Dynamics of membranes with symmetry and projection formalism

Hiroshi Kozaki[1](a), Tatsuhiko Koike[2](b) and Hideki Ishihara[3](b)

(a) *Department of General Education, Ishikawa National College of Technology, Tsubata, Ishikawa 929-0392, Japan*

(b) *Department of Physics, Keio University, Yokohama 223-8522, Japan*

(c) *Department of Mathematics and Physics, Graduate school of Science, Osaka City University, Osaka 558-8585, Japan*

Abstract

We investigate dynamics of membrane with cohomogeneity-one symmetry. The cohomogeneity-one membrane has world surface foliated by two-dimensional orbits of isometry group. We show that the Nambu-Goto equations of motion reduce to geodesic equations in the orbit space. We also clarify the form of the metric provided to the orbit space. The generalization to higher dimensional object with cohomogeneity-one symmetry in higher dimensional spacetime is discussed.

1 Introduction

Extended objects gather much attention in cosmology. The examples are topological defects such as cosmic strings and domain walls which are produced during phase transitions in the early Universe. In the brane-world universe models, the Universe itself is also an extended object embedded in a higher dimensional spacetime.

The dynamics of the extended object is governed by partial differential equations (PDEs) because the trajectory is a surface and its embedding is determined by PDEs. On the other hand, the dynamics of a particle is governed by ordinary differential equations (ODEs). PDEs are much more difficult to solve, and then, exact solutions of extended objects are not known so much.

One way to find exact solutions is to assume symmetry. Stationary strings[1–7] and the generalization called cohomogeneity-one strings[8, 9] are the examples. A cohomogeneity-one string is defined, roughly speaking, as the one whose world surface is homogeneous in one direction. In the case that the homogeneous direction is timelike, the string is stationary. By assuming the cohomogeneity-one symmetry, the Nambu-Goto equations of motion reduces to the geodesic equation in the orbit space endowed with some metric. $U(1)$ membranes are the examples of membranes with symmetry. Hoppe assumed $U(1)$ symmetry on the membrane world surface and observed that the equations of motion for the membrane reduces to those of string[10]. The exact solutions are obtained by Trzetrelewski and Zheltukhin[11]. Higher dimensional objects with symmetry are also investigated as ξ-branes which are the special case of cohomogeneity-one symmetry[12].

In this article, we consider membranes with cohomogeneity-one symmetry and show that the Nambu-Goto equations of motion reduces to geodesic equations in the orbit space. Furthermore, we write down the metric of the orbit space.

2 Cohomogeneity-one membranes

A trajectory of the membrane is a three-dimensional surface which is embedded in a spacetime $(\mathcal{M}, g)$. Hereinafter, we assume that the spacetime $(\mathcal{M}, g)$ admits an isometry group G which spans two-dimensional orbits and that the orbits are not null. The group action on the orbits are divided into two types: simply transitive actions and multiply transitive actions. In the case that the group action is simply transitive,

[1]Email address: kozaki@ishikawa-nct.ac.jp
[2]Email address: koike@phys.keio.ac.jp
[3]Email address: ishihara@sci.osaka-cu.ac.jp

$\dim G = 2$ and two independent Killing vector fields are tangent to the orbit. In the case of the multiply transitive action, $\dim G = 3$ because two-dimensional orbits cannot admit isometry group G such that $\dim G$ is larger than $2(2+1)/2 = 3$. In this case, the orbits are maximally symmetric and three independent Killing vector fields are tangent to the orbits.

We define a cohomogeneity-one membrane such that its world surface, say Σ, is foliated by two-dimensional orbits of G. When we identify the points in $\mathcal{M}$ which are connected by the action of G, $\mathcal{M}$ reduces to an orbit space $\mathcal{M}/G$ and Σ reduces to a curve, say C, in $\mathcal{M}/G$. Let π be a projection from $\mathcal{M}$ to $\mathcal{M}/G$, Σ is represented as a preimage $\pi^{-1}(C)$. Then, the embedding of Σ is completely determined by C, and the problem is reduced to finding C such that $\pi^{-1}(C)$ satisfies the equations of motion. Because a curve is governed by ODEs in general, the equations of motion of cohomogeneity-one membranes are reduced to ODEs. In the following, we show that the Nambu-Goto equations of motion reduces to geodesic equations in $\mathcal{M}/G$ and clarify that what kind of metric are provided to $\mathcal{M}/G$.

2.1 $\dim G = 2$

In the case of $\dim G = 2$, two independent Killing vector fields, say $\boldsymbol{\xi}_I$ $(I = 1, 2)$, are tangent to the orbit. The vector space spanned by $\boldsymbol{\xi}_I$ becomes a Lie algebra with respect to the commutator of vector fields. The structure of two-dimensional Lie algebra is classified into two classes; commutative and solvable. In the case of commutative Lie algebra, it has been already clarified that the Nambu-Goto equations of motion reduce to geodesic equations. In the following, we shall take different method so that we can apply it to solvable Lie algebra.

