Static field configurations in truncated QED

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Abstract

Due to the nonlinearity of QED, a static charge becomes a magnetic dipole if placed in a magnetic field, and a magnetic monopole if the background is a combination of constant electric and magnetic fields. Already without external field, the cubic Maxwell equation for the field of a point charge has a soliton solution with a finite field energy. Equations are given for self-coupling dipole moments. Any theoretically found value for a multipole moment of a baryon or a meson should be subjected to nonlinear renormalization.

1 Introduction

In this talk we give an overview of the results presented in Refs. [1]– [4] and [5]. Yet unpublished results are reported in Subsection 2.2.

The static nonlinear Maxwell equations produced by the Euler-Heisenberg Lagrangian $\mathcal{L}$ truncated at the fourth power of its Taylor expansion in the fields have the form [1]

$$(\boldsymbol{\nabla}\cdot\mathbf{E}\,(\mathbf{x})) = j_0^{\rm lin} + j_0^{\rm nl}, \quad [\boldsymbol{\nabla}\times\mathbf{B}\,(\mathbf{x})] = \mathbf{j}^{\rm lin} + \mathbf{j}^{\rm nl}. \tag{373}$$

Here $j_0^{\rm lin}$and $\mathbf{j}^{\rm lin}$ are external current components, while the nonlinear current $j_\mu^{\rm nl}$ is the one induced by the electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields themselves:

$$j_0^{\rm nl} = \mathcal{L}_{\mathfrak{FF}}\left(\boldsymbol{\nabla}\cdot\mathfrak{F}(\mathbf{x})\mathbf{E}\,(\mathbf{x})\right) - \mathcal{L}_{\mathfrak{GG}}\left(\boldsymbol{\nabla}\cdot\mathbf{B}\,(\mathbf{x})\right)\mathfrak{G}(\mathbf{x})\,, \tag{374}$$

$$\mathbf{j}^{\rm nl} = \mathcal{L}_{\mathfrak{FF}}\left[\boldsymbol{\nabla}\times\mathbf{B}\,(\mathbf{x})\right]\mathfrak{F}(\mathbf{x}) + \mathcal{L}_{\mathfrak{GG}}\left[\boldsymbol{\nabla}\times\mathbf{E}\,(\mathbf{x})\right]\mathfrak{G}(\mathbf{x})\,. \tag{375}$$

Here $\mathcal{L}_{\mathfrak{FF}}$ and $\mathcal{L}_{\mathfrak{GG}}$ are the second derivatives of $\mathcal{L}$ with respect to the field invariants $=\frac{B^2-E^2}{2}$, $\mathfrak{G} = (\mathbf{E}\cdot\mathbf{B})$, taken at constant values of the fields that make up the background above which the expansion of the Lagrangian has been done. In QED $\mathcal{L}_{\mathfrak{FF}}$ and $\mathcal{L}_{\mathfrak{GG}}$ are at least quadratic with respect to the fine-structure constant α. Unlike the cited works, we have disregarded here the third derivative $\mathcal{L}_{\mathfrak{FFG}}$, since it is smaller with respect to α. The differentiaton operators ∇ in (374), (375) act on everything to the right of them.

Equations (373) should be completed with the "first pair" of static Maxwell equations $[\boldsymbol{\nabla}\times\mathbf{E}\,(\mathbf{x})] = (\boldsymbol{\nabla}\cdot\mathbf{B}\,(\mathbf{x})) = 0$.

2 Spheric charge in a static and homogeneous background

In this section the Maxwell equations (373) will be treated perturbatively, with their right-hand sides linearized near an external field. The nonlinear current $\mathbf{j}^{\rm nl}$, its constituents $\mathcal{L}_{\mathfrak{FF}}$ and $\mathcal{L}_{\mathfrak{GG}}$ included,

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will be taken at $\mathfrak{G} = \overline{\mathfrak{G}} \equiv \overline{\mathbf{B}}\cdot\overline{\mathbf{E}}$ $=const,$ $2\mathfrak{F} = 2\overline{\mathfrak{F}} \equiv \overline{B}^2 - \overline{E}^2 = const.$ It depends only on the background time- and space-independent electric, $\overline{\mathbf{E}}$, and magnetic fields $B = |\overline{\mathbf{B}}|$. These constant fields identically satisfy the Maxwell equations (373) without external currents needed to support them: this is a manifestation of the gauge invariance. The electric field will be that produced by a spherically-symmetric charge $j_0^{\rm lin} \neq 0$. Its self-interaction $j_0^{\rm nl}$ is neglected. The magnetic field will be due only to the electric charge, its external source $\mathbf{j}^{\rm lin}$ will be kept equal to zero throughout this section.

