COMPLEX SCALAR FIELDS IN SO(2,1)-INVARIANT BACKGROUNDS: REPRESENTATION OF THE SYMMETRIES IN THE SCHRÖDINGER PICTURE*

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ABSTRACT

We consider complex scalar fields minimally coupled to an SO(2,1)-invariant Maxwell potential in 1+1 de Sitter space and construct an ultraviolet-finite functional representation of the symmetry generators in the Schrödinger picture. We find that there is a unique vacuum state which is strictly SO(2,1)-invariant.

1.INTRODUCTION

For fields coupled to time dependent backgrounds, the concept of an energy ground state does not exist. However, when non-trivial symmetries are present, the vacua might alternatively be defined as the states that are invariant under the corresponding transformations. The implementation of such a definition requires the symmetry generators to be well-defined without reference to any particular field state. This in turn, can be achieved in the Schrödinger picture by demanding that the functional representations of the finite symmetry transformations be free of ultraviolet singularities[1].

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We shall explicitly illustrate how vacuum states can be characterized in this fashion by considering complex scalar fields minimally coupled to an SO(2,1)-invariant Maxwell potential in two-dimensional de Sitter space.

2. SCALAR FIELDS IN SO(2,1)-INVARIANT BACKGROUNDS

In terms of the conformal time t the de Sitter metrics reads [2]

$$g = \frac{1}{h^2t^2}(dt^2 - dx^2) \tag{2.1}$$

with $2h^2$ the curvature. A basis for the Killing vectors $X_f = f^\mu\partial_\mu$ of g is obtained by taking the three operators associated to $f^\mu{}_P = (0,1)$, $f^\mu{}_D = (t,x)$, $f^\mu{}_K = (tx, 1/2(t^2+x^2))$. These vector fields respectively generate infinitesimal translations, dilatations and special conformal transformations and obey SO(2,1) commutation rules: $[X_P,X_D] = X_P$, $[X_P,X_K] = X_D$, $[X_D,X_K] = X_K$.

It can be shown [3,4] that

$$\mathcal{A} = \mathcal{A}_\mu dx^\mu = (\lambda/t)dx \qquad \lambda\in\mathcal{R} \tag{2.2}$$

is the most general Maxwell potential which is invariant up to gauge tranformations [5] under the isometries of the de Sitter space. Indeed one can verify that the Lie derivative of $\mathcal{A}$, $\mathcal{L}_{X_f}\mathcal{A} = d\rho_f$ with $\rho_P = \rho_D = 0$, $\rho_K = \lambda t$.

Let ϕ be a complex saclar field and ϕ^* its conjugate. The canonical momenta are defined by $\pi = \dot{\phi}^*$ and $\pi^* = \dot{\phi}$. (The dot indicate differentiation with respect to t.) The Hamiltonian that governs the classical dynamics of such fields in the SO(2,1)-invariant gravitational and electromagnetic backgrounds (2.1) and (2.2) is given by

$$H = \int\left\{\pi\pi^* + \phi^*\left\{k^2 - 2\frac{\lambda}{t}k + \frac{1}{t}(\lambda^2 + m^2/h^2)\right\}\phi\right\} \tag{2.3}$$

In the above expression we have suppressed the integration variables, adopted an obvious functional matrix notation and introduced the notation

$$k(x,y) \equiv i\delta'(x-y) \tag{2.4}$$

for the derivative of the delta function.

Apart from the electric charge

$$Q_e = i\int(\phi^*\pi^* - \phi\pi) \tag{2.5}$$

three other charges (one for each Killing vector) are conserved owing to the invariance properties of the background fields. They read

$$Q_f = \int\Big\{ \pi f^o\pi^* + \phi^*\Big[kf^0k - \frac{\lambda}{t}(kf^o + f^ok) + \frac{1}{t^2}(\lambda^2 + m^2/h^2)f^o \Big]\phi + i(\phi^* kf'\pi^* - \pi f'k\phi) - i\rho_f(\phi^*\pi^* - \phi\pi) \Big\} \tag{2.6}$$

and it is not difficult to check that they indeed satisfy

$$\frac{d}{dt}Q_f \equiv \frac{\partial}{\partial t}Q_f + \{Q_f, H\} = 0 \tag{2.7}$$

with { } the Poisson bracket.

