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ON A REPRESENTATION OF ENTIRE FUNCTIONS

G.V. Efimov *)

CERN - Geneva

A B S T R A C T

An explicit formula is given for the Fourier transform of an entire function $g(z)$, satisfying the inequality

$$|g(z)| < C e^{h(|z|)}$$

where $h(r)$ is a monotonic, twice differentiable function and $h(r) > 0$, $h'(r) > 0$, $h''(r) > 0$, namely

$$g(z) = \int e^{iz\zeta} d\sigma(\zeta)$$

where $\sigma(\zeta)$ is a complex measure for which there exists the integral

$$\int e^{H(|\zeta|)} |d\sigma(\zeta)| < \infty$$

where

$$H(s) = \max_{\tau > 0} (s\tau - h(\tau))$$

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*) On leave from the Joint Institute for Nuclear Research, Dubna, U.S.S.R.

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We shall consider classes of functions $g(z)$ which depend on one complex variable $z = x+iy$ and are entire analytic functions in the argument z. Let $g(z)$ be an entire function, then

$$M_g(r) = \max_{|z|=r} |g(z)|$$

Let $h(r)$ be a monotonic, twice differentiable function, and

$$h(r) > 0, \quad h'(r) > 0, \quad h''(r) > 0.$$

If

$$\lim_{r \to \infty} \frac{\ln M_g(r)}{h(r)} = 1,$$

then we shall say that the entire function $g(z)$ has an order of growth of $h(r)$. If $h(r) = ar^{\rho}$ we can then say that the entire function $g(z)$ has order ρ and finite type a.

We shall say that the function $g(z)$ belongs to the space $\mathfrak{Z}_h$ if for this function $g(z)$ there exist such constants $\delta > 0$ and $B > 0$, that

$$|g(z)| \leqslant B e^{h((1-\delta)|z|)}. \tag{1}$$

We will prove

<u>Theorem</u>

If $g(z) \in \mathfrak{Z}_h$, then this function can be represented in the form

$$g(z) = \int e^{iz\zeta} d\sigma(\zeta) \tag{2}$$

where $\sigma(\zeta)$ is a complex completely additive measure on the complex plane ζ and for this measure there exists the following integral

$$\int e^{H(|\zeta|)}|d\sigma(\zeta)| < \infty \tag{3}$$

where H(s) is the Young dual function of h(r), i.e.,

$$H(s) = \max_{r>0} (sr - h(r)). \tag{4}$$

Our proof is the generalization of the method given by Gel'fand and Shilov [1] for entire functions with growth of first order, i.e., $h(r) = ar$.

Let Z_H be the linear space of entire functions, for which all the expressions

$$\|f\|_p = \sup_\zeta |f(\zeta)| e^{-H((1+\frac{1}{p})|\zeta|)} \tag{5}$$

are finite. The functions

$$M_p(s) = e^{-H((1+\frac{1}{p})s)}$$

satisfy the inequalities

$$0 < M_1(s) \leqslant M_2(s) \leqslant M_3(s) \leqslant \ldots$$

and the so-called condition (P) :

- For a given $\varepsilon > 0$ and any p, a $p' > p$, and an N can be found such that for all s, for which $|s| > N$ is satisfied, the following inequality is valid

$$M_p(s) < \varepsilon M_{p'}(s)$$

Under these conditions the norms (5) agree, so that the space Z_H is complete, and finally that it is perfect [1].

If $f(\zeta) \in Z_H$, then for any $\varepsilon > 0$ there is a number $C_\varepsilon > 0$ such that

$$|f(\zeta)| < C_\varepsilon e^{H((1+\varepsilon)|\zeta|)}. \tag{6}$$

Let us find the general form of a linear continuous functional in the space Z_H. It is sufficient to find a general linear functional (F,f) in the normed space $Z_H^{(p)}$, the completion of the space Z_H in the norm $\|f\|_p$. The space $Z_H^{(p)}$ consists of some continuous functions $f(\zeta)$. These functions are defined for any ζ and the norm $\|f\|_p$ is finite for them. This space is closed relative to uniform convergence. Continuing the functional F into the space of all continuous functions according to the Hahn-Banach theorem, and applying the Riesz-Radon theorem, we obtain

$$(F,f) = \int f(\zeta)\, d\mu(\zeta) \tag{7}$$

where $\mu(\zeta)$ is a complex, completely additive measure in the complex ζ - plane, for which the integral

$$\int e^{H((1+\frac{1}{p})|\zeta|)} |d\mu(\zeta)| < \infty \tag{8}$$

is finite.

By virtue of the theorem [1)] on the structure of a space conjugate to a countably normed space, Eq. (7) yields the general form of a linear continuous functional in the space Z_H for all possible p.

Furthermore, the Taylor series $f(\zeta) = \sum_{n=0}^{\infty} f_n \zeta^n$ converges in the topology of the space Z_H. In fact, we can obtain by applying the Cauchy formula and Eq. (6) :

$$|f_n| = \left|\frac{1}{2\pi i}\oint \frac{d\zeta\, f(\zeta)}{\zeta^{n+1}}\right| \leq C_\varepsilon \min_{s>0} \frac{e^{H((1+\varepsilon)s)}}{s^n} = C_\varepsilon (1+\varepsilon)^n e^{-B(n)} \tag{9}$$

where

$$\mathcal{B}(n) = \max_{s>0}(n \ln s - H(s)). \tag{10}$$

The norm $\|\zeta^n\|_p$ is equal according to the definition (5) to

$$\|\zeta^n\|_p = \max_{s>0} s^n e^{-H((1+\frac{1}{p})s)} = \frac{e^{\mathcal{B}(n)}}{(1+\frac{1}{p})^n} \tag{11}$$

Then we have the estimation

$$\sum_{n=0}^{\infty} |f_n| \cdot \|\zeta^n\|_p \leqslant C_\varepsilon \sum_{n=0}^{\infty} \left[\frac{(1+\varepsilon)}{(1+\frac{1}{p})}\right]^n \tag{12}$$

Since the number ε can be chosen arbitrarily small, the series (12) converges for each given p.

