STUDIES IN FIELD THEORIES: MHV VERTICES, TWISTOR SPACE, RECURSION RELATIONS AND CHIRAL RINGS

Peter Svrček
Advisor: Edward Witten

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Abstract

In this thesis we study different aspects of four dimensional field theories. In the first chapter we give introduction and overview of the thesis. In the second chapter we review the connection between perturbative Yang-Mills and twistor string theory. Inspired by this, we propose a new way of constructing Yang-Mills scattering amplitudes from Feynman graphs in which the vertices are off-shell continuations of the tree level MHV amplitudes. The MHV diagrams lead to simple formulas for tree-level amplitudes. We then give a heuristic derivation of the diagrams from twistor string theory.

In the third chapter, we explore the twistor structure of scattering amplitudes in theories for which a twistor string theory analogous to the one for $\mathcal{N} = 4$ gauge theory has not yet been proposed. We study the differential equations of one-loop amplitudes of gluons in gauge theories with reduced supersymmetry and of tree level and one-loop amplitudes of gravitons in general relativity and supergravity. We find that the scattering amplitudes localize in twistor space on algebraic curves that are surprisingly similar to the $\mathcal{N} = 4$ Yang-Mills case.

In the next chapter we propose tree-level recursion relations for scattering amplitudes of gravitons. We use the relations to derive simple formulas for all amplitudes up to six gravitons. We prove the relations for MHV and next-to-MHV amplitudes and the eight graviton amplitudes.

In the last two chapters, we concentrate on the nonperturbative aspects of $\mathcal{N} = 1$ gauge theories. Firstly, we find the complete set of relations of the chiral operators of supersymmetric $U(N)$ gauge theory with adjoint scalar. We demonstrate exact correspondence between the solutions of the chiral ring and the supersymmetric vacua of the gauge theory. We discuss the gaugino condensation in the vacua. Finally, we go on to study the nonperturbative corrections to the Konishi anomaly relations. We show that the Wess-Zumino consistency conditions of the chiral rotations of the matter field imply the absence of the corrections for a wide class of superpotentials.

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Contents

1. Introduction

Current physics theories are based on quantum mechanics and general relativity. At energies far below the Planck scale, gravity is negligible. The remaining three forces are described in terms of quantum gauge theory with group $SU(3) \times SU(2) \times U(1)$. The fourth force, gravity, is described by a massless spin two field, the graviton. Early attempts at quantizing gravity led to divergences and ill-defined results. It is believed that general relativity cannot be well defined as a field theory.

These theories have been reconciled in string theory. String theory is the leading candidate for a unified description of the physical world. It naturally incorporates gravity, as the spectrum of strings has a massless spin two excitation. The critical superstring theories are defined on a ten-dimensional manifold. To obtain the four dimensional world, one compactifies string theory on a six dimensional Calabi-Yau manifold.

Although our understanding of string phenomenology is very incomplete, we have a lot of confidence in string theory coming from another direction. String theory teaches us new lessons about established physical theories, like gauge theories and general relativity. Thanks to it, we have learned about black holes, confinement, chiral symmetry breaking and other problems.

Much of this comes from understanding of D-branes. The low energy effective theory of D-branes is gauge theory. This observation lies at the heart of current studies of gauge theories using string theory. The open string excitations living on the D-brane worldvolume describe the D-brane dynamics. At low energies, one keeps only the massless modes. To study gauge theories one usually goes to a corner of parameter space in which the massless closed string modes, that is gravitons, decouple and we are left with open string modes only. Maldacena has implemented this in his AdS/CFT duality. Here, the strongly coupled gauge theory is dual to weakly coupled string theory, which can be approximated by supergravity. This has led to an improved understanding of confinement, chiral symmetry breaking and other aspects of gauge theory.

In twistor string theory we have a novel implementation of gauge string duality. The theory of open strings is a gauge theory of a different kind. Also it lives in six dimensions. This is so called holomorphic Chern-Simons gauge theory, whose gauge fields correspond to fluctuations of holomorphic gauge bundles on the target space. Ward's transform encodes the anti-selfdual gauge configurations in terms of holomorphic bundles on twistor space. To get away from selfduality, one introduces D-instantons that wrap holomorphic curves. As we will see, this leads to novel ways of computing the perturbative S matrix of gauge theory. These scattering amplitudes are useful for eliminating QCD background at LHC, in order to find new physics beyond standard model.

The study of twistor structure of scattering amplitudes has inspired new developments in perturbative Yang-Mills theory itself. At tree level, this has lead to recursion relations for on-shell amplitudes [7]. These recursion relations rest on basic properties of perturbation theory like the factorization of amplitudes. Hence, it is not surprising to find out that they apply to various field theories, including scalar field theories and even perturbative gravity.

Another way to embed four dimensional gauge theories in string theory is as low energy effective field theories of D-branes wrapped on cycles in Calabi-Yau threefolds. This has led to an understanding of dynamics of various $\mathcal{N} = 1$ gauge theories. Dijkgraaf and Vafa conjectured that holomorphic data of the gauge theories can be calculated from an auxiliary matrix model. Cachazo, Douglas, Seiberg and Witten gave a field theory derivation of the results that rests on the analysis of the anomalies and the ring of chiral operators of the field theory. Surprisingly, the chiral ring gives a full description of the low energy nonperturbative vacua.

In the rest of the Introduction, we will discuss each of the remaining chapters in more detail, stressing the general lessons, rather than the particulars for which the reader is referred to the chapters.

1.0.1 Perturbative Gauge Theory and Twistor String Theory

The twistor space is roughly the space of lightrays in Minkowski space. It was proposed by Penrose [8] in an attempt to address the foundational issues of quantum mechanics. He observed that massless fields have natural description in twistor space in terms of certain cohomology classes. For gauge theory, this has been later generalized by Ward [9] who showed that anti-selfdual gauge field configurations correspond to holomorphic vector bundles on twistor space. For gravity, there is a similar construction due to Penrose [10], that encodes anti-selfdual spacetimes in terms of deformations of the complex structure of the twistor space.

Perhaps the main open question in the twistor programme was the description of interactions. In Minkowski space, the standard way to account for interactions is to start with free fields and introduce the S matrix, which is essentially the transition operator from free field configurations in the far past to a free field configurations in the far future. The interactions happen in the middle in a localized region of space time. One accounts for them by doing a perturbative expansion of the S matrix. This is usually packaged in terms of Feynman diagrams, in which the propagators represent free field propagation between the vertices that represent local interactions of the fields. By twistor correspondence a point in Minkowski space is related to a sphere in twistor space, the celestial sphere of all directions in which a lightray can travel from the point. Hence, the interactions between fields in twistor space are related to two dimensional surfaces rather than zero dimensional points.

This hint was taken up by Witten [11] who proposed that these two dimensional surfaces correspond to worldsheets of strings. His proposal is specific to a particular field theory, the maximally supersymmetric Yang-Mills theory. The string theory dual is the open string topological B-model enriched with D-instantons. The free fields of the gauge theory are related to the open strings. The interactions come from the D-strings on which the open strings end. The D-strings wrap holomorphic curves, simplest of which is the celestial sphere that gets related by twistor transform to points in Minkowski space.

In string theory, interactions are organized differently from a local field theory. Hence, evaluating the D-instanton contribution from string theory led to new ways of computing the perturbative S matrix. This has been successfully carried out at tree level. One obtains a new Feynman diagrammatic expansion of the scattering amplitudes [1], which in twistor space corresponds to a collection of D-strings wrapping the celestial spheres with open string connecting them into a tree graph. The more complicated holomorphic curves do not have a natural description in Minkowski space, hence evaluating them does not have a simple interpretation in the gauge theory and is the strongest evidence so far that the twistor string theory is correct [12].

Despite the successes of the twistor string theory at tree level, there are many open questions. The most pressing one is that the B-model has besides the open strings also closed strings. While the open strings give the fields of $\mathcal{N} = 4$ gauge theory, the closed strings give the fields of $\mathcal{N} = 4$ conformal gravity. This is somewhat unwelcome, since conformal gravity theories are generally considered to be unphysical. One would hope to find a string theory that is dual to Yang-Mills, since Yang-Mills theory is known to be consistent without conformal supergravity.

1.0.2 Twistor Structure of Scattering Amplitudes

One of the most general predictions of twistor string theory is that the interactions come from strings that wrap curves in twistor space. We can study this prediction even if we do not have a proper formulation of the twistor string theory.

The curves in questions are algebraic curves of a degree and genus that depends on the amplitudes. The conditions for a scattering amplitude to be supported on the curve are polynomial equations in the twistor coordinates of the external particles. After Fourier transform into Minkowski space, these become differential equations acting on the scattering amplitudes.

For one-loop MHV amplitudes in $\mathcal{N} = 4$ Yang-Mills theory, the differential equations studied in [2] agree with the twistor string picture after one takes into account the holomorphic anomaly of the differential operators [3].

For Yang-Mills theories with reduced supersymmetry we do not have a twistor string proposal. One can get hints of a possible twistor string by studying the differential equations that the amplitudes satisfy. Our results are surprisingly similar to the $\mathcal{N} = 4$ case. We find that the amplitudes are supported on curves whose degree and genus is related to the helicities of the external particles in essentially the same way as in $\mathcal{N} = 4$ Yang-Mills theory. Perhaps the most important difference in the $\mathcal{N} = 0$ case is that the one-loop amplitude with all gluons of positive helicity must be included as a new building block alongside with the MHV amplitude.

We also study the twistor structure of gravity amplitudes. It has been known that graviton amplitudes, just like gluon amplitudes, exhibit remarkable simplicity that cannot be expected from textbook recipes for computing them. The tree level n graviton amplitudes vanish if more than $n-2$ gravitons have the same helicity. The maximally helicity violating (MHV) amplitudes are thus, as in Yang-Mills case, those with $n-2$ gravitons of one helicity and two of the opposite helicity. Some of this simplicity can be explained by relating the graviton amplitudes to gluon amplitudes via the Kawai, Lewellen and Tye (KLT) relations [13]. These express the graviton amplitudes as simple sums of products of gluon amplitudes and momentum invariants [14].

We find that the twistor structure of the graviton amplitudes is remarkably similar to the twistor structure of gluon amplitudes. One difference is that the amplitudes are generically supported in a higher order neighborhood of the curves. Similar behavior was observed for closed strings in the B-model which give the conformal supergravity amplitudes [15].

1.0.3 Tree Level Recursion Relations For Gravity Amplitudes

The discovery of twistor string theory has stimulated renewed progress in computing scattering amplitudes. Among other things, a new set of recursion relations for tree-level amplitudes of gluons have been recently introduced in [7]. A straightforward application of these recursion relations gives new and simple forms for many amplitudes.

A proof of the recursion relations was given in [16]. The proof rests only on generic properties of perturbation theory like the fact that the only poles of tree level amplitudes come from the Feynman propagators and that the tree level amplitudes are rational functions of the kinematic data of the external particles. So one would expect that it extends to other field theories.

In chapter four, we generalize the recursion relations of [7] to tree level amplitudes of gravitons. To write down the recursion relations, we single out two gravitons. Then the recursion relations can be schematically written as follows

$$A_n = \sum_{\mathcal{I},h} A_{\mathcal{I}}^{h} \frac{1}{P_{\mathcal{I}}^{2}} A_{\mathcal{J}}^{-h}, \tag{1.0.1}$$

where the sum is over all channels that divide the external particles into two sets $\mathcal{I}$ and $\mathcal{J}$, such that the marked gravitons are in different groups. The momenta of the marked gravitons and of the intermediate graviton are shifted, so that the subamplitudes are on-shell.

The recursion relations lead to new simple formulas for graviton amplitudes, which explains some of the simplicity of graviton amplitudes that was mentioned in previous subsection. We use them to compute all amplitudes up to six gravitons.

The recursion relations are a precise version of the statement that tree-level amplitudes are uniquely determined by their singularity structure. The tree level amplitudes have factorization, collinear and soft singularities. The recursion relations construct the scattering amplitude from the factorization singularities in the channels that separate the marked gravitons.

There could be ambiguities in the amplitudes that are not fixed by the factorization properties. The ambiguity would have to be a function that is free of singularities, that is a polynomial in the momentum invariants. Yang-Mills amplitudes of n gluons have dimention $4 - n$. Hence for $n \geq 5$ gluons, the amplitudes have negative dimension so they cannot have a polynomial ambiguity, because a polynomial has positive dimension.

For gravitons, all tree-level amplitudes have dimension two, so there could be a polynomial ambiguity in the recursion relations. We were able to prove it absence for some classes of scattering amplitudes which suggests that the recursion relations are valid in general.

1.0.4 Chiral Rings, Vacua of SUSY Gauge Theories

In chapter five we shift our focus from perturbative to nonperturbative aspects of gauge theories. We consider $\mathcal{N} = 1$ $U(N)$ gauge theories with adjoint matter. These gauge theories can be embedded into string theory by wrapping N D5-branes around a two cycle in a local Calabi-Yau manifold. The holomorphic data, that is the F-terms, of the gauge theory can be related to topological observables in the string theory. These in turn are computed from the topological B-model. In the special geometry that we consider, the B-model reduced to a holomorphic matrix model [17] of a single hermitian $N \times N$ matrix. The bosonic potential of the matrix model gets related to the superpotential of the adjoint matter field of the gauge theory.

Cachazo, Douglas, Seiberg and Witten gave a field theory derivation of this result. They studied the chiral operators, which in a sense constitute the topological part of the gauge theory. Indeed, in a supersymmetric vacuum, the chiral operators are independent of their positions and correlation functions of single trace operators factorize. Since the product of chiral operators is chiral, one can consider the ring of chiral operators. The structure of the ring is constrained by various relations. Some of these come from the Konishi anomalies of the chiral rotations of the matter field $\Phi \to e^{i\alpha}\Phi$. Further come from the $\mathcal{N} = 2$ parent theory in which the superpotential of the matter field is turned off.

It has been observed in two dimensions that the ring of chiral operators determines the supersymmetric vacuum structure of the theory. This has been shown in [18] for the $\mathcal{N} = 2$ superconformal field theories and in [19] for the CP^{N-1} supersymmetric sigma model.

In four dimensions, the first example of this correspondence came up in [20] in the case of pure $\mathcal{N} = 1$ $U(N)$ gauge theory. This theory has N low energy confining vacua that break the Z_{2N} chiral symmetry down to Z_2. Here, the only operator in the chiral ring is the gaugino bilinear $S = \mathrm{Tr}\, W_\alpha W^\alpha$. Classically, it satisfies $S^N = 0$. This relation receives one-instanton corrections which deform it to

$$S^N = \Lambda^{3N}. \tag{1.0.2}$$

Each of the solutions of this equation correspond to a vacuum of the gauge theory.

In chapter five, we extend these considerations to the $\mathcal{N} = 1$ gauge theories with adjoint scalar. We study the relations of the chiral ring and find a complete set of the relations. We show that they completely determine the structure of the chiral ring. As in the pure $\mathcal{N} = 1$ case, each vacuum corresponds to an idempotent element of the chiral ring. The rank of the low-energy group is fixed by the dimension of the fermionic part of the operator. We also study generalizations of the equation (1.0.2). Here, the classical equation gets deformed to $S^N = P(\Phi)\Lambda^{2N}$, where $P(\Phi)$ is a degree n polynomial in Φ that depends on the superpotential.

1.0.5 Nonperturbative Exactness of Konishi Anomaly

As we have just seen, the chiral ring of four dimensional $\mathcal{N}=1$ gauge theories with adjoint matter determines the nonperturbative structure of the supersymmetric vacua. To put this argument on a firmer footing, it is necessary to show that the chiral ring relations hold nonperturbatively. A simple argument shows that the Konishi anomalies cannot receive perturbative corrections. The question of nonperturbative corrections can be addressed with the help of the Wess-Zumino consistency conditions that constrain the anomalies.

In the case Konishi anomalies, the rigidity of the Lie algebra of chiral rotations implies that the consistency conditions do not receive nonperturbative correction. In chapter six, we study the consistency conditions and show that they imply nonrenormalization of the Konishi anomalies for a wide class of examples, namely for all gauge theories with a superpotential of degree less than or equal $2l$ where $2l = 3c(Adj) - c(M)$ is the one-loop beta function coefficient.

For superpotential of degree higher than $2l$, nonperturbative corrections are expected due to ambiguities in the definition of the highly nonrenormalizable operators like Tr Φ^n [21], [22] and [23]. The Wess-Zumino consistency conditions can be applied anyway, and we show that the strongly constrain the form of the nonperturbative corrections, so that the corrections can be absorbed into the redefinition of the superpotential. Hence, after the redefinition, these theories have undeformed chiral ring relations as well.

This proof can be applied to other gauge theories as long as the algebra of chiral rotations of the matter fields forms an extension of the partial Virasoro algebra. As an illustration, we study the case of $SO(N)$ and $Sp(N)$ gauge theory with matter in the symmetric or antisymmetric representation. The case of $Sp(N)$ gauge theory with antisymmetric matter is especially interesting in the light of a puzzle raised in the study of the related matrix model in [24], [25] and [26]. Our result confirms that the Dijgraaf-Vafa correspondence works for these theories, as has been demonstrated in [27] and [21].

2. Twistor String Theory and Perturbative Yang-Mills

2.1 Introduction

The idea that a gauge theory should be dual to a string theory goes back to 't Hooft [28]. 't Hooft considered $U(N)$ gauge theory in the large N limit while keeping $\lambda = g_{YM}^2 N$ fixed. He observed that the perturbative expansion of Yang-Mills can be reorganized in terms of Riemann surfaces, which he interpreted as an evidence for a hypothetical dual string theory with string coupling $g_s \sim 1/N$.

In 1997, Maldacena proposed a concrete example of this duality [29]. He considered the maximally supersymmetric Yang-Mills theory and conjectured that it is dual to type IIB string theory on $AdS_5 \times S^5$. This duality led to many new insights from string theory about gauge theories and vice versa. At the moment, we have control over the duality only for strongly coupled gauge theory. This corresponds to the limit of large radius of $AdS_5 \times S^5$ in which the string theory is well described by supergravity. However, QCD is asymptotically free, so we would also like to have a string theory description of a weakly coupled gauge theory.

In weakly coupled field theories, the natural object to study is the perturbative S matrix. The perturbative expansion of S matrix is conventionally computed using Feynman rules. Starting from early studies of de Witt [30], it was observed that the scattering amplitudes show simplicity that is not apparent from the Feynman rules. For example the maximally helicity violating amplitudes can be expressed as simple holomorphic functions.

Recently, Witten proposed a string theory that is dual to a weakly coupled $\mathcal{N} = 4$ gauge theory [11]. The perturbative expansion of the gauge theory is related to D-instanton expansion of the string theory. The string theory in question is the topological open string B-model on a Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$, which is a supersymmetric generalization of Penrose's twistor space.

At tree level, evaluating the instanton contribution has led to new insights about scattering amplitudes. 'Disconnected' instantons give MHV diagram construction of the amplitudes in terms of Feynman diagrams with vertices that are suitable off-shell continuations of the MHV amplitudes [1]. The 'connected' instanton contributions express the amplitudes as integrals over the moduli space of holomorphic curves in twistor space [12]. Surprisingly, the MHV diagram construction and the connected instanton integral can be related via localization on the moduli space [31].

Despite the successes of the twistor string theory at tree level, there are still many open questions. The most pressing issue is perhaps the closed strings that give $\mathcal{N} = 4$ conformal supergravity [15]. At tree level, it is possible to recover the Yang-Mills scattering amplitudes by extracting the single-trace amplitudes. At loop level, the single trace gluon scattering amplitudes receive contributions from internal supergravity states, so it would be difficult to extract the Yang-Mills contribution to the gluon scattering amplitudes. Since, $\mathcal{N} = 4$ Yang-Mills theory is consistent without conformal supergravity, it is likely that there exists a version of the twistor string theory that is dual to pure Yang-Mills theory. Indeed, the MHV diagram construction that at tree level has been derived from twistor string theory seems to compute loop amplitudes as well [32].

The study of twistor structure of scattering amplitudes has inspired new developments in perturbative Yang-Mills theory itself. At tree level, this has lead to recursion relations for on-shell amplitudes [7]. At one loop, unitarity techniques [33,34] have been used to find new ways of computing the $\mathcal{N} = 4$ [35] and $\mathcal{N} = 1$ [36] Yang-Mills amplitudes.

In these lectures we will discuss aspects of the twistor string theory. Along the way we will learn lessons about Yang-Mills scattering amplitudes. The string theory sheds light on Yang-Mills perturbation theory and leads to new methods for computing scattering amplitudes. In the last section, we will describe further developments in perturbative Yang-Mills.

2.2 Helicity Amplitudes

2.2.1 Spinors

Recall[1] that the complexified Lorentz group is locally isomorphic to

$$SO(3,1,\mathbb{C}) \cong Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C}), \tag{2.2.1}$$

hence the finite dimensional representations are classified as (p,q) where p and q are integer or half-integer. The negative and positive chirality spinors transform in the representations $(1/2,0)$ and $(0,1/2)$ respectively. We write generically $\lambda_a, a=1,2$ for a spinor transforming as $(1/2,0)$ and $\widetilde{\lambda}_{\dot{a}}, \dot{a}=1,2$ for a spinor transforming as $(0,1/2)$.

The spinor indices of type $(1/2,0)$ are raised and lowered using the antisymmetric tensors ϵ_{ab} and ϵ^{ab} obeying $\epsilon_{12}=1$ and $\epsilon^{ac}\epsilon_{cb}=\delta^a{}_b$

$$\lambda^a = \epsilon^{ab}\lambda_b \qquad \lambda_a = \epsilon_{ab}\lambda^b. \tag{2.2.2}$$

Given two spinors λ and λ', both of negative chirality, we can form the Lorentz invariant product

$$\langle \lambda, \lambda' \rangle = \epsilon_{ab}\lambda^a\lambda'^b. \tag{2.2.3}$$

It follows that $\langle \lambda, \lambda' \rangle = -\langle \lambda', \lambda \rangle$, so the product is antisymmetric in its two variables. In particular, $\langle \lambda, \lambda' \rangle = 0$ implies that λ equals λ' up to a scaling $\lambda^a = c\lambda'^a$.

Similarly, we lower and raise the indices of positive chirality spinors with the antisymmetric tensor $\epsilon_{\dot{a}\dot{b}}$ and its inverse $\epsilon^{\dot{a}\dot{b}}$. For two spinors $\widetilde{\lambda}$ and $\widetilde{\lambda}'$, both of positive chirality we define the antisymmetric product

$$[\widetilde{\lambda}, \widetilde{\lambda}'] = -[\widetilde{\lambda}', \widetilde{\lambda}] = \epsilon_{\dot{a}\dot{b}}\widetilde{\lambda}^{\dot{a}}\widetilde{\lambda}^{\dot{b}}. \tag{2.2.4}$$

A vector representation of $SO(3,1,\mathbb{C})$ is the $(1/2,1/2)$ representation. Thus a momentum vector $p_\mu, \mu = 0,\ldots,3$ can be represented as a "bi-spinor" $p_{a\dot{a}}$ with one spinor index a and $\dot{a}$ of each chirality. The explicit mapping from p_μ to $p_{a\dot{a}}$ can be

[1] The sections $2.2-2.4$ are based on lectures given by E. Witten at PITP, IAS Summer 2004

made using the chiral part of the Dirac matrices. In signature $+---$, one can take the Dirac matrices to be

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}, \tag{2.2.5}$$

where $\sigma^\mu = (1, \vec{\sigma}), \overline{\sigma}^\mu = (1, -\vec{\sigma})$ with $\vec{\sigma}$ being the 2×2 Pauli matrices. For any vector, the relation between p_μ, and $p_{a\dot{a}}$ is

$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} = p_0 + \vec{\sigma} \cdot \vec{p}. \tag{2.2.6}$$

It follows that,

$$p_\mu p^\mu = \det(p_{a\dot{a}}). \tag{2.2.7}$$

Hence, p_μ is lightlike if the corresponding determinant is zero. This is equivalent to the rank of the 2×2 matrix $p_{a\dot{a}}$ being less or equal to one. So p^μ is lightlike precisely, when it can be written as a product

$$p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}} \tag{2.2.8}$$

for some spinors λ_a and $\widetilde{\lambda}_{\dot{a}}$. For a given null vector p, the spinors λ and $\widetilde{\lambda}$ are unique up to a scaling

$$(\lambda, \widetilde{\lambda}) \to (t\lambda, t^{-1}\widetilde{\lambda}) \qquad t \in \mathbb{C}^*. \tag{2.2.9}$$

There is no continuous way to pick λ as a function p. The λ's form a Hopf line bundle over the sphere S^2 of directions of the lightlike vector p.

For complex momenta, the spinors λ^a and $\widetilde{\lambda}^{\dot{a}}$ are independent complex variables, each of which parameterizes a copy of $\mathbb{CP}^1$. Hence, the complex lightcone $p_\mu p^\mu = 0$ is the connected manifold $\mathbb{CP}^1 \times \mathbb{CP}^1$.

For real null momenta in Minkowski signature $+---$, we can fix the scaling up to a Z_2 by requiring λ^a and $\widetilde{\lambda}^{\dot{a}}$ to be complex conjugates

$$\overline{\lambda}^{\dot{a}} = \pm \widetilde{\lambda}^{\dot{a}}. \tag{2.2.10}$$

Hence, the negative chirality spinors λ are conventionally called 'holomorphic' and the positive chirality 'anti-holomorphic.' In (2.2.10) the $+$ is for a future pointing null vector p^μ, and $-$ is for a past pointing p^μ.

One can also consider other signature. For example in the signature $++--$, the spinors λ and $\widetilde{\lambda}$ are real and independent. Indeed, with signature $++--$, the Lorentz group is $SO(2,2,\mathbb{R})$, which is locally isomorphic to $Sl(2,\mathbb{R}) \times Sl(2,\mathbb{R})$. Hence, the spinor representations are real.

Let us remark, that if p and p' are two lightlike vectors given by $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$ and $p'_{a\dot{a}} = \lambda'_a \widetilde{\lambda}'_{\dot{a}}$ then their scalar product can be expressed as

$$2p \cdot p' = \langle \lambda, \lambda' \rangle [\widetilde{\lambda}, \widetilde{\lambda}']. \tag{2.2.11}$$

Given p, the additional physical information in λ is equivalent to a choice of wavefunction of a helicity $-1/2$ massless particle with momentum p. To see this, we write the chiral Dirac equation for a negative chirality spinor ψ^a

$$0 = i\sigma^{\mu}_{a\dot{a}} \partial_\mu \psi^a. \tag{2.2.12}$$

A plane wave $\psi^a = \rho^a \exp(ip \cdot x)$ satisfies this equation only if $p_{a\dot{a}} \rho^a = 0$. Writing $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$, we get $\lambda_a \rho^a = 0$, that is $\rho^a = c \cdot \lambda^a$ for a constant c, hence the negative chirality fermion has wavefunction

$$\psi^a = c\lambda^a \exp(ix_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}}). \tag{2.2.13}$$

Similarly, $\widetilde{\lambda}$ defines a wavefunction for a helicity $+1/2$ fermion $\psi^{\dot{a}} = c\widetilde{\lambda}^{\dot{a}} \exp(ix_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}})$.

There is an analogous description of wavefunctions of massless particles of spin ± 1. Usually, we describe massless gluons with their momentum vector p^μ and polarization vector ϵ^μ. The polarization vector obeys the constraint

$$p_\mu \epsilon^\mu = 0 \tag{2.2.14}$$

that represents the decoupling of longitudinal modes and is subject to the gauge invariance

$$\epsilon^\mu \to \epsilon^\mu + w p^\mu, \tag{2.2.15}$$

for any constant w. Suppose that instead of being given only a lightlike vector $p_{a\dot{a}}$, one is also given a decomposition $p_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}$. Then we have enough information to determine the polarization vector up to a gauge transformation. For a positive helicity gluon, we take

$$\epsilon^{+}_{a\dot{a}} = \frac{\mu_a \widetilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}, \tag{2.2.16}$$

where μ is any negative helicity spinor that is not a multiple of λ. To get a negative helicity polarization vector, we take

$$\epsilon^-_{a\dot{a}} = \frac{\lambda_a \widetilde{\mu}_{\dot{a}}}{[\widetilde{\lambda}, \widetilde{\mu}]}, \tag{2.2.17}$$

where $\widetilde{\mu}$ is any positive helicty spinor that is not a multiple of $\widetilde{\lambda}$. We will explain the expression for the positive helicity vector. The negative helicity case is similar.

Clearly, the constraint

$$p^\mu \epsilon^+_\mu = p^{a\dot{a}} \epsilon^+_{a\dot{a}} = 0 \tag{2.2.18}$$

holds because $\widetilde{\lambda}^{\dot{a}} \widetilde{\lambda}_{\dot{a}} = 0$. Moreover, ϵ^+ is also independent of μ up to a gauge transformation. To see this, notice that μ lives in a two dimensional space that is spanned with λ and μ. Hence, any change in $\widetilde{\mu}$ is of the form

$$\delta\mu = \alpha\mu + \beta\lambda \tag{2.2.19}$$

for some parameters α and β. The polarization vector (2.2.16) is invariant under the α term, because this simply rescales μ and $\epsilon^+_{a\dot{a}}$ is invariant under the rescaling of μ. The β term amounts to a gauge transformation of the polarization vector

$$\delta\epsilon^+_{a\dot{a}} = \beta \frac{\lambda_a \widetilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}. \tag{2.2.20}$$

Under the scaling $(\lambda, \widetilde{\lambda}) \to (t\lambda, t^{-1}\widetilde{\lambda})$, $t \in C^*$ the polarization vectors scale like

$$\epsilon^- \to t^{+2}\epsilon^- \qquad \epsilon^+ \to t^{-2}\epsilon^+. \tag{2.2.21}$$

This could have been anticipated, since $\widetilde{\lambda}$ gives the wavefunction of a helicity $+1/2$ particle so a helicity $+1$ polarization vector should scale like $\widetilde{\lambda}^2$. Similarly, the helicity -1 polarization vector scales like λ^2.

To show more directly that ϵ^+ describes a massless particle of helicity $+1$, we must show that the corresponding linearized field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is anti-selfdual. Indeed, the field strength written in a bispinor notation has the decomposition

$$F_{a\dot{a}b\dot{b}} = \epsilon_{ab} \widetilde{f}_{\dot{a}\dot{b}} + \epsilon_{\dot{a}\dot{b}} f_{ab}, \tag{2.2.22}$$

where f_{ab} and $\widetilde{f}_{\dot{a}\dot{b}}$ are the selfdual and anti-selfdual part of F. Substituting $A_{a\dot{a}} = \epsilon^+_{a\dot{a}} \exp(ix_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})$ we find that $F_{a\dot{a}b\dot{b}} = \epsilon_{ab}\widetilde{\lambda}_{\dot{a}}\widetilde{\lambda}_{\dot{b}} \exp(ix_{a\dot{a}}\lambda^a\widetilde{\lambda}^{\dot{a}})$.

So far, we have seen that the wavefunction of a massless particle with helicity h scales under $(\lambda, \widetilde{\lambda}) \to (t\lambda, t^{-1}\widetilde{\lambda})$ as t^{-2h} if $|h| \leq 1$. This is true for any h, as can be seen from the following argument. Consider a massless particle moving in the $\vec{n}$ direction. Then a rotation by angle θ around the $\vec{n}$ axis acts on the spinors as

$$(\lambda, \widetilde{\lambda}) \to (e^{-i\theta/2}\lambda, e^{+i\theta/2}\widetilde{\lambda}). \tag{2.2.23}$$

Hence, $\lambda, \widetilde{\lambda}$ carry $-\frac{1}{2}$ or $+\frac{1}{2}$ units of angular momentum around the $\vec{n}$ axis. Clearly, a massless particle of helicity h carries h units of angular momentum around the $\vec{n}$ axis. Hence the wavefunction of the particle gets transformed as $\psi \to e^{ih\theta}\psi$ under the rotation around $\vec{n}$ axis. Hence, the wavefunction obeys the auxiliary condition

$$\left(\lambda^a \frac{\partial}{\partial \lambda^a} - \widetilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}}\right) \psi(\lambda, \widetilde{\lambda}) = -2h\psi(\lambda, \widetilde{\lambda}). \tag{2.2.24}$$

Clearly, this constraint holds for wavefunctions of massless particles of any spin. The spinors $\lambda, \widetilde{\lambda}$ give us a convenient way of writing the wavefunction of massless particle of any spin, as we have seen in detail above for particles with $|h| \leq 1$.

2.2.2 Scattering Amplitudes

Let us consider scattering of massless particles in four dimensions. Consider the situation with n particles of momenta $p_1, p_2, \ldots, p_n$. For scattering of scalar particles, the initial and final states of the particles are completely determined by the momenta. The scattering amplitude is simply a function of the momenta p_i,

$$A_{scalar}(p_1, p_2, \ldots, p_n). \tag{2.2.25}$$

In fact, by Lorentz invariance, it is a function of the Lorentz invariants products $p_i \cdot p_j$ only.

For particles with spin, the scattering amplitude is a function of both the momenta p_i and the wavefunctions ψ_i

$$A(p_1, \psi_1; \ldots; p_n, \psi_n). \tag{2.2.26}$$

Here, A is linear in each of the wavefunctions ψ_i. The description of ψ_i depends on the spin of the particle. As we have seen explicitly above in the case of massless particles of spin $\frac{1}{2}$ or 1, the spinors $\lambda, \widetilde{\lambda}$ give a unified description of the wavefunctions of particles with spin. Hence, to describe the wavefunctions, we specify for each particle the helicity h_i and the spinors λ_i and $\widetilde{\lambda}_i$. The spinors determine the momenta $p_i = \lambda_i \widetilde{\lambda}_i$ and the wavefunctions $\psi_i(\lambda_i, \widetilde{\lambda}_i, h_i)$. So for massless particles with spin, the scattering amplitude is a function of the spinors and helicities of the external particles

$$A(\lambda_1, \widetilde{\lambda}_1, h_1; \ldots; \lambda_n, \widetilde{\lambda}_n, h_n). \tag{2.2.27}$$

In labelling the helicities we take all particles to be outgoing. To obtain an amplitude with incoming particles as well as outgoing particles, we use crossing symmetry, that relates an incoming particle of one helicity to an outgoing particle of the opposite helicity.

It follows from (2.2.24) that the amplitude obeys the conditions

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \widetilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{a}}} \right) A(\lambda_i, \widetilde{\lambda}, h_i) = -2h_i A(\lambda_i, \widetilde{\lambda}_i, h_i) \tag{2.2.28}$$

for each particle i, with helicity h_i. In summary, a general scattering amplitude of massless particles can be written as

$$A = (2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} \right) A(\lambda_i, \widetilde{\lambda}_i, h_i), \tag{2.2.29}$$

where we have written explicitly the delta function of momentum conservation.

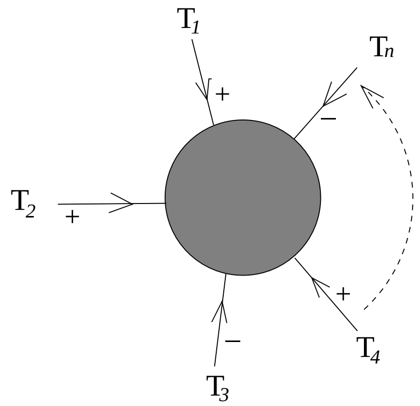

Fig. 1: A scattering amplitude of n gluons in Yang-Mills theory. Each gluon comes with the color factor T_i, spinors $\lambda_i, \widetilde{\lambda}_i$ and helicity label $h_i = \pm 1$.

2.2.3 Maximally Helicity Violating Amplitudes

To make the discussion more concrete, we consider the tree level scattering of n gluons in Yang-Mills theory. These amplitudes are of phenomenological importance. The multi-jet production at LHC will be dominated by tree level QCD scattering.

Consider the Yang-Mills theory with gauge group $U(N)$. Recall that the tree level scattering amplitudes are planar and lead to single trace interactions. In an index loop, the gluons are attached in a definite cyclic order, say $1, 2, \ldots, n$. Then the amplitude comes with a group theory factor Tr $T_1 T_2 \ldots T_n$. It is sufficient to give the amplitude with one cyclic order. The full amplitude is obtained from this by summing over the cyclic orders, to restore Bose symmetry

$$\mathcal{A} = g^{n-2}(2\pi)^4 \delta^4 \left(\sum_i^n p_i \right) A(1, 2, \ldots, n) \mathrm{Tr}\ (T_1 T_2 \ldots T_n) + \mathit{permutations}. \tag{2.2.30}$$

In the rest of the thesis, we will always consider gluons in the cyclic order $1, 2, \ldots, n$ and we will omit the group theory factor and the delta function of momentum conservation in writing the formulas. Hence, we will consider the 'reduced color ordered amplitudes' $A(1, 2, \ldots, n)$.

The scattering amplitude with n outgoing gluons of the same helicity vanishes. So does the amplitude, for $n \geq 3$ with $n - 1$ outgoing gluons of one helicity and one of the opposite helicity. The first nonzero amplitude, the maximally helicity violating (MHV) amplitude has $n - 2$ gluons of one helicity and two gluons of the other helicity. Suppose that gauge bosons r, s have negative helicity and the rest of gluons have positive helicity. Then the tree level amplitude, stripped of the momentum delta function and the group theory factor, is

$$\mathcal{A} = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{k=1}^n \langle \lambda_k, \lambda_{k+1} \rangle}. \tag{2.2.31}$$

Note, that the amplitude has the correct homogeneity in each variable. It is homogeneous of degree -2 in λ_i for positive helicity gluons; and of degree -2 for negative helicity gluons $i = r, s$ as required by the auxiliary condition (2.2.28). The amplitude $\mathcal{A}$ is sometimes called 'holomorphic' because it depends on the 'holomorphic' spinors λ_i only.

2.3 Twistor Space

2.3.1 Conformal Invariance of Scattering Amplitudes

Before discussing twistor space, let us show the conformal invariance of the MHV tree level amplitude. Firstly, we need to construct representation of the conformal group generators in terms of the spinors $\lambda, \widetilde{\lambda}$. We will consider the conformal generators for a single particle. The generators of the n-particle system are given by the sum of the generators over the n particles.

Some of the generators are clear. The Lorentz generators are the first order differential operators

$$\begin{aligned} J_{ab} &= \frac{i}{2}\left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a}\right) \\ \widetilde{J}_{\dot{a}\dot{b}} &= \frac{i}{2}\left(\widetilde{\lambda}_{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{b}}} + \widetilde{\lambda}_{\dot{b}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}}\right) \end{aligned} \tag{2.3.1}$$

The momentum operator is the multiplication operator

$$P_{a\dot{a}} = \lambda_a \widetilde{\lambda}_{\dot{a}}. \tag{2.3.2}$$

The remaining generators are the dilatation operator D and the generator of special conformal transformation $K_{a\dot{a}}$. The commutation relations of the dilatation operator are

$$[D, P] = iP \qquad [D, K] = -iK, \tag{2.3.3}$$

so P has dimension $+1$ and K has dimension -1. We see from (2.3.2) that it is natural to take λ and $\widetilde{\lambda}$ to have dimension $1/2$. Hence, a natural guess for the special conformal generator respecting all the symmetries is

$$K_{a\dot{a}} = \frac{\partial^2}{\partial \lambda^a \partial \widetilde{\lambda}^{\dot{a}}}. \tag{2.3.4}$$

We find the dilatation operator D from the closure of the conformal algebra. The commutation relation

$$[K_{a\dot{a}}, P^{b\dot{b}}] = -i\left(\delta_a{}^b \widetilde{J}_{\dot{a}}{}^{\dot{b}} + \delta_{\dot{a}}{}^{\dot{b}} J_a{}^b + \delta_a{}^b \delta_{\dot{a}}{}^{\dot{b}} D\right) \tag{2.3.5}$$

determines the dilatation operator to be

$$D = \frac{i}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} + \widetilde{\lambda}^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} + 2\right). \tag{2.3.6}$$

Now, let us verify that the MHV amplitude

$$A = (2\pi)^4 \delta^4 \left(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}} \right) \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{k=1}^n \langle \lambda_i, \lambda_{i+1} \rangle} \tag{2.3.7}$$

is invariant under the conformal group. The Lorentz generators are clearly symmetries of the amplitude. The momentum operator annihilates the amplitude thanks to the delta function of momentum conservation.

It remains to verify that the amplitude is annihilated by D and K. For simplicity, we will only consider the dilatation operator D. The numerator contains the delta function of momentum conservation which has dimension $D = -4$ and the factor $\langle \lambda_r, \lambda_s \rangle^4$ of dimension 4. Hence, D commutes with the numerator. So we are left with the denominator

$$\frac{1}{\prod_{k=1}^n \langle \lambda_k, \lambda_{k+1} \rangle}. \tag{2.3.8}$$

This is annihilated by D_k for each particle k, since the -2 coming from the degree of λ_k gets cancelled against the $+2$ from the definition of the dilatation operator.

2.3.2 Transform to Twistor Space

We have demonstrated conformal invariance of the MHV amplitude, however the representation of the conformal group that we have encountered above is quite exotic. The Lorentz generators are first order differential operators, but the momentum is a multiplication operator and the special conformal generator is a second order differential operator.

We can put the action of the conformal group into a more standard form if we make the following transformation

$$\begin{aligned} \widetilde{\lambda}_{\dot{a}} &\to i \frac{\partial}{\partial \mu^{\dot{a}}} \\ \frac{\partial}{\partial \widetilde{\lambda}^{\dot{a}}} &\to i \mu_{\dot{a}}. \end{aligned} \tag{2.3.9}$$

Making this substitution we have arbitrarily chosen to Fourier transform $\widetilde{\lambda}$ rather than l. This choice breaks the symmetry between positive and negative helicities. The amplitude with n_1 positive helicity and n_2 negative helicity gluons has completely different description in twistor space from an amplitude with n_2 positive helicity gluons and n_1 negative helicity gluons.

Upon making this substitution, all operators become first order. The Lorentz generators take the form

$$\begin{aligned} J_{ab} &= \frac{i}{2}\left(\lambda_a \frac{\partial}{\partial \lambda^b} + \lambda_b \frac{\partial}{\partial \lambda^a}\right) \\ \widetilde{J}_{\dot{a}\dot{b}} &= \frac{i}{2}\left(\mu_{\dot{a}} \frac{\partial}{\partial \mu^{\dot{b}}} + \mu_{\dot{b}} \frac{\partial}{\partial \mu^{\dot{a}}}\right). \end{aligned} \tag{2.3.10}$$

The momentum and special conformal operators become

$$\begin{aligned} P_{a\dot{a}} &= i\lambda_a \frac{\partial}{\partial \mu^{\dot{a}}} \\ K_{a\dot{a}} &= i\mu_{\dot{a}} \frac{\partial}{\partial \lambda^a}. \end{aligned} \tag{2.3.11}$$

Finally, the dilatation operator (2.3.6) becomes a *homogeneous* first order operator

$$D = \frac{i}{2}\left(\lambda^a \frac{\partial}{\partial \lambda^a} - \mu^{\dot{a}} \frac{\partial}{\partial \mu^{\dot{a}}}\right). \tag{2.3.12}$$

This representation of the four dimensional conformal group is easy to explain. The conformal group of Minkowski space is $SO(4,2)$ which is the same as $SU(2,2)$. $SU(2,2)$, or its complexification $Sl(4,\mathbb{C})$, has an obvious four-dimensional representation acting on

$$Z^I = (\lambda^a, \mu^{\dot{a}}). \tag{2.3.13}$$

Z^I is called a twistor and the space C^4, spanned by Z^I is called the twistor space. The action of $Sl(4,C)$ on the Z^I is generated by the 15 traceless matrices that correspond to the 15 first order operators $J_{ab}, \widetilde{J}_{\dot{a}\dot{b}}, D, P_{a\dot{a}}, K_{a\dot{a}}$.

If we are in signature $++--$, the conformal group is $SO(3,3) \cong Sl(4,R)$. The twistor space is a copy of $\mathbb{R}^4$ and we can consider λ and μ to be real. In the Euclidean signature $++++$, the conformal group is $SO(5,1) \cong SU^*(4)$, where $SU^*(4)$ is the noncompact version of $SU(4)$, so we must think of twistor space as a copy of $\mathbb{C}^4$.

For signature $++--$, where $\widetilde{\lambda}$ is real, the transformation from momentum space scattering amplitudes to twistor space scattering amplitudes is made by a simple Fourier transform that is familiar from quantum mechanics

$$\widetilde{\mathcal{A}}(\lambda_i, \mu_i) = \int \prod_{j=1}^{n} \frac{d^2\widetilde{\lambda}_j}{(2\pi)^2} \exp(i[\mu_j, \widetilde{\lambda}_j])\mathcal{A}(\lambda_i, \widetilde{\lambda}_i). \tag{2.3.14}$$

The same Fourier transform turns a momentum space wavefunction $\psi(\lambda, \widetilde{\lambda})$ to a twistor space wavefunction

$$\widetilde{\psi}(\lambda, \mu) = \int \frac{d^2\widetilde{\lambda}}{(2\pi)^2} \exp(i[\mu, \widetilde{\lambda}])\psi(\lambda, \widetilde{\lambda}). \tag{2.3.15}$$

Recall that the wave function of a massless particle of helicity h obeys the auxiliary condition

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} - \widetilde{\lambda}_i^{\dot{a}} \frac{\partial}{\partial \widetilde{\lambda}_i^{\dot{a}}}\right) \mathcal{A}(\lambda_i, \widetilde{\lambda}_i, h_i) = -2h_i \mathcal{A}(\lambda_i, \widetilde{\lambda}_i, h_i) \tag{2.3.16}$$

for each particle i, with helicity h_i. In terms of λ_i and μ_i, this becomes

$$\left(\lambda_i^a \frac{\partial}{\partial \lambda_i^a} + \mu_i^{\dot{a}} \frac{\partial}{\partial \mu_i^{\dot{a}}}\right) \mathcal{A}(\lambda_i, \mu_i, h_i) = -(2h_i + 2)\mathcal{A}(\lambda_i, \mu_i, h_i). \tag{2.3.17}$$

There is a similar condition for the twistor wavefunctions of particles. The operator on the left hand side coincides with $Z^I \frac{\partial}{\partial Z^I}$ and generates the scaling of the twistor

$$Z^I \to tZ^I, \qquad t \in \mathbb{C}^*. \tag{2.3.18}$$

So the wavefunctions and scattering amplitudes have known behavior under the $\mathbb{C}^*$ action $Z^I \to tZ^I$. Hence, we can identify the sets of Z^I that differ by the scaling $Z^I \to tZ^I$ and throw away the point $Z^I = 0$. We get the projective space $\mathbb{CP}^3$ or $\mathbb{RP}^3$ if Z^I are complex or real-valued. The Z^I are the homogeneous coordinates on the projective twistor space. It follows from (2.3.17) that, the scattering amplitudes are homogeneous functions of degree $-2h_i - 2$ in the twistor coordinates Z_i^I of each particle particle. In the complex case, this means that scattering amplitudes are sections of the complex line bundle $\mathcal{O}(-2h_i - 2)$ over a $\mathbb{CP}^3$ for each particle. For further details on twistor transform, see any standard textbook, e.g. [37].

2.3.3 Scattering Amplitudes in Twistor Space

In an n-gluon scattering process, after the Fourier transform into twistor space, the external gluons are associated with points P_i in the projective twistor space. The scattering amplitudes are functions of the twistors P_i, that is, they are functions defined on the product of n copies of twistor space, one for each particle.

Let us see what happens to the tree level MHV amplitude with $n-2$ gluons of positive helicity and 2 gluons of negative helicity, after Fourier transform into twistor space.We work in $++--$ signature, for which the twistor space is a copy of $\mathbb{RP}^3$. The advantage of $++--$ signature is that the transform to twistor space is an ordinary Fourier transform and the scattering amplitudes are ordinary functions on a product of $\mathbb{RP}^3$'s, one for each particle. With other signatures, the twistor transform involves $\overline{\partial}$-cohomology and other mathematical machinery.

We recall that the MHV amplitude with negative helicity gluons r, s is

$$A(\lambda_i, \widetilde{\lambda}_i) = (2\pi)^4 \delta^4(\sum_i \lambda_i \widetilde{\lambda}_i) f(\lambda_i), \tag{2.3.19}$$

where

$$f(\lambda_i) = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_k \langle \lambda_k, \lambda_{k+1} \rangle}. \tag{2.3.20}$$

The only property of $f(\lambda_i)$, that we need is that it is a function of the holomorphic spinors λ_i only. It does not depend on the anti-holomorphic spinors $\widetilde{\lambda}_i$.

