (co-)unit, antipode, left and right (co-)action, and braiding are given in Fig. 1, where H is a Hopf algebra and M is an H-module (H-comodule).



a) $\mu_{H\#A}$ b) $\mu^{H\#A}_{r,H^\vee}$ c) $\mu_{H^\vee\#(H\#A)}$

Figure 2:

Duality results

Let H be a Hopf algebra with an invertible antipode in a braided monoidal category $\mathcal{C}$, A be an algebra in the monoidal category $\mathcal{C}_H$ of right H-modules. For these data, one can equip the object $H{\otimes}A$ with a structure of algebra in $\mathcal{C}$ [8]. Multiplication $\mu_{H\#A}$ in this cross product algebra $H\#A$ is given by the diagram in Fig. 2a. The object $H\#A$ equipped with the right $H^\vee$-module structure $\mu^{H\#A}_{r,H^\vee}$ given in Fig. 2b becomes an algebra in the category $\overline{\mathcal{C}}_{(H^\vee)_{\mathrm{op}}}$. Multiplication $\mu_{H^\vee\#(H\#A)}$ in the cross product algebra $H^\vee\#(H\#A)$ is given by the diagram in Fig. 2c.

Let us consider the category $\mathcal{C}^H_{H,A}$ whose objects X are right Hopf H-modules (i.e., right H-modules and right H-comodules satisfying the compatibility condition presented in Fig. 3a) and right A-modules in $\mathcal{C}_H$ (i.e., action $\mu^X_{r,A} : X{\otimes}A \to X$ is an H-module morphism as shown in Fig. 3b) with the additional connection between A-action and H-coaction given in Fig. 3c.

Proposition 1. *There exists an isomorphism between categories $\mathcal{C}^H_{H,A}$ and $\mathcal{C}_{H^\vee\#(H\#A)}$. Functors that set this equivalence are identical on underlying objects and morphism from $\mathcal{C}$. For given $(X, \mu^X_{r,H}, \Delta^X_{r,H}, \mu^X_{r,A}) \in \mathrm{Obj}\big(\mathcal{C}^H_{H,A}\big)$, the structure of the $(H^\vee\#(H\#A))$-module on X is given by the composition*

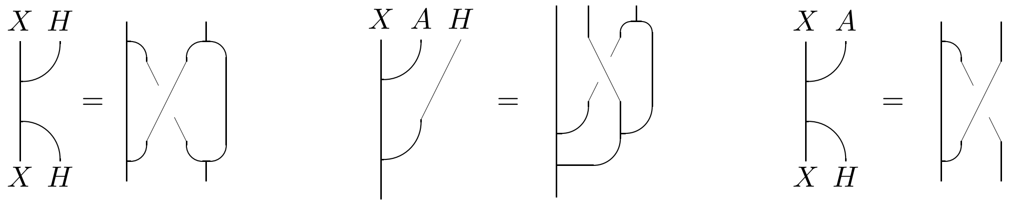
$$\mu^X_{r,H} := \big\{X{\otimes}H^\vee{\otimes}H{\otimes}A \xrightarrow{\Delta^X_{r,H}\otimes\mathrm{id}_{H^\vee\otimes H\otimes A}} X{\otimes}H{\otimes}H^\vee{\otimes}H{\otimes}A \xrightarrow{\mathrm{id}_X\otimes\mathrm{ev}\otimes\mathrm{id}_{H\otimes A}} X{\otimes}H{\otimes}A \xrightarrow{\mu^X_{r,H}\otimes\mathrm{id}_A} X{\otimes}A \xrightarrow{\mu^X_{r,A}} X\big\}\,.$$

An "A-valued matrix algebra" is an object $H^\vee{\otimes}A{\otimes}H$ equipped with multiplication given by the composition

$$H^\vee{\otimes}A{\otimes}H{\otimes}H^\vee{\otimes}A{\otimes}H \xrightarrow{\mathrm{id}_{H^\vee\otimes A}\otimes\mathrm{ev}\otimes\mathrm{id}_{A\otimes H}} H^\vee{\otimes}A{\otimes}A{\otimes}H \xrightarrow{\mathrm{id}_{H^\vee}\otimes\mu_A\otimes\mathrm{id}_H} H^\vee{\otimes}A{\otimes}H\,.$$

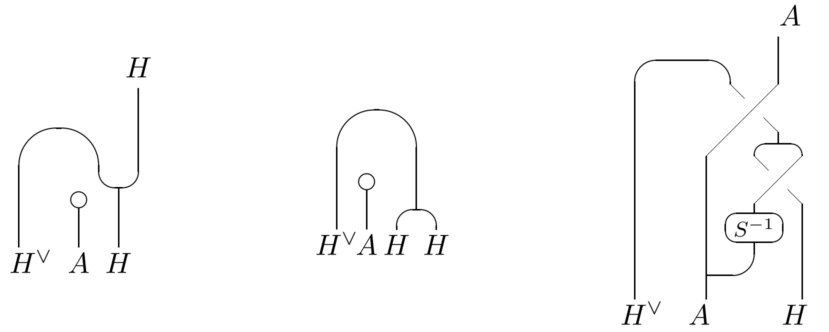
For a Hopf module X, endomorphism

$$\Pi(X) := \big\{X \xrightarrow{\Delta^X_r} X{\otimes}H \xrightarrow{\mathrm{id}_X\otimes S} X{\otimes}H \xrightarrow{\mu^X_r} X\big\}$$



a) Hopf module axiom b) A-module in $\mathcal{C}_H$ c)

Figure 3:



$\phi^H_{H^\vee\otimes A\otimes H}$ $\phi_{H^\vee\otimes A\otimes H,H}$ $\phi^A_{H^\vee\otimes A\otimes H}$

Figure 4:

is an idempotent. This idempotent plays a key role in the theory of Hopf modules [2] and integration on braided Hopf algebras [3].

Proposition 2. *There exists an isomorphism between categories $\mathcal{C}^H_{H,A}$ and $\mathcal{C}_{H^\vee\otimes A\otimes H}$. Functors that set this equivalence are identical on underlying objects and morphism from $\mathcal{C}$. For given $(X,\mu^X_{r,H},\Delta^X_{r,H},\mu^X_{r,A})\in\mathrm{Obj}(\mathcal{C}^H_{H,A})$, the structure of the $(H^\vee\otimes A\otimes H)$-module on X is given by the composition*

$$\mu^X_{r,H^\vee\otimes A\otimes H} := \big\{X\otimes H^\vee\otimes A\otimes H \xrightarrow{\Delta^X_{r,H}\otimes\mathrm{id}_{H^\vee\otimes A\otimes H}} X\otimes H\otimes H^\vee\otimes A\otimes H \xrightarrow{\Pi(X)\otimes\mathrm{ev}\otimes\mathrm{id}_{A\otimes H}} X\otimes A\otimes H \xrightarrow{\mu^X_{r,A}\otimes\mathrm{id}_H} X\otimes H \xrightarrow{\mu^X_{r,H}} X\big\}\,.$$

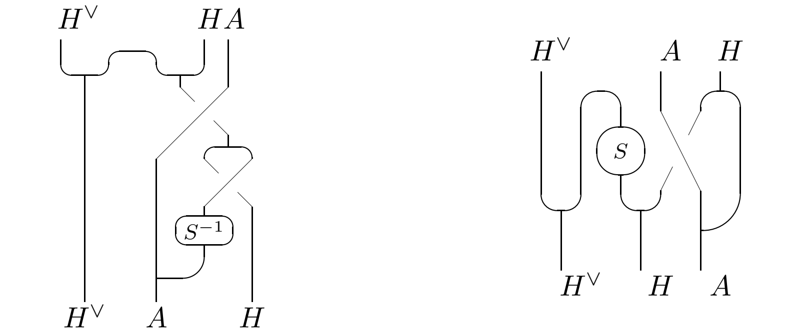
Conversely, for a given right $(H^\vee\otimes A\otimes H)$-module $(X,\mu^X_{r,H^\vee\otimes A\otimes H})$, one can turn X into an object of $\mathcal{C}^H_{H,A}$ equipped with (co)actions

$$\mu^X_{r,H} := \big\{X\otimes H \xrightarrow{\phi^H_{H^\vee\otimes A\otimes H}} X\otimes H^\vee\otimes A\otimes H \xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}} X\big\}\,,$$

$$\mu^X_{r,A} := \big\{X\otimes A \xrightarrow{\phi^A_{H^\vee\otimes A\otimes H}} X\otimes H^\vee\otimes A\otimes H \xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}} X\big\}\,,$$

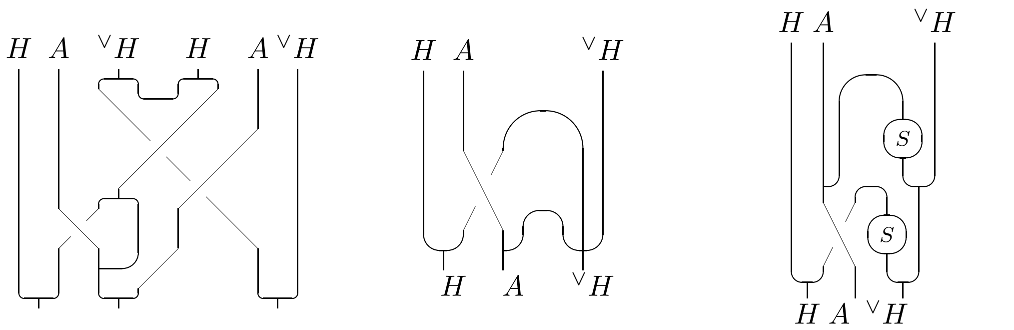
$$\Delta^X_{r,H} := \big\{X \xrightarrow{\mathrm{id}_X\otimes\phi_{H^\vee\otimes A\otimes H,H}} X\otimes H^\vee\otimes A\otimes H\otimes H \xrightarrow{\mu^X_{r,H^\vee\otimes A\otimes H}\otimes\mathrm{id}_H} X\otimes H\big\}\,,$$

where morphisms $\phi^H_{H^\vee\otimes A\otimes H}$, $\phi_{H^\vee\otimes A\otimes H,H}$, $\phi^A_{H^\vee\otimes A\otimes H}$ are presented in Fig. 4.



$\phi : H^{\vee}\#(H\#A) \to H^{\vee}\otimes A\otimes H$ $\phi^{-1} : H^{\vee}\otimes A\otimes H \to H^{\vee}\#(H\#A)$

Figure 5:



a) $\mu_{(H\#A)\#^{\vee}H}$ b) $(H\#A)\#^{\vee}H \to H\otimes A\otimes{}^{\vee}H$ c) $H\otimes A\otimes{}^{\vee}H \to (H\#A)\#^{\vee}H$

Figure 6:

Corollary 3. *There exists an algebra isomorphism $\phi : H^{\vee}\#(H\#A) \to H^{\vee}\otimes A\otimes H$ shown in Fig. 5 such that the corresponding isomorphism of categories $\mathcal{C}_{H^{\vee}\otimes A\otimes H} \overset{\phi^*}{\simeq} \mathcal{C}_{H^{\vee}\#(H\#A)}$ is given by the compositions of functors from Propositions 1, 2.*

Proof. We put

$$\phi := \mu^{(3)}_{H^{\vee}\otimes A\otimes H} \circ \left(\phi^{H^{\vee}}_{H^{\vee}\otimes A\otimes H}\otimes\phi^{H}_{H^{\vee}\otimes A\otimes H}\otimes\phi^{A}_{H^{\vee}\otimes A\otimes H}\right),$$

where

$$\phi^{H^{\vee}}_{H^{\vee}\otimes A\otimes H} := \left\{H^{\vee} \xrightarrow{\phi_{H^{\vee}\otimes A\otimes H,H}\otimes\mathrm{id}_{H^{\vee}}} H^{\vee}\otimes A\otimes H\otimes H\otimes H^{\vee} \xrightarrow{\mathrm{id}_{H^{\vee}\otimes A\otimes H}\otimes\mathrm{ev}_H} H^{\vee}\otimes A\otimes H\right\}$$

and $\mu^{(3)} := \mu\circ(\mu\otimes\mathrm{id})$. Consideration of the regular $(H^{\vee}\otimes A\otimes H)$-module implies that ϕ is an algebra isomorphism.

In the special case $A = \mathbb{1}$, we obtain the braided Heisenberg double $\mathcal{H}(H) := H^{\vee}\#H$, which is isomorphic to the matrix algebra $H^{\vee}\otimes H$ (with multiplication $\mathrm{id}_{H^{\vee}}\otimes\mathrm{ev}_H\otimes\mathrm{id}_H$), and an isomorphism between the category $\mathcal{C}_H^H$ of right Hopf H-modules and the category $\mathcal{C}_{\mathcal{H}(H)}$ of $\mathcal{H}(H)$-modules. See [3] for this special case and connection with integration on braided Hopf algebras.

Remark. In a similar way, one can obtain another variant of the above construction, which does not involve a skew antipode S^{-1}. Let $A \in \mathrm{Obj}(\mathcal{C}_H)$ be a right H-module algebra. One can turn the corresponding cross product algebra $H\#A$ into a left ${}^\vee H$-module algebra with action $\mu_{r,{}^\vee H}^{H\#A} := (\mathrm{ev}\otimes\mathrm{id}_{H\otimes A})\circ(\mathrm{id}_{{}^\vee H}\otimes\Delta_H\otimes\mathrm{id}_A)$. Multiplication in the corresponding cross product algebra $(H\#A)\#{}^\vee H$ is given in Fig. 6a. Isomorphism between this algebra and the "A-valued matrix algebra" $H\otimes A\otimes{}^\vee H$ and its inverse is given in Fig. 6b,c.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 409–417.

On the Peculiarities of Stochastic Invariant Solutions of a Hydrodynamic System Accounting for Non-local Effects

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Abstract

A set of travelling wave solutions of a system of PDE describing nonequilibrium processes in relaxing media is investigated. These solutions satisfy a certain dynamic system obtained from the initial one via the group theory reduction. The dynamic system is shown to possess stochastic oscillatory-type solutions that might play role of a non-trivial intermediate asymptotics for the initial system of PDE. Fine structure of a strange attractor arising in the dynamic system is studied by means of Poincaré sections technique.

The problem of description of a multicomponent medium subjected to high-rate high-intense load actually is far from being solved. The classical continual models are not valid for this purpose since high-rate load induces irreversible processes of energy exchange between components and, besides, different structural changes or chemical reactions initiated by shock wave propagation might take place. A route to the equilibrium in such systems is rather difficult to analyze since the detailed mechanism of relaxation in most cases remains unknown. Yet for the processes that are not far from equilibrium, individual features turn out to be unessential and irreversible thermodynamics formalism may be employed in order to construct constitutive equations [1–4]. In the early 90-th, V.A. Danylenko [5, 6] proposed to describe pulse load afteraction in active and relaxing media with the help of the following equations:

$$\begin{gathered}\rho\frac{du}{dt}+\frac{\partial p}{\partial x}=\Im, \qquad \frac{d\rho}{dt}+\rho\frac{\partial u}{\partial x}=0,\\ \tau\frac{dp}{dt}-\chi\frac{d\rho}{dt}\rho^{n-1}=\kappa\rho^n-p-h\left\{\frac{d^2p}{dt^2}+\rho^{n-1}\left[\frac{2}{\rho}\left(\frac{d\rho}{dt}\right)^2-\frac{d^2\rho}{dt^2}\right]\right\}\end{gathered} \tag{1}$$

The first two equations of system (1) represent the standard balance equations for mass and momentum, taken in hydrodynamic approximation. The third equation called the dynamic equation of state contains the information about relaxing properties of the medium. Its parameters have following physical meaning: τ is the relaxation time, χ is the volume viscosity coefficient, h is the coefficient of structural relaxation, $n=1+\Gamma_{V\infty}$, $\Gamma_{V\infty}$ is the isochoric Gruneisen coefficient [7, 5], $\sqrt{\kappa}$ is proportional to the sound velocity in equilibrium.

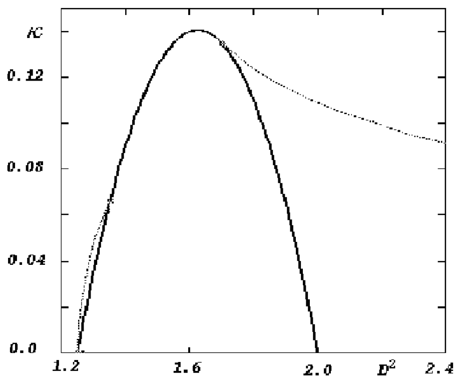


Fig.1. Bifurcation diagram of system (4) in the plane $(D^2,\ \kappa)$, obtained for $\beta = -0.8$ and $\xi = -1.25$. Solid line corresponds to a Hopf bifurcation while the dotted lines correspond to a twin cycle bifurcation

System (1) was investigated by means of asymptotic methods and within this approach it was shown the existence of various non-trivial solutions (periodic, quasiperiodic, soliton-like) [6, 8] in the long wave approximation. In this work, we put up the problem of investigation of some class of exact solutions that might play a role of intermediate asymptotics [9, 10] for the Cauchy (boundary-value) problems, connected with strong pulse load afteraction and to study conditions leading to the wave patterns formation. These solutions satisfy an ODE system obtained from the initial PDE system by group-theoretic reduction. The ODE system is studied both by analytical tools and numerical methods enabling to state the existence of domains in the parametric space corresponding to stochastic autowave solutions.

It is well known that symmetry properties of a given system of PDE can be employed to reduce the number of independent variables [11]. In the case of one spatial variable, this procedure gives rise to an ODE system. By straightforward calculation, one can check that system (1) is invariant under the Galilei algebra $AG(1,1)$ spanned by the following operators:

$$\widehat{P}_0 = \frac{\partial}{\partial t}, \qquad \widehat{P}_1 = \frac{\partial}{\partial x}, \qquad \widehat{G} = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

If $\Im = \gamma\rho^{(n+1)/2}$, then system (1) admits an extra one-parameter group generated by the operator

$$\widehat{\Re} = \frac{n-1}{2}x\frac{\partial}{\partial x} + \rho\frac{\partial}{\partial \rho} + np\frac{\partial}{\partial p}.$$

To the end of this work, we shall analyze the case where $\Gamma_{V\infty}$ is negligible small (and, hence, $n = 1$). Without loss of generality, we may assume that $\tau = \chi = 1$ and the third (constitutive) equation of system (1) contains only two dimensionless parameters.

A passage from system (1) to a subsequent system of ODE will be performed with the help of the following ansatz

$$u = D + U(\omega), \quad \omega = x - Dt, \quad \rho = \exp[\xi t + S(\omega)], \quad p = \rho Z(\omega), \tag{2}$$

built on invariants of the first-order operator

$$\widehat{\mathbf{X}} = \widehat{P}_0 + \widehat{P}_1 + \xi\widehat{\Re}. \tag{3}$$

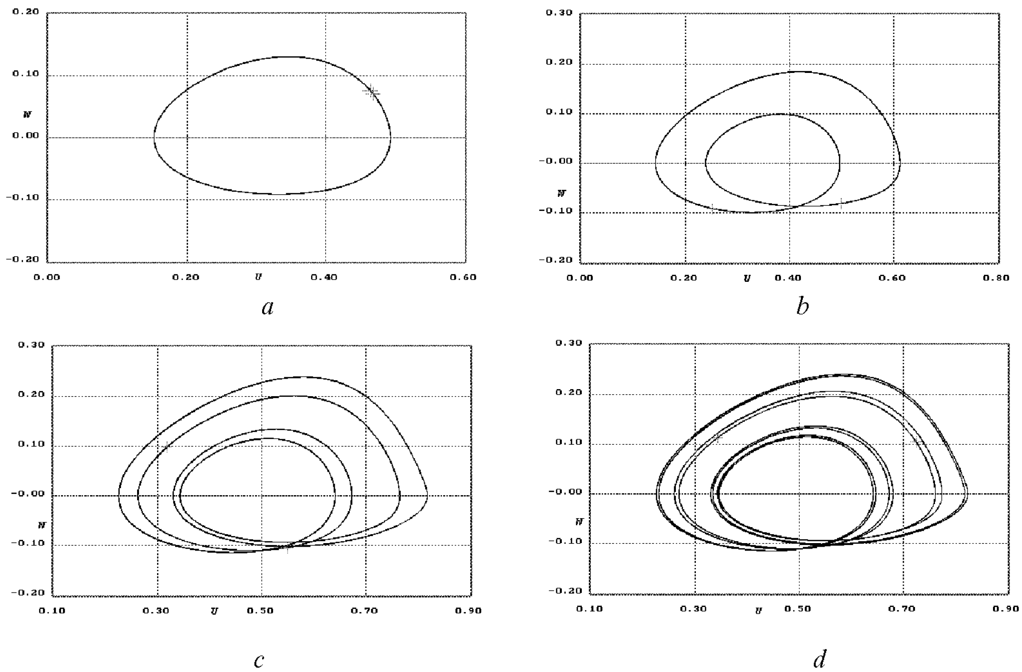


Fig.2. Period doubling bifurcation in system (4) . $\beta = -0.8,\ \xi = -1.25,\ \kappa = 0.05,\ \gamma = \kappa\xi/D$. For case *a*: $D = \sqrt{3}$; for case *b*: $D = \sqrt{3.2}$; for case *c*: $D = \sqrt{3.7}$; for case *d*: $D = \sqrt{3.71}$

Inserting (2) into the formula (1), we obtain an ODE system cyclic with respect to the variable S. If one introduces a new variable $W = dU/d\omega \equiv \dot{U}$, then the following dynamic system is obtained:

$$\begin{aligned} U\dot{U} &= UW, \\ U\dot{Z} &= \gamma U + \xi Z + W(Z - U^2) \equiv \phi, \\ U\dot{W} &= [\beta(1 - U^2)]^{-1}\{M\phi + Z - \kappa + W[1 - MZ]\} - W^2, \end{aligned} \tag{4}$$

where $\beta = -h < 0$, $M = 1 - \beta\xi$.

The only critical point of system (4) belonging to the physical parameter range (i.e., lying in the half-space $Z > 0$ beyond the manifold $U\beta(1 - U^2) = 0$) is a point **A** having the coordinates

$$U_0 = -\kappa\xi/\gamma, \quad Z_0 = \kappa, \quad W_0 = 0.$$

We are going to analyze localized solutions of system (4) in the vicinity of this point.

To begin with, note that the critical point $\mathbf{A}(U_0,\ \kappa,\ 0)$ corresponds, under certain conditions, to an invariant stationary solution of system (1). Indeed, consider a set of time-independent functions belonging to family (2). A simple calculation shows that functions satisfying the above requirements should have the following form

$$u_1 = U_0 + D, \quad U_0 = \text{const}, \quad \rho_1 = \rho_0 \exp(\xi x/D), \quad p_1 = Z_0\rho_1.$$

These functions will satisfy system (1) if $\gamma = \kappa\xi/D$ ($U_0 = -D$) and $Z_0 = \kappa$. Note that the parameter ξ has a clear physical interpretation as defining an inclination of inhomogeneity

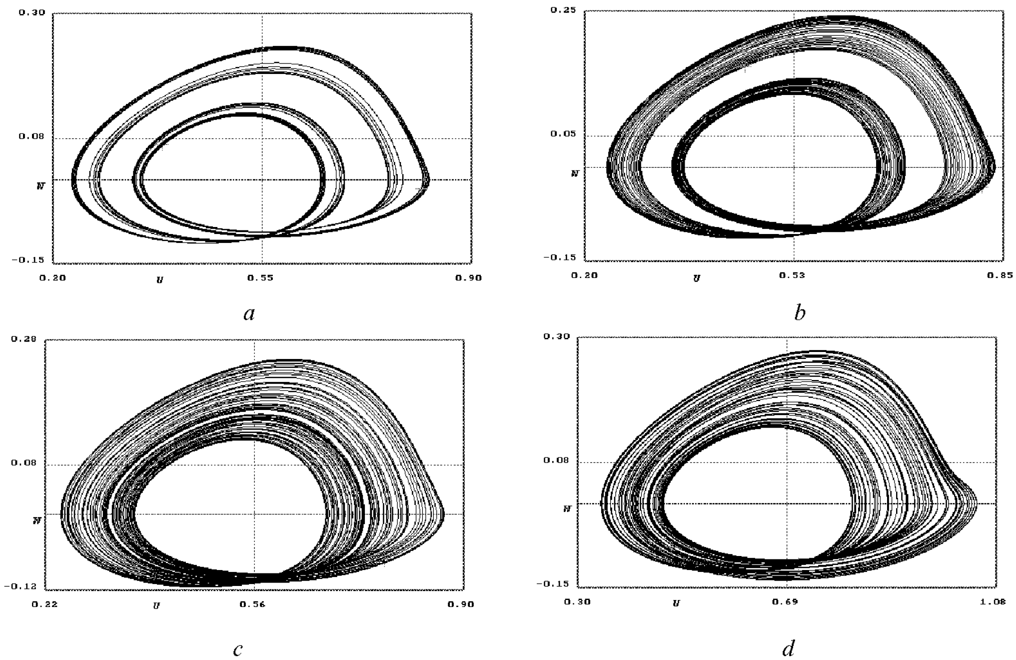


Fig.3. Patterns of chaotic trajectories of system (14). $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$, $\gamma = \kappa\xi/D$. For case *a*: $D = \sqrt{3.73}$; for case —it b: $D = \sqrt{3.75}$; for case *c*: $D = \sqrt{3.83}$; for case *d*: $D = \sqrt{4.299}$

of the time-independent invariant solution represented by the critical point $\mathbf{A}(-D,\ \kappa,\ 0)$. In what follows, we assume the validity of the above conditions.

Let us introduce new variables $X = U + D$, $Y = Z - \kappa$ In the coordinates X, Y, W, system (4) may be rewritten as follows:

$$\frac{d}{dT}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} = \begin{pmatrix} 0, & 0, & U_0 \\ \gamma, & \xi, & \Delta \\ L\gamma, & L\xi + G_1, & \sigma \end{pmatrix}\begin{pmatrix} X \\ Y \\ W \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} \tag{5}$$

where $d(\cdot)/dT = Ud(\cdot)/d\omega$, $L = M/K$, $G_1 = K^{-1}$, $K = \beta(1 - U_0^2)$, $\Delta = \kappa - U_0^2$, $\sigma = (1 - MD^2)/K$,

$$\begin{aligned} &H_1 = WX, \qquad H_2 = W[Y - X(X + 2U_0)], \\ &H_3 = 2U_0X\beta(L\gamma X + L\xi Y + \sigma W)/K - W(2U_0LX + W) - LWX^2(1 + 2U_0\beta/K) + \\ &\qquad \beta\left(1 + 4U_0^2\beta/K\right)X^2(L\gamma X + L\xi Y + \sigma W)/K + O(|X|^3, |Y|^3, |W|^3). \end{aligned} \tag{6}$$

We are going to analyze the case where the matrix $\widehat{M}$ standing on the RHS of equation (5) has one negative eigenvalue $\lambda_1 = a < 0$ and a pair of pure imaginary eigenvalues $\lambda_{2,3} = \pm i\Omega$. Taking into account that function $U(\omega)$ standing on the LHS of equation (4) is negative in the vicinity of the critical point, we can write down the above conditions as follows:

$$a = -[(\xi\beta + 1) - D^2]/K < 0 \tag{7}$$

$$\Omega^2 = (\xi - \kappa + D^2)/K > 0 \tag{8}$$

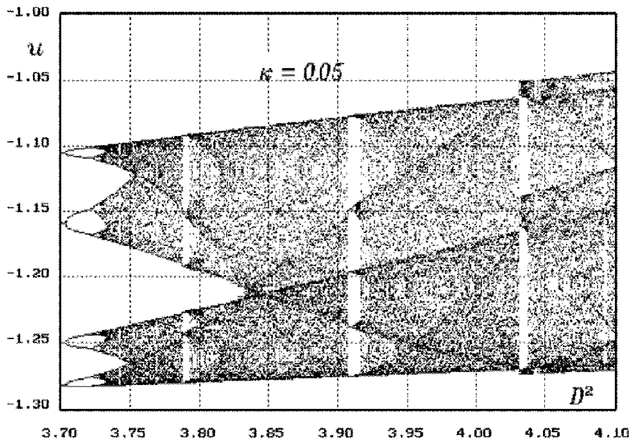


Fig.4. Bifurcation diagram of system (14) in the plane $(D^2,\ U)$. $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$, $\gamma = \kappa\xi/D$.

$$a\Omega^2 = \kappa\xi/K \tag{9}$$

On analyzing relations (7)–(9), one concludes that positiveness of ξ leads to the inequality $\kappa > D^2$. But this relation is unacceptable since it implies instability of the corresponding wave pack in the limiting case where $h = 0$ [12, 13].

So let $\xi < 0$ (and, hence, $K > 0$). We may then rewrite relations (7)–(9) as

$$1 + \xi\beta > D^2 > \kappa - \xi, \tag{10}$$

$$\kappa = \frac{(\xi\beta + 1 - D^2)(\xi + D^2)}{1 + D^2(\xi\beta - 1)}. \tag{11}$$

Equation (11) was solved numerically for $\beta = -0.8$ and $\xi = -1.25$ (see Fig.1, where function $\kappa(D^2)$ is plotted by the solid line). One can easily get convinced that, for the given values of the parameters ξ and β, the segment of curve (11) lying in the positive half-plane $\kappa > 0$ belongs to the open set defined by inequalities (10).

We employed the curve given by equation (11) (a Hopf bifurcation curve [14]) as a starting point for our numerical study. On the opposite sides of this curve stability types of the critical point are different. It is a stable focus beneath the curve and an unstable focus above the curve. The local change of stability can be induced by different global processes: by the unstable limit cycle disappearance (subcritical Hopf bifurcation) or by the stable limit cycle creation (supercritical Hopf bifurcation).

A local stability analysis based on the central manifold theorem and Poincaré normal forms technique [14, 15] shows that the neutral stability curve (11) may be divided into three parts. The domain lying between the values ${D_1}^2 = 1.354076$ and ${D_2}^2 = 1.693168$ corresponds to the subcritical Hopf bifurcation while two other segments correspond to the supercritical one. Thus, the dynamic system's solutions change in a different way with growth of the parameter κ when D^2 is fixed. If $1.354076 < D^2 < 1.693168$, then solutions of the system become globally unstable and tend to infinity after the Hopf bifurcation takes place. In the case where D^2 is beyond this interval, the growth of the parameter κ leads to the self-oscillating solutions appearance. The amplitudes of oscillations grow with parameter's κ growth until one of the twin cycle bifurcation curves (shown on Fig.1 as dotted lines) is attained. Above these curves, system (4) again becomes globally unstable. Numerical experiments show that twin cycle bifurcation curves are attached with one end

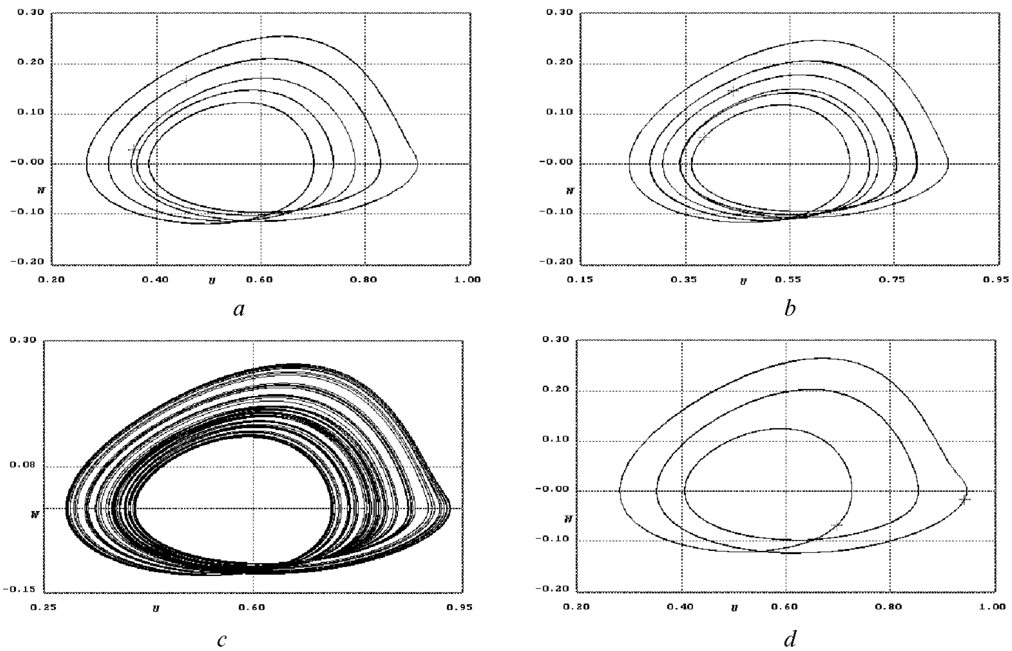
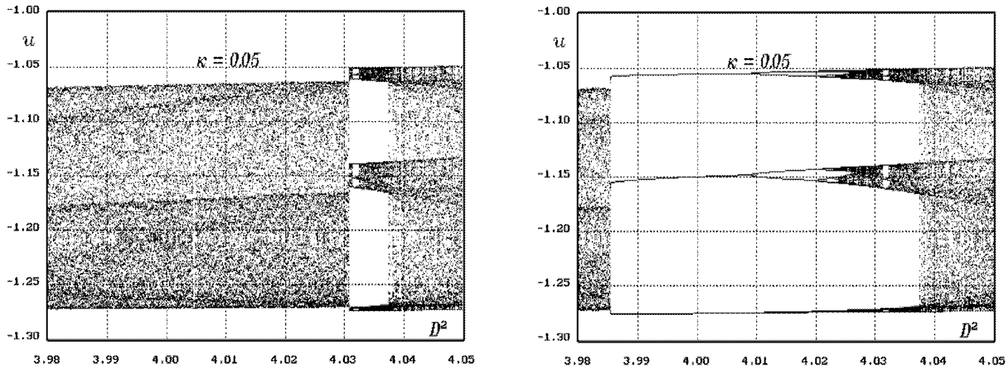


Fig.5. Phase portraits of system (14) obtained *a*: for $D^2 = \sqrt{3.91}$; *b*: for $D^2 = \sqrt{3.789}$; *c*, *d*: for $D^2 = \sqrt{4.005}$

to the Hopf bifurcation curve (11) just at the point where the type of the Hopf bifurcation is changed. In addition, it was shown that system (4) has in these points stable limit cycles surrounded by unstable ones, These unstable attractors cannot be associated with the self-oscillating regimes – in contrast to them, they are not manifested explicitly. On the other hand, the unstable cycles restrict the development of self-oscillations that grow with growth of the parameter κ. The domain of the unstable limit cycle existence in the parameter space (κ, D^2) is restricted by the subcritical branch of the Hopf bifurcation curve from one side and the bifurcation curves of twin cycles (having one of the multipliers equal to unity) from another. On the intersection of the system's parameters of both of these curves, the unstable limiting cycle disappears (note that when the parameters approach the Hopf bifurcation curve, a radius of the periodic trajectory tends to zero, while the cycle disappears with non-zero amplitude on the dotted curves).

