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Reasoning about Quantum Information: An Overview of Quantum Dynamic Logic

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Abstract: This paper provides an overview of quantum dynamic logics, showing how they have been designed and illustrating how these logics can be applied to verify the correctness of quantum protocols. Similar to the advantages of using dynamic logics to reason about the flow of classical information, the quantum analogues of these logics are tailored to the task of reasoning about the flow of quantum information. We present our logical systems in a modular way, starting with the qualitative logic of quantum measurements and unitary evolutions in single quantum systems, which can already express non-classical effects, e.g., the state-changing interference induced by quantum tests, their non-commutativity, etc. We then move on to logics for compound quantum systems that can capture the non-local features of quantum information: separability, entanglement, correlated measurements, Bell states, etc. We then briefly summarize the logic of quantum probabilities and sketch some applications to quantum protocols.

Keywords: quantum dynamic logic; quantum logic; epistemic logic; quantum information; quantum transition systems; quantum protocols; logical verification

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1. Introduction

A number of different dynamic-logical systems [1–8] were developed over the course of the last decade to model the flow of quantum information, each focusing on a specific feature or type of quantum system. In this paper, we bring these different pieces of the puzzle together and provide a guideline through the development of these various logics and their applications.

The first design of the main framework of dynamic quantum logic was introduced in [1–3] by combining two rather different traditions in logic. On the one hand, it uses input from the work on operational quantum logic and in particular the line work on the logical foundations of Quantum Mechanics that was pioneered by J.M. Jauch and C. Piron in the 1960s and 1970s in Geneva [9–12]. On the other hand, it ties in with the "dynamic turn" in logic, as pursued (mainly but not exclusively) by the Dutch school in modal logic [13,14]. A cross-fertilization between these two lines of inquiry has proven to be very fruitful. From the work of J.M. Jauch and C. Piron, we incorporated features in our mathematical framework which belong to the realm of "actions and physical dynamics", stressing the importance of an action-based operational approach to reason about physical properties. From the modal logicians, we borrowed the tools of modeling the dynamics of processes. The standard setting in modal logic makes use of (multi-modal) Kripke models or so-called labeled transition systems, in which the basic ingredients are labeled binary relations on a set of states [14]. In the context of modeling quantum systems, these relations represent actions on the state of a physical system, leading from input states to output states. By combining the Geneva approach and the modal logic approach, one obtains a setting for quantum dynamic logic that is fit to model quantum computational processes. The power of this cross-fertilization comes from the observation that the tools used to model classical computational processes can indeed be adapted to model quantum information flow. We introduced in 2004 first the dynamic version of quantum logic to model the behavior of a single quantum system [2]. Then, we extended in 2004–2006 our framework to model multi-partite (or compound) systems [1,3]. In our follow-up work, we addressed the issue of providing a probabilistic extension of the framework in [7,8].

In this paper, we will take the reader through this development, gradually introducing our logical setting in several stages. In the first stage, we will focus on the logical analysis of quantum actions, following the work in [2,4]. This fragment of our quantum dynamic logic allows us to fully model the global quantum behavior of any quantum system. In the next stage, we zoom in on the local information that is linked to a quantum system, allowing us to express the properties of separability, entanglement, and (non)-locality [1,3]. Such aspects of (non)-locality appear when we focus on local properties of quantum (sub)systems, and their analysis requires specific logical features that come from spatial or epistemic logics. To proceed in this second stage, we explain the epistemic parts of dynamic quantum logic, following the presentation in [5,6]. Via the use of epistemic logical features, we can describe the information that is potentially available at a specific location or stored in a specific quantum (sub)system. We further show how our framework provides a characterization of entanglement and separability in pure epistemic terms. In the third stage of this paper, we focus on the probabilistic features of quantum mechanics and incorporate an operator that allows us to express the fact that quantum measurements are inherently non-deterministic [7,8]. Thus, even if we knew the original state of the system, we may still be unable to predict with certainty the result of the measurement (except in very special cases), yet we can give a probabilistic estimate.