Hence

$$(F, f) = \sum_{n=0}^{\infty} f_n F_n \tag{13}$$

where $F_n = (F, \zeta^n)$ is a fixed sequence of constants. Conversely, every sequence of constants F_n, such that the series (13) converges for any entire function $f(\zeta) \in Z_H$ and defines a continuous linear functional in the Z_H space by means of (13), may be represented as

$$F_n = \int \zeta^n d\sigma(\zeta) \tag{14}$$

which is obtained from the general formula (7) for $f(\zeta) = \zeta^n$.

Let $g(z) = \sum_{n=0}^{\infty} g_n z^n$ be an entire function from the space $\mathcal{Z}_h$, i.e., one satisfying an inequality of the form (1). Let us show that the numbers

$$F_n = (-i)^n g_n n! \tag{15}$$

define a linear functional in the space Z_H when the function $H(s)$ is the Young dual function of $h(r)$ according to (4). Indeed, we have the following estimations :

$$|g_n| = \left|\frac{1}{2\pi i}\oint \frac{dz\, g(z)}{z^n}\right| \leqslant B \min_{r>0} \frac{e^{h((1-\delta)r)}}{r^n} = \tag{16}$$
$$= B\cdot(1-\delta)^n e^{-A(n)}$$

where

$$A(n) = \max_{r>0}\left(n \ln r - h(r)\right) \tag{17}$$

and, according to the well-known Stirling formula :

$$n! = e^{n\ln n - n}\cdot\sqrt{2\pi n}\, E_n \tag{18}$$

where $E_n \to 1$. Hence

$$\sum_{n=0}^{\infty} |f_n F_n| \leqslant \tag{19}$$
$$\leqslant \sqrt{2\pi}\, C_\varepsilon B \sum_{n=0}^{\infty} \sqrt{n}\, E_n (1+\varepsilon)^n (1-\delta)^n \exp\left\{-B(n)-A(n)+n\ln n-n\right\}$$

Let us show that

$$B(n) + A(n) \equiv n \ln n - n\,. \tag{20}$$

Indeed, we have according to (17)

$$A(n) = n \ln r(n) - h(r(n)) \tag{21}$$

where the function $r = r(n)$ is the solution of the following equation

$$n = r(n)\, h'(r(n)) \tag{22}$$

On the other hand, we have, according to (4)

$$H(s) = s\,u(s) - h(u(s)) = u(s)\,h'(u(s)) - h(u(s)) \tag{23}$$

because of

$$s = h'(u(s)). \tag{24}$$

Then

$$\begin{aligned} B(n) &= \max_{s>0}\left(n \ln s - H(s)\right) = \\ &= \max_{s>0}\left(n \ln h'(u(s)) - u(s)\,h'(u(s)) + h(u(s))\right) = \\ &= \max_{u>0}\left(n \ln h'(u) - u\,h'(u) + h(u)\right). \end{aligned} \tag{25}$$

The condition of the maximum is

$$n\,\frac{h''(u)}{h'(u)} - u\,h''(u) = 0$$

or

$$n = u(n)\,h'(u(n)) \tag{26}$$

One can see that equations (26) and (22) are the same. Now we can write

$$B(n) = n \ln \frac{n}{z(n)} - n + h(z(n)). \tag{27}$$

Adding the functions A(n) and B(n) according to (21) and (27), we obtain the equality (20).

Finally, we have for the series (19) :

$$\sum_{n=0}^{\infty} |f_n F_n| \leq \sqrt{2\pi}\, C_\varepsilon B \sum_{n=0}^{\infty} \sqrt{n}\, E_n (1+\varepsilon)^n (1-\delta)^n. \qquad (28)$$

Since δ is a given fixed number and ε can be chosen arbitrarily small, the series (28) converges for any functions $f(\zeta) \in Z_H$. At the same time we obtain boundedness of the functional (13) in the norm $\|\cdot\|_p$ with $p > 1/\delta$, which also means boundedness of the functional (13) in the whole Z_H space.

According to what has been proved, there exists a measure $\sigma(\zeta)$ such that

$$F_n = (-i)^n g_n n! = \int \zeta^n d\sigma(\zeta), \qquad (29)$$

hence

$$g_n = \int \frac{(i\zeta)^n}{n!} d\sigma(\zeta) \qquad (30)$$

Multiplying by z^n and adding, we obtain convergent series on the left and right, and therefore

$$g(z) = \sum_{n=0}^{\infty} g_n z^n = \sum_{n=0}^{\infty} \int \frac{(iz\zeta)^n}{n!} d\sigma(\zeta) = \int e^{iz\zeta} d\sigma(\zeta) \qquad (31)$$

Q.E.D.

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REFERENCE

1) M. Gel'fand and G.E. Shilov, "Generalized Functions", Volume 2, Academic Press, New York-London (1968).