Non-metric fields from quantum gravity

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In this manuscript, we will discuss the construction of covariant derivative operator in quantum gravity. We will find it is more perceptive to use affine connections more general than metric compatible connections in quantum gravity. We will demonstrate this using the canonical quantization procedure. This is valid irrespective of the presence and nature of sources. General affine connections can introduce new scalar fields in gravity.

Keywords: quantization, connections, non-metricity, scalar fields

1. Introduction

In this manuscript, we will discuss a few aspects of spacetime geometry relevant to quantum gravity. In Sec.2, we will discuss the construction of covariant derivative operator in quantum gravity. In this article, we will deal with only affine connections and denote general affine connections by affine connections or connection coefficients.[1,2] We will find it is more perceptive to use affine connections more general than metric compatible connections in quantum gravity. We will use the canonical quantization procedure and the Arnowitt-Deser-Misner (ADM) formalism to show this.[3] This is valid irrespective of the presence and nature of sources. This is a general mathematical issue which will be there, in a theory of quantum gravity which is not a quantum field theory in a fixed background, provided some components of metric can be taken as independent variables in a neighborhood of the spacetime manifold. This can be done around any regular point of the spacetime manifold.[4] We will also use the general metric-metric commutators to illustrate this issue.[2]

2. Quantum Gravity and Covariant Derivatives

We now consider quantization of gravity by using the canonical quantization procedure. Canonical quantization is important to find the particle spectrum when we quantize a classical theory. In the canonical quantization of gravity, metric becomes operator on a Hilbert space. We represent such operators by carets. Affine connections present in the covariant derivatives act on the tensor operators and we represent them also by the symbols: $\hat{\Theta}^{\alpha}{}_{\mu\nu}$. Affine connections will contain components of metric and their spacetime derivatives and also other fields as evident from the previous discussions.

In a Hamiltonian formulation, induced metric on a set of constant time surfaces is used as dynamical variable. The induced metric on a set of constant time surfaces is given by:

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu \tag{1}$$

where n_μ is the unit normal to the constant time surfaces. An expression for conjugate momenta is given by Eq.(4). We presently use the symbols $\hat{h}, \hat{\pi}$ to denote the corresponding collection of canonical operators. In general, the Levi-Civita connections contain metric and time derivative of metric components and hence, will depend on the canonical conjugate variables $(\hat{h}, \hat{\pi})$. We now express covariant derivative operator in the following form:

$$\hat{\nabla}'_\mu \hat{A}_\nu = [\partial_\mu - \hat{\Gamma}^\alpha{}_{\mu\nu}(\partial\hat{g}, \hat{g})]\hat{A}_\alpha = [\partial_\mu - \hat{\Gamma}^\alpha{}_{\mu\nu}(\hat{\pi}, \hat{h})]\hat{A}_\alpha$$

Here, $\hat{\Gamma}^\alpha{}_{\mu\nu}$ are operator version of the Levi-Civita connections. We adopt the following operator ordering in connection coefficients. Whenever there appears a product between partial derivatives of metric and metric itself, the partial derivative is kept as the first term and metric is kept as the second term. The ordering of the operators $(\hat{h}, \hat{\pi})$ in $\hat{\Gamma}^\alpha{}_{\mu\nu}$ is given to be the same as that written in the above equation, *i.e*, $\hat{h}$ is kept as the successor of $\hat{\pi}$.

We next consider the operator: $q^\mu \hat{\nabla}'_\mu q^\nu$, where q^μ is a vector field acting as $q^\mu \hat{I}$ on the Hilbert space. This operator contains canonical conjugate pairs of variables when we choose affine connections to be given by the Levi-Civita connections. In this case, we will have the following expression:

$$[q^\mu \hat{\nabla}'_\mu q^\nu] \left|\Psi\right\rangle \neq 0 \tag{2}$$

remaining valid in a given state $|\Psi\rangle$ with an arbitrary well-behaved vector field q^μ. We will not have a complete set of states for which the expectation value of the operator in the *l.h.s* is zero with negligible fluctuations for all well-behaved vector fields. This will be valid only in the classical limit, and is a subject similar to the familiar Ehrenfest's theorems in non-relativistic quantum mechanics. Similar discussions will remain valid even if we choose affine connections to be given by more general expressions.[5] In general, affine connections will contain canonical pairs of variables from metric sector to have proper classical limit of the Levi-Civita connections, and the concept of geodesics will not remain exact for all vector fields in a quantum state. This will also remain valid for parallel transport and the notion of parallel transport is not exact in a quantum theory of gravity. This is expected and indicates that we can use affine connections more general than the metric compatible connections even in free quantum gravity.

We now consider the metric compatibility conditions. The metric compatibility conditions are to be replaced by the operator identity: $\hat{\nabla}'_\mu[\hat{g}_{\alpha\beta}] \equiv 0$. The action of $\hat{\nabla}'_\mu[\hat{g}_{\alpha\beta}]$ on any state is zero if connection operators are given by the Levi-Civita connection operators and we choose the operator ordering same as that mentioned in.[5] Here, we always keep metric operators as the successors of the partial derivatives of themselves. The same will also remain valid for $\hat{\nabla}'_\mu[\hat{g}_{\alpha\beta}]$. Here, $\hat{g}_{\alpha\beta}$ will be kept at the right of the Levi-Civita connections. We also define the contravariant components of metric as: $\hat{g}^{\alpha\kappa}\hat{g}_{\kappa\beta} = \delta^\alpha_\beta$. This ordering leads to the operator identity: $\hat{\nabla}'_\mu[\hat{g}_{\alpha\beta}] \equiv 0$ irrespective of the ordering of $(\hat{h}, \hat{\pi})$ chosen in the partial derivatives of metric components. However, the operator version of metric compatibility conditions need not be consistent with a canonical quantization condition. We will demonstrate this in the following.

As mentioned above, in a Hamiltonian formulation we use induced metric on a set of constant time surfaces as dynamical variable. Thus, $g_{\mu\nu}$ is replaced by: $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Here, n_μ is given by $(-N, 0, 0, 0)$, N being the lapse function. The contravariant n^μ is given by: $\frac{1}{N}(1, -N^1, -N^2, -N^3)$; where N^i are the shift functions. We have: $g_{00} = N_k N^k - N^2$, $g_{0i} = N_i$ and $N_i = g_{ij}N^j$.[4] The induced metric on the constant time surfaces coincide with the spatial part of $g_{\mu\nu}$ which are expressed as g_{ij}. These fields are taken to dynamical variables in general relativity and we have the following Poisson brackets:

$$\left\{g_{ij}(t, \vec{x}), \pi^{kl}(t, \vec{y})\right\} = \delta^k_{(i}\delta^l_{j)}[\delta(\vec{x}, \vec{y})] \tag{3}$$

Where, $\vec{x}$ refers to the spatial coordinates and the Poisson bracket is evaluated at equal time. The delta function is defined without recourse to metric. The conjugate momentum is a spatial tensor density and is given by:

$$\pi^{pl} = -\sqrt{|g_{ij}|}(K^{pl} - Kg^{pl}) \tag{4}$$

Where $|g_{ij}|$ is the determinant of the spatial metric, $K_{pl} = -\nabla'_p n_l$ is the extrinsic curvature of the spatial sections, g^{pl} is the inverse of g_{pl} and K is the trace of the extrinsic curvature taken *w.r.t* g_{ij}.[4] We can also define the Poisson brackets for the lapse and shift functions although their conjugate momenta vanish giving primary constraints.[4] Thus, in a Hamiltonian formulation with the Einstein-Hilbert action, we have constraints and we can not naively replace the Poisson brackets by commutators when we try to quantize the theory.[4] There are two principal approaches to quantize the theory.[4] In the first approach, gauge fixing conditions are introduced to render the complete set of constraints second class.[4] These conditions also determine the lapse and shift functions. We then pick two components of g_{ij} as independent variables and quantize these components using standard commutation relations. We can solve the constraints to evaluate other commutators. The second approach is similar to the Gupta-Bleuler method used to quantize electrodynamics and was initiated by Dirac.[4] In this approach the classical variables are treated as independent variables and the constraints are imposed on the quantum states. In this case, we can replace the classical Poisson brackets by commutators when we quantize the theory.