3. QUANTIZATION IN THE SCHRÖDINGER PICTURE AND RENORMALIZATION OF THE SYMMETRY GENERATORS.

The implementation of the symmetries at the quantum level requires more care. Quantization is achieved by promoting the dynamical variable to operators $\phi,\phi^*,\pi,\pi^* \rightarrow \Phi,\Phi^\dagger,\Pi,\Pi^\dagger$ and imposing the equal-time canonical commutation rules

$$[\Phi(x), \Pi(y)] = [\Phi^\dagger(x), \Pi^\dagger(y)] = i\delta(x-y) \tag{3.1a}$$

$$[\Phi(x), \Phi(y)] = [\Pi(x), \Pi(y)] = [\Phi(x), \Pi^\dagger(y)] = 0 \tag{3.1b}$$

We shall work in the Schrödinger picture and use states $|\phi\rangle$ on which the time independent field operators Φ and $\Phi^\dagger$ act by multiplication

$$\Phi(x)|\phi\rangle = \phi(x)|\phi\rangle \quad ; \quad \Phi^\dagger(x)|\phi\rangle = \phi^*(x)|\phi\rangle \tag{3.2}$$

$$\langle\phi_1|\phi_2\rangle = \int \mathcal{D}a\mathcal{D}a^* \exp i\left[\int dx\, a(x)(\phi_1(x) - \phi_2(x)) + \text{c.c.}\right] \tag{3.3}$$

The relations (3.1) are realized by taking the momenta to act by functional differentiation:

$$\Pi(x) = -i\delta/\delta\phi(x) \quad ; \quad \Pi^\dagger(x) = -i\delta/\delta\phi^*(x) \tag{3.4}$$

Upon effecting the substitution $(\phi, \phi^*, \pi, \pi^*) \rightarrow (\phi, \phi^*, -i\delta/\delta\phi, -i\delta/\delta\phi^*)$ into (2.6) one arrives at symmetry generators $Q_f(\Phi,\Pi)$ which are ill-defined. The source of the problem resides in the fact that the charges involve products of field operators at the same point. To arrive at well-defined generators one proceeds like this. First, the formal expression for Q is regulated so that no singularities occur: $Q \rightarrow Q^R$. Second, one isolates and eliminates the portions of Q^R that become singular when the regulators are absent. For linear theories a c-number subtraction q^R suffices. Third, the regulators are removed from the subtracted expression leaving well-defined generators which we denote by $:Q: \equiv \lim(Q^R - q^R)$.

The renormalizing subtraction will be determined as follows[1]. Consider the matrix elements of finite transformations

$$U^R(\phi_1, \phi_2; \tau) = \langle\phi_1|e^{-i\tau Q_R}|\phi_2\rangle \tag{3.5}$$

They satisfy a functional Schrödinger-like equation

$$i\partial/\partial\tau U^R = Q^R U^R \quad , \quad U^R(\phi_1, \phi_2; \tau)|_{\tau=0} = \delta(\phi_1 - \phi_2) \tag{3.6}$$

Since Q^R can be renormalized by a subtraction, the infinities in U^R will be confined to a phase. This infinite phase will be identified by analyzing eq. (3.6). We shall then define q^R so that $\exp(i\tau q^R)U^R(\phi_1,\phi_2;\tau) = \langle\phi_1|\exp(-i\tau(Q^R-q^R))|\phi_2\rangle$ has a well-defined local limit. Let us see how this renormalization prescription apply to the case at hand.

We first observe that the momentum Q_P requires no subtraction. The same is also true of the conformal generator Q_K since special conformal transformations correspond to translations in the inverted coordinate system. Only Q_D, the dilatation generator ($f^o=t$, $f^1=x$) therefore needs renormalization. Since Q_D is quadratic in the canonical variables and commutes with the generator of phase transformations, an appropriate Ansatz for $U_D(\phi_1, \phi_2; \tau) = \langle\phi_1| \exp(-i\tau Q_D)|\phi_2\rangle$ is

$$U_D(\phi_1, \phi_2; \tau) = N\exp\{-\int[\phi_1{}^*A\phi_1 - \phi_1{}^*B\phi_2 - \phi_2{}^*C\phi_1 + \phi_2{}^*D\phi_2]\} \quad (3.7)$$

Insertion in the Schrödinger-like equation (3.6) gives the following differential equation for A and N ($X(x,y) \equiv x\delta(x-y)$):

$$-i\frac{\partial}{\partial\tau}\ell n N = t\mathrm{Tr}A \quad (3.8a)$$

$$i\frac{\partial}{\partial\tau}A = tA^2 - tk^2 + 2\lambda k - \frac{1}{t}(\lambda^2 + m^2/h^2) - AXk - kXA \quad (3.8b)$$

The equation for B, C and D will not be of any concern here. Morever, regulators will not play any significant role in the discussion and will therefore be omitted. (See Ref.[3] for details.)