The above analysis suggests that another direction of movement in the parametric space should be tried. Below we describe scenario obtained in the case where the parameter D^2 is varied, while the rest of parameters are fixed as follows: $\beta = -0.8$, $\xi = -1.25$, $\kappa = 0.05$.

So, at $D^2 = 1.927178$, a stable limit cycle appears. The amplitude of the limit cycle grows within some interval when D^2 grows (for $D^2 = 3$, it is shown in Fig.2*a*). But finally, the limit cycle loses stability giving rise to another cycle with approximately two times greater period ($2T$-cycle). The $2T$ cycle obtained for $D^2 = 3.2$ is shown in Fig.2*b*. This cycle also loses stability when D^2 approaches a certain critical value at which the $4T$-period cycle arises (Fig.2*c* shows $4T$ cycle obtained for $D^2 = 3.7$). Further growth of the parameter D^2 is accompanied by creation of the series of cycles having periods $8T$ (Fig.2*d* at $D^2 = 3.71$), $16T$, …, 2^nT. The period-doubling cascade of bifurcation is well known in the theory of non-linear oscillation [15–17]. Very often it leads to the appearance stochastic regimes (strange attractors).



a *b*

Fig.6. Bifurcation diagrams of system (14) in the vicinity of the period $3T$ window obtained (*a*) when the parameter D^2 increases and (*b*) when the parameter D^2 decreases

In fact, system (4) has a strange attractor. Within the interval $3.7288 < D^2 < 3.8328$, it has a band structure. Four bands may be seen in Fig.3*a* corresponding to $D^2 = 3.73$, while in Fig.3*b*, corresponding to $D^2 = 3.75$, trajectories form only two bands. On a further growth of the parameter D^2, narrow chaotic bands join each other (this process is known as a "reverse" period-doubling cascade) and finally give rise to a Rössler-type attractor, covering a certain domain of the phase space (it is seen in Fig.3*c* obtained for $D^2 = 3.833$). An outlook of the strange attractor just before the destruction taking place at $D^2 = 4.3$ is shown in Fig.3*d*. Specific hump on the right edge of the attractor suggests that its destruction results from the interaction with a homoclinic loop.

The detailed investigation of the zone of chaotic oscillations enables us to see (Fig.4) that chaotic patterns are inhomogeneous, including domains with periodic movements where the system has the cyclic solutions with periods that are different from 2^nT. To investigate a fine structure of the chaotic zone, the bifurcation diagrams technique [17, 18] is employed.

A plane $W = 0$ was chosen as a Poincaré section plane in numerical experiments. Only those points were taken into consideration that correspond to the trajectories moving from the half–space $W > 0$ towards its supplement.

Employment of the Poincaré sections technique enables us to conclude that there exist well-known period $5T$ and $6T$ domains (Fig.5*a* and 5*b* obtained for $D^2 = 3.91$ and $D^2 = 3.789$, respectively) predicted by the Sharkovskij theorem [19, 15]. In addition, it was stated another features of the chaotic domain inherent to the system under consideration. The most characteristic property of this attractor is as follows. The period $3T$ domain of system (4) manifests hysteresis features. There exist simultaneously two attractors: a chaotic attractor and a period $3T$ attractor (seen in Fig.5*c* and 5*d*, both obtained for $D^2 = 4.005$). Scenario of the patterns development on this interval depends on the direction of movement along the parameter D^2 values. Parameter's D^2 increasing causes in the period $3T$ a supercritical bifurcation (Fig.6*a*). The attractor created in this way has a band structure. When parameter D^2 decreases, the 3-band chaotic attractor undergoes to the inverse cascade of period doubling bifurcations $3 \cdot 2^n \cdot T$, $3 \cdot 2^{n-1} \cdot T$, ..., $6T$, $3T$, finally giving rise to the twin cycle bifurcation resulting in the stable and unstable period $3T$ attractor creations followed by the passage to the advanced chaotic regime (Fig.6*b*).

Thus, the domain of the coexistence of two attractors is restricted by the twin cycle bifurcation curve from one side and crisis of the strange attractor [20, 21] from another.

For $\kappa = 0.03$ and $\kappa = 0.02$, a similar scenario was observed, this time for significantly greater values of the parameter D^2. It seems that a chaotic attractor does not exist when $\kappa < 0.01$.

Concluding remarks

The goal of this work was to study a certain class of solutions of system (1) describing nonequilibrium processes in multicomponent relaxing media, namely the travelling wave solutions satisfying the ODE system (4). From both qualitative analysis and numerical simulations, it has been observed that a wide variety of regimes can be exhibited by this dynamic system – from multiperiodic to chaotic. The existence of these complicated regimes occurs to be possible due to the complex interaction of non-linear terms with terms describing relaxing properties of the medium.

To realize an influence of relaxing features on the patterns formation, it is useful to compare the results of this work with the investigations undertaken in [22], where processes described by the first-order governing equation were considered and merely periodic solutions were shown to exist. Thus, the existence of stochastic invariant solutions of system (1) is directly linked with the presence of higher-derivative terms in the governing (constitutive) equation.

Of special interest is the fact that system (1) possesses complicated travelling wave solutions provided that an external force of special form is present. Let us note that thus far oscillating invariant solutions in hydrodynamic-type systems have been obtained merely in the presence of mass forces [23, 24] that seem to play the key role in the invariant wave patterns formation.

From the numerical study of dynamic system (4), it is also seen that complex oscillating regimes exist over a wide range of parameter's D values, so the bifurcation phenomena as well as the patterns formation can occur practically at arbitrarily large values of the Mach number. Note, that this conclusion is in agreement with the well-known results obtained by Erpenbeck, Fickett and Wood and several other authors, studying stability conditions for overpressurized detonation waves (for comprehensive survey, see, e.g., [25]).

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 418–428.

Group Method Analysis of the Potential Equation in Triangular Regions

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Dedicated to the memory of Prof. Dr. W. Fushchych, for his valuable contributions in the field and the sincere friendship

Abstract

The group transformation theoretic approach is applied to present an analytic study of the steady state temperature distribution in a general triangular region, Ω, for given boundary conditions, along two boundaries, in a form of polynomial functions in any degree "n", as well as the study of heat flux along the third boundary. The Laplace's equation has been reduced to a second order linear ordinary differential equation with appropriate boundary conditions. Analytical solution has been obtained for different shapes of Ω and different boundary conditions.

1 Introduction

The Laplace's equation arises in many branches of physics attracts a wide band of researchers. Electrostatic potential, temperature in the case of a steady state heat conduction, velocity potential in the case of steady irrotational flow of ideal fluid, concentration of a substance that is diffusing through solid, and displacements of a two-dimensional membrane in equilibrium state, are counter examples in which the Laplace's equation is satisfied.

The mathematical technique used in the present analysis is the parameter-group transformation. The group methods, as a class of methods which lead to reduction of the number of independent variables, were first introduced by Birkhoff [4] in 1948, where he made use of one-parameter transformation groups. In 1952, Morgan [6] presented a theory which has led to improvements over earlier similarity methods. The method has been applied intensively by Abd-el-Malek et al. [1–3].

In this work, we present a general procedure for applying a one-parameter group transformation to the Laplace's equation in a triangular domain. Under the transformation, the partial differential equation with boundary conditions in polynomial form, of any degree, is reduced to an ordinary differential equation with the appropriate corresponding conditions. The equation is then solved analytically for some forms of the triangular domain and boundary conditions.

2 Mathematical formulation

The governing equation, for the distribution of temperature $T(x, y)$, is given

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{2.1}$$

with the following boundary conditions:

$$\begin{aligned} &(i) \quad T(x, y) = \alpha x^n, \quad (x, y) \in L_1, \\ &(ii) \quad T(x, y) = \beta x^n, \quad (x, y) \in L_2, \end{aligned} \tag{2.2}$$

We seek for the distribution of the temperature $T(x, y)$ inside the domain Ω and the heat flux across L_3 with

$$\begin{aligned} &L_1: \; y = x \tan \Phi_1, \\ &L_2: \; y = -x \tan \Phi_2, \\ &L_3: \; y = x \tan \Phi_3 + b, \quad b \neq 0, \end{aligned}$$

$n \in \{0, 1, 2, 3, \ldots\}$, α, β are constants.

Write

$$T(x, y) = w(x, y) q(x), \qquad q(x) \not\equiv 0 \text{ in } \Omega.$$

Hence, (2.1) and (2.2) take the form:

$$q(x) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial x} \frac{dq}{dx} + w \frac{d^2 q}{dx^2} = 0 \tag{2.3}$$

with the boundary conditions:

$$\begin{aligned} &(i) \quad w(x, y) = \frac{\alpha x^n}{q(x)}, \qquad (x, y) \in L_1, \\ &(ii) \quad w(x, y) = \frac{\beta x^n}{q(x)}, \qquad (x, y) \in L_2. \end{aligned} \tag{2.4}$$

3 Solution of the problem

The method of solution depends on the application of a one-parameter group transformation to the partial differential equation (2.1). Under this transformation, two independent variables will be reduced by one and the differential equation (2.1) transforms into an ordinary differential equation in only one independent variable, which is the similarity variable.

3.1 The group systematic formulation

The procedure is initiated with the group G, a class of transformation of one-parameter "a" of the form

$$G: \; \overline{S} = C^s(a) S + K^s(a), \tag{3.1}$$

where S stands for x, y, w, q and the C's and K's are real-valued and at least differentiable in the real argument "a".

3.2 The invariance analysis

To transform the differential equation, transformations of the derivatives are obtained from G via chain-rule operations:

$$\overline{S}_{\overline{i}} = \left(\frac{C^S}{C^i}\right), \qquad \overline{S}_{\overline{i}\,\overline{j}} = \left(\frac{C^S}{C^i C^j}\right) S_{ij}, \qquad i = x, y; \quad j = x, y,$$

where S stands for w and q.

Equation (2.3) is said to be invariantly transformed whenever

$$\overline{q}(\overline{w}_{\overline{x}\,\overline{x}} + \overline{w}_{\overline{y}\,\overline{y}}) + 2\overline{w}_{\overline{x}}\overline{q}_{\overline{x}} + \overline{w}\overline{q}_{\overline{x}\,\overline{x}} = H_1(a)\left[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}\right] \tag{3.2}$$

for some function $H_1(a)$ which may be a constant.

Substitution from equations (3.1) into equation (3.2) for the independent variables, the functions and their partial derivatives yields

$$\begin{aligned} &q\left(\left[\frac{C^q C^w}{(C^x)^2}\right] w_{xx} + \left[\frac{c^q C^w}{(C^y)^2}\right] w_{yy}\right) + 2\left[\frac{C^q C^w}{(C^x)^2}\right] w_x q_x + \left[\frac{C^q C^w}{(C^x)^2}\right] w q_{xx} + \xi_1(a) = \\ &\quad H_1(a)\left[q(w_{xx} + w_{yy}) + 2w_x q_x + w q_{xx}\right], \end{aligned} \tag{3.3}$$

where

$$\xi_1(a) = (K^q C^w)\left(\frac{w_{xx}}{(C^x)^2} + \frac{w_{yy}}{(C^x)^2}\right) + \left[\frac{K^w C^q}{(C^x)^2}\right] q_{xx}.$$

The invariance of (3.3) implies $\xi_1(a) \equiv 0$. This is satisfied by putting

$$K^q = K^w = 0$$

and

$$\left[\frac{C^q C^w}{(C^x)^2}\right] = \left[\frac{C^q C^w}{(C^y)^2}\right] = H_1(a),$$

which yields

$$C^x = C^y.$$

Moreover, the boundary conditions (2.4) are also invariant in form, that implies

$$K^x = K^q = K^w = 0, \qquad \text{and} \quad C^q C^w = (C^x)^n.$$

Finally, we get the one-parameter group G which transforms invariantly the differential equation (2.3) and the boundary conditions (2.4). The group G is of the form

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y + K^y \\ \overline{w} = C^w w \\ \overline{q} = \left[\dfrac{(C^x)^n}{C^w}\right] q \end{cases}$$

3.3 The complete set of absolute invariants

Our aim is to make use of group methods to represent the problem in the form of an ordinary differential equation (similarity representation) in a single independent variable (similarity variable). Then we have to proceed in our analysis to obtain a complete set of absolute invariants. In addition to the absolute invariants of the independent variable, there are two absolute invariant of the dependent variables w and q.

If $\eta \equiv \eta(x, y)$ is the absolute invariant of independent variables, then

$$q_j(x, y; wq) = F_j[\eta(x, y)]; \qquad j = 1, 2,$$

are two absolute invariants corresponding to w and q. The application of a basic theorem in group theory, see [5], states that: *function $g(x, y; w, q)$ is an absolute invariant of a one-parameter group if it satisfies the following first-order linear differential equation*

$$\sum_{i=1}^{4} (\alpha_i S_i + \beta_i) \frac{\partial g}{\partial S_i} = 0, \tag{3.4}$$

where S_i stands for x, y, w and q, respectively, and

$$\alpha_i = \frac{\partial C^{S_i}}{\partial a}(a^0) \qquad \textit{and} \quad \beta_i = \frac{\partial K^{S_i}}{\partial a}(a^0), \qquad i = 1, 2, 3, 4,$$

where a^0 denotes the value of "a" which yields the identity element of the group.

From which we get: $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_3 = \beta_4 = 0$. We take $\beta_2 = 0$.

At first, we seek the absolute invariants of independent variables. Owing to equation (3.4), $\eta(x, y)$ is an absolute invariant if it satisfies the first-order partial differential equation

$$x \frac{\partial \eta}{\partial x} + y \frac{\partial \eta}{\partial y} = 0,$$

which has a solution in the form

$$\eta(x, y) = \frac{y}{x}. \tag{3.5}$$

The second step is to obtain the absolute invariant of the dependent variables w and q. Applying (3.4), we get $q(x) = R(x)\theta(\eta)$.

Since $q(x)$ and $R(x)$ are independent of y, while η is a function of x and y, then $\theta(\eta)$ must be a constant, say $\theta(\eta) = 1$, and from which

$$q(x) = R(x), \tag{3.6}$$

and the second absolute invariant is:

$$w(x, y) = \Gamma(x) F(\eta). \tag{3.7}$$

4 The reduction to an ordinary differential equation

As the general analysis proceeds, the established forms of the dependent and independent absolute invariants are used to an obtain ordinary differential equation. Generally, the absolute invariant $\eta(x,y)$ has the form given in (3.5).

Substituting from (3.6), (3.7) into equation (2.3) yields

$$(\eta^2+1)\frac{d^2F}{d\eta^2} - 2\eta\left[\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right)x - 1\right]\frac{dF}{d\eta} + \left[\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right]x^2F = 0. \tag{4.1}$$

For (4.1) to be reduced to an expression in the single independent invariant η, the coefficients in (4.1) should be constants or functions of η. Thus,

$$\left(\frac{1}{\Gamma}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{dR}{dx}\right)x = C_1, \tag{4.2}$$

$$\left(\frac{1}{\Gamma}\frac{d^2\Gamma}{dx^2} + \frac{2}{R\Gamma}\frac{dR}{dx}\frac{d\Gamma}{dx} + \frac{1}{R}\frac{d^2R}{dx^2}\right)x^2 = C_2. \tag{4.3}$$

It follows, then, from (4.2) that:

$$\Gamma(x)R(x) = C_3x^{C_1}.$$

Also, from (4.2) and (4.3) we can show that

$$C_2 = C_1(C_1-1).$$

By taking $C_3 = 1$ and $C_1 = n$, we get

$$(\eta^2+1)F'' - 2\eta(n-1)F' + n(n-1)F = 0. \tag{4.4}$$

Under the similarity variable η, the boundary conditions are:

$$F(\tan\Phi_1) = \alpha, \qquad F(-\tan\Phi_2) = \beta, \tag{4.5}$$

such that the boundary L_1 or L_2 does not coincide with the vertical axis.

5 Analytic solution

Solution corresponds to: $n = 0$

Equation (4.4) takes the form:

$$(\eta^2+1)F'' + 2\eta F' = 0.$$

Its solution with the aid of boundary conditions (4.5) is presented as

$$F(\eta) = \frac{1}{\Phi_1+\Phi_2}\left[(\alpha-\beta)\tan^{-1}\eta + \beta\Phi_1 + \alpha\Phi_2\right],$$

and from which

$$T(x,y) = \frac{1}{\Phi_1+\Phi_2}\left[(\alpha-\beta)\tan^{-1}\left(\frac{y}{x}\right)+\beta\Phi_1+\alpha\Phi_2\right].$$

Heat flux across L_3:

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = -\frac{\partial T}{\partial x}\sin\Phi_3 + \frac{\partial T}{\partial y}\cos\Phi_3.$$

Hence, we get:

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = -\sin\Phi_3\left(\frac{\beta-\alpha}{\Phi_1+\Phi_2}\right)\left(\frac{y}{x^2+y^2}\right)+\cos\Phi_3\left(\frac{\alpha-\beta}{\Phi_1+\Phi_2}\right)\left(\frac{x}{x^2+y^2}\right).$$

Solution corresponds to: $n \geq 1$

$$F(\eta) = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\eta^{2k}+\frac{b_1}{n}\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\eta^{2k+1} \tag{5.1}$$

and from which we get

$$T(x,y) = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}y^{2k}x^{n-2k}+\frac{b_1}{n}\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}y^{2k+1}x^{n-2k-1}$$

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = b_0[-\sin\Phi_3 M_{0,1}+\cos\Phi_3 M_{0,2}]+\frac{b_1}{n}[-\sin\Phi_3 M_{1,1}+\cos\Phi_3 M_{1,2}].$$

Applying the boundary conditions (4.5), we get:

$$\alpha = b_0 z_{0,1}+\frac{b_1}{n}z_{1,1}, \tag{5.2}$$

$$\beta = b_0 z_{0,2}+\frac{b_1}{n}z_{1,2} \tag{5.3}$$

where

$$M_{0,1} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(n-2k)\binom{n}{2k}y^{2k}x^{n-2k-1},$$

$$M_{1,1} = \sum_{k=0}^{[\frac{n-2}{2}]}(-1)^k(n-2k-1)\binom{n}{2k+1}y^{2k+1}x^{n-2k-2},$$

$$M_{0,2} = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k(2k)\binom{n}{2k}y^{2k-1}x^{n-2k},$$

$$M_{1,2} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(2k+1)\binom{n}{2k+1}y^{2k}x^{n-2k-1},$$

$$z_{0,1} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \tan^{2k} \Phi_1, \qquad z_{1,1} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k+1} \tan^{2k+1} \Phi_1,$$

$$z_{0,2} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} \tan^{2k} \Phi_2, \qquad z_{1,2} = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k+1} \tan^{2k+1} \Phi_2.$$

Solving (5.2) and (5.3) for the given value of "n" we get b_0 and b_1.

6 Special Cases

Case 1: Boundary conditions with different degrees of polynomials

The governing equation is given as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega,$$

with the following boundary conditions:

$$\begin{aligned} (i) \quad & T(x, y) = \alpha x^6, \quad (x, y) \in L_1, \\ (ii) \quad & T(x, y) = \beta x^5, \quad (x, y) \in L_2, \end{aligned}$$

and $\Phi_1 = 60°$, $\Phi_2 = 45°$, $\Phi_3 = 0°$.

From the principle of superposition, write

$$T(x, y) = T_1(x, y) + T_2(x, y),$$

where the boundary conditions for $T_1(x, y)$ are:

$$\begin{aligned} (i) \quad & T(x, y) = \alpha x^6, \quad (x, y) \in L_1, \\ (ii) \quad & T(x, y) = 0, \quad (x, y) \in L_2, \end{aligned}$$

and the boundary conditions for $T_2(x, y)$ are:

$$\begin{aligned} (i) \quad & T(x, y) = 0, \quad (x, y) \in L_1, \\ (ii) \quad & T(x, y) = \beta x^5, \quad (x, y) \in L_2. \end{aligned}$$

Setting $n = 6$ in the general solution (5.1), we get:

$$T_1(x, y) = \frac{\alpha}{64}(x^6 - 15y^2x^4 + 15y^4x^2 - y^6),$$

$$\frac{\partial T_1}{\partial n}(x, y)\Big|_{L_3} = -\frac{3\alpha b}{32}(5x^4 - 10b^2x^2 + b^4).$$

Setting $n = 5$ in the general solution (5.1), we get:

$$T_2(x, y) = \frac{\beta}{4(1 - \sqrt{3})}\left[\sqrt{3}(x^5 - 10y^2x^3 + 5y^4x) + (5yx^4 - 10y^3x^2 + y^5)\right],$$

$$\frac{\partial T_2}{\partial n}(x, y)\Big|_{L_3} = \frac{5\beta}{4(1 - \sqrt{3})}\left[-4b\sqrt{3}(x^3 - b^2x) + (x^4 - 6b^2x^2 + b^4)\right].$$

Hence, the analytic solution has the form

$$T(x,y) = \frac{\alpha}{64}(x^6 - 15y^2x^4 + 15y^4x^2 - y^6) + \frac{\beta}{4(1-\sqrt{3})}\left[\sqrt{3}(x^5 - 10y^2x^3 + 5y^4x) + (5yx^4 - 10y^3x^2 + y^5)\right],$$

and

$$\left.\frac{\partial T}{\partial n}(x,y)\right|_{L_3} = -\frac{3\alpha b}{32}(5x^4 - 10b^2x^2 + b^4) + \frac{5\beta}{4(1-\sqrt{3})}\left[-4b\sqrt{3}(x^3 - b^2x) + (x^4 - 6b^2x^2 + b^4)\right].$$

Case 2: One of the boundaries is vertical

The governing equation is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x,y) \in \Omega \tag{6.1}$$

with the following boundary conditions:

$$\begin{array}{ll} (i) & T(x,y) = \alpha y^n, \quad (x,y) \in L_1, \\ (ii) & T(x,y) = \beta x^n, \quad (x,y) \in L_2, \end{array} \tag{6.2}$$

and $\Phi_1 = \dfrac{\pi}{2}$.

Write

$$T(x,y) = w(x,y)q(y), \qquad q(y) \not\equiv 0 \quad \text{in} \quad \Omega.$$

Hence, (6.1) and (6.2) take the form:

$$q(y)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + 2\frac{\partial w}{\partial y}\frac{dq}{dy} + w\frac{d^2 q}{dy^2} = 0 \tag{6.3}$$

with the boundary conditions:

$$\begin{array}{ll} (i) & w(x,y) = \dfrac{\alpha y^n}{q(y)}, \qquad (x,y) \in L_1, \\ (ii) & w(x,y) = \dfrac{\beta x^n}{q(y)}, \qquad (x,y) \in L_2. \end{array}$$

Applying the invariant analysis, we get:

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y \\ \overline{w} = C^w w \\ \overline{q} = \dfrac{(C^x)^n}{C^w} q \end{cases}$$

and the absolute invariant $\eta(x,y)$ is:

$$\eta(x,y) = \frac{x}{y}. \tag{6.4}$$

The complete set of the absolute invariant corresponding to w and q are:

$$q(y) = R(y), \tag{6.5}$$

$$w(x,y) = \Gamma(y)F(\eta). \tag{6.6}$$

Substituting (6.4)–(6.6) in (6.3), with $\Gamma(y)R(y) = y^n$, we get:

$$(\eta^2+1)F'' - 2\eta(n-1)F' + n(n-1)F = 0. \tag{6.7}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} &F(0) = \alpha, \\ &F(-\cot\Phi_2) = (-\cot\Phi_2)^n\beta. \end{aligned} \tag{6.8}$$

For $n = 0$**:** Solution of (6.7) with the boundary conditions (6.8) is:

$$T(x,y) = \frac{\beta-\alpha}{\frac{\pi}{2}+\Phi_2}\tan^{-1}\left(\frac{x}{y}\right)+\alpha.$$

The heat flux across L_3 is:

$$\frac{\partial T}{\partial n}\Big|_{L_3} = \frac{\alpha-\beta}{\frac{\pi}{2}+\Phi_2}\frac{1}{x^2+y^2}(y\sin\Phi_3 + x\cos\Phi_3).$$

For $n \geq 1$**:** Solution of (6.7) with the boundary conditions (6.8) is:

$$T(x,y) = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\left(\frac{x}{y}\right)^{2k}y^n + \frac{b_1}{n}\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}\left(\frac{x}{y}\right)^{2k+1}y^n,$$

where

$$b_0 = \alpha, \tag{6.9}$$

$$\beta(-\cot\Phi_2)^n = b_0\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}\cot^{2k}\Phi_2 + \frac{b_1}{n}\sum_{k=0}^{[\frac{n}{2-1}]}(-1)^{k+1}\binom{n}{2k+1}\cot^{2k+1}\Phi_2. \tag{6.10}$$

Solving (6.9) and (6.10) for the given value of "n" we get b_0 and b_1.

The heat flux across L_3 is:

$$\frac{\partial T}{\partial n}(x,y)\Big|_{L_3} = b_0\left[-\sin\Phi_3 N_{0,1} + \cos\Phi_3 N_{0,2}\right] + \frac{b_1}{n}\left[-\sin\Phi - 3N_{1,1} + \cos\Phi_3 N_{1,2}\right],$$

where

$$N_{0,1} = \sum_{k=0}^{[\frac{n}{2}]}(-1)^k(2k)\binom{n}{2k}x^{2k-1}y^{n-2k},$$

$$N_{1,1} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(2k+1)\binom{n}{2k+1}x^{2k}y^{n-2k-1},$$

$$N_{0,2} = \sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k(n-2k)\binom{n}{2k}x^{2k}y^{n-2k-1},$$

$$N_{1,2} = \sum_{k=0}^{[\frac{n-2}{2}]}(-1)^k(n-2k-1)\binom{n}{2k+1}x^{2k+1}y^{n-2k-2}.$$

Case 3: $\Phi_1 = \Phi_2 = \dfrac{\pi}{4}$**,** $\Phi_3 = 0$

From (5.2), (5.3) and $\Phi_1 = \Phi_2 = \dfrac{\pi}{4}$, we find that $n = 2$.

The governing equation is given by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x, y) \in \Omega \tag{6.11}$$

with the following boundary conditions:

$$\begin{aligned} &(i) \quad T(x, y) = \alpha x^2, \qquad (x, y) \in L_1, \\ &(ii) \quad T(x, y) = -\alpha x^2, \qquad (x, y) \in L_2. \end{aligned} \tag{6.12}$$

Write

$$T(x, y) = w(x, y) q(x), \qquad q(x) \not\equiv 0 \quad \text{in} \quad \Omega.$$

Hence, (6.11) and (6.12) take the form:

$$q(x) \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) + 2 \frac{\partial w}{\partial x} \frac{dq}{dx} + w \frac{d^2 q}{dx^2} = 0 \tag{6.13}$$

with the boundary conditions:

$$\begin{aligned} &(i) \quad w(x, y) = \frac{\alpha x^2}{q(x)}, \qquad (x, y) \in L_1, \\ &(ii) \quad w9x, y) = -\frac{\alpha x^2}{q(x)}, \qquad (x, y) \in L_2. \end{aligned}$$

Applying the invariant analysis, we get:

$$G : \begin{cases} \overline{x} = C^x x \\ \overline{y} = C^x y + K^y \\ \overline{w} = C^w w \\ \overline{q} = \dfrac{(C^x)^2}{C^w} q \end{cases}$$

and the absolute invariant $\eta(x, y)$ is:

$$\eta(x, y) = \frac{y}{x}. \tag{6.14}$$

The complete set of the absolute invariants corresponding to w and q is:

$$q(x) = R(x), \tag{6.15}$$

$$w(x, y) = \Gamma(x) F(\eta). \tag{6.16}$$

Substituting (6.14)–(6.16) in (6.13), with $\Gamma(x) R(x) = x^2$, we get:

$$(\eta^2 + 1) F'' - 2\eta F' + 2F = 0. \tag{6.17}$$

Under the similarity variable η, the boundary conditions are:

$$\begin{aligned} &F(-1) = -\alpha, \\ &F(1) = \alpha. \end{aligned} \tag{6.18}$$

It is clear that two conditions in (6.18) are identical. Hence, to find the second condition, assume that the heat flux across L_3 takes the form:

$$\left.\frac{\partial T}{\partial y}\right|_{L_3} = \gamma + \alpha x, \tag{6.19}$$

where γ is a constant.

Solution of (6.17) with the boundary conditions (6.18) and (6.19) is:

$$T(x, y) = \frac{\gamma}{2b}(y^2 - x^2) + \alpha x y.$$

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Exact Solutions of the Nonlinear Diffusion Equation $u_0 + \nabla \left[u^{-\frac{4}{5}} \nabla u\right] = 0$

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Abstract

The symmetry reduction of the equation $u_0+\nabla \left[u^{-\frac{4}{5}} \nabla u\right] = 0$ to ordinary differential equations with respect to all subalgebras of rank three of the algebra $A\widetilde{E}(1) \oplus AC(3)$ is carried out. New invariant solutions are constructed for this equation.

1 Introduction

This paper is concerned with some invariant solutions to the three-dimensional nonlinear diffusion equation

$$\frac{\partial u}{\partial x_0} + \nabla \left[u^{-\frac{4}{5}} \nabla u\right] = 0. \qquad (1)$$

Its symmetry properties are know [1, 2]. The maximal Lie point symmetry algebra F of equation (1) has the basis

$$P_0 = \frac{\partial}{\partial x_0}, \quad P_a = -\frac{\partial}{\partial x_a}, \quad J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a},$$
$$D_1 = x_0 \frac{\partial}{\partial x_0} + \frac{5}{4} u \frac{\partial}{\partial u}, \quad D_2 = \sum_{a=1}^{3} x_a \frac{\partial}{\partial x_a} - \frac{5}{2} u \frac{\partial}{\partial u},$$
$$K_a = \left(2x_a^2 - \vec{x}^2\right) \frac{\partial}{\partial x_a} + 2 \sum_{b \neq a} x_a x_b \frac{\partial}{\partial x_b} - 5 x_a u \frac{\partial}{\partial u}$$

with $a, b = 1, 2, 3$. It is easy to see that $F = A\widetilde{E}(1) \oplus AC(3)$, where $A\widetilde{E}(1) = \langle P_0, D_1 \rangle$ is an extended Euclidean algebra and $AC(3) = \langle P_a, K_a, J_{ab}, D_2 : a, b = 1, 2, 3 \rangle$ is a conformal algebra. If we make the transformation $u = v^5$, we obtain the equation

$$\frac{\partial v}{\partial x_0} + v^{-4} \triangle v = 0. \qquad (2)$$

Clearly, we see that

$$5u \frac{\partial}{\partial u} = v \frac{\partial}{\partial v}.$$

In the present article, the symmetry reduction of equation (2) is carried out with respect to all subalgebras of rank three of the algebra F, up to conjugacy with respect to the group of inner automorphisms. Some exact solutions of equation (1) are obtained by means of this reduction (for the concepts and results used here, see also [3, 4]).