First, we shall set up a coordinate in $\mathcal{M}$ by making use of the G orbits. We take an orbit $\mathcal{O}_0$ and a two-dimensional surface $\mathcal{S}$ which intersects with each orbit at one point. Coordinates on $\mathcal{O}_0$ and $\mathcal{S}$ are denoted by x^i $(i = 1, 2)$ and y^p $(p = 3, 4)$ respectively. Because all orbits intersect with $\mathcal{S}$ at different points on $\mathcal{S}$, we can label the orbit as $\mathcal{O}(y^p)$ by using the coordinate y^p of the intersection. The orbits $\mathcal{O}(y^p)$ do not intersect with each other and fill the spacetime. We extend the internal coordinate y^p of $\mathcal{S}$ to spacetime coordinate such a way that y^p is constant on $\mathcal{O}(y^p)$.

Group action moves $\mathcal{S}$ to another surface $\mathcal{S}'$ which also intersects with each orbit at one point. In the case that $\mathcal{S}'$ intersects with $\mathcal{O}_0$ at x^i, we write $\mathcal{S}'$ as $\mathcal{S}(x^i)$. Surfaces $\mathcal{S}(x^i)$ $(x^i \in \mathcal{O}_))$ also do not intersect with each other and fill the spacetime. Then, we can extend x^i to the coordinate of $\mathcal{M}$ so that x^i is constant on $\mathcal{S}(x^i)$. Combining extended x^i and y^p, we set up a coordinate (x^i, y^p) in $\mathcal{M}$. With respect to (x^i, y^p), G action moves is written as $G : (x^i, y^p) \mapsto (x'^i, y^p)$, i.e., y^p is invariant under the G action.

Next, we shall write the spacetime metric with respect to the coordinate (x^i, y^p). Because the group action is simply transitive on the orbit, we have invariant dual basis χ^I in $\mathcal{O}_0$;

$$\mathcal{L}_{\xi_I}\chi^J = 0, \tag{1}$$

where $\mathcal{L}$ denotes the Lie derivative in $\mathcal{O}_0$. With respect to the coordinate x^i in $\mathcal{O}_0$, χ^I is written as

$$\chi^I = \chi^I_i(x)dx^i. \tag{2}$$

We can extend the 1-forms χ^I in $\mathcal{O}_0$ to those defined in $\mathcal{M}$ by taking x^i as spacetime coordinate in Eq.(2). The extended χ^I are also invariant under G action. By using χ^I, we can write the metric as

$$ds^2 = g_{pq}dy^p dy^q + 2g_{pI}dy^p\chi^I + g_{IJ}\chi^I\chi^J. \tag{3}$$

The metric functions g_{pq}, g_{pI} and g_{IJ} do not depend on x^i because isometry group G moves x^i.

The orbits are labeled by y^p. Then, we can also take y^p as a coordinate of $\mathcal{M}/G$. The projection $\pi : \mathcal{M} \to \mathcal{M}/G$ is described as $\pi : (x^i, y^p) \mapsto y^p$. Here, we rewrite the metric as

$$ds^2 = g_{IJ}(\chi^I + N^I_p dy^p)(\chi^J + N^J_q dy^q) + h_{pq}dy^p dy^q, \tag{4}$$

where

$$g_{IJ}N^J_p = g_{Ip}, \tag{5}$$

$$h_{pq} = g_{pq} - g_{IJ}N^I_p N^J_q. \tag{6}$$

constant curvature space(time)	Bianchi type	$[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2]$	$[\boldsymbol{\xi}_2, \boldsymbol{\xi}_3]$	$[\boldsymbol{\xi}_3, \boldsymbol{\xi}_1]$
Euclid space E^2	VII_0	0	$-\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$
sphere S^2	IX	$\boldsymbol{\xi}_3$	$\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$
hyperbolic space H^2	VIII	$\boldsymbol{\xi}_3$	$-\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$
Minkowski spacetime $E^{1,1}$	VI_0	0	$-\boldsymbol{\xi}_2$	$-\boldsymbol{\xi}_1$
de Sitter spacetime dS^2	VIII	$\boldsymbol{\xi}_3$	$-\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$
anti-de Sitter spacetime AdS^2	VIII	$\boldsymbol{\xi}_3$	$-\boldsymbol{\xi}_1$	$\boldsymbol{\xi}_2$

Table 1: Lie algebras of isometry groups of the maximally symmetric space(time)

We can regard h_{pq} as the metric in $\mathcal{M}/G$ so that the projection π is a Riemann submersion. When we identify the points connected by group actions of G, the cohomogeneity-one world surface reduces to a curve C in $\mathcal{M}/G$. The metric h_{pq} measures the length of C in $\mathcal{M}/G$, so that it agrees with the length of a *lift* curve c in $\mathcal{M}$ which are measured orthogonally to the orbits.

