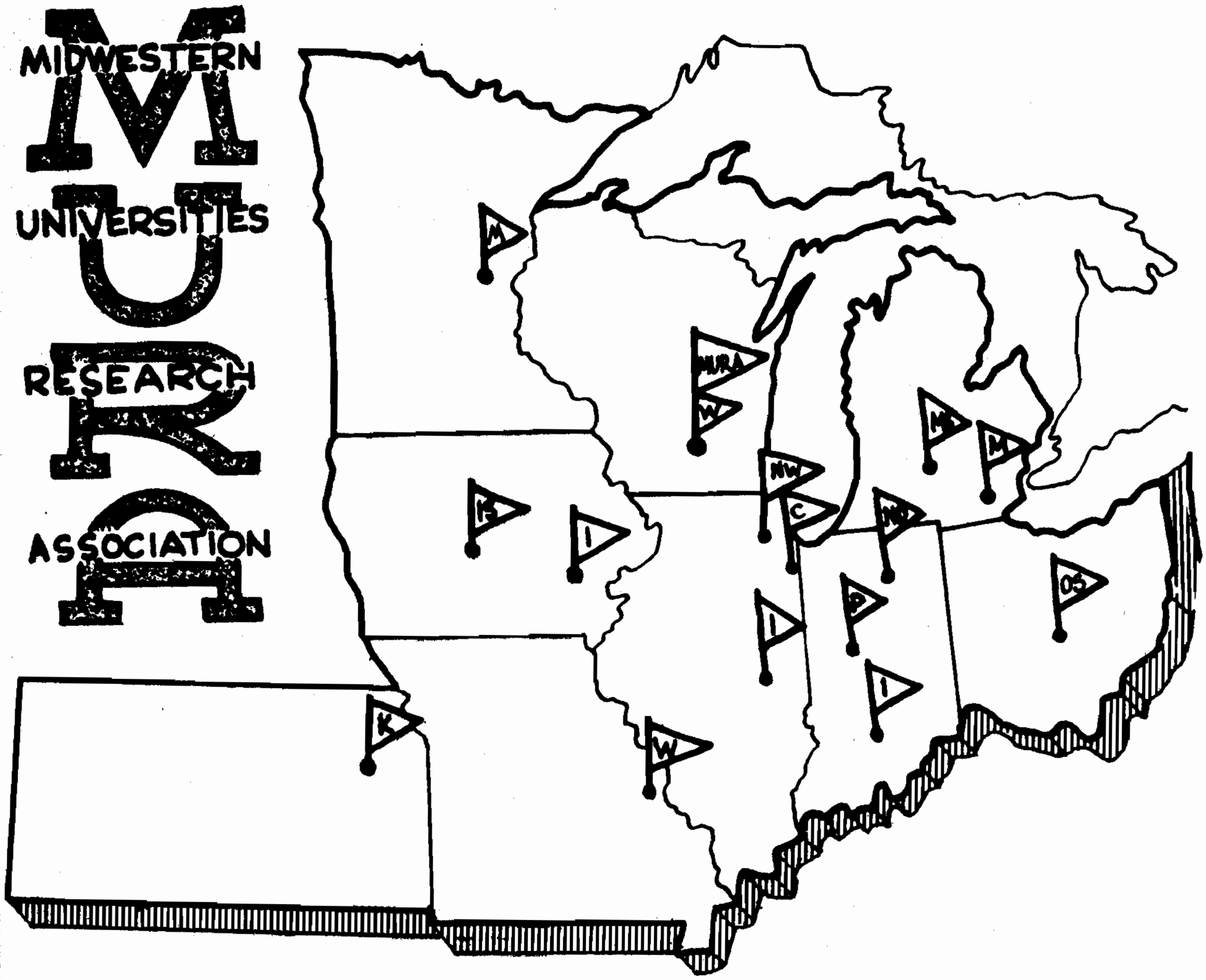

RESONANT BEAM EXTRACTION IN FIXED FIELD ALTERNATING GRADIENT ACCELERATORS

Werner William Shoultz and C. L. Hammer

REPORT NUMBER 666

LEGAL NOTICE

Printed in USA. Price $0.50. Available from the

Office of Technical Services
U. S. Department of Commerce
Washington 25, D. C.

MURA-666
UC-28: Particle Accelerators and
High Voltage Machines
TID-4500 (19th Edition)

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION*

2203 University Avenue, Madison, Wisconsin

RESONANT BEAM EXTRACTION IN FIXED FIELD ALTERNATING GRADIENT ACCELERATORS

Werner William Shoultz and C. L. Hammer
Institute for Atomic Research and Department of Physics
Iowa State University, Ames, Iowa

and

Midwestern Universities Research Association, Madison, Wisconsin

January 2, 1963

ABSTRACT

The resonant beam extraction method proposed by C. L. Hammer and L. J. Laslett for constant gradient and alternating gradient accelerators has been extended to fixed field alternating gradient accelerators. It is possible to find field perturbations which are effective in causing the radial betatron oscillations to grow exponentially at a particular azimuthal position, and which at the same time have an effect upon the equilibrium orbit which greatly enhances ease of extraction. Digital calculations verifying the analytical predictions are presented. It remains necessary, however, to choose field perturbations which do not introduce nonlinear effects having unfavorable extraction characteristics.

*AEC Research and Development Report. Research supported by the U. S. Atomic Energy Commission, Contract No. AT (11-1)-384.

I. INTRODUCTION

Fixed field alternating gradient accelerators as compared to ordinary alternating gradient accelerators have the distinct advantage of a much larger internal beam intensity. This advantage could be exploited in performing high-intensity scattering experiments in the energy machines or storage rings. However, only single-turn extraction has been demonstrated for this machine, whereas many-turn extraction is desirable because its more extended beam burst aids in coincidence studies. Also, when one is using the fixed field accelerator as an injector into a larger machine, a longer beam burst will enable one to fill more completely the phase space of the larger accelerator.

Integral or half-integral resonant beam extraction is deemed particularly suitable for use as an injection process into a higher energy machine for it is anticipated that the resonant extraction process will reduce the radial phase space of the beam. In addition, the perturbation may be applied spatially so that long term extraction can be achieved by slowly moving the beam into the extraction region. Resonant beam extraction has been investigated analytically and checked by digital computations only for constant gradient and alternating gradient synchrotrons.[1] In the above cases the betatron solutions may be written in the Floquet form if the dynamic nonlinearity of the equation of motion of the betatron oscillation is omitted, which is reasonable for these machines. The type of perturbation used in this analysis is an azimuthally dependent perturbation of the azimuthally dependent part of the field. This perturbation drives the operating point into the integral or half integral radially unstable zone, which opens up with a width proportional to the perturbation strength and within which the solution for the exponentially increasing betatron oscillations attains its maximum value at one particular azimuth regardless of the initial conditions of the particle. Therefore, in principle, the beam can be extracted on successive revolutions with no spread in the angle tangent to the equilibrium orbit, thus making the radial phase space of the extracted beam zero.

Because of the high nonlinearity of the guide field of a fixed field alternating gradient accelerator, it is an interesting question whether a fair approximation to a Floquet solution for the betatron oscillations may be made and the resonant beam extraction method applied to exploit its obvious advantages. This paper will consist of a linearization of the equation of motion for the fixed field machine and the application of the resonant beam extraction method to these equations. Determination of the feasibility of the method will be established by the use of digital computations.

II. THEORY

A. Derivation of the Linear Equation of Motion

The equations of motion of particles in a fixed field alternating gradient accelerator are:[2]

$$x' = (1 + x) P_x (1 - p_x^2 - p_y^2)^{-1/2}$$

$$y' = (1 + x) P_y (1 - p_x^2 - p_y^2)^{-1/2}$$

$$(P_x)' = (1 - p_x^2 - p_y^2)^{1/2} + \sigma(1 + x) \left[(B_y/B_o) - (P_y B_\theta/B_o) x (1 - p_x^2 - p_y^2)^{-1/2} \right]$$

$$P_y' = \sigma(1 + x) \left[(P_x B_\theta/B_o) (1 - p_x^2 - p_y^2)^{-1/2} - B_x/B_o \right] \quad (1)$$

where

$x = (r - r_o)/r_o$ $\quad y = z/r_o$ $\quad \sigma = er_oB_o/m_oc = 1$

a prime indicates differentiation with respect to θ

$$P_x = (dx/ds), \qquad P_y = (dy/ds) .$$

One sees that the radial motion and the vertical motion are coupled. For the accelerator considered, it is assumed the perturbation of the radial motion due to this coupling is small so that the radial equation of motion may be written with no vertical motion. The coupling effect will be investigated by digital computations only. Further, to make possible an analytic calculation, the radial equation of motion is linearized and perturbation theory is applied to this linear equation to determine the field perturbations necessary to give the beam an exponential growth and to properly orient the beam so that it will enter the extractor region without loss of the beam to the walls at other azimuths.