2 Classification of subalgebras of the invariance algebra

If $v = v(x_1, x_2, x_3)$ is a solution of equation (2), then v is a solution of the Laplace equation $\triangle v = 0$. In this connection, let us restrict ourselves to those subalgebras of the algebra F that do not contain P_0. Among subalgebras possessing the same invariants, there exists a subalgebra containing all the other subalgebras. We call it I-maximal.

Theorem 1 *Up to the conjugation under to the group of inner automorphisms, the algebra F has 18 I-maximal subalgebras of rank three which do not contain P_0:*

$$L_1 = \langle P_1, P_2, P_3, J_{12}, J_{13}, J_{23}\rangle,\ L_2 = \langle P_0 - P_1, P_2, P_3, J_{23}\rangle,$$

$$L_3 = \langle P_2, P_3, J_{23}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),\quad L_4 = \langle P_0 - P_1, P_3, D_1 + D_2\rangle,$$

$$L_5 = \langle P_3, J_{12}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),\quad L_6 = \langle P_0 - P_3, J_{12}, D_1 + D_2\rangle,$$

$$L_7 = \langle P_3, J_{12} + \alpha P_0, D_2 + \beta P_0\rangle\ (\alpha = 1,\ \beta \in \mathbb{R}\ \textit{or}\ \alpha = 0\ \textit{and}\ \beta = 0, \pm 1),$$

$$L_8 = \langle P_2, P_3, J_{23}, D_1 - P_1\rangle,\ L_9 = \langle P_2, P_3, J_{23}, D_2 + \alpha P_0\rangle\ \ (\alpha = 0, \pm 1),$$

$$L_{10} = \langle P_3, D_1 + \alpha J_{12}, D_2 + \beta J_{12}\rangle\ \ (\alpha > 0,\ \beta \in \mathbb{R}\ \textit{or}\ \alpha = 0,\ \beta \geq 0),$$

$$L_{11} = \langle J_{12}, D_1, D_2\rangle,\ L_{12} = \langle J_{12}, J_{13}, J_{23}, D_1 + \alpha D_2\rangle\ (\alpha \in \mathbb{R}),$$

$$L_{13} = \langle J_{12}, J_{13}, J_{23}, D_2 + \alpha P_0\rangle\ (\alpha = 0, \pm 1),\ L_{14} = \langle J_{12}, K_3 - P_3, D_1\rangle,$$

$$L_{15} = \langle P_1 + K_1, P_2 + K_2, J_{12}, D_1\rangle,$$

$$L_{16} = \langle K_1 - P_1, K_2 - P_2, K_3 - P_3, J_{12}, J_{13}, J_{23}\rangle,$$

$$L_{17} = \langle P_1 + K_1, P_2 + K_2, J_{12}, K_3 - P_3 + \alpha D_1\rangle\ (\alpha \in \mathbb{R}),$$

$$L_{18} = \langle P_1 + K_1, P_2 + K_2, P_3 + K_3, J_{12}, J_{13}, J_{23}\rangle.$$

Proof. Let

$$\Omega_{0a} = \frac{1}{2}(P_a + K_a), \quad \Omega_{a4} = \frac{1}{2}(K_a - P_a),\ \Omega_{ab} = -J_{ab}, \quad \Omega_{04} = D$$

with $a, b = 1, 2, 3$. They satisfy the following commutation relations:

$$[\Omega_{\alpha\beta}, \Omega_{\gamma\delta}] = g_{\alpha\delta}\Omega_{\beta\gamma} + g_{\beta\gamma}\Omega_{\alpha\delta} - g_{\alpha\gamma}\Omega_{\beta\delta} - g_{\beta\delta}\Omega_{\alpha\gamma},$$

where $\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, 4$ and $(g_{\alpha\beta}) = \mathrm{diag}\,[1, -1, -1, -1, -1]$. It follows from this that $AC\,(3) \cong AO\,(1, 4)$.

The classification of subalgebras of the algebra $AO\,(1, 4)$ up to $O\,(1, 4)$-conjugacy is done in [5]. Let L is an I-maximal subalgebra of rank three of the algebra F and $P_0 \notin L$. Denote by K the projection of L onto $AO\,(1, 4)$. If K has an invariant isotropic subspace in the Minkowski space $\mathbb{R}_{1,4}$, then K is conjugate to a subalgebra of the extended Euclidean algebra $A\widetilde{E}(3)$ with the basis $P_a,\ J_{ab},\ D_2\ (a, b = 1, 2, 3)$. In this case, on the basis of Theorem 1 [6], L is conjugate with one of the subalgebras $L_1, \ldots, L_{13}$. If K has no invariant isotropic subspace in the space $\mathbb{R}_{1,4}$, then K is conjugate with one of the following algebras [5]:

$$N_1 = \langle \Omega_{12}, \Omega_{34} \rangle, \quad N_2 = AO(1,2) = \langle \Omega_{01}, \Omega_{02}, \Omega_{12} \rangle,$$

$$N_3 = \langle \Omega_{12} + \Omega_{34}, \Omega_{13} - \Omega_{24}, \Omega_{23} + \Omega_{14} \rangle,$$

$$N_4 = \langle \Omega_{12} + \Omega_{34}, \Omega_{13} - \Omega_{24}, \Omega_{23} + \Omega_{14} \rangle \oplus \langle \Omega_{12} - \Omega_{34} \rangle,$$

$$N_5 = AO(1,2) \oplus AO(2) = \langle \Omega_{01}, \Omega_{02}, \Omega_{12}, \Omega_{34} \rangle,$$

$$N_6 = AO(4) = \langle \Omega_{ab} : \ a, b = 1, 2, 3, 4 \rangle, \quad N_7 = AO(1,3) = \langle \Omega_{\alpha\beta} : \ \alpha, \beta = 0, 1, 2, 3 \rangle,$$

$$N_8 = AO(1,4) = \langle \Omega_{\alpha\beta} : \ \alpha, \beta = 0, 1, 2, 3, 4 \rangle.$$

The rank of N_j equals 2 for $j = 1, 2$; 3 for $j = 3, \ldots, 7$; 4 for $j = 8$. From this we conclude that L is conjugate with one of the subalgebras $L_{14}, \ldots, L_{18}$. The theorem is proved.

3 Reduction of the diffusion equation to the ordinary differential equations

For each of the subalgebras $L_1, \ldots, L_{18}$ obtained in Theorem 1, we point out the corresponding ansatz $\omega' = \varphi(\omega)$ solved for v, where ω and ω' are functionally independent invariants of a subalgebra, as well as the reduced equation which is obtained by means of this ansatz. The numbering of the considered cases corresponds to the numbering of the subalgebras $L_1, \ldots, L_{18}$.

3.1. $v = \varphi(\omega)$, $\omega = x_0$, $\varphi' = 0$. The corresponding exact solution of the diffusion equation (1) is trivial: $u = C$.
3.2. $v = \varphi(\omega)$, $\omega = x_0 - x_1$, then

$$\varphi'' + \varphi^4 \varphi' = 0. \tag{3}$$

The general solution of equation (3) is of the form

$$\int \frac{d\varphi}{C_1 - \varphi^5} = \frac{1}{5}\omega + C_2,$$

where C_1 and C_2 are arbitrary constants. If $C_1 = 0$, then

$$\varphi = \left(\frac{5}{4\omega + C}\right)^{\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is of the form

$$u = \left[\frac{5}{4(x_0 - x_1) + C}\right]^{\frac{5}{4}}.$$

3.3. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = x_1 x_0^{-\alpha}$, then

$$\varphi'' - \alpha\omega\varphi^4\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0. \tag{4}$$

The nonzero function $\varphi = \left(A\omega^2 + B\omega + C\right)^{-\frac{1}{4}}$ is a solution of equation (4) if and only if one of the following conditions is satisfied:

1. $\alpha = 1,\ A = 0,\ C = \frac{5}{4}B^2.$

2. $\alpha = 0,\ A = -\frac{1}{3},\ C = -\frac{3}{4}B^2.$

3. $A = -\frac{1}{3},\ B = C = 0.$

4. $\alpha = \frac{5}{6},\ A = -\frac{1}{3},\ B = 0.$

5. $\alpha = \frac{1}{2},\ A = 0,\ B = 0.$

By means of φ obtained above, we find such invariant solutions of equation (1):

$$u = \left(Bx_1 + \frac{5}{4}B^2x_0\right)^{-\frac{5}{4}}, \quad u = \left[-\frac{12x_0}{(2x_1+3B)^2}\right]^{\frac{5}{4}}, \quad u = \left(-\frac{x_1^2}{3x_0} + Bx_0^{\frac{2}{3}}\right)^{-\frac{5}{4}},$$

where B is an arbitrary constant.

3.4. $v = (x_0 - x_1)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = x_2 (x_0 - x_1)^{-1}$,

$$\left(1+\omega^2\right)\varphi'' + \left(\frac{5}{2} - \varphi^4\right)\omega\varphi' + \frac{5}{16}\varphi - \frac{1}{4}\varphi^5 = 0. \tag{5}$$

The function $\varphi = (A\omega + B)^{-\frac{1}{4}}$, where A and B are constants, satisfies equation (5) if and only if $A^2 = -B\left(B - \frac{4}{5}\right)$. The corresponding invariant solution of equation (1) is of the form $u = [Ax_2 + B(x_0 - x_1)]^{-\frac{5}{4}}$. It is easy to see that the coefficient B can take on any value from the interval $\left(0; \frac{4}{5}\right)$.

3.5. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2\right) x_0^{-2\alpha}$,

$$4\omega\varphi'' + \left(4 - 2\alpha\omega\varphi^4\right)\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0.$$

Integrating this equation for $\alpha = \frac{5}{2}$, we obtain the following equation:

$$4w\varphi' - \omega\varphi^5 = C,$$

where C is an arbitrary constant. If $C = 0$, then

$$\varphi = (-\omega + B)^{-\frac{1}{4}}.$$

Thus, we find the exact solution

$$u = \left(B x_0^4 - \frac{x_1^2 + x_2^2}{x_0} \right)^{-\frac{5}{4}}.$$

3.6. $v = (x_0 - x_3)^{-\frac{1}{4}} \varphi(\omega)\,,\ \omega = \left(x_1^2 + x_2^2\right)(x_0 - x_3)^{-2}$, then

$$4\omega (1 + \omega) \varphi'' + \left(7\omega + 4 - 2\omega\varphi^4\right) \varphi' + \frac{5}{16}\varphi - \frac{1}{4}\varphi^5 = 0.$$

3.7. $v = \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}} \varphi(\omega)\,,\ \omega = \left(x_1^2 + x_2^2\right)^{-\frac{\beta}{2}} \exp\left(x_0 + \alpha \arctan \frac{x_1}{x_2} \right)$, then

$$\left(\alpha^2 + \beta^2\right) \omega^2 \varphi'' + \left(\varphi^4 + \alpha^2 + \beta^2 + \beta\right) \omega \varphi' + \frac{1}{4}\varphi = 0.$$

For $\alpha = \beta = 0$, we have $\varphi = (-\ln\omega - C)^{\frac{1}{4}}$. Consequently, we find the exact solution

$$u = \left(-\frac{x_1^2 + x_2^2}{x_0 + C} \right)^{-\frac{5}{4}}.$$

3.8. $v = x_0^{\frac{1}{4}} \varphi(\omega)\,,\ \omega = x_1 - \ln x_0$,

$$\varphi'' - \varphi^4 \varphi' + \frac{1}{4}\varphi^5 = 0.$$

3.9. $v = x_1^{-\frac{1}{2}} \varphi(\omega)\,,\ \omega = x_1^{-\alpha} \exp(x_0)$, then

$$\alpha^2 \omega^2 \varphi'' + \left(\varphi^4 + \alpha^2 + 2\alpha\right) \omega \varphi' + \frac{3}{4}\varphi = 0.$$

If $\alpha = 0$, then

$$\varphi = \left(-\frac{1}{3\ln\omega + C} \right)^{-\frac{1}{4}}.$$

Whence we obtain the exact solution of (1):

$$u = \left(-\frac{3x_0 + C}{x_1^2} \right)^{\frac{5}{4}}.$$

3.10. $v = x_0^{\frac{1}{4}} \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}} \varphi(\omega)\,,\ \omega = \arctan \frac{x_1}{x_2} + \alpha \ln x_0 + \frac{\beta}{2} \ln\left(x_1^2 + x_2^2\right)$,

$$\left(1 + \beta^2\right) \varphi'' + \left(\alpha\varphi^4 - \beta\right) \varphi' + \frac{1}{4}\left(\varphi + \varphi^5\right) = 0.$$

3.11. $v = x_0^{\frac{1}{4}} \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}} \varphi(\omega)\,,\ \omega = \left(x_1^2 + x_2^2\right) x_3^{-2}$, then

$$4\omega^2 (1 + \omega) \varphi'' + \left(6\omega^2 + 2\omega\right) \varphi' + \frac{1}{4}\left(\varphi + \varphi^5\right) = 0.$$

3.12. $v = x_0^{\frac{1-2\alpha}{4}} \varphi(\omega)$, $\omega = \left(x_1^2 + x_2^2 + x_3^2\right) x_0^{-2\alpha}$, then

$$4\omega\varphi'' + \left(6 - 2\alpha\omega\varphi^4\right)\varphi' + \frac{1-2\alpha}{4}\varphi^5 = 0. \tag{6}$$

Integrating this reduced equation for $\alpha = \frac{5}{2}$, we find the equation

$$4\omega\varphi' + 2\varphi - \omega\varphi^5 = \widetilde{C},$$

where $\widetilde{C}$ is an arbitrary constant. If $\widetilde{C} = 0$, then the general solution of this reduced equation is of the form

$$\varphi = \left(C\omega^2 + \omega\right)^{-\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is a function

$$u = \left[\frac{x_0^6}{C\left(x_1^2 + x_2^2 + x_3^2\right)^2 + x_0^5\left(x_1^2 + x_2^2 + x_3^2\right)}\right]^{\frac{5}{4}}.$$

For $\alpha = -\frac{5}{2}$ equation (6) has a solution given by

$$\varphi = (\omega + C)^{-\frac{1}{4}}.$$

Thus, we find the exact solution

$$u = \left(\frac{x_0^6}{\left(x_1^2 + x_2^2 + x_3^2\right) x_0^5 + C}\right)^{\frac{5}{4}}.$$

3.13. $v = \left(x_1^2 + x_2^2 + x_3^2\right)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = 2x_0 - \alpha \ln\left(x_1^2 + x_2^2 + x_3^2\right)$, then

$$2\alpha^2\varphi'' + \varphi^4\varphi' - \frac{1}{8}\varphi = 0.$$

For $\alpha = 0$ we have $\varphi = \left(\frac{1}{2}\omega + C\right)^{\frac{1}{4}}$. Whence we obtain the exact solution

$$u = \left(\frac{x_0 + C}{x_1^2 + x_2^2 + x_3^2}\right)^{\frac{5}{4}}.$$

3.14. $v = x_0^{\frac{1}{4}} \left(x_1^2 + x_2^2\right)^{-\frac{1}{4}} \varphi(\omega)$, $\omega = \dfrac{x_1^2 + x_2^2 + x_3^2 + 1}{\left(x_1^2 + x_2^2\right)^{\frac{1}{2}}}$,

$$\left(\omega^2 - 4\right)\varphi'' + 2\omega\varphi' + \frac{1}{4}\varphi\left(1 + \varphi^4\right) = 0.$$

3.15. $v = x_0^{\frac{1}{4}} x_3^{-\frac{1}{2}} \varphi(\omega)$, $\omega = \dfrac{x_1^2 + x_2^2 + x_3^2 - 1}{x_3}$,

$$\left(\omega^2 + 4\right)\varphi'' + 3\omega\varphi' + \frac{1}{4}\varphi\left(3 + \varphi^4\right) = 0.$$

3.16. $v = \left(x_1^2 + x_2^2 + x_3^2 + 1\right)^{-\frac{1}{2}} \varphi(\omega)$, $\omega = x_0$, then

$$\varphi^3\varphi' - 3 = 0.$$

In this case, $\varphi = (12\omega + C)^{\frac{1}{4}}$, and therefore

$$u = \left[\frac{12x_0 + C}{\left(x_1^2 + x_2^2 + x_3^2 + 1\right)^2}\right]^{\frac{5}{4}}.$$

3.17. $v = \left[\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2\right]^{-\frac{1}{4}} \exp\left[\frac{\alpha}{8}\arctan\frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3}\right]\varphi(\omega)$,

$$\omega = \alpha \arctan\frac{x_1^2 + x_2^2 + x_3^2 - 1}{2x_3} - 2\ln x_0,$$

$$4\alpha^2\varphi'' + \left[\alpha^2 - 2\varphi^4\exp\left(\frac{\omega}{2}\right)\right]\varphi' + \frac{\alpha^2 + 16}{16}\varphi = 0.$$

We have the exact solution

$$u = \left[\frac{C - 4x_0}{\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2 + 4x_3^2}\right]^{\frac{5}{4}}.$$

3.18. $v = \left(x_1^2 + x_2^2 + x_3^2 - 1\right)^{-\frac{1}{2}} \varphi(\omega)$, $\omega = x_0$, then

$$\varphi^3\varphi' + 3 = 0.$$

Integrating this equation, we obtain

$$\varphi = (-12\omega + C)^{\frac{1}{4}}.$$

The corresponding invariant solution of equation (1) is of the form

$$u = \left[\frac{C - 12x_0}{\left(x_1^2 + x_2^2 + x_3^2 - 1\right)^2}\right]^{\frac{5}{4}}.$$

4 Multiplication of solutions

Solving the Lie equations corresponding to the basis elements of the algebra F, we obtain a one-parameter group of transformations $(x_0, x_1, x_2, x_3, u) \to (x_0', x_1', x_2', x_3', u')$ generated by these vector fields:

$$\exp(\Theta P_0): \quad x_0' = x_0 + \Theta, \quad x_a' = x_a \ (a = 1, 2, 3), \quad u' = u;$$

$$\exp(\Theta P_a): \quad x_0' = x_0, \quad x_a' = x_a - \Theta, \quad x_c' = x_c \text{ for } c \neq a, \quad u' = u;$$

$$\exp(\Theta J_{ab}): \quad x_0' = x_0, \quad x_a' = x_a\cos\Theta - x_b\sin\Theta,$$

$$x_b' = x_a\sin\Theta + x_b\cos\Theta, \ x_c' = x_c, \ u' = u, \text{ where } \{a, b, c\} = \{1, 2, 3\};$$

$$\exp(\Theta D_1): \quad x_0' = x_0 \exp(\Theta), \quad x_a' = x_a\,(a = 1, 2, 3), \quad u' = u \exp\left(\frac{5\Theta}{4}\right);$$

$$\exp(\Theta D_2): \quad x_0' = x_0, \quad x_a' = x_a \exp(\Theta)\,(a = 1, 2, 3), \quad u' = u \exp\left(-\frac{5\Theta}{2}\right);$$

$$\exp(\Theta K_a): \quad x_0' = x_0, \quad x_c' = \frac{x_c - \delta_{ac}\Theta\vec{x}^2}{1 - 2\Theta x_a + \Theta^2\vec{x}^2} \quad (c = 1, 2, 3),$$

$$u' = u\left(1 - 2\Theta x_a + \Theta^2\vec{x}^2\right)^{\frac{5}{2}}, \quad \text{where} \quad \vec{x}^2 = x_1^2 + x_2^2 + x_3^2.$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 437–443.

On Reduction and Q-conditional (Nonclassical) Symmetry

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Abstract

Some aspects of Q-conditional symmetry and of its connections with reduction and compatibility are discussed.

There are a number of examples in the history of science when a scientific theory quickly developed, gave good results and applications, and had no sufficiently good theoretical fundament. Let us remember mathematical analysis in the XVIII-th century. It is true in some respect for the theory of Q-conditional (also called non-classical) symmetry. The pioneer paper of Bluman and Cole [1] where this conception appeared was published 28 years ago. And even now, some properties of Q-conditional symmetry are not completely investigated.

1. Q-conditional symmetry being a generalization of the classical Lie symmetry, we recall briefly some results of the Lie theory.

Let

$$L(x, \underset{(r)}{u}) = 0, \quad L = (L^1, L^2, \ldots, L^q), \tag{1}$$

be a system of q partial differential equations (PDEs) of order r in n independent variables $x = (x_1, x_2, \ldots, x_n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$, satisfying the maximal rank condition. Here $\underset{(r)}{u}$ denotes all derivatives of the function u up to order r.

Consider a differential operator of the first order

$$Q = \sum_{i=1}^{n} \xi^i(x,u)\partial_{x_i} + \sum_{j=1}^{m} \eta^j(x,u)\partial_{u^j}.$$

Actions of the operator Q on the functions u^j, $j = \overline{1,m}$, are defined by the formula

$$Qu^j = \eta^j - \sum_{i=1}^{n} \xi^i u^j_i, \quad j = \overline{1,m}.$$

Definition. *An operator Q is called a Lie symmetry operator of system (1) if system (1) is invariant under the local transformations generated by the operator Q.*

Theorem. *An operator Q is a Lie symmetry operator of system (1) if and only if the condition*

$$\underset{(r)}{Q}\, L(x, \underset{(r)}{u}) \Big|_{\{L(x,\underset{(r)}{u})=0\} =: K} = 0 \tag{2}$$

is satisfied where $\underset{(r)}{Q}$ *denotes the* r*-th prolongation of the operator* Q *and* K *does the set of differential consequences of system (1).*

Theorem. *The set of Lie symmetry operators of system (1) is a Lie algebra under the standard Lie brackets of differential operators of the first order.*

Consider the set of the Lie symmetry operators $Q_1, Q_2, \ldots, Q_s$ $(s < n)$ of system (1) which satisfy the following conditions:

1. $\langle Q_1,\, Q_2, \ldots,\, Q_s \rangle$ is a Lie algebra;
2. $\operatorname{rank} ||\xi^{ki}||\,{}_{k=1}^{s}\,{}_{i=1}^{n} = \operatorname{rank} ||\xi^{ki}, \eta^{kj}||\,{}_{k=1}^{s}\,{}_{i=1}^{n}\,{}_{j=1}^{m} = s.$

Then, a general solution of the system

$$Q^k u^j = 0, \quad k = \overline{1,s}, \quad j = \overline{1,m}, \tag{3}$$

can be written in the form

$$W^j(x,u) = \varphi^j(\omega(x,u)), \quad j = \overline{1,m}, \quad \omega = (\omega_1, \omega_2, \ldots, \omega_{n-s}), \tag{4}$$

where φ^j, $j = \overline{1,m}$, are arbitrary differentiable functions of ω; W^j, $j = \overline{1,m}$, and ω^l, $l = \overline{1, n-s}$, are functionally independent first integrals of system (3) and the Jacobian of W under u does not vanish, that is,

$$\det ||W^j_{u^{j'}}||\,{}_{j,j'=1}^{m} \neq 0. \tag{5}$$

Owing to condition (5) being satisfied, we can solve equation (4) with respect to u. As a result, we obtain a form to find u, also called "ansatz":

$$u = F(x, \varphi(\omega)), \tag{6}$$

where u, F, and φ have m components.

Theorem. *Substituting (6) into (1), we obtain a set of equations that is equivalent to a system for the functions* φ^j $(j = \overline{1,m})$*, depending only on variables* $\omega_1,\, \omega_2,\, \ldots,\, \omega_{n-s}$ *(that is called the reduced system corresponding to the initial system (1) and the algebra* $\langle Q_1,\, Q_2, \ldots,\, Q_s\rangle$*).*

Let us emphasize some properties of Lie symmetry and Lie reduction.

1. The *local approach,* that is, a system of PDEs is assumed a manifold in a prolongated space.
2. To find the maximal Lie invariance algebra of a system of PDEs, we have to consider *all the non-trivial differential consequences* of this system of order less than $r + 1$.
3. *Conditional compatibility,* that is, the system

   $$L(x, \underset{(r)}{u}) = 0, \quad Q^k u^j = 0, \quad k = \overline{1,s}, \quad j = \overline{1,m},$$

   consisting of an inverstigated system od PDEs and the surface invariant conditions is compatible if and only if the corresponding reduced system is compatible.

Note that there are a number of examples when the reduced system is not compatible. For instance, consider the equation $tu_t + xu_x = 1$. It is invariant under the dilatation operator $t\partial_t + x\partial_x$, but the corresponding reduced system $(0 = 1)$ is not compatible.

2. A direct generalization of Lie symmetry is conditional symmetry introduced by Prof. Fushchych in 1983 [2].

Definition. *Consider a system of the form (1) and an appended additional condition*

$$L'(x, \underset{(r')}{u}) = 0. \tag{7}$$

System (1), (7) being invariant under an operator Q, system (1) is called conditionally invariant under this operator.

Conditional symmetry was a favourite conception of Prof. Fushchych.

3. Let us pass to Q-conditional symmetry. Below we take $m = q = 1$, i.e., we consider one PDE in one unknown function.

It can be seemed on the face of it that Q-conditional (non-classical) symmetry is a particular case of conditional symmetry. One often uses the following definition of non-classical symmetry.

Definition 1. *A PDE of the form (1) is called Q-conditionally (or non-classical) invariant under an operator Q if the system of equation (1) and the invariant surface condition*

$$Qu = 0 \tag{8}$$

is invariant under this operator.

It is not a quitely correct definition. Indeed, equation (1) is Q-conditionally invariant under an operator Q in terms of definition 1 if the following condition is satisfied:

$$\underset{(r)}{Q}\, L(x, \underset{(r)}{u})\Bigg|_{\overline{\{L(x, \underset{(r)}{u})=0,\ Qu=0\}} =:\, M} = 0, \tag{9}$$

where M denotes the set of differential consequences of system (1), (8). But the equation $\underset{(r)}{Q}\, L(x, \underset{(r)}{u})$ belongs to M because it is simple to see that the following formula is true:

$$\underset{(r)}{Q}\, L(x, \underset{(r)}{u}) = \sum_{\alpha:\, |\alpha|\le r} \left(\partial_{u_\alpha} L(x, \underset{(r)}{u})\right) \cdot D_\alpha(Qu) + \sum_{i=1}^{n} \xi^i D_{x_i}\left(L(x, \underset{(r)}{u})\right),$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multiindex, D is the full derivation operator. Therefore, equation (9) is an identity for an arbitrary operator Q.

We like the slightly different definition of Q-conditional symmetry that appeared in [3] and was developed in [4].

Following [4], at first, let us formulate two auxiliary definitions.

Definition. *Operators Q^k, $k = \overline{1, s}$ form an involutive set if there exist functions $f^{klp} = f^{klp}(x, u)$ satisfying the condition*

$$[Q^k, Q^l] = \sum_{p=1}^{s} f^{klp} Q^p, \quad k = \overline{1, s}, \quad \textit{where} \quad [Q^k, Q^l] = Q^k Q^l - Q^l Q^k.$$

Definition. *Involutive sets of operators $\{Q^k\}$ and $\{\tilde{Q}^k\}$ are equivalent if*

$$\exists\,\lambda_{kl}=\lambda_{kl}(x,u)\ (\det||\lambda_{kl}||\,^{s}_{k,l=1}\neq 0):\quad \tilde{Q}^k=\sum_{l=1}^{s}\lambda_{kl}Q^l,\quad k=\overline{1,s}.$$

Definition 2 [4]. *The PDE (1) is called Q-conditionally invariant under the involutive sets of operators $\{Q^k\}$ if*

$$Q^k_{(r)}L(x,\underset{(r)}{u})\Big|_{L(x,\underset{(r)}{u})=0,\ \overline{\{Q^lu=0,\quad l=\overline{1,s}\}}=:N}=0,\quad k=\overline{1,s},\tag{10}$$

where N denotes the set of differential consequences of the invariant surface conditions $Q^lu=0,\quad l=\overline{1,s}$, of order either less than or equal to $r-1$.

In fact, namely this definition is always used to find Q-conditional symmetry. Restriction on the maximal order of the differential consequences of the invariant surface conditions is not essential.

Lemma. *If equation (1) is Q-conditionally invariant under an involutive set of operators $\{Q^k\}$ then equation (1) is Q-conditionally invariant under an arbitrary involutive set of operators $\{\tilde{Q}^k\}$ being equivalent to the set $\{Q^k\}$ too.*

It follows from Definition 2 that Q-conditional symmetry conserves the properties of Lie symmetry which are emphasized above, and the reduction theorem is proved in the same way as for Lie symmetry.

The conditional compatibility property is conserved too. Therefore, reduction under Q-conditional symmetry operators does not always follow compatibility of the system from the initial equation and the invariant surface conditions. And vice versa, compatibility of this system does not follow Q-conditional invariance of the investigated equation under the operators Q^k. For instance, the equation

$$u_t+u_{xx}-u+t(u_x-u)=0\tag{11}$$

is not invariant under the translation operator with respect to t (∂_t), and system of the equation (11) and $u_t=0$ is compatible because it has the non-trivial solution $u=Ce^x$ ($C=\mathrm{const}$).

Unfortunately, in terms of the local approach, equation (10) is not a necessary condition of reduction with respect to the corresponding ansatz, although the conceptions of Q-conditional symmetry and reduction are very similar. For example, the equation

$$u_t+(u_x+tu_{xx})(u_{xx}+1)=0$$

is reduced to the equation $\varphi''+1=0$ by means of the ansatz $u=\varphi(x)$ and is not invariant under the operator ∂_t. It is possible that the definition of Q-conditional invariance can be given in another way for the condition analogous to (10) to be a necessary condition of reduction.

4. Last time, Q-conditional symmetry of a number of PDEs was investigated, in particular, by Prof. Fushchych and his collaborators (see [6] for references). Consider two simple examples.

At first, consider the one-dimensional linear heat equation

$$u_t = u_{xx}. \tag{12}$$

Lie symmetries of equation (12) are well known. Its maximal Lie invariance algebra is generated by the operators

$$\begin{aligned}&\partial_t, \quad \partial_x, \quad G = t\partial_x - \tfrac{1}{2}xu\partial_u, \quad I = u\partial_u, \quad D = 2t\partial_t + x\partial_x,\\ &\Pi = 4t^2\partial_t + 4tx\partial_x - (x^2+2t)u\partial_u, \quad f(t,x)\partial_u,\end{aligned} \tag{13}$$

where $f = f(t,x)$ is an arbitrary solution of (12). Firstly, Q-conditional symmetry of (12) was inverstigated by Bluman and Cole in [1].

Theorem 1 [7]. *An arbitrary Q-conditional symmetry operator of the heat equation (12) is equivalent to either the operator*

$$Q = \partial_t + g^1(t,x)\partial_x + (g^2(t,x)u + g^3(t,x))\partial_u, \quad \textit{where}$$

$$g^1_t - g^1_{xx} + 2g^1_x g^1 + 2g^2_x = 0, \quad g^k_t - g^k_{xx} + 2g^1_x g^k = 0, \quad k = 2,3, \tag{14}$$

or the operator

$$Q = \partial_x + \theta(t,x,u)\partial_u, \quad \textit{where}$$

$$\theta_t + \theta_{xx} + 2\theta\theta_{xu} - \theta^2\theta_{uu} = 0. \tag{15}$$

The system of defining equations (14) was firstly obtained by Bluman and Cole [1]. Further investigation of system (14) was continued in [5] where the question of linearization of the first two equations of (14) was studied. The general solution of the problem of linearization of (14) and (15) was given in [7]. We investigated Lie symmetry properties of (14) and (15).

Theorem 2 [7]. *The maximal Lie invariance algebra (14) is generated by the operators*

$$\begin{aligned}&\partial_t, \quad \partial_x, \quad G^1 = t\partial_x + \partial_{g^1} - \tfrac{1}{2}g^1\partial_{g^2} - \tfrac{1}{2}xg^3\partial_{g^3}, \quad I^1 = g^3\partial_{g^3},\\ &D^1 = 2t\partial_t + x\partial_x - g^1\partial_{g^1} - 2g^2\partial_{g^2}, \quad (f_t + f_x g^1 - fg^2)\partial_{g^3},\\ &\Pi^1 = 4t^2\partial_t + 4tx\partial_x - 4(x - tg^1)\partial_{g^1} - (8tg^2 - 2xg^1 - 2)\partial_{g^2} - (10t + x^2)g^3\partial_{g^3},\end{aligned} \tag{16}$$

where $f = f(t,x)$ is an arbitrary solution of (12).