We express the delta function of momentum conservation as an integral

$$(2\pi)^4 \delta^4(\sum_i \lambda_i^a \widetilde{\lambda}_i^{\dot{a}}) = \int d^4 x^{a\dot{a}} \exp\left(i x_{b\dot{b}} \sum_i \lambda_i^b \widetilde{\lambda}_i^{\dot{b}} \right). \tag{2.3.21}$$

Hence, we can rewrite the amplitude as

$$A(\lambda_i, \widetilde{\lambda}_i) = \int d^4 x \exp\left(i x_{b\dot{b}} \sum_i \lambda_i^b \widetilde{\lambda}_i^{\dot{b}} \right) f(\lambda_i). \tag{2.3.22}$$

To transform the amplitude into twistor space, we simply carry out a Fourier transform with respect to all $\widetilde{\lambda}$'s. Hence, the twistor space amplitude is

$$A(\lambda_i, \mu_i) = \int \frac{d^2 \widetilde{\lambda}_1}{(2\pi)^2} \cdots \frac{d^2 \widetilde{\lambda}_n}{(2\pi)^2} \exp\left(i \sum_{j=1}^n \mu_{j\dot{a}} \widetilde{\lambda}_j^{\dot{a}} \right) \int d^4 x \ \exp\left(i x_{b\dot{b}} \sum_j \lambda_j^b \widetilde{\lambda}_j^{\dot{b}} \right) f(\lambda_i). \tag{2.3.23}$$

The only dependence on $\widetilde{\lambda}_i$ is in the exponential factors. Hence the integrals over $\widetilde{\lambda}_j$ can be done trivially, with the result [38]

$$A(\lambda_i, \mu_i) = \int d^4 x \prod_{j=1}^n \delta^2(\mu_{j\dot{a}} + x_{a\dot{a}} \lambda_j^a) f(\lambda_i). \tag{2.3.24}$$

This equation has a simple physical interpretation. Pick some $x^{a\dot{a}}$ and consider the equation

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0. \tag{2.3.25}$$

The solution set for $x = 0$ is a $\mathbb{RP}^1$ or $\mathbb{CP}^1$ depending on whether the variables are real or complex. This is true for any x as the equation lets us solve for $\mu_{\dot{a}}$ in terms of λ^a. So (λ^1, λ^2) are the homogeneous coordinates on the curve.

In real twistor space, which is appropriate for signature $++--$, the curve $\mathbb{RP}^1$ can be described more intuitively as a straight line. Indeed, throwing away the set $Z^1 = 0$, we can describe the rest of $\mathbb{RP}^3$ as a copy of $\mathbb{R}^3$ with the coordinates $x_i = Z_i/Z_1, i = 2, 3, 4$. The equations (2.3.25) determine two planes whose intersection is the straight line in question.

In complex twistor space, the genus zero curve $\mathbb{CP}^1$ is topologically a sphere S^2. The $\mathbb{CP}^1$ is an example of a holomorphic curve in $\mathbb{CP}^3$. The simplest holomorphic curves are defined by vanishing of a pair of homogeneous polynomials in the Z^I

$$\begin{aligned} f(Z^1, \ldots, Z^4) &= 0 \\ g(Z^1, \ldots, Z^4) &= 0. \end{aligned} \tag{2.3.26}$$

If f is homogeneous of degree d_1 and g is homogeneous of degree d_2, the curve has degree $d_1 d_2$. The equations

$$\mu_{\dot{b}} + x_{b\dot{b}}\lambda^b = 0, \quad \dot{b} = 1, 2 \tag{2.3.27}$$

are both linear, $d_1 = d_2 = 1$. Hence the degree of the $\mathbb{CP}^1$ is $d = d_1 d_2 = 1$. Moreover, every degree one genus zero curve in $\mathbb{CP}^3$ is of the form (2.3.27) for some $x^{b\dot{b}}$.

The area of a holomorphic curve of degree d, using the natural metric on $\mathbb{CP}^3$, is $2\pi d$. So the curves we found with $d = 1$ have the minimal area among nontrivial holomorphic curves. They are associated with the minimal nonzero Yang-Mills tree amplitudes, the MHV amplitudes.

Going back to the amplitude (2.3.24), the δ-functions mean that the amplitude vanishes unless $\mu_{j\dot{a}} + x_{a\dot{a}}\lambda_j^a = 0, j = 1, \ldots n$, that is, unless some curve of degree one determined by $x_{a\dot{a}}$ contains all n points (λ_j, μ_j). The result that the MHV amplitudes are supported on a genus zero curves of degree one is equivalent to holomorphy of these amplitudes.

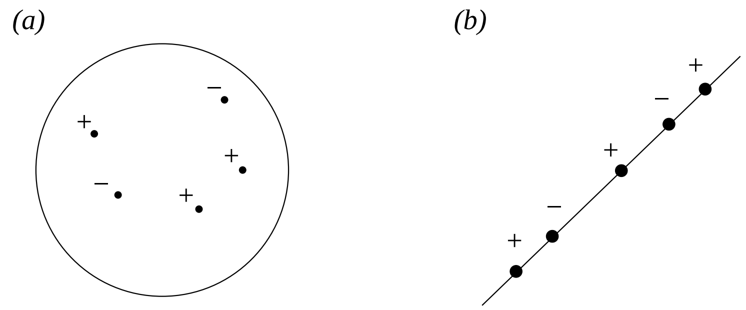

Fig. 2: (a) In complex twistor space CP^3, the amplitude localizes to a $\mathbb{CP}^1$. (b) In the real case, the amplitude is associated to a real line in $\mathbb{R}^3$.

The general conjecture is that an l loop amplitude with p gluons of positive helicity and q gluons of negative helicity is supported on a holomorphic curve in twistor space. The degree of the curve is determined by

$$d = q - 1 + l. \tag{2.3.28}$$

The genus of the curve is bounded by the number of the loops

$$g \leq l. \tag{2.3.29}$$

The MHV amplitude is a special case of this for $q = 2, l = 0$. Indeed the conjecture in this case give that the MHV amplitude is supported in twistor space on a genus zero curve of degree one.

The natural interpretation of this is that the curve is the worldsheet of a string. In some way of describing the perturbative gauge theory, the amplitudes arise from coupling of the gluons to a string. In the next two sections we discuss a proposal for such a string theory due to Witten [11]. There is an alternative version of twistor string theory due to Berkovits [39,40] that seems to give an equivalent description of the scattering amplitudes. Further proposals [41,42] have not yet been used for computing scattering amplitudes.

2.4 Twistor String Theory

In this section, we will describe a string theory that gives a natural framework for understanding the twistor properties of scattering amplitudes discussed in previous section. This is a topological string theory whose target space is a supersymmetric version of the twistor space.

2.4.1 Brief Review of Topological Strings

Firstly, let us consider an $\mathcal{N}=2$ topological field theory in $D=2$ [43]. The $\mathcal{N}=2$ supersymmetry algebra has two supersymmetry generators $Q_i, i=1,2$ that satisfy the anticommutation relations

$$\{Q_{\alpha i}, Q_{\beta j}\} = \delta_{ij}\gamma^{\mu}_{\alpha\beta}P_{\mu}. \tag{2.4.1}$$

In two dimensions, the Lorentz group $SO(1,1)$ is generated by Lorentz boost L. We diagonalize L by going into the light-cone frame $P_{\pm} = P_0 \pm P_1$,

$$\begin{aligned} [L, P_{\pm}] &= \pm P_{\pm} \\ \{L, Q_{\pm}\} &= \pm\frac{1}{2}Q_{\pm}. \end{aligned} \tag{2.4.2}$$

The commutation relations of $\mathcal{N}=2$ supersymmetry algebra become

$$\begin{aligned} \{Q_{+i}, Q_{+j}\} &= \delta_{ij}P_{+} \\ \{Q_{-i}, Q_{-j}\} &= \delta_{ij}P_{-} \\ \{Q_{+i}, Q_{-j}\} &= 0. \end{aligned} \tag{2.4.3}$$

We let

$$Q = Q_{+1} + iQ_{+2} + Q_{-1} \pm iQ_{-2} \tag{2.4.4}$$

with either choice of sign. It follows from (2.4.3) that Q is nilpotent

$$Q^2 = 0, \tag{2.4.5}$$

so we would like to consider Q as a BRST operator.

However Q (2.4.4) is not a scalar so this construction would violate Lorentz invariance. There is a way out if the theory has left and right R-symmetries R_+ and R_-. Under R_+, the combination of supercharges $Q_{+1} \pm iQ_{+2}$ has charge $\pm 1/2$ and $Q_{-1} \pm iQ_{-2}$ is neutral. For R_-, the same is true with 'left' and 'right' interchanged.

Hence, we can make Q scalar if we modify the Lorentz generator L to be

$$L' = L - \frac{1}{2}R_{+} \mp \frac{1}{2}R_{-}. \tag{2.4.6}$$

At a more fundamental level, this change in the Lorentz generator arises if we replace the stress tensor $T_{\mu\nu}$ with

$$\widetilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}(\partial_{\mu}J^{+}_{\nu} + \partial_{\nu}J^{+}_{\mu}) \mp \frac{1}{2}(\partial_{\mu}J^{-}_{\nu} + \partial_{\nu}J^{-}_{\mu}), \tag{2.4.7}$$

where J^+_ν and J^-_μ are the left and right R-symmetry currents. The substitution (2.4.7) is usually referred to as 'twisting' the stress tensor.

We give a new interpretation to the theory by taking Q to be a BRST operator. Hence, a state Ψ is considered to be physical if it is annihilated by Q

$$Q\Psi = 0. \tag{2.4.8}$$

Two states Ψ and Ψ' are equivalent if

$$\Psi - \Psi' = Q\Phi, \tag{2.4.9}$$

for some Φ. Similarly, we take the physical operators to commute with the BRST charge

$$[Q, \mathcal{O}] = 0. \tag{2.4.10}$$

Two operators are equivalent if they differ by an anticommutator of Q,

$$\mathcal{O}' \sim \mathcal{O} + \{Q, \mathcal{V}\}, \tag{2.4.11}$$

for some operator $\mathcal{V}$.

The theory with the stress tensor $\widetilde{T}_{\mu\nu}$ and BRST operator Q is called a topological field theory. The basis for the name is that one can use the supersymmetry algebra to show that the twisted stress tensor is BRST trivial

$$\widetilde{T}_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}. \tag{2.4.12}$$

It follows that in some sense the worldsheet metric is irrelevant. The correlation functions

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\dots\mathcal{O}_n(x_n)\rangle_\Sigma \tag{2.4.13}$$

of physical operators $\mathcal{O}_i$ obeying $[Q, \mathcal{O}_i] = 0$ on a fixed Riemann surface Σ are independent of metric on Σ. Indeed, varying the metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, the correlation function stays the same up to BRST trivial terms

$$\langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\int_\Sigma \delta(\sqrt{g}g^{\mu\nu})\widetilde{T}_{\mu\nu}\rangle = \langle \mathcal{O}_1(x_1)\dots\mathcal{O}_n(x_n)\int_\Sigma \delta(\sqrt{g}g^{\mu\nu})\{Q, \Lambda_{\mu\nu}\}\rangle = 0. \tag{2.4.14}$$

More importantly for us, we can also construct a topological string theory in which one obtains the correlation functions by integrating (2.4.13) over the moduli of the Riemann surface Σ, using $\Lambda_{\mu\nu}$ where the antighost $b_{\mu\nu}$ usually appears in the definition of the string measure.

For an $\mathcal{N} = 2$ supersymmetric field theory in two dimensions with *anomaly-free* left and right R-symmetries we get two topological string theories, depending on the choice of sign in (2.4.4). We would like to consider the case that the $\mathcal{N} = 2$ model is a sigma model with a target space being a complex manifold X. In this case, the two R-symmetries exist classically, so classically we can construct the two topological string theories, called the A-model and the B-model. Quantum mechanically, however, there is an anomaly, and the B-model only exists if X is a Calabi-Yau manifold.

2.4.2 Open String B-model on a Super-Twistor Space

To define open strings in the B-model, one needs BRST invariant boundary conditions. The simplest such conditions are the Neumann boundary conditions [44]. Putting in N space filling D5-branes gives $Gl(n, \mathbb{C})$(whose real form is $U(N)$) gauge symmetry. The physical open string field is a $(0, 1)$ form gauge field $A_{\bar{i}}$. The BRST operator acts as the $\overline{\partial}$ operator and the string $*$ product is just the wedge product. Hence, A is subject to the gauge invariance

$$\delta A = Q\epsilon = \overline{\partial}\epsilon + [A, \epsilon] \tag{2.4.15}$$

and the string field theory action reduces to the action of the holomorphic Chern-Simons theory [44]

$$S = \frac{1}{2}\int \Omega \wedge \mathrm{Tr}\, \left(A \wedge \overline{\partial} A + \frac{2}{3} A \wedge A \wedge A. \right), \tag{2.4.16}$$

where Ω is the Calabi-Yau volume form.

We would like to consider the open string B-model with target space $\mathbb{CP}^3$, but we cannot, since $\mathbb{CP}^3$ is not a Calabi-Yau manifold and the B-model is well defined only on a Calabi-Yau manifold. Otherwise, the R-symmetry that we used to twist the stress tensor is anomalous. A way out is to introduce spacetime supersymmetry. Instead of $\mathbb{CP}^3$, which has homogeneous coordinates $Z^I, I = 1, \ldots, 4$, we consider a supermanifold $\mathbb{CP}^{3|N}$ with bosonic and fermionic coordinates

$$Z^I, \quad \psi^A \qquad I = 1, \ldots, 4, \quad A = 1, \ldots, N, \tag{2.4.17}$$

with identification of two sets of coordinates that differ by a scaling

$$(Z^I, \psi^A) \cong (tZ^I, t\psi^A) \qquad t \in \mathbb{C}^*. \tag{2.4.18}$$

The $\mathbb{CP}^{3|N}$ is a Calabi-Yau supermanifold if and only if the number of fermionic dimensions is $N = 4$. To see this, we construct the holomorphic measure on $\mathbb{CP}^{3|4}$. We start with the $(4|N)$ form on $\mathbb{C}^{4|N}$

$$\Omega_0 = dZ^1 \ldots dZ^4 d\psi^1 \ldots d\psi^N \tag{2.4.19}$$

and study its behavior under the scaling symmetry (2.4.18). For this, recall, that $d\psi$ scales oppositely to ψ

$$(dZ^I, d\psi^A) \to (tdZ^I, t^{-1}d\psi^A). \tag{2.4.20}$$

It follows, that Ω_0 is $\mathbb{C}^*$ invariant if and only if $N = 4$. In this case, we can divide by the $\mathbb{C}^*$ action and get a Calabi-Yau measure on $\mathbb{CP}^{3|4}$

$$\Omega = \frac{1}{4!}\epsilon_{IJKL} Z^I dZ^J dZ^K dZ^L \frac{1}{4!}\epsilon_{ABCD}\psi^A\psi^B\psi^C\psi^D. \tag{2.4.21}$$

The twistor space $\mathbb{CP}^3$ has a natural $Sl(4, \mathbb{C})$ group action. The real form $SU(2, 2)$ of $Sl(4, \mathbb{C})$ is the conformal group of Minkowski space. Similarly, the super-twistor space $\mathbb{CP}^{3|N}$ has a natural $Sl(4|N, \mathbb{C})$ symmetry. The real form $SU(2, 2|N)$ is the superconformal symmetry group with N supersymmetries.

For $N = 4$ the superconformal group $SU(2, 2|4)$ is the symmetry group of $\mathcal{N} = 4$ super-Yang-Mills theory. In a sense, this is the simplest gauge theory in four dimensions. The $\mathcal{N} = 4$ superconformal symmetry uniquely determines the states and interactions of the gauge theory. In particular, the beta function of $\mathcal{N} = 4$ gauge theory vanishes.

Now we know a new reason for $N = 4$ to be special. The topological B-model on $\mathbb{CP}^{3|4}$ exists if and only if $N = 4$. The B-model on $\mathbb{CP}^{3|4}$ has a $SU(2, 2|4)$ symmetry coming from the geometric action of the group on the twistor space. This is related via the twistor transform to the $\mathcal{N} = 4$ superconformal symmetry.

In the topological B-model with space-filling branes on $\mathbb{CP}^{3|4}$, the basic field is the holomorphic gauge field $\mathcal{A} = A_{\overline{I}} dZ^{\overline{I}}$,

$$\begin{aligned}\mathcal{A}(Z, \overline{Z}, \psi) =& A + \psi^A \xi_A(Z, \overline{Z}) + \frac{1}{2!}\psi^A\psi^B\psi_{AB}(Z, \overline{Z}) + \ldots \\ &+ \frac{1}{4!}\epsilon_{ABCD}\psi^A\psi^B\psi^C\psi^D G(Z, \overline{Z}).\end{aligned} \tag{2.4.22}$$

The action is the same as (2.4.16), except that the gauge field $\mathcal{A}$ now depends on ψ

$$S = \frac{1}{2}\int \mathrm{Tr}\ \left(\mathcal{A}\overline{\partial}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)\wedge\Omega, \tag{2.4.23}$$

where the holomorphic three form is (2.4.21). The classical equations of motions obtained from (2.4.23) are

$$\overline{\partial}\mathcal{A} + \mathcal{A}\wedge\mathcal{A} = 0. \tag{2.4.24}$$

Linearizing the equations of motions around the trivial solutions $\mathcal{A} = 0$, they tell us that

$$\overline{\partial}\Phi = 0, \tag{2.4.25}$$

where Φ is any of the components of $\mathcal{A}$. The gauge invariance reduces to $\delta\Phi = \overline{\partial}\alpha$. Hence for each component Φ, the field Φ defines an element of a cohomology group.

This action has the amazing property that its spectrum is the same as that of $\mathcal{N} = 4$ super Yang-Mills theory in Minkowski space. To see this, we need to use that the twistor correspondence relates helicity h free field in Minkowski space to fields in the $(0,1)$ cohomology groups of degree $2h-2$.

To figure out the degrees of various components, notice that the action must be invariant under the $\mathbb{C}^*$ action $Z^I \to tZ^I$. Since the holomorphic measure is also invariant under the scaling, the only way that the action (2.4.23) is invariant, is that the superfield $\mathcal{A}$ is also invariant, in other words, that $\mathcal{A}$ is of degree zero

$$\mathcal{A} \in H^{0,1}(\mathbb{CP}^{3|4}, \mathcal{O}(0)). \tag{2.4.26}$$

Looking back at the expansion (2.4.22) of the superfield, we identify the components, via the twistor correspondence, with fields in Minkowski space of definite helicity. A is is of degree zero, just like the superfield $\mathcal{A}$. Hence, it is related by twistor transform to a field of helicity $+1$. The field G has degree -4 to off-set the degree 4 coming the four ψ, so it corresponds to a field of helicity -1. Continuing in this fashion, we obtain the complete spectrum of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. The twistor fields $A, \xi_A, \phi_{AB}, \widetilde{\xi}_{ABC}, G$ describe, via twistor transform, particles of helicities $1, +\frac{1}{2}, 0, -\frac{1}{2}, -1$ respectively.

The fields also have the correct representations under the $SU(4)$ R-symmetry group. This symmetry is realized in twistor space by the natural geometric action on the fermionic coordinates $\psi^A \to \Lambda^A{}_B\psi^B$. Hence, ψ^A transforms in the **4** of the $SU(4)_R$. The holomorphic gauge superfield $\mathcal{A}(Z,\psi)$ is invariant under the R-symmetry, hence the representations of the components of $\mathcal{A}$ must be conjugate to the representations of the ψ factors that they multiply in eq. (2.4.22). Hence, $A, \xi_A, \phi_{AB}, \widetilde{\xi}_{ABC}$ and G transform in the $\mathbf{1}, \overline{\mathbf{4}}, \mathbf{6}, \mathbf{4}, \mathbf{1}$ representation of $SU(4)_R$ respectively.

2.4.3 D-Instantons

The action (2.4.23) also describes some of the interactions of $\mathcal{N} = 4$ super Yang-Mills, but not all. It cannot describe the full interactions, because an extra $U(1)$ R-symmetry gets in the way. The fermionic coordinates $\psi^A, A = 1, \ldots, 4$ have an extra $U(1)_R$ besides the $SU(4)_R$ symmetry group considered above. Indeed, the full R-symmetry group in twistor space is

$$U(4)_R = SU(4)_R \times U(1)_R, \tag{2.4.27}$$

where we take the extra $U(1)_R$, which we call S, to rotate the fermions by a common phase

$$S:\, Z^I \to Z^I, \psi^A \to e^{i\theta}\psi^A. \tag{2.4.28}$$

However the $\mathcal{N} = 4$ super-Yang-Mills has only an $SU(4)_R$ symmetry. In the B-model, the extra $U(1)_R$ is anomalous, since it does not leave fixed the holomorphic measure $\Omega \sim d^3Z d\psi^1 \ldots d\psi^4$. Under the S transformation, the holomorphic measure transforms as $\Omega \to e^{-4i\theta}\Omega$, so it has charge $S = -4$.

However, as we have set things up so far, the anomaly is too trivial to agree with $\mathcal{N} = 4$ super-Yang-Mills theory. With the normalization (2.4.28), the S charges of fields are given by their degrees. The $\mathcal{N} = 4$ Yang-Mills action is a sum of terms with $S = -4$ and $S = -8$. The action of the open string B-model (2.4.23) has $S = -4$ coming from the anomaly is S from the holomorphic measure. To get the $S = -8$ piece of the Yang-Mills action, we need to enrich the B-model with nonperturbative instanton contributions.

The instantons in question are Euclidean D1-branes wrapped on holomorphic curves in $\mathbb{CP}^{3|4}$ on which open strings can end. The gauge theory amplitudes come from coupling of the open strings to the D1-branes. The massless modes on the worldvolume of a D-instanton are a $U(1)$ gauge field and the modes that describe the motion of the instanton. In the following, we will study tree level amplitudes in the context of string theory. These get contribution from genus zero instantons on which the $U(1)$ line bundles do not have moduli so we will ignore it from now on. The modes describing the motion of the D-instanton simply make up the moduli space $\mathcal{M}$ of holomorphic curves C in the twistor space. To construct scattering amplitudes we need to integrate of $\mathcal{M}$.

2.5 Tree Level Amplitudes from Twistor String Theory

2.5.1 Basic Setup

Recall that the interactions of the full gauge theory come from Euclidean D1-brane instantons on which the open strings can end. The open string are described by the holomorphic gauge field $\mathcal{A}$. Key ingredient in coupling the open strings to the D-instantons is the effective action of the D1-D5 and D5-D1 strings. Quantizing the zero modes of the D1-D5 strings leads to a fermionic zero form field α^i living on the worldvolume of the D-instanton. This transforms in the fundamental representation of the $Gl(n, \mathbb{C})$ gauge group coming from the Chan-Paton factors. The D5-D1 strings are described by a fermion β_i transforming in the anti-fundamental representation. The kinetic operator for the topological strings is the BRST operator Q, which acts as $\overline{\partial}$ on the low energy modes. So the effective action of the D1-D5 strings is

$$S = \int_C \beta(\overline{\partial} + \mathcal{A})\alpha, \tag{2.5.1}$$

where C is the holomorphic curve wrapped by the D-instanton. We read off the vertex operator for an open string with wavefunction $\Psi = \mathcal{A}_{\overline{I}} dZ^{\overline{I}}$

$$V = \int_C J\Psi, \tag{2.5.2}$$

where $J = T_j{}^i \beta_i \alpha^j dz$ is a holomorphic current made from the free fermions α^j, β_i, and $T_j{}^i$ is the group theory factor of the gluon. These currents generate a current algebra on the worldvolume of the D-instanton.

To compute a scattering amplitude, we evaluate the correlation function

$$\mathcal{A} = \int d\mathcal{M}\langle V_1 V_2 \dots V_n\rangle = \int d\mathcal{M} \left\langle \int_C J_1\Psi_1 \dots \int_C J_n\Psi_n \right\rangle. \tag{2.5.3}$$

We can think of this as integrating out the fermions α, β living on the D-instanton. Hence, the generating function for scattering amplitudes is simply the integral of Dirac operator over moduli space

$$\int d\mathcal{M} \det(\overline{\partial} + A). \tag{2.5.4}$$

Here, $d\mathcal{M}$ is the holomorphic measure on the moduli space of holomorphic curves of genus zero and degree d. In topological B-model, the action is holomorphic function of the fields and all path integrals are contour integral. Hence, the integral is actually over a middle-dimensional Lagrangian cycle in the moduli space. This integral is a higher dimensional generalization of the familiar contour integral from complex analysis. To integrate over such a contour, $\mathcal{M}$ must be endowed with a holomorphic measure.

The correlator of the currents on D1-instanton[2]

$$\langle J_1(z_1)J_2(z_2)\dots J_n(z_n)\rangle = \frac{\text{Tr }(T_1T_2\dots T_n)dz_1dz_2\dots dz_n}{(z_1-z_2)(z_2-z_3)\dots(z_n-z_1)} + permutations \tag{2.5.5}$$

follows from the free fermion correlator on a sphere

$$\alpha(z)\beta(z') \sim \frac{1}{z-z'}. \tag{2.5.6}$$

Scattering Wavefunctions

We would like to compute the scattering amplitudes of plane waves $\phi(x) = \exp{(i\,p\cdot x)} = \exp{(i\,\pi^a\widetilde{\pi}^{\dot{a}}x_{a\dot{a}})}$, that are wavefunctions of external particles with definite momentum $p^{a\dot{a}} = \pi^a\widetilde{\pi}^{\dot{a}}$. The twistor wavefunctions corresponding to plane waves are

$$\psi(\lambda,\mu) = \overline{\delta}(\langle\lambda,\pi\rangle)\exp(i[\widetilde{\pi},\mu])g(\psi), \tag{2.5.7}$$

where $g(\psi)$ encodes the dependence on fermionic coordinates. For a positive helicity gluon $g(\psi) = 1$ and for a negative helicity gluon $g(\psi) = \psi^1\psi^2\psi^3\psi^4$. Here, we have introduced the holomorphic delta function

$$\overline{\delta}(f) = \overline{df}\delta^2(f), \tag{2.5.8}$$

[2] Here we write the single trace contribution to the correlation amplitude that reproduces the gauge theory scattering amplitude. As discussed in section 2.5.5, the multitrace contributions correspond to gluon scattering processes with exchange of an internal conformal supergravity state.

which is a closed $(0,1)$ form. We normalize it so that for any function $f(z)$, we have

$$\int dz\, \overline{\delta}(z-a) f(z) = f(a). \tag{2.5.9}$$

The idea of (2.5.7) is that the delta function $\delta(\langle \lambda, \pi \rangle)$ sets λ^a equal to π^a. The Fourier transform of the exponential $\exp(i[\widetilde{\pi}, \mu])$ back into Minkowski space gives another delta function that sets $\widetilde{\lambda}^{\dot{a}}$ equal to $\widetilde{\pi}^{\dot{a}}$. The twistor string computation with these wavefunctions gives directly momentum space scattering amplitudes.

Actually, the wavefunctions should be modified slightly so that they are invariant under the scaling of the homogeneous coordinates of $\mathbb{CP}^{3|4}$. From the basic properties of delta functions, it follows that $\overline{\delta}(\langle \lambda, \pi \rangle)$ is homogeneous of degree -1 in both λ and π. Hence, for positive helicity gluons, the wavefunction is actually

$$\psi^{+}(\lambda, \mu) = \overline{\delta}(\langle \lambda, \pi \rangle)(\lambda/\pi) \exp\big(i[\widetilde{\pi}, \mu](\pi/\lambda)\big). \tag{2.5.10}$$

Here, λ/π is a well defined holomorphic function, since λ is a multiple of π on the support of the delta function. The power of (λ/π) was chosen, so that the wavefunction is homogeneous of degree zero in overall scaling of λ, μ, ψ. Under the scaling

$$(\pi, \widetilde{\pi}) \to (t\pi, t^{-1}\widetilde{\pi}), \tag{2.5.11}$$

the wavefunction is homogeneous of degree -2 as expected for a positive helicity gluon (2.2.28). For negative helicity gluon, the wavefunction is

$$\psi^{-}(\lambda, \mu) = \overline{\delta}(\langle \lambda, \mu \rangle)(\pi/\lambda)^3 \exp\big(i[\widetilde{\pi}, \mu](\pi/\lambda)\big)\psi^1\psi^2\psi^3\psi^4. \tag{2.5.12}$$

Under the scaling (2.5.11), the wavefunction is homogeneous of degree $+2$ as expected. For wavefunctions of particles with helicity h, there are similar formulas with $2-2h$ factors of ψ.

MHV Amplitudes

We saw that the MHV amplitude, after Fourier transform into twistor space, localizes on a genus zero degree one curve, that is, a linearly embedded copy of $\mathbb{CP}^1$. Here we will evaluate the degree one instanton contribution and confirm that it gives the MHV amplitude.

Consider the moduli space of such curves. Each curve can be described by the equations

$$\mu^{\dot{a}} = x^{a\dot{a}}\lambda_a \qquad \psi^A = \theta^{Aa}\lambda_a, \tag{2.5.13}$$

where we parameterize λ^a are the homogeneous coordinates of the $\mathbb{CP}^1$ and $x^{a\dot{a}}$ and θ^{Aa} are the moduli of C. The holomorphic measure on the moduli space is

$$d\mathcal{M} = d^4x d^8\theta. \tag{2.5.14}$$

Hence, the moduli space has 4 bosonic and 8 fermionic dimensions. In terms of the homogeneous coordinate λ^a the current correlator (2.5.5) becomes

$$\langle J_1(\pi_1)J_2(\pi_2)\dots J_n(\pi_n)\rangle = \frac{\prod_i \langle \lambda_i, d\lambda_i\rangle}{\langle \lambda_1, \lambda_2\rangle\langle \lambda_2, \lambda_3\rangle \dots \langle \lambda_n, \lambda_1\rangle}, \tag{2.5.15}$$

which we found by setting $z_i = \lambda_i^2/\lambda_i^1$. We stripped away the color factors and kept only the contribution to the term with $1, 2, \dots, n$ cyclic order. We multiply this with the wavefunctions $\psi_i(\lambda, \mu) = \overline{\delta}(\langle \lambda, \pi_i\rangle) \exp\left(i[\mu, \widetilde{\pi}_i]\right) g_i(\psi_i)$ and integrate over the positions $\lambda_i, \widetilde{\lambda}_i$ over the vertex operators. We perform the integral over the positions of the vertex operators using the formula

$$\int_{\mathbb{CP}^1} \langle \lambda, d\lambda\rangle \ \overline{\delta}(\langle \lambda, \pi\rangle) f(\lambda) = f(\pi), \tag{2.5.16}$$

where $f(\lambda)$ is a homogeneous function of λ^a of degree -1. This is the homogeneous version of definition of holomorphic delta function

$$\int_{\mathbb{C}} dz \ \overline{\delta}(z - b) f(z) = f(b). \tag{2.5.17}$$

Hence, each wavefunction contributes a factor of

$$\int_C \langle \lambda, d\lambda\rangle \psi_i = \exp\left(i[\widetilde{\pi}_i, \mu_i]\right) g_i(\psi_i), \tag{2.5.18}$$

where $\mu_i^{\dot{a}} = x^{a\dot{a}}\lambda_a, \psi_i^A = \theta^{Aa}\lambda_a$. The delta function sets $\lambda_i^a = \pi_i^a$ in the correlation function, so the amplitude becomes

$$\mathcal{A} = \frac{1}{\prod_k \langle \pi_k, \pi_{k+1}\rangle} \int d^4x d^8\theta \ \exp\left(i \sum_k [\widetilde{\pi}_k, \mu_k]\right) \prod_k g_k(\psi_k). \tag{2.5.19}$$

The fermionic part of the wavefunctions is $g_i = 1$ for the positive helicity gluons and $g_i = \psi_i^1\psi_i^2\psi_i^3\psi_i^4$ for the negative helicity gluons. Since we are integrating over eight fermionic moduli $d^8\theta$, we get nonzero contribution to amplitudes with exactly two negative helicities r^-, s^-. Setting $\psi_k^A = \theta^{Aa}\pi_{ka}$, the integral over fermionic dimensions of the moduli space gives the numerator of the MHV amplitude

$$\int d^8\theta \prod_{A=1}^{4} \psi_r^A \prod_{B=1}^{4} \psi_s^B = \langle r, s\rangle^4. \tag{2.5.20}$$

Setting $\mu_k^{\dot{a}} = x^{a\dot{a}}\pi_{ka}$, the integral over bosonic moduli gives the delta function of momentum conservation

$$\int d^4x \exp\left(ix_{a\dot{a}} \sum_i \pi_i^a \tilde{\pi}_i^{\dot{a}}\right) = \delta^4(\sum_{i=1}^{n} \pi_i^a \tilde{\pi}_i^{\dot{a}}). \tag{2.5.21}$$

Collecting the various pieces, we get the familiar MHV amplitude

$$\mathcal{A}(r^-, s^-) = \frac{\langle r, s\rangle^4}{\prod_{i=1}^{n}\langle i, i+1\rangle} \delta^4(\sum_{i=1}^{n} \pi_i^a \tilde{\pi}_i^{\dot{a}}). \tag{2.5.22}$$

2.5.2 Higher Degree Instantons

Instanton Measure

Here we will construct the measure on the moduli space of genus zero degree d curves. Such curves can be described as degree d maps from an abstract $\mathbb{CP}^1$ with homogeneous coordinates (u, v)

$$\begin{aligned} Z^I &= P^I(u, v) \\ \psi^A &= \chi^A(u, v). \end{aligned} \tag{2.5.23}$$

Here P^I, χ^A are homogeneous polynomials of degree d in u, v. The space of polynomials of degree d is a linear space of dimension $d+1$, spanned by $u^d, u^{d-1}v, \ldots, v^d$. Picking a basis $b^\alpha(u, v), \alpha = 1, \ldots, d+1$, we write

$$\begin{aligned} P^I &= \sum_\alpha P^I_\alpha \, b^\alpha \\ \psi^A &= \sum_\alpha \chi^A_\alpha \, b^\alpha. \end{aligned} \tag{2.5.24}$$

A natural measure is

$$d\mathcal{M}_0 = \prod_{\alpha=1}^{d+1} \prod_{I,A=1}^{4} dP^I_\alpha \, d\chi^A_\alpha. \tag{2.5.25}$$

This measure is invariant under a general $Gl(d+1,\mathbb{C})$ transformation of the basis b_α. Since the number of bosonic and fermionic coordinates is the same, the Jacobians cancel between fermionic and bosonic parts of the measure. The description (2.5.23) is redundant. We need to divide by the $\mathbb{C}^*$ action that rescales P^I and ξ^A by a common factor. This reduces the space of curves from $\mathbb{C}^{4d+4|4d+4}$ down to $\mathbb{CP}^{4d+3|4d+4}$. The curve C also stays invariant under an $Sl(2,\mathbb{C})$ transformation on (u,v) so the actual moduli space of genus zero degree d curves is

$$\mathcal{M} = \mathbb{CP}^{4d+3|4d+4}/Sl(2,\mathbb{C}). \tag{2.5.26}$$

As $d\mathcal{M}_0$ is $Gl(2,\mathbb{C})$ invariant, it descends to a holomorphic measure $d\mathcal{M}$ on $\mathcal{M}$. Hence, $\mathcal{M}$ is a Calabi-Yau supermanifold of dimension $(4d|4d+4)$.

We can now understand why amplitudes with different helicities come from holomorphic curves of different degrees. Integrating over the moduli space, the measure absorbs $4d+4$ fermion zero modes. These come from the fermionic factors $g(\psi)$ in the wavefunctions of the gluons (2.5.7). A positive helicity gluon does not contribute any zero modes while a negative helicity gluon with $g^-(\psi) = \psi^1\psi^2\psi^3\psi^4$ gives 4 zero modes. Hence, instantons of degree d contribute to amplitudes with $d+1$ negative helicity gluons.

Alternatively, we can get this from counting the S charge anomaly. Wavefunctions of particles with different helicities violate S by different amount. The positive helicity gluons do not violate S while the negative helicity gluons violate S by -4 units. So, the amplitude with p positive helicity gluons and q negative helicity gluons violates the S charge by $-4q$ units.

In the twistor string, there is a new source of violation of S from the instanton measure. Since the S charge of Z and ψ is 0 and 1 respectively, the charges of the coefficients $P^I_\alpha, \chi^A_\alpha$ are $0, 1$. Hence, the differentials $dP^I_\alpha, d\chi^A_\alpha$ have charges $0, -1$ and the S charge of the $(4d|4d+4)$ dimensional measure $d\mathcal{M}$ is $-4d-4$.

So an instanton can contribute to an amplitude with q negative helicity gluons if and only if

$$d = q - 1. \tag{2.5.27}$$

This is the familiar formula discussed at the end of section 3. For l loop amplitudes, this relation generalizes to $d = q - 1 + l$.

Evaluating the Instanton Contribution

Here we consider the connected instanton contribution along the lines of the MHV calculation. The amplitude is [12,45,46]

$$\mathcal{A} = \int d\mathcal{M}_d \prod_i \int_C \frac{\langle u_i, du_i \rangle}{\prod_k \langle u_k, u_{k+1} \rangle} \overline{\delta}(\langle P(u_i), \pi_i \rangle) \exp\left(i[P(u_i), \widetilde{\pi}_i]\right) g_i(\psi_i). \quad (2.5.28)$$

This is not really an integral. The integral over the $2d+2$ parameters $P^{\dot{a}}_{\alpha}$, $\dot{a} = 1, 2, \alpha = 1, \ldots, d+1$, gives $2d+2$ delta functions because $P^{\dot{a}}$ appears only in the exponential $\exp\left(\sum_i P(u_i)_{\dot{a}} \pi_i^{\dot{a}}\right)$. Hence, we are left with an integral over $4d - (2d+2) + 2n = 2d + 2n - 2$ bosonic variables. Here the $2n$ integrals come from the integration over the positions of the vertex operators. Now there are $2n$ delta functions from the wavefunctions since each holomorphic delta function is really a product of two real delta functions $\overline{\delta}(z) = d\overline{z}\ \delta^2(z)$, and $2d+2$ delta functions from the integral over the exponentials, which gives a total of $2d + 2n + 2$. There are four more delta functions than integration variables. The four extra delta functions impose momentum conservation. Hence, the delta functions localize the integral to a sum of contributions from a finite number of points on the moduli space.

Parity Invariance

In the helicity formalism, the parity symmetry of Yang-Mills scattering amplitudes is apparent. The parity changes the signs of the helicities of the gluons. The parity conjugate amplitude can be obtained by simply exchanging λ_i's with $\widetilde{\lambda}_i$'s.

To go to twistor space, one Fourier transforms with respect to $\widetilde{\lambda}_i$, which breaks the symmetry between λ and $\widetilde{\lambda}$. Indeed, the result (2.5.28) for the scattering amplitude treats λ and $\widetilde{\lambda}$ asymmetrically. An amplitude with p positive helicities and q negative helicities has contribution from instantons of degree $q-1$, while the parity conjugate amplitude has contribution from instantons of degree $p-1$. To show that these two are related by the exchange of λ_i and $\widetilde{\lambda}_i$ requires some amount of work. We refer the interested reader to the original literature [12,45,46,40].

Localization on the Moduli Space

Recall that a tree level amplitude with q negative helicity gluons and arbitrary number of positive helicity gluons receives contribution from instantons wrapping holomorphic curves of degree $d = q-1$. The degree d instanton can consists of several disjoint lower degree instantons whose degrees add up to d. For connected scattering amplitudes the instantons are connected by open strings.

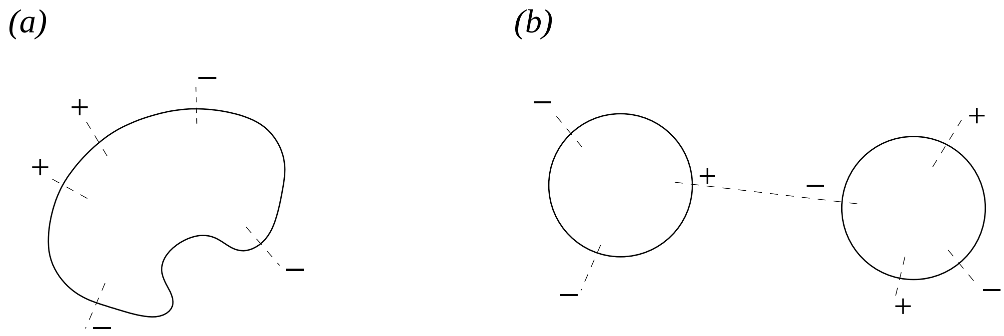

Fig. 3: An amplitude with tree negative helicity gluons has contribution from two configurations: (a) Connected $d = 2$ instanton. (b) Two disjoint $d = 1$ instantons. The dashed line represents an open string connecting the instantons.

A priory, one expects that the amplitude receives contributions from all possible instanton configurations with total degree $q - 1$. So for example an amplitude with three negative helicity gluons has contribution from a connected $d = 2$ instanton and a contribution from two disjoint $d = 1$ instantons, fig. 3.

What one actually finds is that the connected and disconnected instanton contributions reproduce the whole amplitude *separately*. For example, in the case of amplitude with three negative helicity gluons, it seems that there are two different ways to compute the same amplitude. One can either evaluate it from the connected $d = 2$ instantons, fig. 3 (a), [12,45] or alternatively, from two disjoint $d = 1$ instantons, fig. 3 (b), [1].

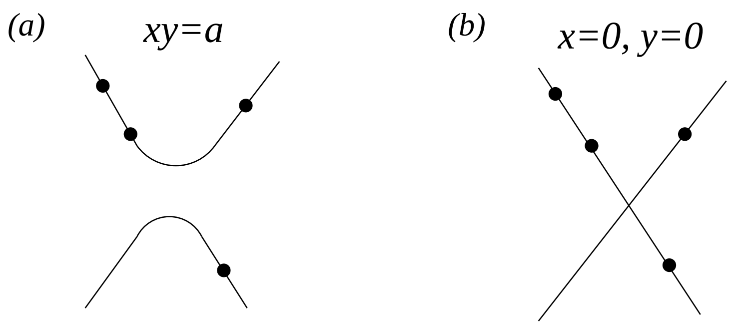

Fig. 4: Localization of the connected instanton contribution to next to MHV amplitude; (a) the integral over the moduli space of connected degree two curves, localizes to an integral over the degenerate curves of (b), that is intersecting complex lines. In the figure, we draw the real section of the curves.

We can explain the equality of various instanton contributions roughly as follows [31]. Consider the connected contribution. The amplitude is expressed as a 'contour' integral over a middle-dimensional Lagrangian cycle in the moduli space of degree two curves . The integrand comes from the correlation function on the worldvolume of the D-instanton and from the measure on the moduli space. It has poles in the region of the moduli space, where the instanton degenerates to two intersecting instantons of lower degrees $d_1 + d_2 = d$, fig. 4. Picking a contour that encircles the pole, the integral localizes to an integral over the moduli space $\mathcal{M}'$ of the intersecting lower degree curves. Similarly, the disconnected contribution has a pole when the two ends of the propagator coincide. This comes from the pole of the open string propagator

$$\overline{\partial} G = \overline{\delta}^3 (Z'^I - Z^I) \delta^4 (\psi'^A - \psi^A). \tag{2.5.29}$$

Hence, the integral over disjoint instantons also localizes on the moduli space of intersecting instantons. It can be shown that the localized integrals coming from either connected or disconnected instanton configurations agree [31] which explains why the separate calculations give the entire scattering amplitude.

Towards MHV Diagrams

Starting with a higher degree instanton contribution, successive localization reduces the integral to the moduli space of intersecting degree one curves. As we will review below, this integral can be evaluated leading to a combinatorial prescription for the scattering amplitudes [1]. Indeed, degree one instantons give MHV amplitudes, so the localization of the moduli integral leads to a diagrammatic construction based on a suitable generalization of the MHV amplitudes.

2.5.3 MHV Diagrams

In this subsection, we start with a motivation of the MHV diagrams construction of Yang-Mills amplitudes from basic properties of twistor correspondence. We then go on to discuss simple examples and extensions to loop amplitudes. In the next subsection, we give a heuristic derivation of the MHV rules from twistor string theory.

Recall that MHV scattering amplitudes are supported on $\mathbb{CP}^1$'s in twistor space

$$\mu_{\dot{a}} + x_{a\dot{a}}\lambda^a = 0. \qquad (2.5.30)$$

Each such $\mathbb{CP}^1$ can be associated to a point $x^{a\dot{a}}$ in Minkowski space[3]. So, in a sense, we can think of MHV amplitudes as *local interaction vertices* [1]. To take this analogy further, we can try to build more complicated amplitudes from Feynman diagrams with vertices that are suitable off-shell continuations of the MHV amplitudes. MHV amplitudes are functions of holomorphic spinors λ_i only. Hence, to use them as vertices in Feynman diagrams, we need to define λ for internal off-shell momenta $p^2 \neq 0$.

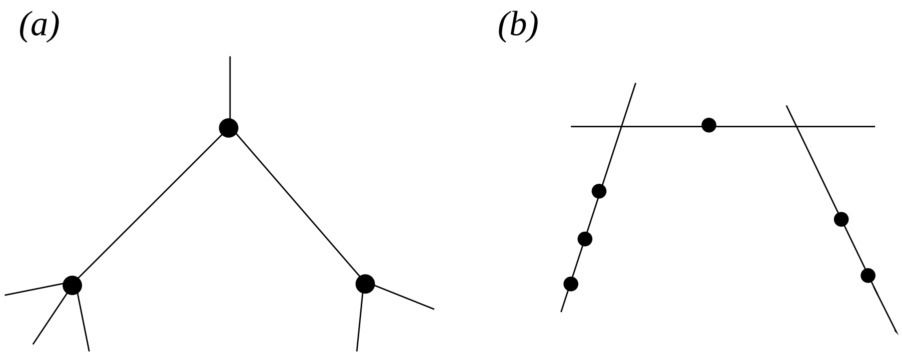

Fig. 5: Two representation of a degree three MHV diagram. (a) In Minkowski space, we the MHV vertices are represented by (b) In twistor space, each MHV vertex corresponds to a line. The three lines pairwise intersect.

To motivate the off-shell continuation, notice that for on-shell momentum $p^{a\dot{a}} = \lambda^a\widetilde{\lambda}^{\dot{a}}$, we can extract the holomorphic spinors λ from the momentum by picking arbitrary anti-holomorphic spinor $\eta^{\dot{a}}$ and contracting it with $p^{a\dot{a}}$. This gives λ^a up to a scalar factor

$$\lambda^a = \frac{p^{a\dot{a}}\eta_{\dot{a}}}{[\widetilde{\lambda}, \eta]}. \qquad (2.5.31)$$

[3] We are being slightly imprecise here. The space of $\mathbb{CP}^1$'s is actually a copy of the complexified Minkowski space $\mathbb{C}^4$. The Minkowski space $\mathbb{R}^{3|1}$ corresponds to $\mathbb{CP}^1$'s that lie entirely in the 'null twistor space', defined by vanishing of the pseudo-hermitian norm $Q(\lambda, \mu) = i(\lambda^a\overline{\mu}_a - \overline{\lambda}^{\dot{a}}\mu_{\dot{a}})$. Indeed, for a $\mathbb{CP}^1$ corresponding to a point in Minkowski space, $x^{a\dot{a}}$ is a hermitian matrix, hence it follows from (2.5.30) that Q vanishes.

For off-shell momenta, this strategy almost works except for the factor $[\widetilde{\lambda}, \eta]$ in the denominator which depends on the undefined spinor $\widetilde{\lambda}$. Fortunately, $[\widetilde{\lambda}, \eta]$ scales out of the Feynman diagrams, so we take as our definition

$$\lambda^a = p^{a\dot{a}}\eta_{\dot{a}}. \tag{2.5.32}$$

This is clearly well-defined for off-shell momentum. We complete the definition of the MHV rules, by taking the simple $1/k^2$ for the propagator connecting the MHV vertices.

Consider an MHV diagram with v vertices. Each vertex gives two negative helicity gluons. To make a connected tree level graph, the vertices are connected with $v-1$ propagators. The propagators absorb $v-1$ negative helicities, leaving $v+1$ negative helicity external gluons. Hence, to find all MHV graphs contributing to a given amplitude, draw all possible tree graphs of v vertices and $v-1$ links, assigning opposite helicities to the two ends of internal lines. The external gluon are distributed among the vertices while preserving cyclic ordering. MHV graphs are those for which each vertex has two negative helicity gluons emanating from it.

Examples

Here we discuss concrete amplitudes to illustrate the MHV diagram construction. Consider first the $+---$ gluon amplitude. This amplitude vanishes in Yang-Mills theory. It has contribution from two diagrams.

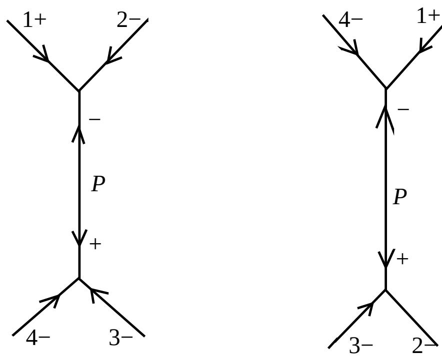

Fig. 6: MHV diagrams contributing to the $+---$ amplitude, which is expected to vanish.

The first of the two diagrams gives

$$\frac{\langle 2,\lambda\rangle^4}{\langle 1,2\rangle\langle 2,\lambda\rangle\langle\lambda,1\rangle}\frac{1}{p^2}\frac{\langle 3,4\rangle^4}{\langle 3,4\rangle\langle 4,\lambda\rangle\langle\lambda,3\rangle}, \tag{2.5.33}$$

where we associate to the internal momentum $p = p_1 + p_2 = -p_3 - p_4$ the holomorphic spinor

$$\lambda^a = (p_1 + p_2)^{a\dot{a}}\eta_{\dot{a}}. \tag{2.5.34}$$

The second diagram can be obtained from the first by exchanging particles 2 and 4

$$\frac{\langle\lambda',4\rangle^4}{\langle 1,\lambda'\rangle\langle\lambda',4\rangle\langle 4,1\rangle}\frac{1}{p'^2}\frac{\langle 2,3\rangle^4}{\langle 2,3\rangle\langle 3,\lambda'\rangle\langle\lambda',2\rangle}, \tag{2.5.35}$$

where $\lambda'^a = (p_1 + p_4)^{a\dot{a}}\eta_{\dot{a}}$. Denoting $\phi_i = \lambda_i^{\dot{a}}\eta_{\dot{a}}$, the first and second diagrams give respectively

$$-\frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 34\rangle}{[21]} - \frac{\phi_1^3}{\phi_2\phi_3\phi_4}\frac{\langle 32\rangle}{[41]}. \tag{2.5.36}$$

The sum of these contributions vanishes, because momentum conservation implies $\langle 32\rangle[21] + \langle 34\rangle[41] = \sum_i \langle 3i\rangle[i1] = 0$.