To illustrate how our formalism works, we apply it to two simple quantum communication protocols, for which we have selected quantum teleportation and quantum secret sharing, and we provide some pointers to the literature for a range of other protocols that can similarly be verified with our logic. By applying this logical setting to the specification and verification of quantum protocols, our work is in line with other recent developments in quantum logic (e.g., the quantum computational logics in [15]), that aim to use the theoretical insights and methods gained from the long quantum-logical tradition and apply them to quantum information theory.[1]

The plan of this paper is as follows. In Section 2, we briefly introduce the syntax of quantum dynamic logic; then, we proceed to present and study its purely dynamic fragment: the logic of quantum actions LQA. In Section 3, we explain how our logic models not only the global information in a quantum system but also local quantum information. More specifically, in Section 3, we focus on the epistemic and probabilistic operators of our framework. In Section 4, we illustrate the power of our formalism on two examples of quantum communication protocols. In Section 5, we end with a short conclusion.

2. The Logic of Quantum Actions

Von Neumann presented his first ideas on quantum logic in [18] and later developed a setting for quantum logic together with Birkhoff in 1936 [19]. The main logical setting in [19] was presented in an algebraic–axiomatic form, but Birkhoff and von Neumann provided a motivation for it that was based on specific semantic ingredients that come directly from the Hilbert-space formalism. At that time, in the early 1930s, the currently known methods of providing a formal semantics had not yet been used in the area of research on quantum logic. Indeed, it still took a while before Tarski's work on the concept of truth in formalized languages (dating from 1930–1931 in Polish and from 1935 in German translation [20]) reached the main logic community. It was W.E. Beth, a logician and philosopher of science in Amsterdam, who proposed to apply Tarski's semantic approach to the formal analysis of physical theories in 1948 and 1949 (see [21–23]). Beth proposed to connect Tarski's semantic method to the logical analysis of classical and quantum mechanics, referring for the latter, among others, to Birkhoff and von Neumann's work. The importance of the semantic method was picked up again later by philosophers of science at the end of the

1960s and early 1970s and was at that time brought to the attention of the wider quantum logic community by B. van Fraassen [24–26]. The semantic analysis of scientific theories certainly helped clarify the ways in which quantum logic can be viewed as a logic in its own right. We stress the importance of this development here for our paper, as it is crucial to understand this background of work when we highlight the important difference between syntax and semantics for our logic. This is the background that also provides insights on the importance of being able to evaluate the truth value of a given sentence in a model. So, here, we work with sentences in a given formal language that express information about the properties of a physical system. The models we work with consist of states at which we evaluate these sentences. Such states refer to the actual realizations of a physical system and belong to the main ingredients coming from physics. The way in which such a logic can be constructed will be made clear further in this section, which starts by introducing first the syntax of the main formalism of Quantum Dynamic Logic QDL and then its quantum semantics.

The design of the logical language below will provide us with enough expressive power to talk about the states and properties of physical systems and comes equipped with operators to model the effect of the main quantum–physical actions (or programs) on these systems, i.e., quantum measurements and unitary evolutions. Zooming in on these quantum actions, we will restrict our attention to a specific type of quantum measurements called "binary measurements". Binary measurements consist of a pair of two so-called "physical tests": one test (or experimental procedure) will come with a specification to check if a given physical property holds, the other one is the test for its negation. Such a test of a physical property (or of its negation) is called successful if, as a result of actually performing the test, the state of the system afterwards will exhibit the tested property. In our logical language, we take such successful tests of physical properties as one of the basic building blocks. Another main building block is given by the unitary evolutions, capturing the specific "quantum logical gates" in quantum information theory such as for instance the Hadamard gate and the controlled NOT gate. Together, these building blocks for quantum actions form the layer of quantum programs in the language below, where such programs are used to express the possible transitions that a quantum system can undergo when it moves from one state to another.