Theorem 3 [7]. *The maximal Lie invariance algebra of (15) is generated by the operators*

$$\begin{aligned}&\partial_t, \quad \partial_x, \quad G^2 = t\partial_x + -\tfrac{1}{2}xu\partial_u - \tfrac{1}{2}(x\theta + u)\partial_\theta, \quad I^2 = u\partial_u + \theta\partial_\theta,\\ &D^2 = 2t\partial_t + x\partial_x + u\partial_u, \quad f\partial_u + f_x\partial_\theta,\\ &\Pi^2 = 4t^2\partial_t + 4tx\partial_x - (x^2+2t)u\partial_u - (x\theta + 6t\theta - 2xu)\partial_\theta,\end{aligned} \tag{17}$$

where $f = f(t,x)$ is an arbitrary solution of (12).

It is easy to see that algebras (16) (17) are similar to algebra (13). Therefore, there can exist trasformations which, in some sense, reduce (14) and (15) to (12).

Theorem 4 [7]. *System (14) is reduced to the system of three uncoupled heat equations* $z^a_t = z^a_{xx}$, $a = \overline{1,3}$, *functions* $z^a = z^a(t,x)$ *by means of the nonlocal transformation*

$$g^1 = -\frac{z^1_{xx}z^2 - z^1 z^2_{xx}}{z^1_x z^2 - z^1 z^2_x}, \quad g^2 = -\frac{z^1_{xx}z^2_x - z^1_x z^2_{xx}}{z^1_x z^2 - z^1 z^2_x}, \quad g^3 = z^3_{xx} + g^1 z^3_x - g^2 z^3, \tag{18}$$

where $z^1_x z^2 - z^1 z^2_x \neq 0$.

Theorem 5 [7]. *Equation (15) is reduced by means of the nonlocal change*

$$\theta = -\Phi_t/\Phi_u, \quad \Phi = \Phi(t,x,u) \tag{19}$$

and the hodograph transformation

$$y_0 = t, \quad y_1 = x, \quad y_2 = \Phi, \quad \Psi = u, \tag{20}$$

to the heat equation for the function $\Psi = \Psi(y_0, y_1, y_2) : \Psi_{y_0} - \Psi_{y_1 y_1} = 0$, *where the variable* y_2 *can be asssumed a parameter.*

Theorem 4 and 5 show that, in some sense, the problem of finding Q-conditional symmetry of a PDE is reduced to solving this equation, although the defining equations for the coefficient of a Q-conditional symmetry operator are more complicated.

To demonstrate the efficiency of Q-conditional symmetry, consider a generalization of the heat equation, which is a linear transfer equation:

$$u_t + \frac{h(t)}{x} + u_{xx} = 0. \tag{21}$$

Theorem 6. [8, 9]. *The maximal Lie invariance algebra of (21) is the algebra*

1) $A^1 = \langle u\partial_u,\ f(t,x)\partial_u\rangle$ *if* $h \neq \mathrm{const}$;

2) $A^2 = A^1 + \langle \partial_t,\ D,\ \Pi\rangle$ *if* $h = \mathrm{const}$, $h \notin \{0,-2\}$;

3) $A^3 = A^2 + \langle \partial_x + \frac{1}{2}hx^{-1}u\partial_u,\ G\rangle$ *if* $h \in \{0,-2\}$.

Here, $f = f(t,x)$ *is an arbitrary solution of (21),* $D = 2t\partial_t + x\partial_x$, $\Pi = 4t^2\partial_t + 4tx\partial_x - (x^2 + 2(1-h)t)u\partial_u$, $G = t\partial_x - \frac{1}{2}(x - htx^{-1})u\partial_u$.

Theorems being like to Theorems 1–5 were proved for equation (21) too.

It follows from Theorem 6 that the Lie symmetry of equation (21) is trivial in the case $h \neq \mathrm{const}$. But for an arbitrary function h, equation (21) is Q-conditionally invariant, for example, under the folowing operators:

$$X = \partial_t + (h(t) - 1)x^{-1}\partial_x, \quad \tilde{G} = (2t + A)\partial_x - xu\partial_u, \quad A = \mathrm{const}. \tag{22}$$

By means of operators (22), we construct solutions of (21):

$$u = C_2(x^2 - 2\int(h(t) - 1)dt + C_1, \quad u = C_1 \exp\left\{-\frac{x^2}{2(2t+A)} + \int \frac{h(t) - 1}{2t + A}dt\right\}$$

that can be generalized in such a way:

$$u = \sum_{k=0}^{n} T^k(t)x^{2k}, \quad u = \sum_{k=0}^{n} S^k(t)\left(\frac{x}{2t+A}\right)^{2k} \exp\left\{-\frac{x^2}{2(2t+A)} + \int \frac{h(t) - 1}{2t + A}dt\right\}.$$

The functions T^K and S^k satisfy systems of ODEs which are easily integrated.

Using Q-conditional symmetry, the nonlocal equivalence transformation in the class of equations of the form (21) are constructed too.

In conclusion, we should like to note that the connection between Q-conditional symmetry and reduction of PDEs is analogous to one between Lie symmetry and integrating ODEs of the first order. The problem of finding Q-conditional symmetry is more complicated than the problem of solving the initial equation. But constructing a Q-conditional symmetry operator by means of any additional suppositions, we obtain reduction of the initial equation. And it is possible that new classes of symmetries, which are as many as Q-conditional symmetry and will be found as simply as the Lie symmetry will be investigated in the future.

The author is grateful to Prof. R.Z. Zhdanov and Prof. I.M. Tsyfra for valuable discussions.

This work is supported by the DFFD of Ukraine (project 1.4/356).

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 444–449.

Lie and Non-Lie Symmetries of Nonlinear Diffusion Equations with Convection Term

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Abstract

Lie and conditional symmetries of nonlinear diffusion equations with convection term are described. Examples of new ansätze and exact solutions are presented.

1. Introduction

In the present paper, we consider nonlinear diffusion equations with convection term of the form

$$U_t = [A(U)U_x]_x + B(U)U_x + C(U), \tag{1}$$

where $U = U(t, x)$ is the unknown function and $A(U)$, $B(U)$, $C(U)$ are arbitrary smooth functions. The indices t and x denote differentiating with respect to these variables. Equation (1) generalizes a great number of the well-known nonlinear second-order evolution equations, describing various processes in physics, chemistry, biology (in paper [1], one can find a wide list of references).

In the case $A = 1$, $B = C = 0$, the classical heat equation

$$U_t = U_{xx} \tag{2}$$

follows from equation (1). S. Lie [2] was the first to calculate the maximal invariance algebra (i.e., the Lie symmetry) of the linear heat equation (2). The algebra found is the generalized Galilei algebra $AG_2(1.1)$ that generates the six-parameter group of time- and space translations and Galilei, scale, and projective transformations (for details, see, e.g., [3, 4]).

In the case $B = C = 0$, the standard nonlinear heat equation

$$U_t = [A(U)U_x]_x \tag{3}$$

follows from equation (1). Lie symmetries of equation (3) were completely described by Ovsyannikov [5]. It has been shown that equation (3) is invariant under a non-trivial algebra of Lie symmetries only in the cases $A = \lambda_0 \exp(mU)$ and $A = \lambda_0(U + \alpha_0)^k$, where $\lambda_0, m, k, \alpha_0$ are arbitrary constants.

Finally, in the case $B(U) = 0$, we obtain the known nonlinear heat equation with a source

$$U_t = [A(U)U_x]_x + C(U). \tag{4}$$

The Lie symmetry of equation (4) was completely described in [6].

An approach to finding a non-classical symmetry (conditional symmetry) of the linear heat equation (2) was suggested in [7] (see also [8]). The conditional symmetry of nonlinear heat equation (4) was studied in [9, 10].

In the present paper (section 2), the Lie symmetries of equation (1) are *completely* described. In particular, such operators of invariance are found that are absent for equations (3) and (4).

In section 3, new operators of conditional symmetry of equations (1) and (4) are constructed. With the help of these operators, new ansätze and exact solutions for some equations of the form (1) are found.

2. Lie symmetries of the nonlinear equation (1)

It is easily shown that equation (1) is reduced by the local substitution (see, e.g., [11])

$$U \to U^* = \int A(U)dU \equiv A_0(U) \tag{5}$$

to the form

$$U^*_{xx} = F_0(U^*)U^*_t + F_1(U^*)U^*_x + F_2(U^*), \tag{6}$$

where

$$F_0 = \frac{1}{A(U)}\Big|_{U=A_0^{-1}(U^*)}, \quad F_1 = -\frac{B(U)}{A(U)}\Big|_{U=A_0^{-1}(U^*)}, \quad F_2 = -C(U)\Big|_{U=A_0^{-1}(U^*)} \tag{7}$$

and A_0^{-1} is the inverse function to $A_0(U)$. Hereinafter, the sign $*$ is omitted, i.e., the equation

$$U_{xx} = F_0(U)U_t + F_1(U)U_x + F_2(U) \tag{8}$$

is considered.

Now let us formulate theorems which give complete information on local symmetry properties of equation (8). Note that we do not consider the cases in which $F_1(U) = 0$ since these cases have been studied in [6].

It is clear that equation (8) is invariant with respect to the trivial algebra

$$P_t = \frac{\partial}{\partial t}, \quad P_x = \frac{\partial}{\partial x} \tag{9}$$

for arbitrary functions $F_0(U), F_1(U), F_2(U)$. Hereinafter, operators (9) are not listed since they are common for all cases.

Theorem 1. *The maximal algebra of invariance of equation (8) in the case $F_0 = 1$ is the Lie algebra with basic operators (9) and*

a) $D_1 = 2mtP_t + mxP_x - UP_U, \quad$ *if* $\quad F_1 = \lambda_1 U^m, \quad F_2 = \lambda_2 U^{2m+1};$

$$b)\quad \mathcal{G}_1 = \exp\left(-\lambda_2 t\right)\left(P_x - \frac{\lambda_2}{\lambda_1}P_U\right), \quad if \quad F_1 = \lambda_1 U, \quad F_2 = \lambda_2 U, \quad \lambda_2 \neq 0;$$

$$c)\quad G_1 = tP_x + \frac{1}{\lambda_1}UP_U, \quad D_2 = 2tP_t + xP_x - UP_U,$$
$$\Pi_1 = t^2 P_t + txP_x + \left(\frac{x}{\lambda_1} - tU\right)P_U \quad if \quad F_1 = \lambda_1 U, \quad F_2 = 0;$$

$$d)\quad G_1 = tP_x + \frac{1}{\lambda_1}UP_U, \quad if \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U;$$

$$e)\quad \mathcal{G}_2 = \exp(-\lambda_3 t)\left(P_x - \frac{\lambda_3}{\lambda_1}UP_U\right),$$
$$if \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U + \lambda_3 U \ln U, \quad \lambda_3 \neq 0;$$

$$f)\quad Y = \exp\left[\left(\frac{\lambda_1^2}{4} - \lambda_3\right)t + \frac{\lambda_1}{2}x\right]UP_U,$$
$$if \quad F_1 = \lambda_1 \ln U, \quad F_2 = \lambda_2 U + \lambda_3 U \ln U - \frac{\lambda_1^2}{4}U \ln^2 U,$$

where $\lambda_1 \neq 0, \lambda_2, \lambda_3$ *and* $m \neq 0$ *are arbitrary constants,* $P_U = \frac{\partial}{\partial U}$.

Theorem 2. *The maximal algebra of invariance of equation (8) in the case* $F_0 = \exp(mU)$ *is the Lie algebra with basic operators (9) and*

$$D = (2n - m)tP_t + nxP_x - P_U$$

if $F_1 = \lambda_1 \exp(nU)$, $F_2 = \lambda_2 \exp(2nU)$, *where* $\lambda_1 \neq 0, \lambda_2, m$ *and* $n \neq 0$ *are arbitrary constants.*

Theorem 3. *The maximal algebra of invariance of equation (8) in the case* $F_0 = U^k$, $k \neq 0$ *is the Lie algebra with basic operators (9) and*

$$a)\quad D_1 = (2m - k)tP_t + mxP_x - UP_U,$$
$$if \quad F_1 = \lambda_1 U^m, \quad F_2 = \lambda_2 U^{2m+1}, \quad m \neq 0, m \neq k, m \neq k/2;$$

$$b)\quad T = \exp(-\lambda_3 kt)\left(P_t - \lambda_3 UP_U\right), \quad if \quad F_1 = \alpha_1, \quad F_2 = \lambda_2 U + \lambda_3 U^{k+1};$$

$$c)\quad X = \exp(-\alpha_1 kx)(P_x + 2\alpha_1 UP_U),$$
$$if \quad F_1 = \lambda_1 U^{k/2} + (k+4)\alpha_1, \quad F_2 = \lambda_2 U^{k+1} - 2\alpha_1\lambda_1 U^{k/2+1} + \lambda_4 U, \quad k \neq -4;$$

$$d)\quad D_2 = ktP_t + UP_U, \quad X, \quad if \quad F_1 = \alpha_1(k+4), \quad F_2 = \lambda_4 U, \quad k \neq -4;$$

$$e)\quad T,\ X, \quad if \quad F_1 = \alpha_1(k+4), \quad F_2 = \lambda_4 U + \lambda_3 U^{k+1}, \quad k \neq -4,$$

where $\alpha_1 \neq 0, \lambda_1 \neq 0, \lambda_2,$ *and* $\lambda_3 \neq 0$ *are arbitrary constants* $\lambda_4 = -2\alpha_1^2(k+2)$.

The proofs of Theorems 1, 2 and 3 are based on the classical Lie scheme (see, e.g., [12, 13]) and here they are omitted. Note that these proofs are non-trivial because equation (8) contains three arbitrary functions $F_0(U), F_1(U), F_2(U)$ (for details see our recently published paper [14]).

Using Theorems 1–3 and [6], one can show that some nonliner convection equations of the form (8) contain the operators G_1, Y and X that are not invariance operators for any nonlinear equation of the form (4).

In particular, it follows from Theorem 1 that the nonlinear convection equation

$$U_t = U_{xx} - \lambda_1(\ln U)U_x - \lambda_2 U$$

is invariant under the Galilei algebra with basic operators P_t, P_x and G_1, in which the unit operator is absent. Note that the Burgers equation is also invariant under the Galilei algebra (see Theorem 1, case (c)), which does not contain a unit operator. All second-order equations, which are invariant with respect to the similar representation of the Galilei algebra, were described in [3].

On the other hand, *all nonlinear* equations of the form (4) do not have the Galilei symmetry [3]. The Galilei algebra of the linear heat equation contains the unit operator $I = U\partial_U$ and is essentially different from that of the Burgers equation. Nonlinear equations and systems of equations, preserving the Galilei algebra of the linear heat equation, have been described in [4], [15], [16].

3. Conditional symmetries of the nonlinear equation (1)

In this section, we study the Q-conditional symmetry (see the definition of conditional symmetry in [13]) of the nonlinear equation (8) if $F_1(U) \neq 0$.

Theorem 4. *Equation (8) is Q-conditional invariant under the operator*

$$Q = \xi^0(t,x,U)P_t + \xi^1(t,x,U)P_x + \eta(t,x,U)P_U$$

if the functions ξ^0, ξ^1, η satisfy the following equations:

case 1.

$$\begin{cases} \xi^0 = 1, \quad \xi^1_{UU} = 0, \quad \eta_{UU} = 2\xi^1_U(F_1 - \xi^1 F_0) + 2\xi^1_{xU}, \\ \eta(F_1 - \xi^1 F_0)_U - (\xi^1_t + 2\xi^1\xi^1_x - 3\xi^1_U\eta)F_0 + \xi^1_x F_1 + 3\xi^1_U F_2 - 2\eta_{xU} + \xi^1_{xx} = 0, \\ \eta(\eta F_0 + F_2)_U + (2\xi^1_x - \eta_U)(\eta F_0 + F_2) + \eta_t F_0 + \eta_x F_1 - \eta_{xx} = 0; \end{cases} \tag{10}$$

case 2.

$$\begin{cases} \xi^0 = 0, \quad \xi^1 = 1, \quad \eta(\eta_x + \eta\eta_U - \eta F_1 - F_2)\dot{F}_0 = \\ = (\eta_{xx} + 2\eta\eta_{xU} + \eta^2\eta_{UU} - \eta^2\dot{F}_1 - \eta_x F_1 - \eta\dot{F}_2 + \eta_U F_2)F_0 + \eta_t F_0^2. \end{cases} \tag{11}$$

The dot above F_0, F_1, F_2 denotes differentiating with respect to the variable U.

One can prove this theorem using [13], §5.7 .

The systems of the nonlinear equations (10) and (11) are very complicated and we did not construct their general solutions. A partial solution of equations (10) has the form

$$\xi^1 = U + \lambda_4, \quad \eta = \mathcal{P}_3(U),$$

$$F_1 = (U + \lambda_4)F_0 + 3\lambda_3 U + \lambda_2, \quad F_2 = -\mathcal{P}_3(U)(F_0 + \lambda_3),$$

where $\mathcal{P}_3(U) = \lambda_0 + \lambda_1 U + \lambda_2 U^2 + \lambda_3 U^3$, $\lambda_\mu \in \mathbf{R}$, $\mu = 0, \ldots, 4$, $F_0(U)$ is an arbitrary smooth function.

A partial solution of equation (11) is the following:

$$\eta = \frac{1}{t}H(U), \quad F_0 = \lambda_0 \dot{H}, \quad F_1 = \lambda_1 \dot{H} + \lambda_0 \dot{H} \int \frac{dU}{H(U)}, \quad F_2 = \lambda_2 H \dot{H},$$

where $H = H(U)$ is an arbitrary smooth function. So, we obtain the following result: the equation

$$U_{xx} = F_0(U)[U_t + (U + \lambda_4)U_x - \mathcal{P}_3(U)] + (3\lambda_3 U + \lambda_2)U_x - \lambda_3 \mathcal{P}_3(U) \tag{12}$$

is a Q-conditional invariant under the operator

$$Q = P_t + (U + \lambda_4)P_x + \mathcal{P}_3(U)P_U \tag{13}$$

and the equation

$$U_{xx} = \dot{H}(U)\left[\lambda_0 U_t + \left(\lambda_1 + \lambda_0 \int \frac{dU}{H(U)}\right) U_x + \lambda_2 H(U)\right] \tag{14}$$

is a Q-conditional invariant under the Galilei operator

$$G = tP_x + H(U)P_U. \tag{15}$$

Using operators (13) and (15), we find the ansätze

$$\int \frac{U + \lambda_4}{\mathcal{P}_3(U)} dU - x = \varphi(\omega), \quad \omega = \int \frac{dU}{\mathcal{P}_3(U)} - t; \tag{16}$$

$$\int \frac{dU}{H(U)} = \varphi(t) + \frac{x}{t}. \tag{17}$$

After the substitution of ansätze (16) and (17) into (12) and (14), some ODEs are obtained that can be solved. Having solutions of these ODEs and using ansätze (16) and (17), we obtain the exact solution

$$\int \frac{dU}{\mathcal{P}_3(U)} - \int \frac{d\tau}{\mathcal{P}_3(\tau)} = t, \qquad \int \frac{U + \lambda_4}{\mathcal{P}_3(U)} dU - \int \frac{\tau + \lambda_4}{\mathcal{P}_3(\tau)} d\tau = x \tag{18}$$

of equation (12) in the parametrical form, and the solution

$$\int \frac{dU}{H(U)} = \frac{x}{t} + \frac{1}{\lambda_0}\left(\frac{1}{t}\ln t - \lambda_1 - \frac{1}{2}\lambda_2 t\right) \tag{19}$$

of equation (14).

Finally, note that operators of conditional symmerty give a possibility to construct ansätze and solutions of PDEs that can not be obtained by the Lie method. A construction of new ansätze and solutions of other types will be considered in the next paper.

Acknowledgments

The main part of this paper was written during the stay of the authors in Mathematisches Forschungsinstitut Oberwolfach and was supported by the Volkswagen-Stiftung (RiP-program). R.Ch. acknowledges also the financial support by the DFFD of Ukraine (project N 1.4/356).

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 450–454.

Conditional Symmetry and Exact Solutions of the Kramers Equation

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Abstract

We research some Q- and Q_1, Q_2-conditional symmetry properties of the Kramers equation. Using that symmetry, we have constructed the well-known Boltzmann solution.

1 Q-conditional symmetry

Let us consider the Kramers equation which describes the motion of a particle in a fluctuating medium [1]

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right), \tag{1}$$

where $u = u(t, x, y)$ is the probability density, γ is a constant and $V(x)$ is an external potential.

The group properties of equation (1) for various potentials were investigated in detail by means of the Lie method [2].

Theorem 1. *The maximal invariance local group of the Kramers equation (1) is 1) a six-dimensional Lie group, when $V'(x) = kx + c$ (k, c are constants), $k \neq -\frac{3}{4}\gamma^2, \frac{3}{16}\gamma^2$; 2) an eight-dimensional Lie group, when $V'(x) = kx + c$, $k = -\frac{3}{4}\gamma^2, \frac{3}{16}\gamma^2$; 3) a two-dimensional Lie group generated by the operators $P_0 = \partial_t$ and I, when $V'(x) \neq kx + c$.*

Investigation of the conditional invariance allows us to obtain new classes of the potential $V(x)$ for which we can find exact solutions of equation (1) [3, 4, 5]. Let us consider an infinitesimal operator of the form

$$Q = \xi^0(t, x, y, u)\frac{\partial}{\partial t} + \xi^1(t, x, y, u)\frac{\partial}{\partial x} + \xi^2(t, x, y, u)\frac{\partial}{\partial y} + \eta(t, x, y, u)\frac{\partial}{\partial u}. \tag{2}$$

We say that equation (1) is Q–conditionally invariant if the system of equations (1) and

$$Qu(t, x, y) = 0 \tag{3}$$

is invariant under the action of operator (2) [5].

Remark. If Q is a Q-conditional operator for a PDE, then the equation is $\bar{Q}$-conditionally invariant under $\bar{Q} = f(t,x,y,u)Q$, where $f(t,x,y,u)$ is an arbitrary function of independent and dependent variables. We say that Q and $\bar{Q}$ are equivalent operators. Then we may consider $\xi^0 = 1$ in (2) if $\xi^0 \not\equiv 0$.

Let us consider equation (2) with the potential

$$V' = kx^{-1/3} + \frac{3}{16}\gamma^2 x, \quad k \neq 0 \tag{4}$$

The operator

$$Q = \partial_t - \frac{3}{4}\gamma x\partial_x + \left(\frac{3}{8}\gamma^2 - \frac{1}{4}\gamma y\right)\partial_y + \gamma u\partial_u \tag{5}$$

is not a Lie symmetry operator of equations (1), (3) as follows from Theorem 1. However the operator (5) gives the invariant solution (ansatz)

$$u(t,x,y) = \exp(\gamma t)\varphi(\omega_1,\omega_2), \quad \omega_1 = \exp\left(\frac{3}{4}\gamma t\right)x, \quad \omega_2 = x^{-1/3}y + \frac{3}{4}\gamma x^{2/3}, \tag{6}$$

which reduces the equation. Indeed, substituting (6) into (1), (3), we obtain the reduced equation

$$\omega_2\varphi_1 - \left(\frac{\omega_2^2}{2} + k\right)\omega_1^{-1}\varphi_2 - \gamma\omega_1^{-1}\varphi_{22} = 0, \quad \varphi_i = \frac{\partial\varphi}{\partial\omega_i}, \quad \varphi_{22} = \frac{\partial^2\varphi}{\partial\omega_2\partial\omega_2}.$$

This equation may be integrated, in particular, when $\varphi = \varphi(\omega_2)$.

Theorem 2. *All Q-conditional operators of equation (1) with $V' \neq kx^{-1/3} + \frac{3}{16}\gamma^2 x$ are eqivalent to ∂_t, I.*

Theorem 3. *Equations (1) and (3) have the following Q-conditional symmetry operators*

$$Q = \partial_t + F(t)x\partial_x + \left(\frac{1}{3}F(t)y + F'(t)x + \frac{2}{3}F^2 x\right)\partial_y + fu\partial_u,$$

where $F(t)$ is an arbitrary solution of the equation

$$\begin{aligned}
&F'' + 2FF' + \frac{4}{9}F^3 - \frac{1}{4}\gamma^2 F = 0,\\
&2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - yx\left(\gamma F' + \frac{2}{3}\gamma F^2 + \frac{1}{2}\gamma^2 F\right) -\\
&\qquad x^2\left(\frac{3}{8}\gamma^2 F' + \frac{1}{4}\gamma^2 F^2 + \frac{3}{16}\gamma^3 F\right) - x^{2/3}k\left(2F' + \frac{4}{3}F^2 + \gamma F\right) +\\
&\qquad C\exp\left\{-\tfrac{2}{3}\int F dt\right\} - \tfrac{4}{3}\gamma F + \gamma^2,
\end{aligned}$$

where $C =$ const.

Theorem 4. *All Q-conditional symmetry operators ($\xi^0 = 1$) of equation (1) with $V' = kx$, $k \neq -\frac{1}{4}\gamma^2, \frac{3}{16}\gamma^2$ are equivalent to the Lie symmetry operators.*

Theorem 5. *All $Q-$conditional symmetry operators ($\xi^0 = 1$) of equation (1) with $V' = kx$, $k = -\frac{1}{4}\gamma^2, \frac{3}{16}\gamma^2$ (including those equivalent to Lie symmetry operators) have the following form*

$$Q = \partial_t + (F(t)x + G(t))\partial_x +$$
$$\left[\frac{1}{3}F(t)y + \left(F'(t) + \frac{2}{3}F(t)^2\right)x + G'(t) + \frac{2}{3}F(t)G(t)\right]\partial_y + f(t,x,y)u\partial_u.$$

Here, the functions $F(t)$, $G(t)$ satisfy the following equations

$$F'' + 2FF' + \frac{4}{9}F^3 + \left(\frac{4}{5}k - \frac{2}{5}\gamma^2\right)F = 0,$$
$$G'' + \frac{4}{3}FG' + \frac{2}{3}F'G + \gamma G' + \frac{2}{3}\gamma FG + \frac{4}{9}F^2G + kG = h(t),$$

where $h(t)$ satisfies the equation

$$h'' + \left(\frac{4}{3}F - \gamma\right)h' + \left(k + \frac{2}{3}F' + \frac{4}{9}F^2 - \frac{2}{3}\gamma F\right)h = 0.$$

The function $f(t,x,y)$ is

$$a)\ k = -\frac{3}{4}\gamma^2,\quad 2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - y\left[x\left(\gamma F' + \frac{2}{3}\gamma F^2\right) + h\right] +$$
$$\frac{3}{4}\gamma^3x^2F + x\left(h' - \gamma h + \frac{2}{3}Fh\right) + s(t),$$
$$b)\ k = \frac{3}{16}\gamma^2,\quad 2\gamma f = -y^2\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) - y\left[x\left(\gamma F' + \frac{2}{3}\gamma F^2 + \frac{1}{2}\right) + h\right] -$$
$$x^2\left(\frac{3}{16}\gamma^3F + \frac{3}{8}\gamma^2F' + \frac{1}{4}\gamma^2F^2\right) + x\left(h' - \gamma h + \frac{2}{3}Fh\right) + s(t),$$

where in both a) and b) the function $s(t)$ satisfies the equation

$$s' + \frac{2}{3}Fs = -2\gamma\left(\frac{2}{3}F' + \frac{4}{9}F^2 + \frac{1}{3}\gamma F\right) + \frac{4}{3}\gamma^2F.$$

2 Q_1,Q_2-conditional symmetry

Let we consider two operators Q_1 and Q_2 which have the form (2).

Definition [6]. *We say that equation (1) is Q_1,Q_2-conditionally invariant if the system*

$$\begin{aligned} &Q_1u = 0,\\ &Q_2u = 0,\\ &\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}\left(yu + \frac{\partial u}{\partial y}\right) \end{aligned} \tag{7}$$

is invariant under the operators Q_1 and Q_2.

Here, we restrict the form of the operators:

$$
\begin{aligned}
Q_1 &= \frac{\partial}{\partial x} - \eta^1(t,x,y,u)\frac{\partial}{\partial u},\\
Q_2 &= \frac{\partial}{\partial y} - \eta^2(t,x,y,u)\frac{\partial}{\partial u}.
\end{aligned} \tag{8}
$$

System (7) for operators (8) can be written as

$$
\begin{aligned}
&\frac{\partial u}{\partial x} = \eta^1(t,x,y,u),\\
&\frac{\partial u}{\partial y} = \eta^2(t,x,y,u),\\
&\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(yu) + \frac{\partial}{\partial y}(V'(x)u) + \gamma\frac{\partial}{\partial y}(yu + \frac{\partial u}{\partial y}).
\end{aligned} \tag{9}
$$

Following the Lie's algorithm [7], we find that

$$
\begin{aligned}
&\frac{\partial \eta^1}{\partial t} + \gamma u\frac{\partial \eta^1}{\partial u} - V''(x)\eta^2 - (\gamma y + V'(x))\frac{\partial \eta^1}{\partial y} + y\frac{\partial \eta^1}{\partial x} - \gamma\eta^1 -\\
&\qquad \gamma\frac{\partial^2 \eta^1}{\partial y \partial y} - 2\gamma\eta^2\frac{\partial^2 \eta^1}{\partial y \partial u} = 0,\\
&\frac{\partial \eta^2}{\partial t} + \gamma u\frac{\partial \eta^2}{\partial u} - \gamma\eta^2 - (\gamma y + V'(x))\frac{\partial \eta^2}{\partial y} + y\frac{\partial \eta^2}{\partial x} - \gamma\eta^2 -\\
&\qquad \gamma\frac{\partial^2 \eta^2}{\partial y \partial y} - 2\gamma\eta^2\frac{\partial^2 \eta^2}{\partial y \partial u} = 0.
\end{aligned} \tag{10}
$$

It is easy to show that $\eta^1 = -V'(x)u$, $\eta^2 = -yu$ is a solution of system (10). According to the algorithm [5] using the operators

$$
Q_1 = \frac{\partial}{\partial x} - V'(x)u\frac{\partial}{\partial u}, \qquad Q_2 = \frac{\partial}{\partial y} - yu\frac{\partial}{\partial u}, \tag{11}
$$

we find the ansatz invariant under operators (8)

$$
u = \varphi(t)\exp\left\{-V(x) - y^2/2\right\}. \tag{12}
$$

Substitution of (12) into (1) gives $\varphi'(t) = 0$. So, we find the solution which is the Boltzmann distribution

$$
u(x,y) = N\exp\left\{-V(x) - y^2/2\right\}
$$

(N is a normalization constant). It is a stationary solution.

From the above example we see that the further work on finding Q- and Q_1, Q_2-conditional symmetry operators is of great interest.

One of the author (SS) is grateful to the DFFD of Ukraine (project 1.4/356) for the financial support.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 455–462.

Symmetry Reduction of a Generalized Complex Euler Equation for a Vector Field

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Abstract

The procedure of constructing linear ansatzes is algorithmized. Some exact solutions of a generalized complex Euler equation for a vector field, invariant under subalgebras of the Poincaré algebra $AP(1,3)$ are found.