Thus, for motion in the median plane, in which case the magnetic field of the accelerator is

$$B_y/B_\theta = -(1+x)^k G(\theta)$$

$$B_\theta = B_x = 0,$$

the equation of motion becomes

$$(1+x)^2 x'' ((1+x)^2 + x'^2)^{-3/2}$$
$$= ((1+x)^2 + 2x'^2)(1+x)((1+x)^2 + x'^2)^{-3/2} - (1+x)^{k+1} G(\theta) . \qquad (2)$$

Ignoring all terms of order xx', x^2, x'^2, and higher order, one obtains

$$x'' = 1 - G(\theta) + x\left[1 - (k+2)G(\theta)\right] . \qquad (3)$$

Making the substitution $x = w - 1/(k+2)$, one obtains

$$w'' + \left[(k+2)G(\theta) - 1\right] w = (k+1)/(k+2) . \qquad (4)$$

B. Derivation of the Betatron Oscillations

If w is separated into $w = w_e + w_b$, with w_e representing the periodic solution to the inhomogeneous equation, (4), i.e., the equilibrium orbit, one is left with the desired linear equation of the betatron oscillations, w_b, about the equilibrium orbit. Thus the equation of motion of the betatron oscillations is

$$w_b'' + \left[(k + 2)\, G(\theta) - 1 \right] w_b = 0 \,. \tag{5}$$

With a G (θ) given by

$$G(\theta) = F(\theta) + \lambda f(\theta)$$

where F (θ) represents the unperturbed field,

$$F(\theta) = A + \left[m_\nu / (k + 2) \right] H(\theta)$$

while H (θ), the field flip-flop, is given by

$$H(\theta) = 1 \quad \text{for} \quad -\pi/2N < (\theta, \bmod 2\pi/N) < \pi/2N,$$

$$H(\theta) = -1 \quad \text{for} \quad \pi/2N < (\theta, \bmod 2\pi/N) < 3\pi/2N,$$

and the perturbation of strength λ has the form

$$f(\theta) = \sum_\sigma \xi_\sigma \cos(\sigma\theta + \delta_\sigma) \,. \tag{6}$$

The solutions to equation (5) may be written in the Floquet form,

$$w_b = C e^{\mu\theta} \Phi_1 + D e^{-\mu\theta} \Phi_2 \tag{7}$$

in which μ is real for operation inside an unstable zone and where Φ_1 and Φ_2 are functions periodic in two revolutions about the machine in the case of the half-integral resonance. These functions are determined by the structure F (θ) and the perturbation $\lambda f(\theta)$.[1]

The problem is to determine f (θ) such that the unstable zone is opened and so that μ has a value which permits the ascending solution to dominate and Φ_1 has the spatial dependence and periodicity required for beam extraction.

Perturbation theory may be used to solve equation (5) conveniently, using the complete set of functions, $x^j_{\nu,\tau}$, where

$$x^j_{\nu,\tau}{}'' + \left[(k + 2) A - 1 + m_\nu H(\theta)\right] x^j_{\nu,\tau} = - a^j_{\nu,\tau} \, x^j_{\nu,\tau} \quad .$$

The latin superscript refers to the parity of the eigenfunction, the subscript τ refers to the fundamental frequency of oscillation in the eigenfunction $x^j_{\nu,\tau}$ and the subscript ν is the fundamental frequency of the unperturbed eigenfunction.

To first order, the exponential growth factor μ is

$$\mu = (m_1^{(2)} - m_1^{(1)})\left(2 \left\langle 2,\nu \left| \frac{d}{d\theta} \right| 1,\nu \right\rangle\right)^{-1} \langle 1,\nu | H | 1,\nu \rangle \left[\alpha(1 - \alpha)\right]^{1/2} \qquad (8)$$

where

$$m_1^j = - \langle j,\nu | \; f \; | j,\nu \rangle \langle j,\nu | H | j,\nu \rangle^{-1}$$

$$\left[\alpha(1 - \alpha)\right]^{1/2} \simeq 1/2$$

$$\langle j,\nu | f, | j,\nu \rangle \; 2^{-1} \, \xi_{2\nu} \cos(\delta_{2\nu} - j\pi) + \xi_0 \cos \delta_0 \qquad (9)$$

$$\langle j,\tau | f | j,\tau \rangle = \int_0^{4\pi} x^j_{\nu,\tau}(\theta) \, f(\theta) \, x^j_{\nu,\tau}(\theta) \, d\theta \; .$$

The stopband is thus opened to first order if

$$\langle 1,\nu \mid f \mid 1,\nu \rangle \neq \langle 2,\nu \mid f \mid 2,\nu \rangle \ .$$

Only a $\xi_{2\nu} \cos(2\nu\theta + \delta_{2\nu})$ perturbation term will accomplish this.

A first-order approximation to Φ_1 is

$$\begin{aligned}\Phi_1 \simeq \cos \nu(\theta - \phi_o/\nu) &+ B_\nu \cos\left[(N-\nu)(\theta - \phi_o/\nu) + N\phi_o/\nu\right] \\ &+ C_\nu \cos\left[(N+\nu)(\theta - \phi_o/\nu) + N\phi_o/\nu\right] \\ &+ (2a^{(1)}_{\nu,\nu+\rho})^{-1}\, \xi_{2\nu+\rho}\Big\{\cos\left[(\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho\right] \\ &+ B_{\nu+\rho} \cos\left[(N-\nu-\rho)(\theta - \phi_o/\nu) - \gamma_\rho + (N\phi_o/\nu)\right] \\ &+ C_{\nu+\rho} \cos\left[(N+\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho + (N\phi_o/\nu)\right] \qquad (10)\end{aligned}$$

where

$$\gamma_\rho = \delta_{2\nu+\rho} + (\rho\phi_o/\nu) + 2\phi_o, \quad \rho = \pm 1, \pm 2, \cdots .$$

The maximum of Φ_1 occurs at $\theta_o = \phi_o/\nu$, with the dominant terms reinforcing the fundamental if one chooses

$$\xi_{2\nu+\rho} = \frac{(|\nu+\rho| - \nu)}{|(\nu+\rho - \nu)|}\, \xi_{2\nu+\rho} \ , \quad N\phi_o/\nu = 2\pi\ell \ , \quad \ell = 1, 2, 3, \cdots . \qquad (11)$$

$\delta_{2\nu+\rho} = -(2\phi_o + \phi_o/\nu) = 2\pi N^{-1}(2\nu+\rho)$. Operation in the center of the stopband gives the auxiliary condition

$\phi_o = (n + \frac{1}{4})\pi$, n = 1, 2, 3, $\cdots$, so that $\ell = (8\nu)^{-1}(4n+1)N$.

A first-order approximation to Φ_2 is

$$
\begin{aligned}
\Phi_2 = & - \cos\left[\nu(\theta - \phi_o/\nu) + 2\phi_o\right] \\
& - B_\nu \cos\left[(N-\nu)(\theta - \phi_o/\nu) - 2\phi_o + N\phi_o/\nu\right] \\
& - C_\nu \cos\left[(N+\nu)(\theta - \phi_o/\nu) + 2\phi_o + N\phi_o/\nu\right] \\
& - (\xi_{2\nu+\rho}/2a_{\nu,\nu+\rho})\Big\{\cos\left[(\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho - 2\phi_o\right] \\
& + B_{\nu+\rho}\cos\left[(N-\nu-\rho)(\theta - \phi_o/\nu) - \gamma_\rho + N\phi_o/\nu + 2\phi_o\right] \\
& + C_{\nu+\rho}\cos\left[(N+\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho + N\phi_o/\nu - 2\phi_o\right]\Big\}.
\end{aligned}
$$

For operation in the center of the stopband, the auxiliary condition holds, and Φ_2 becomes

$$
\begin{aligned}
\Phi_2 = & \sin\nu(\theta - \phi_o/\nu) - B_\nu \sin(N-\nu)(\theta - \phi_o/\nu) \\
& + C_\nu \sin(N+\nu)(\theta - \phi_o/\nu) \\
& - \frac{\xi_{2\nu+\rho}}{2a_{\nu,\nu+\rho}}\Big\{\sin\left[(\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho\right] \\
& - B_{\nu+\rho}\sin\left[(N-\nu-\rho)(\theta - \phi_o/\nu) - \gamma_\rho + N\phi_o/\nu\right] \\
& + C_{\nu+\rho}\sin\left[(N+\nu+\rho)(\theta - \phi_o/\nu) + \gamma_\rho + N\phi_o/\nu\right]\Big\}. \quad (12)
\end{aligned}
$$

The conditions which maximized Φ_1 at $\theta_o = \phi_o/\nu$ thus give Φ_2 a value of zero at $\theta = \phi_o/\nu$.