It is easy to compute more complicated amplitudes. For example, the n gluon $---++\ldots++$ amplitude is a sum of $2n-3$ MHV diagrams, which can be evaluated to give

$$\begin{aligned} A = &\sum_{i=3}^{n-1}\frac{\langle 1\lambda_{2,i}\rangle^3}{\langle\lambda_{2,i}i+1\rangle\langle i+1i+2\rangle\ldots\langle n1\rangle}\frac{1}{q_{2i}^2}\frac{\langle 23\rangle^3}{\langle\lambda_{2,i}2\rangle\langle 34\rangle\ldots\langle i\lambda_{2,i}\rangle} \\ &+\sum_{i=4}^{n}\frac{\langle 12\rangle^3}{\langle 2\lambda_{3,i}\rangle\langle\lambda_{3,i}i+1\rangle\ldots\langle n1\rangle}\frac{1}{q_{3i}^2}\frac{\langle\lambda_{3,i}3\rangle^3}{\langle 3,4\rangle\ldots\langle i-1i\rangle\langle i\lambda_{3,i}\rangle}, \end{aligned} \tag{2.5.37}$$

Loop Amplitudes

Similarly, one can compute loop amplitudes using MHV diagrams. This has been carried out for the one loop MHV amplitude in $\mathcal{N} = 4$ [32] and $\mathcal{N} = 1$ [47] Yang-Mills theory, in agreement with the known answers.

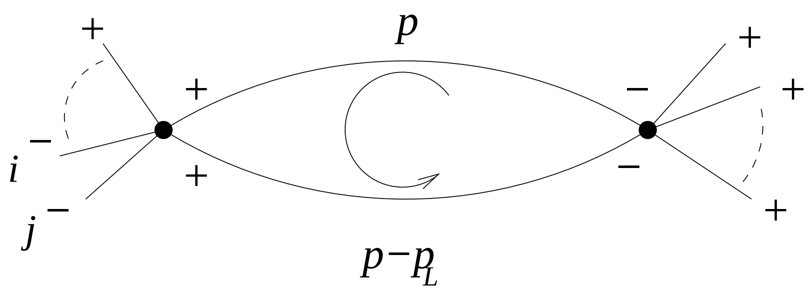

Fig. 7: Schematic representation of MHV diagram computation of one-loop MHV amplitude. The picture shows a diagram in which the negative helicity gluons i^-, j^- are on the same MHV vertex.

The expression for an MHV diagram contributing to the one-loop MHV amplitude is just what one would expect for a one-loop Feynman diagram with MHV vertices, fig. 7. There are two MHV vertices, each coming with two negative helicity gluons. The vertices are connected with two Feynman propagators that absorb two negative helicities, leaving two negative helicity external gluons

$$\mathcal{A}^{loop} = \sum_{\mathcal{D},h} \int \frac{d^4p}{(2\pi)^4} \mathcal{A}_L(\lambda_k, \lambda_p, \lambda_{p-p_L}) \frac{1}{p^2(p-p_L)^2} \mathcal{A}_R(\lambda_k, \lambda_p, \lambda_{p-p_L}). \quad (2.5.38)$$

The off-shell spinors entering the MHV amplitudes $\mathcal{A}_L, \mathcal{A}_R$ are determined in terms of the momenta of the internal lines

$$\lambda_p^a = p^{a\dot{a}}\eta_{\dot{a}}, \qquad \lambda_{p-p_L}^a = (p-p_L)^{a\dot{a}}\eta_{\dot{a}}, \quad (2.5.39)$$

which is the same prescription as for level MHV diagrams. The sum in (2.5.38) is over partitions $\mathcal{D}$ of the gluons among the two MHV diagrams that preserve the cyclic order and over states of the internal particles.

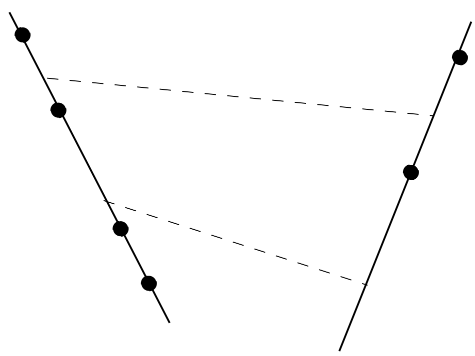

Fig. 8: Schematic representation of a hypothetical twistor string computation of one-loop MHV amplitude. The picture shows a diagram in which the negative helicity gluons i^-, j^- are on the same MHV vertex.

This calculation makes the twistor structure of one-loop MHV amplitudes manifest. The two MHV vertices are supported on lines in twistor space, so the amplitude is a sum of contributions, each of which is supported on a disjoint union two lines. In a hypothetical twistor string theory computation of the amplitude, these two lines are connected by open string propagators, fig. 8.

The $\mathcal{N} = 0$ amplitude is a sum of cut-constructible terms and of rational terms. The cut-constructible terms are correctly reproduced from MHV diagrams [48]. The rational terms are single valued functions of the spinors, hence they are free of cuts in four dimensions. Their twistor structure suggests that they receive contribution from diagrams in which, alongside with MHV vertices, there are new one-loop vertices coming from one-loop all-plus helicity amplitudes [2]. However, a suitable off-shell continuation of the one-loop all-plus amplitude has not been found yet. There has been recent progress in computing the rational part of some one-loop QCD amplitudes using a generalization [49] of the tree level recursion relations [7].

2.5.4 Heuristic Derivation from Twistor String Theory

Here, we will make an analysis of the disconnected twistor diagrams that contribute to tree level amplitudes. Interpreting the vertices in fig fig. 5 (a) as degree one instantons and the lines as twistor propagators, we will evaluate the twistor string amplitude corresponding to this twistor contribution and show how it leads to the MHV diagrammatic rules of the last subsection.

The physical field of the open string B-model is a $(0,1)$-form $\mathcal{A}$ with kinetic operator $\overline{\partial}$ coming from the Chern-Simons action (2.4.16). The twistor propagator for $\mathcal{A}$ a $(0,2)$-form on $\mathbb{CP}^3 \times \mathbb{CP}^3$ that is a $(0,1)$-form on each copy of $\mathbb{CP}^3$. The propagator obeys the equation

$$\overline{\partial} G = \overline{\delta}^3(Z_2^I - Z_1^I)\delta^4(\psi_2^A - \psi_1^A). \qquad (2.5.40)$$

Here, $\overline{\delta}(z) = d\overline{z}\delta(z)\delta(\overline{z})$ is holomorphic delta function $(0,1)$-form. In an axial gauge, the twistor propagator becomes

$$G = \overline{\delta}(\lambda_2^2 - \lambda_1^2)\overline{\delta}(\mu_2^{\dot{1}} - \mu_1^{\dot{1}})\frac{1}{\mu_2^{\dot{2}} - \mu_1^{\dot{2}}}\prod_{A=1}^{4}(\psi_2^A - \psi_1^A), \qquad (2.5.41)$$

where we set $\lambda_1^1 = \lambda_2^1 = 1$.

For simplicity, we evaluate the contribution from two degree one instantons C_1 and C_2 connected by twistor propagator, which is contributing to amplitudes with three negative helicity. The instantons $C_i, i = 1, 2$ are described by the equations

$$\mu_k^{\dot a} = x_i^{a\dot a}\lambda_{ka}, \qquad \psi_k^A = \theta_i^{Aa}\lambda_{ka} \qquad i = 1,2\ , k = 1, \ldots, n. \tag{2.5.42}$$

Here, $x_i^{a\dot a}$ and θ_i^{Aa} are the bosonic and fermionic moduli of C_i.

With our choice of gauge, the twistor propagator is supported on points such that $\lambda_1^a = \lambda_2^a$. Since $\mu_2^{\dot a} - \mu_1^{\dot a} = y^{a\dot a}\lambda_a$, where $y^{a\dot a} = x_2^{a\dot a} - x_1^{a\dot a}$, the condition $\mu_2^{\dot 1} - \mu_1^{\dot 1} = 0$ implies $\lambda^a = y^{a\dot 1}$. Hence, the bosonic part of the propagator becomes $1/y^2$.

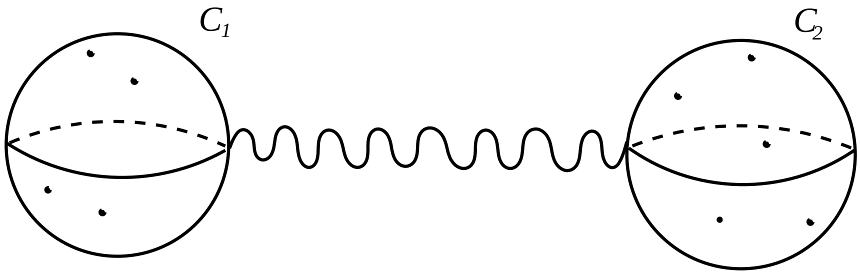

Fig. 9: Twistor string contribution to an amplitude with three negative helicity external gluons. Two disconnected degree one instantons are connected by an open string.

The correlators of the gluon vertex operators on C_1 and C_2 and the integral over θ_i^{Aa} give two MHV amplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ as explained in the $d = 1$ computation. So we are left with the integral

$$\int d^4x_1 d^4x_2 \mathcal{A}_L \frac{1}{(x_2 - x_1)^2}\mathcal{A}_R \prod_{i\in L}\exp(ix_1\cdot p_i)\prod_{j\in R}\exp(ix_2\cdot p_j), \tag{2.5.43}$$

where the integral is over a suitably chosen 4×4 real dimensional 'contour' in the moduli space $\mathbb{C}^4\times\mathbb{C}^4$ of two degree one curves. We rewrite the exponentials as

$$\exp(iy\cdot P)\prod_{j\in L,R}\exp(ix\cdot p_j), \tag{2.5.44}$$

where $x \equiv x_1$ and $P = \sum_{i\in R} p_i$ is momentum of the off-shell line connecting the two vertices. The integral

$$\int d^4x \prod_{i\in L,R}\exp(ix\cdot p_i) = (2\pi)^4\delta^4(\sum_i p_i) \tag{2.5.45}$$

gives the delta function of momentum conservation. We are left with

$$A = \int d^4 y \frac{1}{y^2} \exp(iy \cdot P) \mathcal{A}_L \mathcal{A}_R. \tag{2.5.46}$$

The integrand has a pole at $y^2 = 0$, which is the condition for the curves C_1 and C_2 to intersect. The space y^2 is the familiar conifold. It is a cone over $\mathbb{CP}^1 \times \mathbb{CP}^1$ so we parameterize it as

$$y^{a\dot{a}} = t\lambda^a \widetilde{\lambda}^{\dot{a}}. \tag{2.5.47}$$

Here $\lambda^a \in \mathcal{O}(1,0)$, $\widetilde{\lambda} \in \mathcal{O}(0,1)$, so $t \in \mathcal{O}(-1,-1)$ to make (2.5.47) well-defined. We choose a contour that picks the residue at $y^2 = 0$. The residue is the volume form on the conifold

$$\text{Res}\, \frac{d^4 y}{y^2} = t dt \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}]. \tag{2.5.48}$$

Taking the residue, the integral becomes

$$I = \int t dt \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}] \exp(itP_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}}) \mathcal{A}_L \mathcal{A}_R, \tag{2.5.49}$$

where the MHV vertices depend on the holomorphic spinor λ only. We pick the contour $t \in (-\infty, \infty)$, $\widetilde{\lambda} = \overline{\lambda}$, which is the Minkowski space light-cone. For $t \in (0, \pm\infty)$ we regulate the integral with the prescription $P = (p^0 \pm i\epsilon, \vec{p})$, so

$$\int_{-\infty}^{\infty} t dt \exp(itP_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}}) = -\frac{2}{(P_{a\dot{a}} \lambda^a \widetilde{\lambda}^{\dot{a}})^2}. \tag{2.5.50}$$

Hence we have

$$I = \int \langle \lambda, d\lambda \rangle [\widetilde{\lambda}, d\widetilde{\lambda}] \frac{1}{(P\lambda\widetilde{\lambda})^2} \mathcal{A}_L \mathcal{A}_R(\lambda). \tag{2.5.51}$$

To reduce the integral (2.5.51) to a sum over MHV diagrams, we use the identity

$$\frac{[\widetilde{\lambda}, d\widetilde{\lambda}]}{(P\lambda\widetilde{\lambda})^2} = -\frac{1}{P\lambda\eta} \overline{\partial} \left(\frac{[\widetilde{\lambda}, \eta]}{P\lambda\widetilde{\lambda}} \right), \tag{2.5.52}$$

where $\eta^{\dot{a}}$ is an arbitrary positive helicity spinor, to write the integral as

$$I = \int \langle \lambda, d\lambda \rangle \frac{\mathcal{A}_L \mathcal{A}_R}{(P\lambda\eta)} \overline{\partial} \left(\frac{[\widetilde{\lambda}, \eta]}{(P\lambda\widetilde{\lambda})} \right). \tag{2.5.53}$$

Now we can integrate by parts. The $\overline{\partial}$ operator acting on the holomorphic function on the left gives zero except for contributions coming from poles of the holomorphic function, $\overline{\partial}\,(1/z) = \overline{\delta}(z)$. These evaluate to a sum over residues

$$I = \sum \mathrm{Res}\left(\frac{\mathcal{A}_L \mathcal{A}_R}{P\lambda\eta}\right) \frac{[\widetilde{\lambda}, \eta]}{P\lambda\widetilde{\lambda}}. \tag{2.5.54}$$

The residues of $1/(P\lambda\eta)$ are at

$$\lambda^a = P^{a\dot{a}} \eta_{\dot{a}}. \tag{2.5.55}$$

Substituting this back into (2.5.54), $P\lambda\widetilde{\lambda}$ evaluates to $P^2[\widetilde{\lambda}, \eta]$, so we have

$$I = \frac{1}{P^2} \mathcal{A}_L \mathcal{A}_R(\lambda = P\eta). \tag{2.5.56}$$

But this is simply the contribution from an MHV diagram. Summing over all cyclicly ordered partitions of the gluons among the two instantons gives the sum over MHV diagrams contributing to the scattering amplitude.

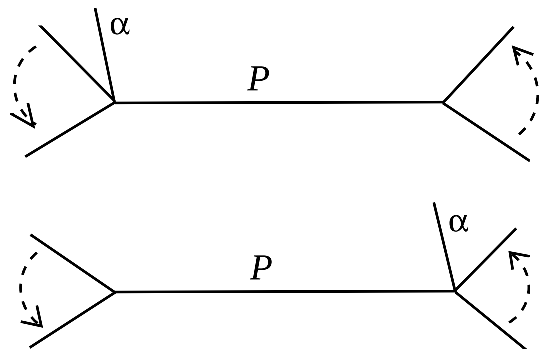

Fig. 10: The graphs contributing to the pole at $\lambda = \lambda_\alpha$. The reversed order of α and the internal line in the two graphs, changes the sign of the residue of the pole.

There are additional additional poles in (2.5.54) that come from the MHV vertices $\mathcal{A}_L \mathcal{A}_R$

$$\frac{1}{\prod_{\alpha=1}^{4} \langle \lambda_\alpha, \lambda \rangle}, \tag{2.5.57}$$

where α runs over the four gluons adjacent to the twistor line. The poles are located at $\lambda = \lambda_\alpha$, which is the condition of the twistor line to meet the gluon vertex operator. Consider the two diagrams, fig. 10 in which the function $\mathcal{A}_L \mathcal{A}_R$

has a pole at $\lambda = \lambda_\alpha$. The graphs differ by whether the gluon α is on the left vertex just after the propagator or on the right vertex just before the propagator. The reversed order of λ and λ_α in the two diagrams changes the sign of the residue. The rest of the residue (2.5.54) stays the same after taking $\lambda = \lambda_\alpha$. The off-shell momenta of the two diagrams differ by $\delta P = \lambda_\alpha \widetilde{\lambda}_\alpha$, so the diagrams have the same value of the denominators $(P\lambda_\alpha \widetilde{\lambda}_\alpha)(P\lambda_\alpha \eta)$. Hence, all poles at $\lambda = \lambda_\alpha$ get cancelled among pairs of diagrams.

This derivation clearly generalizes to several disconnected degree one instantons that contribute to a general tree level amplitude. An amplitude with $d+1$ negative helicity gluons gets contributions from diagrams with d disconnected degree one instantons. The evaluation of the twistor contributions leads to MHV diagrams with d MHV vertices.

Let us remark that the integral (2.5.51) could be taken as the starting point in the study of MHV diagrams. Since (2.5.50) is clearly Lorentz invariant [4], the MHV diagram construction must be Lorentz invariant as well. Although separate MHV diagrams depend on the auxiliary spinor η, the sum of all diagrams contributing to a given amplitude is η independent.

Loops in Twistor Space?

We have just seen that the disconnected instanton contribution leads to tree level MHV diagrams. However, the MHV diagram construction seems to work for loop amplitudes, as discussed in previous subsection. Hence, one would like to generalize the above calculation to higher genus instanton configurations, which contribute to loop amplitudes in Yang-Mills theory. For example, the one-loop MHV amplitude should come from a configuration of two degree one instantons connected by two twistor propagators to make a loop, fig. 8. An attempt to evaluate this contribution runs into difficulties. These are presumably related to the closed string sector of the twistor string theory, that we will now review.

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[4] The Lorentz invariance requires some elaboration, because the choice of contour $\overline{\lambda} = \widetilde{\lambda}$, breaks the complexified Lorentz group $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$ to the diagonal $Sl(2,\mathbb{C})$, the real Minkowski group. It can be argued from the holomorphic properties of the integral that it is invariant under the full $Sl(2,\mathbb{C}) \times Sl(2,\mathbb{C})$

2.5.5 Closed Strings

The closed string of the topological B-model on supertwistor space are related by twistor transform to $\mathcal{N} = 4$ conformal supergravity [15]. Conformal supergravity in four dimension has action

$$S \sim \int d^4x \sqrt{-g} W^2, \tag{2.5.58}$$

where W is the Weyl tensor. This theory is generally considered unphysical. Expanding the action around flat space $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ leads to a fourth order kinetic operator $S \sim \int d^4x\, h\partial^4 h$ for the fluctuations of the metric, and thus to a lack of unitarity.

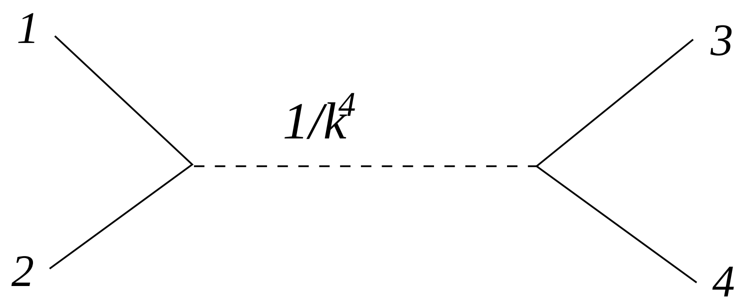

Fig. 11: A double trace $\sim$ Tr T_1T_2Tr T_3T_4 contribution to tree level four gluon scattering amplitude coming from exchange of conformal supergravity particle, which is represented by a dashed line.

We can see a sign of the supergravity already in the tree level MHV amplitude calculation of section 5.1. There we found that the single trace terms agree with the tree level MHV amplitude in gauge theory. We remarked that the current algebra correlators give additional multi-trace contributions. These come from an exchange of an internal conformal supergravity state, which is a singlet under the gauge group. For example, the four gluon MHV amplitude has a contribution Tr T_1T_2Tr T_3T_4 coming from an exchange of supergravity state in the $12 \to 34$ channel, fig. 11. In twistor string theory, this comes from the double trace contribution of the current algebra on the worldvolume of the D-instanton

$$\int_{\mathcal{M}} d\mathcal{M} \langle V_1V_2\rangle \langle V_3V_4\rangle\,. \tag{2.5.59}$$

At tree level, it is possible to recover the pure gauge theory scattering amplitudes by keeping the single-trace terms. However, at the loop level, the diagrams that include conformal supergravity particles can generate single-trace interactions. Hence the presence of conformal supergravity coming from the closed strings puts an obstruction to computation of Yang-Mills loop amplitudes in the present formulation of twistor string theory.

In twistor string theory, the conformal supergravitons have the same coupling as gauge bosons, so it is not possible to remove the conformal supergravity states by going to weak coupling. Since, Yang-Mills theory is consistent without conformal supergravity, it is likely that there is a version of the twistor string theory that does not contain the conformal supergravity states.

3. Twistor Structure of Scattering Amplitudes

3.1 Introduction

We have seen in previous section that the twistor string theory has been successful in description of tree level amplitudes. There has not been comparable progress in understanding the string theory at one-loop. Moreover the present versions of the twistor string seems to describe $\mathcal{N} = 4$ Yang-Mills coupled to $\mathcal{N} = 4$ conformal supergravity [50].

Putting the issue of conformal supergravity aside, the expectation from twistor string theory is that the amplitudes are localized on algebraic curves of appropriate genus and degree. These are interpreted as the worldvolumes of D1-branes in Witten's version of the twistor string and as worldsheets of open strings in the Berkovits's version. Hence, one of the simplest predictions is that a twistor amplitude vanishes unless all particles lie on the curve. The conditions for localization, after Fourier transform into Minkowski space, correspond to certain differential operators that annihilate the scattering amplitudes.

For one-loop MHV amplitudes in $\mathcal{N} = 4$ Yang-Mills theory, the differential equations studied in [2] agree with the twistor string picture discussed in previous section, after one takes into account the holomorphic anomaly of differential operators [3].

For amplitudes in Yang-Mills theories with reduced supersymmetry and in gravity theories we do not have a twistor string proposal. Even if one does not know the twistor string theory appropriate for description of the amplitudes, one can gain insight by studying differential equations that the amplitudes satisfy. In this section, we study differential equations of various $\mathcal{N} = 1, 0$ Yang-Mills amplitudes and also of some gravity amplitudes. Our results are surprisingly similar to the

$\mathcal{N} = 4$ case. Perhaps the most important difference in the $\mathcal{N} = 0$ case is that the one-loop amplitude with all gluons of positive helicity must be included as a new building block alongside with the MHV amplitude. We hope that these results may serve as a clue in a search for twistor string theories that would generate these amplitudes.

Summary of Results

In section 3.2 we review differential operators that test for twistor structure of scattering amplitudes. In particular we derive compact formulas for operators of genus zero curves of degree two and three. In section 3.3 we use these operator to study twistor structure of the one-loop $\mathcal{N} = 1$ MHV amplitudes. In section 3.4 we perform a similar analysis in the nonsupersymmetric case. Unlike the supersymmetric case, the nonsupersymmetric n-gluon amplitudes of n or $n-1$ gluons of the same helicity do not vanish. We discuss how this might be useful in a hypothetical MHV diagrams construction of nonsupersymmetric one-loop amplitudes. Finally, in section 3.5 we discuss the twistor structure of graviton scattering amplitudes in general relativity and $\mathcal{N} = 8$ supergravity and note the similarity to the Yang-Mills case.

3.2 Review of Differential Equations

There are various differential equations that the scattering amplitudes can satisfy [11]. All these can be expressed in terms of so called collinear and coplanar operators that we will now describe.

The differential equations correspond via twistor transform $\partial/\partial\widetilde{\lambda}^{\dot{a}} \to i\mu_{\dot{a}}$ to geometrical conditions on sets of points in $\mathbb{CP}^3$. Given three points $P_i, P_j, P_k \in \mathbb{CP}^3$ with coordinates Z_i^I, Z_j^I, and Z_k^I, the condition that they lie on a line, that is a on a degree one genus zero curve, is that $F_{ijkL} = 0$, where

$$F_{ijkL} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K. \tag{3.2.1}$$

This condition translates, via twistor transform into a differential equation. For example, the choice $L = \dot{a}$ leads to

$$F_{ijk\dot{a}} = \langle \lambda_i, \lambda_j \rangle \frac{\partial}{\partial \lambda_k^{\dot{a}}} + \langle \lambda_j, \lambda_k \rangle \frac{\partial}{\partial \lambda_i^{\dot{a}}} + \langle \lambda_k, \lambda_i \rangle \frac{\partial}{\partial \lambda_j^{\dot{a}}}. \tag{3.2.2}$$

Given four points $P_i, P_j, P_k, P_l \in \mathbb{CP}^3$, the condition that they are all contained in a plane, that is a hyperplane in $\mathbb{CP}^3$, is that the vectors Z_i^I, $i, I = 1, \ldots, 4$ are linearly dependent. This amounts to $K_{ijkl} = 0$, where

$$K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L . \tag{3.2.3}$$

Via twistor transform, this goes into a second order differential operator in $\widetilde{\lambda}^{\dot{a}}$. The coplanar operator can be related to the collinear operators

$$K_{ijkl} = \sum_L F_{ijkL} Z_l^L , \tag{3.2.4}$$

which expresses the elementary fact that if the points ijk are collinear then for any point l, the points $ijkl$ are coplanar.

Given a scattering amplitude $A(\lambda_1, \widetilde{\lambda}_1; \ldots; \lambda_n, \widetilde{\lambda}_n)$, the condition that the twistor space amplitude has a support where the points i, j, k are collinear is that $F_{ijkL} A = 0$. Similarly, the condition that the points i, j, k, l are coplanar is $K_{ijkl} A = 0$, [11].

There are a few simple criteria for an amplitude to be annihilated by F_{ijkL}. Firstly, F_{ijkL} annihilates the amplitude if it depends only on the holomorphic spinors λ^a of the particles i, j, k. For $L = a$, F_{ijka} is a second order differential operator in $\widetilde{\lambda}^{\dot{a}}$ that annihilates the amplitude. Secondly, the amplitude is annihilated by F_{ijkL} if it depends on the spinors of particles i, j, k only through the sum of their momenta $P^{a\dot{a}} = p_i^{a\dot{a}} + p_j^{a\dot{a}} + p_k^{a\dot{a}}$. This follows from the application of the Schoutens's identity

$$\langle \lambda_i, \lambda_j \rangle \lambda_k^a + \langle \lambda_j, \lambda_k \rangle \lambda_i^a + \langle \lambda_k, \lambda_i \rangle \lambda_j^a = 0. \tag{3.2.5}$$

3.2.1 Higher Degree Curves

Holomorphic curves of genus zero in $\mathbb{CP}^3$ have a simple description. Indeed, the curve C is a copy of a $\mathbb{CP}^1$ that can be described by homogeneous coordinates u, v. Any curve of genus zero and degree d has parametric description

$$Z^I = f^I(u, v), \qquad I = 1, \ldots, 4, \tag{3.2.6}$$

where Z^I are the homogeneous coordinates of $\mathbb{CP}^3$ and $f^I(u, v)$ are homogeneous polynomials of degree d.

Degree Two

For $d = 2$, f^I must be linear combinations of the tree quadratic monomials u^2, uv, v^2. Since, there are four f^I's one combination of them vanishes, so the curve C lies in a $\mathbb{CP}^2 \subset \mathbb{CP}^3$, say in the $\mathbb{CP}^2$ characterized by

$$\sum_{I=1}^{4} a_I Z^I = 0. \tag{3.2.7}$$

A degree two curve in $\mathbb{CP}^2$ can be described as a zero set of a homogeneous polynomial of degree two

$$F = \sum_{I,J=1}^{4} a_{IJ} Z^I Z^J. \tag{3.2.8}$$

Hence, the plane conic is a 'complete intersection', which means that it is the solution set to a collection of homogeneous polynomials.

Using an $Sl(4)$ transformation, we identify the plane with the set $Z^4 = 0$. This can be achieved by ie. projection on the $Z^4 = 0$ plane. The condition that the n points P_i all lie in C becomes

$$\sum_{I,J=1}^{3} a_{IJ} Z_i^I Z_i^J = 0 \qquad i = 1, \ldots, n. \tag{3.2.9}$$

We can view this as n linear conditions on the six coefficient a_{IJ}. For $n \leq 5$, there is always a solution, so any five points lie in a (possibly singular) conic. Given six points in $\mathbb{CP}^2$ are contained in a conic if the 6×6 matrix with entries

$$M^{IJ}{}_i = Z_i^I Z_i^J, \qquad i, IJ = 1, \ldots, 6 \tag{3.2.10}$$

has zero determinant. In momentum space, the determinant becomes a fourth order differential operator that we denote V [11].

In the $\mathbb{CP}^2$, the condition for P_i, P_j, P_k to be collinear is that the operator

$$F_{ijk} = F_{ijk4} = \epsilon_{IJK} Z_i^I Z_j^J Z_k^K \tag{3.2.11}$$

vanishes. We now express the condition for $P_1, P_2, \ldots, P_6$ to lie on a conic in terms of these operators. Since, F_{123} is the only invariant of the group $Sl(3)$ of transformation preserving the $\mathbb{CP}^2$, we use $Sl(3)$ to set $P_1 = (1,0,0), P_2 = (0,1,0), P_3 = (0,0,F_{123})$. Then the coordinates of the remaining points are $P_i =$

$(F_{i23}/F_{123}, F_{1i3}/F_{123}, F_{12i})$. The conic operator V is the determinant of the 6×6 matrix $M^{IJ}{}_i = Z_i^I Z_i^J$. Substituting in the choices for coordinates we made, V reduces to the determinant of the 3×3 matrix

$$V = F_{123}^{-2} \det \begin{pmatrix} F_{234}F_{314} & F_{234}F_{124} & F_{314}F_{124} \\ F_{235}F_{315} & F_{235}F_{125} & F_{315}F_{125} \\ F_{236}F_{316} & F_{236}F_{126} & F_{316}F_{126} \end{pmatrix}. \tag{3.2.12}$$

We simplify this using the identity

$$F_{ijk}F_{ilm} + F_{ijl}F_{imk} + F_{ijm}F_{ikl} = 0. \tag{3.2.13}$$

For example, the upper left 2×2 minor of the determinant is $F_{234}F_{235}(F_{314}F_{125} - F_{124}F_{315}) = F_{234}F_{235}F_{123}F_{145}$. Other minors are obtained from this by making a cyclic permutation of indices $4, 5, 6$. Hence we get

$$V = F_{123}^{-1}(F_{316}F_{216}F_{234}F_{235}F_{145} + F_{314}F_{214}F_{235}F_{236}F_{156} + F_{315}F_{215}F_{236}F_{234}F_{164}). \tag{3.2.14}$$

In the middle term, we use $F_{214}F_{235} + F_{213}F_{254} + F_{215}F_{243} = 0$, to get

$$\begin{aligned} V = &- F_{245}F_{314}F_{236}F_{156} \\ &+ F_{123}^{-1}F_{234}(F_{215}F_{314}F_{236}F_{156} + F_{316}F_{216}F_{235}F_{145} + F_{315}F_{215}F_{236}F_{164}). \end{aligned} \tag{3.2.15}$$

After substituting $F_{216}F_{235} + F_{213}F_{256} + F_{215}F_{263} = 0$ into the middle term in the parenthesis, we get three terms proportional to F_{215} that cancel thanks to the identity (3.2.13). We are left with a polynomial in F's

$$V = F_{234}F_{316}F_{145}F_{256} - F_{245}F_{314}F_{236}F_{156}. \tag{3.2.16}$$

Since V is invariant, up to a minus sign, under permutations of the points P_i, there are many equivalent formulas obtained by permuting the right hand side. Further expressions for V follow from projecting the points on a CP^2 in generic position $\sum_{I=1}^4 b_I Z^I = 0$. Then the F's in the expression for V must be taken to be

$$F_{ijk} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K b^L. \tag{3.2.17}$$

Degree Three

A genus zero degree three curve in $\mathbb{CP}^3$ is called 'twisted cubic curve'. Degree three is the first case for which the curve is not contained in any hyperplane $\mathbb{CP}^2$. Also this curve is not a complete intersection, which means that any set of equations defining the curve is redundant. Hence, the parametric description of C (3.2.6) is the most convenient one.

Recall that a twisted cubic curve is described parametrically as

$$Z^I = f^I(u,v), \tag{3.2.18}$$

where f^I are homogeneous cubic polynomials of the homogeneous coordinates u, v of an abstract $\mathbb{CP}^1$. The f^I have 16 coefficients so, after taking into account the action of $Gl(2)$ on u, v, the space of twisted cubics is 12 dimensional. For a point to lie on the twisted cubic entails two conditions, so we guess that any 6 points lie on some twisted cubic. The condition for seven points $P_1, P_2, \ldots, P_7$ to lie on a twisted cubic can be described as follows[5].

Since a twisted cubic is not contained in any $\mathbb{CP}^2$ we can assume that P_1, P_2, P_3, P_4 are not contained in any $\mathbb{CP}^2$ as well. Then, by a $Gl(4)$ transformation, we set $P_1 = (0,0,0,1), P_2 = (0,1,0,0), P_3 = (0,0,1,0)$ and $P_4 = (0,0,0,1)$. The coordinates of the remaining points can be expressed in terms of $K_{ijkl} = \epsilon_{IJKL} Z_i^I Z_j^J Z_k^K Z_l^L$ as $P_i = (K_{j234}, K_{1j34}, K_{12j4}, K_{123j})$. Hence the twisted cubic operator can be expressed in terms of the coplanar operators[6] K. If the point P_1 corresponds to (u_1, v_1), we have $f^2(u_1,v_1) = f^3(u_1,v_1) = f^4(u_1,v_1) = 0$, so f^2, f^3 and f^4 are all divisible by by the linear function $g_1(u,v) = uv_1 - vu_1$. Applying the same arguments to P_2, P_3, P_4, we end up with four linear function $g_I(u,v), I = 1, \ldots, 4$, such that g_I divides f^J for $I \neq J$. So, up to rescalings that can be absorbed into g_I, the twisted cubic curve C is described by

$$f^J(u,v) = \prod_{I \neq J} g_I(u,v). \tag{3.2.19}$$

Introducing the new coordinates $W^I = (Z^1 Z^2 Z^3 Z^4)^{1/3}/Z^I$, the equation for C becomes

$$W^I(u,v) = g_I(u,v). \tag{3.2.20}$$

[5] We thank M. Atiyah and S. Popescu for explaining us this construction

[6] Clearly, this argument applies to higher degree curves as well. Hence, the operators for higher degree curves can be expressed as polynomials in K's.

Hence, in terms of the dual coordinates W^I, the curve C is just a straight line. Given two points P_5, P_6, there is always a straight line going through them, whence any six points $P_1, \ldots, P_6$ are contained in some cubic curve. This construction actually shows that six generic points are contained in a unique twisted cubic.

For seven points $P_1, \ldots, P_7$, the condition that they are on a twisted cubic is that the points P_5, P_6, P_7 in the dual $\mathbb{CP}^3$ lie on a line. This happens when the operators

$$T_L = F_{567L} = \epsilon_{IJKL} W_5^I W_6^J W_7^K \qquad L = 1, \ldots, 4 \tag{3.2.21}$$

vanish. The simplification of the twisted cubic operators T_L is similar to the planar conic case. Using the identity

$$K_{ijkl}K_{ijmn} + K_{ijkm}K_{ijnl} + K_{ijkn}K_{ijlm} = 0, \tag{3.2.22}$$

we find

$$\begin{aligned}
T_1 &= K_{1245}K_{1356}K_{1467}K_{1237} - K_{1235}K_{1456}K_{1367}K_{1247} \\
T_2 &= K_{1245}K_{2356}K_{2467}K_{1237} - K_{1235}K_{2456}K_{2367}K_{1247} \\
T_3 &= K_{1345}K_{2356}K_{3467}K_{1237} - K_{1235}K_{3456}K_{2367}K_{1347} \\
T_4 &= K_{1345}K_{2456}K_{3467}K_{1247} - K_{1245}K_{3456}K_{2467}K_{1347}.
\end{aligned} \tag{3.2.23}$$

In momentum space, T_L's become eighth order differential operators in $\widetilde{\lambda}_i$'s. Again, many equivalent formulas can be obtained by permutations of indices and the use of (3.2.22).

3.3 Twistor Structure of One-Loop $\mathcal{N}=1$ MHV Amplitude

In this section, we study twistor structure of one-loop amplitudes in gauge theory with reduced supersymmetry. In internal particles running in the loop have spin $0, 1/2, 1$. The amplitudes are conveniently described in terms of supersymmetric multiplets running in the loop. One considers the $\mathcal{N}=4$ amplitude, the amplitude $\mathcal{A}^{\mathcal{N}=1}_{chiral}$ in which the particles in the loop form an $\mathcal{N}=1$ chiral multiplet and $\mathcal{A}_{scalar}$ with a scalar in the loop.

3.3.1 Twistor Structure of $\mathcal{N}=1$ Amplitude

In this basis, the one-loop $\mathcal{N}=1$ amplitude can be written as a sum of the contributions of $\mathcal{N}=4$ multiplet minus thrice the $\mathcal{N}=1$ chiral multiplet

$$A^{\mathcal{N}=1} = A^{\mathcal{N}=4} - 3A^{\mathcal{N}=1}_{chiral}. \tag{3.3.1}$$

The contribution from the $\mathcal{N}=1$ chiral multiplet to MHV amplitude with negative helicity gluons i^- and j^- is [51]

$$\begin{aligned} A^{\mathcal{N}=1}_{chiral} = A^{tree} \times \Bigg\{ & \sum_{p=i+1}^{j-1} \sum_{q=j+1}^{i-1} b^{i,j}_{p,q} B(t^{[q-p]}_{p+1}, t^{[q-p]}_{p}; t^{[q-p-1]}_{p+1}, t^{[p-q-1]}_{q+1}) \\ & + \sum_{p=i+1}^{j-1} \sum_{a=j}^{i-1} c^{i,j}_{p,a} \frac{\ln(t^{[a-p]}_{p+1}/t^{[a-p+1]}_{p})}{t^{[a-p]}_{p+1} - t^{[a-p+1]}_{p}} \\ & + \sum_{p=j+1}^{i-1} \sum_{a=i}^{j-1} c^{i,j}_{p,a} \frac{\ln(t^{[p-a]}_{a+1}/t^{[p-a-1]}_{a+1})}{t^{[p-a]}_{a+1} - t^{[p-a-1]}_{a+1}} \\ & + \frac{c^{i,j}_{i+1,i-1}}{t^{[2]}_{i}} K_0(t^{[2]}_{i}) + \frac{c^{i,j}_{i-1,i}}{t^{[2]}_{i-1}} K_0(t^{[2]}_{i-1}) + \frac{c^{i,j}_{j+1,j-1}}{t^{[2]}_{j}} K_0(t^{[2]}_{j}) + \frac{c^{i,j}_{j-1,j}}{t^{[2]}_{j-1}} K_0(t^{[2]}_{j-1}) \Bigg\}, \end{aligned} \tag{3.3.2}$$

where,

$$\begin{aligned} & t^{[k]}_{i} = t^{[n-k]}_{i} \qquad \text{for} \quad k<0 \\ & \sum_{k=i}^{j} = \sum_{k=i}^{n} + \sum_{k=1}^{j} \qquad \text{for} \quad j<i \end{aligned}. \tag{3.3.3}$$

We sum only over a satisfying

$$|a-p|>1 \quad \text{and} \quad |a+1-p|>1, \tag{3.3.4}$$

so that the logarithms on the second and third line have a finite nonzero argument.

The coefficients in front of the integral functions are

$$\begin{aligned} b^{i,j}_{p,q} &= 2\frac{\langle i,p\rangle\langle p,j\rangle\langle i,q\rangle\langle q,j\rangle}{\langle i,j\rangle^2\langle p,q\rangle^2} \\ c^{i,j}_{p,a} &= \frac{(\mathrm{tr}_+[\not k_i \not k_j \not k_p \not q_{p,a}] - \mathrm{tr}_+[\not k_i \not k_j \not q_{p,a} \not k_p])}{(k_i+k_j)^2} \frac{\langle i,p\rangle\langle p,j\rangle}{\langle i,j\rangle} \frac{\langle a,a+1\rangle}{\langle a,p\rangle\langle p,a+1\rangle} \end{aligned}, \tag{3.3.5}$$

where $q_{i,j} = \sum_{l=i}^{j} k_l$ and

$$\mathrm{tr}_+[\not k_{a_1} \not k_{a_2} \not k_{a_3} \not k_{a_4}] = \frac{1}{2}\mathrm{tr}\left[(1+\gamma_5)\not k_{a_1} \not k_{a_2} \not k_{a3} \not k_{a4}\right] = [a_1 a_2]\langle a_2 a_3\rangle[a_3 a_4]\langle a_4 a_1\rangle. \tag{3.3.6}$$

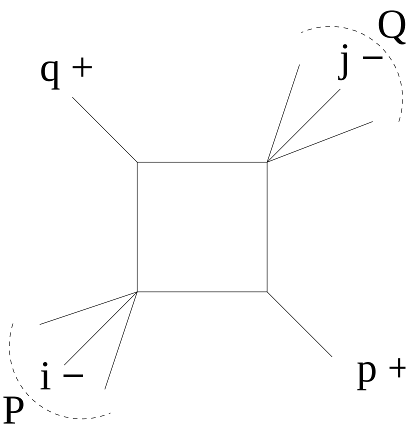

Fig. 12: The scalar box integral contributing to the amplitude. Two of the vertices, p and q are light-like. P and Q are sums of several light-like momenta. We pick conventions so that $i \in P$ and $j \in Q$.

The function B comes from two mass scalar integral, fig. 12. Using the conventions shown in the figure, $p = p_p, q = p_q$ while $P = p_p + p_{p+1} + \ldots + p_{q-1}$ and $Q = p_{q+1} + p_{q+2} + \ldots + p_{p-1}$ so that p, q, P, Q are the four incoming momenta of the box diagram. As discussed in Appendix A, the scalar function B is the finite part of the $\mathcal{N} = 4$ scalar box function

$$\begin{aligned} B(p,q,P,Q) &= F^{finite}(p,q,P,Q) \\ &= \mathrm{Li}_2\left(1 - \frac{P^2}{(P+p)^2}\right) + \mathrm{Li}_2\left(1 - \frac{P^2}{(P+q)^2}\right) \\ &+ \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+q)^2}\right) + \mathrm{Li}_2\left(1 - \frac{Q^2}{(Q+p)^2}\right) \\ &- \mathrm{Li}_2\left(1 - \frac{P^2Q^2}{(P+p)^2(P+q)^2}\right) + \frac{1}{2}\log^2\left(\frac{(P+p)^2}{(P+q)^2}\right). \end{aligned} \tag{3.3.7}$$

On the second and third line of (3.3.2) is the contribution from the triangle functions, fig. 13. Here $p = p_m$, the momenta P, Q are the sums $p_{p+1} + p_{p+2} + \ldots + p_a$ and $p_{a+1}, p_{a+2}, \ldots, p_{p-1}$. We choose P to be the momentum containing i and Q to be the momentum containing j. Using the variables p, P, Q, we rewrite the triangle function in the convenient form

$$T(p,P,Q) = \frac{\ln(Q^2/P^2)}{Q^2 - P^2}. \tag{3.3.8}$$

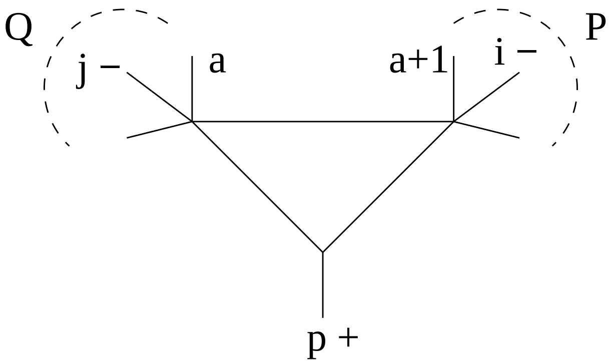

Fig. 13: Triangle diagram contributing to the amplitude. p is a lightlike momentum, P is a sum of light-like momenta containing i and Q is a sum of momenta containing j.

The coefficient $c^{i,j}_{p,a}$ (3.3.5) can be simplified using the definition (3.3.6)

$$c^{i,j}_{p,a} = \frac{\langle i,p\rangle\langle p,j\rangle}{\langle i,j\rangle^2}\frac{\langle a,a+1\rangle}{\langle a,p\rangle\langle a,p+1\rangle} \times \begin{cases} \big(\langle j,p\rangle\langle i|P|p] + \langle i,p\rangle\langle j|P|p]\big) & \text{p=j+1,...,i-1} \\ \big(\langle j,p\rangle\langle i|Q|p] + \langle i,p\rangle\langle j|Q|q]\big) & \text{p=i+1, ...,j-1.} \end{cases} \tag{3.3.9}$$

The main feature that we will use in remainder of the discussion is that the anti-holomorphic dependence of the coefficients (3.3.9) is captured in p, P and Q. In particular, these coefficients are holomorphic in i, j, a.

The amplitude (3.3.2) diverges when i^- or j^- becomes collinear with one of the adjacent positive helicity gluons. The piece that diverges when p_i and p_p become collinear, where $p = i-1, i+1$, comes from a scalar bubble diagram, with $P = p_i+p_p$ and $Q = -P$. It can be simplified to (3.3.2)

$$\frac{c^{i,j}_{p,a}}{s_{i,p}}K_0(s_{i,p}) = -\frac{\langle i,p\rangle\langle p,j\rangle}{\langle i,j\rangle}\frac{\langle a,a+1\rangle}{\langle a,p\rangle\langle p,a+1\rangle}\frac{1}{\epsilon(1-2\epsilon)}(-P^2)^{-\epsilon}, \tag{3.3.10}$$

where $a = i, i-1$ for $p = i-1, i+1$ respectively.

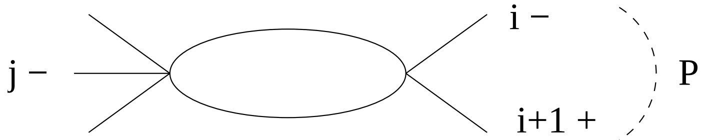

Fig. 14: The scalar bubble diagram giving the divergent part $K_0(P^2)$ of the amplitude.

Collecting the different pieces we can write the amplitude schematically as the sum of box, triangle and bubble contributions

$$A^{\mathcal{N}=1chiral} = A^{tree} \times \left(\sum_{p,q;i\in P,j\in Q} b^{i,j}_{p,q} B(p,q,P,Q) + \sum_{p,a;i\in P,j\in Q} c^{i,j}_{p,a} T(p,P,Q) + A_{IR}. \right) \tag{3.3.11}$$

3.3.2 Interpretation

Box Diagrams

Let us firstly discuss the contribution to the amplitude (3.3.11) from box functions

$$b^{i,j}_{p,q} B(p,q,P,Q). \tag{3.3.12}$$

The coefficient $b^{i,j}_{p,q}$ (3.3.5), is a holomorphic function in the momentum spinors so it does not affect the localization in twistor space. Hence, the localization properties of the box diagrams are determined by the the box function $B(p,q,P,Q)$. This is the finite part of the scalar box function, whose twistor inspired decomposition was found in [2]. The gluons in P and Q are localized on intersecting lines. Moreover, the gluons p,q are localized either on the lines or in a first order neighborhood of the CP^2 containing the intersecting lines. At most one gluon can be localized away from the lines.

There are some differences between the $\mathcal{N}=4$ and the $\mathcal{N}=1$ amplitude. For the $\mathcal{N}=4$ amplitude, there is no restriction on the position of the negative helicity gluons. For the $\mathcal{N}=1$ chiral amplitude, the negative helicity gluons are always localized on the lines. Moreover, one of negative helicity gluons is localized on one line and the other gluon on the other line.

Triangle Diagrams

We can see part of the localization of the triangle diagram without any additional work. The diagram contributes

$$c^{i,j}_{p,a} T(p,P,Q) \tag{3.3.13}$$

to the amplitude (3.3.11). The coefficient $c^{i,j}_{p,a}$ is holomorphic in the spinors of gluons a,i,j (3.3.9). The anti-holomorphic dependence of (3.3.13) is captured via the dependence on P, Q and p. Hence, the vertices of the triangle diagram behave effectively as local vertices in Minkowski space. In the twistor space, the particles of each vertex are supported on a line. The gluons whose momenta add up to P are localized on one line and the gluons whose momenta add up to Q are localized on another line lines.

Furthermore, we found using a computer program that the square of the coplanar operator annihilates the triangle function. Hence, all gluons are in a first order neighborhood of a plane, that is a CP^2. The two lines supporting gluons in P and Q, are intersecting up to first order and the gluon p is localized in the first order neighborhood of the CP^2.

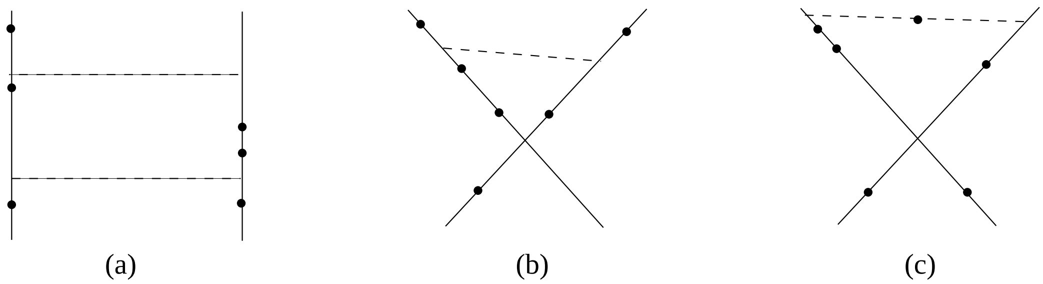

Fig. 15: The twistor configurations contributing to one-loop MHV amplitude as found by studying the differential equations. In (a), the gluons are supported on two disjoint lines that are connected by two twistor propagators. In (b), the lines intersect at a point. In (c), the lines intersect at a point and one gluon is not supported on the lines but rather in the plane spanned by the two lines.