In this article, we consider the equation

$$\frac{\partial \Sigma_k}{\partial x_0} + \Sigma_l \frac{\partial \Sigma_k}{\partial x_l} = 0, \quad \Sigma_k = E_k + iH_k \quad (k,l = 1,2,3). \tag{1}$$

It was proposed by W. Fushchych [1] to describe vector fields. This equation can be considered as a complex generalization of the Euler equation for ideal liquid [2]. Equation (1) is equivalent to the system of real equations for $\vec{E} = (E_1, E_2, E_3)$ and $\vec{H} = (H_1, H_2, H_3)$:

$$\left\{ \begin{array}{l} \dfrac{\partial E_k}{\partial x_0} + E_l \dfrac{\partial E_k}{\partial x_l} - H_l \dfrac{\partial H_k}{\partial x_l} = 0, \\ \dfrac{\partial H_k}{\partial x_0} + H_l \dfrac{\partial E_k}{\partial x_l} + E_l \dfrac{\partial H_k}{\partial x_l} = 0. \end{array} \right. \tag{2}$$

It was established in paper [1] that the maximal invariance algebra of system (2) is a 24-dimensional Lie algebra containing the affine algebra $AIGL(4,\mathbb{R})$ with the basis elements

$$\begin{array}{l} P_\alpha = \dfrac{\partial}{\partial x_\alpha} \quad (\alpha = 0,1,2,3), \quad \Gamma_{a0} = -x_0 \dfrac{\partial}{\partial x_a} - \dfrac{\partial}{\partial E_a}, \\ \Gamma_{00} = -x_0 \dfrac{\partial}{\partial x_0} + E_l \dfrac{\partial}{\partial E_l} + H_l \dfrac{\partial}{\partial H_l} \quad (l = 1,2,3), \\ \Gamma_{aa} = -x_a \dfrac{\partial}{\partial x_a} - E_a \dfrac{\partial}{\partial E_a} - H_a \dfrac{\partial}{\partial H_a} \ (\text{no sum over } a), \\ \Gamma_{0a} = -x_a \dfrac{\partial}{\partial x_0} + (E_a E_k - H_a H_k) \dfrac{\partial}{\partial E_k} + (E_a H_k + H_a E_k) \dfrac{\partial}{\partial H_k}, \\ \Gamma_{ac} = -x_c \dfrac{\partial}{\partial x_a} - E_c \dfrac{\partial}{\partial E_a} - H_c \dfrac{\partial}{\partial H_a} \quad (a \neq c;\ a,c = 1,2,3). \end{array} \tag{3}$$

The algebra $AIGL(4,\mathbb{R})$ contains as a subalgebra the Poincaré algebra $AP(1,3)$ with the basis elements

$$J_{0a} = -\Gamma_{0a} - \Gamma_{a0}, \quad J_{ab} = \Gamma_{ba} - \Gamma_{ab}, \quad P_\alpha \quad (a,b = 1,2,3;\ \alpha = 0,1,2,3).$$

The purpose of our investigation is to construct invariant solutions to system (2) by reducing this system to systems of ordinary differential equations on subalgebras of the algebra $AP(1,3)$.

On the basis of Proposition 1 [3] and the necessary existence condition for nondegenerate invariant solutions [4], we obtain that to perform the reduction under consideration, we need the list of three-dimensional subalgebras of the Poincaré algebra $AP(1,3)$ with only one main invariant of the variables x_0, x_1, x_2, x_3. We can consider subalgebras of the algebra $AP(1,3)$ up to affine conjugacy.

Let us denote $G_a = J_{0a} - J_{a3}$ $(a = 1,2)$.

Proposition 1. *Up to affine conjugacy, three-dimensional subalgebras of the algebra $AP(1,3)$, having only one main invariant depending on the variables x_0, x_1, x_2, x_3, are exhausted by the following subalgebras:*

$$\langle P_1,P_2,P_3\rangle,\ \langle J_{12}+\alpha J_{03},P_0,P_3\rangle,\ \langle J_{12}+\alpha J_{03},P_1,P_2\rangle\ (\alpha\neq 0),\ \langle J_{03},P_1,P_2\rangle,$$
$$\langle G_1,P_0+P_3,P_2+\alpha P_1\rangle,\ \langle G_1,G_2,P_0+P_3\rangle,\ \langle G_1,J_{03},P_2\rangle,\ \langle J_{12},J_{03},P_0+P_3\rangle,$$
$$\langle J_{03}+P_1,P_0,P_3\rangle,\ \langle G_1,G_2,J_{12}+\alpha J_{03}\rangle\ (\alpha>0),\ \langle J_{12}+P_0,P_1,P_2\rangle,\ \langle G_1,G_2,J_{03}\rangle,$$
$$\langle J_{03}+\gamma P_1,P_0+P_3,P_2\rangle\ (\gamma=0,1),\ \langle G_1+P_2,P_0+P_3,P_1\rangle,$$
$$\langle G_1+P_0-P_3,P_0+P_3,P_2\rangle,\ \langle G_1+P_0-P_3,P_0+P_3,P_1+\alpha P_2\rangle,$$
$$\langle G_1,G_2+P_2,P_0+P_3\rangle,\ \langle G_1,J_{03}+\alpha P_1+\beta P_2,P_0+P_3\rangle,$$
$$\langle G_1+P_2,G_2-P_1+\beta P_2,P_0+P_3\rangle,\ \langle G_1,G_2,J_{12}+P_0+P_3\rangle.$$

To obtain this list of subalgebras, we should apply the affine conjugacy to the list of subalgebras of the algebra $AP(1,3)$, considered up to $P(1,3)$-conjugacy [5]. In so doing, in particular, we may identify all one-dimensional subspaces of the translation space $\langle P_0,P_1,P_2,P_3\rangle$.

The linear span Q of a system of operators, obtained from basis (3) by excluding the operators Γ_{0a} $(a = 1,2,3)$, forms a Lie subalgebra of the algebra $AIGL(4,\mathbb{R})$. Each operator $Y\in Q$ can be presented as

$$Y = a_\alpha(x)\frac{\partial}{\partial x_\alpha} + b_{ij}\left(E_j\frac{\partial}{\partial E_i} + H_j\frac{\partial}{\partial H_i}\right) + c_i\frac{\partial}{\partial E_i}, \tag{4}$$

where $x=(x_0,x_1,x_2,x_3)$; b_{ij}, c_i are real numbers; $\alpha=0,1,2,3$; $i,j=1,2,3$.

Definition. An invariant of a subalgebra L of the algebra Q, that is a linear function in the variables E_a, H_a $(a=1,2,3)$, is called *linear.* A vector function $\vec{F}(\vec{E},\vec{H})$ is called a *linear invariant* of a subalgebra L if its components are linear invariants of this subalgebra.

Let

$$B=\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix},\ \vec{C}=\begin{pmatrix} c_1\\ c_2\\ c_3\end{pmatrix},\ U=\begin{pmatrix} u_{11}(x) & u_{12}(x) & u_{13}(x)\\ u_{21}(x) & u_{22}(x) & u_{23}(x)\\ u_{31}(x) & u_{32}(x) & u_{33}(x)\end{pmatrix},\ \vec{V}=\begin{pmatrix} v_1(x)\\ v_2(x)\\ v_3(x)\end{pmatrix}.$$

Theorem. *The vector function $U\vec{E}+\vec{V}$ is a linear invariant of the operator Y if and only if*

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0,\quad a_\alpha(x)\frac{\partial \vec{V}}{\partial x_\alpha} + U\vec{C} = \vec{0}.$$

The vector function $U\vec{H}$ *is a linear invariant of the operator* Y *if and only if*

$$a_\alpha(x)\frac{\partial U}{\partial x_\alpha} + UB = 0.$$

Proposition 2. *Let*

$$Y_j = a_\alpha^{(j)}(x)\frac{\partial}{\partial x_\alpha} + \sum_{i,k=1}^{3} b_{ik}^{(j)}\left(E_k\frac{\partial}{\partial E_i} + H_k\frac{\partial}{\partial H_i}\right) + \sum_{i=1}^{3} c_i^{(j)}\frac{\partial}{\partial E_i} \quad (j = 1, 2, 3) \tag{5}$$

be operators of the form (4) and their corresponding matrices B_1, B_2, B_3 *be linearly independent and satisfy the commutation relations*

$$[B_3, B_j] = B_j \quad (j = 1, 2), \quad [B_1, B_2] = 0.$$

The vector function $U\vec{H}$ *with the matrix* $U = \prod\limits_{i=1}^{3} \exp[f_i(x)B_i]$ *is a linear invariant of the algebra* $\langle Y_1, Y_2, Y_3\rangle$ *if and only if*

$$a_\alpha^{(i)}(x)\frac{\partial f_j}{\partial x_\alpha} + g_{ij}(x) = 0, \tag{6}$$

where $i, j = 1, 2, 3$ *and* $(g_{ij}) = diag\,[\mathrm{e}^{f_3}, \mathrm{e}^{f_3}, 1]$.

Proof. On the basis of the Campbell-Hausdorff formula, we have

$$\exp(\theta B_3)\cdot B_j\cdot\exp(-\theta B_3) = B_j + \frac{\theta}{1!}B_j + \frac{\theta^2}{2!}B_j + \cdots = \mathrm{e}^{\theta}\, B_j \quad (j = 1, 2).$$

Therefore,

$$B_j \exp(\theta B_3) = \mathrm{e}^{-\theta}\exp(\theta B_3)B_j \quad (j = 1, 2).$$

For this reason

$$a_\alpha^{(j)}(x)\frac{\partial U}{\partial x_\alpha} = a_\alpha^{(j)}(x)\frac{\partial f_1}{\partial x_\alpha}\exp(f_1B_1)B_1\exp(f_2B_2)\exp(f_3B_3)+$$
$$+a_\alpha^{(j)}(x)\frac{\partial f_2}{\partial x_\alpha}\exp(f_1B_1)\exp(f_2B_2)B_2\exp(f_3B_3)+$$
$$+a_\alpha^{(j)}(x)\frac{\partial f_3}{\partial x_\alpha}\exp(f_1B_1)\exp(f_2B_2)\exp(f_3B_3)fB_3 =$$
$$= a_\alpha^{(j)}(x)\frac{\partial f_1}{\partial x_\alpha}\mathrm{e}^{-f_3}UB_1 + a_\alpha^{(j)}(x)\frac{\partial f_2}{\partial x_\alpha}\mathrm{e}^{-f_3}UB_2 + a_\alpha^{(j)}(x)\frac{\partial f_3}{\partial x_\alpha}UB_3.$$

It follows from this that $a_\alpha^{(1)}(x)\dfrac{\partial U}{\partial x_\alpha} + UB_1 = 0$ if and only if

$$a_\alpha^{(1)}(x)\frac{\partial f_1}{\partial x_\alpha}\mathrm{e}^{-f_3} + 1 = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(1)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Similarly, $a_\alpha^{(2)}(x)\dfrac{\partial U}{\partial x_\alpha} + UB_2 = 0$ if and only if

$$a_\alpha^{(2)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_2}{\partial x_\alpha}\mathrm{e}^{-f_3} + 1 = 0, \quad a_\alpha^{(2)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Finally, $a^{(3)}_\alpha(x)\dfrac{\partial U}{\partial x_\alpha}+UB_3=0$ if and only if

$$a^{(3)}_\alpha(x)\frac{\partial f_1}{\partial x_\alpha}=0,\quad a^{(3)}_\alpha(x)\frac{\partial f_2}{\partial x_\alpha}=0,\quad a^{(3)}_\alpha(x)\frac{\partial f_3}{\partial x_\alpha}+1=0.$$

The proposition is proved.

Proposition 3. *Let Y_j $(j=1,2,3)$ be operators (5) and their corresponding matrices B_1, B_2, $B_3=B_3'+B_3''$ be linearly independent and satisfy the commutation relations*

$$[B_3',B_j]=\rho B_j\quad(j=1,2),\quad [B_3'',B_1]=-B_2,\quad [B_3'',B_2]=B_1,$$

$$[B_3',B_3'']=0,\quad [B_1,B_2]=0.$$

The vector function $U\vec{H}$ with the matrix $U=\prod\limits_{i=1}^{3}\exp[f_i(x)B_i]$ is a linear invariant of the algebra $\langle Y_1,Y_2,Y_3\rangle$ if and only if functions $f_1(x)$, $f_2(x)$, $f_3(x)$ satisfy system (6), where

$$(g_{ij})=\begin{pmatrix}\mathrm{e}^{\rho f_3}\cos f_3 & -\mathrm{e}^{\rho f_3}\sin f_3 & 0\\ \mathrm{e}^{\rho f_3}\sin f_3 & \mathrm{e}^{\rho f_3}\cos f_3 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Proof. Let us apply the Campbell-Hausdorff formula:

$$\begin{aligned}&\exp(\theta B_3)(\gamma B_1+\delta B_2)\exp(-\theta B_3)=\\ &\quad=\exp(\theta B_3'')\exp(\theta B_3')(\gamma B_1+\delta B_2)\exp(\theta B_3')\exp(\theta B_3'')=\\ &\quad=\mathrm{e}^{\rho\theta}\left\{\gamma B_1+\delta B_2+\frac{\theta}{1!}(-\gamma B_2+\delta B_1)+\frac{\theta^2}{2!}(-\gamma B_1-\delta B_2)+\cdots\right\}=\\ &\quad=\mathrm{e}^{\rho\theta}\left\{(\gamma B_1+\delta B_2)\cos\theta+(-\gamma B_2+\delta B_1)\sin\theta\right\}=\\ &\quad=\mathrm{e}^{\rho\theta}\left\{(\gamma\cos\theta+\delta\sin\theta)B_1+(\delta\cos\theta-\gamma\sin\theta)B_2\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&(\gamma B_1+\delta B_2)\exp(\theta B_3)=\\ &\qquad\qquad=\mathrm{e}^{-\rho\theta}\exp(\theta B_3)\left\{(\gamma\cos\theta-\delta\sin\theta)B_1+(\delta\cos\theta+\gamma\sin\theta)B_2\right\}.\end{aligned}$$

On the basis of the formula obtained, we get

$$\begin{aligned}a^{(j)}_\alpha(x)\frac{\partial U}{\partial x_\alpha}&=a^{(j)}_\alpha(x)\frac{\partial f_1}{\partial x_\alpha}U\,\mathrm{e}^{-\rho f_3}(\cos f_3B_1+\sin f_3B_2)+\\ &+a^{(j)}_\alpha(x)\frac{\partial f_2}{\partial x_\alpha}U\,\mathrm{e}^{-\rho f_3}(-\sin f_3B_1+\cos f_3B_2)+a^{(j)}_\alpha(x)\frac{\partial f_3}{\partial x_\alpha}UB_3.\end{aligned}$$

It follows from this that

$$a^{(1)}_\alpha(x)\frac{\partial U}{\partial x_\alpha}+UB_1=0$$

if and only if

$$\begin{cases} a_\alpha^{(1)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a_\alpha^{(1)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 + \mathrm{e}^{\rho f_3} = 0, \\ a_\alpha^{(1)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a_\alpha^{(1)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 = 0, \\ a_\alpha^{(1)}(x)\dfrac{\partial f_3}{\partial x_\alpha} = 0. \end{cases}$$

If we consider the obtained system as a linear inhomogeneous system in the variables $a_\alpha^{(1)}(x)\dfrac{\partial f_1}{\partial x_\alpha}$, $a_\alpha^{(1)}(x)\dfrac{\partial f_2}{\partial x_\alpha}$, $a_\alpha^{(1)}(x)\dfrac{\partial f_3}{\partial x_\alpha}$, then it is equivalent to the system

$$a_\alpha^{(1)}(x)\frac{\partial f_1}{\partial x_\alpha} = -\mathrm{e}^{\rho f_3}\cos f_3, \quad a_\alpha^{(1)}(x)\frac{\partial f_2}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\sin f_3, \quad a_\alpha^{(1)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

Reasoning similarly, we obtain that

$$a_\alpha^{(2)}(x)\frac{\partial U}{\partial x_\alpha} + UB_2 = 0$$

if and only if

$$\begin{cases} a_\alpha^{(2)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a_\alpha^{(2)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 = 0, \\ a_\alpha^{(2)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a_\alpha^{(2)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 + \mathrm{e}^{\rho f_3} = 0, \\ a_\alpha^{(2)}(x)\dfrac{\partial f_3}{\partial x_\alpha} = 0 \end{cases}$$

or

$$a_\alpha^{(2)}(x)\frac{\partial f_1}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\sin f_3, \quad a_\alpha^{(2)}(x)\frac{\partial f_2}{\partial x_\alpha} = \mathrm{e}^{\rho f_3}\cos f_3, \quad a_\alpha^{(2)}(x)\frac{\partial f_3}{\partial x_\alpha} = 0.$$

The equality

$$a_\alpha^{(3)}(x)\frac{\partial U}{\partial x_\alpha} + UB_3 = 0$$

holds if and only if

$$\begin{cases} a_\alpha^{(3)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\cos f_3 - a_\alpha^{(3)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\sin f_3 = 0, \\ a_\alpha^{(3)}(x)\dfrac{\partial f_1}{\partial x_\alpha}\sin f_3 + a_\alpha^{(3)}(x)\dfrac{\partial f_2}{\partial x_\alpha}\cos f_3 = 0, \\ a_\alpha^{(3)}(x)\dfrac{\partial f_3}{\partial x_\alpha} + 1 = 0 \end{cases}$$

or

$$a_\alpha^{(3)}(x)\frac{\partial f_1}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_2}{\partial x_\alpha} = 0, \quad a_\alpha^{(3)}(x)\frac{\partial f_3}{\partial x_\alpha} + 1 = 0.$$

The proposition is proved.

Using subalgebras from Proposition 1, we construct ansatzes of the form

$$U\vec{E}+\vec{V}=\vec{M}(\omega),\quad U\vec{H}=\vec{N}(\omega) \tag{7}$$

or

$$\vec{E}=U^{-1}\vec{M}(\omega)-U^{-1}\vec{V},\quad \vec{H}=U^{-1}\vec{N}(\omega), \tag{8}$$

where $\vec{M}(\omega)$, $\vec{N}(\omega)$ are unknown three-component functions, the matrices U, $\vec{V}$ are known, and $\det U\neq 0$ in some domain of the point x space.

Ansatzes of the form (7) or (8) are called *linear*.

Since the generators G_1, G_2, J_{03} are nonlinear differential operators, we act on subalgebras containing them by the inner automorphism corresponding to the element $g=\exp\left(\frac{\pi}{4}X\right)$, where

$$X=-\Gamma_{03}+\Gamma_{30}=x_3\frac{\partial}{\partial x_0}-x_0\frac{\partial}{\partial x_3}+(H_3H_k-E_3E_k)\frac{\partial}{\partial E_k}-(E_3E_k+H_3H_k)\frac{\partial}{\partial H_k}-\frac{\partial}{\partial E_3}.$$

Let us denote $J'_{\alpha\beta}=gJ_{\alpha\beta}g^{-1}$, $P'_\alpha=gP_\alpha g^{-1}$, $G'_\alpha=\frac{\sqrt{2}}{2}gG_\alpha g^{-1}$. It is not difficult to verify that

$$G'_a=J'_{0a}-J'_{a3}=x_0\frac{\partial}{\partial x_a}+x_a\frac{\partial}{\partial x_3}+E_a\frac{\partial}{\partial E_3}+H_a\frac{\partial}{\partial H_3}+\frac{\partial}{\partial E_a}\quad (a=1,2),$$

$$J'_{12}=J_{12}=x_2\frac{\partial}{\partial x_1}-x_1\frac{\partial}{\partial x_2}+E_2\frac{\partial}{\partial E_1}-E_1\frac{\partial}{\partial E_2}+H_2\frac{\partial}{\partial H_1}-H_1\frac{\partial}{\partial H_2},$$

$$J'_{03}=-x_0\frac{\partial}{\partial x_0}+x_3\frac{\partial}{\partial x_3}+\sum_{k=1}^{2}\left(E_k\frac{\partial}{\partial E_k}+H_k\frac{\partial}{\partial H_k}\right)+2E_3\frac{\partial}{\partial E_3}+2H_3\frac{\partial}{\partial H_3},$$

$$P'_0=\frac{\sqrt{2}}{2}(P_0+P_3),\quad P'_3=-\frac{\sqrt{2}}{2}(P_0-P_3),\quad P'_1=P_1,\quad P'_2=P_2.$$

Employing some subalgebras from Proposition 1, let us perform the reduction of system (2) to systems of ODEs, on solutions of which we construct the corresponding solutions of system (2). For each of these subalgebras, we write out its corresponding ansatz, the reduced system, particular or general solution of the reduced system, and the corresponding solution of system (2).

1. $\langle G'_1, G'_2, J'_{03}\rangle:\ E_1=\frac{x_1}{x_0}+\frac{1}{x_0}M_1(\omega),\ E_2=\frac{x_2}{x_0}+\frac{1}{x_0}M_2(\omega),$

$$E_3=\frac{x_1^2+x_2^2}{2x_0^2}+\frac{x_1}{x_0^2}M_1(\omega)+\frac{x_2}{x_0^2}M_2(\omega)+\frac{1}{x_0^2}M_3(\omega),\ H_1=\frac{1}{x_0}N_1(\omega),$$

$$H_2=\frac{1}{x_0}N_2(\omega),\ H_3=\frac{x_1}{x_0^2}N_1(\omega)+\frac{x_2}{x_0^2}N_2(\omega)+\frac{1}{x_0^2}N_3(\omega),\ \omega=x_1^2+x_2^2-2x_0x_3,$$

$$\begin{cases}\dot{M}_1(\omega-2M_3)+2N_3\dot{N}_1=0,\quad \dot{M}_2(\omega-2M_3)+2N_3\dot{N}_2=0,\\ \dot{M}_3(\omega-2M_3)-2M_3+M_1^2+M_2^2-N_1^2-N_2^2+2N_3\dot{N}_3=0,\\ \dot{N}_1(\omega-2M_3)-2N_3\dot{M}_1=0,\quad \dot{N}_2(\omega-2M_3)-2N_3\dot{M}_2=0,\\ \dot{N}_3(\omega-2M_3)-2N_3+2M_1N_1+2M_2N_2-2N_3\dot{M}_3=0.\end{cases}$$

The reduced system has the solution

$$M_1 = C_1, \quad M_2 = C_2, \quad M_3 = \frac{1}{2}(C_1^2 + C_2^2 - C_3^2 - C_4^2),$$
$$N_1 = C_3, \quad N_2 = C_4, \quad N_3 = C_1C_3 + C_2C_4,$$

where C_i $(i = \overline{1,4}\,)$ are arbitrary constants. The corresponding invariant solution of system (2) is of the form

$$E_1 = \frac{x_1 + C_1}{x_0}, \quad E_2 = \frac{x_2 + C_2}{x_0},$$
$$E_3 = \frac{x_1^2 + x_2^2 + 2(C_1x_1 + C_2x_2) + C_1^2 + C_2^2 - C_3^2 - C_4^2}{2x_0^2},$$
$$H_1 = \frac{C_3}{x_0}, \quad H_2 = \frac{C_4}{x_0}, \quad H_3 = \frac{C_3x_1 + C_4x_2 + C_1C_3 + C_2C_4}{x_0^2}.$$

2. $\langle G_1', P_3, P_2 + \alpha P_1\rangle : \; E_1 = \dfrac{x_1 - \alpha x_2}{x_0} + M_1(\omega), \; E_2 = M_2(\omega),$

$$E_3 = \frac{(x_1 - \alpha x_2)^2}{2x_0^2} + \frac{x_1 - \alpha x_2}{x_0} M_1(\omega) + M_3(\omega), \quad H_1 = N_1(\omega), \quad H_2 = N_2(\omega),$$

$$H_3 = \frac{x_1 - \alpha x_2}{x_0} N_1(\omega) + N_3(\omega), \; \omega = x_0,$$

$$\begin{cases} \omega\dot{M}_1 + M_1 - \alpha M_2 = 0, \quad \dot{M}_2 = 0, \quad \omega\dot{M}_3 + M_1^2 - \alpha M_1M_2 - N_1^2 + \alpha N_1N_2 = 0, \\ \omega\dot{N}_1 + N_1 - \alpha N_2 = 0, \quad \dot{N}_2 = 0, \quad \omega\dot{N}_3 + 2M_1N_1 - \alpha M_1N_2 - \alpha M_2N_1 = 0. \end{cases}$$

The general solution of the reduced system is

$$M_1 = \frac{C_3}{\omega} + \alpha C_1, \quad M_2 = C_1, \quad M_3 = \frac{C_3^2 - C_4^2}{2\omega^2} + \frac{\alpha(C_1C_3 - C_2C_4)}{\omega} + C_5,$$
$$N_1 = \frac{C_4}{\omega} + \alpha C_2, \quad N_2 = C_2, \quad N_3 = \frac{C_3C_4}{\omega^2} + \frac{\alpha(C_1C_4 + C_2C_3)}{\omega} + C_6,$$

where C_i $(i = \overline{1,6}\,)$ are arbitrary constants.

The corresponding invariant solution of system (2) is of the form

$$E_1 = \frac{x_1 - \alpha x_2 + C_3}{x_0} + \alpha C_1, \quad E_2 = C_1, \quad H_1 = \frac{C_4}{x_0} + \alpha C_2, \quad H_2 = C_2,$$
$$E_3 = \frac{(x_1 - \alpha x_2 + C_3)^2 - C_4^2}{2x_0^2} + \frac{\alpha[C_1(x_1 - \alpha x_2 + C_3) - C_2C_4]}{x_0} + C_5,$$
$$H_3 = \frac{C_4(x_1 - \alpha x_2 + C_3)}{x_0^2} + \frac{\alpha[C_2(x_1 - \alpha x_2 + C_3) + C_1C_4]}{x_0} + C_6.$$

3. $\langle G_1', G_2', J_{12}' + P_3\rangle : \; E_1 = \dfrac{x_1}{x_0} + M_1(\omega)\cos f_3 - M_2(\omega)\sin f_3,$

$$E_2 = \frac{x_2}{x_0} + M_1(\omega)\sin f_3 + M_2(\omega)\cos f_3, \; E_3 = \frac{x_1^2 + x_2^2}{2x_0^2} +$$

$$+\frac{1}{x_0}(x_1\cos f_3 + x_2\sin f_3)M_1(\omega) - \frac{1}{x_0}(x_1\sin f_3 - x_2\cos f_3)M_2(\omega) + M_3(\omega),$$

$$H_1 = N_1(\omega)\cos f_3 - N_2(\omega)\sin f_3,\ \ H_2 = N_1(\omega)\sin f_3 + N_2(\omega)\cos f_3,$$

$$H_3 = \frac{1}{x_0}(x_1\cos f_3 + x_2\sin f_3)N_1(\omega) - \frac{1}{x_0}(x_1\sin f_3 - x_2\cos f_3)N_2(\omega) + N_3(\omega),$$

$$\omega = x_0, \quad f_3 = \frac{x_1^2 + x_2^2}{2x_0} - x_3.$$

This ansatz reduces system (2) to the system

$$\begin{cases} \dot{M}_1 + M_2M_3 - N_2N_3 + \dfrac{M_1}{\omega} = 0, \quad \dot{M}_2 - M_1M_3 + N_1N_3 + \dfrac{M_2}{\omega} = 0, \\ M_1^2 + M_2^2 - N_1^2 - N_2^2 = 0, \quad \dot{N}_3 = 0, \\ \dot{N}_1 + M_3N_2 + M_2N_3 + \dfrac{N_1}{\omega} = 0, \quad \dot{N}_2 - M_3N_1 - M_1N_3 + \dfrac{N_2}{\omega} = 0. \end{cases}$$

The reduced system has the following particular solution:

$$M_1 = N_1 = \frac{C_1}{\omega}, \quad M_2 = N_2 = \frac{C_2}{\omega}, \quad M_3 = N_3 = 0,$$

where C_1, C_2 are integration constants.

The corresponding solution of system (2), invariant with respect to the subalgebra $\langle G_1', G_2', J_{12}' + P_3\rangle$, is of the form

$$E_1 = \frac{x_1}{x_0} + \frac{C_1}{x_0}\cos f_3 - \frac{C_2}{x_0}\sin f_3,\ \ E_2 = \frac{x_2}{x_0} + \frac{C_1}{x_0}\sin f_3 + \frac{C_2}{x_0}\cos f_3,$$

$$E_3 = \frac{x_1^2 + x_2^2}{2x_0^2} + \frac{C_1}{x_0^2}(x_1\cos f_3 + x_2\sin f_3) - \frac{C_2}{x_0^2}(x_1\sin f_3 - x_2\cos f_3),$$

$$H_1 = \frac{C_1}{x_0}\cos f_3 - \frac{C_2}{x_0}\sin f_3,\ \ H_2 = \frac{C_1}{x_0}\sin f_3 + \frac{C_2}{x_0}\cos f_3,$$

$$H_3 = \frac{C_1}{x_0^2}(x_1\cos f_3 + x_2\sin f_3) - \frac{C_2}{x_0^2}(x_1\sin f_3 - x_2\cos f_3),$$

where $f_3 = \dfrac{x_1^2 + x_2^2}{2x_0} - x_3$.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 463–466.

Symmetry Properties of a Generalized System of Burgers Equations

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Abstract

The classification of symmetry properties of a generalized system of Burgers equations $u_0^a + u_{11}^a = F^{ab}(\vec{u})u_1^a$ is investigated in those cases where the invariance with respect to the Galilean algebra retains.

The symmetry of the classical Burgers equation

$$u_0 + \lambda u u_1 + u_{11} = 0,$$

is well known (see, for example, [1]). Its widest algebra of invariance in the class of the Lie's operators is given by the following basic elements

$$\partial_0 = \frac{\partial}{\partial x_0}, \quad \partial_1 = \frac{\partial}{\partial x_1}, \quad G = x_0\partial_1 + \frac{1}{\lambda}\partial_u,$$
$$D = 2x_0\partial_0 + x_1\partial_1 - u\partial_u, \quad \Pi = x_0^2\partial_0 + x_0x_1\partial_1 + \left(\frac{x_1}{\lambda} - x_0u\right)\partial_u.$$

In the paper [2], the classification of symmetry properties of the generalized Burgers equation

$$u_0 + u_{11} = F(u, u_1), \tag{1}$$

was investigated depending on a nonlinear function $F(u, u_1)$ with which the invariance of equation (1) with respect to the Galilean algebra retains.

A problem of a wider generalization of equation (1) is stated. In this paper, the symmetry properties of the generalized system of the Burgers equations

$$u_0^a + u_{11}^a = F^{ab}(\vec{u})u_1^a. \tag{2}$$

are studied. In these equations, $\vec{u}(u^1, u^2)$, $\vec{u} = \vec{u}(x)$, $x = (x_0, x_1)$, $u_\mu = \partial_\mu u$, $\partial_\mu = \dfrac{\partial}{\partial x_\mu}$, $\mu = 0, 1$; F^{ab} are arbitrary differentiable functions.

The following statements are true.