2.1 Magnetic dipole solution produced by electric charge in a magnetic background [2], [3], [4].

In this subsection the constant electric field is not included, $\overline{\mathbf{E}}$= $\mathbf{0}$. The electric field $\mathbf{E}$ is only that of an non-selfinteracting external spherical charge distribution.Then the nonlinear current (374), (375) is: $j_0^{\rm nl}(\mathbf{x}) = 0\,,\ \ j_k^{\rm nl}(\mathbf{x}) = \epsilon_{ijk}\nabla_i\mathfrak{h}_k$, where

$$\mathfrak{h}_i(\mathbf{x}) = -\frac{\overline{B}_i}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}E^2 + E_i\left(\overline{\mathbf{B}}\cdot\mathbf{E}\right)\mathcal{L}_{\mathfrak{G}\mathfrak{G}} \tag{376}$$

is an auxiliary magnetic field. From this current, the part containing the magnetic field $\delta\mathbf{B} = \mathbf{B} - \overline{\mathbf{B}}$ produced by the electric field is omitted as containing higher-order corrections with respect to α. The magnetic excitation of the background results from equations (373) and from equation $(\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{x})) = 0$ to be:

$$\delta B_i(\mathbf{x}) = \left(\delta_{ik} - \frac{\nabla_i\nabla_k}{\nabla^2}\right)\mathfrak{h}_k(\mathbf{x}) = \mathfrak{h}_i(\mathbf{x}) + \frac{\partial_i\partial_k}{4\pi}\int d^3y\frac{\mathfrak{h}_k(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\,. \tag{377}$$

Let us take the electric field $\mathbf{E}$ as the one produced – disregarding the higher-order effect of the linear electrization in the magnetic field – by the charge distributed with the constant density $j_0^{\rm lin}(\mathbf{x}) = \rho(r) = \left(\frac{3}{4\pi}\frac{q}{R^3}\right)\theta(R-r)$ inside a sphere with the radius R. The long-range asymptotic behavior of (377) proves to be that of a magnetic dipole:

$$\delta B_i^{\rm LR}(\mathbf{x}) = \frac{3(\mathbf{x}\cdot\mathbf{M})\,x_i}{r^5} - \frac{M_i}{r^3}\,, \tag{378}$$

with $\mathbf{M}$ being an equivalent magnetic dipole moment. It cannot be otherwise, because Eq. (378) is the only axial-symmetric magnetic field satisfying the free Maxwell equations $[\boldsymbol{\nabla}\times\mathbf{B}(\mathbf{x})] = (\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{x})) = 0$, bearing in mind that the electric field contribution into (375) decreases at large distances as r^{-5}, while the deviation from zero of any axial-symmetric field other than (378) would produce a larger extra addition to the current of the order of r^{-4}. The value of the magnetic moment was calculated in ([3], [4]) to be

$$M_i = \left(\frac{q}{4\pi}\right)^2\frac{1}{5R}\left(3\mathcal{L}_{\mathfrak{F}\mathfrak{F}} - 2\mathcal{L}_{\mathfrak{G}\mathfrak{G}}\right)\overline{B}_i\,. \tag{379}$$

The extension beyond the spherical symmetry of the electric field is also available [4].