2.1. Syntax of Quantum Dynamic Logic

Given a list (ordered set) Ω of atomic sentences $p, q, \ldots$, and a list $\mathcal{U}$ of actions symbols (meant to denote unitary evolutions), the full language of quantum dynamic logic consists of two layers: a layer of well-formed formulas φ and a layer of programs π defined by mutual induction as follows in BNF form:

$$\begin{array}{lllllll} \varphi & ::= & p & \mid \neg\varphi & \mid \varphi \wedge \varphi & \mid [\pi]\varphi & \mid K_I\varphi \mid P^{\geq r}\varphi \\ \pi & ::= & U & \mid \varphi? & \mid \pi \cup \pi & \mid \pi;\pi & \end{array}$$

where $p \in \Omega$ and $U \in \mathcal{U}$.

We will explain the language of this logic in different stages by focussing on specific fragments. We start first with the dynamic fragment of this logic, which is called the *logic of quantum actions* LQA, and it is obtained by leaving out the epistemic operators K_I and the probabilistic statements $P^{\geq r}\varphi$ (which will be explained in the next section).

The Language of *LQA*. More concretely, the fragment LQA consists of the following two layers, which are defined by mutual induction[2]:

$$\begin{array}{llllll} \varphi & ::= & p & \mid \neg\varphi & \mid \varphi \wedge \varphi & \mid [\pi]\varphi \\ \pi & ::= & U & \mid \varphi? & \mid \pi \cup \pi & \mid \pi;\pi \end{array}$$

The notation φ is used for arbitrary formulas, i.e., well-formed statements about the properties and states of physical systems. The formulas are built from atomic sentences $p, q, \ldots$

coming from a given set Ω, using logical connectives. The state in which a system can be realized is itself a physical property and can similarly be expressed via an atomic sentence.

Propositional Constructs: negation, conjunction and dynamic modalities. Among our logical connectives, we have a classical *negation*: intuitively, $\neg\varphi$ expresses the fact that property φ does *not* hold in the current state of the system. We also have a classical *conjunction* $\varphi \wedge \psi$, intuitively expressing the fact that *both* properties φ and ψ hold in the current state. Finally, the dynamic modality $[\pi]\varphi$ uses ingredients from both the layer of well-formed formulas and the layer of programs. Intuitively, the dynamic modality $[\pi]\varphi$ is the *weakest precondition* that ensures that a property φ will hold after the execution of the action/program π. In other words, $[\pi]\varphi$ expresses the fact that if action π is successfully performed in the current state of the statement, then property φ will definitely hold after that (in any of the possible output states that this action may yield).

Program constructs of LQA. The basic building-blocks of our quantum program constructs π in the language of LQA are *quantum tests* $\varphi?$ of a property φ (where φ comes from the layer of well-formed formulas), as well as *unitary evolutions* U of the physical system. The first intuitively denotes the successful measurement of a property φ (whose formal semantics will be later given in terms of projectors), while the second corresponds to the so-called "quantum logical gates" used in Quantum Computing. More complex programs are built using *concatenation* (sequential composition) of programs $\pi;\pi'$, and the *non-deterministic choice* (union) of programs $\pi \cup \pi'$. Concatenation makes it possible to express, e.g., the performance of a test after we let the system evolve via a logical gate. The arbitrary choice of programs makes it possible to express binary measurements as non-deterministic choices between two successful tests (and more generally, arbitrary measurements in a finite basis as choices between finitely many tests).

Abbreviations: classical disjunction and implication, tautology and contradiction. Classical *disjunction* can be introduced as an abbreviation, namely as the De Morgan dual of conjunction: $\varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi)$. Classical material *implication* is also defined in the standard way as $\varphi \Rightarrow \psi := \neg\varphi \vee \psi$. Finally, we can define the *tautological proposition* ('top') $\top := p \vee \neg p$ and the *contradictory proposition* ("bottom") $\bot := p \wedge p$, where p is the first atomic sentence in our list Ω.