Theorem 1. *System (2) is invariant with respect to the Galilean algebra $AG = \langle \partial_0, \partial_1, G\rangle$ in the next cases:*
a) If $G = G_1 = x_0\partial_1 + k_1\partial_{u^1} + k_2\partial_{u^2}$, then

$$F^{11} = -\frac{u^1}{k_1} + \varphi^{11}(\omega), \quad F^{12} = \varphi^{12}(\omega),$$
$$F^{21} = \varphi^{21}(\omega), \quad F^{22} = -\frac{u^2}{k_2} + \varphi^{22}(\omega),$$

where $\omega = k_2u^1 - k_1u^2$, φ^{ab} are arbitrary differentiable functions; k_i are arbitrary constants.

b) If $G = G_2 = x_0\partial_1 + (k_1u^1 + k_2)\partial_{u^1} + (k_3u^2 + k_4)\partial_{u^2}$, *then*

$$F^{11} = -\frac{1}{k_1}\ln\omega_1 + \varphi^{11}(\omega_2), \quad F^{12} = \omega_1^{1-k_3/k_1}\varphi^{12}(\omega_2),$$
$$F^{21} = \omega_1^{1-k_3/k_1}\varphi^{21}(\omega), \qquad F^{22} = -\frac{1}{k_1}\ln\omega_1 + \varphi^{22}(\omega),$$

where $\omega_1 = k_2u^1 + k_2$, $\omega_2 = \dfrac{(k_1u^1 + k_2)^{k_3/k_1}}{k_3u^2 + k_4}$, φ^{ab} *are arbitrary differentiable functions;* k_i *are arbitrary constants.*

c) If $G = G_3 = x_0\partial_1 + k_1u^2\partial_{u^1}$, *then*

$$F^{11} = \frac{u^1}{u^2}\left(\varphi^{21}(u^2) - \frac{1}{k_1}\right) + \varphi^{11}(u^2),$$
$$F^{12} = \frac{u^1}{u^2}\left(\varphi^{22}(u^2) - \varphi^{11}(u^2)\right) - \varphi^{21}(u^2)\left(\frac{u^1}{u^2}\right)^2 + \varphi^{12}(u^2),$$
$$F^{21} = \varphi^{21}(u^2), \qquad F^{22} = -\frac{u^1}{u^2}\left(\varphi^{21}(u^2) + \frac{1}{k_1}\right) + \varphi^{22}(u^2),$$

where φ^{ab} *are arbitrary differentiable functions;* k_1 *is an arbitrary constant.*

d) If $G = G_4 = x_0\partial_1 + k_2u^1\partial_{u^2}$, *then*

$$F^{11} = -\frac{u^2}{u^1}\left(\varphi^{12}(u^1) + \frac{1}{k_2}\right) + \varphi^{11}(u^1), \qquad F^{12} = \varphi^{12}(u^1),$$
$$F^{21} = \frac{u^2}{u^1}\left(\varphi^{11}(u^1) - \varphi^{22}(u^1)\right) - \varphi^{12}(u^1)\left(\frac{u^2}{u^1}\right)^2 + \varphi^{21}(u^1),$$
$$F^{22} = \frac{u^2}{u^1}\left(\varphi^{12}(u^1) - \frac{1}{k_1}\right) + \varphi^{22}(u^1),$$

where φ^{ab} *are arbitrary differentiable functions;* k_2 *is an arbitrary constant.*

Theorem 2. *System (2) is invariant with respect to the Galilean algebra* $AG_1 = \langle\partial_0, \partial_1, G, D\rangle$ *in the next cases:*
1) $G = G_1$, $\quad D = D_1 = 2x_0\partial_0 + x_1\partial_1 + \left(-u^1 + m_1\omega\right)\partial_{u^1} + \left(-u^2 + m_2\omega\right)\partial_{u^2}$;

$$F^{11} = F^{22} = m_1u^2 - m_2u^1, \qquad F^{12} = F^{21} = 0,$$

where $\omega = k_2u^1 - k_1u^2$, φ^{ab} *are arbitrary differentiable functions;* k_i, m_i *are arbitrary constants.*

2) $G = G_4$, $\quad D = D_2 = 2x_0\partial_0 + x_1\partial_1 + Au^1\partial_{u^1} + (A-1)u^2\partial_{u^2}$, *where* A *is an arbitrary constant (AC).*

a) $A \neq 0$

$$F^{11} = -\frac{u^2}{u^1}\left(\frac{1}{k_2} + c_{12}\right) + c_{11}(u^1)^{-1/A}, \qquad F^{12} = c_{12},$$
$$F^{21} = c_{21}(u^1)^{-2/A} + (c_{11} - c_{22})u^2(u^1)^{-1-1/A} - c_{12}\left(\frac{u^2}{u^1}\right)^2,$$
$$F^{22} = \frac{u^2}{u^1}\left(-\frac{1}{k_2} + c_{12}\right) + c_{22}(u^1)^{-1/A},$$

where c_{ab}, k_2 *are arbitrary constants.*

b) $A = 0$

$$F^{11} = -\frac{u^2}{u^1}\left(\frac{1}{k_2} + \varphi^{12}(u^1)\right), \quad F^{12} = \varphi^{12}(u^1),$$

$$F^{21} = -\varphi^{12}(u^1)\left(\frac{u^2}{u^1}\right)^2, \qquad F^{22} = \frac{u^2}{u^1}\left(-\frac{1}{k_2} + \varphi^{12}(u^1)\right),$$

where φ^{ab} *are arbitrary differentiable functions.*

3) $G = G_3$, $\quad D = D_3 = 2x_0\partial_0 + x_1\partial_1 + (B-1)u^1\partial_{u^1} + Bu^2\partial_{u^2}$, *where* B *is an arbitrary constant.*

a) $B \neq 0$

$$F^{11} = \frac{u^1}{u^2}\left(-\frac{1}{k_1} + c_{12}\right) + c_{11}(u^2)^{-1/B},$$

$$F^{12} = c_{12}(u^2)^{-2/B} + (c_{22} - c_{11})u^1(u^2)^{-1-1/B} - c_{21}\left(\frac{u^1}{u^2}\right)^2,$$

$$F^{21} = c_{12}, \qquad F^{22} = -\frac{u^1}{u^2}\left(\frac{1}{k_1} + c_{12}\right) + c_{22}(u^2)^{-1/B},$$

where c_{ab}, k_1 *are arbitrary constants.*

b) $B = 0$

$$F^{11} = \frac{u^1}{u^2}\left(-\frac{1}{k_1} + \varphi^{21}(u^2)\right), \quad F^{12} = -\varphi^{21}(u^2)\left(\frac{u^1}{u^2}\right)^2,$$

$$F^{21} = \varphi^{21}(u^2), \qquad F^{22} = -\frac{u^1}{u^2}\left(\frac{1}{k_1} + \varphi^{21}(u^2)\right),$$

where φ^{ab} *are arbitrary differentiable functions.*

Theorem 3. *System (2) is invariant with respect to the Galilean algebra* $AG_2 = \langle \partial_0, \partial_1, G, D, \Pi\rangle$, *when*

$$G = G_1, \quad D = D_1, \quad \Pi = x_0^2\partial_0 + x_0x_1\partial_1 - u^a\partial_{u^a} + x_0\omega(\alpha_2\partial_{u^1} - \alpha_1\partial_{u^2}) - x_1(\beta_2\partial_{u^1} - \beta_1\partial_{u^2}),$$

where $\omega = \vec{\beta}\vec{u}$, $F^{ab} = \delta_{ab}\vec{\alpha}\vec{u}$, α_i, β_i *are arbitrary constants, and* $\begin{vmatrix}\beta_1 & \beta_2\\ \alpha_1 & \alpha_2\end{vmatrix} = 1$.

Proof of Theorems 1–3. Let an infinitesimal operator X (see [3]) be

$$X = \xi^0(x, \vec{u})\,\partial_0 + \xi^1(x, u)\,\partial_1 + \eta^1(x, u)\,\partial_{u^1} + \eta^2(x, u)\,\partial_{u^2}. \tag{3}$$

Using the invariance condition of equation (2) with respect to operator (3), we obtain defining equations for the coordinates of operator (3) and functions $F^{ab}(u^1, u^2)$:

$$\begin{aligned}
&\xi^0_1 = \xi^0_{u^a} = 0, \quad \xi^1_{u^a} = 0, \quad \xi^0_0 = 2\xi^1_1, \quad \eta^a_{u^bu^c} = 0,\\
&-F^{11}\eta^1_1 - F^{21}\eta^2_1 + \eta^1_0 + \eta^1_{11} = 0, \quad -F^{21}\eta^1_1 - F^{22}\eta^2_1 + \eta^2_0 + \eta^2_{11} = 0,\\
&F^{11}_{u^1}\eta^1 + F^{11}_{u^2}\eta^2 + F^{12}\eta^1_{u^1} + \xi^1_0 - 2\eta^1_{1u^1} + F^{11}\xi^1_1 - F^{21}\eta^1_{u^2} = 0,\\
&F^{22}_{u^1}\eta^1 + F^{22}_{u^2}\eta^2 + F^{21}\eta^1_{u^22} + \xi^1_0 - 2\eta^2_{1u^2} + F^{22}\xi^1_1 - F^{12}\eta^2_{u^1} = 0,\\
&F^{12}_{u^1}\eta^1 + F^{12}_{u^2}\eta^2 + F^{11}\eta^1_{u^2} - F^{12}(\eta^1_{u^1} - \eta^2_{u^2} - \xi^1_1) - 2\eta^1_{1u^2} - F^{22}\eta^1_{u^2} = 0,\\
&F^{21}_{u^1}\eta^1 + F^{21}_{u^2}\eta^2 + F^{22}\eta^2_{u^1} - F^{21}(\eta^2_{u^2} - \eta^1_{u^1} - \xi^1_1) - 2\eta^2_{1u^1} - F^{11}\eta^2_{u^1} = 0.
\end{aligned} \tag{4}$$

Solutions of system (4) define a form of the nonlinear functions F^{ab} and invariance algebra.

Theorem 4. *The widest invariance algebra of the equation*

$$\vec{u}_0 + (\vec{\lambda}\vec{u})\vec{u}_1 + \vec{u}_{11} = 0 \tag{5}$$

consists of the operators

$$\begin{gathered}\partial_0, \quad \partial_1, \quad G_a = \lambda_a x_0 \partial_1 + \partial_{u^a}, \quad D_a = \lambda_a(2x_0\partial_0 + x_1\partial_1) - \vec{\lambda}\vec{u}\partial_{u^a}, \\ \Pi_a = \lambda_a(x_0^2\partial_0 + x_0x_1\partial_1 +) + \left(x_1 - x_0\vec{\lambda}\vec{u}\right)\partial_{u^a}, \quad Q_{ij} = u^i(\lambda_n\partial_{u^j} - \lambda_j\partial_{u^n}),\end{gathered} \tag{6}$$

where $a = 1, \ldots, n$*;* $i, j = 1, \ldots, n-1$*.*

This theorem is proved by the standard Lie's method.

Commutation relations of the operators of algebra (6) are as follows:

$$\begin{array}{lll}
[\partial_0, \partial_1] = 0; & [\partial_0, G_a] = \lambda_a\partial_1; & [\partial_0, D_a] = 2\lambda_a\partial_0; \\
[\partial_0, \Pi_a] = D_a; & [\partial_0, Q_{ij}] = 0; & [\partial_1, G_a] = 0; \\
[\partial_1, D_a] = \lambda_a\partial_1; & [\partial_1, \Pi_a] = G_a; & [\partial_1, Q_{ij}] = 0; \\
[Q_{ij}, Q_{ab}] = 0; & [D_a, D_b] = \lambda_a D_b - \lambda_b D_a; & [G_a, D_b] = -\lambda_a G_b; \\
[G_a, \Pi_b] = 0; & [G_a, Q_{ij}] = \delta_{ai}(\lambda_n G_j - \lambda_j G_n); & [G_a, G_b] = 0; \\
[D_a, \Pi_b] = \lambda_a \Pi_b + \lambda_b \Pi_a; & [D_a, Q_{ij}] = \delta_{ai}(\lambda_j D_n - \lambda_n D_j); & [\Pi_a, \Pi_b] = 0; \\
[\Pi_a, Q_{ij}] = \delta_{ai}(\lambda_n \Pi_j - \lambda_j \Pi_n). & &
\end{array}$$

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 467–469.

Symmetry of Burgers-type Equations with an Additional Condition

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Abstract

We study symmetry properties of Burgers-type equations with an additional condition.

It is well known that the Burgers equation

$$u_t + uu_x + \lambda u_{xx} = 0, \qquad \lambda = \text{const} \tag{1}$$

can be reduced by means of the Cole-Hopf non-local transformation

$$u = 2\lambda \frac{\psi_x}{\psi} \tag{2}$$

to the linear heat equation

$$\psi_t - \lambda \psi_{xx} = 0. \tag{3}$$

We note that the symmetry of the heat equation (3) is wider than the symmetry of the Burgers equation (1) [1].

In [2], the symmetry classification of the following generalization of the Burgers equation

$$u_t + uu_x = F\left(u_{xx}\right) \tag{4}$$

is carried out. In the general case, equation (4) with an arbitrary function $F(u_{xx})$ is invariant with respect to the Galilei algebra $AG(1,1)$. Equation (4) admits a wider symmetry only in the following cases [2]:

$$F(u_{xx}) = \lambda u_{xx}^k, \tag{5}$$

$$F(u_{xx}) = \ln u_{xx}, \tag{6}$$

$$F(u_{xx}) = \lambda u_{xx}, \tag{7}$$

$$F(u_{xx}) = \lambda u_{xx}^{1/3}, \tag{8}$$

where k, λ are arbitrary constants.

This paper contains the symmetry classification of equation (4), where $F(U_{xx})$ is determined by relations (6)-(8), with the additional condition which is a generalization of (2) of the following form

$$\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \tag{9}$$

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \lambda u_{xx} = 0, \\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{10}$$

Theorem 1. *Maximal invariance algebras of system (10) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1\rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary, where

$$P_0 = \partial_t, \qquad P_1 = \partial_x, \qquad X_1 = b(t)\psi\partial_\psi;$$

2) $\langle P_0, P_1, X_1, X_2\rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$, where

$$X_2 = h(t)\partial_\psi;$$

3) $\langle P_0, P_1, X_1, X_3, X_4, X_5\rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$X_3 = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi \qquad X_4 = 2t\partial_t + x\partial_x - u\partial_u - \frac{1}{2}bx\psi\partial_\psi,$$
$$X_5 = t^2\partial_t + tx\partial_x + (x - tu)\partial_u - \frac{1}{4}(2btx + ax^2)\psi\partial_\psi,$$

4) $\langle P_0, P_1, X_1, X_3, X_4, X_5, R_1, R_2\rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants), where

$$\left.\begin{aligned} R_1 &= C_1(t)\exp\left(-\frac{1}{2}(b + \sqrt{b^2 - 4d})x\right), \\ R_2 &= C_2(t)\exp\left(-\frac{1}{2}(b - \sqrt{b^2 - 4d})x\right), \end{aligned}\right. \qquad if \quad b^2 - 4d > 0,$$

$$\left.\begin{aligned} R_1 &= C_1(t)\exp\left(-\frac{b}{2}x\right), \\ R_2 &= xC_2(t)\exp\left(-\frac{b}{2}x\right), \end{aligned}\right. \qquad if \quad b^2 - 4d = 0,$$

$$\left.\begin{aligned} R_1 &= C_1(t)\exp\left(-\frac{b}{2}x\right)\cos\frac{\sqrt{4d - b^2}}{2}x, \\ R_2 &= C_2(t)\exp\left(-\frac{b}{2}x\right)\sin\frac{\sqrt{4d - b^2}}{2}x, \end{aligned}\right. \qquad if \quad b^2 - 4d < 0.$$

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \ln u_{xx} = 0, \\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{11}$$

Theorem 2. *Maximal invariance algebras of system (11) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1\rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary;
2) $\langle P_0, P_1, X_1, X_2\rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$;
3) $\langle P_0, P_1, X_1, Q\rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$Q = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi;$$

4) $\langle P_0, P_1, X_1, X_3, X_4, X_5, R_1, R_2\rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants).

Let us consider the system

$$\begin{aligned} &u_t + uu_x + \lambda(u_{xx})^{1/3} = 0,\\ &\psi_{xx} + f^1(u)\psi_x + f^2(u)\psi = 0. \end{aligned} \tag{12}$$

Theorem 3. *Maximal invariance algebras of system (12) depending on functions $f^1(u)$ and $f^2(u)$ are the following Lie algebras:*
1) $\langle P_0, P_1, X_1, Y_1\rangle$ if $f^1(u)$, $f^2(u)$ are arbitrary, where

$$Y_1 = u\partial_x;$$

2) $\langle P_0, P_1, X_1, X_2, Y_1\rangle$ if $f^1(u)$ is arbitrary, $f^2 = 0$;
3) $\langle P_0, P_1, X_1, Y_1, Y_2, Y_3, Y_4\rangle$ if $f^1(u) = au + b$, $f^2(u) = \frac{1}{4}a^2u^2 + \frac{1}{2}abu + d$, where

$$\begin{aligned} &Y_2 = t\partial_x + \partial_u - \frac{1}{2}ax\psi\partial_\psi,\\ &Y_3 = (t^2u - tx)\partial_x + (tu - x)\partial_u + \frac{1}{2}btx\psi\partial_\psi,\\ &Y_4 = \frac{8}{15}t\partial_t + (x - \frac{2}{3}u)\partial_x - \frac{1}{5}u\partial_u - \frac{1}{2}bx\psi\partial_\psi; \end{aligned}$$

4) $\langle P_0, P_1, X_1, Y_1, Y_2, Y_3, Y_4, R_1, R_2\rangle$ if $f^1 = b$, $f^2 = d$ (b, d are arbitrary constants).

The proof of Theorems 1-3 is carried out by means of the classical Lie algorithm [1].

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 470–472.

The Symmetry of a Generalized Burgers System

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Abstract

The following system of equations in the form $u_0^a + u^b u_b^a = F^a\left(\triangle u^1, \triangle u^2\right)$, $a, b = 1, 2$ is considered as a generalization of the classical Burgers equation. The symmetry properties of this system are investigated.

In a medium with dissipation, the Burgers equation

$$u_0 + uu_1 = u_{11}, \tag{1}$$

where $u_0 = \dfrac{\partial u}{\partial x_0}$, $u_1 = \dfrac{\partial u}{\partial x_1}$, $u_{11} = \dfrac{\partial^2 u}{\partial x_1^2}$, $u = u(x_0, x_1)$, describes a quasisimple wake behaviour. It's worth noticing, that equation (1), as is well known, reduces to the linear heat equation [1] by a nonlocal Cole-Hopf substitution. The algebra of invariance of the heat equation is well known.

The nonlinear generalization of equation (1)

$$u_0 + uu_1 = F\left(u_{11}\right), \tag{2}$$

where $F\left(u_{11}\right)$ is an arbitrary smooth function, $F \neq \mathrm{const}$, is investigated too. In paper [2], symmetry properties of equation (2) were studied in detail depending on a form of a function $F\left(u_{11}\right)$.

Let us generalize equation (2) to the case of two functions $u^1 = u^1\left(x_0, x_1, x_2\right)$, $u^2 = u^2\left(x_0, x_1, x_2\right)$ by the following system

$$\begin{cases} u_0^1 + u^1 u_1^1 + u^2 u_2^1 = F^1\left(\triangle u^1, \triangle u^2\right), \\ u_0^2 + u^1 u_1^2 + u^2 u_2^2 = F^2\left(\triangle u^1, \triangle u^2\right), \end{cases} \tag{3}$$

where $F^1\left(\triangle u^1, \triangle u^2\right)$, $F^2\left(\triangle u^1, \triangle u^2\right)$ are arbitrary smooth functions, $F^1 \neq \mathrm{const}$, $F^2 \neq \mathrm{const}$, $\triangle u^a = u_{11}^a + u_{22}^a$, $a = 1, 2$. We'll study a symmetry of system (3) depending on forms of the functions F^1, F^2.

Theorem 1. *The maximum algebra of invariance of system (3) is given by operators:*

1. $\langle P_\alpha = \partial_\alpha, G_a = x_0\partial_a + \partial_{u^a}\rangle$, *when* $F^1\left(\triangle u^1, \triangle u^2\right)$, $F^2\left(\triangle u^1, \triangle u^2\right)$ *are arbitrary functions;*

2. $\langle P_\alpha, G_a, J_{12} = x_2\partial_1 - x_1\partial_2 + u^2\partial_{u^1} - u^1\partial_{u^2}\rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \triangle u^1 \varphi^1(\omega) - \triangle u^2 \varphi^2(\omega),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = \triangle u^1 \varphi^2(\omega) + \triangle u^2 \varphi^1(\omega);$$

3. $\langle P_\alpha, G_a, D_1 = (n+1)x_0\partial_0 + (2-n)x_a\partial_a + (1-2n)u^a\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \left(\triangle u^1\right)^n \varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = \left(\triangle u^2\right)^n \varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right), \qquad n \neq 1;$$

4. $\langle P_\alpha, G_a, D_2 = x_0\partial_0 + \left(2x_a - \frac{3}{2}m_a x_0^2\right)\partial_a + (u^a - 3m_a x_0)\,\partial_{u^a}\rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = m_1 \ln \triangle u^1 + \varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = m_2 \ln \triangle u^2 + \varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right),$$

m_a are arbitrary constants, which must not be equal to zero at the same time;

5. $\langle P_\alpha, G_a, J_{12}, D_3 = 2(n+1)x_0\partial_0 + (1-2n)x_a\partial_a + (4n+1)u^a\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \omega^n\left(\lambda_2\triangle u^1 - \lambda_1\triangle u^2\right),$$
$$F^2\left(\triangle u^1, \triangle u^2\right) = \omega^n\left(\lambda_1\triangle u^1 + \lambda_2\triangle u^2\right), \qquad n \neq 0;$$

6. $\langle P_\alpha, G_a, D = 2x_0\partial_0 + x_a\partial_a - u^a\partial_{u^a}, \Pi = x_0^2\partial_0 + x_0x_a\partial_a + (x_a - x_0u^a)\,\partial_{u^a}\rangle$, *if*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \triangle u^1\varphi^1\left(\frac{\triangle u^1}{\triangle u^2}\right), \qquad F^2\left(\triangle u^1, \triangle u^2\right) = \triangle u^2\varphi^2\left(\frac{\triangle u^1}{\triangle u^2}\right);$$

7. $\langle P_\alpha, G_a, J_{12}, D, \Pi\rangle$, *when*

$$F^1\left(\triangle u^1, \triangle u^2\right) = \lambda_2\triangle u^1 - \lambda_1\triangle u^2, \qquad F^2\left(\triangle u^1, \triangle u^2\right) = \lambda_1\triangle u^1 + \lambda_2\triangle u^2,$$

under the condition of the theorem $\omega = \left(\triangle u^1\right)^2 + \left(\triangle u^2\right)^2$, φ^a are arbitrary smooth functions, n, λ_a are arbitrary constants, $\alpha = 0, 1, 2$.

Proof. The symmetry classification of system (3) is carried out in a class of the first order differential operators

$$X = \xi^0\left(x_0, \vec{x}, u^1, u^2\right)\partial_0 + \xi^a\left(x_0, \vec{x}, u^1, u^2\right)\partial_a + \eta^a\left(x_0, \vec{x}, u^1, u^2\right)\partial_{u^a}. \tag{4}$$

As system (3) includes the second order equations, finding a generation of operator (4), we may write a condition of the Lie's invariance in the form:

$$\begin{cases} \tilde{X}\left[u_0^1 + u^1u_1^1 + u^2u_2^1 - F^1\left(\triangle u^1, \triangle u^2\right)\right]\Big|_{u_0^a + u^bu_b^a = F^a(\triangle u^1, \triangle u^2)} = 0, \\ \tilde{X}\left[u_0^2 + u^1u_1^2 + u^2u_2^2 - F^2\left(\triangle u^1, \triangle u^2\right)\right]\Big|_{u_0^a + u^bu_b^a = F^a(\triangle u^1, \triangle u^2)} = 0, \end{cases}$$

where $b = 1, 2$.

Using Lie's algorithm [3], we get a system of defining equations to find the functions ξ^0, ξ^a, η^a and F^1, F^2. Solving this system, we obtain:

$$\begin{cases} 3C_2\left(\triangle u^1 F^1_{\triangle u^1} + \triangle u^2 F^1_{\triangle u^2} - F^1\right) + C_6 = 0, \\ 3C_2\left(\triangle u^1 F^2_{\triangle u^1} + \triangle u^2 F^2_{\triangle u^2} - F^2\right) + C_{10} = 0, \\ w^1 F^1_{\triangle u^1} + w^2 F^1_{\triangle u^1} + (C_3 - 2C_4)F^1 + C_1 F^2 + C_7 = 0, \\ w^1 F^2_{\triangle u^1} + w^2 F^2_{\triangle u^1} + (C_3 - 2C_4)F^2 - C_1 F^1 + C_{11} = 0, \end{cases} \tag{5}$$

$$\xi^0 = C_2 x_0^2 + C_4 x_0 + C_5,$$

$$\xi^1 = (C_2 x_0 + C_3)\, x_1 + C_1 x_2 + \frac{C_6 x_0^3}{6} + \frac{C_7 x_0^2}{2} + C_8 x_0 + C_9,$$

$$\xi^2 = (C_2 x_0 + C_3)\, x_2 - C_1 x_1 + \frac{C_{10} x_0^3}{6} + \frac{C_{11} x_0^2}{2} + C_{12} x_0 + C_{13},$$

$$\eta^1 = (-C_2 x_0 + C_3 - C_4)\, u^1 + C_1 u^2 + C_2 x_1 + \frac{C_6 x_0^2}{2} + C_7 x_0 + C_8,$$

$$\eta^2 = (-C_2 x_0 + C_3 - C_4)\, u^2 - C_1 u^1 + C_2 x_2 + \frac{C_{10} x_0^2}{2} + C_{11} x_0 + C_{12},$$

where $w^1 = \left((C_3 + C_4)\triangle u^1 - C_1 \triangle u^2\right)$, $w^2 = \left(C_1 \triangle u^1 + (C_3 + C_4)\triangle u^2\right)$, C_i are arbitrary constants, $i = 1, \ldots, 13$.

Analyzing system (5), we receive the condition of Theorem.

Notice. Following Theorem, the system

$$\begin{cases} u^1_0 + u^1 u^1_1 + u^2 u^1_2 = \lambda_2 \triangle u^1 - \lambda_1 \triangle u^2, \\ u^2_0 + u^1 u^2_1 + u^2 u^2_2 = \lambda_1 \triangle u^1 + \lambda_2 \triangle u^2, \end{cases} \tag{6}$$

has the widest symmetry among all systems of the form (3). If $\lambda_1 = 0$, then system (6) transforms to the classical Burgers system

$$\begin{cases} u^1_0 + u^1 u^1_1 + u^2 u^1_2 = \lambda_2 \triangle u^1, \\ u^2_0 + u^1 u^2_1 + u^2 u^2_2 = \lambda_2 \triangle u^2, \end{cases} \tag{7}$$

which is invariant with respect to the Galilean algebra. This system is used to describe real physical processes. Since (6) and (7) have the same symmetry, system (6) may be also used to describe these processes.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 473–474.

A Galilei-Invariant Generalization of the Shiff Equations

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Abstract

A Galilei-invariant generalization of the Shiff rotating frame electrodynamics is suggested.

To describe the charge and electromagnetic field distributions in a conductor rotating with a constant angular velocity (the reference frame is in rest with respect to the conductor), one uses the system of Shiff equations (see, for example, [1])

$$\begin{aligned} &\mathrm{rot}\,\mathbf{E} + \mathbf{B}_t = \mathbf{0}, \qquad \mathrm{div}\,\mathbf{B} = 0, \\ &\mathrm{rot}\,\mathbf{B} - \mathbf{E}_t = \mu_0 \left\{ \mathbf{v} \times \mathrm{rot}\,\mathbf{E} + \frac{1}{\varepsilon_0}\mathrm{rot}\,[\mathbf{v} \times (\mathbf{E} - \mathbf{v} \times \mathbf{B})] + \mathbf{j} \right\}, \\ &\mathrm{div}\,\mathbf{E} = \frac{1}{\varepsilon_0}(\mathrm{div}\,(\mathbf{v} \times \mathbf{B}) + j_0). \end{aligned} \tag{1}$$

Here,

$$\mathbf{v} = \omega \times \mathbf{x}, \tag{2}$$

$\mathbf{E} = \mathbf{E}(t, \mathbf{x})$, $\mathbf{H} = \mathbf{H}(t, \mathbf{x})$ are three-vectors of electromagnetic field; $\omega = (\omega_1, \omega_2, \omega_3)$ is the vector of angular velocity, $\omega_a = \mathrm{const}$; ε_0, μ_0 are the electric and magnetic permittivities; $j_0, \mathbf{j}$ are the charge and current densities, respectively.

In a sequel, we will consider system (1) under $j_0 = 0$, $\mathbf{j} = \mathbf{0}$.

As is well known, the system of Shiff equations (1), (2) is not invariant with respect to the Galilei group $G(1, 3)$. However, if we suppose that $\mathbf{v}(t, \mathbf{x})$ is an arbitrary vector field, then system (1) admits the Galilei group having the following generators:

$$\begin{aligned} &P_0 = \frac{\partial}{\partial t}, \qquad P_a = \frac{\partial}{\partial x_a}, \\ &J_{ab} = x_a \frac{\partial}{\partial x_b} - x_b \frac{\partial}{\partial x_a} + E_a \frac{\partial}{\partial E_b} - E_b \frac{\partial}{\partial E_a} + B_a \frac{\partial}{\partial B_b} - B_b \frac{\partial}{\partial B_a} + v_a \frac{\partial}{\partial v_b} - v_b \frac{\partial}{\partial v_a}, \\ &G_a = t \frac{\partial}{\partial x_a} - \varepsilon_{abc} B_b \frac{\partial}{\partial B_c} - \varepsilon_0 \frac{\partial}{\partial x_a}, \end{aligned} \tag{3}$$

where $a, b, c = 1, 2, 3$ and

$$\varepsilon_{abc} = \begin{cases} 1, & (a, b, c) = \mathrm{cycle}(1, 2, 3), \\ -1, & (a, b, c) = \mathrm{cycle}(2, 1, 3), \\ 0, & \text{for other cases.} \end{cases}$$

Consequently, the additional condition (2) breaks the Galilei invariance of system (1). The principal aim of the present paper is to suggest a generalization of system (1), (2) such that

- its solution set includes the set of solutions of system (1), (2) as a subset, and
- it is Galilei-invariant.

To this end, we replace the additional condition (2) by the following equation:

$$F(\operatorname{div}\mathbf{v}, \mathbf{v}^2) = 0, \tag{4}$$

where F is a smooth function of absolute invariants of the group $O(3)$ which is the subgroup of the Galilei group $G(1,3)$.

Applying the infinitesimal Lie approach (see, e.g., [2]), we obtain the following assertion.

Theorem. *The system of partial differential equations (1), (4) is invariant with respect to the group* $G(1,3)$ *having generators (3) if and only if*

$$F = \operatorname{div}\mathbf{v} + C, \tag{5}$$

where C *is an arbitrary real constant.*

Consequently, equation (4) now reads as $\operatorname{div}\mathbf{v} + C = 0$. Inserting into this equation $\mathbf{v} = \omega \times \mathbf{x}$ yields $C = 0$.

Thus, we have proved that system (1) considered together with the equation

$$\operatorname{div}\mathbf{v} = 0 \tag{6}$$

fulfills the Galilei relativity principle. Furthermore, its set of solutions contains all solutions of the Shiff system (1), (2).

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 475–482.

Fermionic Symmetries of the Maxwell Equations with Gradient-like Sources

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Abstract

The Maxwell equations with gradient-like sources are proved to be invariant with respect to both bosonic and fermionic representations of the Poincar'e group and to be the kind of Maxwell equations with maximally symmetric properties. Nonlocal vector and tensor-scalar representations of the conformal group are found, which generate the transformations leaving the Maxwell equations with gradient-like sources being invariant.

1. Introduction

The relationship between the massless Dirac equation and the Maxwell equations attracts the interest of investigators [1–21] since the creation of quantum mechanics. In [8, 11], one can find the origin of the studies of the most interesting case where mass is nonzero and the interaction in the Dirac equation is nonzero too. As a consequence, the hydrogen atom can be described [11, 19–21] on the basis of the Maxwell equations. Starting from [10], the Maxwell equations with gradient-like sources have appeared in consideration. From our point of view, it is the most interesting kind of the Maxwell equations especially in studying the relationship with the massless Dirac equation. Below we investigate the symmetry properties of this kind of the Maxwell equations.

2. The Maxwell equations with gradient-like sources

Let us choose the γ^μ matrices in the massless Dirac equation

$$i\gamma^\mu\partial_\mu\Psi(x)=0;\quad x\equiv(x^\mu)\in R^4,\quad \Psi\equiv(\Psi^\mu),\quad \partial_\mu\equiv\frac{\partial}{\partial x^\mu},\quad \mu=0,1,2,3, \tag{1}$$

obeying the Clifford-Dirac algebra commutation relations

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu},\quad \gamma^{\mu\dagger}=g^{\mu\nu}\gamma_\nu,\quad \mathrm{diag}\, g=(1,-1,-1,-1), \tag{2}$$

in the Pauli-Dirac representation (shortly: PD-representation):

$$\gamma^0=\begin{vmatrix}1&0\\0&-1\end{vmatrix},\quad \gamma^k=\begin{vmatrix}0&\sigma^k\\-\sigma^k&0\end{vmatrix};$$
$$\sigma^1=\begin{vmatrix}0&1\\1&0\end{vmatrix},\quad \sigma^2=\begin{vmatrix}0&-i\\i&0\end{vmatrix},\quad \sigma^3=\begin{vmatrix}1&0\\0&-1\end{vmatrix}. \tag{3}$$

Note first of all that after the substitution of ψ by the following column

$$\psi = \text{column}\left|E^3 + iH^0, \quad E^1 + iE^2, \quad iH^3 + E^0, \quad -H^2 + iH^1\right|, \tag{4}$$

the Dirac equation (1) is transformed into equations for the system of electromagnetic $\left(\vec{E}, \vec{H}\right)$ and scalar $\left(E^0, H^0\right)$ fields:

$$\begin{aligned} &\partial_0 \vec{E} = \text{curl}\,\vec{H} - \text{grad}\,E^0, \qquad \partial_0 \vec{H} = -\text{curl}\,\vec{E} - \text{grad}\,H^0, \\ &\text{div}\,\vec{E} = -\partial_0 E^0, \qquad \text{div}\,\vec{H} = -\partial_0 H^0 \end{aligned} \tag{5}$$

(three other versions of the treatment of these equations are given in [16], the complete set of linear homogeneous connections between the Maxwell and the Dirac equations is given in [14, 17, 18]). In the notations $(E^\mu) \equiv \left(E^0, \vec{E}\right), (H^\mu) \equiv \left(H^0, \vec{H}\right)$, Eqs.(5) have a manifestly covariant form

$$\partial_\mu E_\nu - \partial_\nu E_\mu + \varepsilon_{\mu\nu\rho\sigma}\partial^\rho H^\sigma = 0, \qquad \partial_\mu E^\mu = 0, \qquad \partial_\mu H^\mu = 0. \tag{6}$$

In terms of the complex 4-component object

$$\mathcal{E} = \text{column}\left|E^1 - iH^1, \quad E^2 - iH^2, \quad E^3 - iH^3, \quad E^0 - iH^0\right|, \tag{7}$$

Eqs.(5) have the form

$$\partial_\mu \mathcal{E}_\nu - \partial_\nu \mathcal{E}_\mu + i\varepsilon_{\mu\nu\rho\sigma}\partial^\rho \mathcal{E}^\sigma = 0, \qquad \partial_\mu \mathcal{E}^\mu = 0. \tag{8}$$

The free Maxwell equations are obtained from Eqs. (5), (6), (8) in the case of $E^0 = H^0 = 0$.