2.2 Magnetic monopole solution produced by electric charge in a magnetic plus electric background

All is the same as in the previous item, except that now both electric and magnetic fields in the constant background are different from zero. They are taken to be parallel to each other in the Lorentz frame, where the electric charge is at rest, $\overline{\mathbf{B}}\parallel\overline{\mathbf{E}}$. Their common direction is presented by a unit vector μ, $\mu = 1$. We are going to consider the deviations of the electric $\delta\mathbf{E}(\mathbf{x}) = \overline{\mathbf{E}} - \mathbf{E}(\mathbf{x})$ and magnetic $\delta\mathbf{B}(\mathbf{x}) = \overline{\mathbf{B}} - \mathbf{B}(\mathbf{x})$ fields to be small as compared to their constant parts $\delta\mathbf{E} << \overline{\mathbf{E}}$, $\delta\mathbf{B} << \overline{\mathbf{B}}$. Omitting the deviations squared and also neglecting $\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\mathfrak{F}$ and $-\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\overline{B}^2 + \overline{E}^2\mathcal{L}_{\mathfrak{G}\mathfrak{G}}$ as compared to unity[34] (this reduces to omitting from the current (375) the contribution of $\delta\mathbf{B}$, analogous to what has been done in the previous subsection, when passing to Eq. (376)) equation (373) becomes

$$[\boldsymbol{\nabla}\times\delta\mathbf{B}(\mathbf{x})]-\mathcal{L}_{\mathfrak{FF}}\left[\overline{\mathbf{B}}\times\boldsymbol{\nabla}\right](\overline{\mathbf{E}}\cdot\delta\mathbf{E}))+\mathcal{L}_{\mathfrak{GG}}\left[\overline{\mathbf{E}}\times\nabla\right]\left(\delta\mathbf{E}\cdot\overline{\mathbf{B}}\right)=0 \tag{380}$$

For the special case of the Coulomb field $\delta\mathbf{E}=\frac{q\mathbf{x}}{4\pi r^3}$ (outside the charge) this equation is compatable with the ansatz $\delta\mathbf{B}=\mathbf{x}\frac{1}{r^3}f(\frac{z}{r})$, where $z=(\mu\cdot\mathbf{x})=r\cos\theta$ is the coordinate component along the common direction of the constant fields, $r\equiv|\mathbf{x}|$. This ansatz formally obeys the other Maxwell equation not included into (373) $(\boldsymbol{\nabla}\cdot\delta\mathbf{B})=0$ outside the charge, which may be violated only inside the charge or in the point $r=0$, where the charge is located, when the charge is pointlike. Equation (380) reduces to a linear first-order differential equation for the function $f(\frac{z}{r})$, ready to solve. In this way the magnetic field is found

$$\delta\mathbf{B}=-\frac{q}{8\pi}(\mathcal{L}_{\mathfrak{FF}}+\mathcal{L}_{\mathfrak{GG}})\overline{\mathfrak{G}}\frac{\mathbf{x}}{r^5}z^2=-\frac{q}{8\pi}(\mathcal{L}_{\mathfrak{FF}}+\mathcal{L}_{\mathfrak{GG}})\overline{\mathfrak{G}}\frac{\mathbf{x}\cos^2\theta}{r^3},$$

whose lines of force are directed along the radius-vector $\mathbf{x}$. Note that in the limit $\overline{\mathbf{E}}=0$ this solution disappears not to turn into the solution of the previous subsection, which is of completely different character. To determine the magnetic charge q_M it is necessary to integrate $\delta\mathbf{B}$ over the surface of a sphere with arbitrary radius r

$$q_M=\int(\delta\mathbf{B}\cdot\mathrm{d}\mathbf{S})=-\frac{q}{6}(\mathcal{L}_{\mathfrak{FF}}+\mathcal{L}_{\mathfrak{GG}})\overline{\mathfrak{G}}$$

Hence, for the pointlike electric charge, its magnetic charge density in the constant background is

$$(\nabla\cdot\delta\mathbf{B})=q_M\delta^3(\mathbf{x}).$$

3 Cubic self-interaction of electro- and magneto-static fields in blank vacuum

In this section no background field will be present. Unlike the previous section, now the nonlinearity in the Maxwell equation will not be taken as small, but will be treated seriously. In the two subsections below we include only the cases, where either only electric, E, or only magnetic, B, field is present, and not the both fields simultaneously. Then the nonlinear current (374, 375) is

$$j_0^{\rm nl}(x)=\frac{1}{2}\mathcal{L}_{\mathfrak{FF}}\partial_i\left[\left(B^2-E^2\right)E_i\right],\qquad j_i^{\rm nl}(x)=-\frac{1}{2}\mathcal{L}_{\mathfrak{FF}}\partial_j\left[\left(B^2-E^2\right)B_k\right]\epsilon_{ijk}. \tag{381}$$

In the present section the derivative $\mathcal{L}_{\mathfrak{FF}}\equiv\gamma$ is understood as taken at $\mathfrak{F}=\mathfrak{G}=0$.