Abbreviation: quantum implication as a dynamic operator. Applying the weakest precondition operator $[\pi]\psi$ to the quantum test $\pi := \varphi?$ of a formula φ, we can define a *quantum implication* as an abbreviation in our logic

$$\varphi \stackrel{S}{\to} \psi := [\varphi?]\psi$$

The notation is due to the fact that quantum implication is also known as 'Sasaki Hook' in quantum logic. Our definition has the advantage that it makes clear that this is not a static conditional, but a *dynamic* operator, whose semantics can best be described in terms of successful tests [2]: a system satisfies $\varphi \stackrel{S}{\to} \psi$ if after any successful test of property φ, it will necessarily satisfy property ψ. This dynamic view of the main quantum implication can be traced back to the analysis of the Sasaki hook as a Stalnaker conditional presented in [28,29], which is a view that is also further reflected upon in, e.g., [30,31].

Abbreviations: quantum negation and quantum disjunction. The dynamic modality $[\pi]\varphi$ also allows us to define the notion of *orthocomplement* (also known as 'quantum negation') in our language as follows:

$$\sim \varphi := [\varphi?]\bot$$

This characterization of orthocomplement indicates that it is a strong, 'dynamic' form of negation, capturing the *impossibility* of successfully testing φ. In standard Hilbert space terms, $\sim \varphi$ means that the system is in a state that is orthogonal to (the linear subspace generated by the set of states satisfying) φ, so that any test of φ will surely yield a negative answer. We can also define the *quantum disjunction* $\sqcup$ of properties as the De Morgan dual via the orthocomplement of the classical conjunction, i.e., as

$$\varphi \sqcup \psi := \sim (\sim \varphi \wedge \sim \psi)$$

Intuitively, $\varphi \sqcup \psi$ says that the system is either in a φ state, or in a ψ state, or in a *superposition* of φ-states and ψ-states.

2.2. Semantics of LQA

The language of LQA can be interpreted in terms of quantum frames or also called 'quantum transition systems'. We illustrate how this works for the concrete case starting from a given physical system and its given Hilbert space $\mathcal{H}$. As standard, a Hilbert space $\mathcal{H}$ is taken to be a complex vector space with an inner product $< -|- >$, which is complete in the induced metric.

Quantum Transition Systems. Any given Hilbert space $\mathcal{H}$ of a physical system can be canonically structured as a *quantum transition system* (also called 'quantum frame') $\mathbf{\Sigma}$ with the following signature:

$$\mathbf{\Sigma} = (\Sigma, \{\stackrel{P?}{\to}\}_{P \in \mathcal{L}}, \{\stackrel{U}{\to}\}_{U \in \mathcal{U}})$$

This structure consists of a set Σ of states and two families of binary relations $\{\stackrel{P?}{\to}\}_{P \in \mathcal{L}} \subseteq \Sigma \times \Sigma$ and $\{\stackrel{U}{\to}\}_{U \in \mathcal{U}} \subseteq \Sigma \times \Sigma$ as explained below. In the remainder of this paragraph, we explain how a Hilbert space $\mathcal{H}$ can be structured into a quantum frame $\mathbf{\Sigma}$.

State space. Given any set of vectors $X \subseteq \mathcal{H}$ in a Hilbert space $\mathcal{H}$, we denote by $\overline{X}$ the closed linear subspace of $\mathcal{H}$ generated by X. In particular, for a singleton $\{x\}$ consisting of only non-zero one-vector $x \in \mathcal{H} \setminus \{0\}$, we can write $\overline{x}$ for $\overline{\{x\}}$, denoting the 'ray' (one-dimensional subspace) generated by x. Our state space $\Sigma := \{\overline{x} : x \in \mathcal{H}\}$ is the set of all such rays (one-dimensional subspaces).