The unitary operator V

$$\begin{aligned} &V \equiv \begin{vmatrix} 0 & C_+ & 0 & C_- \\ 0 & iC_- & 0 & iC_+ \\ C_+ & 0 & C_- & 0 \\ C_- & 0 & C_+ & 0 \end{vmatrix}, \quad V^\dagger \equiv \begin{vmatrix} 0 & 0 & C_+ & C_- \\ C_+ & iC_+ & 0 & 0 \\ 0 & 0 & C_- & C_+ \\ C_- & iC_- & 0 & 0 \end{vmatrix}; \\ &C_\pm \equiv \frac{C \pm 1}{2}; \quad VV^\dagger = V^\dagger V = 1. \end{aligned} \tag{9}$$

(in the space where the Clifford-Dirac algebra is defined as a real one, this operator is unitary) transforms the ψ from (4) into the object $\mathcal{E}$ (7):

$$\psi \equiv \begin{vmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \\ \psi^4 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} \mathcal{E}^3 + \mathcal{E}^{*3} - \mathcal{E}^0 + \mathcal{E}^{*0} \\ \mathcal{E}^1 + \mathcal{E}^{*1} + i\mathcal{E}^2 + i\mathcal{E}^{*2} \\ \mathcal{E}^0 + \mathcal{E}^{*0} - \mathcal{E}^3 + \mathcal{E}^{*3} \\ -i\mathcal{E}^2 + i\mathcal{E}^{*2} - \mathcal{E}^1 + \mathcal{E}^{*1} \end{vmatrix} \equiv V^{-1}\mathcal{E}, \qquad \mathcal{E} \equiv \begin{vmatrix} \mathcal{E}^1 \\ \mathcal{E}^2 \\ \mathcal{E}^3 \\ \mathcal{E}^0 \end{vmatrix} = V\psi, \tag{10}$$

and the γ^μ matrices (3) in the PD-representation into the bosonic representation (shortly: B-representation)

$$\gamma^\mu \longrightarrow \widetilde{\gamma}^\mu \equiv V\gamma^\mu V^\dagger. \tag{11}$$

Here C is the operator of complex conjugation: $C\Psi = \Psi^*$. The unitarity of the operator V can be proved by means of relations

$$Ca = (aC)^* = a^*C, \quad (AC)^\dagger = CA^\dagger = A^T C \tag{12}$$

for an arbitrary complex number a and a matrix A.

Let us write down the explicit form of the Clifford-Dirac algebra generators in the B-representation

$$\widetilde{\gamma}^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix} C, \quad \widetilde{\gamma}^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix} C,$$
$$\widetilde{\gamma}^2 = \begin{vmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix} C, \quad \widetilde{\gamma}^3 = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix} C. \tag{13}$$

In the B-representation, the complex number i is represented by the following matrix operator

$$\widetilde{i} = ViV^{\dagger} = i\Gamma, \quad \Gamma \equiv \begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{vmatrix} = \Gamma^{\dagger} = \Gamma^{-1}, \quad \Gamma^2 = 1. \tag{14}$$

The above-mentioned facts mean that the Maxwell equations (5), (6), (8) can be rewritten in the Dirac-like form

$$\widetilde{i}\widetilde{\gamma}^{\mu}\partial_{\mu}\mathcal{E}(x) = 0, \quad \mathcal{E} = V\Psi^{III}. \tag{15}$$

Even from the explicit form of equations (5), (6), (8), (15), one can suppose that Eqs.(6) and (8) are equations for a vector field, Eqs.(5) are those for the system of tensor and scalar field and Eqs.(15) are for a spinor field. From the electrodynamical point of view, one can interpret Eqs.(5) as the Maxwell equations with fixed gradient-like sources. However, before going from assumptions to assertions, we must investigate the transformation properties of the object $\mathcal{E}$ and symmetry properties of equations (5), (6), (8), (15). The mathematically well-defined assertion that substitutions (4) transform the Dirac equation into the Maxwell equations (5) for the electromagnetic and scalar fields is impossible without the proof that the object $\mathcal{E}$ can be transformed as electromagnetic and scalar fields, i.e., we need the additional arguments in order to have the possibility to interpret the real and imaginary parts of spinor components from (4) as the components of electromagnetic and scalar fields – such arguments can be taken from the symmetry analysis of the corresponding equations.

3. Symmetries

For the infinitesimal transformations and generators of the conformal group $C(1,3) \supset P$, we use the notations:

$$f(x) \to f'(x) \stackrel{i}{=} \left(1 - a^{\rho}\partial_{\rho} - \frac{1}{2}\omega^{\rho\sigma}\widehat{j}_{\rho\sigma} - \kappa\widehat{d} - b^{\rho}\widehat{k}_{\rho}\right) f(x) \tag{16}$$

$$\partial_{\rho} \equiv \frac{\partial}{\partial x^{\rho}}, \quad \widehat{j}_{\rho\sigma} = M_{\rho\sigma} + S_{\rho\sigma}, \quad \widehat{d} = d + \tau = x^{\mu}\partial_{\mu} + \tau,$$
$$\widehat{k}_{\rho} = k_{\rho} + 2S_{\rho\sigma}x^{\sigma} - 2\tau x_{\rho} \equiv 2x_{\rho}\widehat{d} - x^2\partial_{\rho} + 2S_{\rho\sigma}x^{\sigma}, \tag{17}$$

The generators (17) obey the commutation relations

$$\begin{aligned}
&[\partial_\mu, \partial_\nu] = 0, \quad \left[\partial_\mu, \widehat{j}_{\nu\sigma}\right] = g_{\mu\nu}\partial_\sigma - g_{\mu\sigma}\partial_\nu, && \text{(a)}\\
&\left[\widehat{j}_{\mu\nu}, \widehat{j}_{\lambda\sigma}\right] = -g_{\mu\lambda}\widehat{j}_{\nu\sigma} - g_{\nu\sigma}\widehat{j}_{\mu\lambda} + g_{\mu\sigma}\widehat{j}_{\nu\lambda} + g_{\nu\lambda}\widehat{j}_{\mu\sigma}, && \text{(b)}\\
&\left[\partial_\mu, \widehat{d}\right] = \partial_\mu, \quad \left[\partial_\rho, \widehat{k}_\sigma\right] = 2(g_{\rho\sigma}\widehat{d} - \widehat{j}_{\rho\sigma}), \quad \left[\partial_\mu, \widehat{j}_{\mu\sigma}\right] = 0, \\
&\left[\hat{k}_\rho, \hat{j}_{\sigma\nu}\right] = g_{\rho\sigma} \hat{k}_\nu - g_{\rho\nu} \hat{k}_\sigma, \quad \left[\hat{d}, \hat{k}_\rho\right] = \hat{k}_\rho, \quad \left[\hat{k}_\rho, \hat{k}_\sigma\right] = 0.
\end{aligned} \qquad (18)$$

with an arbitrary number τ (the conformal number) and $S_{\rho\sigma}$ matrices being the generators of the 4-dimensional representation of the Lorentz group $SL(2,C)$, e.g., for symmetries of the Dirac (m=0) equation, the conformal number $\tau = 3/2$ and 4×4 $S_{\rho\sigma}$ matrices are chosen in the form

$$S^I_{\rho\sigma} \equiv -\frac{1}{4}\left[\gamma_\rho, \gamma_\sigma\right] \in D\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right), \qquad (19)$$

where $S^I_{\rho\sigma}$ are generators of the spinor representation $D\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$ of the $SL(2,C)$ group. Let us define the following six operators from the Pauli-Gursey-Ibragimov symmetry [22]:

$$S^{II}_{\rho\sigma} = -S^{II}_{\sigma\rho}: \quad \left\{ \begin{array}{ccc} S^{II}_{01} = \frac{i}{2}\gamma^2 C, & S^{II}_{02} = -\frac{1}{2}\gamma^2 C, & S^{II}_{03} = -\frac{i}{2}\gamma^4, \\ S^{II}_{12} = -\frac{i}{2}, & S^{II}_{31} = -\frac{1}{2}\gamma^2\gamma^4 C, & S^{II}_{23} = \frac{i}{2}\gamma^2\gamma^4 C \end{array} \right\}. \qquad (20)$$

It is easy to verify that operators (20) obey same commutation relations as generators (19) and, as a consequence, they form an another realization of the same spinor representation $D\left(0, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0\right)$ of the $SL(2,C)$ group. But, in contradiction to operators (19), they are themselves (without any differential angular momentum part) the symmetry operators of the massless Dirac equation (1), i.e., they leave this equation being invariant.

We prefer to use the Dirac-like form (15) of the Maxwell equations with gradient-like sources for the symmetry analysis. The operator equality $V\widehat{Q}_\psi V^\dagger = q_E$ allows one to find the connections between the symmetries of the Dirac equation (1) and those of equation (15) for the field $\mathcal{E} = (\mathcal{E}^\mu)$. It was shown in [15, 16] that the massless Dirac equation is invariant (in addition to the standard spinor representation) with respect to two bosonic representations of Poincaré group being generated by the $D\left(\frac{1}{2}, \frac{1}{2}\right)$ and $D(1,0) \oplus (0,0)$ representations of the Lorentz group (for the 8-component form of the Dirac equation, the similar representations of Poincar'e group being generated by the reducible $D\left(\frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, \frac{1}{2}\right)$ and $D(1,0) \oplus (0,0) \oplus (0,0) \oplus (0,1)$ representations of the Lorentz group were found in [13]). Of course, Eqs.(15) for the field $\mathcal{E}$ (the Maxwell equations with gradient-like sources) are also invariant with respect to these three different representations of the Poincar'e group that is evident due to the unitary connection (9) between the fields ψ and $\mathcal{E}$. Let us prove this fact directly.

Let us write down the explicit form of the $S^I_{\rho\sigma}$ and $S^{II}_{\rho\sigma}$ operators (19) and (20) in the B-representation

$$\widetilde{S}^I_{\rho\sigma} = -\frac{1}{4}\left[\widetilde{\gamma}_\rho, \widetilde{\gamma}_\sigma\right]: \quad \widetilde{S}^I_{jk} = -i\varepsilon^{jkl}\widetilde{S}^I_{0l}, \quad \widetilde{S}^I_{0l} = -\frac{1}{2}\widetilde{\gamma}^{0l}, \qquad (21)$$

$$\widetilde{S}^{II}_{\rho\sigma} \equiv V S^{II}_{\rho\sigma} V^{\dagger}: \quad \widetilde{S}^{II}_{jk} = -i\varepsilon^{jkl}\widetilde{S}^{II}_{0l}, \quad \widetilde{S}^{II}_{01} = \frac{1}{2}\begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix},$$
$$\widetilde{S}^{II}_{02} = \frac{1}{2}\begin{vmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -1 \\ i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{II}_{03} = \frac{1}{2}\begin{vmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \tag{22}$$

We consider in addition to the sets of generators (19), (20) the following two sets of matrices $S_{\rho\sigma}$:

$$S^{III}_{0k} = S^{I}_{0k} - S^{II}_{0k}, \quad S^{III}_{mn} = S^{I}_{mn} + S^{II}_{mn}, \tag{23}$$

$$S^{IV}_{\rho\sigma} = S^{I}_{\rho\sigma} + S^{II}_{\rho\sigma}. \tag{24}$$

Theorem 1. *The commutation relations (18b) of the Lorentz group are valid for each set $S^{I-IV}_{\rho\sigma}$ of $S_{\rho\sigma}$ matrices. Sets (19), (20) (or (21), (22)) are the generators of the same (spinor) representation $D\left(0,\frac{1}{2}\right)\oplus\left(\frac{1}{2},0\right)$ of the $SL(2,C)$ group, set (23) consists of generators of the $D(0,1)\oplus(0,0)$ representation and set (24) consists of the generators of the irreducible vector $D\left(\frac{1}{2},\frac{1}{2}\right)$ representation of the same group.*

Proof. The fact that matrices (19) are the generators of the spinor representation of the $SL(2,C)$ group is well known (for matrices (21) this fact is a consequence of the operator equality $V\widehat{Q}_{\psi}V^{\dagger} = q_E$ which unitarily connects operators in the PD- and B-representations). It is better to fulfil the proof of nontrivial assertions of Theorem 1 in the B-representation where their validity can be seen directly from the explicit form of the operators $S_{\rho\sigma}$ even without the Casimir operators calculations. In fact, using the explicit forms of matrices (21), (22), we find

$$\widetilde{S}^{II}_{\rho\sigma} = C\widetilde{S}^{I}_{\rho\sigma}C \Longleftrightarrow \widetilde{S}^{I}_{\rho\sigma} = C\widetilde{S}^{II}_{\rho\sigma}C, \tag{25}$$

$$\widetilde{S}^{IV}_{01} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{IV}_{02} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}, \quad \widetilde{S}^{IV}_{03} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \tag{26}$$

$$\widetilde{S}^{IV}_{mn} = \begin{vmatrix} s_{mn} & 0 \\ 0 & 0 \end{vmatrix} = \widetilde{S}^{III}_{mn}; \quad \widetilde{S}^{III}_{0k} = \begin{vmatrix} s_{0k} & 0 \\ 0 & 0 \end{vmatrix}, \tag{27}$$

where

$$s_{0k} = \frac{i}{2}\varepsilon^{kmn}s_{mn} = -s_{k0}, \quad s_{12} = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix},$$
$$s_{23} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}, \quad s_{31} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}. \tag{28}$$

The unitarity of the C operator in relations (25), the direct calculation of correspondings commutators and the Casimir operators $S_{\pm}$ of the $SL(2,C)$ group

$$S^2_{\pm} = \frac{1}{2}(\tau_1 \pm i\tau_2), \quad \tau_1 \equiv -\frac{1}{2}S^{\mu\nu}S_{\mu\nu}, \quad \tau_2 \equiv -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}S_{\mu\nu}S_{\rho\sigma} \tag{29}$$

accomplish the proof of the theorem. ■

It is interesting to mark the following. Despite the fact that the matrices $\widetilde{S}^I_{\rho\sigma}$ and $\widetilde{S}^{II}_{\rho\sigma}$ are unitarily interconnected according to formulae (25), which in the PD-representation have the form

$$S^I_{\rho\sigma} = \hat{C}\, S^{II}_{\rho\sigma}\, \hat{C}, \quad \hat{C} \equiv V^{\dagger}CV = \begin{vmatrix} C & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{30}$$

the matrices $\widetilde{S}^I_{\rho\sigma}$ (19) (or (21)), as well as the matrices (23), (24) (or (26), (27)), in contradiction to the matrix operators (20) (or (22)) being taken themselves, are not the invariance transformations of equation (1) (or (15)). It is evident because the C operator does not commute (or anticommute) with the Diracian $\gamma^{\mu}\partial_{\mu}$. Nevertheless, due to the validity of relations

$$\left[S^{II}_{\mu\nu}, S^{I,III,IV}_{\rho\sigma}\right] = 0, \quad \mu,\nu,\rho,\sigma = (0,1,2,3), \tag{31}$$

not only the generators $(\partial, \widehat{j}^I)$ of the well-known spinor representation P^S of the Poincar'e group, but also the following generators

$$j^{III,IV}_{\rho\sigma} = M_{\rho\sigma} + S^{III,IV}_{\rho\sigma}. \tag{32}$$

are the transformations of invariance of equation (1) (or (15)). It means the validity of the following assertion.

Theorem 2. *The Maxwell equations with gradient-like sources are invariant with respect to the three different local representations P^S P^T P^V of the Poincar'e group P given by the formula*

$$\Psi(x) \longrightarrow \Psi^{'}(x) = F^{I-III}(\omega)\Psi(\Lambda^{-1}(x-a)), \tag{33}$$

where

$$\begin{aligned} &F^I(\omega) \in D\left(0,\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2},0\right), \quad (P = P^S), \\ &F^{II}(\omega) \in D\left(0,1\right) \oplus \left(0,0\right), \quad (P = P^{Ts}), \\ &F^{III}(\omega) \in D\left(\tfrac{1}{2},\tfrac{1}{2}\right) \quad (P = P^V). \end{aligned} \tag{34}$$

Proof of the theorem for equations (15) follows from Theorem 1 and the above-mentioned consideration. It is only a small technical problem to obtain the explicit form of corresponding symmetry operators for the form (5) of these equations, having their explicit form for equations (15). ■

It is easy to construct the corresponding local $C(1,3)$ representations of the conformal group, i.e., C^S, C^T and C^V, but only one of them (the well-known local spinor representation C^S) gives the transformations of invariance of the massless Dirac equation and, therefore, of the Maxwell equations with gradient-like sources.

The simplest of Lie-Bäcklund symmetries are the transformations of invariance generated by the first-order differential operators with matrix coefficients. The operators of the Maxwell and Dirac equations also belong to this class of operators. In order to complete the present consideration, we shall recall briefly our result [14, 15].

Theorem 3. *The 128-dimensional algebra A_{128}, whose generators are*

$$Q_a = (\partial, \widehat{j}^I, \widehat{d}^I, \widehat{k}^I), \quad Q_b = (\gamma^2 C, i\gamma^2 C, \gamma^2\gamma^4 C, i\gamma^2\gamma^4 C, i, i\gamma^4, \gamma^4, I), \tag{35}$$

where $\gamma^4 \equiv \gamma^0\gamma^1\gamma^2\gamma^3$, and all their compositions Q_aQ_b, $a = (1, 2, 3, ..., 15)$, $b = (1, ..., 8)$, is the simplest algebra of invariance of the Maxwell equations with gradient-like sources in the class of first-order differential operators with matrix coefficients.

Proof. For the details of the proof via the symmetries of Eqs.(15) see [14, 15]. ■

In the class of nonlocal operators, we are able to represent here a new result. In spite of the fact that the above-mentioned local representations C^T and C^V are not the symmetries of equation (15), the corresponding symmetries can be constructed in the class of nonlocal operators.

Theorem 4. *The Maxwell equations with gradient-like sources (15) (or (5)) are invariant with respect to the representations $\widetilde{C}^T$, $\widetilde{C}^V$ of the conformal group $C(1,3)$. The corresponding generators are $(\partial, \widehat{j}^{III,IV})$ together with the following nonlocal operators*

$$\begin{gathered} d^{III,IV} = \frac{1}{2}\left\{\frac{\partial_0, \partial_k}{\Delta}, j_{0k}^{III,IV}\right\}, \\ k_0^{III,IV} = \frac{1}{2}\left\{\frac{\partial_0}{\Delta}, (j_{0k}^{III,IV})^2 + \frac{1}{2}\right\}, \quad k_m^{III,IV} = \left[k_0^{III,IV}, j_{0m}^{III,IV}\right], \end{gathered} \tag{36}$$

where $\Delta = \partial_k^2$.

Proof. The validity of this theorem follows from the above-mentioned Theorem 2 and Theorem 4 in [23]. ■

4. Conclusions

We prove that the object $\mathcal{E}$ of Eqs.(8), (15) can be interpreted as either (i) the spinor field, or (ii) the complex vector field, or (iii) the tensor-scalar field. Moreover, each of equations (5), (6), (8), (15) can be interpreted as either (i) the Maxwell equations with an arbitrary fixed gradient-like 4-current $j_\mu = \partial_\mu \mathcal{E}^0(x)$, being determined by a scalar function $\mathcal{E}^0(x)$, or (ii) the Dirac equation in the bosonic representation, or (iii) the Maxwell equations for the tensor-scalar field, or (iV) the equations for the complex vector field.

Let us underline that the Maxwell equations with gradient-like sources (5), (6), (8), (15) are the kind of Maxwell equations with maximally wide possible symmetry properties – they have both Fermi and Bose symmetries and can describe both fermions and bosons, see their corresponding quantization in [18].

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 483–487.

One Geometric Model for Non-local Transformations

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Abstract

A geometric formulation of an important class of non-local transformations is presented.

The contemporary geometric theory of Bäcklund transformations (BT) [1–4] is based on certain integrability conditions for a system of differential equations which are determine it. These equations are written in terms of connections in the submanifold of a jet-bundle foliation. One of the first nontrivial examples of BT in three dimensions was described in paper [1]. In this paper, a geometric formulation of an important class of non-local transformations is presented. The generalization is obtained due to the use of non-local transformations of dependent variables and integrability conditions of a more general type.

1. In the fibered submanifold (E^j, M', ρ') with the k-jet bundle $J^1(M'N')$, we set an equation of order $t = 2 \le l$:

$$L_2^q\left(y, v, \underset{1}{v}, \underset{2}{v}\right) = 0,$$
$$v = \left\{v^B\right\}, \quad M' = R\left(1, \ldots, n-1\right), \quad \left(q = 1, \ldots, m',\ B = 1, \ldots, m'\right). \tag{1}$$

We use here such notations:

$$\partial_\mu u = \frac{\partial u}{\partial x_\mu}, \qquad \{x_\mu\} = (x_0, x_1, \ldots, x_{n-1}).$$

Let now equation (1) be written in the form of n-th order exterior differential forms (n-forms)

$$\alpha^c = \frac{1}{n!}\alpha^c_{\mu_0\cdots\mu_{n-1}}(y, v)dy^{\mu_0} \wedge dy^{\mu_1} \wedge \ldots \wedge dy^{\mu_{n-1}}, \quad (c = 1, \ldots, r). \tag{2}$$

When this system of forms α^c is equal to zero, then (1) is fulfiled. So, the system of $\alpha^c = 0$ forms generates an ideal $I = \{\alpha^c\}$. The condition for system (2) to be closed is $d\alpha^c \subset I$. Let now consider 1-forms of connections which generate the standard basis of contact forms

$$\omega^A = du^A - H^A_\mu\left(x, v, \underset{1}{v}; u\right) dx^\mu, \qquad \left(\mu = 0, \ldots, n-1,\ A = 1, \ldots, m\right). \tag{3}$$

Let

$$x^\mu = y^\mu, \qquad (\mu = 0, \ldots, n-1).$$

The differential prolongation of (3) gives us such a system of contact forms:

$$\begin{aligned} \omega^A_{\mu_1} &= du^A_{\mu_1} - H^A_{\mu_1\mu}\left(x, v, \underset{1}{v}, \underset{2}{v}; u, \underset{1}{u}\right) dx^\mu, \ldots, \\ \omega^A_{\mu_1\ldots\mu_k} &= du^A_{\mu_1\ldots\mu_k\mu} - H^A_{\mu_1\ldots\mu_k\mu}\left(x, v, \underset{1}{v}, \underset{2}{v}; u, \ldots, \underset{k}{u}\right) dx^\mu. \end{aligned} \tag{4}$$

Let us construct the set of $(n-1)$-forms

$$\Omega^B = \beta^B_A \wedge \omega^A, \quad (B = 1, \ldots, m), \tag{5}$$

where β^B_A are some $(n-2)$-forms of the type

$$\beta^B_A = \frac{1}{(n-2)!}\beta^B_{A\mu_1\ldots\mu_{n-2}}(x,u)dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_{n-2}}.$$

An extended ideal I' is obtained by adding forms (5) to I:

$$I' = \{\alpha^A, \Omega^B\}.$$

Demand Ω^B be closed:

$$d\Omega^B \subset I'. \tag{6}$$

This condition is given by

$$d\Omega^B = \alpha^A f^B_A + \theta \wedge \Omega^B, \tag{7}$$

where θ is some 1-form such as, for example,

$$\theta = \theta_\mu dx^\mu, \qquad \theta_\mu = \text{const}.$$

Equation (7) is the basic for determining the functions $H^A_{\mu_1\ldots\mu_k\mu}$ from connections (3) and for finding β^B_A and f^B_A coefficients . In [1], $\beta^B_{A\mu_1\ldots\mu_{n-2}}$ satisfy such commuting conditions, from which it follows

$$\begin{aligned} I^{0A} &= -\beta^{0A}_c H^c_2 + \beta^{2A}_c H^c_1, \\ I^{1A} &= -\beta^{1A}_c H^c_1 + \beta^{0A}_c H^c_0, \\ I^{2A} &= -\beta^{2A}_c H^c_0 + \beta^{1A}_c H^c_2. \end{aligned} \tag{8}$$

It makes us possible to write the equation $L^q_2(x,v) = 0$ in the form of the conservation law

$$\partial_\mu I^{\mu A} = 0. \tag{9}$$

H^c_k have a special structure, which allows us to write

$$I^{\mu A} = X_a(x,v)[v^A] \cdot f^{\mu a}\left(x, u, \underset{1}{u}\right), \qquad (\mu = 0,1,2;\ a = 1, \ldots, r'), \tag{10}$$

and then obtain the incomplete Lie algebra

$$X_a(x,v) = X^c_a(x,v)\partial_{v^c}. \tag{11}$$

Additional conditions on X_a allow it to become a complete Lie algebra.

2. In the space $E^1(y,v)$, let an invariance Lie algebra of infinitesimal operators for the equation $L_2^q(x,v)=0$ be given:

$$X_a = \xi_a^\mu(y,v)\partial_\mu + \eta_a^B(y,v)\partial_{v^B}, \quad [X_a, X_b] = \lambda_{ab}^c X_c, \ (a=1,\ldots,r;\ B=1,\ldots,m) \tag{12}$$

and the projection $\rho' : E^1 \to M'$ in a fibre bundle(E', M', ρ') to each operator X_a set the corresponding shorted operator

$$X_a^{\rho'} = \xi_a^\mu(y,v)\partial_\mu. \tag{13}$$

Operator (12) acting in E^1 is closely connected with an operator, which acts in M' [5]

$$Q_a(y,v) = \xi_a^\mu(y,v)\partial_\mu - \eta_a^*(y,v), \quad \eta_a^* v^B = \eta_a^B. \tag{14}$$

Determine now Q_a-invariant solutions v^{inv} of the equation $L_1^q(y,v)=0$:

$$Q_a(y,v)[v^B] = \xi_a^\mu(y,v)\partial_\mu v^B - \eta_a^B(y,v) = 0. \tag{15}$$

If $u^A(x) \neq v^B(y)$, then it follows from (15) that

$$Q_a(y,v)[u^A] = \Gamma_{ac}^A\left(y, v, \underset{1}{v}, \ldots\right) u^c \neq 0. \tag{15a}$$

In more general case, it is

$$Q_a(y,v)[u^A] = \Gamma_a^A\left(y, v, \underset{1}{v}, \ldots; u\right) \neq 0. \tag{15b}$$

Let now consider 1-forms of connections (3)

$$\omega^A = (\partial_{x^\mu} u^A)dx^\mu - H_\mu^A\left(x, v, \underset{1}{v}; u\right) dx^\mu. \tag{16}$$

Interior product of ω^A and the vector field

$$W_\nu = D_{y^\nu} h^\mu \partial_{x^\mu}, \tag{17}$$

where

$$\|W\| = \|Dh\| \cdot \left\|\underset{1}{\partial}\right\|, \qquad \underset{1}{\partial} \equiv \|\partial_0, \partial_1, \ldots, \partial_{n-1}\|^T,$$

is obtained [2–4] in the form

$$W_\nu \lrcorner \omega^A = D_{y^\nu} h^\mu \partial_{x^\mu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A. \tag{18}$$

Here, we use the notation

$$D_{y^\nu} h^\mu \partial_{x^\mu}|_M = \partial_{y^\nu}|_{M'}. \tag{19}$$

So

$$W_\nu \lrcorner \omega^A = \partial_{y^\nu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A. \tag{20}$$

For each fixed ν, we multiply the scalar equation (20) by a 1-form dy^ν and then find the sum

$$\tilde{\omega}^A = \left[\partial_{y^\nu} u^A - D_{y^\nu} h^\mu \cdot H_\mu^A\right] dy^\nu = du^A - D_{y^\nu} h^\mu \cdot H_\mu^A \cdot dy^\mu. \tag{21}$$

An interior product of the vector field (13) and the form (21) is

$$X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A = \xi_a^\nu(y,v)\partial_{y^\nu} u^A - \eta_a^*(y,v)u^A - \left[\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A\right]. \quad (22)$$

Let us set

$$\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A \equiv \Gamma_{ac}^A\left(y,v,\underset{1}{v},\ldots\right)u^c. \quad (23)$$

In a more general case, we have

$$\xi_a^\nu(y,v)D_{y^\nu}h^\mu \cdot H_\mu^A - \eta_a^*(y,v)u^A \equiv \Gamma_a^A\left(y,v,\underset{1}{v},\ldots,u\right). \quad (24)$$

In new notations, (22) will have the form

$$X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A = Q_a(y,v)[u^A] - \Gamma_a^A\left(y,v,\underset{1}{v},\ldots,u\right). \quad (25)$$

When (25) is equal to zero, it determines a linear connection along the vector field $X_a^{\rho'}$.

We calculate now the exterior differential of scalar (25) and mark a result as $\overset{\sim}{\omega}^A$. The interior product of this form and the vector field $X_a^{\rho'}$ is

$$X_b^{\rho'} \lrcorner \overset{\sim}{\omega}^A = \xi_b^\mu(y,v)\partial_{y^\mu}\left(X_a^{\rho'} \lrcorner \overset{\sim}{\omega}^A\right) = \xi_b^\mu(y,v)\partial_{y^\mu}(Q_a[u^A]) - \xi_b^\mu(y,v)\partial_{y^\mu}\Gamma_a^A =$$
$$= \left(\xi_b^\mu(y,v)\partial_{y^\mu} - \eta_b\right)Q_a[u^A] + \eta_b Q_a[u^A] - \xi_b^\mu(y,v)\partial_{y^\mu}\Gamma_a^A = Q_bQ_a[u^A] - Q_b\Gamma_a^A, \quad (26)$$
$$\left(X_b^{\rho'} \lrcorner \overset{\sim}{\omega}^A = 0\right).$$

With (26), let construct an equality

$$\{[Q_a,Q_b] - \lambda_{ab}^c Q_c\}\,u^A = \widetilde{Q}_{[a}\,\Gamma_{b]}^A + \Gamma_{[a|u^c|}^A\Gamma_{b]}^c - \lambda_{ab}^c\Gamma_c^A\,|_{L_2(y,v)} = 0. \quad (27)$$

Here, $\widetilde{Q}_a$ is the projection of the operator Q_a on E' (not differentiate with respect to u-variables)

$$\Gamma_{au^c}^A \equiv \partial_{u^c}\Gamma_a^A.$$

If the connection is linear, equation (27) is of the form

$$\{Q_{[a}\Gamma_{b]c}^A + \Gamma_{[a|K|}^A\Gamma_{b]c}^K - \lambda_{ab}^p\Gamma_{pc}^A\}u^c\,|_{L_2^q} = 0. \quad \text{(27a)}$$

So the equation, which determines the reducing of a non-local transformation system to the equation $L_2^q(y,v) = 0$, is ($X_a^{\rho'} \equiv X_a'$):

$$*\left\{\underset{X'_{[a}}{L}\,\underset{X'_{b]}}{L}\,u^A - \lambda_{ab}^c\,\underset{X'_c}{L}\,u^A\right\} \subset I'. \quad (28)$$

Here, we use the notation [2, 3]:

$$\underset{X}{L}\,u = X \lrcorner du, \qquad \Gamma = \left\{\alpha^c, X \lrcorner \overset{\sim}{\omega}^A\right\}.$$

The system of exterior differential equations $\alpha^c = 0$ gives us the representation of $L_2^q(y, v) = 0$. Condition (28) shows that

$$*\left\{ X^{\rho'}_{[a} \lrcorner d(X^{\rho'}_{b]} \lrcorner \widetilde{\omega}^A) - \lambda^c_{ab}(X^{\rho'}_c \lrcorner \widetilde{\omega}^A) \right\} - f^A_c \cdot \alpha^c - *g^{As}_c \left(X^{\rho'}_s \lrcorner \widetilde{\omega}^A \right) = 0. \qquad (28a)$$

$*$ is the Hodge operator in $R(0, n)$.