The infinities in U_D are all contained in N in the form of a phase $e^{-i\tau q_D}$. From (3.8a) wee see that the divergent part of lnN which is linear in τ is to be gotten from the τ-independent divergent part of TrA. Once (3.8b) has been solved for A, this singular quantity can be obtained by computing the ultraviolet divergence that occurs in $A(x, y; \tau)$ as $x \to y$ after τ has been continued to

The solution of (3.8b) which takes care of the initial condition $U_D(\phi_1,\phi_2;0)=\delta(\phi_1-\phi_2)$ is given by

$$A(x,y;\tau) = i\int\frac{dp}{2\pi}e^{-ip(x-y)}p\left[\frac{\Theta''_{\lambda,\mu}(tp)}{\Theta'_{\lambda,\mu}(tp)} - \Theta'_{\lambda,\mu}(tp)\cot\left[\Theta_{\lambda,\mu}(tp) - \Theta_{\lambda,\mu}(tpe^{-\tau})\right]\right] \quad (3.9)$$

Here $\Theta_{\lambda,\mu}(tp)$ is the phase of the Whittaker function $W_{-i\lambda,\mu}(-2itp)$:

$$W_{-i\lambda,\mu}(-2itp) = W^*_{i\lambda,\mu}(2itp) \equiv M_{\lambda,\mu}(tp)e^{i\Theta_{\lambda,\mu}(tp)} \qquad (3.10a)$$

$$M^2_{\lambda,\mu}(tp)\Theta'_{\lambda,\mu}(tp) = e^{-\pi\lambda sign(tp)} \qquad (3.10b)$$

and $\mu = [1/4 - (\lambda^2 + m^2/h^2)]^{1/2}$. A prime denotes differentiation with respect to the argument. The asymptotic behavior of $\Theta_{\lambda,\mu}(tp)$ for large $|tp|$ is

$$\Theta_{\lambda,\mu}(tp) \sim tp - \lambda \ell n 2|tp| + \frac{v}{2tp} - \frac{\lambda(1-v)}{4t^2p^2} + O\left(\frac{1}{t^3p^3}\right) \qquad (3.11)$$

with $v = \mu^2+\lambda^2-1/4$. Using this result we find that the symmetric part A_S of A behaves like

$$A_S(x,y;\tau) \sim \int \frac{dp}{2\pi} e^{-ip(x-y)}\left[|p| + \frac{m^2}{2|p|t^2h^2}\right] \qquad (3.12)$$

for $x \sim y$; this is thus the renormalizing subtraction. The renormalized generators can now be written as

$$\tilde{Q}_f = Q_f - \mathrm{Tr} f^o \omega^m \qquad (3.13)$$

$$\omega^m(x,y) = \int \frac{dp}{2\pi} e^{-ip(x-y)}\left[p^2 + \frac{m^2}{t^2h^2}\right]^{1/2} \qquad (3.14)$$

Observe that ω^m differs only by finite terms from the expression on the r.h.s. of (3.12). At this point the finite part of the subtraction is arbitrary, it will be fixed by conservation requirements as we shall soon see. Finally note that (3.13) can be used for the translation and conformal generators as well since the subtraction actually vanishes in these cases.

4. THE VACUUM STATE

As stated in the Introduction, the vacua should here be defined as the states whose wave functionals are Gaussian solutions to the time-dependent Schrödinger equation and in addition invariant under the symmetry transformations of the theory. Since no reference to any preselected state has been made in defining the SO(2,1) generators, the characterization of these vacuum states can be carried out for the scalar field system that we have been studying. This was done in Ref.[3] and here is a

We have found that there exists a unique SO(2,1)-invariant solution of the form $\Psi(\phi,\phi^*;\,t) = N\exp(-\int\phi^*\Omega\phi)$ to the Schrödinger equation $i\partial/\partial t\Psi = H\Psi$ with H the Hamiltonian corresponding to (2.3). The covariance $\Omega = \Omega_R = i\Omega_I$ is given explicitely by

$$\Omega_R = |k|\,\Theta'_{\lambda,\mu}(tk) \tag{4.1a}$$

$$\Omega_I = |k|\left\{\frac{1}{2}\,\frac{\Theta''_{\lambda,\mu}(tk)}{\Theta'_{\lambda,\mu}(tk)} + \frac{\Theta'_{\lambda,\mu}(tk)\,tan\left[\Theta_{\lambda,\mu}(tk) - \chi\right]}{1+tan^2\left[\Theta_{\lambda,\mu}(tk) - \chi\right]}\right\} \tag{4.1b}$$

and N is fixed by the normalization condition $\int\mathcal{D}\phi^*\mathcal{D}\phi\Psi^*\Psi = 1$. The remaining arbitrary real constant χ which occurs in the above expression just accounts for the arbitrariness in the choice of the time origin. In obtaining these results, we observe that the renormalized charges of Section 3 are not conserved. (Recall that at the time, we did not want to invoke any dynamics.) This problem is easily remedied by redefining the generators as follows:

$$:Q_f: \; = Q_f - \mathrm{Tr}f^o\,|k|\,\Theta'_{\lambda,\mu}(tk) \tag{4.2}$$

Such a redefinition amounts to a finite renormalization as is seen by comparing (4.2) with the expression (3.13) for Q_f.

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