Orthogonality and Bi-orthogonal Closure. Two states in Σ are said to be *orthogonal*, written as $s \perp t$, if every two vectors $x \in s$, $y \in t$ are orthogonal, that is, if their inner product $< x|y >$ equals to 0. For a given set of states $P \subseteq \Sigma$, we write $P^{\perp} := \{t \in \Sigma \,|\, t \perp s \text{ for all } s \in P\}$ for the set of its orthogonal states and we denote by $P^{\perp\perp} := (P^{\perp})^{\perp}$ the *bi-orthogonal closure* of P.

Testable Property. Any arbitrary set of states $P \subseteq \Sigma$ is called a *logical property*, or 'proposition'. Moreover, a set of states $P \subseteq \Sigma$ is called a (*quantum*) *testable property* if and only if it is bi-orthogonally closed, i.e., if $P = P^{\perp\perp}$. We use the notation $\mathcal{L} \subseteq \mathcal{P}(\Sigma)$ to denote the family of all quantum-testable properties. Note that there is a natural bijective correspondence between the family $\mathcal{L}$ of all testable properties and the family $\mathcal{W}$ of all closed linear subspaces W of $\mathcal{H}$, given by the bijection $P \to W_P =: \bigcup P$. The image under this map of the bi-orthogonal closure $P^{\perp\perp}$ of a set $P \subseteq \Sigma$ coincides with the closed linear subspace generated by the union of its states: $\bigcup P^{\perp\perp} = \overline{\bigcup P}$. It is this construction that will be crucial for us later in this section when providing an interpretation to the elements of our syntax of LQA, which makes essential use of these properties.

Quantum Tests. Recall that for any closed linear subspace $W \subseteq \mathcal{H}$, the *projector* $Proj_W : \mathcal{H} \to \mathcal{H}$ onto W is a linear operator given by: $Proj_W(x + y) = x$, for all $x \in W$, $y \in W^{\perp}$ (where $W^{\perp} = \{y \in \mathcal{H} :< x, y >= 0 \text{ for all } x \in W\}$ is the orthogonal space. Projector operators are idempotent ($Proj_W \circ Proj_W = P$) and self-adjoint ($Proj_W^{\dagger} = Proj_W$). Now, to every testable property $P \in \mathcal{L} \subseteq P(\Sigma)$, we assign a partial map $P?$ on Σ, that we call the *quantum test* of P. If $W = \bigcup P$ is the corresponding subspace of $\mathcal{H}$, then the quantum test is the map induced on states by the projector $Proj_W$ onto the subspace W, which is given by:

$$P?(\overline{x}) := \overline{Proj_W(x)} \in \Sigma, \text{ if } \overline{x} \notin P^{\perp} (\text{i.e., if } Proj_W(x) \neq 0)$$

$$P?(\overline{x}) := \text{undefined, otherwise.}$$

We use the notation $\stackrel{P?}{\rightarrow} \subseteq \Sigma \times \Sigma$ to denote the binary relation corresponding to the partial map $P?$ that is given by:

$$s \stackrel{P?}{\rightarrow} t \text{ if and only if } P?(s) = t$$

In our semantics, we work with a family of such binary relations $\{\stackrel{P?}{\rightarrow}\}_{P \in \mathcal{L}}$ indexed by the testable properties $P \in \mathcal{L}$ expressing the state transitions that a successful test of a physical property induces. More specifically, the action of a quantum test $s \stackrel{P?}{\rightarrow} t$ represents the *successful test* of a testable property P on an input state s. A "successful test" indicates that if the state is such that P happens to be successfully tested, then the answer of the binary measurement of P will be "yes", and as a result, the output state t of the system will satisfy property P.