The coefficients of equations (10) and (12) are

$$B_\tau = \frac{2\, m_\nu}{\pi}\left[(N - \tau)^2 - (k + 2) A + 1 - a^j_{\nu,\tau}\right]^{-1}$$

$$C_\tau = \frac{2\, m_\nu}{\pi}\left[(N + \tau)^2 - (k + 2) A + 1 - a^j_{\nu,\tau}\right]^{-1}$$

$$a^j_{\nu,\tau} \simeq \left[\tau^2 - \nu^2\right]\left[1 + 4\,(\nu/N)^2\right]^{-1}$$

$$m_\nu^2 = \frac{\pi^2}{8}\left[\nu^2 + 1 - (k + 2) A\right]\left[(N - \nu)^2 - (k + 2) A + 1\right] \times$$

$$\left[(N + \nu)^2 - (k + 2) A + 1\right]\left[N^2 + \nu^2 - (k + 2) A + 1\right]^{-1} .$$

C. Derivation of the Equilibrium Orbit Solution

Since the homogeneous equation (5) does not contain first derivatives of w_b, the Wronskian is equal to a constant. That is, for properly normalized independent solutions $w_b^{(1)}(\theta)$ and $w_b^{(2)}(\theta)$ of the homogeneous equation,

$$w_b^{(1)}(\theta) = N_1\, e^{\mu\theta}\, \Phi_1(\theta) \quad \text{and} \quad w_b^{(2)}(\theta) = N_2\, e^{-\mu\theta}\, \Phi_2(\theta) ,$$

$$\begin{vmatrix} w_b^{(1)}(\theta) & w_b^{(2)}(\theta) \\ w_b^{(1)'}(\theta) & w_b^{(2)'}(\theta) \end{vmatrix} = 1 .$$

Hence, one can construct the periodic solution of equation (4) using the Green's function,

$$G(\theta, \theta') = w_b^{(1)}(\theta')\, w_b^{(2)}(\theta), \quad 0 < \theta' < \theta, \quad G(\theta, \theta')$$

$$= w_b^{(1)}(\theta)\, w_b^{(2)}(\theta'), \quad \theta < \theta' < 4\pi .$$

The general solution to the inhomogeneous equation therefore becomes

$$w_e = A\, e^{\mu\theta}\, \Phi_1(\theta) + B\, e^{-\mu\theta}\, \Phi_2(\theta)$$

$$+ \frac{k+1}{k+2} N_1 N_2\, e^{-\mu\theta}\, \Phi_2(\theta) \int_0^{\theta} e^{+\mu\theta'}\, \Phi_1(\theta')\, d\theta'$$

$$+ \frac{k+1}{k+2} N_1 N_2\, e^{\mu\theta}\, \Phi_1(\theta) \int_{\theta}^{4\pi} e^{-\mu\theta'}\, \Phi_1(\theta')\, d\theta' .$$

To obtain the equilibrium orbit, the constants A and B must be chosen so that the nonperiodic parts of w_e vanish. Thus, an approximation to w_e to first order in the pertur bation is

$$w_e \left(\frac{k+1}{k+2} N_1 N_2\right)^{-1} = \frac{1}{\nu} - \frac{B_\nu^2}{N-\nu} + \frac{C_\nu^2}{N+\nu} + B_\nu\left[\frac{(N-2\nu)}{(N-\nu)\nu} \tag{13}$$

$$+ C_\nu \left(\frac{N+2\nu}{(N+\nu)\nu}\right)\right] \cos N\theta' - \frac{2 B_\nu C_\nu}{N^2-\nu^2} \cos 2N\theta'$$

$$+ \frac{\xi_{2\nu+\rho}}{2 a_{\nu,\nu+\rho}} \left\{\left[\frac{\rho}{(\nu+\rho)\nu} + C_\nu B_{\nu+\rho} \frac{(2N-\rho)}{(N+\nu)(N-\nu-\rho)}\right.\right.$$

$$\left. - B_\nu C_{\nu+\rho} \frac{2N+\rho}{(N+\nu+\rho)(N-\nu)}\right] \cos (2\nu+\rho)\theta'$$

$$+ \left[\frac{(N-\rho) B_{\nu+\rho}}{\nu(N-\nu-\rho)} - \frac{(N+\rho) B_\nu}{(\nu+\rho)(N-\nu)}\right] \cos (N-2\nu-\rho)\theta'$$

$$+ \left[\frac{(N+\rho) C_{\nu+\rho}}{(N+\nu+\rho)} - \frac{(N-\rho) C_\nu}{(\nu+\rho)(N+\nu)}\right] \cos (N+2\nu+\rho)\theta'$$

$$+ \frac{\rho B_\nu B_{\nu+\rho}}{(N-\nu-\rho)(N-\nu)} \cos (2N-2\nu-\rho)\theta'$$

$$+ \frac{\rho C_\nu C_{\nu+\rho}}{(N+\nu+\rho)(N+\nu)} \cos (2N+2\nu+\rho)\theta'$$

$$+ \frac{\mu}{\nu^2} \sin 2\nu\theta' - \mu B_\nu\left[\frac{1}{\nu^2} + \frac{1}{(N-\nu)^2}\right] \sin (N-2\nu)\theta'$$

$$+ \mu C_\nu\left[\frac{1}{\nu^2} + \frac{1}{(N+\nu)^2}\right] \sin (N+2\nu)\theta'$$

$$- \frac{B_\nu^2 \mu}{(N-\nu)^2} \sin (2N-2\nu)\theta' + \frac{C_\nu^2 \mu}{(N+\nu)^2} \sin (2N+2\nu)\theta'$$

where

$$\theta' = \theta - \phi_0/\nu .$$

It is evident that the coefficients of the harmonics added to the equilibrium orbit which contain the factor $1/(\nu+\rho)$ are large if $(\nu+\rho)$ is small. Thus the largest harmonic added to the equilibrium orbit is that for which $(\nu+\rho) = -\frac{1}{2}$, in the case of the half integral resonance. One always has two choices of ρ in picking the field perturbation introducing the harmonic with the fundamental frequency $(\nu+\rho)$ into the ascending betatron solution. That is, one can choose such that $(\nu+\rho) = \pm\left|(\nu+\rho)\right|$. The choice of $\rho < 0$ such that $(\nu+\rho) < 0$ always has a much more favorable effect on the equilibrium orbit than the other choice, because it always adds harmonics to the equilibrium orbit which have their maximum at the extraction azimuth. In the case of $\rho < 0$, the choice $(\nu+\rho) > 0$ has a deleterious effect since the largest harmonic added to the equilibrum orbit has its minimum at the extraction azimuth. The choice, for $\rho > 0$, $(\nu+\rho) > 0$ adds its largest perturbation harmonic to the equilibrium orbit such that its maximum is at the extraction azimuth, but its maximum is usually much smaller than the choice $\rho < 0$, $(\nu+\rho) < 0$. In addition, some of the smaller harmonics added have their minima at the extraction azimuth.

Although it may not be possible to make $(\nu+\rho)$ small in introducing certain harmonics into the ascending betatron solution, it should be possible to choose a perturbation so that a favorable effect on the equilibrium orbit is obtained.

The terms in the equilibrium orbit which are multiplied by the growth factor μ have, in general, a deleterious effect upon the extraction properties of the equilibrium orbit since these terms take on the value 0 at the extraction azimuth, while they may add to peaks at other azimuths. Hence, one should usually avoid large values of μ.

III. APPLICATION OF THE THEORY TO THE MURA 50 MEV RADIAL SECTOR MACHINE

The MURA machine tuned for one-way operation operates on the $\nu_x = 4.5$ resonance and has an unperturbed magnetic field in the median plane, $G(\theta) = 0.57015 + 5.65325 \cos 16\,\theta$ plus small harmonics of $16\,\theta$, $k = 9.3$.

It is desired to choose a perturbation for the field that enhances Φ_1 at some particular azimuth. From equation (11), choosing $\ell = 4$, one obtains $\phi_o/\nu = \pi/2$ as the extraction azimuth. By equation (10) the unperturbed function is

$$\Phi_1 \simeq \cos 4.5\,(\theta - \pi/2) + 0.298 \cos 11.5\,(\theta - \pi/2)$$

$$+ 0.091 \cos 20.5\,(\theta - \pi/2)\,.$$

To aid in extraction, it was decided to add harmonics to the function Φ_1 such that its magnitude at any azimuth other than the extraction azimuth is no greater than 0.9 of the magnitude of Φ_1 at the extraction azimuth. In particular, the harmonics added to the unperturbed function must reduce the absolute value of the function at azimuths other than the extraction azimuth where it takes on large values. It remains, therefore, to find an economical combination of harmonics which will accomplish this. In addition, one must recognize that the magnetic field perturbations used to produce these harmonics in Φ_1 may affect the equilibrium orbit in a way which may greatly enhance ease of extraction or may have a deleterious effect. Keeping in mind that perturbation harmonics closest to the resonant frequency have the greatest effect on the oscillation amplitude, one obtains by trial and error a sum of the harmonics $\cos 0.5 (\theta - \pi/2)$ and $\cos 5.5 (\theta - \pi/2)$ as a combination which gives the proper shaping of Φ_1. The perturbations required to introduce these harmonics are found from equations (6), (10), and (11). Thus the perturbation added to $F(\theta)$ to introduce $\cos 0.5 (\theta - \pi/2)$ into Φ_1 is either $-|\xi_4| \cos 4\theta$ or $-|\xi_5| \sin 5\theta$. The perturbation introducing a harmonic of $\cos 5.5 (\theta - \pi/2)$ is either $-|\xi_{10}| \cos 10\theta$ or $-|\xi_1| \sin \theta$.

