New identity for Green functions in N=1 supersymmetric theories

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Investigation of quantum corrections in supersymmetric theories is an interesting and sometimes nontrivial problem. For example, in $N = 1$ supersymmetric theories it is possible to suggest [1] the form of exact β-function

$$\beta(\alpha) = -\frac{\alpha^2\Big[3C_2 - C(R)\Big(1-\gamma(\alpha)\Big)\Big]}{2\pi(1 - C_2\alpha/2\pi)}. \qquad (1)$$

To derive it by usual methods of the perturbation theory is a complicated problem. Here we will show that a derivation of this β-function (more exactly, the matter contribution to this function) can be made if a special new identity for the Green functions of the matter superfield takes place. We consider $N = 1$ supersymmetric Yang–Mills theory, which is described by the action

$$S = \frac{1}{2e^2}\mathrm{Re}\,\mathrm{tr}\int d^4x\, d^2\theta\, W_a C^{ab} W_b + \frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi^+ e^{2V}\phi + \tilde\phi^+ e^{-2V^t}\tilde\phi\Big). \qquad (2)$$

Quantization of this model can be made by standard methods. In particular, we use the background field method, which allows preserving the background gauge invariance and considerably simplifies the calculations of quantum corrections. In order to regularize model (2) we add to its action the higher derivative term

$$S_\Lambda = \frac{1}{2e^2}\mathrm{tr}\,\mathrm{Re}\int d^4x\, d^4\theta\, V\frac{(\boldsymbol{D}_\mu^2)^{n+1}}{\Lambda^{2n}}V. \qquad (3)$$

(In our notation bold letters denote background covariant derivatives.) This term is invariant under the background gauge invariance, but breaks the BRST-invariance. Therefore, calculating quantum corrections it is necessary to use a special subtraction scheme, which restores the Slavnov-Taylor identities in each order [2]. In order to cancel the remaining one-loop divergences we should also insert to generating functional the Pauli–Villars determinants [3].

We will calculate a matter contribution to the Gell-Mann–Low function. If $\mathbf{V}$ denotes the background gauge field and

$$\Gamma^{(2)} = -\frac{1}{16\pi}\mathrm{tr}\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,\mathbf{V}(-p)\,\partial^2\Pi_{1/2}\mathbf{V}(p)\,d^{-1}(\alpha,\mu/p) +$$
$$+\frac{1}{4}\int\frac{d^4p}{(2\pi)^4}\,d^4\theta\,\Big(\phi^+(-p,\theta)\,\phi(p,\theta)\tilde\phi^+(-p,\theta)\,\tilde\phi(p,\theta)\Big)\,ZG(\alpha,\mu/p), \qquad (4)$$

where α is a renormalized coupling constant and Z is the renormalization constant for the matter superfield, then the Gell-Mann–Low function $\beta(\alpha)$ and the anomalous dimension $\gamma(\alpha)$ are defined by

$$\beta\Big(d(\alpha,\mu/p)\Big) = \frac{\partial}{\partial \ln p} d(\alpha,\mu/p); \qquad \gamma\Big(d(\alpha,\mu/p)\Big) = -\frac{\partial}{\partial \ln p} \ln ZG(\alpha,\mu/p). \tag{5}$$

Calculation of the matter contribution to the Gell-Mann–Low function can be made substituting solutions of Slavnov–Taylor identities to the Schwinger–Dyson equation for the two-point Green function of the gauge superfield [4]. Graphically this equation can be presented as a sum of two diagrams:

The result (without subtraction diagrams) can be written as

$$-\frac{1}{16\pi}\frac{d}{d\ln\Lambda} d_0^{-1}\Big|_{p=0} = \ldots - C(R)\int \frac{d^4q}{(2\pi)^4}\frac{d}{d\ln\Lambda}\left\{\frac{1}{2q^2}\frac{d}{dq^2}\ln\Big(q^2G^2\Big) - \frac{8f}{q^2G}\right\} - (PV) \tag{6}$$

where dots denote contributions of the gauge field and ghosts and (PV) denotes a contribution of the Pauli- Villars fields. The function G is defined by Eq. (4) and the function f is related with the three-point function

$$\frac{\delta^3\Gamma}{\delta \mathbf{V}_y^a \delta\phi_{0z}^+ \delta\phi_x}\Bigg|_{p=0} = e\Bigg[-2\partial^2\Pi_{1/2y}\Big(\bar D_y^2\delta^8_{xy}\delta^8_{yz}\Big)F(q^2) + \frac{1}{8}D^bC_{bc}\bar D_y^2\Big(\bar D_y^2\delta^8_{xy}D_y^c\delta^8_{yz}\Big)f(q^2) -$$
$$-\frac{1}{16}q^\mu G'(q^2)\bar D\gamma^\mu\gamma_5 D_y\Big(\bar D_y^2\delta^8_{xy}\delta^8_{yz}\Big) - \frac{1}{4}\bar D_y^2\delta^8_{xy}\delta^8_{yz}G(q^2)\Bigg]T^a. \tag{7}$$

Here ϕ_0 and $\tilde\phi_0$ are scalar superfields, introduced in the generating functional by adding to the action the term

$$S_{\phi_0} = \frac{1}{4}\int d^4x\, d^4\theta\,\Big(\phi_0^+ e^{\Omega^+}e^{2V}e^{\Omega}\phi + \text{similar terms}\Big). \tag{8}$$

Actually the Green function (7) is very similar to the usual Green function, but one of the matter ends is not chiral.