Equation (28a) can be represented in terms of covariant derivatives ∇_{X_a} along the vector field X_a. If connections are linear, the covariant and Lie derivatives are identical. So, from

$$\{Q_{[a}\Gamma^A_{b]c} + \Gamma^A_{[a|K|}\Gamma^K_{b]c} - \lambda^p_{ab}\Gamma^A_{pc}\}u^c \mid_{L_2^q} = 0,$$

we obtain

$$\left\{ \left[\nabla_{X^\rho_a}, \nabla_{X^\rho_b} \right] + \nabla_{[X_a, X_b]^\rho} - R(X_a, X_b) \right\} u^c = 0. \qquad (29)$$

Here,

$$R(X_a, X_b)u^c = \lambda^p_{ab}\Gamma^A_{pc}u^c \qquad (30)$$

is the tensor field of curvature of the corresponding connections [4].

Theorem. *The non-local transformation represented by system (25) via integrability conditions (27) for variables u^A has, as a consequence, the equation $L_2^q(y, v) = 0$ when (28a) is fulfiled.*

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 488–491.

On One Problem of Automatic Control with Turning Points

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Abstract

An algorithm for solutions of one problem of automatic control is suggested for the case of an interval containing turning points.

Let us consider an equation of the form

$$\varepsilon^2\frac{d^2x(t,\varepsilon)}{dt^2}+\varepsilon a(\tau,\varepsilon)\frac{dx(t,\varepsilon)}{dt}+b(\tau,\varepsilon)x(t,\varepsilon)=\varepsilon h(\tau)\int\limits_{-\infty}^{t}G(t-t',\tau')x(t',\varepsilon)dt', \qquad (1)$$

where ε is a small parameter, $\varepsilon\in(0;\varepsilon_0]$, $\varepsilon_0\le 1$, $\tau=\varepsilon t$, $t\in[0,L]$. The left-hand side of the equation defines automatic control, where $G(t-t',\tau')$ is the impulse transfer function of a regulator. Functions $a(\tau,\varepsilon)$, $b(\tau,\varepsilon)$ admit the following decomposition:

$$a(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i a_i(\tau), \qquad b(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i b_i(\tau). \qquad (2)$$

In papers [1–2], automatic control systems described by equation (1) were studied in detail, but turning points were not considered. It was assumed that the characteristic equation, i.e., the equation

$$\lambda_0^2(\tau)+a_0(\tau)\lambda_0(\tau)+b_0(\tau)=0, \qquad (3)$$

has two roots for all $\tau\in[0,\varepsilon L]$.

For the case where equation (1) is considered on the interval $[\alpha,\beta]$, which does not contain any turning points, the following theorem is true:

Theorem 1. *If functions $a_i(\tau)$, $b_i(\tau)$ are continuous and differentiable infinite number of times on $[\varepsilon\alpha,\varepsilon\beta]$ and $a_0^2(\tau)-4b_0(\tau)\ne 0$, then, on the interval $[\alpha,\beta]$, equation (1) has two linearly independent formal solutions of the form*

$$x_1(t,\varepsilon)=\exp\left(\frac{1}{\varepsilon}\int\limits_{\alpha}^{t}\varphi_1(\tau',\varepsilon)dt'\right), \qquad \varphi_1(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\varphi_{1i}(\tau), \qquad (4)$$

$$x_2(t,\varepsilon)=\exp\left(\frac{1}{\varepsilon}\int\limits_{\alpha}^{t}\varphi_2(\tau',\varepsilon)dt'\right), \qquad \varphi_2(\tau,\varepsilon)=\sum_{i=0}^{\infty}\varepsilon^i\varphi_{2i}(\tau), \qquad (5)$$

where $\varphi_{1i}(\tau)$, $\varphi_{2i}(\tau)$ *are continuous and infinitely differentiable on* $[\varepsilon\alpha, \varepsilon\beta]$.

An algorithm for determination of functions $\varphi_{1i}(\tau)$, $\varphi_{2i}(\tau)$ is adduced in [2].

Along with equation (1), we consider also the following auxiliary equation

$$\frac{d^2 z(\xi,\mu)}{d\xi^2} + \widetilde{a}(\eta,\mu)\frac{dz(\xi,\mu)}{d\xi} + \widetilde{b}(\eta,\mu)z(\xi,\mu) = = \mu\widetilde{h}(\eta,\mu)\int\limits_{-\infty}^{\xi} G\left((\xi-\xi')\mu, \mu^2 t_0 + \mu\eta'\right) z(\xi',\mu)d\xi', \tag{6}$$

where $\xi \in [\alpha;\beta]$, $\mu \in (0;\mu_0)$, $\eta = \xi\mu^2$, $t_0 \in [\alpha\mu;\beta\mu]$, and functions $\widetilde{a}(\eta,\mu)$, $\widetilde{b}(\eta,\mu)$, $\widetilde{h}(\eta,\mu)$ admit the following decomposititon:

$$\widetilde{a}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\widetilde{a}_i(\eta), \qquad \widetilde{b}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\widetilde{b}_i(\eta), \qquad \widetilde{h}(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i h_i(\eta).$$

Theorem 2. *If functions* $\widetilde{a}_i(\eta)$, $\widetilde{b}_i(\eta)$, $\widetilde{h}_i(\eta)$ *are infinitely differentiable on the interval* $[\alpha\mu^2\beta\mu^2]$ *and* $\widetilde{a}_0^2(\eta) - 4\widetilde{b}_0(\eta) \neq 0$, *then, on the interval* $[\alpha,\beta]$, *equation (6) has two linearly independent formal solutions of the form*

$$z_1(\xi,\mu) = \exp\left(\int\limits_{\alpha}^{\xi}\lambda_1(\eta',\mu)d\xi'\right), \qquad \lambda_1(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\lambda_{1i}(\eta),$$

$$z_2(\xi,\mu) = \exp\left(\int\limits_{\alpha}^{\xi}\lambda_2(\eta',\mu)d\xi'\right), \qquad \lambda_2(\eta,\mu) = \sum_{i=0}^{\infty}\mu^i\lambda_{2i}(\eta),$$

where $\lambda_{1i}(\tau)$, $\lambda_{2i}(\tau)$ *are infinitely differentiable functions on* $[\mu^2\alpha, \mu^2\beta]$.

The proof of this theorem is similar to that of Theorem 1.

Let us go back to the consideration of equation (1) under condition of the existence of a zero turning point. Let functions $a_0(\tau)$, $b_0(\tau)$ have the form $a_0 = \tau a_{10}(\tau)$, $b_0 = \tau^2 b_{10}(\tau)$. Then roots of the characteristic equation have the form

$$\lambda_{1,2}(\tau) = -\frac{1}{2}\tau\left(a_{10}(\tau) \pm \sqrt{a_{10}^2(\tau) - 4b_{10}(\tau)}\right). \tag{7}$$

If $a_{10}^2(\tau) - 4b_{10}(\tau) \neq 0$ $\forall t \in (0;L]$, then $t = 0$ is a zero turning point.

Let us consider the Cauchy problem for equation (1) with the initial conditions

$$x(\beta\sqrt{\varepsilon},\varepsilon) = y_{10}, \qquad \left.\frac{dx}{dt}\right|_{t=\beta\sqrt{\varepsilon}} = y_{20}, \tag{8}$$

where β is a real number, y_{10} and y_{20} are quantities which do not depend on ε.

First using Theorem 1 for the case $t \in [\beta\sqrt{\varepsilon}, L]$, we construct the m-approximation for a general solution of equation (1) using the formula

$$x_m^{(2)}(t,\varepsilon) = a_1^{(2)}(\varepsilon)x_{1m}(t,\varepsilon) + a_2^{(2)}(\varepsilon)x_{2m}(\varepsilon),$$

where

$$x_m^{(2)}(t,\varepsilon) = \exp\left(\frac{1}{\varepsilon}\int\limits_{\beta\sqrt{\varepsilon}}^{t}\left(\sum_{i=0}^{m}\varepsilon^i\varphi_{ki}(\tau)\right)dt'\right), \qquad k=1,2,$$

$a_k^{(2)}(\varepsilon)$, $k=1,2$ are arbitrary constants which are determined from condition (8).

Finally we construct a solution of equation (1) within the vicinity of a turning point or when $t \in [0, \beta\sqrt{\varepsilon}]$.

After the substitution $t = \xi\sqrt{\varepsilon}$ ($\xi \in [0,\beta)$) using the notations $\mu = \sqrt{\varepsilon}$, $\xi\mu^2 = \eta$, $x = z(\xi,\mu)$ equation (1) takes the form

$$\begin{aligned}\frac{d^2 z(\xi,\mu)}{d\xi^2} + \eta a_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu)\frac{dz(\xi,\mu)}{d\xi} + \\ + \left(\eta^2 b_{10}(\eta\mu) + \sum_{i=1}^{\infty} b_i(\eta\mu)\mu^{2i-2}\right) z(\xi,\mu) = \\ = \mu h(\eta\mu)\int\limits_{-\infty}^{\xi} G\left((\xi-\xi')\mu, \eta'\mu\right) z(\xi',\mu)d\xi'.\end{aligned} \tag{9}$$

Let us assume that the coefficients $a_{10}(\tau)$, $b_{10}(\tau)$, $a_i(\tau), b_i(\tau)$, $i=1,\ldots,8$, and function $h(\tau)$ can be decomposed into a Taylor series within the neighbourhood of the point $\tau = 0$. Then

$$a_{10}(\eta\mu) = a_{10}(0) + \sum_{i=1}^{\infty}\mu^i\frac{a_{10}^{(i)}(0)}{i!}\eta^i, \qquad a_i(\eta\mu) = a_i(0) + \sum_{j=1}^{\infty}\mu^j\frac{a_i^{(j)}(0)}{j!}\eta^j,$$

$$\sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu) = \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\mu^{2i+j-1}\frac{a_i^{(j)}(0)}{j!}\eta^j = \sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta),$$

whence

$$\eta a_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-1}a_i(\eta\mu) = \eta a_{10}(0) + \sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta). \tag{10}$$

Similarly, we get

$$\eta^2 b_{10}(\eta\mu) + \sum_{i=1}^{\infty}\mu^{2i-2}b_i(\eta\mu) = b_{10}(0)\eta^2 + \sum_{i=1}^{\infty}\mu^i\widetilde{b}_i(\eta) + b_1(0). \tag{11}$$

Finally, for $h(\eta\mu)$, we have

$$h(\eta\mu) = \sum_{i=1}^{\infty}\frac{h^{(i)}(0)}{i!}(\eta\mu)^i = \sum_{i=1}^{\infty}\mu^i\widetilde{h}_i(\eta). \tag{12}$$

Substituting (10)–(12) into (9), we get

$$\frac{d^2z(\xi,\mu)}{d\xi^2}+\left(\eta a_{10}(0)+\sum_{i=1}^{\infty}\mu^i\widetilde{a}_i(\eta)\right)\frac{dz(\xi,\mu)}{d\xi}+ \\ +\left(b_{10}(0)\eta^2+b_1(0)+\sum_{i=1}^{\infty}\mu^i\widetilde{b}_i(\eta)\right)z(\xi,\mu)= \\ =\mu\sum_{i=1}^{\infty}\mu^i\widetilde{h}_i(\eta)\int\limits_{-\infty}^{\xi}G\left((\xi-\xi')\mu,\eta'\mu\right)z(\xi',\mu)d\xi'. \tag{13}$$

If $\forall\xi\in[0,\beta)$ $\eta^2(a_{10}^2(0)-4b_{10}(0))-4b_1(0)\neq 0$, then, according to Theorem 2, equation (13) has the general solution, the m-approximation of which can be constructed using the formula

$$x_m^{(1)}(t,\varepsilon)=a_1^{(1)}(\xi)x_{1m}^{(1)}(t,\varepsilon)+a_1^{(1)}(\xi)x_{2m}^{(1)}(t,\varepsilon),\qquad t\in[0,\beta\sqrt{\varepsilon}],$$

where

$$x_{km}^{(1)}(t,\varepsilon)=\exp\left(\frac{1}{\sqrt{\varepsilon}}\int\limits_{\beta\sqrt{\varepsilon}}^{t}\lambda_{km}\left(\frac{\tau'}{\sqrt{\varepsilon}},\sqrt{\varepsilon}\right)dt'\right),\qquad k=1,2,$$

$$\lambda_{km}\left(\frac{\tau}{\sqrt{\varepsilon}},\sqrt{\varepsilon}\right)=\sum_{i=m}^{m}(\sqrt{\varepsilon})^i\lambda_{ki}\left(\frac{\tau}{\sqrt{\varepsilon}}\right).$$

The constants $a_1^{(1)}(\varepsilon)$, $a_2^{(1)}(\varepsilon)$ are determined as solutions of the system $x_m^{(1)}(\beta\sqrt{\varepsilon},\varepsilon)=y_{10}$, $dx_m^{(1)}\Big|_{t=\beta\sqrt{\varepsilon}}=y_{20}$. Thus, for equation (1) with a zero turning point, we have constructed a continuous (together with its derivative) m-approximation of the solution

$$x_m(t,\varepsilon)=\begin{cases}x_m^{(1)}(t,\varepsilon), & t\in[0,\beta\sqrt{\varepsilon}],\\ x_m^{(2)}(t,\varepsilon), & t\in[\beta\sqrt{\varepsilon},L],\end{cases}$$

which satisfies the initial conditions (8).

Asymptotic character of the solution obtained (as defined in [3]) is established by using the methods suggested in [4].

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 492–494.

Nonlinear Boundary-Value Problem for the Heat Mass Transfer Model of W. Fushchych

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Abstract

We find a numerical-analytic solution of a nonlinear boundary-value problem for the biparabolic differential equation, which describes the mass and heat transfer in the model of W. Fushchych.

We consider the following generalized law of heat conduction:

$$q + \tau_q \frac{\partial q}{\partial t} = -\lambda \frac{\partial}{\partial x}\left(T + \tau_T \frac{\partial T}{\partial t}\right), \tag{1}$$

where λ is the heat conduction coefficient, T is the temperature, q is the heat flux, τ_q, τ_T are the relaxation coefficients.

Using (1), we obtain the following partial differential equation of heat conduction in relaxing media [1]:

$$\left(L_1 + R\frac{\partial}{\partial t}L_2\right) T(x,t) = 0, \tag{2}$$

where

$$L_1 \equiv \frac{\partial}{\partial t} - a\frac{\partial^2}{\partial x^2}, \qquad L_2 \equiv \frac{\partial}{\partial t} - \frac{\tau_T}{\tau_q}\frac{\partial^2}{\partial x^2},$$

a is the temperature conduction coefficient.

Let us take note of the fact that equation (2) satisfies no requirements of symmetry [2].

A more accurate mathematical model of heat and mass transfer consider in [2]. This model based on the following law of heat conduction:

$$q + \tau_r \frac{\partial q}{\partial t} = -\lambda \frac{\partial}{\partial x}(T + 2\tau_r L_1 T), \tag{3}$$

where τ_r is the relaxation time.

In accordance with (3), we obtain the biparabolic differential equation for heat conduction [2–4]

$$(L_1 + \tau_r L_1^2) T(x,t) = 0. \tag{4}$$

It is well known that equation (4) is invariant with respect to the Galilei group $G(1,3)$ [2–4].

We shall consider the process of burning in active media in accordance with the bi-parabolic equation (4). Solution of this problem reduces to solution of a boundary–value problem of the following form:

$$(L+\tau_r L^2)T(x,t)=\left(1+\tau_r\frac{\partial}{\partial t}\right)Q(T), \tag{5}$$

$$T(0,t)=T''_{xx}(0,t)=T(l,t)=T''_{xx}(l,t)=0, \tag{6}$$

$$T(x,0)=\theta(x), \qquad T'_t(x,0)=\psi(x), \tag{7}$$

where $\theta(x),\psi(x)$ are known functions, $Q(T)=T^m (m\geq 1)$ is the potential of heat sources, l is the scale length.

Introducing the integral transform

$$\overline{T}_n(t)=\int_0^l T(x,t)\sin(\lambda_n x)\,dx \qquad \left(\lambda_n=\frac{n\pi}{l}x\right), \tag{8}$$

we obtain the following Cauchy prroblem:

$$\begin{aligned}&\tau_r\frac{d^2\overline{T}_n(t)}{dt^2}+\nu_n^{(1)}\frac{d\overline{T}_n(t)}{dt}+\nu_n^{(2)}\overline{T}_n(t)=\overline{\Phi}(t),\\ &\overline{T}_n(0)=\alpha_n, \qquad \overline{T}_n{}'(0)=\beta_n,\end{aligned} \tag{9}$$

where

$$\nu_n^{(1)}=1+2\tau_r\lambda_n^2, \qquad \nu_n^{(2)}=\lambda_n^{(2)}(1+\tau_r\lambda_n^2), \tag{10}$$

$$\left\{\begin{array}{c}\alpha_n\\ \beta_n\end{array}\right\}=\int_0^l\left\{\begin{array}{c}\varphi(x)\\ \psi(x)\end{array}\right\}\sin(\lambda_n x)\,dx, \tag{11}$$

$$\overline{\Phi}_n(t)=\int_0^l\left(1+\tau_r\frac{\partial}{\partial\tau}\right)Q(T(x,t))\sin(\lambda_n x)\,dx. \tag{12}$$

A solution of the system of equations (9) may be written in the form

$$T(x,t)=q(x,t)+\int_0^t\int_0^l\left(1+\tau_r\frac{\partial}{\partial\tau}\right)Q(T(\xi,\tau))K(\xi,x;\ t-\tau)\,d\xi d\tau, \tag{13}$$

where

$$q(x,t)=\frac{2}{l}\sum_{n=1}^{\infty}\exp(-\lambda_n^2 t)\left(1+\tau_r\left(\frac{\beta_n}{\alpha_n}+\lambda_n^2\right)\left(1-e^{-\frac{t}{\tau_r}}\right)\right)\alpha_n\sin(\lambda_n x), \tag{14}$$

$$K(\xi,x;t-\tau)=\frac{\tau_r}{l}\left(1-\exp\left(-\frac{t-\tau}{\tau_r}\right)\right)\sum_{n=1}^{\infty}\exp\left(-\lambda_n^2(t-\tau)\right)\sin(\lambda_n x)\sin(\lambda_n\xi). \tag{15}$$

In order to construct solution (13), we can use the projective method [5]. Finally, we have the system of nonlinear algebraic equations

$$T_{j\mu} = F_{j\mu} + \sum_{i=1}^{N}\sum_{k=1}^{M} c_{ikj\mu} T_{ik}^{m}, \qquad j = \overline{1,N};\ \mu = \overline{1,M}, \tag{16}$$

where

$$F_{j\mu} = \frac{1}{\Delta x \Delta t}\int\limits_{t_{\mu-1}}^{t_\mu} dt \int\limits_{x_{j-1}}^{x_j} \mu(x,t)dx, \qquad c_{ikj\mu} = \frac{1}{\Delta t}\int\limits_{t_{\mu-1}}^{t_\mu} dt \int\limits_{t_{k-1}}^{t_k} G_{ij}(t-\tau)d\tau,$$

$$\mu(x,t) = \frac{2}{l}\sum_{n=1}^{\infty} \exp(-\lambda_n^2 t)\sin(\lambda_n x)\left(1 + \tau_r\left(1 - e^{-\frac{t}{\tau_r}}\right)\left(\frac{\beta_n}{\alpha_n} + \lambda_n^2 - \frac{\gamma_n}{2}\right)\right),$$

$$\gamma_n = \int\limits_0^l \theta_i^m(\xi)\sin(\lambda_n\xi)d\xi,$$

$$G_{ij}(t-\tau) = \frac{\tau_r}{l}\int\limits_{x_{j-1}}^{x_j} dx \int\limits_{x_{i-1}}^{x_i} \sum_{n=1}^{\infty}\left(1 - \lambda_n^2 + \left(\frac{1}{\tau_r} - 1 + \lambda_n^2\right)\exp\left(-\frac{t-\tau}{\tau_r}\right)\right) \times$$
$$\times \exp(-\lambda_n^2(t-\tau))\sin(\lambda_n x)\sin(\lambda_n\xi)d\xi.$$

After having solved the system of equations (16), we get the solution of the problem by (5)–(7) . The results of calculations show that the values of temperature determined by (5)–(7) essentially differ from values determined by (2), (6), (7). In so doing, a regular change of temperature is similar in both modells. In particular, the blow–up regime [6] exists.

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Symmetry in Nonlinear Mathematical Physics 1997, V.2, 495–498.

Discrete Symmetries and Supersymmetries – Powerful Tools for Studying Quantum Mechanical Systems

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Abstract

Dicrete symmetries (DS) of the Schrödinger-Pauli equation are applied to reduction of this equation and search for its hidden supersymmetries. General problems of using of DS in quantum mechanics are discussed.

1 Introduction

It is well-known that quantum mechanical systems are usually described in terms of differential equations and that symmetries of these equations form powerful tools for their studies. They are used to separate variables, to find out solutions of linear and nonlinear differential equations as well as to solve associated labelling problems, to derive spectra and related complete sets of functions of linear differential operators, to derive the corresponding conservation laws, to guide constructions of new theories, i.e., to figure out differential equations invariant with respect to a given symmetry, and so on.

Let us recall that in quantum mechanics the statement: "The physical system S has a symmetry group G" means that there is a group of transformations leaving the equation of motion of system S as well as the rules of quantum mechanics invariant. In particular no transformation from symmetry group G is allowed to produce an observable effect. Thus if system S is described by an observable A in states $|\psi>$, $|\phi>$, ..., then the system S' obtained by a symmetry transformation $g \in G$, $g : S \to S'$, is desribed by the corresponding observable A' in the states $|\psi'>$, $|\phi'>$, ... and the equality

$$\left|<\psi'|A'|\phi'>\right|^2 = |<\psi|A|\phi>|^2 \tag{1.1}$$

holds. Thus, as shown by E. Wigner [1], to any symmetry g there exists a unitary or antiunitary operator U_g (representing g in the Hilbert space H of the system S) such that

$$\begin{array}{l} |\psi'>= U_g|\psi> \quad \text{and} \\ A' = U_g A U_g^+ \end{array} \tag{1.2}$$

describe the effect of g, i.e., the change $S \to S'$.

There are two types of symmetries: *continuous* (e.g., rotations) and *discrete* (e.g., parity transformation). For continuous symmetries any $g \in G$ is a function of one or more continuous parameters α^i, $i = 1, 2, \ldots, n$, $g(\alpha^1, \alpha^2, \ldots, \alpha^n)$ and any U_g can be expressed in terms of Hermitian operators B_1, B_2, ... via $e^{i\alpha^j B_j}$, where each of B_j is an observable, i.e., a constant of motion, due to continuity of parameters α^j, since for a given quantum-mechanical system described by Hamiltonian H

$$\left[e^{i\alpha^j B_j}, H\right] = 0 \;\Leftrightarrow\; \sum_{n=0} \frac{(i\alpha^j)^n}{n!}\left[B_j^n, H\right] = 0 \;\Leftrightarrow\; [B_j, H] = 0. \tag{1.3}$$

Now, if $g \in G$ is a discrete symmetry, it does not depend on continuous parameters. The corresponding operator U_g can still be written as e^{iB} or Ke^{iB}, where K is an anti-unitary operator, but in fact $[B,H] = 0$ is only a sufficient condition for $\sum_{n=0} \frac{(i)^n}{n!}[B^n, H] = 0$ but not necessary. However, all discrete symmetries in physics fulfil the condition $U_g^2 = 1$. Thus if U_g is unitary ($U_g U_g^+ = U_g^+ U_g = 1$) it is also Hermitian $U_g^+ = U_g$ and therefore an observable. This is not true for $U_g^2 \neq 1$.

Now we are ready to review some results derived by A. G. Nikitin and myself [2, 3, 4].

2 Involutive symmetries and reduction of the physical systems

Consider the free Dirac equation

$$L_0\psi = (i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.1}$$

with

$$\gamma_0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \qquad \gamma_a = \begin{pmatrix} 0 & -\sigma_a \\ \sigma_a & 0 \end{pmatrix}, \quad a = 1,2,3, \qquad \gamma_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix}.$$

It is invariant w.r.t the complete Lorentz group. Involutive symmetries form a finite subgroup of the Lorentz group consisting of 4 reflections of x_μ, 6 reflections of pairs of x_μ, 4 reflections of triplets of x_μ, reflection of all x_μ and the identity transformation.

If the coordinates x_μ in (2.1) are transformed by these involutive symmetries, function $\psi(x)$ cotransforms according to a projective reprezentation of the symmetry group, i.e., either via $\psi(x) \to R_{kl}\psi(x)$ or via $\psi(x) \to B_{kl}\psi(x)$. Here R_{kl} and $B_{kl} = CR_{kl}$ are linear and antilinear operators respectively which commutes with L_0 and consequently transform solutions of (2.1) into themselves. The operators $R_{kl} = -R_{kl}$ form a reprezentation of the algebra $so(6)$ and C is the operator of charge conjugation $C\psi(x) = i\gamma_2\psi^*(x)$. Among the operators B_{kl} there are six which satisfy the condition that $(B_{kl})^2 = -1$ and nine for which $(B_{kl})^2 = 1$. We shall consider further only B_{kl} fulfilling the last condition (for the reason mentioned in the Introduction and since otherwise B_{kl} cannot be diagonalized to real γ_5 and consequently used for reduction). As shown in [2] the operators R_{kl}, B_{kl} and C form a 25–dimensional Lie algebra. It can be extended to a 64-dimensional real Lie algebra or via non-Lie symmetries (for details see [3]).

Let us discuss now only one example how to use discrete symmetries to reduce a physical system into uncoupled subsystems (for the other examples see [2]). Let the system be a spin $\frac{1}{2}$ particle interacting with a magnetic field described by the Dirac equation

$$L\psi(x) = (\gamma^\mu(i\partial_\mu - eA_\mu) - m)\,\psi(x) = 0. \tag{2.2}$$

Eq. (2.2) is invariant w.r.t. discrete symmetries provided $A_\mu(x)$ cotransforms appropriety. For instance,

$$A_\mu(-x) = -A_\mu(x) \tag{2.3}$$

for $x \to -x$ and $\psi(x) \to \widehat{R}\psi(x) = \gamma_5\widehat{\theta}\psi(x) = \gamma_5\psi(-x)$. Then, diagonalizing symmetry operator $\widehat{R}$ by means of the operator

$$W = \frac{1}{\sqrt{2}}(1+\gamma_5\gamma_0)\frac{1}{\sqrt{2}}(1+\gamma_5\gamma_0\widehat{\theta}), \tag{2.4}$$

the equation (2.2) is reduced to the block diagonal form:

$$(-\mu(i\partial_0 - eA_0) - \vec{\sigma}(i\vec{\partial} - e\vec{A})\widehat{\theta} - m)\psi_\mu(x) = 0, \tag{2.5}$$

where $\mu = \pm 1$ and ψ_μ are two-component spinor satisfying $\gamma_5\psi_\mu = \mu\psi_\mu$.

If equations (2.5) admit again a discrete symmetry then they can further be reduced to one-component uncoupled subsystems.

3 Discrete symmetries and supersymmetries

It was shown in [4] that extended, generalized and reduced supersymmetries appear rather frequently in many quantum-mechanical systems. Here I illustrate only one thing – appearance of extended supersymmetry in the Schrödinger-Pauli equation describing a spin $\frac{1}{2}$ particle interacting with a constant and homogeneous magnetic field $\vec{H}$:

$$\widehat{H}\psi(x) = \left[(-i\vec{\partial} - e\vec{A})^2 - \frac{1}{2}eg\vec{\sigma}.\vec{H}\right]\psi(x) = 0 \tag{3.1}$$

This system is exactly solvable (for details see [4]). One standard supercharge of this equation is

$$\begin{aligned} Q_1 &= \vec{\sigma}(-i\vec{\partial} - e\vec{A}), \\ Q_1^2 &= \widehat{H}. \end{aligned} \tag{3.2}$$

Three other supercharges can be constructed due to the fact that (3.1) is invariant w.r.t. space reflections R_a of x^a, $a = 1, 2, 3$. It was found in [4] that they are of the form:

$$\begin{aligned} Q_2 &= iR_3\vec{\sigma}.(-i\vec{\partial} - e\vec{A}), \\ Q_3 &= iCR_4\vec{\sigma}.(-i\vec{\partial} - e\vec{A}), \\ Q_4 &= iCR_2\vec{\sigma}.(-i\vec{\partial} - e\vec{A}). \end{aligned}$$

They are integrals of motion for (3.1) (notice that without the usual "fermionic" operators) and responsible for degeneracy of the energy spectrum of the system. For many other examples see [4].

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Remarks for Closing Ceremony

This has been a wonderful conference. I have enjoyed listening to the talks about the fine work in mathematics that you are doing. It is good for Americans to learn about the excellent research done here in Ukraine and also in Russia and other European countries. Thank you for that. A student of mine said once that going to conferences was like taking vitamins! But it is more than that. It is a time to make friends and to learn about each other and to love each other. Mathematics is a common language that has brought us together – despite the different languages that we speak. I have made new friends here and I have learned much.

It is helpful to have a list of participants with university addresses, also E-mail, so we can communicate with each other. The conference organizers kindly provided that. Professor Nikitin and all who helped have done much work to organize this conference. They have been very kind and have helped us, visitors, much-they have picked us up at the airport, have provided meals, have arranged the talks, have taken us to the river – so Professor Goldin can swim! – they have made this a very pleasant experience. They have helped us with language – we who do not speak Ukrainian or Russian. It has been good to see beautiful places in this historic city, Kyiv.

This conference could not have been without Professor Fushchych. He produced much research and pioneered so much work in mathematics. I looked in Volume 4 of the Journal of Nonlinear Mathematical Physics and many articles referred to papers that Professor Fushchych and his collaborators had written. He organized that journal and he started this conference. That is a great legacy that he has left us. I extend my best wishes to his wife and his family. I believe that we will be able to see him again and thank him ourselves for the great work he started and that now will continue. The journal will continue and so will the conference.

So maybe we will have the opportunity to be together again in two years. But for now, I wish all of you a safe trip to home and say "thank you" again to the conference organizers!

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Conclusion

Only now, when you hold in your hands the last volume of the Proceedings of the Conference, we may say that we did everything possible to make the Conference happen.

Unfortunately, not all participants were able to present their papers for publication them in the Proceedings. Here are the titles of their talks from our Conference not submitted to the Proceedings

1. *O. Batsula*, "Duality, Conformal Invariance and Local Conservation Laws in Relativistic Field Theories"
2. *E. Belokolos*, "Integrable Systems of Classical Mechanics and Algebraic Geometry"
3. *G. Goldin*, "Nonlinear Gauge Transformation and their Physical Implications"
4. *K. Jones*, "Spectra of Self-Gravitating Bose Einstein Condensate"
5. *A. Pavlyuk*, "On Self-Duality for N=2 Yang-Mills Theory"
6. *V. Rosenhaus*, "Infinite Symmetries, Differential Constraints, and Solutions for PDE's"

Finally, we would like to express our sincere gratitude to all participants of the Conference "Symmetry in Nonlinear Mathematical Physics" that was dedicated to the memory of Professor Wilhelm Fushchych. And also we invite everybody to participate in the next Conference planned for July, 1999.

We wish to thank especially those people without help of whom the Conference would be impossible: Professor Yuri Mytropolsky, Professor Anatoly Samoilenko, Professor Mykola Shkil, Professor Petro Dmytrenko.

Our special thanks are addressed to the members of Professor Fushchych's family – to Olga Fushchych, Bohdan Fushchych, Marianna Euler and Norbert Euler for invaluable assistance, support and understanding.

We are indebted to Professor Peter Bassarab-Horwath for collaboration and support.

We express deep gratitude to our colleagues at the Department of Applied Research of the Institute of Mathematics of the National Academy of Sciences of Ukraine and other teaching and research institutions, Lyudmyla Barannyk, Vyacheslav Boyko, Violeta Buchynchyk, Roman Cherniha, Olexander Galkin, Valentyna Komisarova, Halyna Lahno, Victor Lahno, Valentyn Leifura, Maxim Lutfullin, Olena Panchak, Halyna Popovych, Roman Popovych, Victor Popovych, Olena Roman, Arthur Sergeyev, Mykola Serov, Stanislav Spichak, Zoya Symenoh, Ivan Tsyfra, Alla Vorobyova, Irina Yehorchenko, Renat Zhdanov.

We would like to thank Dr. Valery Kukhtin for editing the Proceedings.

We also thank staff of the Ukrainian Pedagogical University for invaluable help in solving all large and small problems during the Conference, and everybody from the printing-shop of the Institute of Mathematics, where these Proceedings were published.

Anatoly NIKITIN,
October, 1997.