It remains to choose between the alternate perturbations introducing the desired harmonics into Φ_1 that combination of perturbations which has the most favorable effect on the equilibrium orbit. One sees, from equation (13), that the two alternate perturbations of equal perturbation strength which introduce identical harmonics into Φ_1, introduce into the equilibrium orbit different harmonics which may be of greatly different magnitudes and may even be of opposite sign. To clarify this, consider the principal harmonic of the equilibrium orbit

$$\frac{\xi_{2\nu+\rho}}{2a_{\nu,\,\nu+\rho}} \quad \frac{\rho}{\nu(\nu+\rho)} \cos(2\nu+\rho)(\theta - \pi/2)$$

added by a perturbation term of the form

$$\xi_{2\nu+\rho} \quad \cos\left[(2\nu+\rho)\,\theta + \delta_{2\nu+\rho}\right].$$

Here only the main coefficient of the principal harmonic is included. Thus, the perturbation $-\left|\xi_4\right| \cos 4\,\theta$ adds a positive harmonic of $\cos 4\,(\theta - \pi/2)$ to the equilibrium orbit, of magnitude 0.65 times the magnitude of the principal harmonic $\cos 16\,\theta$ for $\left|\xi_4\right| = 0.05$. On the other hand, the perturbation $-\left|\xi_5\right| \sin 5\,\theta$ adds a negative harmonic of $\cos 5\,(\theta - \pi/2)$ of the same order of magnitude. Hence, the perturbation $-\left|\xi_4\right| \cos 4\,\theta$ is chosen since it enhances the maximum at the extraction azimuth, while the $-\left|\xi_5\right| \sin 5\,\theta$ perturbation depresses the maximum at the extraction azimuth with respect to maxima at other azimuths. Similarly, $-\left|\xi_{10}\right| \cos 10\,\theta$ adds a $\cos 10\,(\theta - \pi/2)$ harmonic and a $\cos 6\,(\theta - \pi/2)$ harmonic to the equilibrium orbit, both of magnitude 0.05 times the magnitude of the principal harmonic, for $\left|\xi_{10}\right| = 0.08$. The perturbation $-\left|\xi_{10}\right| \cos 10\,\theta$ produces an equilibrium orbit in which the extraction azimuth maxima is enhanced over all other maxima by a magnitude

0.2 times the principal harmonic except for the azimuth $\theta = 3\pi/2$, at which point the maximum is equal to the extraction azimuth maximum. Since the ascending betatron oscillation amplitude is near zero at $\theta = 3\pi/2$, this equilibrium orbit is also very favorable for extraction purposes.

IV. DIGITAL CALCULATIONS

A. Radial Motion

The perturbed azimuthal field component, $G(\theta)$, used in the digital calculation is

$$G(\theta) = 0.57015 + 5.5 \cos 16\theta - 0.05 \cos 4\theta - 0.08 \cos 10\theta - 0.0113 \cos 9\theta \,. \qquad (14)$$

The cos 9θ term is used to open the stopband. The approximate normalized function Φ_1 then becomes

$$\Phi_1 = 0.678 \cos 4.5\theta' + 0.194 \cos 11.5\theta' + 0.061 \cos 20\theta' + 0.012 \cos 0.5\theta' + 0.040 \cos 5.5\theta' - 0.004 \cos 7.5\theta' + 0.015 \cos 10.5\theta' + 0.003 \cos 21\theta' \qquad (15)$$

$$\theta' = \theta - \pi/2 \,.$$

This function has the property of having in no region other than $\theta' = 0$, mod. 2, an absolute value greater than 0.9.

The approximate equilibrium orbit is obtained from equation (13),

$$w_e = \left[N_1 N_2 \frac{(k+1)}{(k+2)}\right]\left[0.215 + 0.064 \cos 16\,\theta' \right.$$

$$+ 0.042 \cos 4\,\theta' + 0.011 \cos 12\,\theta' + 0.004 \cos 10\,\theta' \qquad (16)$$

$$\left. + 0.003 \cos 6\,\theta'\right].$$

The factor $N_1\, N_2$ could be obtained by using the approximate solutions of Φ_1 and Φ_2 in normalizing the Wronkskian,

$$\left| \Phi_1 (\theta' = 0) \quad \Phi_2' (\theta' = 0) \right| = (N_1\, N_2)^{-1} . \qquad (17)$$

This is impractical since the higher frequency harmonics, which have small coefficients and therefore are neglected in the approximate solutions, have a large effect on the normalization of the Wronskian. Therefore, an empirical determination of $N_1\, N_2 \frac{(k+1)}{(k+2)}$ is made. This is done by equating the average value of the equilibrium orbit calculated digitally to the average value of the equilibrium orbit,

$$\overline{(x_e)}_{digital} = 0.215 \left[N_1\, N_2\ 10.3/11.3\right] - 1/11.3 . \qquad (18)$$

One obtains

$$N_1\, N_2\ 10.3/11.3 = 0.34 . \qquad (19)$$

This is in contrast to a value of 0.23, obtained from equation (16).

Figure 1 represents the analytic calculation of $e^{\mu\theta'} \Phi_1 (\theta')$, and the digital calculation of the displacement from the equilibrium orbit for the initial conditions

$$u(\theta' = 0) = 1 \times 10^{-3}, \quad u'(\theta = 0) = -2.86 \times 10^{-5}, \quad u = x - x_e .$$

Figure 2 represents the digital calculation of $u(\theta')$ for the larger initial conditions,

$$u(\theta' = 0) = 1 \times 10^{-2}, \quad u'(\theta = 0) = +2.86 \times 10^{-4} .$$

The initial conditions of u used in the digital calculations were chosen to pick out only the ascending solutions. The growth factor of the analytic solution portrayed in Fig. 1 is chosen to match the digital results. Figure 3 represents the digital and the analytic calculations of the equilibrium orbit. The total motion for the initial displacement

$$u(\theta' = 0) = 1 \times 10^{-2}$$

is shown in Fig. 4.

B. Vertical Motion Effects

In all the perturbation fields investigated, care was taken to keep the vertical tune as far from an integral or half-integral vertical resonance as practical. This was done, when necessary, by adjusting the relative strengths of the field perturbation terms, thus putting a restriction on the form of the perturbation field. For the perturbation field

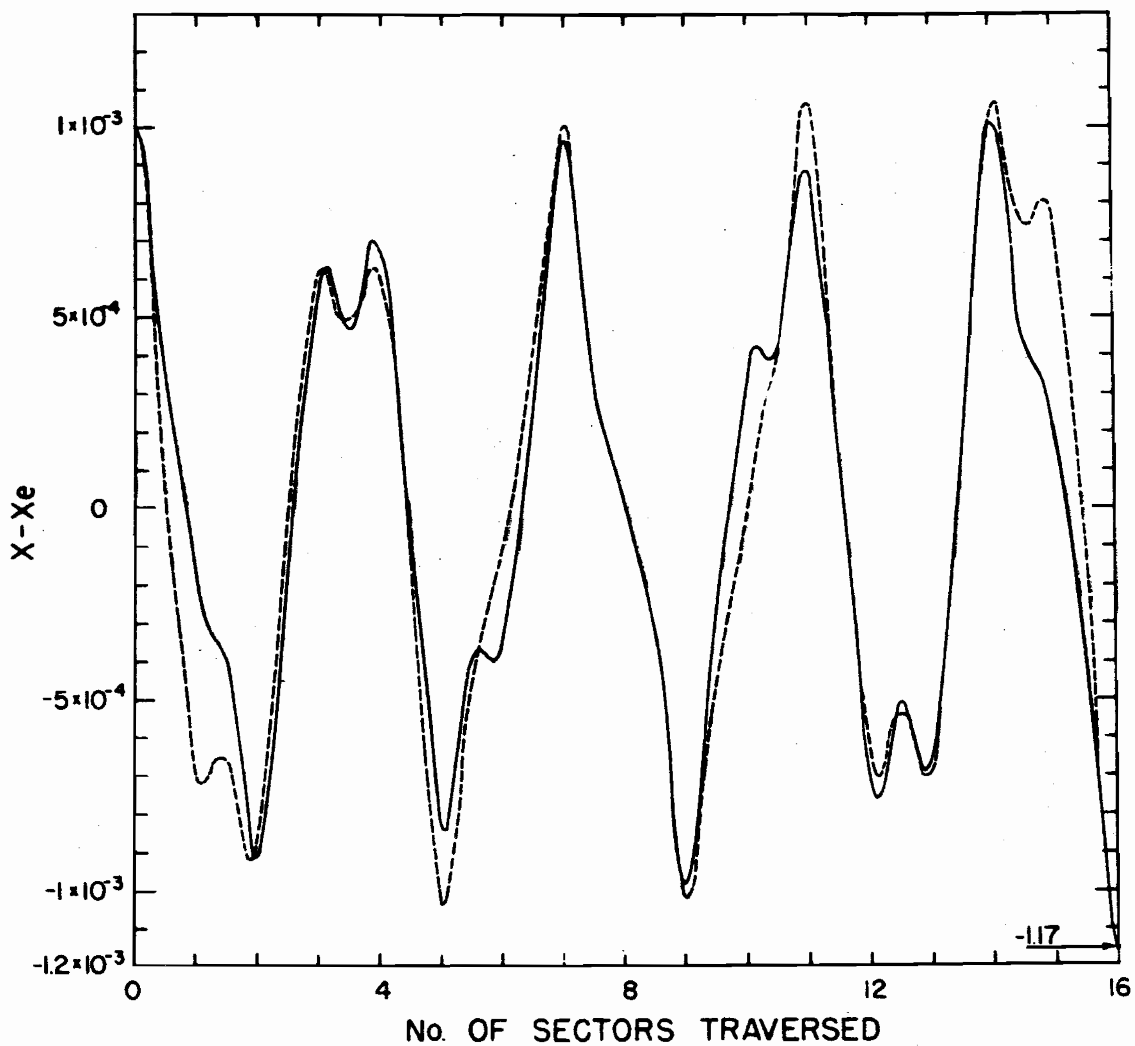

Fig. 1. Displacement from the equilibrium orbit, $u\ (\theta' = 0) = 1 \times 10^{-3}$.