Unitary Evolutions. A *unitary transformation* (or 'quantum evolution') is a linear map U on $\mathcal{H}$ such that $U \circ U^{\dagger} = U^{\dagger} \circ U = id$ (where id is the identity on $\mathcal{H}$ and $U^{\dagger}$ is the adjoint of U). Unitary evolutions preserve inner products. For every such unitary transformation U on $\mathcal{H}$, we consider the corresponding binary relation on the state space $U \subseteq \Sigma \times \Sigma$ given by

$$s \stackrel{U}{\rightarrow} t \text{ if and only if } U(x) = y \text{ for some non-zero vectors } x \in s \text{ and } y \in t$$

In our semantics, we work with a family of such binary relations $\{\stackrel{U}{\rightarrow}\}_{U \in \mathcal{U}}$ indexed by the unitary quantum logical gates U to express the state transitions that are due to unitary evolutions that let the system evolve in a specific way.

The binary relations labeled by the successful tests of testable properties or by unitary evolutions are all state transition relations, expressing how the action of performing a test or letting the system evolve via a logical gate changes the state of a physical system.

Semantic Interpretation of LQA. As standard in logic, we define a quantum–dynamic *model* as a pair $(\mathbf{\Sigma}, \| \bullet \|)$, consisting of a quantum frame (transition system) $\mathbf{\Sigma}$, together with a valuation $\| \bullet \| : \Omega \rightarrow \mathcal{P}(\Sigma)$, assigning to every atomic sentence $p \in \Omega$ some proposition $\|p\| \subseteq \Sigma$. As standard, we can recursively extend the valuation to an *interpretation* function, that maps each well-formed sentence φ to a proposition $\|\varphi\| \subseteq \Sigma$, and it also maps each program π to a binary relation $\stackrel{\pi}{\rightarrow} \subseteq \Sigma \times \Sigma$ on states. The definition of the interpretation map is provided by induction in terms of its compositional clauses for well-formed sentences:

$$s \in \|\neg\varphi\| \text{ iff } s \notin \|\varphi\|$$

$$s \in \|\varphi \wedge \psi\| \text{ iff } s \in \|\varphi\| \text{ and } s \in \|\psi\|$$

$$s \in [\pi]\varphi \text{ iff } t \in \|\varphi\| \text{ whenever } s \stackrel{\pi}{\rightarrow} t$$

As usual, we will sometimes use the *satisfaction relation* $s \models \varphi$ as an alternative notation for $s \in \|\varphi\|$. For programs, the relations $\stackrel{U}{\rightarrow}$ are already given as part of the structure of our quantum transition system, while for tests $\varphi?$, we put

$$s \stackrel{\varphi?}{\rightarrow} t \text{ iff } \overline{\bigcup \|\varphi\|}?(s) = \|\varphi\|^{\perp\perp}?(s) = t.$$

In addition, we have the following recursive clauses for programs:

$$s \stackrel{\pi_1;\pi_2}{\longrightarrow} t \text{ iff } s \stackrel{\pi_1}{\rightarrow} u \stackrel{\pi_2}{\rightarrow} t \text{ for some } u \in \Sigma$$

$$s \stackrel{\pi_1 \cup \pi_2}{\longrightarrow} t \text{ iff } s \stackrel{\pi_1}{\rightarrow} t \text{ or } s \stackrel{\pi_2}{\rightarrow} t$$

These clauses use the fact that the set $P(\Sigma)$ of all the logical properties of a physical system is closed under the standard set-theoretic operations. However, note that the quantum testable properties in $\mathcal{L}$ are only a *subset* of the set $P(\Sigma)$ of all logical properties, corresponding to the bi-orthogonally-closed sets of states. Indeed, to give an interpretation to the quantum negation, quantum disjunction, and Sasaki hook in this semantics, the quantum state transitions that are labeled by quantum testable properties do play a crucial role. Following our earlier definitions, one observes that:

$$s \in \| \sim \varphi\| \text{ iff } s \in \|\varphi\|^{\perp}$$

$$s \in \|[\psi?]\varphi\| \text{ iff } t \in \|\varphi\| \text{ whenever } s \stackrel{\|\psi\|}{\rightarrow} t$$