There are further differential equations satisfied by the triangle contribution

$$K_{PPQQ}F_{pPP}F_{pQQ}\big(c^{i,j}_{p,a}T(p,P,Q)\big) = 0, \tag{3.3.14}$$

which complicate the picture. Here, K_{PPQQ} represents a coplanar operator K_{ijkl} with $i,j \in P$ and $k,l \in Q$. Similarly F_{pPP} is a collinear operator F_{pij} with $i,j \in P$. This collection of differential equations means roughly that the triangle function is a sum of contributions that are annihilated by either of the three differential operators (3.3.14). Hence, either the lines P and Q are strictly coplanar or one of the lines contains p.

Divergent Part

The infrared divergent part of the amplitude (3.3.2) is (3.3.10)

$$\frac{1}{\epsilon(1-2\epsilon)}(-P^2)^{-\epsilon} \tag{3.3.15}$$

times a holomorphic function of spinors. As discussed before, this localizes on a disjoint union of two lines. The gluons whose momenta add up to P are on one line and the remaining gluons are on the second line.

3.3.3 Holomorphic Anomaly

In a hypothetical twistor string theory dual to perturbative $\mathcal{N} = 1$ gauge theory, we would expect that all gluon are supported on an algebraic curve, which is a worldsheet of a string that generates the interaction. This is what we find for the first two contributions of figure fig. 15. The first contribution can be interpreted as coming from two degree one D-instantons connected by two open strings. Similarly the second comes when one of the propagators degenerates or equivalently, when a degree two instanton degenerates to two intersecting degree one instantons.

However, for the finite part of the amplitude we find that the amplitude also has a contribution, fig. 15 (c), where the gluons are localized on two intersecting lines except for one, which is in the plane spanned by the lines. This configuration is not expected from the twistor string theory.

In order to resolve this discrepancy, we need to recall from section 3.1 the differential equations that tests whether external gluons are supported on a line. For gluons i, j, k with momenta $p_l^{a\dot{a}} = \lambda_l^a \widetilde{\lambda}_l^{\dot{a}}$, the differential operator that should annihilate the amplitude is

$$F_{ijk} = \langle \lambda_i, \lambda_j \rangle \frac{\partial}{\partial \widetilde{\lambda}_k} + \langle \lambda_j, \lambda_k \rangle \frac{\partial}{\partial \widetilde{\lambda}_i} + \langle \lambda_k, \lambda_i \rangle \frac{\partial}{\partial \widetilde{\lambda}_j}. \tag{3.3.16}$$

For example the MHV amplitude

$$A(\lambda_i, \widetilde{\lambda}_i) = \frac{\langle \lambda_r, \lambda_s \rangle^4}{\prod_{k=1}^n \langle \lambda_m, \lambda_{m+1} \rangle}, \tag{3.3.17}$$

is manifestly annihilated by $F_{ijk},\ i, j, k = 1, \ldots, n$ in agreement with the discussion of previous chapter.

This is actually true only for generic momenta, as there is a delta function contribution when two of the momenta become collinear. To see this, recall that in Minkowski space λ^a and $\widetilde{\lambda}^a$ are complex conjugates. Hence, the $\partial/\partial\widetilde{\lambda}$ operator acts on λ the same way as $\overline{\partial}$ acts on z. So we get a nonzero contribution when the operator acts on a pole

$$d\widetilde{\lambda}^{\dot{a}} \frac{\partial}{\partial\widetilde{\lambda}^{\dot{a}}} \frac{1}{\langle\lambda, \lambda'\rangle} = -2i\pi\overline{\delta}(\langle\lambda, \lambda'\rangle), \tag{3.3.18}$$

where $\overline{\delta}(z) = d\overline{z}\delta^2(z)$ is the holomorphic delta function. Hence, $F_{ijk}A$ is actually a sum of delta function when gluons become collinear with the gluons i, j, k. At tree level, we can always pick the momenta of the external particles so that no two are collinear so the delta functions can be safely ignored.

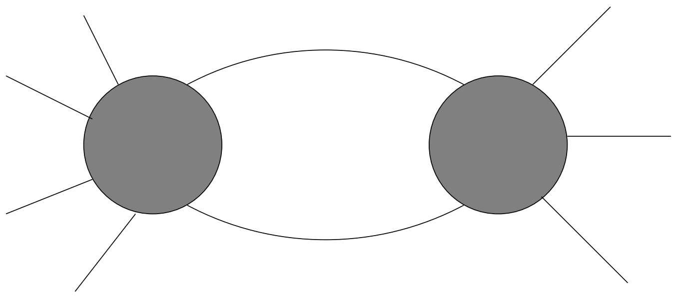

Fig. 16: The cut diagram computing the imaginary part of the one-loop MHV amplitude. The left and right amplitudes are the on-shell MHV amplitudes.

Consider now the similar argument for loop amplitudes. For clarity, we focus on the imaginary part of one-loop MHV amplitude. From unitarity, this can be obtained from a 'cut' diagram of fig. 16 where the cut propagators are on-shell and the scattering amplitudes on the left and right are the MHV amplitudes. Naively, $F_{ijk}A$ is zero if i, j, k are gluons coming from one MHV amplitude. However, the MHV amplitudes develop a pole when one of the external gluons becomes collinear with one of the internal gluons. The condition for the internal gluon to be collinear with a given external gluon fixes the momentum of the internal gluon. Hence, F_{ijk} acting on A gets a delta function contribution that localizes the integral over the momentum of the internal gluon. One would naively interpret the nonvanishing of $F_{ijk}\mathcal{A}$ as a sign of one gluon being localized away from the lines supporting the MHV vertices. For generic external momenta, the internal gluon can be collinear with only one external gluon which explains why in the previous subsection we found only one external gluon in the bulk of the twistor space.

3.4 Twistor Structure of Nonsupersymmetric One-Loop Amplitudes

In this section, we discuss nonsupersymmetric scattering amplitudes of gluons. As discussed at the beginning of previous section, it is convenient to decompose the the contributions of internal particles into supersymmetric multiplets

$$\mathcal{A}^{QCD} = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1}_{chiral} + A^{scalar}. \tag{3.4.1}$$

The supersymmetric contribution $\mathcal{A}^{\mathcal{N}=1}_{chiral}$ was studied in previous section. In this section we consider the contribution from an internal scalar running in a loop.

Unlike in the supersymmetric case, the nonsupersymmetric n-gluon amplitudes with n or $n-1$ gluons of the same helicity do not vanish. We begin with the discussion of these amplitudes and then go on to discussing the amplitudes with two negative helicity gluons.

3.4.1 All Plus Helicity One-Loop Amplitude

The one-loop scattering amplitude of $n \geq 4$ gluons of positive helicity is [52], [53]

$$A_n^{1-loop}(+,\ldots,+) = -\frac{i}{48\pi^2} \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{\langle i_1 i_2\rangle [i_2 i_3]\langle i_3 i_4\rangle [i_4 i_1]}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{3.4.2}$$

For future reference, we rewrite the amplitude in terms of the momenta and holomorphic spinors of the external particles

$$A = -\frac{i}{96\pi^2} \sum_{1\leq i_1<i_2<i_3<i_4\leq n} \frac{s_{i_1i_2}s_{i_3i_4} - s_{i_1i_3}s_{i_2i_4} + s_{i_1i_4}s_{i_2i_4} - 4i\epsilon_{\mu\nu\lambda\rho}p^{\mu}_{i_1}p^{\nu}_{i_2}p^{\rho}_{i_3}p^{\sigma}_{i_4}}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}. \tag{3.4.3}$$

The amplitude is a single valued function of spinors, hence it is free of cuts. Indeed, cutting the amplitude into two parts, the cut is proportional to product of two tree level amplitudes, at least one of which has less than two negative helicities so it vanishes.

The twistor structure of the amplitude is clear. The product of any three collinear operators annihilates the amplitude, because the amplitude is quadratic in $\widetilde{\lambda}_i^{\dot{1}}$. Hence the external gluons are all supported in a second order neighborhood of a line , that is a $\mathbb{CP}^1$. In analogy with MHV vertices, the all-plus amplitudes are a twistor space analogs of local interaction vertices [54]. Hence, it is tempting to guess, that the nonsupersymmetric amplitudes can be constructed from these two types of vertices.

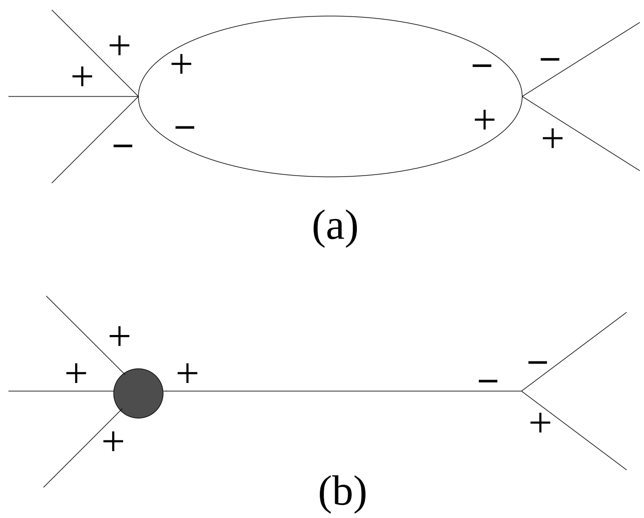

Fig. 17: (a) A one-loop diagram with MHV vertices. Each vertex has two negative helicity gluons. Out of the four negative helicity gluons, two are absorbed by the propagators. (b) A diagram that also contains a one-loop all-plus vertex. The vertex is drawn as a disk to indicate that it contains a loop.

Before we go on to study further amplitudes, let us discuss the hypothetical Feynman diagrams construction of nonsupersymmetric one-loop amplitudes using MHV and all-plus vertices. Consider first a diagram with d MHV vertices. Each vertex contains two negative helicity gluons. To make a connected diagram, the vertices are connected with $d-1$ propagators, each of which absorbs one negative helicity gluon, leaving $d+1$ negative helicity external gluons. An l-loop diagram contains l additional propagators, hence the diagram has

$$q = d - l + 1 \tag{3.4.4}$$

negative helicity gluons.

Each all-plus vertex contains a hidden loop inside, hence adding p such vertices we need to remove p propagators leaving us with

$$q = d + 1 - l - p \tag{3.4.5}$$

negative helicity gluons. An l-loop amplitude can have up to l all-plus vertices, hence it has contributions from quivers of degrees

$$q - 1 + l \leq d \leq q - 1 + 2l. \tag{3.4.6}$$

3.4.2 The $- + + + \ldots +$ One Loop Amplitude

We will now discuss the n gluon amplitude with $n-1$ gluons of positive helicity and compare it to the expectations from previous subsection. We find that the twistor structure agrees with our expectations. However we have not been able to find an off-shell continuation of the one-loop all plus vertex that would give the right amplitude.

The one loop nonsupersymmetric scattering amplitude of all but one gluon of the same helicity has been derived using recursive techniques by Mahlon [55], [56].

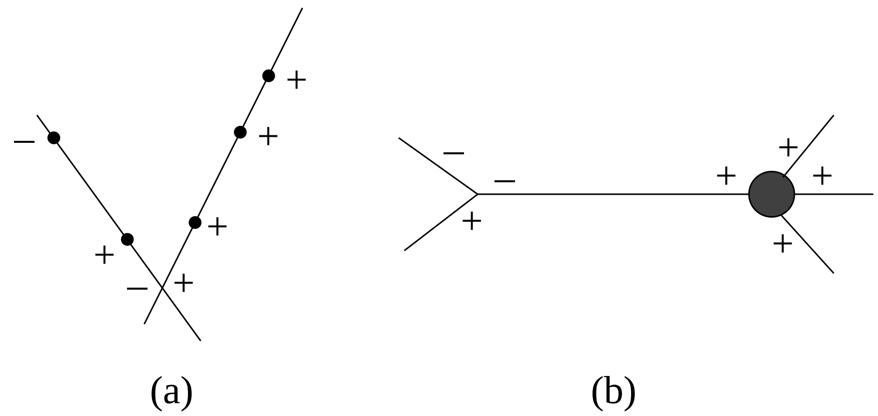

Fig. 18: Two representations of a diagram contributing to a one loop non-supersymmetric amplitude. (a) The geometry of the diagram in twistor space, as found from differential equations. (b) Minskowski space representation of the diagram in terms of local vertices, a four-valent all-plus vertex and a three-valent MHV vertex.

For example, consider the five gluon $- + + + +$ amplitude [57]

$$A = \frac{i}{48\pi^2}\frac{1}{\langle 34\rangle^2}\left[-\frac{[25]^3}{[12][51]} + \frac{\langle 14\rangle^3[45]\langle 35\rangle}{\langle 12\rangle\langle 23\rangle\langle 45\rangle^2} - \frac{\langle 13\rangle^3[32]\langle 42\rangle}{\langle 15\rangle\langle 54\rangle\langle 32\rangle^2}\right]. \qquad (3.4.7)$$

We find that the product of any three coplanar operators annihilates the amplitude

$$K^3 A = 0. \qquad (3.4.8)$$

We also find that the amplitude is annihilated by

$$F^3_{123}F^3_{234}A = 0. \qquad (3.4.9)$$

The differential equations have the following interpretation. (3.4.8) implies that all gluons are coplanar. Furthermore, it follows from (3.4.9) that three of the gluons lie on a line. Drawing a line through the other two gluons, we find that the two lines intersect as indicated in fig. 18 (a).

We have studied Mahlon's amplitude up to eight gluons and found that the differential equations satisfied by the amplitude are in agreement with the proposed twistor configurations. The amplitude has contribution from twistor diagrams with one MHV vertex and one all-plus vertex. The MHV vertex is localized on a line while the all-plus vertex is localized in a second order neighborhood of a line. The two lines have a second order intersection, hence the amplitude is supported in a second order neighborhood of a plane defined by the two lines.

Towards the Construction of All-Plus Vertex

On the other hand, we have not been successful in finding an off-shell continuation of the all-plus one-loop amplitude to use in the diagrams fig. 18(b). One approach is to use the second form of the all-plus amplitude (3.4.3) that depends on the holomorphic spinors λ_i and the momenta p_i only. Hence, one can adopt the off-shell continuation used in [54] $\lambda_a = p_{a\dot{a}}\eta^{\dot{a}}$, but one finds that it does not lead to the right amplitudes.

3.4.3 Nonsupersymmetric $--+\ldots+$ Amplitude

Here we consider the nonsupersymmetric amplitudes with two negative helicity gluons. The part of the amplitudes that contains cuts can be computed via unitarity. According to [51], the cut-constructible part of the scalar loop amplitude with two adjacent negative helicity gluons is

$$\begin{aligned}\mathcal{A}_{scalar\ cut} =& \frac{1}{3}\mathcal{A}^{\mathcal{N}=1}_{chiral} - \frac{c_\Gamma}{3}A^{tree}\sum_{p=4}^{n-1}\frac{L_2\left(t_2^{[p-2]}/t_2^{[p-1]}\right)}{\left(t_1^{[2]}t_2^{[p-1]}\right)^3}\\ &\times \mathrm{tr}_+[\not{k}_1\not{k}_2\not{k}_p\not{q}_{p,1}]\,\mathrm{tr}_+[\not{k}_1\not{k}_2\not{q}_{p,1}\not{k}_p]\left(\mathrm{tr}_+[\not{k}_1\not{k}_2\not{k}_p\not{q}_{p,1}] - \mathrm{tr}_+[\not{k}_1\not{k}_2\not{q}_{p,1}\not{k}_p]\right),\end{aligned} \tag{3.4.10}$$

where $\mathcal{A}^{tree}$ is the tree level MHV amplitude and

$$L_2(x) = \frac{\ln(x) - (x-1/x)/2}{(1-x)^3}. \tag{3.4.11}$$

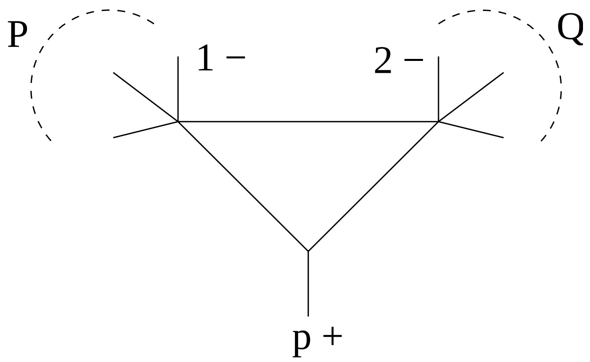

Fig. 19: A triangle diagram contributing to the scalar loop amplitude with adjacent negative helicity gluons.

Setting $P = p_{p+1} + p_{p+2} + \ldots + p_1$ and $Q = p_2 + p_3 + \ldots + p_{p-1}$, the scalar loop amplitude becomes

$$\begin{aligned} \mathcal{A}_{scalar\ cut} = \frac{1}{3}\mathcal{A}^{\mathcal{N}=1}_{chiral} - \frac{c_\Gamma}{3}\frac{\mathcal{A}^{tree}}{\langle 1,2\rangle^3}\sum_{p=4}^{n-1}\frac{L_2(P^2/Q^2)}{(Q^2)^3} \\ \times \langle 1,p\rangle\langle 2,p\rangle\langle 1|P|p]\langle 2|P|p]\big(\langle 1,m\rangle\langle 2|P|p] - \langle 2,m\rangle\langle 1|P|m]\big). \end{aligned} \quad (3.4.12)$$

Now we have two types of triangle functions

$$T(p,P,Q) = \frac{\ln(Q^2/P^2)}{Q^2-P^2}, \qquad \widetilde{T}(p,P,Q) = \frac{L_2(P^2/Q^2)}{(Q^2)^3}. \quad (3.4.13)$$

Schematically, the amplitude is a sum of triangle diagrams

$$A^{scalar} = \sum_p \frac{1}{3}c^{1,2}_{1,p}T(p,P,Q) + \sum_p \widetilde{c}^{1,2}_p\widetilde{T}(p,P,Q). \quad (3.4.14)$$

The part containing $T(p,P,Q)$ has been studied in previous section. The part containing the nonsupersymmetric triangle function $\widetilde{T}(p,P,Q)$ localizes on almost the same configurations as the $\mathcal{N}=1$ triangle function. The gluons whose momenta add up to P are localized on a line and the gluons whose momenta add up to Q are localized on another line. The study of the differential equations shows that the amplitude is annihilated by the square of the coplanar operator K^2, so all gluons are coplanar, they lie in a second order neighborhood of a plane. The amplitude satisfies further differential equations analogous to (3.3.14)

$$K_{PPQQ}F^2_{pPP}F^2_{pQQ}\big(\widetilde{c}^{i,j}_{p,a}\widetilde{T}(p,P,Q)\big) = 0. \quad (3.4.15)$$

3.4.4 Comparison of Amplitudes with Two Negative Helicity Gluons

The surprising result of the analysis in preceding sections is that the $\mathcal{N} = 1$ chiral and the cut-constructible part of scalar MHV amplitudes localize on the same type of twistor configurations. In all cases, the infrared divergent part of the amplitude localizes on two disjoint lines. The finite part of the amplitude is localized on two intersecting lines. One gluon can have a distinguished position. It can be supported away from the lines, in a first order neighborhood of the plane defined by the two lines. At first this twistor picture would seem at odds the discussion of previous chapter, where we noted that the MHV loop computation of the one-loop MHV amplitude [47] makes manifest that the amplitude is supported on a disjoint union of lines, with all gluons localized on the lines. This apparent discrepancy has been reinterpreted in terms of holomorphic anomaly in the differential equations [3] as discussed in previous section.

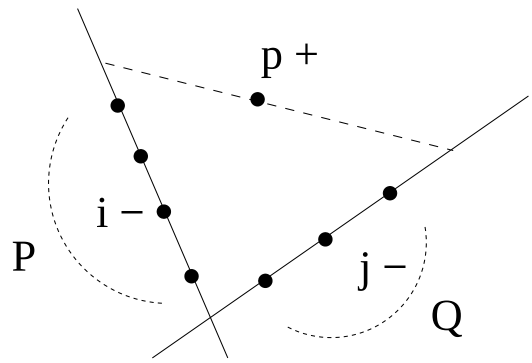

Fig. 20: A twistor configuration contributing to the $\mathcal{N} = 1$ chiral amplitude. One gluon is in the plane containing the lines P, Q.

3.4.5 Cut-free Part of $- - + + \ldots +$ Amplitude

The cut-constructible terms do not give the whole $- - + \ldots +$ amplitude. In particular, they lack singularities in some multiparticle channels. The amplitude also contains cut-free rational functions. For five gluons the rational part of the 'MHV' amplitudes have been computed via string inspired methods [57]. Studying the differential equations, we find the possible diagrams contributing to the rational terms, fig. 21.

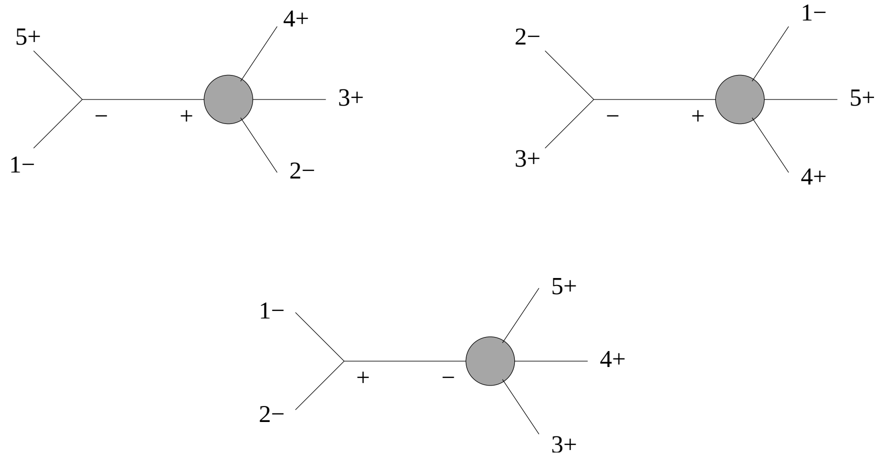

Fig. 21: The diagrams contributing to rational function part of the $--+++$ loop amplitude.

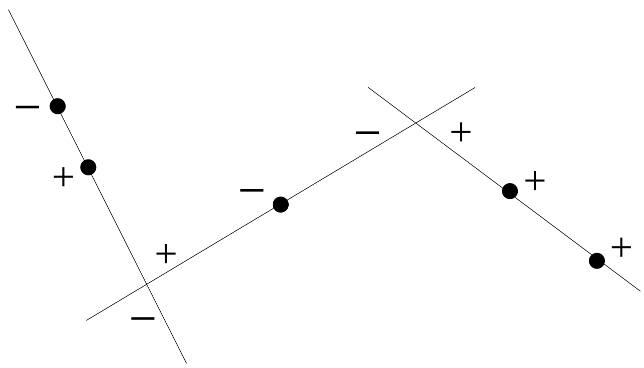

Fig. 22: A hypothetical diagram that could contribute to the rational function part of the $--+++$ loop amplitude.

Unfortunately, the cut-free part of the MHV amplitude has been computed only for amplitudes with five gluons [57]. So we were unable to analyze the cut-free part of the $n \geq 6$ gluon amplitude. These amplitudes are expected to receive contribution from quivers drawn in fig. 22 (with additional positive helicity gluons added on the lines.) The amplitudes should satisfy differential equations reflecting the structure of the quivers.

3.5 Twistor Structure of Gravitational Amplitudes

The study of graviton amplitudes has been initiated in [11] where the tree level n graviton maximally helicity violating amplitude was shown to be supported on a genus zero degree one curve in twistor space. Here, we continue the empirical study of graviton scattering amplitudes. In analogy with gauge theory amplitudes, we conjecture that the maximally supersymmetric supergravity amplitudes localize on the same type twistor configurations as the gauge theory amplitudes. We propose that the l-loop scattering amplitude with q gravitons of negative helicity is supported on a algebraic curve of degree

$$d = q - 1 + l \tag{3.5.1}$$

in the twistor space, but now with a 'higher derivative delta function support' in the normal directions to the curve. The genus of the curve is bounded by the number of loops

$$g \leq l. \tag{3.5.2}$$

In a hypothetical twistor string theory that is dual to perturbative $\mathcal{N} = 8$ supergravity, one would expect to write the scattering amplitudes as integrals over the moduli space of curves of appropriate degree and genus. As in gauge theory case, the differential equations obeyed by the amplitudes suggest that the amplitudes are supported on singular degenerations of the curves. The singular curves in question are collections of pairwise intersecting genus zero degree one curves, that is $\mathbb{CP}^1$'s. Taking the analogy with Yang-Mills theory further, we would conjecture that each of the $\mathbb{CP}^1$'s is related to an MHV graviton amplitude in a hypothetical MHV diagram construction of graviton amplitudes.

In the following subsections, we undertake an empirical study of the differential equations obeyed by the graviton scattering amplitudes to support the above conjecture. We give additional evidence that graviton scattering amplitudes are supported on intersecting lines in the twistor space as in gauge theory case, but now with a multiple derivative delta function support in the normal directions. We study both tree level and one loop graviton scattering amplitudes in general relativity and in $\mathcal{N} = 8$ supergravity. This section has some overlap with [58,59].

3.5.1 Tree Level Graviton Amplitudes

In analogy with Yang-Mills theory, we conjectured that the n graviton tree level scattering amplitude with q gravitons of negative helicity is supported on a configuration of $q-1$ pairwise intersecting degree one curves in twistor space. Since the minimal degree of an algebraic curve is one, it follows from this proposal that the amplitudes with zero or one negative helicity gravitons vanish[7]. This is indeed the case. The vanishing of these amplitudes is a consequence of supersymmetric Ward identities. The Ward identities hold for tree level scattering amplitudes in gravity theory with no supersymmetry as well, because the tree level amplitudes are not sensitive to supersymmetry.

MHV Amplitude

The first nonvanishing tree level amplitude, the Maximally Helicity Violating amplitude, has two gravitons of negative helicity and any number of positive helicity gravitons. It has been computed by Berends, Giele and Kuijf [60] using the KLT relations [61]. As discussed in [11], after factoring out the delta function of energy-momentum conservation, the amplitude is a rational function in λ_i^a times a polynomial in $\widetilde{\lambda}_i^{\dot{a}}$

$$\mathcal{A}(\lambda_i, \widetilde{\lambda}_i) = f(\lambda_i^a) P(\widetilde{\lambda}_i^{\dot{a}}). \tag{3.5.3}$$

After Fourier transform into twistor space, the polynomial $P(\widetilde{\lambda}_i^{\dot{a}})$ becomes a differential operator

$$\widetilde{\mathcal{A}}(\lambda_i, \mu_i) = f(\lambda_i) P(\frac{\partial}{\partial \mu_{i\dot{a}}}) \prod_{i=1}^{n} \delta^2(\mu_{i\dot{a}} + x_{a\dot{a}}\lambda_i^a). \tag{3.5.4}$$

Hence, the MHV amplitude is supported on genus zero degree one curve, that is a $\mathbb{CP}^1$, as in the Yang-Mills case. Since the polynomial $P(\widetilde{\lambda}_i)$ is of degree $n-3$, the amplitude has $n-3^{rd}$ derivative delta function support in the normal directions to the curve. Hence, the product of any $n-2$ collinear operators F_{ijk} annihilates the amplitude

$$F^{n-2}\mathcal{A}_n = 0. \tag{3.5.5}$$

Amplitudes with Three Negative Helicities

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[7] The three graviton amplitude with two gravitons of the same helicity is an exception. This amplitude is nonzero of on-shell complex momenta, but becomes zero for real momenta in Minkowski space. This follows from analogous remarks for the gauge theory case [11], since the three graviton amplitude is simply the square of the three gluon color ordered amplitude.

The next case to consider are amplitudes with three negative helicity gravitons. From (3.5.1) we expect that these amplitudes are supported on two intersecting lines in twistor space.

The simplest amplitude with three negative helicity gravitons is the five graviton googly MHV amplitude. The amplitude with gravitons 1 and 2 of positive helicity and the rest of negative helicity is [60]

$$\mathcal{A}(+,+,-,-,-) = i[12]^8 \frac{[12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41]}{N(5)}, \tag{3.5.6}$$

where

$$N(5) = \prod_{i=1}^{4} \prod_{j=i+1}^{n} [ij]. \tag{3.5.7}$$

The amplitude can be derived from the gauge theory MHV amplitude using the five particle KLT relation[8]

$$\begin{aligned}\mathcal{A}(1,2,3,4,5) = {} & is_{12}s_{34}A(1,2,3,4,5)A(2,1,4,3,5) \\ & + is_{13}s_{24}A(1,3,2,4,5)A(3,1,4,2,5).\end{aligned} \tag{3.5.8}$$

A computer assisted study of the differential equations shows that

$$K^2 M = 0, \tag{3.5.9}$$

hence the five gravitons are contained in a higher order neighborhood of a plane, $\mathbb{CP}^2$. Furthermore the amplitude is annihilated by the product of the squares of all collinear operators but F_{345}

$$\prod_{ijk \neq 345} F_{ijk}^2 M = 0. \tag{3.5.10}$$

This has a simple interpretation in a hypothetical MHV diagram construction of the amplitude. According to (3.5.1) the amplitude is supported in an infinitesimal neighborhood of a singular planar conic composed of two intersecting linearly embedded $\mathbb{CP}^1$'s. The configuration with gravitons ijk on one line and the remaining gravitons on the other corresponds to an MHV diagram with one three-valent and one four-valent MHV vertex. The gravitons ijk are contained in the four-valent

[8] See Appendix 3.B for a discussion of KLT relations.

MHV vertex, which is annihilated by F^2_{ijk} (3.5.5), so the diagram is annihilated by F^2_{ijk} as well. (3.5.10) is simply the product of these operators over all twistor diagrams contributing to the amplitude. The condition $ijk \neq 345$ comes in because an MHV vertex has exactly two negative helicities.

Next is the six graviton amplitude with three negative helicities, $+ + + - - -$. Following [60], we define the amplitude from KLT relations

$$\begin{aligned}\mathcal{A}(1,2,3,4,5,6) &= -is_{12}s_{45}A(1,2,3,4,5,6)[s_{35}A(2,1,5,3,4,6) + (s_{34}+s_{35})A(2,1,5,4,3,6) \\ &\quad + \text{permutations of } (234)] \\ &= -is_{12}s_{45}A(1,2,3,4,5,6)[s_{13}A(2,3,1,5,4,6) + (s_{13}+s_{23})A(3,2,1,5,4,6) \\ &\quad + \text{permutations of } (234)].\end{aligned} \tag{3.5.11}$$

According to our conjecture, the amplitude is supported on a pair of intersecting lines in twistor space. We were able to verify with computer assistance that the third power of coplanar operator and of the operator for plane conic annihilate the amplitude

$$\begin{aligned} K^3\mathcal{A} &= 0 \\ V^3\mathcal{A} &= 0. \end{aligned} \tag{3.5.12}$$

Hence the external gravitons are contained in a plane conic. The condition for the plane conic degenerates to a pair of intersecting lines is a differential operator made from products of collinear operators. A short analysis of the quivers contributing to the amplitude shows, that one of the two $\mathbb{CP}^1$'s of the quiver always contains at least two gravitons of positive helicity and one of negative helicity. This configuration is annihilated by F^3_{ijk} where P_i, P_j, P_k are the gravitons in question.

Hence, the following degree 27 operator annihilates all quivers, fig. 23

$$\mathcal{O} = \prod_{h_i=h_j=-h_k=+} F^3_{ijk}, \tag{3.5.13}$$

so according to our conjecture, $\mathcal{O}$ annihilates the scattering amplitude. We have not been able to verify this conjecture in a reasonable amount of computational time.

KLT Relations vs. Twistor Structure of Amplitudes

A keen reader might ask whether the twistor structure of the graviton amplitudes is a consequence of the twistor structure of gluon amplitudes. After all,

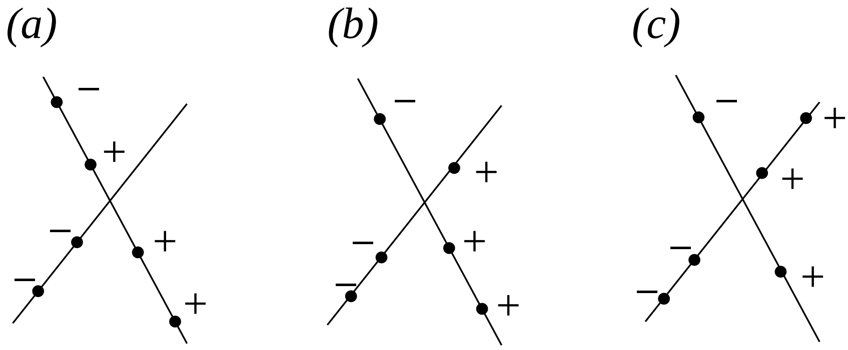

Fig. 23: The three classes of quivers contributing to the $+++---$ six graviton amplitude. $+$ points denote any permutation of the three positive helicity gravitons, so altogether there are $3+9+9$ diagrams coming from the classes (a),(b) and (c) respectively. Each of the diagrams is annihilated by at least one of the 9 operators F^3_{++-}.

we conjecture that the graviton amplitudes are supported on quivers of the same degree (3.5.1) as the gluon amplitudes of the same helicity configuration. The KLT relations (3.5.8), (3.5.11) express the graviton scattering amplitudes as simple sums of squares of gauge theory amplitudes up to some factors of s_{ij}.

The simplest way that the KLT relations could imply the localization of graviton amplitudes would be that each of the terms in the KLT relations (3.5.8) and (3.5.11) localizes on the same configurations as the entire gravity amplitude.

We find a counterexample in the operator K^3 acting on the six graviton non-MHV amplitude. The operator does not annihilate the separate terms on the right hand side of (3.5.11). Only the whole graviton amplitude, which is a sum of terms, is annihilated by K^3. Hence, the separate summands in the KLT relations do not have a straightforward interpretation in the twistor space. This suggests that localization of the graviton amplitudes is independent of the KLT relations. It is an intrinsic property of gravity amplitudes that gives a hint of a twistor string theory whose instanton expansion would naturally lead to the quiver picture discussed in this section.

3.5.2 One-Loop Graviton Amplitudes

The Four Graviton One-Loop Amplitudes

The four graviton amplitudes with arbitrary particle content running in the loop have been calculated using string based methods in [62]. The one-loop

$(\pm,+,+,+)$ supergravity amplitudes vanish due to supersymmetric Ward identities. In the nonsupersymmetric case, the amplitudes are finite rational functions of the spinors $\lambda_i, \widetilde{\lambda}_i$,

$$\begin{aligned}\mathcal{A}(+,+,+,+) &= -N_s \frac{i\kappa^4}{(4\pi)^2}\left(\frac{st}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}\right)^2 \frac{s^2+st+t^2}{1920},\\ \mathcal{A}(-,+,+,+) &= \frac{i\kappa^4}{(4\pi)^2}\frac{N_s}{5760}\frac{s^2t^2}{u^2}(u^2-st)\left(\frac{[24]^2}{[12]\langle 23\rangle\langle 34\rangle[41]}\right)^2,\end{aligned} \tag{3.5.14}$$

where N_s is the number of bosonic states circulating in the loop minus the number of fermionic states and $s = s_{12}, t = s_{14}, u = s_{13}$ are the Mandelstam variables. For General Relativity $N_s = 2$ because the graviton has two helicity states. These amplitudes have a polynomial dependence on $\widetilde{\lambda}_i$, whence they are supported in a higher order neighborhood of a $\mathbb{CP}^1$.

The the maximally helicity violating $(-,-,+,+)$ amplitude is nonzero in gravity theory with or without supersymmetry. It is convenient to consider the partial amplitudes which receive contributions from $\mathcal{N} = 1,2,4,6,8$ chiral multiplets and a scalar running in the loop. The general relativity amplitude receives contribution only from internal gravitons. It can be decomposed as

$$\mathcal{A}^{GR} = \mathcal{A}^{N=8} - 8\mathcal{A}^{N=6} + 20\mathcal{A}^{N=4} - 16\mathcal{A}^{N=1} + \mathcal{A}_{scalar}. \tag{3.5.15}$$

The $\mathcal{N} < 8$ chiral multiplets give finite contributions. Moreover, they are polynomial in $\widetilde{\lambda}_i$, so they are supported in an infinitesimal neighborhood of a $\mathbb{CP}^1$. The $\mathcal{N} = 8$ chiral multiplet contains graviton which gives infrared divergent contribution to the scattering amplitude

$$\begin{aligned}\mathcal{A}^{N=8} =& \frac{2F}{\epsilon}\left(\frac{\ln(-u)}{st} + \frac{\ln(-t)}{su} + \frac{\ln(-s)}{tu}\right)\\ &+ 2F\left(\frac{\ln(-t)\ln(-s)}{st} + \frac{\ln(-u)\ln(-t)}{tu} + \frac{\ln(-s)\ln(-u)}{us}\right),\end{aligned} \tag{3.5.16}$$

where

$$F = i\kappa^2 stu A^{tree}, \tag{3.5.17}$$

is a polynomial in $\widetilde{\lambda}$ times the tree level amplitude. We expect that this contribution, in analogy with gauge theory, comes from two disjoint lines in twistor space. In a hypothetical twistor string dual to $\mathcal{N} = 8$ supergravity, the lines are connected by two twistor propagators.

$\mathcal{N}=8$ *Five Graviton MHV amplitude*

The $\mathcal{N}=8$ one-loop five graviton MHV amplitude with gravitons i and j of negative helicity and the remaining 3 gravitons of positive helicity is [63]

$$\begin{aligned} M_5^{1-loop} = -\frac{1}{2}\langle ij\rangle^8 [s_{12}^2 s_{23}^2 h(1,\{2\},3) h(3,\{4,5\},1)\mathcal{I}_4^{123(45)} \\ + \, permutations], \end{aligned} \tag{3.5.18}$$

where the sum is over 30 distinct permutations and

$$\begin{aligned} h(a,\{c\},b) &= \frac{1}{\langle a1\rangle^2\langle 1b\rangle^2}, \\ h(a,\{cd\},b) &= \frac{[cd]}{\langle cd\rangle\langle ac\rangle\langle cb\rangle\langle ad\rangle\langle da\rangle}. \end{aligned} \tag{3.5.19}$$

$\mathcal{I}^{123(45)}$ is the one-mass scalar box integral with momenta p_1, p_2, p_3 and p_4+p_5 flowing out of the four corners of the 'box'

$$\begin{aligned} \mathcal{I}_4^{123(45)} = \frac{i}{s_{12}s_{23}}\Bigg\{ \frac{2}{\epsilon}\left[\left(\frac{\mu^2}{-s_{12}}\right)^{\epsilon} + \left(\frac{\mu^2}{-s_{23}}\right)^{\epsilon} - \left(\frac{\mu^2}{-s_{45}}\right)^{\epsilon}\right] \\ - 2\mathrm{Li}_2\left(1-\frac{s_{45}}{s_{12}}\right) - 2\mathrm{Li}_2\left(1-\frac{s_{34}}{s_{23}}\right) - \ln^2\left(\frac{s_{12}}{s_{23}}\right)\Bigg\}. \end{aligned} \tag{3.5.20}$$

The the infrared divergent part of the amplitude is a sum over two particle terms times a polynomial in the anti-holomorphic spinors[9] $P(\widetilde{\lambda}_i)$. The term

$$\frac{2}{\epsilon}\left(\frac{\mu^2}{-s_{mn}}\right)^{\epsilon} P_{mn}(\widetilde{\lambda}_i) \tag{3.5.21}$$

is supported on a pair of skew lines with a multiple derivative delta function behavior in the normal directions. The particles m,n lie on one line and the remaining particles are on the other line.

We expect the finite part of the scattering amplitude to be supported on the same configuration as the Yang-Mills amplitude, but with a higher derivative delta function behavior in normal directions. We recall that in Yang-Mills theory, after taking into account holomorphic anomaly, the finite part of the one-loop MHV amplitude was supported on a pair of intersecting $\mathbb{CP}^1$'s. To test this conjecture in

[9] The IR terms are also multiplied by rational functions of holomorphic spinors λ_i, which do not affect the twistor structure.

the gravity case, we verified that the graviton amplitude localizes to a higher order neighborhood of a $\mathbb{CP}^2$, as the fifth power of the coplanar operator annihilates it

$$K^5 M_5^{1-loop} = 0. \tag{3.5.22}$$

Each of the permutations in (3.5.18) is annihilated by K^5 separately.

One Loop $+ + \ldots +$ Graviton Amplitude

The only known infinite series of one loop scattering amplitudes in General Relativity are the amplitudes of n gravitons with the same helicity. It has been computed both from KLT relations and from the soft and collinear properties of the amplitude [63]. Since the tree-level amplitude of the same helicity gravitons is zero, the one-loop amplitude is a rational function of the spinor variables.

The interesting feature of this series of amplitudes is that, just like the gluon amplitudes of the same helicity structure and the MHV graviton amplitudes, they are a product of a rational function in λ_i times polynomial in $\widetilde{\lambda}$. By the same reasoning as in the MHV case, the amplitude is supported in a higher order neighborhood of a $\mathbb{CP}^1$.

3.A The Integral Functions

The box function $F^{2\mathrm{m}\ e}_{n:r;i}$ is one of a set of functions constructed from the scalar box integrals. The latter form a complete list of the possible integrals that can appear in a Feynman diagrammatic computation of one-loop amplitudes in $\mathcal{N} = 4$ gauge theory. In one-loop amplitudes with reduced supersymmetry, the triangle and bubble functions can appear as well.

These integrals are known as the scalar box integrals because they would arise in a one-loop computation of a scalar field theory with four internal propagators.

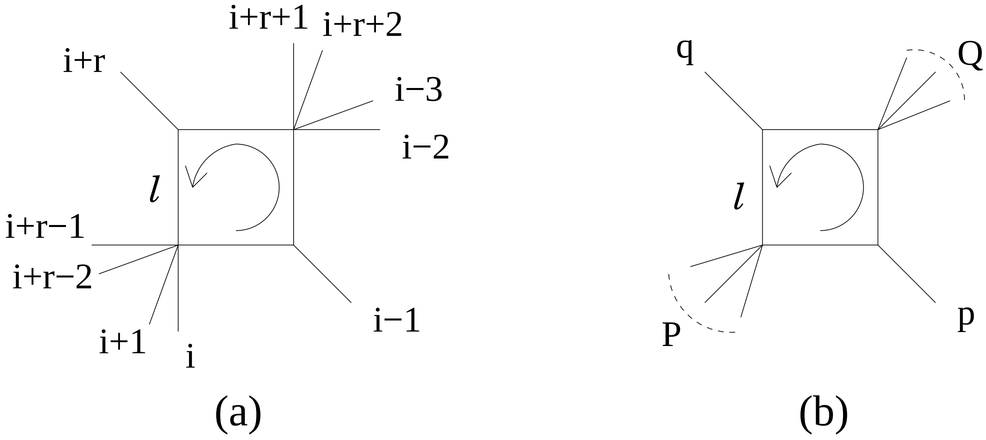

Fig. 24: Scalar Box Integrals used in the definition of: (a) The box function $F^{2\text{m e}}_{n:r;i}$. (b) The generic box function $F(p,q,P)$.

The scalar box integral is defined as follows:

$$I_4 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell-K_1)^2(\ell-K_1-K_2)^2(\ell+K_4)^2}. \tag{3.A.1}$$

The incoming external momenta at each of the vertices are K_1, K_2, K_3, K_4. The labels are given in consecutive order following the loop. Momentum conservation implies that $K_1+K_2+K_3+K_4=0$ and this is why (3.A.1) only depends on three momenta. We are interested in the case when $K_1 = p_{i-1}$, $K_2 = p_i + \ldots + p_{i+r-1}$ and $K_3 = p_{i+r}$. The scalar box function is then defined as follows,

$$F^{2me}_{n:r;i} = \left(t^{[r+1]}_{i-1} t^{[r+1]}_{i} - t^{[r]}_{i} t^{[n-r-2]}_{i+r+1} \right) I^{2me}_{4:r;i} \tag{3.A.2}$$

We also use the finite scalar function

$$F(t^{[r+1]}_{i-1}, t^{[r+1]}_{i}; t^{[r]}_{i}, t^{[n-r-2]}_{i+r+1})_{finite} = \frac{1}{2r_\Gamma}(t^{[r+1]}_{i-1} t^{[r+1]}_{i} - t^{[r]}_{i} t^{[n-r-2]}_{i+r+1}) I^{2me}_{4;r;i} + \frac{1}{\epsilon^2}((-t^{[r]}_{i})^{-\epsilon} + (-t^{[n-r-2]}_{i+r+1})^{-\epsilon} + (-t^{[r+1]}_{i-1})^{-\epsilon} + (-t^{[r+1]}_{i})^{-\epsilon}). \tag{3.A.3}$$

This can be expressed as

$$B = F(i,r)_{finite} = \mathrm{Li}_2\left(1 - \frac{t^{[r]}_{i}}{t^{[r+1]}_{i-1}}\right) + \mathrm{Li}_2\left(1 - \frac{t^{[r]}_{i}}{t^{[r+1]}_{i}}\right) + \mathrm{Li}_2\left(1 - \frac{t^{[n-r-2]}_{i+r+1}}{t^{[r+1]}_{i-1}}\right) + \mathrm{Li}_2\left(1 - \frac{t^{[n-r-2]}_{i+r+1}}{t^{[r+1]}_{i}}\right) - \mathrm{Li}_2\left(1 - \frac{t^{[r]}_{i} t^{[n-r-2]}_{i+r+1}}{t^{[r+1]}_{i-1} t^{[r+1]}_{i}}\right) + \frac{1}{2}\ln^2\left(\frac{t^{[r+1]}_{i-1}}{t^{[r+1]}_{i}}\right). \tag{3.A.4}$$

The divergent part of the $\mathcal{N}=1$ amplitude (3.3.2) is expressed in terms of scalar bubble function

$$K_0(s) = \frac{1}{\epsilon(1-2\epsilon)}(-s)^{-\epsilon} = \frac{1}{\epsilon} + 2 - \ln(-s) + \mathcal{O}(\epsilon). \tag{3.A.5}$$

The finite part of the $\mathcal{N}=1$ chiral amplitude is a sum of scalar box and triangle integrals. The triangle integral is defined as follows:

$$I_3 = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon}\,\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell-K_1)^2(\ell+K^3)^2}. \tag{3.A.6}$$

The incoming momenta at each vertex are K_1, K_2, K_3. The labels are given in consecutive order following the loop. The incoming momenta satisfy momentum conservation $K_1 + K_2 + K_3 = 0$. In the integral (3.A.6), we set $K_1 = p_{i-1}$, $K_2 = p_i + p_{i+1} + \ldots + p_{i+r-1}$ and $K_3 = p_{i+r} + p_{i+r+1} + \ldots + p_{i-2}$

$$I^{2m}_{3:r,i} = \frac{r_\Gamma}{\epsilon} \frac{(-t_i^{[r]})^{-\epsilon} - (-t_{i+r}^{[n-r-1]})^{-\epsilon}}{(-t_i^{[r]}) - (-t_{i+r}^{[n-r-1]})}. \tag{3.A.7}$$

The triangle function T (3.4.13) enters the amplitudes as

$$\frac{L_0\left(t_i^{[r]}/t_i^{[r+1]}\right)}{t_i^{[r+1]}} = \frac{\ln(t_i^{[r]}) - \ln(t_i^{[r+1]})}{t_i^{[r]} - t_i^{[r+1]}} = \frac{1}{r_\Gamma} I^{2m}_{3:r,i}[a_2]. \tag{3.A.8}$$

This is a Feynman parameter integral for a two mass triangle integral $I^{2m}_{3:r,i}$ where $t_i^{[r]}$ and $t_i^{[r+1]}$ are squares of the momenta of the massive legs and a_2 is the Feynman parameter for the light-like leg. This representation arises when one carries out the calculation of the $\mathcal{N}=1$ chiral multiplet amplitude in a manner analogous to the $\mathcal{N}=4$ calculation.

3.B KLT Relations

The Kaway, Lewellen and Tye (KLT) relations of string theory [61] relate closed string amplitudes to the open string amplitudes. They arise from representing each closed string vertex operator C as a product of two open string vertex operators O

$$C(z_i, \overline{z}_i) = O(z_i)\overline{O}(\overline{z}_i). \tag{3.B.1}$$

In the infinite tension limit of the closed strings, only the massless gravity states survive.

In the infinite tension limit, the closed string states reduce to the states of $\mathcal{N} = 8$ supergravity and the open string states reduce to the states of the $\mathcal{N} = 4$ Yang-Mills theory. The particle content of $\mathcal{N} = 8$ supergravity is 1 graviton, 8 gravitinos, 28 vectors, 56 Majorana spinors and 70 scalars. As a consequence of the factorization of the closed string vertex operator into the product of two open string vertex operators, the $\mathcal{N} = 8$ supergravity multiplet can be thought of as a tensor product of two $\mathcal{N} = 4$ gauge theory multiplets. The infinite tension limit of the KLT relations relates the $\mathcal{N} = 8$ gravity amplitudes to $\mathcal{N} = 4$ gauge theory amplitudes [60]. The tree level four particle KLT relations are

$$\mathcal{A}(1,2,3,4) = -is_{12}A(1,2,3,4)A(1,2,4,3). \tag{3.B.2}$$

Here, $\mathcal{A}$ is the gravity amplitude and A is the color ordered gauge theory amplitude. Each of the gravity states on the left hand side is a product of two gauge theory states on the right hand side. At tree level, supersymmetry does not affect scattering amplitudes, whence the KLT relations hold for tree level scattering amplitudes in theories with reduced or no supersymmetry. In the past, KLT relations have been the main computational tool used to derive gravity scattering amplitudes.