The volume element of Σ is given as a product of the area element of the orbits and the line element of c which are orthogonally measured to the orbits. The orthogonally measured line element of c agrees with the line element of C, and then, it is given as

$$dL = \sqrt{|h_{pq}(y)dy^p dy^q|}. \tag{7}$$

The area element on the orbit is given as

$$dA = \sqrt{|\det g_{IJ}(y)|}\chi^1 \wedge \chi^2. \tag{8}$$

Hence, the Nambu-Goto action is

$$S = \iint_\Sigma dAdL = \mathcal{E}\int_C \sqrt{|(\det g_{IJ})h_{pq}dy^p dy^q|}, \tag{9}$$

where

$$\mathcal{E} = \int_{\text{orbit}} \chi^1 \wedge \chi^2. \tag{10}$$

Therefore, the problem is reduced to solving the geodesic equations in the orbit space endowed with the metric $(\det g_{IJ})h_{pq}$.

2.2 $\dim G = 3$

In the case of $\dim G = 3$, three independent Killing vector fields, say $\boldsymbol{\xi}_I$ $(I = 1, 2, 3)$, are tangent to the two-dimensional orbit. Then, the two-dimensional orbit is a space of constant curvature. Let $\mathfrak{g}$ be a Lie algebra spanned by $\boldsymbol{\xi}_I$. The structure of $\mathfrak{g}$ is same as the constant curvature space. Following the Bianchi classification of three dimensional Lie algebras, we summarize the possible structure of $\mathfrak{g}$ in Table 1.

In the case that the spacetime admits two-dimensional orbits with constant curvature K, the metric is written as[13, 14]

$$ds^2 = Y^2[(dx^1)^2 + \epsilon\Sigma^2(x^1, k)(dx^2)^2] + e^{2\lambda}(dy^3)^2 - \epsilon e^{2\nu}(dy^4)^2. \tag{11}$$

where Y, λ and ν are functions of y^3 and y^4,

$$\epsilon = \begin{cases} +1 & \text{(spacelike orbit)} \\ -1 & \text{(timelike orbit)} \end{cases}, \tag{12}$$

and

$$\Sigma(x^1, k) = (\sin x^1, x^1, \sinh x^1) \text{ for } k = KY^2 = (1, 0, -1). \tag{13}$$

We can consider x^1 and x^2 as the coordinates in the orbits and y^3 and y^4 as the parameters which distinguish the orbits. Then, the map $\pi : (x^1, x^2, y^3, y^4) \mapsto (y^3, y^4)$ is a projection from $\mathcal{M}$ to $\mathcal{M}/G$.

The last two terms of Eq.(11) is the 2-metric in $\mathcal{M}/G$ such that the projection is Riemann submersion. Following the similar derivation in $\dim G = 2$ case, the Nambu-Goto action is

$$S = \int_\Sigma dv = \int_{\rm orbit} \sqrt{|\epsilon\Sigma^2(x^1,k)|}dx^1dx^2 \int_C \sqrt{|Y^4(y)(e^{2\lambda}(dy^3)^2 - \epsilon e^{2\nu}(dy^4)^2)|}. \tag{14}$$

Therefore, the problem is reduced to solving the geodesic equations in $\mathcal{M}/G$ with the metric

$$ds^2_{\mathcal{M}/G} = Y^4(e^{2\lambda}(dy^3)^2 - \epsilon e^{2\nu}(dy^4)^2). \tag{15}$$

3 Conclusion

We have studied dynamics of the Nambu-Goto membrane with cohomogeneity-one symmetry in four dimensional spacetime $(\mathcal{M}, g)$. The cohomogeneity-one symmetry means that the world surface is foliated by two-dimensional orbits of isometry group G in $(\mathcal{M}, g)$. By virtue of the symmetry on the world surface, the equations of motion reduces to the geodesic equations in the orbit space $\mathcal{M}/G$. In this case, the metric is of the form $\det(g_{IJ})h_{pq}$ where g_{IJ} is the metric on the G orbit and h_{pq} is a metric in $\mathcal{M}/G$ such that the projection $\pi : \mathcal{M} \to \mathcal{M}/G$ is a Riemann submersion.

In this article, we have considered dynamics of cohomogeneity-one membrane in four dimensional spacetime. The results may be valid for higher dimensional cohomogeneity-one objects in higher dimensional spacetime in two cases. One is the case that the group action is simply transitive on orbit and the other is that the orbit is maximally symmetric. In the other cases, the equations of motion may reduce to geodesic equations but we do not know the metric form.

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