———— analytic calculation
------- digital calculation

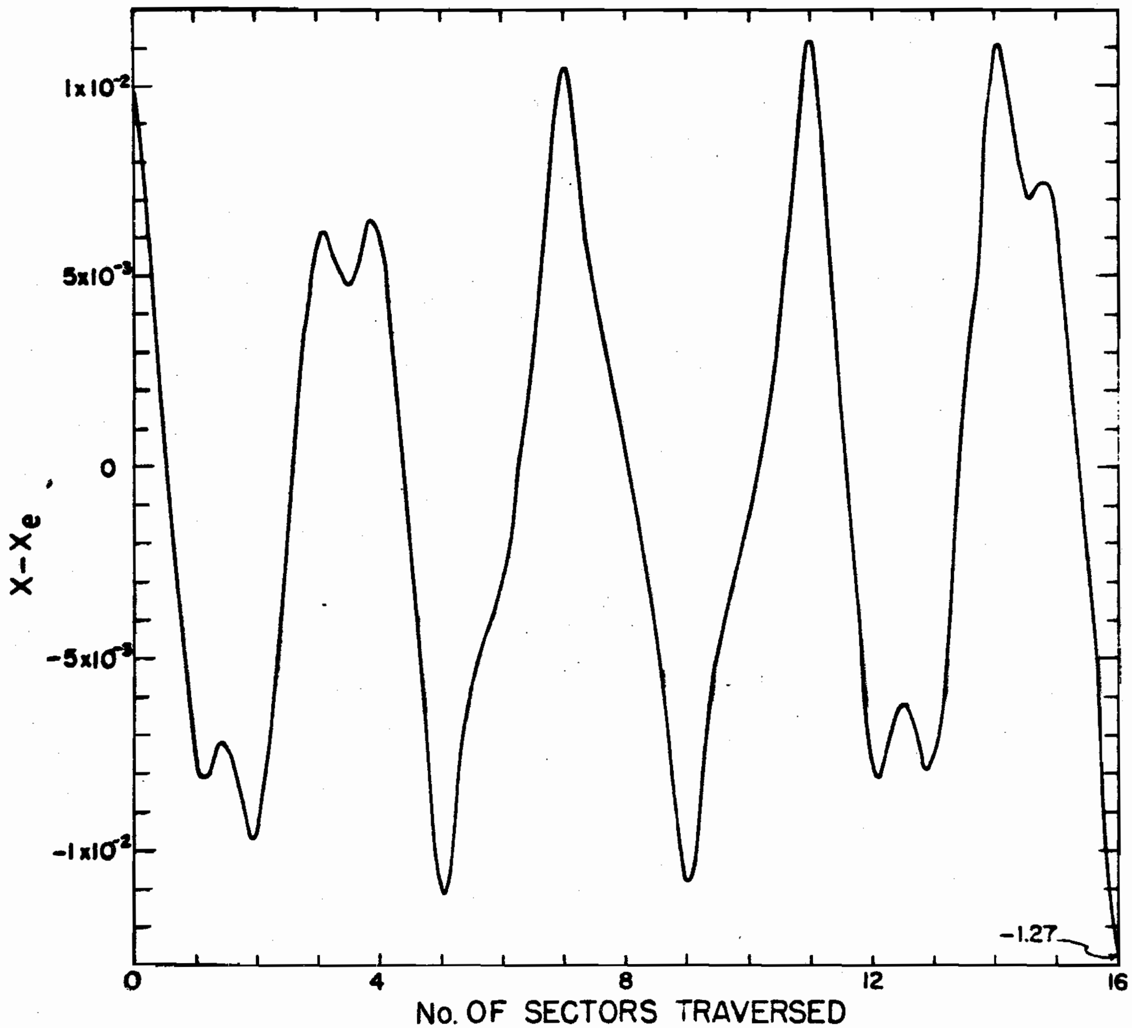

Fig. 2. Displacement from the equilibrium orbit, $u(\theta' = 0) = 1 \times 10^{-2}$.

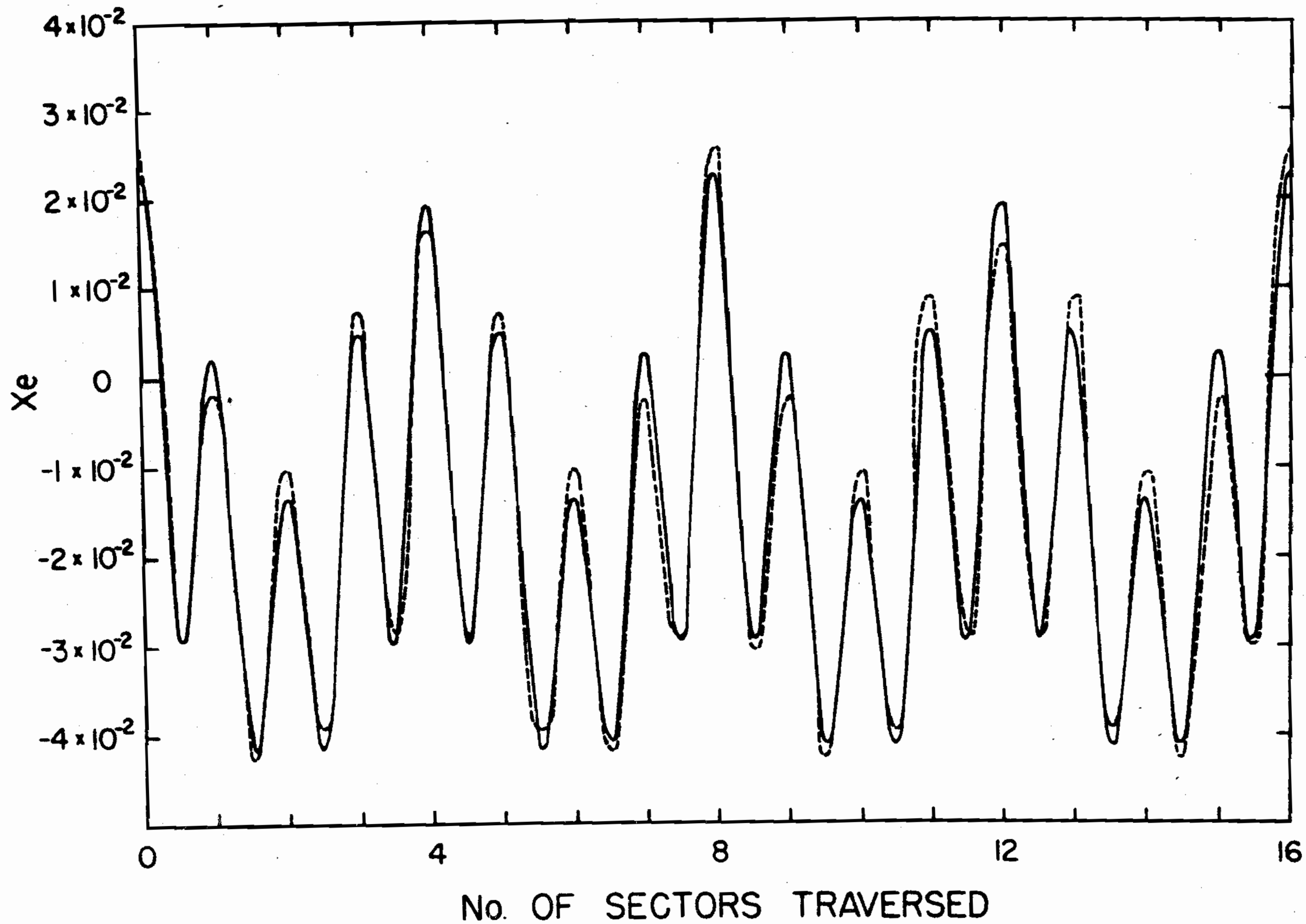

Fig. 3. Equilibrium orbit.