Conditions on Quantum Transition Systems. A quantum transition system comes equipped with a number of conditions, which are also called 'frame conditions'. These conditions include, e.g., that unitary transformations preserve the orthogonality relation between states or that a successful test of a property that is already true in a state $s \in \Sigma$ will not change the state s, i.e.,

$$\text{if } s \in P \text{ then } s \stackrel{P?}{\rightarrow} s$$

Such conditions ensure that our relational interpretation does capture the state transitions of not just any system but of a quantum system. All these conditions are naturally satisfied in the concrete case where we start from a given Hilbert space as we did in this section. Yet, in case we work on a more abstract level and hence do not necessarily start from a given Hilbert space, we need to regulate the allowed state transitions on the state space Σ and impose such frame conditions to model the behavior of quantum tests and quantum logical gates. For a list of all these conditions and their explanations, we refer the reader to [2,3]. On the proof-theoretic side, we impose in [2,3] a number of corresponding axioms that similarly regulate the behavior of such actions. For instance, the above frame condition that states that testing an already actual property will not change the state will correspond to the axiom:

$$(p \wedge q) \Rightarrow [p?]q$$

When focusing on the logic with quantum tests only, an axiomatization can also be offered in terms of quantum dynamic algebras, as was done in [2]. Moreover, in [2], we also provided Representation Theorems that show these axiomatizations for LQA to be complete with respect to the natural Hilbert-space semantics. These theorems can be viewed as abstract logical completeness results, showing that all qualitative features of single quantum systems (ignoring for now their more fine-grained composition and division into subsystems) are indeed captured by our axioms and that the logical system at hand is well-equipped to provide an intuitive dynamic-operational meaning to key quantum postulates.

3. Epistemic and Probabilistic Operators

We now proceed to extend the dynamic fragment LQA in two stages: first, we add spatial–epistemic modalities; then, we add probabilistic features.

3.1. Quantum Epistemic Features

In this section, we look at *compound* (multi-partite) quantum systems, i.e., systems that are composed of subsystems. In our papers [1,3,5,6], we have proposed a number of different logical formalisms that can capture both the global and the local features of such systems. Here, we focus on the approach taken in [5,6], which uses a "*General Epistemic Logic*" GEL (constructed as an extension of traditional epistemic logic [32]) to give informational–logical characterizations of locality, separability, and entanglement.

Compound Systems. We start from a complex system $N = \{1, \ldots, n\}$ that is composed of n basic *components*. We can think of these components as different "subsystems" of a physical system that can carry information, or we can think of them as "locations" where information can be stored. In standard epistemic logic and in work on multi-agent systems in AI, it is custom to refer to these components in terms of (artificial or human) "agents". In this paper, we will think of them as quantum subsystems. We label each basic component of a given system with an element coming from a given (finite) set $N = \{1, \ldots, n\}$ of components. In case we impose specific restrictions on one or another component, then we will make this explicit in the text. For now, it is important to note that epistemic logic is ideal to reason about the spatial features of such systems that are composed of different components. The added value of epistemic logic is that it comes equipped with epistemic operators that can capture the actual and potentially available information, which is viewed from the perspective of one component (or location) $i \in N$ or from the perspective of a group of components (or a complex subsystem) $I \subseteq N$. The largest group of components is the entire system N, while the smallest groups are singletons $\{i\}$ consisting of an individual component.

Syntax of the Quantum Epistemic Fragment. To reason about compound physical systems, we extend the dynamic fragment LQA of our language with an epistemic operator:

$$\begin{array}{lllllll} \varphi & ::= & p & \mid \neg\varphi & \mid \varphi \wedge \varphi & \mid [\pi]\varphi & \mid K_I\varphi \\ \pi & ::= & U & \mid \varphi? & \mid \pi \cup \pi & \mid \pi;\pi & \end{array}$$

The syntax of our epistemic extension includes an "information" (or "implicit knowledge") operator K_I for each subsystem I of a given physical system. As stressed in [5], the term knowledge is used here only in the *implicit, external* sense as "information that is in principle available" via local observations at a given location. Informally, proposition $K_I\varphi$ refers to the fact that the subsystem I (potentially) carries the information that φ is the case. In the case of one component or individual agent, we write $K_{\{i\}}$ or simply K_i.