4. Tree Level Recursion Relations For Gravity Amplitudes

4.1 Introduction

The twistor string has inspired a lot renewed progress in understanding the tree-level and one-loop gluon scattering amplitudes in Yang-Mills theory. Among other things, a new set of recursion relations for computing tree-level amplitudes of gluons have been recently introduced in [7]. A proof of the recursion relations was given in [16]. A straightforward application of these recursion relations gives new and simple forms for many amplitudes. Many of these have been obtained recently using somewhat related methods [64,65,66].

It has been known that tree level graviton amplitudes have remarkable simplicity that cannot be expected from textbook recipes for computing them. The tree level n graviton amplitudes vanish if more than $n-2$ gravitons have the same helicity. The maximally helicity violating (MHV) amplitudes are thus, as in Yang-Mills case, those with $n-2$ gravitons of one helicity and two of the opposite helicity. These have been computed by Berends, Giele, and Kuijf (BGK) [14] from the Kawai, Lewellen and Tye (KLT) relations [13]. The four particle case was first computed by DeWitt [30].

The simplicity of amplitudes raises the question whether there are analogous recursion relations for amplitudes of gravitons. The possibility of such recursion relations has been recently raised in [59].

In this chapter, we propose tree-level recursion relations for amplitudes of gravitons. The recursion relations can be schematically written as follows

$$A_n = \sum_{\mathcal{I},h} A_{\mathcal{I}}^{h} \frac{1}{P_{\mathcal{I}}^2} A_{\mathcal{J}}^{-h}. \qquad (4.1.1)$$

In writing a recursion relation for n graviton amplitude A_n, one marks two gravitons and sums over products of subamplitudes with external gravitons partitioned into sets $\mathcal{I} \cup \mathcal{J} = (1, 2, \ldots, n)$ among the two subamplitudes so that $i \in \mathcal{I}$ and $j \in \mathcal{J}$. $P_{\mathcal{I}}$ is the sum of the momenta of gravitons in the set $\mathcal{I}$ and h is the helicity of the internal graviton. The momenta of the internal and the marked gravitons are shifted so that they are on-shell.

We use the recursion relations to derive new compact formulas for all amplitudes up to six gravitons. In particular, we give the first published result for the six graviton non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

We attempt to prove the recursion relations along the lines of [16]. The first part of the proof that rests on basic facts about tree-level diagrams, such as the fact that their singularities come only from the poles of the internal propagators can be easily adapted to the gravity case. To have a complete proof of the recursion relations, it is necessary to prove that certain auxiliary function $A(z)$ constructed from the scattering amplitude vanishes as $z \to \infty$.

We are able to prove this fact from the KLT relations for all amplitudes up to eight gravitons. For amplitudes with nine or more gravitons, the KLT relations suggest that the function $A(z)$ does not vanish at infinity unless there is an unexpected cancellation between different terms in the KLT relations.

While we are not able to prove that $A(z)$ vanishes at infinity for a general n graviton scattering amplitude, we show that $A(z)$ does vanish at infinity for MHV amplitudes with arbitrary number of gravitons from the BGK formula. Hence, the recursion relations are valid for all MHV amplitudes contrary to the expectation from KLT relations.

Finally, we introduce an auxiliary set of recursion relations for NMHV amplitudes which are easier to prove but give more complicated results for the amplitudes. This auxiliary recursion relation is then used to prove the vanishing of $A(z)$ for any NMHV amplitudes.

This raises the hope, that the recursion relations hold for other scattering amplitudes of gravitons as well.

Summary of Results

In section 4.2 we present the BCF relations to the case of gravity. In section 4.3, we discuss explicit examples of computations of graviton scattering amplitudes using our recursion relations. We derive formulas for all amplitudes up to six gravitons. In section 4.4 we derive the graviton recursion relations and in section 4.5 we study the large z behavior of $A(z)$ using several tools, including Feynman diagrams, KLT relations and 'auxiliary recursion relations.' In the appendix, we give a refined version of the KLT relations approach to vanishing of $A(z)$.

4.2 Recursion Relations

Just like gauge theory scattering amplitudes, the graviton scattering amplitudes are efficiently written in terms of spinor-helicity formalism. The polarization tensors of the gravitons can be expressed in terms of gluon polarization vectors

$$\epsilon^{+}_{a\dot{a},b\dot{b}} = \epsilon^{+}_{a\dot{a}}\epsilon^{+}_{b\dot{b}} \qquad \epsilon^{-}_{a\dot{a},b\dot{b}} = \epsilon^{-}_{a\dot{a}}\epsilon^{-}_{b\dot{b}}. \tag{4.2.1}$$

The polarization vectors of positive and negative helicity gluons are respectively

$$\epsilon^{-}_{a\dot{a}} = \frac{\lambda_a \widetilde{\mu}_{\dot{a}}}{[\widetilde{\lambda}, \widetilde{\mu}]} \qquad \epsilon^{+}_{a\dot{a}} = \frac{\mu_a \widetilde{\lambda}_{\dot{a}}}{\langle \mu, \lambda \rangle}, \tag{4.2.2}$$

where μ and $\widetilde{\mu}$ are fixed reference spinors.

Consider a tree level graviton scattering amplitude $A(1, 2, \dots, n)$. The amplitude is invariant under any permutations of the gravitons because there is no color ordering.

To write down the recursion relations, we single out two gravitons. Without loss of generality, we call these gravitons i and j. Define the shifted momenta $p_i(z)$ and $p_j(z)$, where z is a complex parameter, to be

$$p_i(z) = \lambda_i(\widetilde{\lambda}_i + z\widetilde{\lambda}_j) \qquad p_j(z) = (\lambda_j - z\lambda_i)\widetilde{\lambda}_j. \tag{4.2.3}$$

Note that $p_i(z)$ and $p_j(z)$ are on-shell for all z and that $p_i(z) + p_j(z) = p_i + p_j$. Hence, the following function

$$A(z) = A(p_1, \dots, p_i(z), \dots, p_j(z), \dots, n) \tag{4.2.4}$$

is a physical on-shell scattering amplitude for all values of z.

Consider the partitions of the gravitons $(1, 2, \ldots, i, \ldots, j, \ldots, n) = \mathcal{I} \cup \mathcal{J}$ into two groups such that $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Then the recursion relation for a tree-level graviton amplitude is

$$A(z) = \sum_{\mathcal{I},\mathcal{J}} \sum_h A_L(\mathcal{I}, -P^h_{\mathcal{I}}(z_{\mathcal{I}}), z_{\mathcal{I}}) \frac{1}{P^2_{\mathcal{I}}(z)} A_R(\mathcal{J}, P^{-h}_{\mathcal{I}}(z_{\mathcal{I}}), z_{\mathcal{I}}), \tag{4.2.5}$$

where

$$\begin{aligned} P_{\mathcal{I}}(z) &= \sum_{k \in \mathcal{I}, k \neq i} p_k + p_i(z) \\ z_{\mathcal{I}} &= \frac{P^2_{\mathcal{I}}}{\langle i|P_{\mathcal{I}}|j]}. \end{aligned} \tag{4.2.6}$$

The sum in (4.2.5) is over the partitions of gravitons and over the helicities of the intermediate gravitons. The physical amplitude is obtained by taking z in equation (4.2.5) to be zero

$$A(1, 2, \ldots, n) = A(0). \tag{4.2.7}$$

We will give evidence below that the recursion relation is valid for gravitons i and j of helicity $(+,+), (-,-)$ and $(-,+)$ respectively.

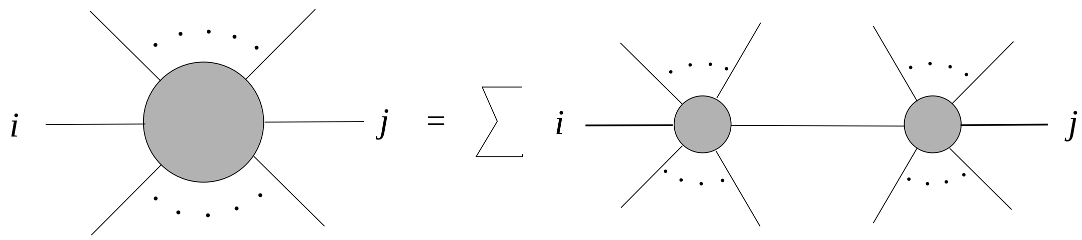

Fig. 25: This is a schematic representation of the recursion relations (4.2.5). The thick lines represent the reference gravitons. The sum here is over all partitions of the gravitons into two groups with at least two gravitons on each subamplitude and over the two choices of the helicity of the internal graviton.

4.3 Explicit Examples

In this section, we compute all tree-level amplitudes up to six gravitons to illustrate the use of the recursion relations (4.2.5).

Consider first the four-graviton MHV amplitude $A(1^-, 2^-, 3^+, 4^+)$. The amplitude is invariant under arbitrary permutations of external gravitons so the order of gravitons does not matter. Hence, this is the only independent four graviton amplitude. In contrast, in gauge theory, there are two independent amplitudes $A_{YM}(1^-, 2^-, 3^+, 4^+)$ and $A_{YM}(1^-, 3^+, 2^-, 4^+)$ because the Yang-Mills scattering amplitudes are color ordered.

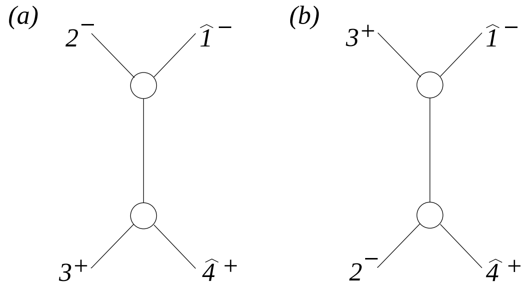

Fig. 26: Two configurations contributing to the four graviton amplitude $A(1^-, 2^-, 3^+, 4^+)$. Notice that the diagrams are related by the interchange $2 \leftrightarrow 3$.

We single out gravitons 1^- and $4+$. Then, there are two possible configurations contributing to the recursion relations (4.2.5), see fig. 26. We refer to the configuration from fig. 2(a) as $(2, \hat{1}|\hat{4}, 3)$ and from fig. 2(b) as $(3, \hat{1}|\hat{4}, 3)$. To evaluate the diagrams we use the known form of three graviton scattering amplitudes

$$A(1^-, 2^-, 3^+) = \frac{\langle 12\rangle^6}{\langle 23\rangle^2\langle 31\rangle^2}, \qquad A(1^+, 2^+, 3^-) = \frac{[12]^6}{[23]^2[31]^2}. \tag{4.3.1}$$

The sum of the two contributions from fig. 2 is

$$A(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12\rangle^5[34]^2}{[12]\langle 23\rangle^2\langle 14\rangle^2} + \frac{\langle 12\rangle^8[24]^2}{\langle 13\rangle^3[13]\langle 23\rangle^2\langle 14\rangle^2}. \tag{4.3.2}$$

A short calculation shows that (4.3.2) equals to the known result [14] obtained from KLT relations

$$A(1^-, 2^-, 3^+, 4^+) = \frac{\langle 12\rangle^8[12]}{\langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle^2}. \tag{4.3.3}$$

We picked the reference gravitons to have opposite helicity because this leads to most compact expressions for graviton scattering amplitudes. We could have chosen reference gravitons of the same helicity, ie. 1^- and 2^-. This leads to a longer expression because there are more diagrams contributing to the scattering amplitude. In the rest of the chapter, we will always choose reference gravitons of opposite helicity. The actual choice of reference gravitons does not matter, because the amplitude is invariant under permutations that preserve the sets of positive and negative helicity gravitons. All choices lead to the same answer up to relabelling of the gravitons.

The next amplitude to consider is the five graviton MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+)$. Just as in the four graviton example, this is the only independent five graviton amplitude. All other five graviton amplitudes are related to it by permutation and/or conjugation symmetry.

The amplitude has contribution from three diagrams $(1, 4, \widehat{2}|\widehat{3}, 5), (1, 5, \widehat{2}|\widehat{3}, 4), (4, 5, \widehat{2}|\widehat{3}, 1)$. These contributions give the following three terms

$$A(1^-, 2^-, 3^+, 4^+, 5^+) = \frac{\langle 12\rangle^7}{\langle 14\rangle\langle 15\rangle\langle 23\rangle^2\langle 45\rangle}\left(\frac{[14][35]}{\langle 24\rangle\langle 35\rangle} - \frac{[15][34]}{\langle 25\rangle\langle 34\rangle} - \frac{\langle 12\rangle[13][45]}{\langle 13\rangle\langle 24\rangle\langle 25\rangle}\right). \tag{4.3.4}$$

This expression agrees with the BGK result [14]

$$A(1^-, 2^-, 3^+, 4^+, 5^+) = \langle 12\rangle^7 \frac{([12]\langle 23\rangle[34]\langle 41\rangle - \langle 12\rangle[23]\langle 34\rangle[41])}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 23\rangle\langle 24\rangle\langle 25\rangle\langle 34\rangle\langle 35\rangle\langle 45\rangle}. \tag{4.3.5}$$

At six gravitons, there are two independent scattering amplitudes, the MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ and the first non-MHV amplitude $A(1^-, 2^-, 3^-, 4^+, 5^+, 6^+)$.

The MHV amplitude $A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+)$ has contribution from four configurations, $(4, \widehat{3}|\widehat{2}, 1, 5, 6), (5, \widehat{3}|\widehat{2}, 1, 4, 6), (6, \widehat{3}|\widehat{2}, 1, 4, 5)$ and $(1, \widehat{3}|\widehat{2}, 4, 5, 6)$. Notice that the first three diagrams are related by interchange of $4, 5, 6$ gravitons, so there are only two diagrams to compute.

The first configuration $(4, \widehat{3}|\widehat{2}, 1, 5, 6)$ evaluates to

$$D_1 = \langle 12\rangle^7[34]\frac{\langle 2|3+4|5]\langle 4|2+3|1]\langle 51\rangle - \langle 12\rangle p_{234}^2\langle 45\rangle[51]}{\langle 14\rangle\langle 15\rangle\langle 16\rangle\langle 23\rangle^2\langle 25\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}. \tag{4.3.6}$$

The last configuration $(1, \widehat{3}|\widehat{2}, 4, 5, 6)$ gives

$$D_2 = \langle 12\rangle^8[13]\frac{\langle 14\rangle[45]\langle 52\rangle p_{123}^2 - \langle 45\rangle\langle 2|1+3|4]\langle 1|2+3|5]}{\langle 13\rangle\langle 14\rangle\langle 15\rangle\langle 16\rangle\langle 23\rangle^2\langle 24\rangle\langle 25\rangle\langle 26\rangle\langle 45\rangle\langle 46\rangle\langle 56\rangle}. \tag{4.3.7}$$

Adding all four contributions, we get

$$A(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = D_1 + D_1(4 \leftrightarrow 5) + D_1(4 \leftrightarrow 6) + D_2. \tag{4.3.8}$$

(4.3.8) agrees with the known result for the six graviton MHV amplitude.

The non-MHV amplitude $A(1^-,2^-,3^-,4^+,5^+,6^+)$ has contribution from six classes of diagrams $D_1 = (2,\widehat{3}|\widehat{4},5,6,1) + (1 \leftrightarrow 2), D_2 = (1,6,\widehat{3}|\widehat{4},2,5) + (1 \leftrightarrow 2) + (5 \leftrightarrow 6) + (1 \leftrightarrow 2, 5 \leftrightarrow 6), D_3 = (2,5,6,\widehat{3}|\widehat{4},1) + (1 \leftrightarrow 2), D_4 = \overline{D}_3^{flip}, D_5 = \overline{D}_1^{flip}$ and $D_6 = (5,6,\widehat{3}|\widehat{4},1,2)$. The 'conjugate flip' $\overline{D}^{flip}$ exchanges the spinor products $\langle\rangle \leftrightarrow []$ and the labels $i \leftrightarrow 7-i$.

The first class of diagrams $D_1 : (2,\widehat{3}|\widehat{4},5,6,1) + (1 \leftrightarrow 2)$ evaluates to

$$D_1 = \frac{\langle 23\rangle\langle 1|2+3|4]^7\big(\langle 1|2+3|4]\langle 5|3+4|2][51] + [12][45]\langle 51\rangle p_{234}^2\big)}{\langle 15\rangle\langle 16\rangle[23][34]^2\langle 56\rangle p_{234}^2\langle 1|3+4|2]\langle 5|3+4|2]\langle 5|2+3|4]\langle 6|3+4|2]\langle 6|2+3|4]} + (1 \leftrightarrow 2). \tag{4.3.9}$$

The second group, $D_2 : (1,6,\widehat{3}|\widehat{4},2,5) +$ permutations, gives

$$D_2 = -\frac{\langle 13\rangle^7\langle 25\rangle[45]^7[16]}{\langle 16\rangle[24][25]\langle 36\rangle p_{245}^2\langle 1|2+5|4]\langle 6|2+5|4]\langle 3|1+6|5]\langle 3|1+6|2]} + (1 \leftrightarrow 2) + (5 \leftrightarrow 6) + (1 \leftrightarrow 2, 5 \leftrightarrow 6). \tag{4.3.10}$$

The third class $D_3 : (2,5,6,\widehat{3}|\widehat{4},1) + (1 \leftrightarrow 2)$ is

$$D_3 = \frac{\langle 13\rangle^8[14][56]^7\big(\langle 23\rangle\langle 56\rangle[62]\langle 1|3+4|5] + \langle 35\rangle[56]\langle 62\rangle\langle 1|3+4|2]\big)}{\langle 14\rangle[25][26]\langle 34\rangle^2 p_{134}^2\langle 1|3+4|2]\langle 1|3+4|5]\langle 1|3+4|6]\langle 3|1+4|2]\langle 3|1+4|5]\langle 3|1+4|6]} + (1 \leftrightarrow 2). \tag{4.3.11}$$

The fourth and fifth group are related by conjugate flip to the third and first group respectively. The last group to evaluate consists of a single diagram $D_6 : (5,6,\widehat{3}|\widehat{4},1,2)$

$$D_6 = \frac{\langle 12\rangle[56]\langle 3|1+2|4]^8}{[21][14][24]\langle 35\rangle\langle 36\rangle\langle 56\rangle p_{124}^2\langle 5|1+2|4]\langle 6|1+2|4]\langle 3|5+6|1]\langle 3|5+6|2]}. \tag{4.3.12}$$

Adding the pieces together, the six graviton non-MHV amplitude reads

$$A(1^-,2^-,3^-,4^+,5^+,6^+) = D_1 + \overline{D}_1^{flip} + D_2 + D_3 + \overline{D}_3^{flip} + D_6. \tag{4.3.13}$$

4.4 Derivation of the Recursion Relations

The derivation of the tree-level recursion relations (4.2.5) goes, with few modifications, along the same lines as the derivation of the tree-level recursion relations for scattering amplitudes of gluons [16], so we will be brief.

We start with the scattering amplitude $A(z)$ defined at shifted momenta, see (4.2.4) and (4.2.3). $A(z)$ is a rational function of z because the z dependence enters the scattering amplitude only via the shifts $\widetilde{\lambda}_i \to \widetilde{\lambda}_i + z\widetilde{\lambda}_j$ and $\lambda_j \to \lambda_j - z\lambda_i$ and because the original tree-level scattering amplitude is a rational function of the spinors.

Actually, for generic momenta, $A(z)$ has only single poles in z. These come from the singularities of the propagators in Feynman diagrams. To see this, recall that for tree level amplitudes, the momentum through a propagator is always a sum of momenta of external particles $P_{\mathcal{I}} = p_{i_1} + p_{i_2} + \ldots + p_{i_l}$, where $\mathcal{I}$ is a group of *not* necessarily adjacent gravitons. At nonzero z, the momentum becomes $P_{\mathcal{I}}(z) = p_{i_1}(z) + p_{i_2}(z) + \ldots + p_{i_l}(z)$. Here, $p_k(z)$ is independent of z for $k \neq i, j$ and $p_i(z) + p_j(z)$ is independent of z. Hence, $P_{\mathcal{I}}(z)$ is independent of z if both i and j are in $\mathcal{I}$ or if neither of them is in $\mathcal{I}$. In the remaining case, one of i and j is in the group $\mathcal{I}$ and the other is not. Without loss of generality, we take $i \in \mathcal{I}$. Then $P_{\mathcal{I}}(z) = P_{\mathcal{I}} + z\lambda_i\widetilde{\lambda}_j$ and $P_{\mathcal{I}}^2(z) = P_{\mathcal{I}}^2 - z\langle i|P_{\mathcal{I}}|j]$. Clearly, the propagator $1/P_{\mathcal{I}}(z)^2$ has a simple pole for

$$z_{\mathcal{I}} = \frac{P_{\mathcal{I}}^2}{\langle i|P_{\mathcal{I}}|j]}. \tag{4.4.1}$$

For generic momenta, $P_{\mathcal{I}}$'s are distinct for distinct groups $\mathcal{I}$, hence the $z_{\mathcal{I}}$'s are distinct. So all singularities of $A(z)$ are simple poles.

To continue the argument, we need to assume that $A(z)$ vanishes as $z \to \infty$. In the next section we will argue that the tree level graviton amplitudes obey this criterium. A rational function $A(z)$ that has only simple poles and vanishes at infinity can be expressed as

$$A(z) = \sum_{\mathcal{I}} \frac{\operatorname{Res} A(z_{\mathcal{I}})}{z - z_{\mathcal{I}}}, \tag{4.4.2}$$

where $\operatorname{Res}A(z_{\mathcal{I}})$ are the residues of $A(z)$ at the simple poles $z_{\mathcal{I}}$. The physical scattering amplitude is simply $A(0)$

$$A = -\sum_{\mathcal{I}} \frac{\operatorname{Res} A(z_{\mathcal{I}})}{z_{\mathcal{I}}}. \tag{4.4.3}$$

It follows from the above discussion that the sum is over $\mathcal{I}$ such that i is in $\mathcal{I}$ while j is not.

The residue Res $A(z_\mathcal{I})$ has contribution from Feynman diagrams which contain the propagator $1/P^2_\mathcal{I}$. The propagator divides the tree diagram into "left" part containing gravitons in $\mathcal{I}$ and "right" part containing gravitons in $\mathcal{J} = (1, 2, \ldots, n) - \mathcal{I}$. For $z \to z_\mathcal{I}$, the propagator with momentum $P_\mathcal{I}$ goes on-shell and the left and right part of the diagram approach tree-level diagrams for on-shell amplitudes. The contribution of these diagrams to the pole is

$$\sum_h A^h_L(z_\mathcal{I}) \frac{1}{P^2_\mathcal{I}(z)} A^{-h}_R(z_\mathcal{I}), \tag{4.4.4}$$

where the sum is over the helicity $h = \pm$ of the intermediate graviton. This gives the recursion relation (4.2.5).

4.5 Large z Behavior of Gravity Amplitudes

To complete the proof, it remains to show that the amplitude $A(z)$ goes to zero as z approaches infinity. We were able to obtain only partial results in this direction, which we now discuss.

4.5.1 Vanishing of the MHV Amplitudes

Let us firstly consider the large z behavior of the n graviton MHV amplitude [14]

$$A(1^-, 2^-, 3^+, \ldots, n) = \langle 12\rangle^8 \Big\{ \frac{[23]\langle n|P_{2,3}|4]\langle n|P_{2,4}|5] \ldots \langle n|P_{2,n-2}|n-1]}{\langle 12\rangle\langle 23\rangle \ldots \langle n-2, n-1\rangle\langle n-1, 1\rangle\langle 1n\rangle^2\langle 2n\rangle\langle 3n\rangle \ldots \langle n-1, n\rangle} + \text{permutations of } (3, 4, \ldots, n-1)\Big\}, \tag{4.5.1}$$

where $P_{i,j} = \sum_{k=i}^{j} p_k$. The formula is valid for $n \geq 5$. It follows from supersymmetric Ward identities that the expression in the bracket is totally symmetric, although this is not manifest.

The terms in the curly brackets are completely symmetric so they contribute the same power of z independently of i and j. To find the contribution of the terms in the brackets, we pick a convenient value $(i, j) = (1, n)$ for the reference gravitons. Recall that $\widetilde{\lambda}_i(z)$ and $\lambda_j(z)$ are linear in z while λ_i and $\widetilde{\lambda}_j$ do not depend on z. It follows that the numerator of each term in the brackets (4.5.1) goes like z^{n-4} and the denominator gives a factor of z^{n-2}. Hence, the terms in the brackets give a factor of $1/z^2$. This factor is the same for all choices of reference momenta by the complete symmetry of the terms in the brackets. For the helicity configurations $(h_i, h_j) = (-,+), (+,+)$ and $(-,-)$, the factor $\langle 12 \rangle^8$ does not contribute, so the amplitude vanishes at infinity as $A_{MHV} \sim 1/z^2$.

A recent paper [67] relates the MHV amplitudes to current correlators on curves in twistor space. This raises the possibility of a twistor string description of perturbative $\mathcal{N} = 8$ supergravity [11]. In the gauge theory case, the twistor string leads to an MHV diagrams construction for the tree level scattering amplitudes [1]. One computes the tree-level amplitudes from tree-level Feynman diagrams in which the vertices are MHV amplitudes, continued off-shell in a suitable manner, and the propagators are ordinary Feynman propagators.

The vanishing of the gluon scattering amplitude $A(z)$ at infinity follows very easily from the vanishing of the MHV diagrams via the MHV diagrams construction [16]. We would like to speculate, that it might be possible to prove the vanishing of graviton scattering amplitude $A(z)$ along the same lines using the hypothetical MHV diagrams construction.

4.5.2 Analysis of the Feynman Diagrams

In this section we study the large z behavior of Feynman diagrams contributing to $A(z)$ following [16].

Recall that any Feynman diagram contributing to $A(z)$ is linear in the polarization tensors $\epsilon_{a\dot{a},b\dot{b}}$ of the external gravitons. The polarization tensors of all but the i^{th} and j^{th} graviton are independent of z. To find the z dependence of the polarization tensors of the reference gravitons, recall that $\widetilde{\lambda}_i(z), \lambda_j(z)$ are linear in z and $\lambda_i, \widetilde{\lambda}_j$ do not depend on z. It follows from (4.2.1) and (4.2.2) that the polarization tensors of the reference gravitons give a factor of $z^{\pm 2}$ depending on their helicities. Hence, the polarization tensors can suppress $A(z)$ by at most a factor of z^4.

The remaining pieces in Feynman diagrams are constructed from vertices and propagators that connect them. Perturbative gravity has infinite number of vertices coming from the expansion of the Einstein-Hilbert Lagrangian

$$\mathcal{L} = -\sqrt{-g}\, R \tag{4.5.2}$$

around the flat vacuum $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The graviton vertices have two powers of momenta coming from the two derivatives in the Ricci scalar.

The z dependence in a tree level diagram "flows" along a unique path of Feynman propagators from the i^{th} to the j^{th} graviton. In a path composed of k propagators, there are $k+1$ vertices. Each propagator contributes a factor of $1/z$ and each vertex contributes a factor of z^2. Altogether, the propagators and vertices give a factor of z^{k+2}.

The product of polarization tensors vanishes at best as $1/z^4$, so the contribution of individual Feynman diagrams to $A(z)$ seems to grow at infinity as z^{k-2}, where k is the number of propagators from the i^{th} to the j^{th} graviton. Clearly, in a generic Feynman diagram, this number grows with the number of external gravitons. So this analysis suggests that $A(z)$ grows at infinity with a power of z that grows as we increase the number of external gravitons.

This is in contrast to the above analysis of MHV amplitude that vanishes at infinity as $1/z^2$. The vanishing of $A(z)$ at infinity depends on unexpected cancellation between Feynman diagrams.

4.5.3 KLT Relations and the Vanishing of Gluon Amplitudes

A different line of attack is to express the graviton scattering amplitudes via the KLT relations in terms of the gluon scattering amplitudes. One then infers behavior of $A(z)$ at infinity from the known behavior [16] of the gauge theory amplitudes. The KLT relations have been used in past to show that $\mathcal{N}=8$ supergravity amplitudes [68,69] have better than expected [70,71] ultraviolet behavior so we expect that the KLT relations give us a better bound on $A(z)$ than the analysis of Feynman diagrams. Indeed, we will use them to prove the vanishing of $A(z)$ at infinity up to six gravitons and up to eight gravitons in the appendix 4.A.

As discussed in previous chapter, the tree level KLT relations up to six gravitons are

$$\begin{aligned}
A(1,2,3) &= \mathcal{A}(1,2,3)^2 \\
A(1,2,3,4) &= s_{12}\mathcal{A}(1,2,3,4)\mathcal{A}(1,2,4,3) \\
A(1,2,3,4,5) &= s_{12}s_{34}\mathcal{A}(1,2,3,4,5)\mathcal{A}(2,1,4,3,5) + s_{13}s_{24}\mathcal{A}(1,3,2,4,5)\mathcal{A}(3,1,4,2,5) \\
A(1,2,3,4,5,6) &= s_{12}s_{45}\mathcal{A}(1,2,3,4,5,6)\big\{s_{35}\mathcal{A}(2,1,5,3,4,6) + (s_{34}+s_{35})\mathcal{A}(2,1,5,4,3,6)\big\} \\
&\quad + \text{permutations of } (234),
\end{aligned} \tag{4.5.3}$$

where $s_{ij} = (p_i + p_j)^2$. $A(1, 2, \ldots, n)$ is the n graviton scattering amplitude and $\mathcal{A}(i_1, i_2, \ldots, i_n)$ is the color ordered gauge theory amplitude. The KLT relations for any number of gravitons are written down in Appendix A of [72] and schematically in the appendix 4.A.

The KLT relations express an n graviton scattering amplitude as a sum of products of two gluon scattering amplitudes and $n-3$ s_{ij} invariants. The gluon scattering amplitudes vanish at infinity as $1/z$ or faster [16]. Hence, KLT relations imply the vanishing at infinity of the graviton amplitudes as long as the products of s_{ij}'s in (4.5.3) grow at most linearly with z.

For $n \leq 6$ gravitons, a quick glance at (4.5.3) shows that this is the case. We rename the gravitons so that the reference gravitons are 1 and n. The products of s_{ij}'s in (4.5.3) are independent of p_n and linear in p_1. Hence they give one power of z because $p_1(z)$ and $p_n(z)$ are linear in z and p_k for $k \neq 1, n$ is independent of z. It follows that $A(z)$ vanishes as $1/z$ or faster as $z \to \infty$ for less than seven gravitons.

For seven or more gravitons, an analysis of the general KLT relations shows that on the right hand side of KLT relations, there are always some products of $n-3$ s_{ij}'s that have more than one power of the reference momenta. The corresponding terms in the KLT relations are not expected to vanish at infinity. Hence, the function $A(z)$ does not vanish at infinity unless there is an unexpected cancellation between different terms in the KLT relations.

In the appendix we present a more careful study of KLT relations that reveals that $A(z)$ vanishes for $n \leq 8$.

4.5.4 Proof of Vanishing of $A(z)$ for NMHV Amplitudes

NMHV amplitudes are those with three negative helicity gravitons and any number of plus helicity gravitons, $A(p_1^-, p_2^-, p_3^-, p_4^+, \ldots, p_n^+)$. Consider the following function of z, $A_a(p_1^-(z), p_2^-, p_3^-, p_4^+(z), \ldots, p_n^+(z))$, where

$$p_1(z) = \lambda_1 \left(\widetilde{\lambda}_1 + z \sum_{i=4}^{n} \widetilde{\lambda}_i \right), \qquad p_k(z) = (\lambda_k - z\lambda_1)\widetilde{\lambda}_k \tag{4.5.4}$$

for $k = 4, \ldots, n$. The subscript a in $A_a(z)$ stands for auxiliary. The idea is to derive an new set of recursion relations for $A_a(z)$ which we use later on to prove that $A(z)$ vanishes at infinity.

In order to get the auxiliary recursion relations we start by proving from Feynman diagrams that $A_a(z)$ vanishes as $z \to \infty$. Note that $(n-2)$ polarization tensors depend on z and with the choice made in (4.5.4) all of them vanish as $1/z^2$. The most dangerous Feynman diagram is the one with the largest number of vertices. Such a diagram must only have cubic vertices. For n gravitons there are $n-2$ vertices. Each vertex contributes a factor of z^2. Altogether, the polarization tensors contribute a factor of $1/z^{2(n-2)}$ and the vertices contribute a factor of $z^{2(n-2)}$ which gives a constant for large z. Now we have to consider propagators. Each propagator that depends on z goes like $1/z$. Therefore, all we need is that in every diagram at least one propagator depends on z. From (4.5.4) it is easy to see that the only propagator that does not depend on z is $1/(p_2+p_3)^2$. A diagram with only this propagator has exactly two vertices and therefore our proof is complete for $n > 4$.

The shift in (4.5.4) can be thought of as iterating the shift introduced in [16].[10] Now we can follow the same steps as in section 4 to derive recursion relations based on the pole structure of $A_a(z)$.

We find

$$A_a(z) = \sum_{\mathcal{I}} \sum_h A_{\mathcal{I}}(z_{\mathcal{I}}, P^h_{\mathcal{I}}(z_{\mathcal{I}})) \frac{1}{P^2_{\mathcal{I}}(z)} A_{\mathcal{J}}(z_{\mathcal{I}}, -P^{-h}_{\mathcal{I}}(z_{\mathcal{I}})). \tag{4.5.5}$$

where the sum is over all possible sets of two or more gravitons $\mathcal{I} \neq \{2,3\}$, such that the graviton 1 is not in $\mathcal{I}$. Here, $\mathcal{J}$ is the complement of $\mathcal{I}$.

The main advantage of choosing the same negative helicity graviton in (4.5.4) to pair up with all plus helicity gravitons is that $P^2_{\mathcal{I}}(z)$ is a linear function of z. Therefore, the location of the poles $z_{\mathcal{I}}$ is easily computed to be of the form

$$z_{\mathcal{I}} = \frac{P^2_{\mathcal{I}}}{\sum_j \langle 1|P_{\mathcal{I}}|j]} \tag{4.5.6}$$

where the sum in j runs over all gravitons in $\mathcal{I}$ that depend on z.

Setting z to zero in (4.5.5) gives us a new representation of the original amplitude, i.e.,

$$A(p_1^-, p_2^-, p_3^-, p_4^+, \ldots, p_n^+) = \sum_{\mathcal{I}} \sum_h A_L(z_{\mathcal{I}}, P^h_{\mathcal{I}}(z_{\mathcal{I}})) \frac{1}{P^2_{\mathcal{I}}} A_R(z_{\mathcal{I}}, -P^{-h}_{\mathcal{I}}(z_{\mathcal{I}})). \tag{4.5.7}$$

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[10] This iteration procedure was used recently in [49] to find recursion relations for all plus one-loop amplitudes of gluons.

This is a new set of recursion relations for NMHV amplitudes. However, the expressions obtained from (4.5.7) are naturally more complicated than the ones obtained from the one introduced in section 2. Instead of computing amplitudes with (4.5.7), the idea is to use it to prove that $A(z)$ of section 2 vanishes for large z.

Consider $A(z)$ constructed from (4.5.7) by defining

$$p_1(z) = \lambda_1(\widetilde{\lambda}_1 + z\widetilde{\lambda}_4), \qquad p_4(z) = (\lambda_4 - z\lambda_1)\widetilde{\lambda}_4. \tag{4.5.8}$$

There are two different kind of terms in (4.5.7). One class consists of those where p_1 and p_4 are on the same side. This implies that neither $P_{\mathcal{I}}$ nor $z_{\mathcal{I}}$ depends on z. Therefore, the z dependence is confined into one of the amplitudes, say A_L. But this is an amplitude with less gravitons and by induction we assume that it vanishes for large z.

The second class of terms is more subtle. Since $p_1(z)$ and $p_4(z)$ are on different sides, both $P_{\mathcal{I}}$ and $z_{\mathcal{I}}$ become functions of z.

It turns out that $z_{\mathcal{I}}$ is a linear function of z. More explicitly[11],

$$z_{\mathcal{I}}(z) = \frac{P_{\mathcal{I}}^2 + z\langle 1|P_{\mathcal{I}}|4]}{\sum_j \langle 1|P_{\mathcal{I}}|j]}. \tag{4.5.9}$$

Recall that $P_{\mathcal{I}}$ denotes $P_{\mathcal{I}}(0)$.

Now we are left with $A_{\mathcal{I}}$ and $A_{\mathcal{J}}$ in (4.5.7), one with n_1+1 and the other with n_2+1 gravitons. Note that $n = n_1 + n_2$. In a Feynman diagram expansion of each of them we can single out the most dangerous diagrams and multiply them to get the most dangerous terms in (4.5.7). Each diagram contributes a factor of $z^{2(n_i-1)}$ from the vertices. Therefore we find $z^{2(n-2)}$. From the polarization tensors we find $z^{-2(n-2)}$, this comes from the z dependence of $z_{\mathcal{I}}$. The polarization tensors for the internal gluons give a factor of

$$\sum_h \epsilon^h_{\mu\nu}\epsilon^{-h}_{\lambda\rho} = d_{\mu\rho}d_{\nu\lambda} + d_{\mu\lambda}d_{\nu\rho} - d_{\mu\nu}d_{\rho\lambda}, \tag{4.5.10}$$

where

$$d_{\mu\nu} = \eta_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}. \tag{4.5.11}$$

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[11] Had we chosen a different negative helicity graviton in (4.5.8), we would have found that $z_{\mathcal{I}}$ becomes a rational function of z.

Here, $k = P_{\mathcal{I}}(z)$ is the momentum of the internal propagator and $n_{a\dot{a}} = \mu_a \widetilde{\mu}_{\dot{a}}$ is an auxiliary vector used in the definition (4.2.2). $n_{a\dot{a}}$ is taken no collinear with k. For large z, the tensor $d_{\mu\nu}$ does not depend on z so the polarization tensors of the internal gravitons do not contribute a factor of z.

Finally, the propagator in (4.5.7) is $1/P^2_{\mathcal{I}}(z)$, which vanishes as $1/z$. Therefore, the most dangerous term in $A(z)$ vanishes as $1/z$.

This completes the proof of the recursion relations of section 2 for next-to-MHV amplitudes of gravitons.

While we are not able to prove that $A(z)$ vanishes at infinity for general amplitudes with more than eight gravitons, we showed above that $A(z)$ vanishes at infinity for MHV and NMHV amplitudes with arbitrary number of gravitons. Hence, the recursion relations are valid for all MHV and NMHV amplitudes contrary to the expectations from KLT relations. This raises the hope, that the recursion relations are valid for other scattering amplitudes of gravitons as well. In particular, one might expect that by considering more general auxiliary recursion relations one could prove that $A(z)$ vanishes at infinity for general gravity amplitudes.

4.A Proof of Vanishing of $A(z)$ up to Eight Gravitons.

In this appendix we provide further evidence for validity of the recursion relations (4.2.5). We show that the recursion relations hold for any graviton amplitude up to eight gravitons. Recall that we need to prove that the auxiliary function $A(z)$ (4.2.4) vanishes at infinity. We demonstrate this using the KLT relations.

The basic fact we will use is that the function $\mathcal{A}(z)$ for a gluon scattering amplitude goes like $1/z^2$ at infinity for non-adjacent marked gluons. Hence, picking the marked gluons so that they are non-adjacent in all terms in KLT relations, the product of two Yang-Mills amplitudes in each term goes like $1/z^4$.

Hence, $A(z)$ vanishes at infinity as long as the products of s_{ij}'s in the KLT relations do not contribute more than a factor of z^3. An inspection of the KLT relations will show that this holds at least up to eight gravitons which will complete the proof.

Let us begin by showing that the gluon amplitudes go as $1/z^2$ as $z \to \infty$ for non-adjacent marked gluons with helicities $(h_i, h_j) = (+,+), (-,-), (-,+)$. The argument uses MHV rules and is a simple generalization of the argument given in [16], which showed that the amplitudes vanish as $1/z$. We assume that $h_j = +$. For $h_j = -$ one makes the same argument using the opposite helicity MHV rules.

Firstly, consider the n gluon MHV amplitude

$$\mathcal{A}(r^-, s^-) = \frac{\langle r, s\rangle^4}{\prod_{k=1}^{n}\langle k, k+1\rangle}. \tag{4.A.1}$$

Recall that $\lambda_j(z) = \lambda_j - z\lambda_i$ is linear in z and $\lambda_i(z) = \lambda_i$ is independent of z. For $h_j = +$, λ_j does not occur in the numerator. In the denominator it appears in the two factors $\langle\lambda_{j-1}, \lambda_j\rangle$ and $\langle\lambda_j, \lambda_{j+1}\rangle$, both of which are linear in z for i not adjacent to j. Hence for $h_i = +$ and $|i-j| > 1$, the MHV amplitude goes like $1/z^2$ at infinity.

For general amplitudes, we use MHV diagram constructions. In this construction, the amplitudes are built from Feynman vertices which are suitable off-shell continuations of the MHV amplitudes. The vertices are connected with ordinary scalar propagators.

The Feynman vertices are the MHV amplitudes (4.A.1), where we take $\lambda^a = P^{a\dot{a}}\eta_{\dot{a}}$ for an off-shell momentum P. Here η is an arbitrary positive helicity spinor. The physical amplitude, which is a sum of MHV diagrams, is independent of the choice of η [1].

The internal momentum P can depend on z only through a shift by the null vector $z\lambda_i\widetilde{\lambda}_j$. Taking $\eta = \widetilde{\lambda}_j$, $\lambda^a = P^{a\dot{a}}\widetilde{\lambda}_{j\,\dot{a}}$ becomes independent of z. Hence, the internal lines do not introduce additional z dependence into the MHV vertices. The MHV vertices give altogether a factor of $1/z^2$ from the two powers of $\lambda_j(z)$ in the denominator of one of the vertices. The propagators $1/k^2$ are either independent of z or contribute a factor of $1/z$. So, a general gluon amplitude goes like $1/z^2$ at infinity for non-adjacent marked gluons.

The KLT relations [72] for n gravitons are

$$\begin{aligned} A(1,2,\ldots,n) =& \Big(\mathcal{A}(1,2,\ldots,n) \sum_{perm} f(1,i_1,\ldots,i_j)\overline{f}(n-1,l_1,\ldots,l_{j'}) \\ & \times \mathcal{A}(i_1,\ldots,i_j,1,n-1,l_1,\ldots,l_{j'},n)\Big) \\ & + \mathcal{P}(2,\ldots,n-2), \end{aligned} \tag{4.A.2}$$

where $j = \lfloor n/2\rfloor - 1, j' = \lfloor (n-1)/2\rfloor - 1$ and the permutations are $(i_1,\ldots,i_j) \in \mathcal{P}(2,\ldots,\lfloor n/2\rfloor)$ and $(l_1,\ldots,l_{j'}) \in \mathcal{P}(\lfloor n/2\rfloor + 1,\ldots,n-2)$. The exact form of the functions f and $\overline{f}$ does not concern us here. The only property we need is that f and $\overline{f}$ are homogeneous polynomials of degree j and j' in the Lorentz invariants $p_m \cdot p_n$ with $m, n \in (1, i_1, \ldots, i_j)$ or $m, n \in (l_1, \ldots, l_{j'}, n-1)$ respectively.

Consider $A(z)$ with marked gravitons n and k where k is any label from the set $(2, \ldots, n-2)$. In the KLT relations (4.A.2), the gluon amplitude $\mathcal{A}(1, 2, \ldots, n)$ contributes a factor of $1/z^2$ since k and n are non-adjacent. For $k \in (i_1, \ldots, i_j)$ the second gluon amplitude gives a factor of $1/z^2$ and f gives at most a factor of $z^j, j = \lfloor n/2 \rfloor - 1$. Hence, the terms with $k \in (i_1, \ldots, i_j)$ are bounded at infinity by z^α where $\alpha = \lfloor n/2 \rfloor - 5$. For $k \in (l_1, \ldots, l_{j'})$ the second gluon amplitude gives a factor of $1/z$ because k and n might be adjacent. $\overline{f}$ contributes $z^{j'}, j' = \lfloor (n-1)/2 \rfloor - 1$ so the graviton amplitude is bounded by $z^{\alpha'}, \alpha' = \lfloor (n-1)/2 \rfloor - 4$. The exponents α, α' are negative for $n \leq 8$, which completes the proof of the recursion relations up to eight gravitons.

5. Chiral Rings and Vacua of SUSY Gauge Theories

5.1 Introduction

Recently there has been a progress in understanding the dynamics of a wide class of supersymmetric field theories. Embedding of the gauge theories in string theory as low energy effective field theories of D-branes wrapped on cycles in Calabi-Yau threefolds led to the conjecture of Dijkgraaf and Vafa that holomorphic data of the field theories can be calculated from an auxiliary matrix model. The bosonic potential of the matrix model is the superpotential of the gauge theory. Identifying the generating function for the glueball moments with the matrix model resolvent, the effective superpotential of the gauge theory gets related to the planar matrix model free energy. For the $U(N)$ gauge theory, the nonperturbative part of the superpotential comes from the measure of the matrix model and is given by a sum of Veneziano-Yankielowicz superpotentials of the $U(N_i)$ subgroups. The perturbative part is given by a sum of planar diagrams of the matrix model. Cachazo, Douglas, Seiberg and Witten gave a field theory derivation of the results. The derivation rests on the analysis of the anomalies and of the ring of chiral operators of the field theory.

It has been known for over a decade that the chiral ring of two dimensional field theories determines the structure of its supersymmetric vacua. The chiral operators obey relations that hold in every supersymmetric vacuum of the theory. It has been shown in [18] for the $\mathcal{N}=2$ superconformal field theories and in [19] for the CP^{N-1} supersymmetric sigma model that there is an exact correspondence between the solutions to the chiral ring relations and the supersymmetric vacua of the theory.

The authors of [20] showed that this continues to hold in four dimensions for the $\mathcal{N}=1$ pure $U(N)$ gauge theory. In this article we will extend this correspondence to $\mathcal{N} = 1$ $U(N)$ gauge theories with matter field Φ in the adjoint representation. The adjoint field has superpotential

$$W(\Phi) = \sum_{k=0}^{n} \frac{g_k}{k+1} \text{Tr } \Phi^{k+1}. \tag{5.1.1}$$

We can view this theory as a deformation of the $\mathcal{N} = 2$ gauge theory by the superpotential (5.1.1) for the scalar Φ of the $\mathcal{N} = 2$ vector superfield.

We will show that solving the chiral ring equations is equivalent to factorization of the $\mathcal{N} = 2$ curve. The factorization was originally derived by a strong coupling analysis of the gauge theory [73] based on monopole condensation.

Summary of Results

In section 5.2, we review the general properties of chiral rings, their relation to supersymmetric vacua and discuss the chiral ring relations both on the classical and quantum level. In section 5.3, we solve the chiral ring relations and demonstrate exact correspondence between the supersymmetric vacua and the roots of the chiral ring relations. In section 5.4, we use the chiral ring relations to give a brief discussion of the intersection of the vacua. In section 5.5, we study the chiral ring relations obeyed by the gaugino condensate and in section 5.6 we treat examples that illustrate the results from previous sections.

5.2 The Chiral Ring

Chiral operators are the operators that are annihilated by the anti-chiral supersymmetry generators $\overline{Q}_{\dot{\alpha}}$. Instead of chiral operators we will consider the set of equivalence classes of chiral operators where two operators are in the same equivalence class if they differ by a term of the form $\{\overline{Q}_{\dot{\alpha}}, \ldots\}$. This set is a ring because the product of two equivalence classes of chiral operators is another equivalence class. The expectation value of a chiral operator in a supersymmetric vacuum depends only on its equivalence class because the vacuum is annihilated by the supersymmetry generator $\overline{Q}_{\dot{\alpha}}$. It follows from $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$ that momentum, which is the generator of translations, annihilates chiral operators. Hence, chiral operators are independent of position. The chiral ring keeps only the information about the zero modes. In a product of chiral operators, we can put the operators far apart without changing the value of the product. Then the product factorizes into individual operators by cluster decomposition principle. Hence, we need to consider only the single trace operators. To classify the single trace operators we notice the identity [20]

$$[\overline{Q}^{\dot{\alpha}}, D_{\alpha\dot{\alpha}}\mathcal{O}\} = [W_\alpha, \mathcal{O}\} \tag{5.2.1}$$

which holds for any adjoint valued chiral superfield. Substituting Φ for $\mathcal{O}$, we see that Φ commutes with W_α

$$[\Phi, W_\alpha] = 0, \tag{5.2.2}$$

so it suffices to consider only operators where all Φ's are grouped together. Taking $\mathcal{O} = W_\alpha$ in (5.2.1) we learn that W_α's anticommute

$$\{W_\alpha, W_\beta\} = 0. \tag{5.2.3}$$

It follows that the single trace operators with three or more gaugino operators are descendants because the fermionic index α takes two values. The single trace chiral operators are

$$\begin{aligned} u_k &= \mathrm{Tr}\ \Phi^k, \\ w_{\alpha,k} &= \frac{1}{4\pi}\mathrm{Tr}\ \Phi^k W_\alpha, \\ r_k &= \frac{-1}{32\pi^2}\mathrm{Tr}\ \Phi^k W_\alpha W^\alpha. \end{aligned} \tag{5.2.4}$$

We assemble these operators into the resolvents

$$\begin{aligned} T(z) &= \mathrm{Tr}\ \frac{1}{z-\Phi} = \sum_{k\geq 0} u_k z^{-1-k}, \\ w_\alpha(z) &= \frac{1}{4\pi}\mathrm{Tr}\ W_\alpha \frac{1}{z-\Phi} = \sum_{k\geq 0} w_{\alpha,k} z^{-1-k}, \\ R(z) &= -\frac{1}{32\pi^2}\mathrm{Tr}\ W_\alpha W^\alpha \frac{1}{z-\Phi} = \sum_{k\geq 0} r_k z^{-1-k}. \end{aligned} \tag{5.2.5}$$

The single trace operators $u_k, w_{\alpha,k}$ and r_k generate the chiral ring. Formally, the chiral ring is a polynomial ring over the field of complex number with the single trace operators as indeterminates

$$\mathcal{C} = C[u_k, w_{\alpha,k}, r_k]. \tag{5.2.6}$$

Our interest is in the relations that the chiral operators satisfy. These relations are operator statements that hold in any supersymmetric vacuum. Taking an expectation value of a chiral ring relation in a given vacuum and using the fact that the expectation value of a product of chiral operators factorizes we get a relation for the expectation values of the chiral operators in that particular vacuum. By solving the chiral ring we mean finding the solutions to these chiral ring equations. The vacuum expectation values of $u_k, w_{\alpha,k}, r_k$ in a supersymmetric vacuum solve the chiral ring relations by definition. In principle, the chiral ring relations could have additional "unphysical" solutions for $u_k, w_{\alpha,k}, r_k$ which do not correspond to any supersymmetric vacuum. We will show that this is not the case. The roots of the chiral ring relations are in exact correspondence with the supersymmetric vacua of the gauge theory.