The first term in Eq. (6) is an integral of the total derivative and can be easily calculated using the identity

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{d}{dk^2}f(k^2) = \frac{1}{16\pi^2}\Big(f(k^2=\infty) - f(k^2=0)\Big). \tag{9}$$

However, the explicit calculations [5,6] always show that the second term in Eq. (6) is also an integral of the total derivative and is always equal to 0. This allows suggesting existence of the new identity

$$\int \frac{d^4q}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{f}{q^2 G} = 0, \tag{10}$$

which is not a consequence of supersymmetric or gauge Slavnov–Taylor identities. (In the massive case the corresponding identity is presented in [4].) In functional form the new identity can be written as

$$\mathrm{tr} \int d^8x\, d^8y\, \mathbf{V}_y D^a \mathbf{V}_x \frac{d}{d\ln\Lambda} \frac{D_{az}\bar{D}_z^2}{\partial^2} \frac{\delta^3\Gamma}{\delta j_z^+ \delta \mathbf{V}_y \delta \phi_{0x}^+}\Big|_{z=x,p=0} = 0, \tag{11}$$

where the derivative with respect to the sources should be expressed via derivative with respect to the fields. The condition $p = 0$ means that the background field $\mathbf{V}$ depends only on θ and is independent of x^μ. The derivative with respect to $\ln\Lambda$ is needed in order to make all integrals well defined.

The new identity is nontrivial starting from the three-loop approximation (or the two-loop approximation for the Green function (7)). Its verification for the Abelian theory was made in three- and partially four-loop approximation [6]. A sketch of a possible proof exactly to all orders in Abelian case is made in Ref. [7]. However, it is necessary to verify if the new identity takes place in the non-Abelian case. For this purpose we considere [8] the three-loop diagram

and construct the corresponding function f, calculating the diagrams, which are obtained from it by cutting the matter line and attaching an external line of the background gauge field by all possible ways. After substituting the result to the left hand side of identity (10), we obtain

$$\int \frac{d^4q}{(2\pi)^4} \frac{d}{d\ln\Lambda} \frac{f(q^2)}{q^2\, G(q^2)} =$$
$$= \alpha^2\pi^2 C_2\Big(C_2(R) - \frac{1}{2}C_2\Big) \int \frac{d^4q\, d^4k\, d^4l}{(2\pi)^{12}} \frac{\partial}{\partial q^\mu}\Bigg\{\Lambda\frac{d}{d\Lambda}\Bigg[\frac{(k+q+l)^\mu}{q^2\,(k+q)^2\,(k+q+l)^2} \times$$
$$\times \frac{1}{k^2\Big(1+k^{2n}/\Lambda^{2n}\Big)\, l^2\Big(1+l^{2n}/\Lambda^{2n}\Big)\,(k+l)^2\Big(1+(k+l)^{2n}/\Lambda^{2n}\Big)}\Bigg]\Bigg\} = 0. \tag{12}$$

Therefore, the new identity and the factorization of integrands to total derivatives also take place in the non-Abelian case.

In order to check if the factorization of integrands to total derivatives is a general feature of supersymmetric theories, we calculate the two-loop β-function for the $N = 1$ supersymmetric Yang-Mills theory without matter. It well known that in this case

$$d_0^{-1}(\alpha, \Lambda/p) = d_2 \ln\frac{\Lambda}{p} + \mathrm{const} + O(\alpha^2). \tag{13}$$

Calculating two-loop diagrams, defining the Gell-Mann–Low function (so far without diagrams with insertions of counterterms), in the limit $p \to 0$ we find (in the Euclidean space after the Weak rotation)

$$d_2 = 8\pi \cdot 6\pi\,\alpha_0 \frac{d}{d\ln\Lambda} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \frac{d}{dk^2} \int \frac{d^4q}{(2\pi)^4} \Big(q^2(1+q^{2n}/\Lambda^{2n})\Big)^{-1} \Bigg\{ \Big((q+k)^2 \times \\ \times (1+(q+k)^{2n}/\Lambda^{2n})\Big)^{-1} \Bigg[2(n+1)\Big(1+k^{2n}/\Lambda^{2n}\Big)^{-1} - 2n\Big(1+k^{2n}/\Lambda^{2n}\Big)^{-2} \Bigg] \Bigg\}. \quad (14)$$

This integrals can be calculated by Eq. (9). Then the two-loop Gell-Mann–Low function agrees with Eq. (1). After taking into account diagrams with counterterms insertions [9] with the higher derivative regularization we find that divergences are only in the one-loop approximations similar to the supersymmetric electrodynamics. This is in agreement with the results of [10]. However, the physical Gell-Mann–Low function has corrections in all loops.

Therefore, factorization of integrands to the total derivatives seems to be a general feature of all supersymmetric theories. However, the reason is so far unclear. Actually new identity for Green functions (10) is a consequence of this fact.

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