——— analytic calculation

------ digital calculation

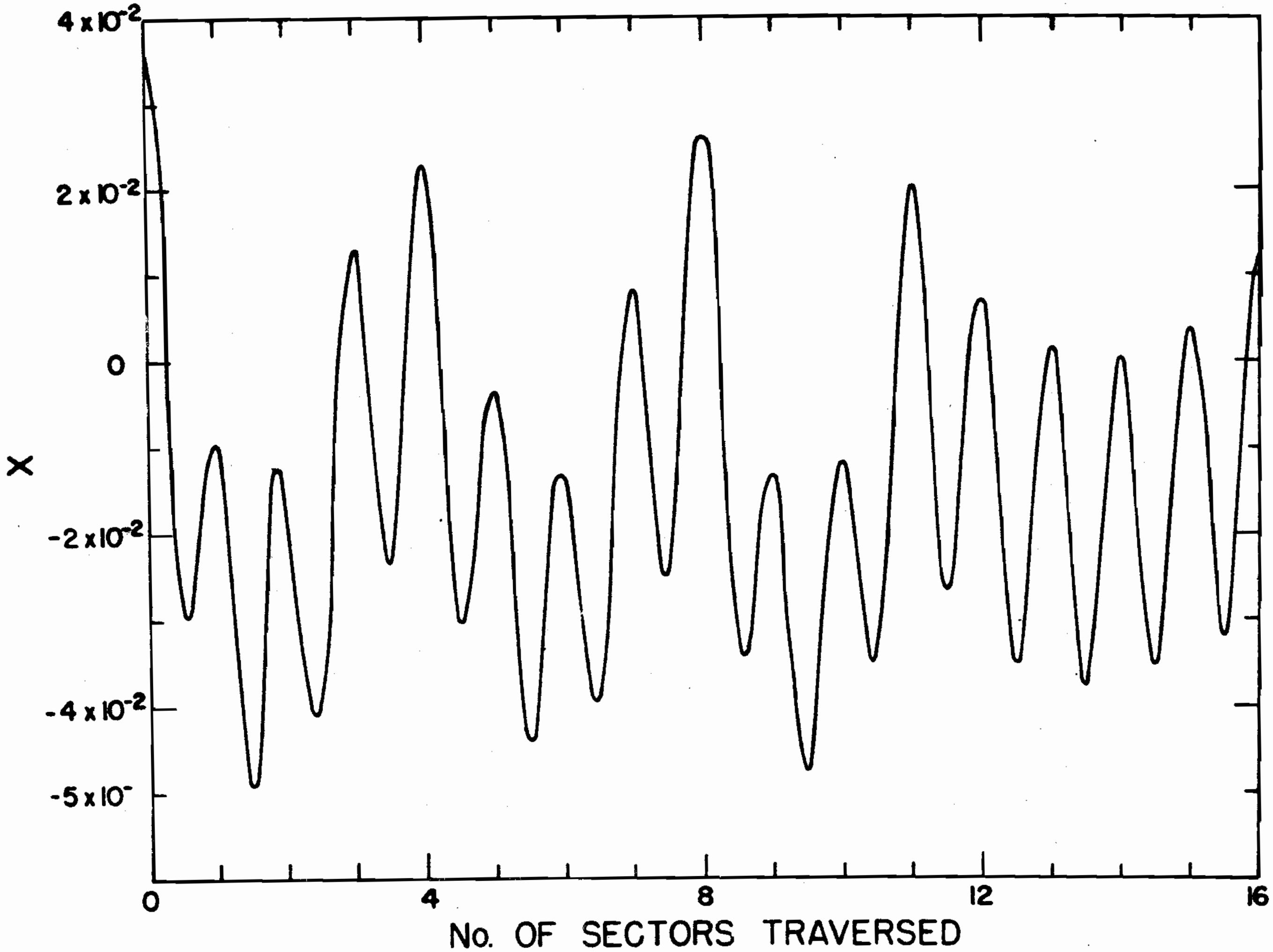

Fig. 4. Total displacement, $u(\theta' = 0) = 1 \times 10^{-2}$.

under present consideration, $\;_{y} = 2.80$, well between the integral and half-integral vertical resonances.

Coupling effects due to the vertical motion were investigated for two ranges of vertical amplitude. For a total vertical amplitude, y_t, of magnitude 0.004 and 0.0048, corresponding to initial radial amplitudes of 0.001 and 0.01, respectively, the coupling effects were entirely negligible. For a 200 cm radius machine, these vertical amplitudes are 0.8 cm and 0.96 cm. For total vertical amplitudes of 0.04 and 0.05, corresponding to radial amplitudes of 0.001 and 0.01, respectively, coupling effects were quite pronounced. This range of vertical amplitudes is much larger than would be encountered practically, however. Table I compares the radial oscillations for the uncoupled cases, $y_t = 0$, and the coupled cases.

C. An Additional Perturbation Field

One additional perturbation field studied is of the form

$$G(\theta) = 0.57015 + 5.91 \cos 16\theta - 0.01 \cos 9\theta - 0.1 \cos 10\theta - 0.2 \sin 11\theta \,. \qquad (20)$$

With this perturbation field the approximate function ϕ_1 is

$$\phi_1 = 0.632 \cos 4.5\theta' + 0.190 \cos 11.5\theta' + 0.058 \cos 20.5\theta' + 0.042 \cos 5.5\theta' + 0.017 \cos 10.5\theta' + 0.038 \cos 6.5\theta' + 0.023 \cos 9.5\theta'$$

TABLE I. TOTAL RADIAL OSCILLATION

Total Radial Oscillation, $x = \frac{r - r_o}{r_o}$, multiplied by 10

$\theta/2$	$y_t = 0.004$ $x - x_e = 0.001$	$y_t = 0$ $x - x_e = 0.001$	$y_t = 0.04$ $x - x_e = 0.001$	$y_t = 0.0048$ $x - x_e = 0.01$	$y_t = 0$ $x - x_e = 0.01$	$y_t = 0.05$ $x - x_e = 0.01$
0.2500	0.26773	0.26773	0.26773	0.35773	0.35773	0.35773
.3125	- .02705	- .02689	- .04350	- .09743	- .09724	- .11677
.3750	- .11011	- .11006	- .11501	- .19681	- .19673	- .20453
.4375	.08099	.08098	.08109	.13331	.13332	.13194
.5000	.17343	.17379	.13830	.22939	.22990	.18133
.5625	.06464	.06452	.07546	- .03479	- .03506	- .01112
.6250	- .10212	- .10208	- .10731	- .12877	- .12877	- .13379
.6875	- .01050	- .01010	- .04612	.08409	.08481	.02266
.7500	.25787	.25786	.26080	.26022	.26005	.27745
.8125	- .03671	- .03684	- .03142	- .13408	- .13416	- .13752
.8750	- .10488	- .10473	- .11770	- .11902	- .11868	- .14772
.9375	.10066	.10084	.08870	.20134	.20151	.19411
1.0000	.14116	.14110	.13485	.07027	.07001	.07033
1.0625	.08352	.08350	.08758	.01422	.01428	.01165
1.1250	- .09480	- .09464	- .10736	.00382	.00414	- .01679
1.1875	- .1942	- .01920	- .04521	.04037	.04048	.01604
1.2500	.24612	.24605	.25324	.13049	.13027	.15088

The analytic calculations for the betatron oscillations and for the equilibrium orbit drastically fail to agree with the digital calculation. Figures 5, 6, 7, 8, and 9 represent the digital calculations of the equilibrium orbit and of the ascending solution with the initial amplitudes u = 0.001, 0.003, 0.006, 0.01. The form of the solution is dependent on the magnitude of the initial conditions, which is an undesirable result. Figure 8 demonstrates an extreme nonlinear effect. The large peak is very smooth when the equilibrium orbit is added in. It is possible that this large oscillation can be used as a feasible extraction process. At the present time this behavior is not understood and should be investigated further.