3.1 Self-coupling of a charge. Finiteness of the point-charge electrostatic field-energy

Let there be a point charge e placed at the origin $r=0$. We are looking for a spherically symmetric solution of the Maxwell equation (373) with $B=0$, which, given the nonlinear current (381), takes the form

$$\nabla\left[\left(1+\frac{\gamma}{2}E^2\right)\mathbf{E}\right]=0, \tag{382}$$

valid everywhere outside the origin $\mathbf{x}=0$, since $j_0=0$ there. At large r the standard Coulomb field of the point charge e

$$\frac{e}{4\pi r^2}\frac{\mathbf{x}}{r}, \tag{383}$$

[34] This disregard is not necessary. Solutions can be found without referring to it within the same ansatz (see below). These are, however, a bit more complicated.

should be implied as the boundary condition. Then with the spherically symmetric Ansatz $E(r)\frac{\mathbf{x}}{r} = \mathbf{E}(\mathbf{x})$ equation (382) is solved as

$$\left(1+\frac{\gamma}{2}E^2(r)\right)E(r)=\frac{e}{4\pi r^2}. \tag{384}$$

This cubic equation is readily solved by the Cardano formula (see [5] for the explicit representation), but the most important thing about its solution is clear without solving it: at short distances $r\to\infty$ the field E also infinitely grows, hence one can neglect the unity in (384) to immediately obtain $E(r)\sim\left(\frac{e}{2\pi\gamma}\right)^{\frac{1}{3}}\left(\frac{1}{r}\right)^{\frac{2}{3}}$. This behavior of the electrostatic field , produced by the point charge e via the nonlinear field equations, is essentially less singular in the vicinity of the charge than the standard Coulomb field $\frac{e}{4\pi r^2}$. This is an extension of elecrodynamics to the domain of short distances compatable with the traditional theory of electromagnetism surely established for larger distances.

Let us see that this suppression of the singularity is enough to provide convergence of the integrals giving the energy of the field configuration that solves equation (382). To this end note that on the subclass of electromagnetic field we are considering here, the equations of motion (382) are generated by the quartic Lagrangian

$$-F(x)+\mathcal{L}=-F(x)+\frac{\gamma}{2}(F(x))^2. \tag{385}$$

With this Lagrangian, the energy density calculated on spherically-symmetric electric field configuration following the Noether theorem is

$$\Theta^{00}=\frac{E^2}{2}+\frac{3\gamma E^4}{8}. \tag{386}$$

The behaviour $E(r)\sim\left(\frac{1}{r}\right)^{\frac{2}{3}}$ obtained provides the ultraviolet, near $|\mathbf{x}|=0$, convergence of the electrostatic field energy $\int\Theta^{00}d^3x$ of the point charge. As for the convergence of this integral at $|\mathbf{x}|\to\infty$, it is provided by the standard long-range Coulomb behaviour (383) of the solution to equation (382) obtained by neglecting the second term inside the bracket as compared to the unity in the far-off region.

The explicit use of the Cardano formula in (386) allows to calculate the integral for the field energy. If the value $\mathcal{L}_{\mathfrak{F}\mathfrak{F}}=\frac{e^4}{45\pi^2m^4}$,where e and m are the electron charge and mass, is accepted – referring to the Euler-Heisenberg effective Lagrangian – for γ, the result for the "rest mass of the electron," understood as a point charge, is about twice the true electron mass: $\int\Theta^{00}d^3x=2.09m$.

The conclusion about finiteness of the electrostatic field energy of a point charge can be extended [6] to any nonlinear electrodynamics with the effective Lagrangian growing faster than $\mathfrak{F}^{3/2}$, any-power polynomial of the field invariants included, thereby also to QED truncated at any finite term of its Taylor expansion in powers of the field in place of (373).