We can make the correspondence more precise. We introduce further algebraic construct, the coordinate chiral ring, which is the quotient of the chiral ring by the ideal generated by the chiral ring relations. Two chiral operators are considered to be the same elements of the coordinate chiral ring if their difference is a chiral ring relation. Hence, the coordinate chiral ring encodes the information about chiral operators that is invariant under addition of chiral ring relations. There is a natural correspondence between the roots of the chiral ring relations and the elements of the coordinate chiral ring. For semisimple coordinate chiral ring, all the roots are single and isolated, the only information that the coordinate ring encodes is the value of the operators at the solutions of the chiral ring relations. The solutions correspond to idempotent elements of the coordinate chiral ring. An idempotent is an operator that squares to itself. The idempotent associated to a particular vacuum takes expectation value one in that vacuum and vanishes in other vacua. In the general case, the roots can be multiple or have massless fermionic directions for the $U(1)$ photinos. Then a root corresponds to an ideal, called local ring, of the coordinate ring generated by the idempotent element above. The local ring is the set of elements obtained by multiplying the idempotent by all chiral operators. The dimension of the local ring equals the multiplicity of the corresponding vacuum. The basis of the local ring consists of the idempotent together with nilpotent elements. The coordinate chiral ring is a direct sum of the local rings. Any operator can be expanded as

$$\mathcal{O} = \sum_i o_i \Pi_i + n_i \tag{5.2.7}$$

where Π_i are the idempotents corresponding to i^{th} vacuum and n_i is the nilpotent part of $\mathcal{O}$ in the i^{th} local ring. The nilpotent elements correspond to different intersecting vacua or to vacuum with different value of the nilpotent $U(1)$ photinos Tr $W_\alpha \Phi^k$. The expectation value of an operator does not depend on these parameters hence it does not depend on the nilpotent part n_i. The expectation value of $\mathcal{O}$ in the i^{th} group of vacua is o_i.

Hence, each supersymmetric vacuum corresponds to a solution of the chiral ring relations which naturally corresponds to local ring which is generated by an idempotent element together with its nilpotents. This allows us to calculate the expectation values of the chiral operators from the knowledge of the idempotents.

A simple example that illustrates the above discussion is the polynomial ring in one indeterminate $C[x]$. This is the case of $U(1)$ gauge theory. x is the 1×1 matrix Φ in the adjoint representation of $U(1)$ which is trivial. The n vacua of the theory are at the critical points of the superpotential $W'(\Phi) = \prod_{i=1}^{n}(\Phi - \lambda_i)$, where we assume $\lambda_i \neq \lambda_j$ for $i \neq j$. Hence, the indeterminate x satisfies the polynomial relation of n^{th} degree $W'(x) = 0$. The coordinate chiral ring

$$C[x]/(W'(x) = 0) \tag{5.2.8}$$

has dimension n. The n distinguished idempotents are

$$\Pi_i(x) = \prod_{j \neq i}(x - \lambda_j) / \prod_{j \neq i}(\lambda_i - \lambda_j). \tag{5.2.9}$$

Clearly Π_i takes value one at λ_i and vanishes at λ_j for $j \neq i$. Any polynomial of degree less than n can be expressed as a linear combination of $\Pi(x)$. Polynomials of a higher degree can be reduced to polynomials of degree less than n using the relation $W'(x) = 0$. This completes the proof that the idempotents $\Pi_i(x)$ form an n dimensional basis of the coordinate chiral ring. The expansion coefficients of a polynomial

$$S(x) = \sum s_i \Pi_i(x) \tag{5.2.10}$$

in the idempotents are the values that the polynomial takes at the n roots of $W'(x)$

$$S(\lambda_i) = \sum_k s_k \Pi_k(\lambda_i) = s_i \tag{5.2.11}$$

in agreement with our general discussion.

To illustrate the correspondence when the coordinate ring has nilpotent elements, we consider the polynomial ring in one indeterminate x which satisfies $x^n = 0$. This is the case of n intersecting vacua of the field Φ. The coordinate chiral ring

$$C[x]/(x^n = 0) \tag{5.2.12}$$

is an n dimensional complex vector space. The basis consists of the idempotent 1 and of the nilpotents $x, x^2, \ldots, x^{n-1}$. Any polynomial can be expanded in this basis modulo the relation $x^n = 0$ which eliminates the powers of x^k for $k \geq n$. The value of the polynomial at the root $x = 0$ equals the zeroth order coefficient, which is coefficient the idempotent 1 in the expansion of the polynomial in terms of the above basis. Hence, to find the expectation value of a chiral operator we expand it in the basis of the coordinate chiral ring and read off the coefficient at the idempotent element. Chiral operators have the same expectation value in each of the intersecting vacua. The n intersecting vacua correspond to the multiple root which in turn corresponds to the n dimensional coordinate chiral ring that is spanned by the idempotent and nilpotent elements.

We can view the quantum relations as deformations of the classical relations. The classical relations can receive both perturbative and nonperturbative corrections. Quantum generalization of the classical equations of motion are the perturbative Ward identities coming from the one-loop Konishi anomaly. The Tr Φ^k with $k > N$ can be expressed as a polynomial in $u_1, \ldots, u_N$ because an $N \times N$ matrix is specified by the N independent gauge invariant operators $u_1, \ldots, u_N$. The classical relations for Tr Φ^k are deformed nonperturbatively by instanton corrections.

5.2.1 Perturbative Corrections

In this subsection we will find the classical chiral ring relations that follow from equations of motion and review the anomaly that corrects these relations. We start by multiplying the classical equation of motion for Φ,

$$\partial_\Phi W(\Phi) = \overline{D}_{\dot\alpha} \overline{D}^{\dot\alpha} \overline{\Phi} \tag{5.2.13}$$

with $A/(z - \Phi)$ where $A = 1, \frac{1}{4\pi} W_\alpha$ or $-\frac{1}{32\pi^2} W_\alpha W^\alpha$ and take the trace

$$\text{Tr } A \frac{W'(\Phi)}{z - \Phi} = 0. \tag{5.2.14}$$

We used the fact that $\overline{D}_{\dot{\alpha}}$ is conjugate to $\overline{Q}_{\dot{\alpha}}$ hence the right hand side of (5.2.14) can be written as $\{\overline{Q}_{\alpha}, \ldots\}$ and is a chiral ring descendant. To express these equations in terms of the resolvents (5.2.5), we notice the following identity

$$\begin{aligned}\text{Tr } A\frac{W'(\Phi)}{z-\Phi} &= W'(z)\text{Tr } \frac{A}{z-\Phi} - \text{Tr } A\frac{(W'(z)-W'(\Phi))}{z-\Phi} \\ &= W'(z)\text{Tr } \frac{A}{z-\Phi} - a(z).\end{aligned} \tag{5.2.15}$$

The function $a(z)$ is a polynomial in z of degree $n-1$ because $W'(z) - W'(\Phi)$ is a polynomial in z of degree n that vanishes when z equals to one of the eigenvalues of Φ. We define the polynomials

$$\begin{aligned}f(z) &= \frac{4}{32\pi^2}\text{Tr } W_\alpha W^\alpha \frac{W'(z)-W'(\Phi)}{z-\Phi}, \\ \rho_\alpha(z) &= \frac{1}{4\pi}\text{Tr } W_\alpha \frac{W'(z)-W'(\Phi)}{z-\Phi}, \\ c(z) &= \text{Tr } \frac{W'(z)-W'(\Phi)}{z-\Phi}\end{aligned} \tag{5.2.16}$$

and rewrite (5.2.14) with the help of (5.2.15) in the form

$$\begin{aligned}0 &= W'(z)R(z) + \frac{1}{4}f(z), \\ 0 &= W'(z)w_\alpha(z) - \rho_\alpha(z), \\ 0 &= W'(z)T(z) - c(z).\end{aligned} \tag{5.2.17}$$

To find the quantum corrections to (5.2.17) we recall that the classical equations of motions are derived by varying Φ. We will now review the anomaly in the variation which corrects the above relations quantum mechanically. Varying Φ by a general holomorphic function $\delta\Phi = f(\Phi, W_\alpha)$ gives anomaly of the current

$$J_f = \text{Tr } \overline{\Phi}e^{\text{ad}V} f(\Phi, W_\alpha) \tag{5.2.18}$$

which generates the variation of Φ. We find

$$\overline{D}_\alpha \overline{D}^\alpha J_f = \text{Tr } f(\Phi, W_\alpha)\frac{\partial W(\Phi)}{\partial \Phi} + \textit{anomaly} + \overline{D}(\ldots). \tag{5.2.19}$$

The first term on right hand side is the classical variation. The anomaly comes from one-loop diagrams involving $\overline{\Phi}$ and a single Φ from $f(\Phi, W_\alpha)$. To find the generalized Konishi equations expressed in terms of the resolvents (5.2.5) we make the variation

$$\delta\Phi_{ij} = f(\Phi, W_\alpha)_{ij} = \left(\frac{A}{z-\Phi}\right)_{ij}. \tag{5.2.20}$$

Computing the anomaly and setting $\{\overline{D}_{\dot{\alpha}}, \ldots\}$ terms to zero gives [20]

$$\begin{aligned} R^2(z) &= W'(z)R(z) + \frac{1}{4}f(z), \\ 2R(z)w_\alpha(z) &= W'(z)w_\alpha(z) - \rho_\alpha(z), \\ 2R(z)T(z) + w_\alpha(z)w^\alpha(z) &= W'(z)T(z) - c(z). \end{aligned} \tag{5.2.21}$$

On the right hand side of the anomaly equations are the classical equations of motion (5.2.17) and on the left hand side are the perturbative corrections coming from the one-loop Konishi anomaly. We can solve the anomaly equations (5.2.21) for the resolvents $R(z), w_\alpha(z)$ and $T(z)$ in terms of the superpotential and the auxiliary polynomials

$$\begin{aligned} R(z) &= \frac{1}{2}\left(W'(z) - \sqrt{W'^2(z) + f(z)}\right), \\ w_\alpha(z) &= \frac{\rho_\alpha(z)}{\sqrt{W'^2(z) + f(z)}}, \\ T(z) &= \frac{c(z) + w_\alpha(z)w^\alpha(z)}{\sqrt{W'^2(z) + f(z)}}. \end{aligned} \tag{5.2.22}$$

Throughout most of the article we will neglect the quadratic term $w_\alpha(z)w^\alpha(z)$ in the relation for $T(z)$.

5.2.2 Nonperturbative Corrections

The gauge invariant operators $u_k = \text{Tr } \Phi^k$ obey relations coming from the fact that Φ is an $N \times N$ matrix. Φ is determined up to a gauge transformation by the independent gauge invariant operators $\text{Tr } \Phi^l$ with $l = 1, \ldots, N$. The operators $\text{Tr } \Phi^k$ with $k > N$ can be expressed as polynomials in the first N traces. We will find the classical formulas for $\text{Tr } \Phi^k$ and then we will show how they get modified by nonperturbative instanton corrections.

Φ can be specified by the characteristic polynomial of N^{th} degree

$$P(z) = \det(z - \Phi) = \prod_{i=1}^{N}(z - \lambda_i). \tag{5.2.23}$$

The roots of $P(z)$ are the classical eigenvalues λ_i of Φ. We refer the reader to appendix 5.A for more details on $P(z)$. To derive the relations for u_k we write the generating function $T(z)$ in terms of eigenvalues of Φ

$$T(z) = \text{Tr}\ \frac{1}{z-\Phi} = \sum_{i=1}^{N} \frac{1}{z-\lambda_i} \tag{5.2.24}$$

and notice that this is the same as $P'(z)/P(z)$. Hence we have

$$T(z) = \frac{P'(z)}{P(z)}. \tag{5.2.25}$$

Notice that the left hand side depends on all traces Tr $\Phi^k = u_k$ while the right hand side depends only on $u_1, \ldots, u_N$. Expanding (5.2.25) in powers of $1/z$ and comparing the coefficients of the z^{-k-1} we get an expression for u_k from left hand side as a polynomial in $u_1, \ldots, u_N$ from the right hand side. We give a few examples of the resulting formulas in the appendix 5.A.

The operators

$$m_k = \text{Tr}\ M\Phi^k = \sum_{i=1}^{N} M_{ii}\lambda_i^k \tag{5.2.26}$$

where M is an arbitrary $N \times N$ matrix depend on $2N$ parameters, the N eigenvalues of Φ and the N diagonal elements of M. Hence, $m_0, m_1, \ldots m_{N-1}$ together with $u_1, u_2, \ldots, u_N$ are independent variables that determine m_k for $k \geq N$. To find the relations for m_k we make first order variation of (5.2.25) as $\Phi' = \Phi + \epsilon M$

$$\text{Tr}\ \frac{M}{(z-\Phi)^2} = -\frac{P_m'(z)}{P(z)} + \frac{P_m(z)P'(z)}{P^2(z)} = -\left(\frac{P_m(z)}{P(z)}\right)'. \tag{5.2.27}$$

The characteristic polynomial $P_m(z)$ of degree $N-1$ comes from the first variation of $P(z)$ in M

$$P_m(z) = -\partial_\epsilon \det(z - \Phi - \epsilon M)|_{\epsilon=0} = \sum_{i=1}^{N} M_{ii} \prod_{j\neq i}(z-\lambda_j). \tag{5.2.28}$$

We integrate (5.2.27) to get

$$\text{Tr}\ \frac{M}{z-\Phi} = \frac{P_m(z)}{P(z)}. \tag{5.2.29}$$

We fixed the constant of integration by requiring both sides of the equation to fall off to zero for z going to infinity. The left hand side of (5.2.29) depends on m_k while the right hand side depends only on $m_0, m_1, \ldots, m_{N-1}$ and $u_1, u_2, \ldots, u_N$. Expanding (5.2.29) in $1/z$ and comparing the coefficients of z^{-k-1} term we find formula for m_k as a polynomial in $m_0, m_1, \ldots, m_{N-1}$ and $u_1, u_2, \ldots, u_N$ from the right hand side.

We find the classical relations for $w_{\alpha,k}$ and r_k by substituting $\frac{1}{4\pi}W_\alpha$ and $-\frac{1}{32\pi^2}W_\alpha W^\alpha$ for M in (5.2.29)

$$w_\alpha(z) = \frac{P_\alpha(z)}{P(z)} \tag{5.2.30}$$

$$R(z) = \frac{P_R(z)}{P(z)}. \tag{5.2.31}$$

We have mentioned that the classical relations for Tr Φ^k with $k > N$ receive instanton corrections. We will determine the quantum modified relations by strong coupling analysis. We consider the $\mathcal{N} = 1$ theory as a deformation of the $\mathcal{N} = 2$ theory by a superpotential for the adjoint scalar field [20,73]. We will closely follow [20] but we will consider a vacuum with nonzero expectation value of the $U(1)$ photinos Tr $W_\alpha \Phi^k$ which break the $\mathcal{N} = 2$ supersymmetry to $\mathcal{N} = 1$ even before we turn on the superpotential. The superpotential takes the form

$$W'(z) = gP(z) = g\prod_{i=1}^{N}(z - \lambda_i) \tag{5.2.32}$$

with all λ_i different. We consider the maximally Higgsed vacuum in which the eigenvalues of Φ to occupy the N different critical points of the superpotential. Φ breaks the $U(N)$ gauge group down to $U(1)^N$. To find the resolvents (5.2.22)

$$\begin{aligned} R(z) &= \frac{1}{2}\left(gP(z) - \sqrt{g^2P^2(z) + f(z)}\right), \\ w_\alpha(z) &= \frac{\rho_\alpha(z)}{\sqrt{g^2P^2(z) + f(z)}}, \\ T(z) &= \frac{c(z)}{\sqrt{g^2P^2(z) + f(z)}} \end{aligned} \tag{5.2.33}$$

we need to determine the auxiliary polynomials $f(z), c(z), \rho_\alpha(z)$. The polynomial $c(z)$ depends on the operators u_k for $k = 1, \ldots, N-1$

$$c(z) = \text{Tr }\frac{W'(z) - W'(\Phi)}{z - \Phi} = g\sum_{i=1}^{N}\frac{P(z) - P(\lambda_i)}{z - \lambda_i} = gP'(z). \tag{5.2.34}$$

The third equality follows from the special form of the superpotential (5.2.32). We find $f(z)$ by comparing the $\mathcal{N}=2$ curve [20]

$$y^2 = P^2(z) - 4\Lambda^{2N} \tag{5.2.35}$$

with the matrix model curve

$$y_m^2 = W'^2(z) + f(z). \tag{5.2.36}$$

In the maximally Higgsed case, each of the occupied critical points of the superpotential gets smeared into a cut. Hence, the matrix model curve has single roots only. We find the curve from the $\mathcal{N}=2$ curve by factoring out the double roots

$$P^2(z) - 4\Lambda^{2N} = Q^2(z)(g^2P^2(z) + f(z)). \tag{5.2.37}$$

In the present case, there are no double roots so $Q(z) = 1/g, f(z) = -4g^2\Lambda^{2N}$. Substituting $c(z)$ and $f(z)$ into (5.2.33) we get the quantum modified versions of the formula (5.2.25) for $T(z)$

$$T(z) = \frac{P'(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}. \tag{5.2.38}$$

This relation is valid for the $\mathcal{N}=2$ gauge theory because it does not depend on g so we might as well set g to zero restoring $\mathcal{N}=2$ supersymmetry.

For general superpotential, we argue that (5.2.38) continues to hold. We again use the factorization of the $\mathcal{N}=2$ curve. In general, some number, say h of the critical points of the superpotential are unoccupied. The corresponding roots of the curve $y^2 = W'^2(z) + f(z)$ do not get smeared into cuts, they remain double roots. The matrix model curve sees only the $N-h$ occupied critical points, hence we factor out the double roots

$$H^2(z)y_m^2 = W'^2(z) + f(z). \tag{5.2.39}$$

The roots of $H(z)$ are near the unoccupied critical points of the superpotential. They are moved from the classical critical points by gaugino condensation, which is encoded in the polynomial $f(z)$. Factoring out the double roots of the $\mathcal{N}=2$ curve we get the matrix model curve

$$(P^2(z) - 4\Lambda^{2N}) = Q^2(z)y_m^2(z). \tag{5.2.40}$$

Taking first derivative of this equation we see that $P'(z)$ is divisible by $Q(z)$. Hence, we write $P'(z) = Q(z)\widetilde{P}(z)$. Furthermore, $c(z)$ must be divisible by $H(z)$. Otherwise

$$T(z) = \frac{c(z)}{H(z)y_m(z)} \tag{5.2.41}$$

would have poles at the roots of $H(z)$ which is a contradiction. The number of eigenvalues of Φ at the i^{th} critical point is

$$N_i = \frac{1}{2\pi i}\oint_{C_i} dz\, T(z) = \frac{1}{2\pi i}\oint_{C_i} dz\, \frac{c(z)}{H(z)y_m(z)} \tag{5.2.42}$$

where C_i is a curve going counterclockwise around the i^{th} critical point. The occupation number vanishes for the unoccupied critical points. Hence, $T(z)$ cannot have a pole at the unoccupied critical point because the contour integral would pick out the residue $T(z)$ and give a nonzero answer for N_i. So, $c(z) = H(z)\widetilde{c}(z)$.

The N_i's can be also calculated as the logarithmic residues of $P(z)$. For small Λ, N_i roots of $P(z) \sim \prod_{i=1}^{N}(z-\lambda_i)$ are near the classical critical points of the superpotential. For small Λ, the integral

$$\oint dz\, \frac{P'(z)}{P(z)} = \oint dz\, \ln'(P(z)) \tag{5.2.43}$$

counts the number of roots of $P(z)$ near the i^{th} classical critical point of the superpotential. This is the same as the number of eigenvalues of Φ at that critical point. When we turn on Λ, and deform the contour C_i so that none of the roots of $P(z)$ cross it, (5.2.43) is still valid. We can deform this formula even more by turning on Λ^{2N} to

$$N_i = \oint_{C_i} dz\, \frac{P'(z)}{\sqrt{P^2(z)-\Lambda^{2N}}} = \oint_{C_i} dz\, \ln'(P(z) + \sqrt{P^2(z)-\Lambda^{2N}}) \tag{5.2.44}$$

again making sure that the contour C_i does not cross the cuts of the square root in the denominator. We get an integer answer which must be N_i by continuity. Hence, we have two equivalent formulas for the occupation numbers

$$N_i = \oint_{C_i} dz\, \frac{\widetilde{c}(z)}{y_m(z)} = \oint_{C_i} dz\, \frac{\widetilde{P}(z)}{y_m(z)}. \tag{5.2.45}$$

The polynomials $\widetilde{P}(z)$ and $\widetilde{c}(z)$ have the same degree $N-h-1$. The $N-h$ equations coming from (5.2.45) for the $N-h$ coefficients of $\widetilde{c}(z)$ or $\widetilde{P}(z)$ uniquely determine these two polynomials. Hence they are the same as polynomials in z but at the same time they depend on the vacuum nontrivially through (5.2.45). Hence, (5.2.38) holds every vacuum of the gauge theory which means that it is a chiral ring relation.

To fix the formula for $w_\alpha(z)$ we still need to find the fermionic polynomial $\rho_\alpha(z)$. This goes through as in (5.2.34) thanks to the special form of the superpotential

$$\rho_\alpha(z) = \mathrm{Tr}\ W_\alpha \frac{(W'(z) - W'(\Phi))}{z - \Phi} = g \sum_{i=1}^{N} W_{ii} \frac{P(z) - P(\lambda_i)}{z - \lambda_i} = g P_\alpha(z). \qquad (5.2.46)$$

$\rho_\alpha(z)$ has coefficients which are linear $w_{\alpha,k}$ and polynomial in u_k with $k = 1, \dots, N-1$. For more details on $P_\alpha(z)$ we refer the reader to appendix 5.A. Substituting $\rho_\alpha(z)$ and $f(z)$ into (5.2.33) we get the quantum modified version of (5.2.30)

$$w_\alpha(z) = \frac{P_\alpha(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}. \qquad (5.2.47)$$

The formula for $T(z)$ has been derived in [20,74] while the relation for $w_\alpha(z)$ is new. One can show that (5.2.47) holds for arbitrary superpotential similarly as we showed the validity of (5.2.38) in previous paragraph. By taking the superpotential to zero we learn that (5.2.47) is valid for the $\mathcal{N} = 2$ gauge theory. We need to keep in mind that the chiral operators Tr $W_\alpha \Phi^k$ are descendants of the $\mathcal{N} = 2$ chiral ring [20] hence the formula for photinos makes sense only in the $\mathcal{N} = 1$ chiral ring. A different reason for considering the formulas as an $\mathcal{N} = 1$ chiral ring relation for the $\mathcal{N} = 2$ gauge theory is that the VEV's of photinos break the $\mathcal{N} = 2$ gauge symmetry down to $\mathcal{N} = 1$. For the lack of a better name, we will call the relations coming from (5.2.47) and (5.2.38) the $\mathcal{N} = 2$ relations. A suitable linear combination of the coeffients of $P_\alpha(z)$ are the $\mathcal{N} = 2$ low energy photinos. Physically, the relations for gaugino operators describe the expectation value of the $w_{\alpha,i}$'s when we turn on a coherent state of zero modes of the photinos.

It is easy to find now the formula for $R(z)$. We divide the third equation in (5.2.21) by $2T(z)$ to get

$$R(z) = -\frac{w_\alpha(z) w^\alpha(z)}{2T(z)} + \frac{1}{2} W'(z) - \frac{c(z)}{2T(z)}. \qquad (5.2.48)$$

Substituting (5.2.38) and (5.2.47) into (5.2.48) we get

$$R(z) = -\frac{P_\alpha(z) P^\alpha(z)}{2P'(z)\sqrt{P^2(z) - 4\Lambda^{2N}}} + \frac{W'(z)}{2} - c(z)\frac{\sqrt{P^2(z) - 4\Lambda^{2N}}}{2P'(z)}. \qquad (5.2.49)$$

The first term on the right hand side represent the quadratic response of $R(z)$ to nonzero vacuum expectation value of the $\mathcal{N} = 2$ photinos. The next terms are linear in the coefficients of the superpotential, as expected. They give the gaugino condensate of the $\mathcal{N} = 1$ gauge theory with nonzero superpotential. Taking $\Lambda = 0$ we get the classical relation

$$R(z) = -\frac{P_\alpha(z)P^\alpha(z)}{2P'(z)P(z)}, \tag{5.2.50}$$

where we used that

$$W'(z)P'(z) = c(z)P(z) \tag{5.2.51}$$

holds in the classical chiral ring. This follows from combining the two relations (5.2.17) and (5.2.25) for $T(z)$. We have derived from classical considerations that $R(z) = P_R(z)/P(z)$. This agrees with (5.2.50) only if

$$P_R(z)P'(z) = -\frac{1}{2}P_\alpha(z)P^\alpha(z). \tag{5.2.52}$$

We have not been able to verify this relation.

We recast the $\mathcal{N} = 2$ relations into a different form that is more convenient for some applications. We integrate both sides of the equation (5.2.38) to get

$$\int T(z) = \ln\frac{1}{2}\left(P(z) + \sqrt{P^2(z) - 4\Lambda^{2N}}\right), \tag{5.2.53}$$

where the integral means that we expand $T(z)$ in inverse powers of z and then integrate the resulting series

$$\int T(z) = \int dz \sum_{l=0}^{\infty} \frac{u_l}{lz^{l+1}} = N\ln(z) + \sum_{l=1}^{\infty}\frac{u_l}{z^l}. \tag{5.2.54}$$

The constant of integration in (5.2.53) was determined by matching the $N\ln(z)$ terms on both sides of the equation. Finally, we can find $P(z)$ from (5.2.53),

$$P(z) = e^{\int T(z)} + \Lambda^{2N}e^{-\int T(z)} \tag{5.2.55}$$

which we use to find $P_\alpha(z)$ from (5.2.47)

$$P_\alpha(z) = w_\alpha(z)\left(e^{\int T(z)} - \Lambda^{2N}e^{-\int T(z)}\right). \tag{5.2.56}$$

The constraints on u_k and $w_{\alpha,k}$ come from imposing that the coefficients of the negative powers of z in the Laurent series on the left hand side of (5.2.55) or (5.2.56) vanish. Since the coefficient of z^{N-k} of (5.2.55) is linear in u_k and does not depend on u_l with $l > k$, setting the coefficient to zero gives a recursion relation for u_k in terms of $u_1, u_2, \ldots, u_{k-1}$. We can solve the recursion relations to find u_k as a polynomial in $u_1, \ldots, u_N$. Similarly, the coefficient of z^{N-1-l} of (5.2.56) is linear in $w_{\alpha,k}$ and is independent of $w_{\alpha,l}$ with $l > k$. Hence, we get recursion relations for $w_{\alpha,k}$ with $k \geq N$ which determine $w_{\alpha,k}$ in terms of $w_{\alpha,0}, \ldots, w_{\alpha,N-1}$ and $u_1, \ldots, u_N$.

We recast the formula (5.2.49) as

$$-\frac{P_\alpha(z)P^\alpha(z)}{2P(z)} = \left(R(z) - W'(z)/2 + \frac{c(z)}{2T(z)}\right)\left(e^{\int T(z)} - \Lambda^{2N}e^{-\int T(z)}\right). \quad (5.2.57)$$

We will not use this formula except for next subsection where we relate it to (5.2.55) and (5.2.56).

5.2.3 $U(1)_{free}$ and Shift Symmetry

We decompose the $U(N)$ gauge symmetry as $SU(N) \times U(1)_{free}$ on the level of Lie algebras. We embed the $U(1)_{free}$ photino $\mathcal{W}_\alpha$ into the $U(N)$ gauge theory as $\mathcal{W}_\alpha \times 1_{N\times N}$. All fields are in the adjoint representation of the $U(N)$ gauge symmetry, hence they are neutral under the diagonal $U(1)_{free}$ which gets decoupled from the rest of the theory. It is described completely by the free $\mathcal{W}_\alpha\mathcal{W}^\alpha$ action. In the chiral ring, the $U(1)_{free}$ photino is described by an anticommuting number ψ_α because the chiral operator $\mathcal{W}_\alpha$ is independent of position. Hence, the $U(1)_{free}$ part of the gaugino generating functions are

$$\begin{aligned} w_\alpha(z) &= \frac{1}{4\pi}\mathrm{Tr}\ \frac{\psi_\alpha}{z-\Phi} = \frac{1}{4\pi}\psi_\alpha T(z) \\ R(z) &= -\frac{\psi_\alpha}{4\pi}w^\alpha(z) - \frac{1}{32\pi^2}\psi_\alpha\psi^\alpha T(z). \end{aligned} \quad (5.2.58)$$

It follows from the decoupling of $U(1)_{free}$ that the theory has an exact symmetry $W_\alpha \to W_\alpha + 4\pi\psi_\alpha 1_{N\times N}$ that shifts W_α by an anticommuting c-number. This symmetry acts on the chiral operators by

$$\begin{aligned} \delta r_k &= -w_{\alpha,k}\psi^\alpha - \frac{1}{2}\psi_\alpha\psi^\alpha u_k, \\ \delta w_{\alpha,k} &= u_k\psi_\alpha, \\ \delta u_k &= 0. \end{aligned} \quad (5.2.59)$$

We define the field $\widetilde{W}_\alpha = W_\alpha + 4\pi\psi_\alpha$, then the shift symmetry is generated by $\partial/\partial\psi_\alpha$. Invariance under this symmetry implies that the chiral ring relations do not depend on ψ_α when they are expressed in terms of $\widetilde{W}_\alpha$. We substitute $\widetilde{W}_\alpha$ instead of W_α into the definitions (5.2.5), (5.2.16), (5.2.23) and (5.2.28) and find the shift symmetric resolvents and polynomials

$$\begin{aligned} \widetilde{R}(z) &= R(z) - w_\alpha(z)\psi^\alpha - \frac{1}{2}\psi_\alpha\psi^\alpha T(z), \\ \widetilde{f}(z) &= f(z) - 4\psi_\alpha\rho^\alpha(z) - 2\psi_\alpha\psi^\alpha c(z), \\ \widetilde{w}_\alpha(z) &= w_\alpha(z) + \psi_\alpha T(z), \\ \widetilde{P}_\alpha(z) &= P_\alpha(z) + \psi_\alpha P'(z). \end{aligned} \tag{5.2.60}$$

Finally, we can write down the shift invariant form of the anomaly relations (5.2.21)

$$\widetilde{R}^2(z) = W'(z)\widetilde{R}(z) + \frac{1}{4}\widetilde{f}(z) \tag{5.2.61}$$

and the shift invariant $\mathcal{N} = 2$ equations

$$\widetilde{R}(z) = -\frac{\widetilde{P}_\alpha(z)\widetilde{P}^\alpha(z)}{2P'(z)\sqrt{P^2(z) - 4\Lambda^{2N}}} + \frac{W'(z)}{2} - c(z)\frac{\sqrt{P^2(z) - 4\Lambda^{2N}}}{2P'(z)}. \tag{5.2.62}$$

Here, the second and the third term are independent of ψ_α, whence they contribute only to the lowest component of $\widetilde{R}(z)$ which is $R(z)$ itself. To get the relations for $\mathcal{N} = 2$ gauge theory we set the superpotential to zero

$$\widetilde{R}(z) = -\frac{\widetilde{P}_\alpha(z)\widetilde{P}^\alpha(z)}{2P'(z)\sqrt{P^2(z) - 4\Lambda^{2N}}}. \tag{5.2.63}$$

The shift invariant $\mathcal{N} = 2$ relation that combines the formula for $T(z)$ and $w_\alpha(z)$ is

$$\widetilde{w}_\alpha(z) = \frac{\widetilde{P}_\alpha(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}. \tag{5.2.64}$$

This relation holds for any superpotential unlike (5.2.63) which is valid only for zero superpotential.

The equation (5.2.64) is the unique shift symmetric completion of the $\mathcal{N} = 2$ formula for $T(z)$. Each term in formula for $w_\alpha(z)$ depends on W_α and hence gives a nonzero contribution by shift symmetry to the formula for $T(z)$. Barring unexpected cancellations, the formula for $w_\alpha(z)$ is fixed by requiring that it shifts to the correct relation for $T(z)$. The formula for $R(z)$ is not fixed by shift symmetry from the formula for $w_\alpha(z)$. It can have additional terms that are independent of W_α which get shifted to zero, hence they are not constrained by the formula for w_α. These terms are absent for the $\mathcal{N}=2$ gauge theory, where the formula (5.2.63) for $R(z)$ gives the response to nonzero vacuum expectation value of the $U(1)$ photinos. There are such terms when we turn on the superpotential as is manifest from (5.2.62).

The shift invariant integral relation that combines (5.2.55), (5.2.56) and (5.2.57) is

$$-\frac{\widetilde{P}_\alpha(z)\widetilde{P}^\alpha(z)}{2P'(z)} = \widetilde{R}(z)\left(e^{\int T(z)} - \Lambda^{2N}e^{-\int T(z)}\right). \qquad (5.2.65)$$

In this form, the formula for $-P_\alpha(z)P^\alpha(z)/P'(z)P(z)$ goes by shift symmetry into the formula for $P_\alpha(z)$ which goes to the first derivative of the formula for $P(z)$. The $\mathcal{N}=2$ relation for $P(z)$ and $P_\alpha(z)$ combines is

$$\widetilde{P}_\alpha(z) = \widetilde{w}_\alpha(z)\left(e^{\int T(z)} - \Lambda^{2N}e^{-\int T(z)}\right). \qquad (5.2.66)$$

Similarly to (5.2.64) this formula holds for any superpotential.

5.3 Solution of the Chiral Ring

5.3.1 $U(2)$ Gauge Theory with Cubic Superpotential

Before giving a general proof that the chiral ring determines all the vacua of the theory we will illustrate this in detail in the case of the $U(2)$ gauge theory with cubic superpotential

$$W'(\Phi) = \frac{1}{3}\text{Tr }\Phi^3 - \frac{1}{2}\text{Tr }\Phi^2. \qquad (5.3.1)$$

We can always put a cubic superpotential into this simple form by rescaling and shifting Φ and W_α. Let us count the number of chiral operators that we need to consider after taking into account the recursion relations for the moments. A 2×2 matrix Φ is described by two independent gauge invariant chiral operators u_1 and u_2 which determine the remaining u_i's from (5.2.38). There are two independent gaugino operators $w_{\alpha,0}$ and $w_{\alpha,1}$ which determine the remaining $w_{\alpha,i}$'s from (5.2.47). For cubic superpotential, the r_i's are determined by r_0 and r_1 from the first anomaly equation (5.2.21). Hence, we have already reduced the number of chiral operators that generate the ring down to six.

To get started, we solve for vacua in the classical case. We treat u_k's as numbers and ignore the nilpotents $w_{\alpha,k}$ and r_k. The superpotential has two critical points, $\lambda = 0, 1$. Hence, the theory has three vacua corresponding to different arrangements of the eigenvalues of Φ among the critical points $\Phi = diag(0,0), diag(0,1), diag(1,1)$. The vacuum $\Phi = diag(1,0)$ is gauge equivalent to the vacuum $\Phi = diag(0,1)$. The vacua are described by the gauge invariant operators

$$u_1 = u_2 = u_3 = \ldots = 0, 1, 2. \tag{5.3.2}$$

These values of u_k obey all chiral ring relations by definition. We will now show that there are no additional solutions to the chiral ring relations. We expand the equation for Φ (5.2.14) in $1/z$ to get

$$\text{Tr } \Phi^k W'(\Phi) = u_{k+2} - u_{k+1} = 0. \tag{5.3.3}$$

Hence the equations of motion set all moments of $T(z)$ equal to u_1

$$u_k = u_1 \tag{5.3.4}$$

giving us one dimensional family of solutions parameterized by u_1. However, we know from above that only three solutions of this family correspond to supersymmetric vacua of the theory. Hence, the relations coming from the equations of motions are not restrictive enough. Fortunately, $u_3, u_4, \ldots$ are determined by u_1, u_2 from (5.2.38), so we have additional constraints which we can impose on the above one dimensional family of solutions. Substituting (5.3.4) into the relation (A.7) $u_3 = -\frac{1}{2}u_1^3 + \frac{3}{2}u_1 u_2$, we find

$$u_1(u_1 - 1)(u_1 - 2) = 0. \tag{5.3.5}$$

The solutions of this equation are $u_1 = 0, 1, 2$ which are the expectation values of u_1 in the three supersymmetric vacua discussed above. The idempotents corresponding to these vacua are $\frac{1}{2}(u_1 - 1)(u_1 - 2), -u_1(u_1 - 2), \frac{1}{2}u_1(u_1 - 1)$. Each solution of the chiral ring corresponds to a supersymmetric vacuum of the gauge theory.

The calculation in the quantum case is similar except that we need to keep track of r_k's which get nonzero expectation value from gaugino condensation. We will take into account the infinitesimal $w_\alpha(z)$ to find the low energy gauge group. We take the last anomaly equation (5.2.21)

$$2R(z)T(z) = \text{Tr } \frac{W'(\Phi)}{z-\Phi} \tag{5.3.6}$$

and expand it in $1/z$ to find the recursion relations for u_k

$$u_{k+2} = u_{k+1} + 2\sum_{i=1}^{k} u_i r_{k-i}. \tag{5.3.7}$$

We compare these with the equations (A.7) for u_3 and u_4 in terms of u_1 and u_2. This allows us to express u_2, r_0 and r_1 in terms of u_1

$$\begin{aligned} u_2 &= u_1 \\ r_0 &= -\frac{1}{8}u_1(u_1-1)(u_1-2) \\ r_1 &= -\frac{1}{16}u_1^2(u_1-1)(u_1-2) + \Lambda^4. \end{aligned} \tag{5.3.8}$$

We use that $r_2 = r_1 + r_0^2$, which comes from the $1/z^2$ term of the first equation (5.2.21)

$$R^2(z) = -\frac{1}{32\pi^2}\text{Tr } W_\alpha W^\alpha \frac{W'(\Phi)}{z-\Phi}, \tag{5.3.9}$$

and plug into this (5.3.8) to find

$$(u_1 - 1)[u_1^2(u_1-2)^2 - 64\Lambda^4] = 0, \tag{5.3.10}$$

which determines the location of the roots of the chiral ring relations in the complex u_1 plane. The equation (5.3.8) has five roots for u_1.

Quantum corrections do not move the vacuum at $u_1 = 1$ from the classical position in $\mathcal{N} = 2$ moduli space because all monopoles are massive and the instanton corrections to the moduli space move the classical vacua only for superpotential of degree five or higher. The vacuum has zero total gaugino condensate (5.3.8)

$$S = S_1 + S_2 = 0. \tag{5.3.11}$$

Instantons generate gaugino condensation in each of the $U(1)$ factors leading to $r_1 = \Lambda^4$. There are two vacua with $u_1 = 1+\sqrt{1 \pm 8\Lambda^2}$ near the classical critical point $\Phi_{cl} = diag(1,1)$, from the strongly coupled $SU(2)$ and two more vacua with $u_1 =$

$1-\sqrt{1\pm 8\Lambda^2}$ near $\Phi_{cl}=diag(0,0)$. The vacua have nonzero gaugino condensation $r_0=\pm\sqrt{1\pm 8\Lambda^2}$.

To find the rank of the low energy gauge group we solve the linear equations for $w_{\alpha,k}$ and count the dimension of the space of solutions. We will justify this prescription in the next section. The gaugino operators $w_{\alpha,0}$ and $w_{\alpha,1}$ obey relations that come from expanding (5.2.47) in powers of $1/z$. This gives a single constraint

$$(u_1-1)(u_1 w_{\alpha,0}-2w_{\alpha,1})=0. \tag{5.3.12}$$

At the vacuum with $u_1=1$, the constraint becomes trivial hence $w_{\alpha,0}$ and $w_{\alpha,1}$ are independent. The vacuum has $U(1)^2$ low energy gauge symmetry. The vacua with $u_1\neq 1$ have only one independent photino because (5.3.12) has a one dimensional family of solutions $w_{\alpha,1}=\frac{u_1}{2}w_{\alpha,0}$. Hence these vacua have $U(1)$ low energy gauge group.

5.3.2 Classical Case

We will now show that the supersymmetric vacua are in one to one correspondence with the solutions of the chiral ring relations. We will warm up on the classical case.

We have found two different formulas for the resolvents in terms of the first n or N moments. Comparing the formulas for resolvents from (5.2.17) with (5.2.25), (5.2.30) and (5.2.31) we obtain nontrivial relations for the first $\max(N,n)$ moments

$$\begin{aligned} T(z) &= \frac{c(z)}{W'(z)} = \frac{P'(z)}{P(z)}, \\ w_\alpha(z) &= \frac{\rho_\alpha(z)}{W'(z)} = \frac{P_\alpha(z)}{P(z)}, \\ R(z) &= -\frac{f(z)}{4W'(z)} = \frac{P_R(z)}{P(z)}. \end{aligned} \tag{5.3.13}$$

Expanding these equations in $1/z$, we would get an infinite number of equations for the moments. Instead, we rewrite the equations as

$$\begin{aligned} P'(z)W'(z) &= P(z)c(z), \\ P_\alpha(z)W'(z) &= P(z)\rho_\alpha, \\ P_R(z)W'(z) &= -\frac{1}{4}P(z)f(z). \end{aligned} \tag{5.3.14}$$

Then expanding in z we get a finite number of chiral relations to solve. Assume that the superpotential $W'(z) = \prod_{i=1}^{n}(z-\lambda_i)$ has n distinct critical points. The most general solution of (5.3.14) can be expressed in terms of auxiliary polynomials $F(z), H(z), Q(z)$ and $\widetilde{c}(z)$

$$\begin{aligned} W'(z) &= Q(z)F(z), \\ c(z) &= Q(z)\widetilde{c}(z), \\ P(z) &= H(z)F(z), \\ P'(z) &= H(z)\widetilde{c}(z), \end{aligned} \tag{5.3.15}$$

where $F(z) = \prod_{i=1}^{k}(z-\lambda_i)$ is a polynomial of degree k. $F(z)$ has only single roots. They are the k occupied critical points of the superpotential. The resolvent $T(z)$ is (5.3.13)

$$T(z) = \frac{\widetilde{c}(z)}{F(z)} = \sum_{j=1}^{k} \frac{N_j}{z-\lambda_j}. \tag{5.3.16}$$

The second equality holds because the polynomial $\widetilde{c}(z)$ of degree $k-1$ can be expressed as a linear combination of the k linearly independent polynomials $F_i(z) = \prod_{j\neq i}(z-\lambda_j)$. The N_i's are integers being the logarithmic residues of $P(z)$. They give the multiplicity of the eigenvalue λ_i in Φ. The solution is completely specified by N_i's. It corresponds to the vacuum in which Φ breaks the $U(N)$ gauge symmetry to $U(N_1) \times U(N_2) \times \ldots \times U(N_k)$. Taking different N_i gives all vacua of the gauge theory. The expectation values of u_k's in a particular vacuum are generated by $T(z)$. The roots of the chiral ring relations are in one to one correspondence with the vacua of the theory.

The equations (5.3.14) linear in W_α determine the number of unbroken $U(1)$'s. Their general solution

$$\begin{aligned} P_\alpha(z) &= H(z)\sigma_\alpha(z), \\ \rho_\alpha(z) &= Q(z)\sigma_\alpha(z) \end{aligned} \tag{5.3.17}$$

is written in terms of the arbitrary polynomial $\sigma_\alpha(z)$ of degree $k-1$ which has k independent anticommuting coefficients. We have

$$w_\alpha(z) = \frac{\sigma_\alpha(z)}{F(z)}, \tag{5.3.18}$$

whence the vacuum has k $U(1)$ gauginos coming from the $U(1)$ factors of $U(N_i)$

$$\widehat{w}_{\alpha,i} = \frac{1}{4\pi}\mathrm{Tr}\ W_\alpha P_i \tag{5.3.19}$$

where P_i is the projector on the subspace $\Phi = \lambda_i$ preserved by the $U(N_i)$ gauge symmetry. We use a hat to distinguish $\widehat{w}_{\alpha i}$ from $w_{\alpha,i}$ Similarly, $R(z)$ is given in terms of an arbitrary polynomial $q(z)$ of degree $k-1$

$$R(z) = \frac{q(z)}{F(z)} \tag{5.3.20}$$

which indicates that the vacuum has k independent r_i's. Linear combinations of r_i give the gaugino bilinears

$$S_i = \frac{-1}{32\pi^2}\mathrm{Tr}\ W_\alpha W^\alpha P_i \tag{5.3.21}$$

of the $U(N_i)$ gauge group.

5.3.3 Quantum Case

The solution of the quantum case is similar to the classical case. We compare the perturbative formulas (5.2.22) for the resolvents $T(z)$ and $w_\alpha(z)$ with the nonperturbative formulas (5.2.38) and (5.2.47). We find the chiral ring relations

$$\begin{aligned} T(z) &= \frac{c(z)}{\sqrt{W'^2(z)+f(z)}} = \frac{P'(z)}{\sqrt{P^2(z)-4\Lambda^{2N}}}, \\ w_\alpha(z) &= \frac{\rho_\alpha(z)}{\sqrt{W'^2(z)+f(z)}} = \frac{P_\alpha(z)}{\sqrt{P^2(z)-4\Lambda^{2N}}}. \end{aligned} \tag{5.3.22}$$

Expanding both sides of (5.3.22) in $1/z$ and comparing the coefficients of the two Laurent series we obtain an infinite number of relations for the first $\max(n, N)$ moments of the resolvents.

We rewrite (5.3.22) as

$$\begin{aligned} P'^2(z)(W'^2(z)+f(z)) &= (P^2(z)-4\Lambda^{2N})c^2(z), \\ P_\alpha(z)c(z) &= P'(z)\rho_\alpha(z), \end{aligned} \tag{5.3.23}$$

to get a finite number of equations. We have eliminated the square roots in the second equation using the first equation. Let us focus now on the first equation in (5.3.23). Expanding the equation in z and comparing the coefficients we obtain a finite number of chiral ring relations that can be solved to find expectation values $u_1, \ldots, u_{max(N,n)}, r_0, \ldots, r_{n-1}$ in all vacua. We obtain $2N+2n-1$ equations while the number of independent variables is $\max(N,n)+n$. Generically, the number of independent equations is larger than the number of variables.

To solve the quantum chiral ring relations, assume that the matrix curve $y_m^2 = F_{2g}(z)$ has genus g. Hence, the $\mathcal{N}=2$ curve has $N-g$ double roots

$$P^2(z) - 4\Lambda^{2N} = H_{N-g}^2(z)F_{2g}(z). \tag{5.3.24}$$

Taking derivative of (5.3.24) we find that $P'(z) = H_{N-g}(z)\widetilde{c}_{g-1}(z)$ is divisible by $H_{N-g}(z)$. To match single roots on both sides of (5.3.23) we must have

$$W'^2(z) + f(z) = Q_{n-g}^2(z)F_{2g}(z), \tag{5.3.25}$$

hence $c(z) = Q_{n-g}\widetilde{c}_{g-1}(z)$. The equation (5.3.25) is the generalized condition for finding vacua for arbitrary degree of the superpotential [74]. We remark that even though the relation (5.3.25) has a direct physical interpretation in terms of condensation of $N-g$ massless monopoles and factorization of the matrix model curve, it is *not* a chiral ring relation because it does not hold in all vacua of the theory. The equations (5.3.23) are chiral ring operator statements valid in every vacuum of the gauge theory. Substituting the solution (5.3.25) into (5.3.22) we get the relation for $T(z)$ in terms of the matrix model curve

$$T(z) = \frac{\widetilde{c}(z)}{\sqrt{F_{2g}(z)}}. \tag{5.3.26}$$

To find the position of the supersymmetric vacua in the Φ moduli space we have set to zero the $U(1)$ photinos. We were allowed to do this because the expectation value of the photinos moves the vacua by an infinitesimally small amount because of the nilpotent nature of the photino operators $w_{\alpha,i}$.