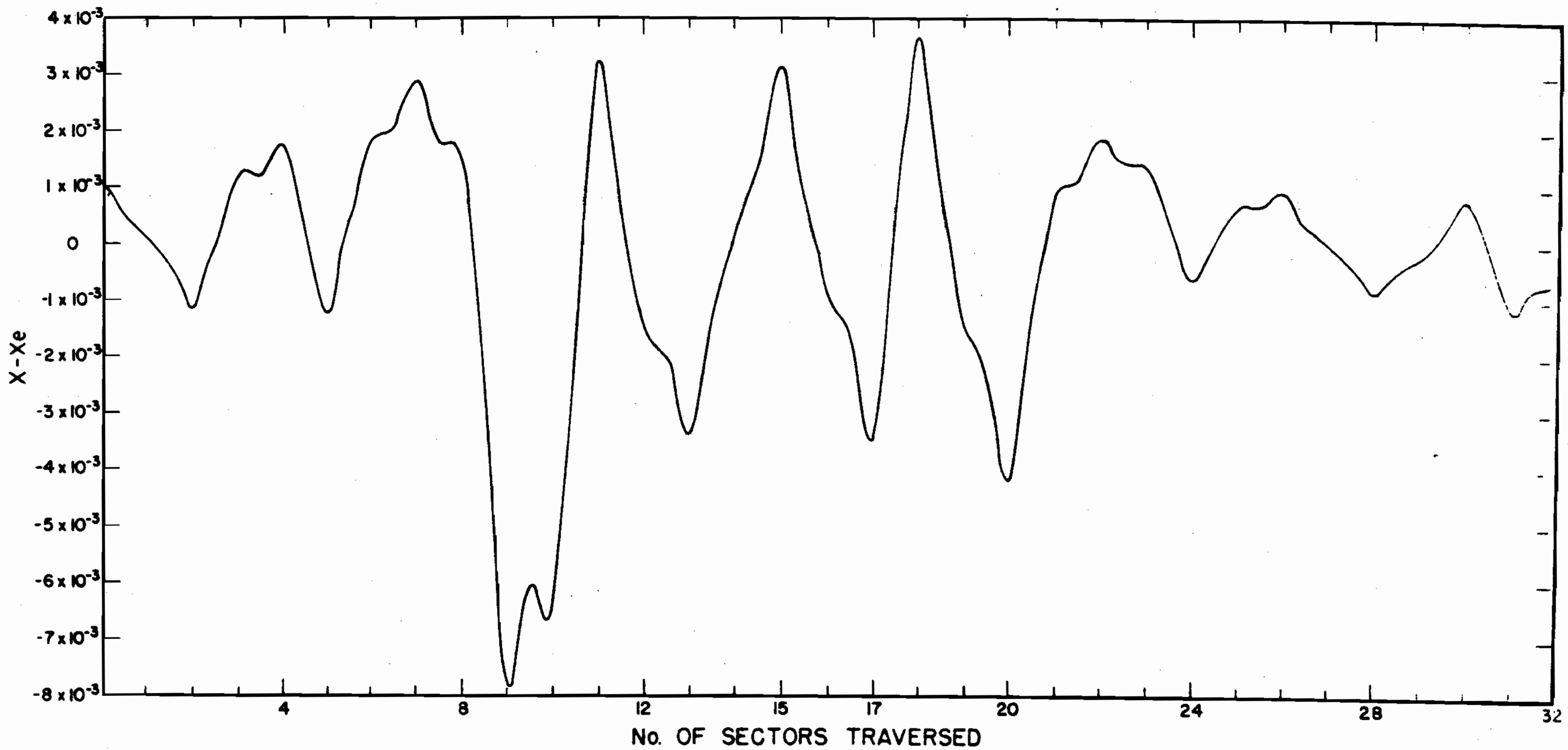

Fig. 5. Displacement from the equilibrium orbit, u (θ' = 0) = 1×10^{-3}.

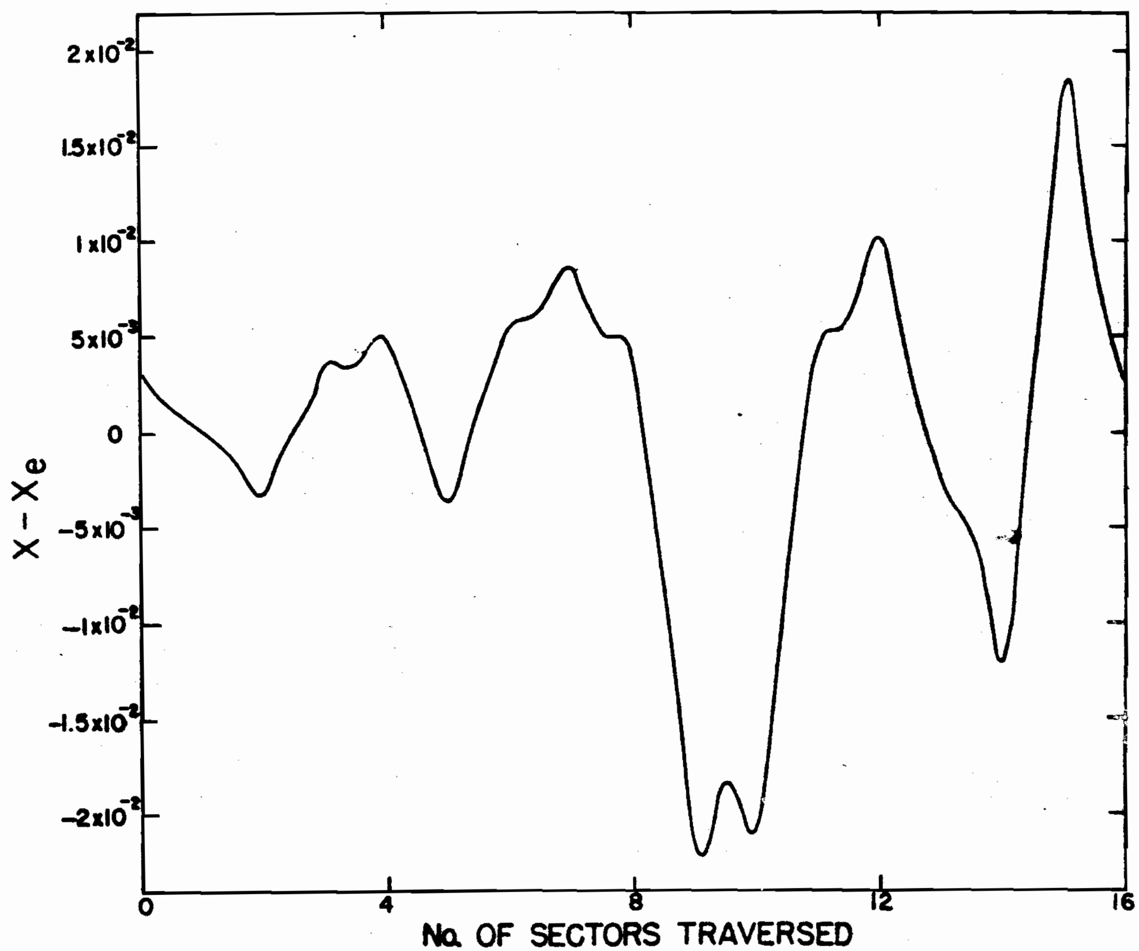

Fig. 6. Displacement from the equilibrium orbit, $x(\theta = 0) = 3 \times 10^{-3}$.

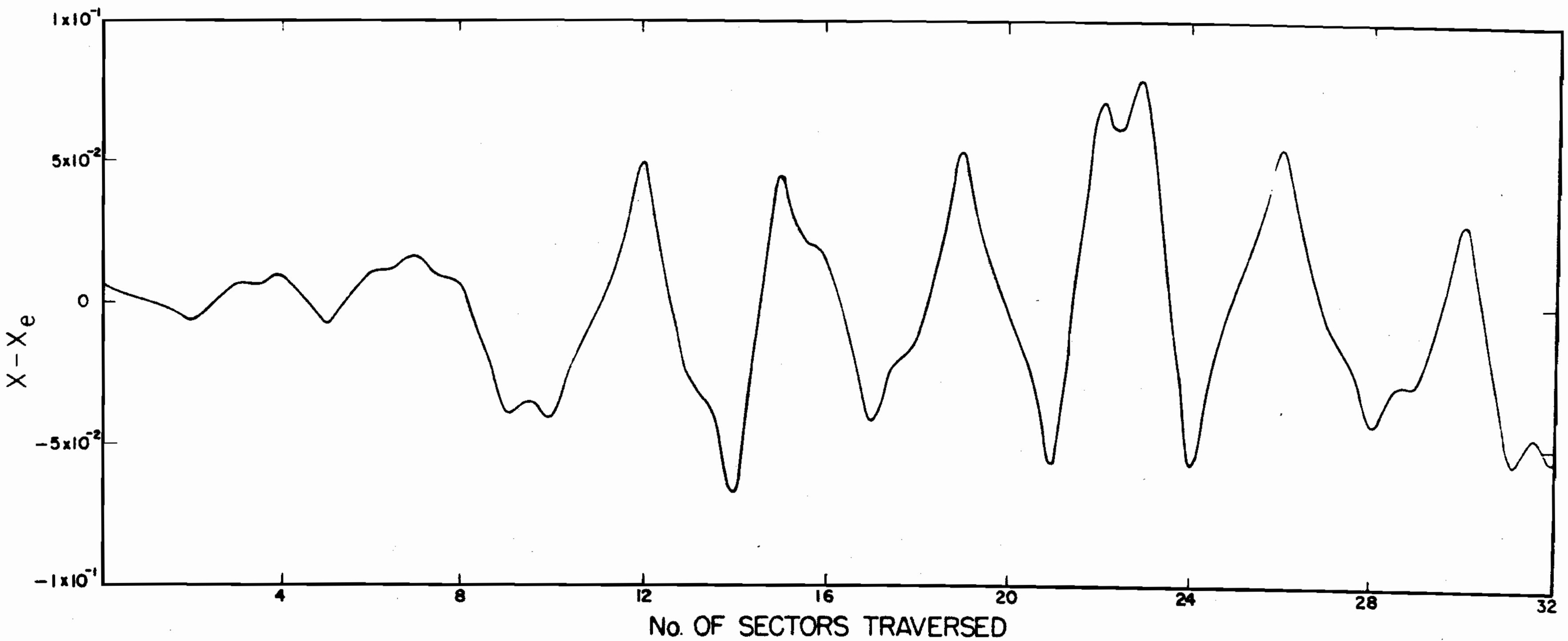

Fig. 7. Displacement from the equilibrium orbit, $u(\theta' = \theta) = 6 \times 10^{-3}$.