We now demonstrate that it is appropriate to extend the Levi-Civita connections and metric compatibility conditions as long as we can regard components of spatial metric on the constant time surfaces as independent physical variables subjected to usual canonical quantization conditions. We will also find that we can not have a Hilbert space on which we can impose the metric compatibility conditions when such quantization conditions remain valid. In the following, we will restrict our attention to a neighborhood around a regular point $'x'$. We can extend the neighborhood to the complete spacetime manifold leaving away singularities and other possible irregular points associated with the constraints.[4] We pick a component of spatial metric, say g_{pl}, as an independent physical variable. We then have the following equal time commutator:

$$[\hat{g}_{pl}(t,\vec{x}), \hat{\pi}^{pl}(t,\vec{y})] = i\delta^{p}_{(p}\delta^{l}_{l)}[\delta(\vec{x},\vec{y})] \tag{5}$$

Where, the point $'y'$ belongs to the above mentioned neighborhood of $'x'$. There will be another such commutator for the other independent variable. The *r.h.s* of the commutator is taken to be a distribution that is a spatial tensor density in the spatial coordinates of $\hat{\pi}^{pl}(t,\vec{y})$. The *r.h.s* will be replaced by different expressions when we replace $\hat{g}_{pl}(t,\vec{x})$ by any dependent component of metric including $g_{0\mu}$. These terms are determined by the secondary constraints, gauge fixing conditions, definitions of the lapse and shift functions given before and the fundamental commutators given by the above equation. Covariant derivatives give changes in a tensor when we move from one point to a neighbouring point. If the Levi-Civita connections are consistent with the commutators obtained above, corresponding spatial covariant derivatives of both sides of any of the commutators *w.r.t* the arguments of metric will agree since both sides are equal for all components of metric. We now consider Eq.(5). The action of $\hat{\nabla}'_{xk}$ on the *r.h.s* is same as that on a second rank covariant tensor and will contain spatial partial derivatives of the delta function. It will also contain additional terms dependent on connections and metric that can explicitly depend on time due to explicit time dependence of the gauge fixing conditions.[4] This covariant derivative is not vanishing in general for all values of $\vec{x}$. The left hand side vanishes as can be found from the following expression:

$$\hat{\nabla}'_{xk}\left\{\hat{g}_{pl}(t,\vec{x})\right\}\hat{\pi}^{pl}(t,\vec{y}) - \hat{\pi}^{pl}(t,\vec{y})\hat{\nabla}'_{xk}\left\{\hat{g}_{pl}(t,\vec{x})\right\} = 0 \tag{6}$$

This follows since we are imposing the operator versions of the metric compatibility conditions. This can also be seen by applying the *l.h.s* of the above equation to any state, introducing a sum over a complete set of states between the products of the operators and using the fact that the action of $\hat{\nabla}'_{\mu}[\hat{g}_{\alpha\beta}]$ on any state is zero if connection operators are given by the Levi-Civita connection operators with operator ordering chosen below Eq.(2). Thus, the Levi-Civita connections are not consistent with the canonical commutators given by Eq.(5). We will consider other operator ordering in the Levi-Civita symbols in Appendix:A and we will find that similar inconsistency arise in these cases also. Similar situation will remain valid for any point in the manifold where we can introduce constant time surfaces and assume the existence of a metric component as an independent field in a neighborhood around that point. The above inconsistency will also arise with a different choice of constant time surfaces. Thus, there will be a multitude of coordinate systems where we can not use the Levi-Civita connections as connection coefficients if we impose the quantization condition given by Eq.(5). This also indicates that we can not use the Levi-Civita connections as connection coefficients in all coordinate systems that are diffeomorphic to these coordinate systems due to the tensorial character of $C^{\alpha}_{\ \mu\nu}$.