3.2 Self-coupling of magnetic and electric dipoles

Consider first a magnetic dipole. This means that only B is kept in the nonlinear current (381), hence $j_0^{\mathrm{n}l}=0$. As for the nonlinear 3-current, it is expressed as

$$j_i^{\mathrm{nl}}(\mathbf{x})=\epsilon_{ijk}\nabla_j\eta_k(\mathbf{x}),\qquad \eta_i(\mathbf{x})=-\frac{1}{2}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}B_i(\mathbf{x})B^2(\mathbf{x}) \tag{387}$$

in terms of the auxiliary magnetic field $\mathfrak{h}$ analogous to (376). Eq. (377) is again valid for the magnetic field induced by the nonlinear current, this time without the reservations made in the previous section about the disregard of the linear magnetization. This field is to be added to the initial magnetic field $\mathbf{h}^{\mathrm{nl}}$ (linearly produced by the current $\mathbf{j}$) to make the total resulting magnetic field $\mathbf{h}^{\mathrm{tot}}=\mathbf{h}^{\mathrm{nl}}+\mathbf{h}$.

Let there be a sphere with the radius R, and a time-independent current $\mathbf{j}(\mathbf{x})$ concentrated on its surface:

$$\mathbf{j}(\mathbf{x})=\frac{[\mathbf{M}^{(0)}\times\mathbf{x}]}{r^4}\delta(r-R). \tag{388}$$

Here $\mathbf{M}^{(0)}$ is a constant vector directed, say, along the axis 3. The current density (388) obeys the continuity condition $\nabla \mathbf{j} = 0$, its flow lines are circular in the planes parallel to the plane (1,2). The magnetic field produced by this current via the Maxwell equation $\nabla \times \mathbf{h}^{\text{lin}}(\mathbf{x}) = \mathbf{j}(\mathbf{x})$ is

$$\mathbf{h}^{\text{lin}}(\mathbf{x}) = \theta\,(R-r)\,\frac{2\mathbf{M}^{(0)}}{R^3} + \theta\,(r-R)\left(-\frac{\mathbf{M}^{(0)}}{r^3} + 3\frac{(\mathbf{x}\cdot\mathbf{M}^{(0)})}{r^5}\mathbf{x}\right). \tag{389}$$

Outside the sphere this is the magnetic dipole field with the constant vector density $\mathbf{M}^{(\mathbf{0})}$ playing the role of its magnetic moment. Using this expressioin in the r.-h. side of Eq. (377), after a lengthy calculation the nonlinear correction h to the field (389) of the magnetic dipole (388) was obtained in [1] both inside and outside the sphere. At large distances the resulting field reproduces the original magnetic dipole behaviour:

$$\mathbf{h}^{tot}\left(\mathbf{x}\right)\big|_{r>>R} = \mathbf{h}^{\text{lin}}\left(\mathbf{x}\right)\left(1-\frac{7}{5}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\frac{M^{(0)2}}{R^6}\right). \tag{390}$$

Once we want to treat the nonlinearity seriously, and not just as a perturbation, we should for self-consistency demand that the magnetic field forming the nonlinear current (387) be not (389), but its final result, which is again the magnetic dipole field, but with the bare magnetic moment $\mathbf{M}^{(\mathbf{0})}$ replaced by the final magnetic moment to be denoted as $\mathbf{M}$. Then in the long range for the total field we obtain

$$-\frac{\mathbf{M}}{r^3} + 3\frac{\mathbf{x}\cdot\mathbf{M}}{r^5}\mathbf{x} = -\frac{\mathbf{M}^{(0)}}{r^3} + 3\frac{\mathbf{x}\cdot\mathbf{M}^{(0)}}{r^5}\mathbf{x} - \left(\frac{\mathbf{M}}{r^3} + 3\frac{\mathbf{x}\cdot\mathbf{M}}{r^5}\mathbf{x}\right)\left(\frac{7}{5}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\frac{M^2}{R^6}\right). \tag{391}$$

From this the equation for self-coupling of the magnetic moment follows to be:

$$\mathbf{M}\left(1+\frac{7}{5}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\frac{M^2}{R^6}\right) = \mathbf{M}^{(0)} \tag{392}$$

Analogous equation for the electric moment is

$$\mathbf{p}\left(1+\frac{1}{10}\mathcal{L}_{\mathfrak{F}\mathfrak{F}}\frac{p^2}{R^6}\right) = \mathbf{p}^{(0)}. \tag{393}$$

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