Substituting $c(z)$ and $P'(z)$ into (5.3.23) gives

$$P_\alpha(z)Q_{n-g}(z) = H_{N-g}(z)\rho_\alpha(z). \tag{5.3.27}$$

The general solution of this equation is

$$\begin{aligned} P_\alpha(z) &= H_{N-g}(z)\sigma_{\alpha,g-1}(z) \\ \rho_\alpha(z) &= Q_{n-g}(z)\sigma_{\alpha,g-1}(z) \end{aligned} \tag{5.3.28}$$

where $\sigma_{\alpha,g-1}(z)$ in an arbitrary polynomial of $g-1$'st degree. So $w_\alpha(z)$ is determined by the g independent fermionic coefficients of $\sigma_{\alpha,g-1}(z)$

$$w_\alpha(z) = \frac{\sigma_{\alpha,g-1}(z)}{\sqrt{F_{2g}(z)}}. \tag{5.3.29}$$

Along these directions, the photinos can take vacuum expectation values. Hence, each massive vacuum has massless fermionic moduli directions parameterized by the magnitude of the photino condensate. The photons that are supersymmetric partners of the massless photinos are massless as well. These are the freely propagating photons of the low energy effective gauge group. Hence, the number of $U(1)$ photons is equal to the number of massless photinos which is equal to the dimension of the fermionic moduli space. To find the dimension, it is enough to consider equations linear in $w_\alpha(z)$ and count the number of parameters describing their solution. This justifies the calculation in the cubic superpotential example and implies that the vacuum corresponding to genus g matrix model curve have $U(1)^g$ low energy gauge symmetry.

5.3.4 Perturbative Chiral Ring

Finally, let us consider the chiral ring that incorporates the perturbative corrections only. We turn off the nonperturbative corrections by setting the strong coupling scale Λ to zero in chiral ring relations. The ideal of relations is generated by (5.3.23)

$$P'^2(z)(W'^2(z) + f(z)) = P^2(z)c^2(z) \tag{5.3.30}$$

and

$$P_\alpha(z)c(z) = P'(z)\rho_\alpha(z). \tag{5.3.31}$$

As a simple consequence of (5.3.30), we observe that $\langle f(z)\rangle = 0$ in every vacuum, because $W'^2(z) + f(z)$ is a square of a polynomial if and only if $f(z) = 0$ or $\deg f(z) \geq \deg W'(z)$, but $f(z)$ has degree one smaller than $W'(z)$. So we see from (5.2.22) that

$$\langle R(z)\rangle = 0, \tag{5.3.32}$$

the gaugino condensate is vanishes to all orders in perturbation theory. The r_k's are nilpotent operators of the perturbative chiral ring because $\langle r_k\rangle = 0$ for each solution of the perturbative relations (5.3.30). The nilpotency follows from Hilbert's

Nullstellensatz, p. 412 of [75], which states that if a polynomial g vanishes at every solution of an ideal $\mathcal{I}$ of polynomial relations then g^k for sufficiently large k is an element of the ideal $\mathcal{I}$. We will discuss in section on gaugino condensation that the actual nilpotency condition on r_k is that the product of any N r_k's is zero in the perturbative chiral ring.

To find the positions of the vacua in the Φ moduli space, we set $f(z)$ to zero in (5.3.30). The chiral ring reduces to the classical chiral ring. Hence, the perturbative corrections do not shift the positions of the vacua in the Φ moduli space and the equations (5.3.31) give the correct number of $U(1)$ gauge symmetries. To account for the correct multiplicity of the vacua we need to retain the nilpotent $f(z)$. The multiplicity of the solution equals the multiplicity of the supersymmetric vacua. The multiple root splits into single roots and the supersymmetric vacua separate in the Φ moduli space when we make Λ nonzero.

In the example the chiral ring of $U(2)$ gauge theory with cubic superpotential, the classical ring is $u_1(u_1-1)(u_1-2)=0$ while the perturbative ring is

$$u_1^2(u_1-1)(u_1-2)^2=0, \tag{5.3.33}$$

which can be obtained from (5.3.10) by setting $\Lambda=0$. The double roots correspond to the pairs of vacua that come from the strongly coupled $SU(2)$ vacua.

5.4 Intersection of Vacua

Generically, all vacua are located at different points in the Φ moduli space. By tuning the superpotential, we can make two or more vacua intersect. We will consider only the intersections at which mutually local monopoles are massless. The chiral ring relations will have a multiple root. Its multiplicity equals to the number of intersecting vacua. We notice that $R(z)$ is determined by the location of the vacua in the $N=2$ moduli space from (5.2.49). Hence, the intersecting vacua have the same expectation value of the moments of gaugino condensate.

Let us investigate the the low energy gauge group of the intersecting vacua. $T(z)$ determines the linear constrains (5.2.56) for $w_\alpha(z)$. So the rank of the low energy gauge group depends only on the position of the vacuum in the Φ moduli space. The intersecting vacua have the same low energy gauge group $U^g(1)$ with $g\geq g_i$ where $U(1)^{g_i}$ is the the gauge group of i^{th} vacuum near the intersection.

The lower bound of the rank of the gauge group follows, because when tuning the superpotential to make the vacua intersect, the dimension of the space solution to (5.2.56) can suddenly jump up as some of the constraints for $w_{\alpha,k}$ become satisfied on a submanifold of the Φ moduli space.

Physically, the increase in the rank of the gauge group is connected with vanishing of the condensate of monopoles at the vacuum. As the vacua approach each other, the dual Meissner effects of the confined $U(1)$'s turns off. At the intersection the monopole has zero expectation value and the dual electric $U(1)$ is free. We will investigate monopole condensation using the low energy effective lagrangian [76,77] that includes the monopole fields

$$\mathcal{L}_{eff} = \sum_{k=0}^{n} \frac{g_k}{k+1} \mathrm{Tr}\ \Phi^{k+1} + \sum_{i=1}^{N} M_i(\Phi) m_i \widetilde{m}_i. \tag{5.4.1}$$

The mass of the i^{th} monopole $M_i(\Phi)$ is a function on the $\mathcal{N} = 2$ moduli space. We can find the monopole condensate by varying these equations with respect to $u_1, \dots u_N, m_1, \dots, m_N$ and $\widetilde{m}_1, \dots, \widetilde{m}_N$. For present purposes it is enough to notice that the monopole condensates depends continuously on the superpotential and the u_i's. Thus, the monopole condensate associated with the deconfining $U(1)$ turns off continuously on the approach of the intersection. This follows from the formula (3.16) of [78] for the value of monopole condensates.

We will now illustrate this behavior for $U(2)$ gauge theory with the cubic superpotential $W'(z) = z^2 - z$ which we analyzed in previous section. When

$$8\Lambda^2 = 1 \tag{5.4.2}$$

the two vacua at $u_1 = 1 \pm \sqrt{1 - 8\Lambda^2}$ intersect with the $u_1 = 1$ vacuum. Ignoring the photinos for a moment,we see that the chiral ring is generated by u_1, which obeys the constraint (5.3.10)

$$(u_1 - 1)^3 (u_1^2 - 2u_1 - 1) = 0. \tag{5.4.3}$$

The equation (5.4.3) has a triple root at $u_1 = 1$. The local ring at the triple root is three dimensional. The basis elements behave as $1, (u_1 - 1), (u_1 - 1)^2$ near the root and vanish at the two other roots of (5.4.3). To find the expectation value of a chiral operator in the intersecting vacua, we expand it in the local ring and read off the coefficient at the idempotent element.

We see from (5.3.12) that the gauge group of each of them gets enlarged to $U(1)^2$. We can see the increase in the rank of the gauge group directly from the low energy effective action of the theory near the intersection point (5.4.1)

$$W(\Phi) = \frac{u_3}{3} - \frac{u_2}{2} + m(2u_2 - u_1^2 \pm \Lambda^2)q\widetilde{q}. \tag{5.4.4}$$

The monopole condensate in the vacua with $U(1)_{free}$ gauge symmetry is

$$q\widetilde{q} = m'(u_1 - 1), \tag{5.4.5}$$

where m' is a constant. Near $u_1 = 1$, the condensate which confines the second $U(1)$ goes to zero and the dual Meissner effect continuously turns off. Some of these results were previously derived in [79].

5.5 Gaugino Condensation

We have seen that chiral ring determines all the supersymmetric vacua together with the expectation values of all chiral operators. The chiral ring can be used to extract general statements about the properties of the vacua as well. For example, we showed above that chiral ring encodes the low energy gauge group of the vacua. The dimension of the gauge group was shown to be equal to the number of the fermionic moduli parameterizing the condensate of the $U(1)$ photinos. We will now analyze the chiral ring relations satisfied by the gaugino bilinears r_i and their implications for gaugino condensation. For simplicity, we will assume throughout this section that the photino expectation values vanish.

5.5.1 Classical case

Classically, W_α is an $N \times N$ matrix of two component grassmannian numbers. The operators $r_i \sim \mathrm{Tr}\ \Phi^i W_\alpha W^\alpha$ are bosonic operators constructed from fermionic operators. These operators are nilpotent because of the anticommutativity of W_α. We have

$$r_{i_1} \ldots r_{i_{N^2+1}} = 0, \tag{5.5.1}$$

because W_α consists of N^2 two component fermions. In the chiral ring, the relations

$$\begin{aligned} \{W_\alpha, W_\beta\} &= 0 \\ [W_\alpha, \Phi] &= 0 \end{aligned} \tag{5.5.2}$$

imply a more powerful result. These identities generate the ideal $\mathcal{I}$ which is the subideal of the full ideal of classical relations. The remaining the classical relations have been discussed in the section 2. We denote W_1 and W_2 by A and B. Then the ideal $\mathcal{I}$ is generated by A^2, B^2 with both A and B commuting with Φ and anticommuting with each other. For example the authors of [20] showed that

$$r_0^N = S^N = 0 \tag{5.5.3}$$

holds in the chiral ring of the pure $U(N)$ gauge theory. This relation continues to be valid when we add the adjoint field Φ because

$$r_0 = -\frac{1}{32\pi^2}\mathrm{Tr}\ W_\alpha W^\alpha = -\frac{1}{16\pi^2}\mathrm{Tr}\ AB \tag{5.5.4}$$

does not depend on Φ, so the proof from [20] for the pure $U(N)$ gauge theory is still valid. There is a similar relation for the product of arbitrary N moments of $R(z)$

$$r_{k_1} r_{k_2} \dots r_{k_N} = 0. \tag{5.5.5}$$

To derive (5.5.5), we closely follow [80]. We construct the tensor $F(A)$ from A

$$F^{i_1 i_2 \dots i_N}(A) = \epsilon^{j_1 j_2 \dots j_N} A^{i_1}_{j_1} A^{i_2}_{j_2} \dots A^{i_N}_{j_N}. \tag{5.5.6}$$

The epsilon tensor on the right hand side picks out the completely antisymmetric part in the j indices of A, hence by anticommutativity of A, F is completely symmetric in the i indices. We will show later that $F(A)$ is contained in the ideal $\mathcal{I}$. We also define a complementary tensor from B and Φ

$$G_{i_1 i_2 \dots i_N}(B) = \left(-\frac{1}{16\pi^2}\right)^N \epsilon_{l_1 l_2 \dots l_N} (\Phi^{k_1} B)^{l_1}_{i_1} (\Phi^{k_2} B)^{l_2}_{i_2} \dots (\Phi^{k_N} B)^{l_N}_{i_N}. \tag{5.5.7}$$

Since $F(A)$ is contained in the ideal $\mathcal{I}$, so is its contraction with $G(B)$

$$F(A) \cdot G(B) = \epsilon^{i_1 i_2 \dots i_N} A^{i_1}_{j_1} A^{i_2}_{j_2} \dots A^{i_N}_{j_N} \epsilon_{l_1 l_2 \dots l_N} (\Phi^{k_1} B)^{l_1}_{i_1} (\Phi^{k_2} B)^{l_2}_{i_2} \dots (\Phi^{k_N} B)^{l_N}_{i_N}. \tag{5.5.8}$$

We arrange the right hand side of (5.5.8) using the identity

$$\epsilon^{j_1 j_2 \dots j_N} \epsilon_{l_1 l_2 \dots l_N} = \delta^{j_1}_{l_1} \delta^{j_2}_{l_2} \dots \delta^{j_N}_{l_N} \pm permutations. \tag{5.5.9}$$

The delta tensors contract the indices between $F(A)$ and $G(B)$ making $F(A)\cdot G(B)$ into a sum of terms

$$\text{Tr }\ \Phi^{p_1}(AB)^{s_1}\text{Tr }\ \Phi^{p_2}(AB)^{s_2}\dots\text{Tr }\ \Phi^{p_N}(AB)^{s_N} \tag{5.5.10}$$

with various p_i and s_i. In writing (5.5.10) we used the fact that Φ commutes with A and B to collect Φ's to the left of each trace. The term coming from the trivial permutation in (5.5.9) is

$$r_{k_1}r_{k_2}\dots r_{k_N}=\left(-\frac{1}{16\pi^2}\right)^N\text{ Tr }\ \Phi^{k_1}AB\,\text{Tr }\ \Phi^{k_2}AB\dots\text{Tr }\ \Phi^{k_N}AB. \tag{5.5.11}$$

The remaining permutations give terms with some $s_i>1$ hence they are contained in the ideal $\mathcal{I}$.

To complete the proof, we will show that $F(A)$ is in the ideal $\mathcal{I}$. Since $F(A)^{i_1i_2\dots i_N}$ is symmetric in its indices, we can set them to the same value. We will show that

$$F(A)^{NN\dots N}=\epsilon^{j_1j_2\dots j_N}A^N_{j_1}A^N_{j_2}\dots A^N_{j_N} \tag{5.5.12}$$

is proportional to

$$\epsilon^{Nj_1j_2\dots j_N}(A^2)^N_{j_1}A^N_{j_2}\dots A^N_{j_{N-1}}, \tag{5.5.13}$$

which is in the ideal $\mathcal{I}$ because it is a multiple of A^2. We can write (5.5.13) as

$$\sum_{x=1}^N\epsilon^{Nj_1j_2\dots j_{N-1}}A^N_xA^x_{j_1}A^N_{j_2}\dots A^N_{j_{N-1}}. \tag{5.5.14}$$

The expression

$$A^N_xA^N_{j_2}A^N_{j_3}\dots A^N_{N-1} \tag{5.5.15}$$

is antisymmetric in its $N-1$ indices $x,j_2,j_3,\dots,j_{N-1}$, hence it is a nonzero multiple of

$$\epsilon_{xj_2j_3\dots j_{N-1}k}\epsilon^{kl_1l_2\dots l_{N-1}}A^N_{l_1}A^N_{l_2}\dots A^N_{l_{N-1}}. \tag{5.5.16}$$

We substitute this into (5.5.14)

$$\sum_{x=1}^N\epsilon^{Nj_1j_2\dots j_{N-1}}\epsilon_{xj_2j_3\dots j_{N-1}k}\epsilon^{kl_1l_2\dots l_{N-1}}A^x_{j_1}A^N_{l_1}A^N_{l_2}\dots A^N_{l_{N-1}} \tag{5.5.17}$$

and use (5.5.9) to express the product of the first two epsilon tensors as a multiple of $\delta^N_x\delta^{j_1}_k-\delta^N_k\delta^{j_1}_x$. We find that (5.5.14) is a nonzero multiple of

$$(\delta^N_x\delta^{j_1}_k-\delta^N_k\delta^{j_1}_x)\epsilon^{kl_1l_2\dots l_{N-1}}A^x_{j_1}A^N_{l_1}A^N_{l_2}\dots A^N_{l_{N-1}}. \tag{5.5.18}$$

The term contracted with $\delta^N_k\delta^{j_1}_x$ are proportional to the $U(1)$ photino Tr A which we took to be zero. The term contracted with $\delta^N_x\delta^{j_1}_k$ is $F^{NN\dots N}(A)$ as promised.

5.5.2 Quantum case

Quantum mechanically, all vacua of the theory have nonzero gaugino condensation. This follows because all solutions of the equation (5.3.23)

$$P'^2(z)(W'^2(z) + f(z)) = (P^2(z) - 4\Lambda^{2N})c^2(z) \tag{5.5.19}$$

have $f(z) \neq 0$. We must have $f(z) \neq 0$ to insure that the left hand side of (5.5.19) is not a square of a polynomial, since the right hand side is cannot be written as a square of a polynomial when $\Lambda \neq 0$. Nonzero $f(z)$ is equivalent to nonzero gaugino condensation which can be seen from the equation (5.2.22)

$$R(z) = \frac{1}{2}\left(W'(z) - \sqrt{W'^2(z) + f(z)}\right) \tag{5.5.20}$$

for the generating function $R(z)$.

For a generic shape of the superpotential, we expect that the coefficients of the polynomial $f(z)$ are generic and nonzero. Hence, generically, all the moments r_k are nonzero. In a particular vacuum, the first few moments can vanish if the gaugino condensates S_i of the $U(N_i)$ subgroups cancel among each other when added up to make the gauge invariant operators

$$r_0 \sim \sum S_i,\; r_1 \sim \sum \lambda_i S_i,\; \ldots. \tag{5.5.21}$$

In this case, some of the higher traces r_k must be nonzero. Actually, infinitely many moments r_k do not vanish. This follows from the fact that expanding the square root in (5.5.20) in powers of $1/z$ we obtain Laurent series with infinite number of nonzero terms.

We would like to find the quantum version of the classical formulae (5.5.5). The product of N gaugino bilinears is generated by instantons. We expect the l instanton contribution to be proportional to the exponential of the l instanton action $e^{-lS_{inst}} = \Lambda^{2lN}$. The zero instanton term is absent. This expresses the absence of perturbative contribution to the gaugino condensation (5.3.32). The coefficient in front of the exponential is a polynomial in u_k because the expectation value of the gaugino condensate depends on the position of the vacuum in the Φ moduli space. In summary, nonperturbative effects correct (5.5.5) to

$$r_{k_1} r_{k_2} \ldots r_{k_N} = \sum_{l>0} \Lambda^{2lN} Q_{l,k_1k_2\ldots k_N}(u_1, u_2, \ldots, u_N). \tag{5.5.22}$$

The dimension of the left hand side is $3N + \sum_i k_i$ hence the dimension of the polynomial $Q_{l,k_1 k_2 \ldots k_N}$ is $(3-2l)N + \sum_i k_i$. Recalling that the dimension of u_k and Λ^{2N} is k and $2N$ respectively, dimensional analysis gives us a simple constraint on Q. For example, $r_0^N = S^N$ can have only one instanton contribution, since $Q_{l,00\ldots0}$ for $l > 1$ would have negative dimension $-(l-1)$, which is a contradiction. Q, being a polynomial in u_k, has always positive dimension. The general form of $Q_{l,k_1 k_2 \ldots k_N}(u_1, u_2, \ldots, u_N)$ is not know. It is a complicated polynomial in u_i that depends on the superpotential in a nontrivial way. Also, the Q_l's are not uniquely defined. The chiral ring has often relations that express a polynomial in u_k as Λ^{2N} times another polynomial of dimension $2N$ less than the original polynomial. Adding Λ^{2lN} times this relation to the right hand side of (5.5.22) we change Q_l and Q_{l+1} without affecting the total sum. This is related to the fact that Λ^{2N} has the same quantum numbers as Φ^{2N}.

For the example of $U(2)$ gauge theory with cubic superpotential from section (3.1), the formulas for the product of two r_0 and r_1 are

$$\begin{aligned} r_0^2 &= (u_1 - 1)^2 \Lambda^4, \\ r_0 r_1 &= \frac{\Lambda^4}{8} u_1 (u_1 - 1)(3u_1 - 2), \\ r_1^2 &= \frac{\Lambda^4}{8} u_1^3 (u_1 - 1) + \Lambda^8. \end{aligned} \tag{5.5.23}$$

We obtained these by multiplying the formulas (5.3.8) that express r_0 and r_1 in terms of u_1. To get the overall Λ^4 factor we have used the quintic equation (5.3.10) for u_1. We see that r_1^2 has also a two instanton contribution proportional to Λ^8. The product of any two moments r_i and r_j can be easily worked out from (5.5.23) because the higher moments can expressed as polynomials in r_0 and r_1 with the help of recursion relations obtained by expanding (5.3.9) in $1/z$

$$r_{k+2} = r_{k+1} + \sum_{i=0}^{k-1} r_i r_{k-i-1}. \tag{5.5.24}$$

We find that the product can be written as a sum of terms that are polynomials of degree two or higher in r_0 and r_1. We rewrite these polynomials with the Λ^{4l} prefactor using (5.5.23)

$$r_i r_j = \sum_{l>0} \Lambda^{4l} Q_{l,ij}(u_1). \tag{5.5.25}$$

5.6 Examples

In this section we give additional examples to illustrate in detail how the chiral ring determines the vacua of the gauge theory.

5.6.1 Unbroken Gauge Group

In our first example we consider the $U(N)$ gauge theory with unbroken gauge group. For simplicity we will assume that the superpotential has one critical point $W(\Phi) = \frac{1}{2}m\Phi^2$. The theory with quadratic superpotential for the adjoint field was solved first by Douglas and Shenker [81]. It has been recently studied in [82,83,84,85]. Semiclassically, Φ is a massive scalar field with zero expectation value preserving the $U(N)$ gauge symmetry. The $SU(N)$ subgroup of the $U(N)$ gauge group gets strongly coupled by nonperturbative effects and the low energy gauge group is the decoupled $U(1)_{free}$. There are N strongly coupled massive confining vacua with nonzero gaugino condensation. They are symmetrically distributed around the origin of the S plane.

We will now study the full chiral ring of the gauge theory keeping both linear and quadratic terms in $w_\alpha(z)$. We substitute $c(z) = mN$, $f(z) = -4mS$ and $\rho_\alpha(z) = mw_{\alpha,0}$ into the expressions (5.2.22) for the resolvents

$$\begin{aligned} T(z) &= \frac{N + w_\alpha(z)w^\alpha(z)/m}{\sqrt{z^2 - 4S/m}} = \frac{N}{\sqrt{z^2 - 4S/m}} + \frac{w_{\alpha,0}w_0^\alpha}{m(z^2 - 4S/m)^{\frac{3}{2}}}, \\ w_\alpha(z) &= \frac{w_{\alpha,0}}{\sqrt{z^2 - 4S/m}}, \\ R(z) &= \frac{m}{2}\left(z - \sqrt{z^2 - 4S/m}\right). \end{aligned} \tag{5.6.1}$$

We can write $T(z)$ more compactly in terms of $\widetilde{S} = S + \frac{1}{2N}w_{\alpha,0}w_0^\alpha$, the $SU(N)$ part of S

$$T(z) = \frac{N}{\sqrt{z^2 - 4\widetilde{S}/m}}. \tag{5.6.2}$$

Hence, $T(z)$ does not depend on the $U(1)$ photinos. It is easy to check (5.6.2) by expanding in $w_{\alpha,0}w_0^\alpha$ and using the fact that higher order terms are zero by anticommutativity since $w_{\alpha,0}$ is a two-component spinor. We substitute

$$\int T(z) = N \ln\left(\frac{z + \sqrt{z^2 - 4\widetilde{S}/m}}{2}\right) \tag{5.6.3}$$

into (5.2.55) to find

$$P(z) = \left(\frac{z + \sqrt{z^2 - 4\widetilde{S}/m}}{2} \right)^N + \frac{\Lambda^{2N}}{\left(\frac{z+\sqrt{z^2-4\widetilde{S}/m}}{2} \right)^N}. \tag{5.6.4}$$

The chiral ring relations come from setting the negative powers of z in the right hand side of (5.6.4) to zero. These relations are generated by

$$\widetilde{S}^N = m^N \Lambda^{2N}, \tag{5.6.5}$$

or equivalently in terms of the $U(N)$ gaugino bilinear

$$S^N + \frac{1}{2} S^{N-1} w_{\alpha,0} w_0^\alpha = m^N \Lambda^{2N}. \tag{5.6.6}$$

Hence we find that $P(z)$ is the Chebychev polynomial

$$P(z) = \left(\frac{z + \sqrt{z^2 - 4\widetilde{S}/m}}{2} \right)^N + \left(\frac{z - \sqrt{z^2 - 4\widetilde{S}/m}}{2} \right)^N, \tag{5.6.7}$$

in agreement with [81].

The quantum relation (5.6.5) implies that in any vacuum $\langle \widetilde{S}^N \rangle = m^N \Lambda^{2N}$. Since the expectation values of products of chiral operators factorize

$$\langle \widetilde{S} \rangle^N = m^N \Lambda^{2N}. \tag{5.6.8}$$

Solving for $\langle S \rangle$ we get $\langle S \rangle = \omega m \Lambda^2$ where ω is an N^{th} root of unity. We see that each of the N solutions to the chiral ring relations corresponds to a supersymmetric vacuum with nonzero gaugino condensate, as claimed. The equations for photinos $w_{\alpha,i}$ depend on one independent fermion $w_{\alpha,0}$, whence each of the massive vacua can have an arbitrary coherent state of the $U(1)$ photinos. The photon is massless by supersymmetry and the low energy gauge group is $U(1)$.

To find the expectation values of operators in each vacuum, we expand (5.6.1) in powers of $1/z$. The odd moments vanish by the symmetry $\Phi \to -\Phi$ while the even moments are nonzero,

$$\begin{aligned} u_{2k} &= N \binom{2k}{k} (\widetilde{S}/m)^k, \\ w_{\alpha,2k} &= w_{\alpha,0} \binom{2k}{k} (S/m)^k, \\ r_{2k} &= \frac{S}{k+1} \binom{2k}{k} (S/m)^k. \end{aligned} \tag{5.6.9}$$

The vacua are symmetrically distributed around zero in the complex S plane. This pattern is reminiscent of the pure $\mathcal{N}=1$ supersymmetric $U(N)$ gauge theory. Indeed, we recover the chiral ring of the $\mathcal{N}=1$ $U(N)$ gauge theory by taking the mass m of the adjoint field to infinity while holding the gaugino condensate S and the strong coupling scale of the pure $U(N)$ gauge theory fixed

$$\Lambda_p^{3N} = m^N \Lambda^{2N}. \tag{5.6.10}$$

The higher moments of $T(z), w_\alpha(z)$ and $R(z)$ vanish in the $m \to \infty$ limit (5.6.9). The ring of the pure $U(N)$ gauge theory is generated by S and $w_\alpha = w_{\alpha,0}$ which satisfy the relation

$$S^N + \frac{1}{2} S^{N-1} w_\alpha w^\alpha = \Lambda_p^{3N}. \tag{5.6.11}$$

Let us now determine the classical ring. We see from the classical equations of motion

$$W'(\Phi) = m\Phi = 0 \tag{5.6.12}$$

that Φ is a zero matrix. It follows that $u_k, w_{\alpha,k}$ and r_k are zero in the ring for $k \geq 1$ because they contain Φ. Hence S and $w_{\alpha,0}$ are the only nonzero operators. They are not constrained by the equations of motion. S satisfies the relation

$$S^N + \frac{1}{2} S^{N-1} w_{\alpha,0} w_0^\alpha = 0, \tag{5.6.13}$$

which is the generalization of (5.5.3) when Tr W_α is nonzero. It can be obtained by substituting $\widetilde{S}$ for S in (5.5.3). We notice the different origin of the formula for S in the classical and the quantum chiral ring. Classically, (5.6.13) follows from the fermionic character of W_α while quantum mechanically (5.6.5) is a consequence of the anomaly equations together with the $\mathcal{N}=2$ relations.

5.6.2 $U(3)$ Gauge Theory with Cubic Superpotential

We will now solve the chiral ring of the $U(3)$ gauge theory with cubic superpotential $W'(z) = z^2 - az - b$ and show that it determines all the vacua of the theory. This example has been studied by [74,79] and [86] using different approach. We will see that all vacua have nonzero gaugino condensation and that the chiral ring predicts the correct low energy gauge group. In this subsection, we will ignore the quadratic terms in the $U(1)$ photinos.

The polynomials $c(z), f(z)$ and $\rho_\alpha(z)$ are

$$\begin{aligned} c(z) &= 2(z-a)+u_1, \\ f(z) &= -4((z-a)r_0+r_1), \\ \rho_\alpha(z) &= (z-a)w_0+w_1. \end{aligned} \tag{5.6.14}$$

For $U(3)$, the polynomials $P(z)$ and $P_\alpha(z)$ become

$$\begin{aligned} P(z) &= z^3 - z^2 u_1 + z\left(\frac{u_1^2-u_2}{2}\right) + \frac{3u_1u_2 - u_1^3 - 2u_3}{6}, \\ P_\alpha(z) &= \left(z^2 - u_1 z + \frac{u_1^2-u_2}{2}\right) w_{\alpha,0} + (z-u_1)w_{\alpha,1} + w_{\alpha,2}. \end{aligned} \tag{5.6.15}$$

We get chiral ring relations by expanding (5.3.23) in z. Firstly, we express u_2, r_0 and r_1 in terms of u_1 and u_3

$$\begin{aligned} u_2 =& 3b + au_1, \\ r_0 =& -\frac{1}{6}(3ab + (a^2+b)u_1 - u_3), \\ r_1 =& -\frac{1}{36}\big(-u_1^4 + 6au_1^3 + u_1^2(-5a^2+16b) \\ & - 6u_1(3ab+u_3) + 6au_3 - 9b^2\big). \end{aligned} \tag{5.6.16}$$

Then u_3 can be found in terms of u_1 from the following equations

$$\begin{aligned} &(2a^2 - b - 3au_1 + u_1^2)(27bu_1 + 9au_1^2 - 2u_1^3 - 9u_3) = 0, \\ &\big((-91a^2+77b)u_1^4 + 39au_1^5 - 5u_1^6 + 3u_1^3(25a^3 - 98ab - 6u_3) \\ &+27u_1^2(13a^2b - 5b^2 + 2au_3) - 9u_1(-45ab^2 + 8a^2u_3 + 2bu_3) \\ &+9(9b^3 - 72\Lambda^6 - 12abu_3 + 2u_3^2)\big) = 0. \end{aligned} \tag{5.6.17}$$

u_1 is a solutions of the eight order polynomial equation

$$(2a^2 - b - 3au_1 + u_1^2)(5832\Lambda^6 + (9b + 3au_1 - u_1^2)^3) = 0. \tag{5.6.18}$$

The relations for the gaugino operators $w_{\alpha,0,1,2}$ are

$$\begin{aligned} (au_1 - u_2)w_{\alpha,0} - 3aw_{\alpha,1} + 3w_{\alpha,2} = 0, \\ \frac{1}{2}(u_1^2 - u_2)(aw_{\alpha,0} - w_{\alpha,1}) + (u_1 - 3a)(\frac{1}{2}(u_1^2-u_2)w_{\alpha,0} - u_1w_{\alpha,1} + w_{\alpha,2}) = 0. \end{aligned} \tag{5.6.19}$$

After some algebraic manipulations using the equations (5.6.16) to (5.6.18), we find that product of any three gaugino bilinears can be written with the Λ^6 prefactor

$$\begin{aligned} r_0^3 =& \frac{1}{6}\Lambda^6(6a^2 - 6ab - (10a^2 + b)u_1 + 3au_1^2 + u_3), \\ r_0^2 r_1 =& -\frac{1}{18}\Lambda^6(a - u_1)(9a^2u_1 - 16au_1^2 + 5u_1^3 + 3u_3), \\ r_0 r_1^2 =& \frac{1}{36}\Lambda^6\big(14u_1^5 - 77au_1^4 + 2(74a^2 - 25b)u_1^3 + (-121a^3 + 146ab + 6u_3)u_1^2 \\ &+ 6(6a^4 - 22a^2b + 5b^2 - 2au_3)u_1 + 3a(12a^2b - 9b^2 + 2au_3)\big), \\ r_1^3 =& \frac{1}{24}\Lambda^6\big(11u_1^6 - 73au_1^5 + 5(37a^2 - 9b)u_1^4 \\ &+ (-227a^3 + 190ab + 4u_3)u_1^3 + (136a^4 - 293a^2b + 33b^2 - 12au_3)u_1^2 \\ &+ a(-32a^4 + 196a^2b - 69b^2 + 12au_3)u_1 - 3b(16a^4 - 12a^2b + b^2)\big). \end{aligned} \tag{5.6.20}$$

Hence, by (5.5.24) and (5.6.20) , the product of any three r_i's can be written as

$$r_{i_1}r_{i_2}r_{i_3} = \Lambda^{6l}Q^l_{i_1,i_2,i_3}. \tag{5.6.21}$$

To keep the equations simple, we will continue the discussion for the superpotential $W'(z) = z^2 - z$. The equation for u_1 becomes

$$(u_1 - 1)(u_1 - 2)(5832\Lambda^6 - u_1^3(u_1 - 3)^3) = 0. \tag{5.6.22}$$

We will now discuss in detail all the roots to show each of them gives a supersymmetric vacuum of the gauge theory.

We see from (5.6.19) that the constrains for the expectation values of photinos are

$$\begin{aligned} w_{\alpha,2} - w_{\alpha,1} = 0 \\ u_1(u_1 - 1)(u_1 - 2)w_{\alpha,0} - 3(u_1 - 1)(u_1 - 2)w_1 = 0. \end{aligned} \tag{5.6.23}$$

For the vacua with $u_1 = 1, 2$ $w_{\alpha,1}, w_{\alpha,2}$ can take arbitrary expectation values, they are massless. Hence, by supersymmetry the corresponding photons are massless as well and we have $U^2(1)$ low energy gauge group. The $w_{\alpha,0}$ must have zero expectation value, it is massive. The remaining six vacua have only one massless direction for the photinos. Their low energy gauge group is $U(1)_{free}$.

The theory has four vacua coming from the confined $SU(2)$. Two of them are at $u_1 = 1$ and the other two at $u_1 = 2$. The two vacua with the same u_1 differ by the sign of gaugino condensate

$$r_0 = S = \pm\Lambda^2. \tag{5.6.24}$$

Their positions in the $\mathcal{N}=2$ moduli space are distinguished by $u_3 = u_1 - 6r_0$. The two vacua at $u_1 = 1$ have $r_1 = \mathrm{Tr}\ W^2\Phi = 0$ because the gaugino condensation is in the $SU(2)$ part of the gauge group which is preserved $(0,0)$ block of $\Phi = diag(0,0,1)$. The vacua $u_1 = 2$ have $r_1 = r_0$ since the gauginos condense in the $(1,1)$ block of $\Phi = diag(0,1,1)$. There are six vacua with confined $SU(3)$ that are symmetrically distributed in the u_1 plane around $u_1 = 0$ and around $u_1 = 3$,

$$u_1 = 3/2 \pm \sqrt{3/2 + 18\omega\Lambda^2} \tag{5.6.25}$$

where ω is a third root of unity. As discussed in previous paragraph, these vacua have $U(1)_{free}$ gauge symmetry. All these vacua have nonzero gaugino condensation

$$r_0 \sim \Lambda^2 \tag{5.6.26}$$

with dominant one instanton contribution for small Λ. The vacua near $u_1 = 3$ with $\Phi = diag(1,1,1)$ have the first moment of gaugino condensate of the same order $r_1 \sim \Lambda^2$ as r_0. The vacua near $u_1 = 0$ with $\Phi_{class} = diag(0,0,0)$ have vanishing one instanton contribution but nonzero second instanton contribution to $r_1 \sim \Lambda^4$.

5.A $\mathcal{N}=2$ Chiral Ring Relations

In this appendix, we will write down for illustration the first few of the $\mathcal{N}=2$ recursion formulas. These relations are expressed in terms of the characteristic polynomials $P(z)$ and $P_m(z)$. The coefficients of

$$P(z) = \det(z - \Phi) = \prod_{i=1}^{N}(z - \lambda_i) = \sum_{i=0}^{N} p_i z^{N-i} \tag{5.A.1}$$

are $p_0 = 1$ and $p_k = \sum_{i=1}^{N}(-1)^k\lambda_i^k$ for $k = 1\dots N$, which can be expressed in terms of $u_1,\dots,u_N$ from the recursion relations

$$p_k = -\sum_{i=1}^{k}\frac{u_i}{k}p_{k-i}. \tag{5.A.2}$$

The first few p_i's are

$$\begin{aligned} p_0 &= 1, \\ p_1 &= -u_1, \\ p_2 &= -\frac{u_2}{2} + \frac{u_1^2}{2}, \\ p_3 &= -\frac{u_3}{3} + \frac{u_2 u_1}{3} - \frac{1}{6}(u_1^2 - u_2)u_1. \end{aligned} \tag{5.A.3}$$

The characteristic polynomial $P_m(z)$ (5.2.28) comes from the first variation of $\delta\Phi = \epsilon M$ of $P(z)$

$$P_m(z) = -\partial_\epsilon \det(z - \Phi - \epsilon M)|_{\epsilon=0} = \sum_{i=1}^{N} M_{ii} \prod_{i \neq j} (z - \lambda_i) = \sum_{i=0}^{N-1} p_{m,i} z^{N-1-i}. \tag{5.A.4}$$

We find the recursion relations for the coefficients $p_{m,k} = \sum_{i=1}^{N} M_{ii}\lambda_i^k$ by making the first order variation $\delta p_k = -p_{m,k-1}$ and $\delta u_k = k m_{k-1}$ of the recursion relation (5.A.2)

$$p_{m,k} = \sum_{i=0}^{k} \frac{i}{k} m_i p_{k-i} - \sum_{i=1}^{k} \frac{1}{k} p_{m,k-i} u_i. \tag{5.A.5}$$

The recursion relations together with first coefficient $p_{m,0} = m_0$ determine $p_{m,k}$ in terms of $m_1, \ldots, m_k$ and $u_1, \ldots, u_k$. We write down the first few coefficients $p_{m,i}$ that are used in the examples

$$\begin{aligned} p_{m,0} &= m_0, \\ p_{m,1} &= m_1 - u_1 m_0, \\ p_{m,2} &= m_2 - u_1 m_1 + \frac{1}{2}(u_1^2 - u_2)m_0. \end{aligned} \tag{5.A.6}$$

We are ready to show first few $\mathcal{N} = 2$ relations obtained by expanding (5.2.38) and (5.2.49) in powers of $1/z$

$$\begin{aligned} T(z) &= \frac{P'(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}, \\ w_\alpha &= \frac{P_\alpha(z)}{\sqrt{P^2(z) - 4\Lambda^{2N}}}, \\ R(z) &= -\frac{P_\alpha(z)P^\alpha(z)}{2P'(z)\sqrt{P^2(z) - 4\Lambda^{2N}}}. \end{aligned} \tag{5.A.7}$$

Let us note that in the last formula we are ignoring the part of the gaugino condensate that depends on the superpotential. The classical formulas are obtained by setting Λ to zero in the quantum formulas.

For $U(2)$, all u_i's can be written as polynomials in Λ^4 and u_1, u_2 which are the two independent chiral operators that we can make from a 2×2 matrix Φ. We have

$$\begin{aligned} u_3 &= -\frac{1}{2}(u_1 - 3u_1 u_2), \\ u_4 &= 4\Lambda^4 - \frac{1}{2}(u_1^4 - 2u_1^2 u_2 - u_2^2), \\ u_5 &= 10 u_1 \Lambda^4 - \frac{1}{4}(u_1^5 - 5u_1 u_2^2). \end{aligned} \tag{5.A.8}$$

For $U(3)$, the first three u_1, u_2 and u_3 are independent. The higher moments are polynomials in these and in Λ^6

$$\begin{aligned} u_4 &= \frac{1}{6}(u_1^4 - 6u_1^2 u_2 + 4u_2^2 + 8u_1 u_3), \\ u_5 &= \frac{1}{6}(u_1^5 - 5u_1^3 u_2 + 5u_1^2 u_3 + 5u_2 u_3), \\ u_6 &= 6\Lambda^6 + \frac{1}{12}(u_1^6 - 3u_1^4 u_2 - 9u_1^2 u_2^2 + 3u_2^3 + 4u_1^3 u_3 + 12 u_1 u_2 u_3 + 4u_3^2). \end{aligned} \tag{5.A.9}$$

Taking $M = \frac{1}{4\pi} W_\alpha$ in (5.A.4) to (5.A.6) we can read off the formulae for $w_{\alpha,i}$ from the $1/z$ expansion of the generating relation (5.A.7). For $U(2)$, we find $w_{\alpha,i}$'s as polynomials in $w_{\alpha,0}, w_{\alpha,1}$ and u_1, u_2

$$\begin{aligned} w_{\alpha,2} &= -\frac{1}{2}(u_1^2 - u_2) w_{\alpha,0} + u_1 w_{\alpha,1}, \\ w_{\alpha,3} &= -\frac{1}{2}(u_1^3 - u_1 u_2) w_{\alpha,0} + \frac{1}{2}(u_1^2 + u_2) w_{\alpha,1}, \\ w_{\alpha,4} &= 2\Lambda^4 w_{\alpha,0} - \frac{1}{4}(u_1^4 - u_2^2) w_{\alpha,0} + u_1 u_2 w_{\alpha,1}. \end{aligned} \tag{5.A.10}$$

The first few relations for $U(3)$ are

$$\begin{aligned} w_{\alpha,3} &= \frac{1}{6}(u_1^3 - 3u_1 u_2 + 2u_3) w_{\alpha,0} - \frac{1}{2}(u_1^2 - u_2) w_{\alpha,1} + u_1 w_{\alpha,2}, \\ w_{\alpha,4} &= \frac{1}{6}(u_1^4 - 3u_1^2 u_2 + 2u_1 u_3) w_{\alpha,0} - \frac{1}{3}(u_1^2 - u_3) w_{\alpha,1} + \frac{1}{2}(u_1^2 + u_2) w_{\alpha,2}. \end{aligned} \tag{5.A.11}$$

The relations for $R(z)$ give the infinitesimal gaugino condensate coming from the vacuum expectation value of photinos. Notice that are ignoring here the finite gaugino condensate that is induced by the superpotential (5.2.49). For $U(2)$, we have

$$\begin{aligned} r_0 &= -\frac{1}{4}w_{\alpha,0}w_0^{\alpha}, \\ r_1 &= \frac{1}{8}u_1 w_{\alpha,0}w_0^{\alpha} - \frac{1}{2}w_{\alpha,0}w_1^{\alpha}, \\ r_2 &= \frac{1}{16}(u_1^2 - 3u_2)w_{\alpha,0}w_0^{\alpha} - \frac{1}{4}u_1 w_{\alpha,0}w_1^{\alpha} - \frac{1}{4}w_{\alpha,1}w_1^{\alpha}. \end{aligned} \tag{5.A.12}$$

The first few cases for $U(3)$ are

$$\begin{aligned} r_0 &= -\frac{1}{6}w_{\alpha,0}w_0^{\alpha}, \\ r_1 &= \frac{1}{18}u_1 w_{\alpha,0}w_0^{\alpha} - \frac{1}{3}w_{\alpha,0}w^{\alpha,1}, \\ r_2 &= -\frac{1}{54}(u_1^2 - 3u_2)w_{\alpha,0}w_0^{\alpha} + \frac{1}{9}u_1 w_{\alpha,0}w_1^{\alpha} - \frac{1}{6}w_{\alpha,1}w_1^{\alpha} - \frac{1}{3}w_{\alpha,0}w_2^{\alpha}. \end{aligned} \tag{5.A.13}$$

6. Nonperturbative Exactness of Konishi Anomaly

6.1 Introduction

As was discussed in last chapter, the Dijkgraaf-Vafa conjecture can be studied without recourse to string theory arguments. For a pedagogical introduction to the gauge theory methods used to study the Dijkgraaf-Vafa conjecture, see [87]. The authors of [88] gave a field theory argument showing that the Feynman diagrams contributing to the perturbative part of the glueball superpotential reduce to matrix model diagrams. A different approach was pursued in [20] using the chiral ring of the gauge theory. The generalized Konishi anomalies of the chiral rotations of the adjoint field imply constraints between chiral operators. These constraints have the same form as the loop equations of the matrix model in the planar limit. Hence the effective superpotential can be expressed in terms of the matrix model free energy up to a coupling independent term which can by seen to be a sum of Veneziano-Yankielowicz superpotentials by taking the limit of large couplings of the superpotential.

To complete the above argument it is necessary to verify that the generalized Konishi anomaly equations remain valid nonperturbatively and that the low energy effective description of the gauge theory in terms of the glueball fields is correct. In [20] it was suggested that one can prove the absence of corrections to the generalized Konishi anomaly by showing that the algebra of chiral rotations of the matter field does not have nonperturbative corrections and then arguing along the lines of Wess-Zumino consistency conditions that the anomalies do not have nonperturbative corrections. In this chapter we carry out this proposal. We show that the Konishi anomaly does not have nonperturbative corrections for superpotentials of degree less than $2l + 1$ where $2l = 3c(Adj) - c(R)$ is the one-loop beta function coefficient.

The consistency conditions do not completely fix the nonperturbative corrections to anomaly for superpotentials of a degree higher than $2l$. Such corrections are expected due to ambiguities in the definition of highly nonrenormalizable operators like Tr Φ^n [21], [22] and [23]. We show that all the ambiguities can be absorbed into the nonperturbative redefinition of the superpotential.

There are additional UV ambiguities for gauge theories which are not asymptotically free coming the freedom in their UV completion. For these theories our proof does not apply because Λ^{2l} has zero or negative dimension, whence there are infinitely many types of corrections of a given dimension. The consistency conditions are not powerful enough to constrain these corrections uniquely. In summary, in this chapter we prove the absence of nonperturbative corrections to the generalized Konishi anomaly that come from strong coupling dynamics and determine the form of corrections for high degree superpotentials.

The proof can be applied to gauge theories whose algebra of chiral rotations of matter fields forms an extension of a partial Virasoro algebra. For example it is possible to consider matter in other than adjoint representation. In particular we show nonrenormalization of the generalized Konishi anomaly for $SO(N)$ and $Sp(N)$ gauge theories with matter in the symmetric or antisymmetric representation. The nonrenormalization of the generalized Konishi anomaly for $Sp(N)$ with the antisymmetric tensor is expected in the light of recent results [27] and [21] that demonstrated agreement between the effective superpotential obtained using Konishi anomalies with the dynamically generated superpotential approach [89], [90]. The papers [27] and [21] resolved a puzzle raised in [24], [25] and [26] about the application of Dijkgraaf-Vafa correspondence for $Sp(N)$ with antisymmetric matter.

Organization and Results of the Chapter

In section 6.2 we introduce the algebra of chiral rotations of the matter field and show that it is an $\mathcal{N} = 1$ extension of a partial Virasoro algebra. We consider the $U(N)$ gauge theory with adjoint scalar to keep the discussion concrete. In section 6.3 we discuss the generalized Konishi anomalies of the chiral rotations and use the Virasoro symmetry to derive Wess-Zumino consistency conditions for the anomalies. In section 6.4 we use $U(1)$ symmetries of the gauge theory to determine the form of the nonperturbative corrections. In section 6.5 we use the Lie algebraic structure of the algebra of chiral rotations to prove that the algebra cannot get deformed nonperturbatively. This implies that the Wess-Zumino consistency conditions derived in section 6.4 are exact nonperturbatively. We use them to show for $U(N)$ in section 6.6 and for $SO(N)$ and $Sp(N)$ in section 6.7 that the generalized Konishi anomaly cannot have nonperturbative corrections except for nonperturbative renormalization of superpotentials of degree greater than $2l = 3c(Adj) - c(Matter)$. In section 6.8 we review the loop equations of the planar matrix model, considering them as anomalies of the matrix model free energy under reparameterization of the matrix M to highlight their similarity with gauge theory anomalies. In section 6.9 we discuss the implications of the results for the Dijkgraaf-Vafa conjecture.

6.2 The Algebra of Chiral Rotations

In [20], a series of constraints for the chiral operators of $\mathcal{N} = 1$ gauge theory were derived by considering the possible anomalies of the chiral rotations $\delta\Phi = f(\Phi, W_\alpha)$ of the adjoint scalar field. These chiral rotations are generated by the operators

$$\begin{aligned} L_n &= \Phi^{n+1}\frac{\delta}{\delta\Phi}, \\ Q_{n,\alpha} &= \frac{1}{4\pi}W_\alpha\Phi^{n+1}\frac{\delta}{\delta\Phi}, \\ R_n &= -\frac{1}{32\pi^2}W_\alpha W^\alpha\Phi^{n+1}\frac{\delta}{\delta\Phi}. \end{aligned} \tag{6.2.1}$$

The action of the operators (6.2.1) on the single trace chiral operators $u_k = \mathrm{Tr}\ \Phi^k, w_{k,\alpha} = \frac{1}{4\pi}\mathrm{Tr}\ W_\alpha\Phi^k$ and $r_k = -\frac{1}{32\pi^2}\mathrm{Tr}\ W_\alpha^2\Phi^k$ is

$$\begin{aligned} L_n u_k &= k u_{k+n}, \\ Q_{n,\alpha}u_k &= k w_{k+n,\alpha}, \\ &\ldots. \end{aligned} \tag{6.2.2}$$

The classical commutation relations of the generators follow from the definitions (6.2.1)

$$\begin{aligned} [L_m, L_n] &= (n-m)L_{m+n}, \\ [L_m, Q_{n,\alpha}] &= (n-m)Q_{n+m,\alpha}, \\ [L_m, R_n] &= (n-m)R_{m+n}, \\ \{Q_{m,\alpha}, Q_{n,\beta}\} &= -\epsilon_{\alpha\beta}(n-m)R_{n+m}, \\ [Q_{m,\alpha}, R_n] &= 0, \\ [R_m, R_n] &= 0, \end{aligned} \tag{6.2.3}$$

where $m, n \geq -1$. The last two commutators are trivially zero in the chiral ring because the third and higher powers of W_α are chiral ring descendants. The L_n's form a partial Virasoro subalgebra which is extended by $Q_{n,\alpha}$'s and R_n's into a partial $\mathcal{N} = 1$ super-Virasoro algebra.