In Dirac's approach to quantize gravity, all the classical variables are independent and we quantize them accordingly. We can find out $[\hat{g}_{0\beta}(t,\vec{x}),\hat{\pi}^{pl}(t,\vec{y})]$ from the definitions of the lapse and shift functions and $[\hat{N}^{\lambda}(t,\vec{x}),\hat{\pi}^{pl}(t,\vec{y})]=0$, where $N^0=N$.[4] All spatial components g_{ij} satisfy canonical commutation relations given by Eq.(5). We will again have the inconsistency mentioned above when we use the Levi-Civita connections. In this case, the action of $\hat{\nabla}'_{xk}$ on the *r.h.s* of Eq.(5) is given by expressions like: $i[\partial_{xk}\delta(\vec{x},\vec{y})-2\hat{\Gamma}^{0}_{\ kp}(t,\vec{x})\hat{N}^{p}(t,\vec{x})\delta(\vec{x},\vec{y})-2\hat{\Gamma}^{p}_{\ kp}(t,\vec{x})\delta(\vec{x},\vec{y})]$, where we have taken $p=l$ and there is no sum over the repeated indices. Also, we can not make the Levi-Civita connections consistent with the commutation relation given by Eq.(5) by introducing additional constraints on the physical Hilbert space. If we demand that the action of $\hat{\nabla}'_{xk}$ to the *r.h.s* of Eq.(5) vanishes on the physical Hilbert space, we will have the constraint:

$$[\partial_{xk}\delta(\vec{x},\vec{y})-2\hat{\Gamma}^{0}_{\ kp}(t,\vec{x})\hat{N}^{p}(t,\vec{x})\delta(\vec{x},\vec{y})-2\hat{\Gamma}^{p}_{\ kp}(t,\vec{x})\delta(\vec{x},\vec{y})]|\Psi\rangle=0 \tag{7}$$

Where, we have again taken $p=l$ and there is no sum over the repeated indices. There will be other similar constraints associated with other commutators. These states also satisfy the secondary constraints. We have used the operator identities $\hat{\nabla}'_{\mu}\hat{g}_{\alpha\beta}=0$, and the Levi-Civita connections. All the above conditions will lead to singular expressions involving $\delta(\mathbf{0})$ and the partial derivatives $\delta'(\mathbf{0})$ for expectation values of some of the variables: $\hat{g}_{0\mu},\hat{g}_{ij},\hat{N}^{p}$ and $\hat{\Gamma}^{\alpha}_{\ \mu\nu}$. This is valid for all physical states and is physically undesirable. Lastly, the above problems will arise if we use any set of metric compatible connections.

In the first approach to quantize the theory, it is unlikely that there will exist a set of gauges that is time dependent, render the complete set of constraints second class and also remove the inconsistency mentioned above. It is not possible to remove the inconsistency in the second approach to quantize the theory. Also, it is expected that $[\hat{g}_{\alpha\beta}(x^{\mu}),\hat{g}_{\alpha\beta}(y^{\nu})]$ will depend on (x^{μ},y^{ν}) non-trivially with non-vanishing covariant derivatives.[2] We can consider semiclassical theories like quantum fields in curved spaces to assume so. Thus, it is more appropriate to use connections more general than metric compatible connections in quantum gravity. The above discussions are valid irrespective of the presence and nature of sources. We can analyze this issue further in the following way. The Levi-Civita connections and metric compatibility conditions are taken as basic assumptions to calculate the scalar curvature when we use the Einstein-Hilbert action to describe classical and quantum gravity. It is better to discuss quantization and non-metricity using the metric-affine action or Palatini action where metric and affine connections are independent variables. We have found that nen-metricity can give scalar fields in the theory. One of the scalar fields can give negative stress-tensor.[5]

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