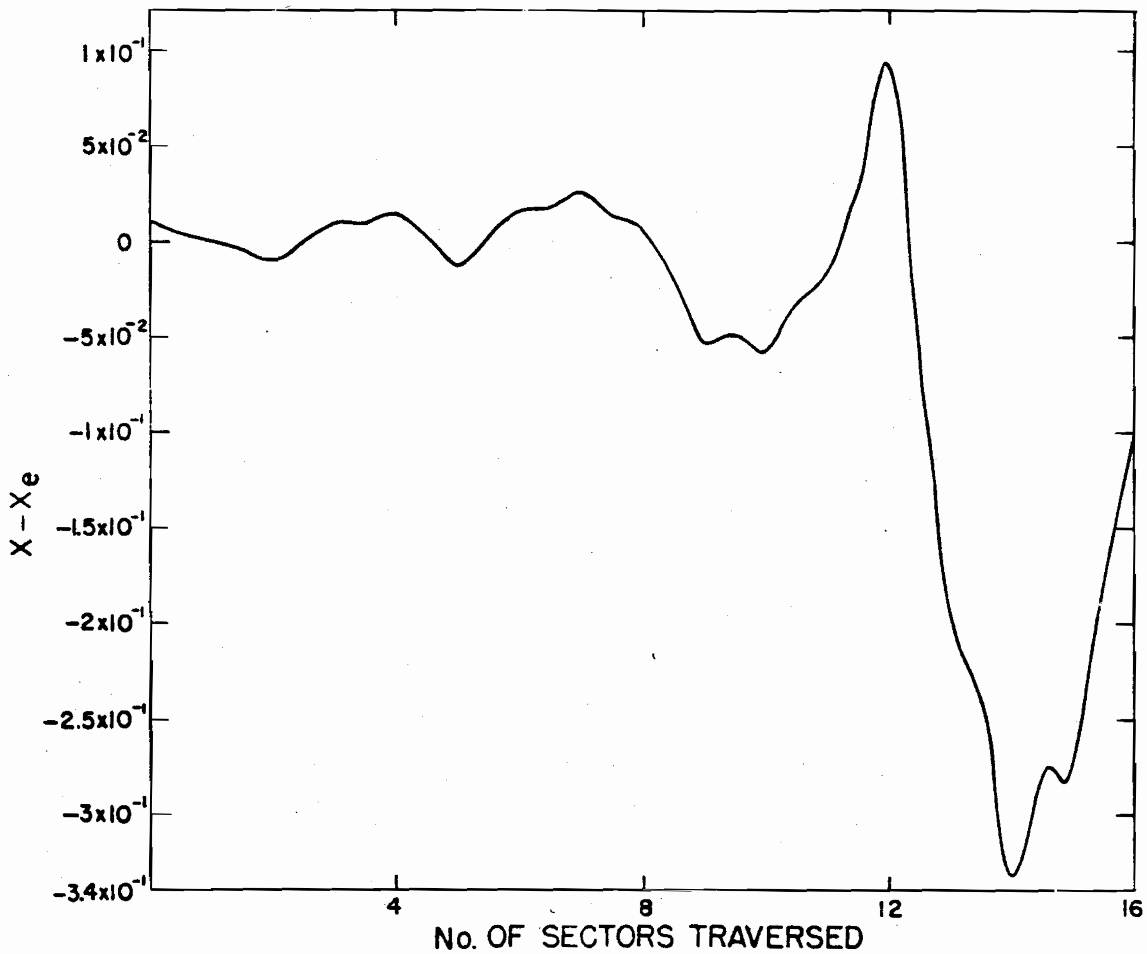

Fig. 8. Displacement from the equilibrium orbit, $u\ (\theta' = 0) = 1 \times 10^{-2}$.

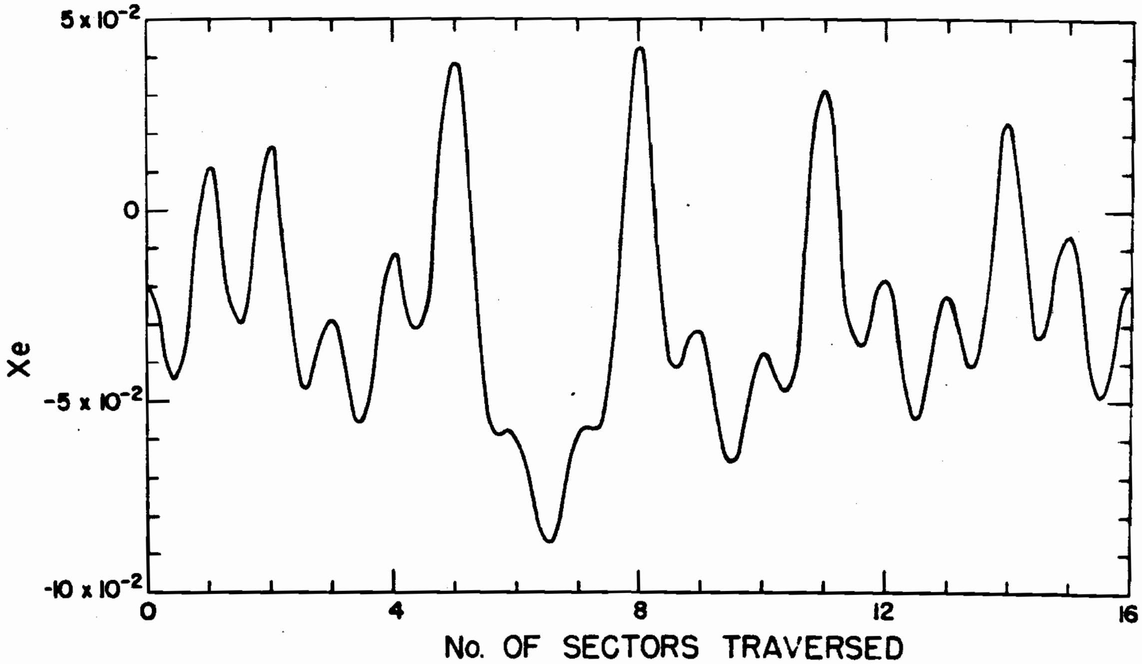

Fig. 9. Equilibrium orbit.

V. CONCLUSIONS

The first field perturbation used is very favorable for extraction from the particular machine considered. The favorable equilibrium orbit shape relaxes the requirements on the ascending betatron solution, requiring only that the ascending solution have an amplitude at the extraction azimuth significantly larger than the amplitude at the azimuth $\theta = 3\pi/2$ and have their extraction azimuth peaks enhanced over all other peaks. In addition, but for the difference in the exponential growth, these two ascending solutions are almost exactly the same, the one of larger initial amplitude being almost a photographic enlargement of the other when the exponential growth are factored out. This enlargement characteristic suggests strongly that all amplitudes between the two amplitudes investigated will similarly be enlargements, and the particles will be brought from small amplitudes to extraction amplitude without any beam loss.

In addition, it has been shown that for some fields the process of linearizing the equation of motion and applying the perturbation theory developed by C. L. Hammer and L. Jackson Laslett produces results in remarkable agreement with the digital calculations of the ascending betatron oscillation solutions when one considers the appreciable nonlinearities inherent in the problem. Thus, the linear theory provides an excellent guide to the important features of the betatron oscillations of a guide field with azimuthal perturbations present.

The analytic solution derived in the text for the equilibrium orbit provides a convenient method of finding the approximate values of the magnitudes of the various harmonics added to the equilibrium orbit by a given perturbation. However, as pointed out in the text, the average value of the equilibrium orbit as determined by the Wronkskian was inaccurate by approximately 30%.

The magnitude of the growth factor μ may be estimated analytically from equation (8) using the approximation

$$\langle 2,\nu \mid d/d\theta \mid 1,\nu \rangle = \nu - B_{\nu}^{2} (N - \nu) + C^{2} (N + \nu) .$$

This gives $\mu = 0.055/2\pi$, as compared to a digitally calculated value using small oscillations, of $\mu = 0.063/2\pi$. The width of the stopband may also be calculated from equation (8). The analytic calculation of the width is 3.01, as compared to the digital result of 1.95.

A complete breakdown occurred for the linear theory in the case of the second guide field studied. It is therefore apparent that careful consideration of the extraction problem is necessary when the operating point is chosen for the accelerator. That is, the operating point should be close to a half-integral resonance that does not introduce severe nonlinear characteristics.

REFERENCES

1. C. L. Hammer and L. Jackson Laslett, Rev. Sci. Instr. 32, 144-149 (1962).

2. E. Z. Chapman, "Flexible Fiver (Program 280)," Internal MURA Report No. 604. Midwestern Universities Research Association, Madison, Wisconsin.