As discussed in previous chapter, the scalar Φ and the gauge field are in the adjoint representation of the $U(N)$ gauge group so they do not couple to the diagonal $U(1)$ gauge field. Hence shifting W_α by an anticommuting number is a symmetry of the full gauge theory [20]. If we define the field $\widetilde{W}_\alpha = W_\alpha + 4\pi\psi_\alpha$ where ψ_α is an anticommuting c-number spinor then the generator of the shift symmetry is $\partial/\partial\psi_\alpha$. Hence all expressions are independent of ψ_α when expressed in terms of $\widetilde{W}_\alpha$ and Φ. The shift symmetry combines the single trace chiral operators into

$$\widetilde{r}_k = -\frac{1}{32\pi^2}\mathrm{Tr}\ \widetilde{W}_\alpha^2\Phi^k = r_k - \psi_\alpha w_k^\alpha - \frac{1}{2}\psi_\alpha\psi^\alpha u_k. \tag{6.2.4}$$

The shift symmetric generators of the chiral rotations are L_n and

$$\begin{aligned} \widetilde{Q}_{n,\alpha} &= \frac{1}{4\pi}\widetilde{W}_\alpha\Phi^{n+1}\frac{\partial}{\partial\Phi} = Q_{n,\alpha} + \psi_\alpha L_n, \\ \widetilde{R}_n &= -\frac{1}{32\pi^2}\widetilde{W}_\alpha^2\Phi^{n+1}\frac{\partial}{\partial\Phi} = R_n - \psi_\alpha Q_n^\alpha - \frac{1}{2}\psi_\alpha\psi^\alpha L_n. \end{aligned} \tag{6.2.5}$$

Shift invariance implies that the commutation relations can be written in terms of $L_n, \widetilde{Q}_{n,\alpha}$ and $\widetilde{R}_n$. We find that the shift invariant commutation relations are

$$\begin{aligned} [L_m, \widetilde{R}_n] &= (n-m)\widetilde{R}_{m+n}, \\ \{\widetilde{Q}_{m,\alpha}, \widetilde{Q}_{n,\beta}\} &= -\epsilon_{\alpha,\beta}(n-m)\widetilde{R}_{m+n}, \\ [\widetilde{R}_m, \widetilde{R}_n] &= 0. \end{aligned} \tag{6.2.6}$$

We did not write down the $[L, \widetilde{Q}]$ and $[\widetilde{Q}, \widetilde{R}]$ commutators because they are contained in the $[L, \widetilde{R}]$ and $[\widetilde{R}, \widetilde{R}]$ commutators respectively. For future reference let us show that the first and the third commutation relation in (6.2.6) imply the remaining relation. The first commutator contains the $[L, L], [L, Q]$ and $[L, R]$ commutators. If we expand the last commutator in ψ_α, all commutators are trivially zero except for the commutator multiplying $\psi_\alpha\psi^\alpha$ which is

$$[L_m, R_n] + [R_m, L_n] + \epsilon_{\alpha\beta}\{Q_m^\alpha, Q_n^\beta\} = 0. \tag{6.2.7}$$

We use this equation together with the $[L, R]$ commutator to get the $\{Q, Q\}$ commutator. Hence, the first and third commutator in (6.2.6) contain all commutation relations of the partial $\mathcal{N} = 1$ super-Virasoro algebra.

6.3 Wess-Zumino Consistency Conditions for the Konishi Anomaly

Assume that the adjoint scalar has the tree level superpotential

$$W(\Phi) = \sum_{i=1}^{n+1} \frac{g_i}{i} \mathrm{Tr}\ \Phi^i. \tag{6.3.1}$$

The effective superpotential of the gauge theory is

$$\exp\left(-\int d^4x d^2\theta W_{eff}\right) = \left\langle \exp\left(-\int d^4x d^2\theta W(\Phi)\right)\right\rangle, \tag{6.3.2}$$

where the path integral is over the massive fields in the presence of a slowly varying background gauge field. The effective superpotential has an anomaly under the chiral rotations generated by $L_n, Q_{n,\alpha}, R_n$

$$\begin{aligned} L_n W_{eff} &= \mathcal{L}_n, \\ Q_{n,\alpha} W_{eff} &= \mathcal{Q}_{n,\alpha}, \\ R_n W_{eff} &= \mathcal{R}_n. \end{aligned} \tag{6.3.3}$$

The perturbative anomaly of the effective superpotential under the chiral rotations $\widetilde{R}_n$ were derived in [20]

$$\widetilde{\mathcal{R}}_k = \sum_{i=1}^{n+1} g_i \widetilde{r}_{k+i} - \sum_{i=0}^{k} \widetilde{r}_i \widetilde{r}_{k-i}. \tag{6.3.4}$$

The equation (6.3.4) is obtained from the $1/z^{k+2}$ term of the equation (4.14) of [20] for the generating function for the generalized Konishi anomaly, remembering that the g_i in this chapter is g_{i-1} of [20]. The first part of $\widetilde{\mathcal{R}}_k$ is the classical variation of the superpotential and the second part comes from the anomalous transformation of the measure of Φ under the chiral rotations. The anomalous divergence of the currents generating the chiral rotations is the Konishi anomaly

$$\begin{aligned} \overline{D}_\alpha \overline{D}^\alpha J_{L_n} &= \mathcal{L}_n, \\ &\ldots . \end{aligned} \tag{6.3.5}$$

The generalized Konishi anomaly, being $\overline{D}_\alpha$ exact, is a chiral ring descendant. Setting (6.3.4) to zero gives nontrivial relations between the chiral operators, which enabled the authors of [20] to study the dynamics of the gauge theory and to give a partial proof of the Dijkgraaf-Vafa conjecture. We will return to this in more detail in the last section.

The Lie algebra structure of the chiral rotations implies relations between anomalies of different chiral rotations. These conditions were first discussed by Wess and Zumino [91]. They express the closure of the Lie algebra under commutation relations. For two chiral rotations R_1 and R_2 the anomaly of the effective superpotential under $R_1R_2 - R_2R_1$ must be the same as the anomaly under $R_3 = [R_1, R_2]$

$$R_1\mathcal{R}_2 - R_2\mathcal{R}_1 = \mathcal{R}_{[R_1,R_2]}. \tag{6.3.6}$$

The Wess-Zumino consistency conditions for the algebra of chiral rotations (6.2.3) are

$$\begin{aligned} L_m\mathcal{L}_n - L_n\mathcal{L}_m &= (n-m)\mathcal{L}_{n+m}, \\ L_m\mathcal{Q}_{n,\alpha} - Q_{n,\alpha}\mathcal{L}_m &= (n-m)\mathcal{Q}_{n,\alpha}, \\ &\dots . \end{aligned} \tag{6.3.7}$$

In the shift invariant notation, we have

$$\begin{aligned} L_m\widetilde{\mathcal{R}}_n - \widetilde{R}_n\mathcal{L}_m &= (n-m)\widetilde{\mathcal{R}}_{m+n}, \\ \widetilde{Q}_{m,\alpha}\widetilde{\mathcal{Q}}_{n,\beta} + \widetilde{Q}_{n,\beta}\widetilde{\mathcal{Q}}_{m,\alpha} &= -\epsilon_{\alpha\beta}(n-m)\widetilde{\mathcal{R}}_{m+n}, \\ \widetilde{R}_m\widetilde{\mathcal{R}}_n - \widetilde{R}_n\widetilde{\mathcal{R}}_m &= 0. \end{aligned} \tag{6.3.8}$$

Let us verify that the perturbative anomaly (6.3.4) satisfies the Wess-Zumino consistency conditions. The calculations are routine so we will check only the first equation in (6.3.7). Expanding (6.3.4) with respect to ψ_α we find using (6.2.4) and (6.2.5)

$$\mathcal{L}_k = \sum_{i=1}^{n+1} g_i u_{k+i} - 2\sum_{i=0}^{k} u_i r_{k-i}. \tag{6.3.9}$$

The action of L_k on $\mathcal{L}_l$ is

$$L_k\mathcal{L}_l = \sum_{i=1}^{n+1}(k+i)g_i u_{l+k} - 2\sum_{i=0}^{l}\left(i\widetilde{u}_{i+k}r_{l-i} + (l-i)u_i r_{k+l-i}\right). \tag{6.3.10}$$

Subtracting from this the analogous expression for $L_l\mathcal{L}_k$ we get

$$L_k\mathcal{L}_l - L_l\mathcal{L}_k = (l-k)\mathcal{L}_{k+l} \tag{6.3.11}$$

which is the Wess-Zumino consistency condition (6.3.7) for the Virasoro subalgebra.

6.4 Nonpertubative Corrections

In this section we review the argument for the absence of the multi-loop corrections to the generalized Konishi anomaly and then discuss the structure of nonperturbative corrections. For this, it is instrumental to study the $U(1)$ symmetries of the gauge theory. The gauge theory has two continuous symmetries, a standard $U(1)_R$ symmetry and a symmetry $U(1)_\Phi$ under which the entire superfield Φ undergoes a rotation

$$\Phi \to e^{i\alpha}\Phi. \tag{6.4.1}$$

We also introduce a linear combination of these, $U(1)_\theta$, which is convenient in certain arguments. These symmetries are symmetries of the theory with nonzero superpotential if we assign nonzero $U(1)$ charges to the couplings g_k.

$$\begin{array}{ccccc} & \Delta & Q_\Phi & Q_R & Q_\theta \\ \Phi & 1 & 1 & 2/3 & 0 \\ W_\alpha & 3/2 & 0 & 1 & 1 \\ g_k & 1-k & -k & \frac{2}{3}(1-k) & 2 \\ \Lambda^{2l} & 2l & 2l & 4l/3 & 0 \\ \widetilde{\mathcal{R}}_k & 6+k & k & 4+2k/3 & 4 \end{array} \tag{6.4.2}$$

The one-loop beta function coefficient is $2l = 3c(Adj) - c(R)$ where $c(R)$ is the index of the representation R of the matter field

$$\begin{array}{ccccccc} R & U_{Adj}(N) & SO(N)_A & SO(N)_S & Sp(N)_A & Sp(N)_S \\ c(R) & N & N-2 & N+2 & N-1 & N+1 \\ l & N & N-2 & N-3 & N+2 & N+1 \end{array} \tag{6.4.3}$$

The shift invariant $\widetilde{W}_\alpha$ and the anticommuting shift c-number ψ_α have the same $U(1)$ charges as W_α. These symmetries are violated at one loop. In the last line of the table (6.4.2) we have written the charges by which the anomaly $\widetilde{\mathcal{R}}_k$ violates the $U(1)$ symmetries. The higher loop computations are finite and the $U(1)$ symmetries leave them invariant.

We are now ready to analyze the corrections to the generalized Konishi anomaly (6.3.4). The corrections must have the same $U(1)$ charges as $\widetilde{\mathcal{R}}_k$. They are polynomial in the chiral operators. Furthermore, the corrections that depend on g_k must vanish for the theory with zero superpotential and the nonperturbative corrections that depend on Λ^{2l} vanish when we take the strong coupling scale Λ to zero. Hence, the corrections to the anomaly are also polynomial in g_k and Λ^{2l}.

Referring to the table (6.4.2) we see that the only polynomials in g_k, Φ and W_α with the quantum numbers of $\widetilde{\mathcal{R}}_k$ are the ones already present in the one loop expression (6.3.4). Hence the anomaly does not have higher loop contributions, as claimed at the end of the previous paragraph. The nonperturbative corrections are polynomial in Λ^{2l}. The possible j instanton corrections to $\widetilde{\mathcal{R}}_k$ are of the form $\Lambda^{2jl} g_{i+2jl} \widetilde{r}_{k+i}$ and $\Lambda^{2jl} \widetilde{r}_{i-2jl} \widetilde{r}_{k-i}$.

We can similarly derive the possible form of corrections to the extended Virasoro algebra (6.2.3). The corrections to the $[L, L]$ commutator are linear the Virasoro generators L_n and polynomial in g_k and Λ^{2h}. The Virasoro generator L_n (6.2.1) increases the $U(1)$ charges of a chiral operators by the same value as multiplication by Φ^n. Hence, the commutator $[L_m, L_n]$ fixes Q_θ and increases the dimension by $m + n$. Consulting the table (6.4.2) we see that g_i has $Q_\theta = 2$ charge so the there are no corrections that depend on the superpotential. The nonperturbative l instanton corrections have the form $\Lambda^{2jl} L_{m+n-2jl}$. Similar corrections contribute to the $[L, Q], [L, R]$ and Q, Q. The commutators that shift Q_θ by two can also have corrections proportional to g_i. Counting the $U(1)$ charges we see that the $[L_m, R_n]$ commutator has corrections $\Lambda^{2jl} R_{m+n-2jl}$ and $\Lambda^{2jl} g_i L_{m+n+i-2jl}$. There are similar corrections to $\{Q, Q\}$. The $[Q, R]$ and $[R, R]$ commutators cannot have corrections because they map chiral operators into chiral ring descendants.

6.5 Nonrenormalization of the Algebra of Chiral Rotations

In this section we prove the nonrenormalization of the algebra (6.2.3) of chiral rotations of the $U(N)$ adjoint scalar. Firstly we analyze in detail the corrections to the partial Virasoro subalgebra

$$[L_m, L_n] = (n - m) L_{m+n} + \sum_{j>0} \Lambda^{2jN} b^j_{m,n} L_{m+n-2jN}, \tag{6.5.1}$$

where the coefficients $b^j_{m,n}$ are antisymmetric in m and n by antisymmetry of the commutator (6.5.1). The coefficient $b^j_{m,n}$ is in front of $L_{m+n-2jN}$ hence it vanishes if $m + n - 2jN < -1$ because L_{-1} is the lowest nonzero generator. We will prove that all the coefficients $b^j_{m,n}$ can be absorbed into nonperturbative redefinition of the Virasoro generators

$$\mathrm{L}_n = L_n + \sum_{j>0} a^j_n \Lambda^{2jN} L_{n-2jN}, \tag{6.5.2}$$

where a_n^j is the coefficient of the j-instanton correction to L_n. The Virasoro generators are corrected which is natural considering that they act on the nonperturbatively corrected chiral operators $\widetilde{r}_k$. In terms of the new basis of generators L_n the commutations relations of the partial Virasoro algebra remain valid nonperturbatively

$$[\mathrm{L}_m, \mathrm{L}_n] = (n-m)\mathrm{L}_{m+n}. \tag{6.5.3}$$

Calculating the coefficients a_n^j which parameterize the nonperturbative corrections to L_n's is beyond the scope of the present chapter. We will show instead that there is a choice of a_n^j's for which the Virasoro algebra takes the standard form (6.5.3). This shows that the algebra itself is not corrected even though the Virasoro operators might receive corrections. We make induction in the instanton number of the nonperturbative corrections. The coefficients $b_{m,n}^j$ obey equations that follow from the Jacobi identity

$$[L_l, [L_m, L_n]] + [L_n, [L_l, L_m]] + [L_m, [L_n, L_l]] = 0. \tag{6.5.4}$$

On the zero instanton level the identity reduces to the Jacobi identity for the Virasoro algebra which is satisfied. On the one instanton level, we evaluate the commutators in (6.5.4) using (6.5.1) to find the coefficient of the $\Lambda^{2N} L_{l+m+n-2N}$ term which has to be zero

$$(n-m)b_{l,m+n}^1 + (m+n-l-2N)b_{m,n}^1 + \textit{cyclic permutations} = 0. \tag{6.5.5}$$

The one instanton corrections can be absorbed into one instanton corrections to L_n's (6.5.2). The new commutation relations are

$$[\mathrm{L}_m, \mathrm{L}_n] = (n-m)\mathrm{L}_{m+n} + \mathrm{b}_{m,n}^1 \Lambda^{2N} \mathrm{L}_{m+n-2N} + \dots, \tag{6.5.6}$$

where $\mathrm{b}_{\mathrm{m,n}}^1$'s are the redefined nonperturbative corrections

$$\mathrm{b}_{m,n}^1 = b_{m,n}^1 + (n-m-2N)a_n^1 + (n-m+2N)a_m^1 - (n-m)a_{m+n}^1. \tag{6.5.7}$$

We show that $b_{m,n}^1$ can be set to zero by redefinition $\mathrm{L}_{m+n} = L_{m+n} + a_{m+n}^1 \Lambda^{2N} \mathrm{L}_{m+n-2N}$ by induction on $m+n$. The first step of the induction holds because $b_{m,n}^1$ vanishes for $m+n < 2N-1$. By induction hypothesis we assume that we have redefined L_{m+n} for $m+n < M$ so that $b^1_{m,n} = 0$. Setting l, m, n in equation (6.5.5) equal to $0, m, M-m$ respectively, we find for $0 < m < M$

$$(M-2m)b^1_{0,M} + (M-2N)b^1_{m,M-m} + (m-M)b^1_{m,M-m} + mb^1_{M-m,m} = 0. \quad (6.5.8)$$

Using antisymmetry of $b^1_{m,n}$ in m and n we rewrite this as

$$2Nb^1_{m,M-m} = (M-2m)b^1_{0,M}. \quad (6.5.9)$$

From (6.5.7) the redefined nonperturbative corrections are

$$\begin{aligned} \mathrm{b}^1_{0,M} &= b^1_{0,M} - 2Na^1_M, \\ \mathrm{b}^1_{m,M-m} &= b^1_{m,M-m} - (M-2m)a^1_M. \end{aligned} \quad (6.5.10)$$

We see from (6.5.9) that taking $a_M = b^1_{0,M}/2N$ we set $\mathrm{b}^1_{m,n} = 0$ for $m+n = M$. This completes the induction in $m+n$ and shows that there are no one instanton corrections to the Virasoro algebra. We can now proceed with the induction in the instanton number by assuming absence of nonperturbative corrections to the Virasoro algebra for instanton number less than k. We also assume that we have redefined the the Virasoro operators L_n up to instanton number $k-1$ to set $b^j_{m,n} = 0$ for $j < k$. The proof that the k instanton corrections to the Virasoro algebra can be absorbed into k instanton redefinition of the operators L_n goes exactly as the above calculation in the one instanton case because the necessary equations at the Λ^{2kN} order are identical to the equations (6.5.5), (6.5.7) – (6.5.10) we found at Λ^{2N} order after substituting N for kN in all equations. The additional terms in (6.5.5) and (6.5.7) that would come from lower instanton corrections vanish by the induction hypothesis.

Now it remains to show that the commutation relations of $\mathrm{L}_{-1} = L_{-1}$ with L_n do not get corrected. Firstly consider one instanton corrections. Notice that $b^1_{-1,0}$ vanishes on dimensional grounds as noted below (6.5.1). Taking l, m, n in (6.5.5) to be $-1, 0, n$ for $n > 0$ we find $2Nb^1_{1,n} = 0$ which completes the proof of the absence of one instanton corrections. We prove the absence of k instantons corrections the same way after substituting N for kN in (6.5.5).

We give two different proofs of the nonrenormalization of the remaining commutators of the algebra of chiral rotations. The first one is simpler and uses the shift symmetry of the commutations relations. The second one does not use the $U(1)$

shift symmetry and hence is applicable for the $SO(N)$ and $Sp(N)$ gauge theories as well. We postpone it to the appendix 6.A because it is more technical. From now on we do not use roman font to distinguish the nonperturbatively defined generators.

Let us outline the first argument. We use shift symmetry to fix the nonperturbative definitions $\mathrm{Q}_{n,\alpha}, \mathrm{R}_n$ for $n \geq 2N$ using the nonperturbatively defined L_n (6.5.2). The last commutator in (6.2.6)

$$[\widetilde{R}_m, \widetilde{R}_n] = 0 \tag{6.5.11}$$

cannot receive nonperturbative corrections. Its lowest ψ_α component is the $[\mathrm{R}, \mathrm{R}] = 0$ commutator which has to vanish in the chiral ring because the commutator shifts Φ by a chiral operator containing the fourth power of W_α. But the third and higher powers of W_α are chiral ring descendants, so the commutator has trivial action in the chiral ring. The nonperturbative corrections to the first commutator in (6.2.6) that are allowed by shift symmetry are

$$[L_m, \widetilde{R}_n] = (n-m)\widetilde{R}_{m+n} + \sum_{j=1}^{\infty}\sum_{i=1}^{n+1} \Lambda^{2jN} g_i c^{i,j}_{m,n} L_{m+n+i-2jN} \tag{6.5.12}$$

because the ψ_α^2 component of (6.5.12) is the $[L, L]$ commutator, which does not have nonperturbative corrections. The nonperturbative corrections (6.5.12) contribute to the $[L, R]$ commutator only. To prove that these corrections vanish we evaluate the L, Q, R Jacobi identity

$$\begin{aligned}
[Q_{l,\alpha}, [L_m, R_n]] + [L_m, [R_n, Q_{l,\alpha}]] + [R_n, [Q_{l,\alpha}, L_m]] = \\
= [Q_{l,\alpha}, \sum_{j=1}^{\infty}\sum_{i=1}^{n+1} \Lambda^{2jN} g_i c^{i,j}_{m,n} L_{m+n+i-2jN}] = \\
= \sum_{j=1}^{\infty}\sum_{i=1}^{n+1} \Lambda^{2jN} g_i (m+n-l+i-2jN) c^{i,j}_{m,n} Q_{m+n+i+l-2jN,\alpha} = 0.
\end{aligned} \tag{6.5.13}$$

In simplifying (6.5.13) we used the $[L, Q]$ commutator (6.5.12) which is nonrenormalized by shift symmetry and the $[R, Q] = 0$ commutator. Clearly, the only way to satisfy the Jacobi identity (6.5.13) is that $c^{i,j}_{m,n} = 0$. All corrections to (6.5.12) vanish. Hence, none of the commutation relations of the extended Virasoro algebra get nonperturbative corrections because as we noted below (6.2.6) the above two commutators imply the remaining one.

6.6 Nonperturbative Corrections to the Konishi Anomaly

Let us now consider nonperturbative corrections to the anomaly. The anomaly $\widetilde{\mathcal{R}}_k$ (6.3.4) differs from its perturbative value implicitly through the dependence of the chiral operators $\widetilde{r}_k$ on nonperturbative physics. In this section we ask the question whether there are additional nonperturbative corrections that depend explicitly on Λ^{2jN}. We can easily introduce terms proportional to Λ^{2jN} into the expression for $\mathcal{R}_k$ by redefining the chiral operators

$$\widetilde{r}_k = \widetilde{r}_k + \alpha\Lambda^{2N}\widetilde{r}_{k-2N} + \ldots . \tag{6.6.1}$$

Notice that r_k for $k > 1$ are nonrenormalizable operators so their value depends on the renormalization scheme. It is natural to expect terms of the form (6.6.1) to relate the definitions of r_k coming from different renormalization schemes. Hence we expect that the anomaly has generically terms proportional to Λ^{2jN} if we take some arbitrary prescription for $\widetilde{r}_k$.

However, there is a natural definition of the higher moments $\widetilde{r}_k$. In the previous section we showed that there is a preferred basis for the generators of the chiral rotations $\widetilde{R}_k$ in terms of which the partial super-Virasoro algebra takes the standard form (6.2.6). We can use their action on the chiral operators to give a nonperturbative definition of nonrenormalizable operators $\widetilde{r}_k$ for $k > 1$ in terms of the the first moment $\widetilde{r}_k = L_k\widetilde{r}_1$. It follows from the commutation relations (6.2.6) that remaining operators $\widetilde{R}_k$ act on the chiral operators as before (6.2.2).

Having defined $\widetilde{r}_k$ nonperturbatively, we can now show using the Wess-Zumino consistency conditions that the one-loop anomaly $\sum \widetilde{r}_i\widetilde{r}_{k-i}$ in the path integral measure for Φ does not have nonperturbative corrections. We will also show that the consistency conditions allow nonperturbative renormalization of the superpotential. The consistency conditions of the full gauge theory (6.3.8) do not have nonperturbative corrections because their derivation rested only on the commutation relations of the super-Virasoro algebra (6.2.6) which are nonrenormalized. We deduced in section 4 using $U(1)$ symmetries that the general form of nonperturbative corrections to $\widetilde{\mathcal{R}}_n$ is

$$\begin{aligned}\widetilde{\mathcal{R}}_k = &\sum_i (g_i + \Lambda^{2N} g_{i+2N} c^1_{k,i} + \ldots)\widetilde{r}_{k+i} + \\ &- \sum_{i=0}^{k} \widetilde{r}_i\widetilde{r}_{k-i} - \Lambda^{2N} \sum_{i=2N}^{k} d^1_{k,i}\widetilde{r}_{i-2N}\widetilde{r}_{k-i} + \ldots .\end{aligned} \tag{6.6.2}$$

In writing (6.6.2) we take $g_k = 0$ for $k < 1$ and $k > n+1$ to simplify the notation. We can consider the corrections to the the superpotential separately from the corrections to the one-loop anomaly. The corrections to the superpotential are proportional to $\Lambda^{2jN} g_{i+2jN}$ which have the same quantum numbers as $\widetilde{r}_{-i}$ which does not exist. Hence the two types of corrections do not mix.

Firstly we show that all the nonperturbative corrections to the one-loop part of $\widetilde{\mathcal{R}}_k$ vanish. Notice, that the lowest dimensional correction is $\widetilde{r}_0\widetilde{r}_0\Lambda^{2N}$ which contributes to $\widetilde{\mathcal{R}}_{2N}$, hence the one-loop parts $\widetilde{\mathcal{R}}_k$ for $k = -1, 0, \ldots, 2N-1$ does not have nonperturbative corrections. The first consistency condition (6.3.8) with $m = 0$ simplifies to $L_0\widetilde{\mathcal{R}}_k = k\widetilde{\mathcal{R}}_k$ because $R_k\widetilde{r}_0\widetilde{r}_0 = 0$. In other words L_0 acting on $\widetilde{\mathcal{R}}_k$ gives k times the anomaly. But L_0 acting on a j-instanton correction $\Lambda^{2jN}\widetilde{r}_{i-2jN}\widetilde{r}_{k-i}$ gives back $k-2jN$ multiple of the correction, whence all nonperturbative corrections to the one-loop part of the anomaly vanish.

It remains to consider the corrections to the classical part of $\widetilde{\mathcal{R}}_n$. We find from (6.6.2) that the first consistency condition (6.3.8) becomes

$$L_k\widetilde{\mathcal{R}}_l - \widetilde{R}_l\mathcal{L}_k = (l-k)\widetilde{\mathcal{R}}_{l+k} + \sum_{j\geq 1}\Lambda^{2jN}\sum_{i=-2jN}^{n-2jN}[(l+1)c^j_{l,i} - (k+1)c^j_{k,i} - (l-k)c^j_{k+l,i}]g_{i+2jN}\widetilde{r}_{k+l}. \tag{6.6.3}$$

But the Wess-Zumino consistency conditions do not have nonperturbative corrections whence we set the terms in the square brackets to zero

$$(l+1)c^j_{l,i} - (k+1)c^j_{k,i} = (l-k)c^j_{k+l,i}. \tag{6.6.4}$$

Taking $l = 0$ we have $c^j_{k,i} = c^j_{0,i}$. Clearly, this solves all the constraints coming from (6.6.4). Notice that the terms $\Lambda^{2jN}c^j_{-1,i}\widetilde{r}_{i-1}$ in $\widetilde{\mathcal{R}}_{-1}$ are absent for $i < 1$ because $\widetilde{r}_k \sim \mathrm{Tr}\,\widetilde{W}^2\Phi^k$ is defined only for positive k. Hence $c^j_{k,i} = 0$ for $i < 1$. In conclusion, the general form of the anomaly is

$$\widetilde{\mathcal{R}}_k = \sum_{i=1}^{n+1} \mathrm{g}_i\widetilde{r}_{k+i} - \sum_{i=0}^{k}\widetilde{r}_i\widetilde{r}_{k-i} \tag{6.6.5}$$

where

$$\mathrm{g}_i = g_i + \Lambda^{2N}c^1_{0,i}g_{i+2N} + \Lambda^{4N}c^2_{0,i}g_{i+4N} + \ldots. \tag{6.6.6}$$

are the nonperturbatively renormalized coefficients of the superpotential. Hence, all corrections to the classical part of the anomaly allowed by the Wess-Zumino consistency conditions can be absorbed into nonperturbative renormalization of the superpotential

$$W(\Phi) = \sum_{i=1}^{n+1} \frac{\mathrm{g}_i}{i} \mathrm{Tr}\ \Phi^i. \tag{6.6.7}$$

The superpotentials of degree less than $2N+1$ cannot have nonperturbative corrections. This is the only ambiguity that is not fixed by the consistency conditions. We could have anticipated it from the observation that both g_i and Λ^{2N} are invariant under the chiral rotations hence substituting for g_i any polynomial $g_i(g_k, \Lambda^{2N})$ with the correct quantum numbers cannot spoil the Wess-Zumino consistency conditions whose validity depends only on the Lie algebraic structure of the chiral rotations. As noted around (6.6.1) the nonperturbative corrections depend on the scheme used to define the single trace operators $\widetilde{r}_k$. Using a different UV completion of the gauge theory changes the definition of the chiral operators hence it redefines the superpotential. For further discussion of Dijkgraaf-Vafa conjecture for high degree superpotentials, see [21] [22] and [23].

6.7 $SO(N)$ and $Sp(N)$ Gauge Theories

In this section we show that the previous analysis applies with minor modifications to the $SO(N)$ and $Sp(N)$ gauge theories. It follows that the generalized Konishi anomaly in these gauge theories does not have nonperturbative corrections for superpotentials of degree less than $2l+1$. Superpotentials of higher degree might get nonperturbatively renormalized.

The gauge group do not have a decoupled diagonal $U(1)$ subgroup hence the arguments based on the shift symmetry do not carry over from the $U(N)$ case. That is the main reason why we gave a separate proof of the nonrenormalizability of the extended Virasoro algebra which did not use shift symmetry. For simplicity, we do not consider the fermionic generators and chiral operators. The $SO(N)$ adjoint can be represented by an $N \times N$ antisymmetric matrix $\Phi^{\mathrm{T}} = -\Phi$. The gauge field transforms in the adjoint representation hence it is antisymmetric as well $W_\alpha^{\mathrm{T}} = -W_\alpha$. The $Sp(N)$ has adjoint which can be represented as $2N \times 2N$ matrix that satisfies the condition $\Phi^{\mathrm{T}} = -J\Phi J^{-1}$ where J is the invariant antisymmetric tensor of $Sp(N)$. A matrix in the adjoint representation of $Sp(N)$ can be written as a product of a symmetric matrix S and the invariant tensor $\Phi = SJ$, which explains why this representation is called symmetric in the literature. The single trace chiral operators for both gauge groups are u_{2k} and r_{2k} because the remaining chiral operators vanish by antisymmetry. Hence the odd coefficients of the superpotential (6.3.1) vanish $g_{2k+1} = 0$. Similarly the nonvanishing generators of the algebra of chiral rotations are L_{2k} and R_{2k} which form a closed subalgebra of the partial $\mathcal{N} = 1$ super-Virasoro algebra (6.2.3). Our method also applies to the symmetric tensor $\Phi^{\mathrm{T}} = \Phi$ of $SO(N)$ and the antisymmetric tensor $\Phi^{\mathrm{T}} = J\Phi J^{-1}$ of $Sp(N)$. The definitions of the representations do not restrict the chiral operators nor the chiral rotations.

The generalized Konishi anomaly for the $SO(N)$ and $Sp(N)$ gauge theories has been derived in [92], [26] and [25]

$$\begin{aligned} \mathcal{L}_k &= \sum_{i=1}^{n+1} g_i u_{i+k} - \sum_{i=0}^{k} u_i r_{k-i} + c_k(R) r_k, \\ \mathcal{R}_k &= \sum_{i=1}^{n+1} g_i r_{i+k} - \frac{1}{2}\sum_{i=0}^{k} r_i r_{k-i} \end{aligned} \tag{6.7.1}$$

where $c_k(R)$ depends on the representation R of the matter field

$$\begin{array}{ccccc} R & SO_A(N) & SO_S(N) & Sp_A(N) & Sp_S(N) \\ c_k(R) & 2 & -k-1 & k+1 & -2. \end{array} \tag{6.7.2}$$

In section 5 we proved that the algebra generated by L_k's and R_k's where $k \geq -1$ does not get renormalized. This is the algebra for symmetric $SO(N)$ and antisymmetric $Sp(N)$ matter, hence the algebra of chiral rotations of these gauge theories does not receive nonperturbative corrections. The proof for the adjoint representation works exactly as before if we substitute for all subscripts of the generators in the equations of section 5 twice their value. The proof of the nonrenormalization of the $\mathcal{R}_k$ anomaly also carries over because the only difference in the anomaly compared to the $U(N)$ gauge theory is the $c_k r_k$ term in $\mathcal{L}_k$ which has the same form as $u_0 r_k$ so it cannot receive corrections. The proof for $\mathcal{L}_k$ follows the same pattern but instead of using the Wess-Zumino consistency condition coming from $[L, R]$ commutator we use the condition coming from $[L, L]$ commutator.

6.8 Virasoro Constraints for the One-Matrix Model

In this section we review the exact constraints for the planar level free energy F_m of the one-matrix model [93,94]. We consider the $U(N)$ matrix model that is related to the $U(N)$ gauge theory with the adjoint scalar. The $SO(N)$ and $Sp(N)$ matrix models are treated similarly. We derive the loop equations by considering the Virasoro algebra of redefinitions of the matrix M. This highlights the similarity of the algebraic structure of the loop equations with the gauge theory anomalies. The partition function of the matrix model is

$$Z_m = \exp\left(-\frac{\widehat{N}^2}{g_m^2}F_m\right) = \int d^{\widehat{N}^2}M \exp\left(-\frac{\widehat{N}}{g_m}W(M)\right), \tag{6.8.1}$$

where $W(M) = \sum_{i=1}^{n+1}\frac{g_i}{i}\mathrm{Tr}\ M^i$ is the potential of the matrix model and F_m is the matrix model free energy. The partition function is invariant under arbitrary redefinition of the integration variable $M \to f(M)$. These redefinitions are symmetries of the matrix model. The generators of the redefinitions annihilate the partition function and the free energy

$$R_{m,k} = M^{k+1}\frac{\delta}{\delta M}. \tag{6.8.2}$$

They form a partial Virasoro algebra

$$[R_{m,k}, R_{m,l}] = (l-k)R_{m,k+l}, \tag{6.8.3}$$

where $k, l \geq -1$. Acting with $\epsilon R_{m,k}$ on the free energy F_m we obtain the following identity

$$\begin{aligned} 0 &= \epsilon R_{m,k}F_m \equiv \epsilon\mathcal{R}_{m,k} \\ &= -\frac{g_m^2}{\widehat{N}^2 Z_m}\delta\int d(M+\epsilon M^{n+1})\exp\left(-\frac{\widehat{N}}{g_m}\sum_{i=1}^{n+1}\frac{g_i}{i}\mathrm{Tr}\ (M+\epsilon M^{k+1})^i\right). \end{aligned} \tag{6.8.4}$$

Expanding (6.8.4) to first order in ϵ we have

$$\mathcal{R}_{m,k} = \frac{-g_m^2}{\widehat{N}^2 Z_m}\int dM\left(-\frac{\widehat{N}}{g_m}\sum_{i=1}^{n+1}g_i\mathrm{Tr}\ M^{i+k} + \mathrm{Tr}\ \frac{\delta M^{k+1}}{\delta M}\right)\exp\left(-\frac{\widehat{N}}{g_m}W(M)\right). \tag{6.8.5}$$

To evaluate the Jacobian we write

$$\begin{aligned}\text{Tr}\ \frac{\delta M^{n+1}}{\delta M} &= \frac{\delta M^{k+1}_{ij}}{\delta M_{ij}} = \sum_{i=l}^{k} \frac{(M^l \delta M M^{k-l})_{ij}}{\delta M_{ij}} \\ &= \sum_{l=0}^{k} M^l_{il} \frac{\delta M_{lm}}{\delta M_{ij}} M^{k-l}_{mj} = \sum_{l=0}^{k} \text{Tr}\ M^l \text{Tr}\ M^{k-l}.\end{aligned} \tag{6.8.6}$$

Hence the variation of the free energy is

$$\mathcal{R}_{m,k} = R_{m,k} F_m = \sum_{i=1}^{n+1} g_i \langle \text{Tr}\ M^{i+k} \rangle - \sum_{i=0}^{k} \langle \text{Tr}\ M^i \text{Tr}\ M^{k-i} \rangle. \tag{6.8.7}$$

In the large $\widehat{N}$ limit the expectation values of products $U(N)$ invariant operators factorize $\langle \text{Tr}\ M^i \text{Tr}\ M^{k-i} \rangle = \langle \text{Tr}\ M^i \rangle \langle \text{Tr}\ M^{k-i} \rangle$. Defining $r_{m,k} = \frac{g_m}{\widehat{N}} \langle \text{Tr}\ M^k \rangle$ we rewrite (6.8.7) in the large $\widehat{N}$ limit as

$$\mathcal{R}_{m,k} = \sum_{i=1}^{n+1} g_i r_{m,i+k} - \sum_{i=0}^{k} r_{m,i} r_{m,k-i} \tag{6.8.8}$$

which takes the same form as the as the Konishi anomaly (6.3.4). The loop equations are obtained by setting $\mathcal{R}_{m,k} = 0$. They are recursion relations for $r_{m,k}$ in terms of the first n moments $r_{m,0}, \ldots, r_{m,n-1}$. Equivalently, the loop equations determine the matrix model curve $y^2(z) = W'^2(z) + f(z)$ where $y(z) = \frac{g_m}{\widehat{N}} \langle \text{Tr}\ \frac{1}{z-M} \rangle$ is the resolvent. The consistency conditions for $\mathcal{R}_{m,k}$ are derived the same way as for the gauge theory (6.3.6)

$$R_{m,k} \mathcal{R}_{m,l} - R_{m,l} \mathcal{R}_{m,k} = (l-k) \mathcal{R}_{m,k+l}. \tag{6.8.9}$$

It is easy to verify that (6.8.8) satisfies the consistency conditions (6.8.9). Similarly one can show that the full matrix model loop equations (6.8.7) satisfy (6.8.9).

6.9 Implications for the Dijkgraaf-Vafa conjecture

Let us discuss the implications of the above results for the relation between the matrix models and the supersymmetric gauge theories. We will consider the $U(N)$ gauge theory with adjoint matter to keep the discussion concrete. The anomalous variation of the free energy of the gauge theory under R_k (6.3.4) has the same form as the variation of the matrix model free energy under $R_{m,k}$ (6.8.2) if we identify the expectation values [20]

$$r_k = r_{m,k}. \tag{6.9.1}$$

The equations (6.3.4) $\widetilde{R}_k = 0$ can be considered as recursion relations for higher moments $\widetilde{r}_i$ in terms of the first n moments $\widetilde{r}_0, \widetilde{r}_1 \ldots \widetilde{r}_{n-1}$. Hence it is enough to identify the first n moments in (6.9.1). The matrix model then determines the expectation values of all chiral operators r_i.

The expectation values of the moments of the scalar depend also on the gauge symmetry breaking pattern $U(N) \to \otimes_{i=1}^{r} U(N_i)$ [95]. The $U(1)$ photinos of the $U(N_i)$ subgroups can have arbitrary vacuum expectation value. These values determine all moments of the gaugino field Tr $\Phi^k W_\alpha$ [96]. Hence the isolated massive vacua come with a $2r$-dimensional fermionic moduli space where r is the rank of the low energy gauge group. In conclusion, matrix model determines the expectation values of all chiral operators up to the choice of the gauge symmetry breaking pattern and k independent expectation values of the $U(1)$ photino condensates.

The generalized Konishi anomaly can be viewed as the equation of the curve

$$y^2 = W'^2(z) + f(z) \tag{6.9.2}$$

where y is the generating function of the glueball moments [20]. This curve is identified with the matrix model curve using (6.9.1) which is the same as identifying the polynomials $f(z) = f_m(z)$. The results from section 6 on nonperturbative corrections to the Konishi anomaly imply that the gauge theory curve does not have nonperturbative deformations for superpotentials of degree less than $2N+1$. Hence for these superpotentials the curve of the full gauge theory agrees with matrix model curve. For higher degree of the superpotential the curve can get deformed. We have identified that the only possible deformation of the curve comes from the nonperturbative renormalization of the superpotential. This is so because the form of the curve is uniquely fixed from the Virasoro symmetry and we know from section 5 that the extended Virasoro symmetry is exact in the full gauge theory. For given $f(z) = f_m(z)$, the coefficients of the superpotential are the only parameters of the curve.

The effective superpotential and the matrix model free energy are generating functions for chiral operators and for the moments of M respectively

$$\begin{aligned} \frac{\partial}{\partial g_k} W_{eff} &= \left\langle \frac{\mathrm{Tr}\ \Phi^k}{k} \right\rangle, \\ \frac{\partial F_m}{\partial g_k} &= \left\langle \frac{\mathrm{Tr}\ M^k}{k} \right\rangle. \end{aligned} \tag{6.9.3}$$

To relate W_{eff} and F_m, we use shift symmetry to generalize the first equation in (6.9.3) to a generating function for Tr $\widetilde{W}^2 \Phi^k$. The effective superpotential is invariant under shift symmetry so it can be written as

$$W_{eff} = \int d^2\psi\, \mathcal{F}(\widetilde{r}_i) \tag{6.9.4}$$

for some function $\mathcal{F}$. We use (6.9.4) to rewrite the first equation in (6.9.3) as

$$\frac{\partial}{\partial g_k} \mathcal{F} = \langle \frac{\widetilde{r}_k}{k} \rangle. \tag{6.9.5}$$

Hence we have the relation [20]

$$F_m(S_i, g_k) = \mathcal{F}(\widetilde{S}_i, g_k)|_{\psi=0} + \mathcal{H}(\widetilde{S}_i)|_{\psi=0} \tag{6.9.6}$$

where $\mathcal{H}(\widetilde{S}_i)$ is a coupling independent function. Similar relations for the $Sp(N)$ and $SO(N)$ gauge theory are given in [25] and [21]. The derivation of the relation (6.9.6) rests on the Konishi anomaly equations and on the validity of low energy description of the gauge theory in terms of the glueball fields S_i. The nonrenormalization of the Konishi anomaly implies that $\mathcal{F}$ does not have additional nonperturbative corrections, whence the relation (6.9.6) is valid nonperturbatively. The derivation of the nonperturbative exactness of the Konishi anomaly is the first step in a full proof of the Dijkgraaf-Vafa correspondence.

6.A Second Proof of the Nonrenormalization

In this appendix we give a proof of absence of nonperturbative corrections to the extended Virasoro algebra without using the shift symmetry. This proof is applicable to $SO(N)$ and $Sp(N)$ gauge theories which do not posses shift symmetry. We assume from section 5 the nonrenormalization of the Virasoro subalgebra generated by L_n's because we did not use shift symmetry to prove it. We use the nonperturbatively defined Virasoro generators L_n to fix the nonperturbative definition of the remaining generators by recursively commuting $Q_{n,\alpha}$ and R_n with the raising operator L_1. Having defined the generators, let us show that the nonperturbative corrections to the $[L, Q]$ commutator vanish

$$[L_m, Q_{n,\alpha}] = (n-m)Q_{m+n,\alpha} + \sum_{j=1}^{\infty} \Lambda^{2jN} c^j_{m,n} Q_{m+n-2jN,\alpha}. \tag{6.A.1}$$

Firstly, we prove nonrenormalization of $[L_0, Q_{n,\alpha}]$ using mathematical induction. The lowest dimensional correction to the commutators is $\Lambda^{2N} Q_{-1,\alpha}$ hence the first step of induction is valid because the commutator of L_0 with $Q_{-1,\alpha}, \ldots, Q_{2N-2,\alpha}$ does not have nonperturbative corrections. Assuming the induction hypothesis is valid for $Q_{-1,\alpha}, \ldots, Q_{n,\alpha}$ we calculate

$$\begin{aligned} [L_0, Q_{n+1,\alpha}] &= \frac{1}{n-1}[L_0, [L_1, Q_{n,\alpha}]] \\ &= \frac{1}{n-1}[[L_0, L_1], Q_{n,\alpha}] + \frac{1}{n-1}[L_1, [L_0, Q_{n,\alpha}]] \\ &= (n+1)\frac{[L_1, Q_{n,\alpha}]}{n-1} = (n+1)Q_{n+1,\alpha}, \end{aligned} \tag{6.A.2}$$

where the first equality comes from the recursive definition of $Q_{n+1,\alpha}$, the second from Jacobi identity, the third from the induction hypothesis and the nonrenormalization of the Virasoro algebra and the last equality is again from the recursive definition of $Q_{n+1,\alpha}$. We show the absence of corrections to the remaining $[L, Q]$ commutators by commuting them with L_0 and then using Jacobi identity and the commutators we showed above to be nonrenormalized

$$[L_0, [L_m, Q_{n,\alpha}]] = [[L_0, L_m], Q_{n,\alpha}] + [L_m, [L_0, Q_{n,\alpha}]] = (m+n)[L_m, Q_{n,\alpha}]. \tag{6.A.3}$$

But the $[L_m, Q_{n,\alpha}]$ commutator is a linear combination of $Q_{k,\alpha}$'s which are eigenvectors of the adjoint action of L_0 with eigenvalue k, whence the commutator is proportional to $Q_{m+n,\alpha}$ so all corrections to the commutator vanish. Let us show the absence of corrections to the $[L, R]$ commutator

$$[L_m, R_n] = (n-m)R_{m+n} + \sum_{j=1}^{\infty} \Lambda^{2jN} c^j_{m,n} R_{m+n-2jN} + \sum_{j=1}^{\infty} \sum_{i=1}^{n+1} \Lambda^{2jN} g_i d^{i,j}_{m,n} L_{m+n+i-2jN}. \tag{6.A.4}$$

We commute (6.A.4) with $Q_{l,\alpha}$ to get

$$\begin{aligned}[Q_{l,\alpha},[L_m,R_n]]+[L_m,[R_n,Q_{l,\alpha}]]+[R_n,[Q_{l,\alpha},L_m]] &= \\ =[Q_{l,\alpha},\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\Lambda^{2jN}g_i d^{i,j}_{m,n}L_{m+n+i-2jN}] &= \\ =\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\Lambda^{2jN}g_i(m+n-l+i-2jN)d^{i,j}_{m,n}Q_{m+n+i+l-2jN,\alpha} &= 0.\end{aligned} \tag{6.A.5}$$

In simplifying (6.A.5) we used the $[L,Q]$ commutator which we proved above to be nonrenormalized and the $[R,Q]=0$ commutator. Clearly, the only way to satisfy the Jacobi identity (6.A.5) is that $d^{i,j}_{m,n}=0$. All g_i dependent corrections vanish. The remaining corrections have the same algebraic structure as the corrections (6.A.5) to the $[L,Q]$ commutator so the nonrenormalization proof for that commutator works for the $[L,R]$ commutator as well.

It remains to consider the $\{Q,Q\}$ anticommutator. The nonperturbative corrections are proportional to $\epsilon_{\alpha\beta}$

$$\begin{aligned}\{Q_{\alpha,m},Q_{\beta,n}\}= &-\epsilon_{\alpha\beta}(n-m)R_{m+n}-\epsilon_{\alpha,\beta}\sum_{j=1}^{\infty}\Lambda^{2jN}c^{j}_{m,n}R_{m+n-2jN} \\ &-\epsilon_{\alpha,\beta}\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\Lambda^{2jN}g_i d^{i,j}_{m,n}L_{m+n+i-2jN}.\end{aligned} \tag{6.A.6}$$

Consider the following Jacobi identity

$$\begin{aligned}0=[L_m,\{Q_{0,\alpha},Q_{n,\beta}\}]+\{Q_{0,\alpha},[Q_{n,\beta},L_m]\}-\{Q_{n,\beta},[L_m,Q_{0,\alpha}]\} &= \\ -\epsilon_{\alpha,\beta}\sum_{j=1}^{\infty}\Lambda^{2jN}R_{m+n-2jN}[(n-m-2jN)c^{j}_{0,n}+(m-n)c^{j}_{0,m+n}-mc^{j}_{n,m}] & \\ -\epsilon_{\alpha,\beta}\sum_{j=1}^{\infty}\sum_{i=1}^{n+1}\Lambda^{2jN}g_i L_{m+n+i-2jN}[(n-m+i-2jN)d^{i,j}_{0,n}+(m-n)d^{i,j}_{0,m+n}-md^{i,j}_{n,m}]. &\end{aligned} \tag{6.A.7}$$

Setting $m=0$ we get $c^{j}_{0,n}=0$ and $d^{i,j}_{0,n}=0$ unless $i=2jN$. Substituting this back into (6.A.7) we see that all $c^{j}_{m,n}$ vanish and $d^{i,j}_{m,n}=0$ unless $i=2jN$. To prove that the remaining corrections vanish we evaluate the R,Q,Q Jacobi identity

$$\begin{aligned}[R_0,\{Q_{m,\alpha},Q_{n,\beta}\}]+\{Q_{m,\alpha},[Q_{n,\beta},R_0]\}-\{Q_{n,\beta},[R_0,Q_{m,\alpha}]\} &= \\ [R_0,\{Q_{m,\alpha},Q_{n,\beta}\}] &= \\ -\epsilon_{\alpha,\beta}[R_0,\sum_{j>0}\Lambda^{2jN}g_{2jN}d^{j}_{m,n}L_{m+n}]=-\epsilon_{\alpha,\beta}\sum_{j>0}\Lambda^{2jN}g_{2jN}(m+n)d^{j}_{m,n}R_{m+n} &= 0.\end{aligned} \tag{6.A.8}$$

Hence, $d^j_{m,n} \equiv d^{2jN,j}_{m,n} = 0$ and the $\{Q, Q\}$ anticommutator is nonrenormalized.

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