Key Semantic Notions. Before we can provide an interpretation to the main epistemic ingredient in our language, we need to introduce a number of key semantic concepts. As the system can be composed of different subsystems, this requires a new concept to refer to the state of a specific subsystem. Similarly, we need to introduce new notions to refer to the actions on a system that affect only a subsystem.

***I*-Local States.** Let us consider a concrete quantum system composed of N subsystems $1, \ldots, n$ as described in the Hilbert space formalism of quantum theory so that its state space $\Sigma = \Sigma_1 \otimes \cdots \otimes \Sigma_n$ is generated from the *tensor product* $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ of n Hilbert spaces. In quantum theory, the "state" of subsystem I can be obtained by taking the partial trace $tr_{N \setminus I}$ (with respect to the subsystem's environment $N \setminus I$) of the density operator [3] associated to the global state $s \in \Sigma$. So, if a global system is in state s, having an associated density operator ρ_s, then the state s_I of any of its subsystems I (possibly entangled with its environment $N \setminus I$ in the state s) is given by :

$$s_I := tr_{N \setminus I}(\rho_s)$$

We call s_I the I-local state of a system that captures the state of information about the global system viewed from the perspective of (sub)system $I \subseteq N$. We denote by Σ_I the set of all local states s_I.

***I*-Local Actions.** In addition to the notion of an I-local state, we introduce the notion of an I-local action such as an I-local test or I-remote unitary evolution. On a given system N, an action can be described as a map $T : \Sigma \to \Sigma$ that transforms one state into another state. For a subsystem $I \subseteq N$, *I-local actions* are maps from I-local states to I-local states, i.e., maps $T_I : \Sigma_I \to \Sigma_I$. In the quantum case, these maps are required to be linear. There are two types of I-local actions that are important for us: tests and unitary evolutions. What we call quantum I-local-tests are actions denoted by S_I? that test an I-local property. In the Hilbert space formalism, these I-local tests correspond to the projectors onto a subspace generated by S_I of $\mathcal{H}_I = \bigotimes_I \mathcal{H}_i$. The I-local evolutions are actions U_I induced on Σ_I by unitary evolutions $U : \mathcal{H}_I \to \mathcal{H}_I$. We denote global actions as the N-actions $T : \Sigma \to \Sigma$ on the global state space.

Given a (global) N-action $T : \Sigma \to \Sigma$, we call this action *I-local* if there exists some I-local action $T_I : \Sigma_I \to \Sigma_I$ such that the global action T can be written as follows:

$$T(s_I \otimes s_{N \setminus I}) = T_I(s_I) \otimes s_{N \setminus I} \quad \text{for all } s_I \in \Sigma_I, s_{N \setminus I} \in \Sigma_{N \setminus I}.$$

Here, we use $\otimes$ as the notation for product states as usual in quantum mechanics.

Observational Equivalence. The above abstract description of I-local states in terms of the mixed states of a quantum system serves us well to extract a relation of 'observational equivalence' $\sim_I \subseteq \Sigma \times \Sigma$ on global states as follows:

$$s \sim_I s' \text{ iff } tr_{N\setminus I}(\rho_s) = tr_{N\setminus I}(\rho'_s)$$

This relation of observational equivalence $s \sim_I s'$ expresses that for any two possible states s and s', if the implicit information carried by a system I is the same in these two states, then they are "observationally equivalent", or "indistinguishable" from the perspective of system I. More specifically, this means that the observations that can be made from the perspective of I in states s and s' are exactly the same.

Concretely, an I-local state satisfies the following:

$$s_I := \{s' \in \Sigma : s \sim_I s'\}$$

In other words, we can express the *local state* s_I of a system (which is in a global state $s \in \Sigma$) as an $\sim_I$-equivalence class consisting of a set of global states that are informationally consistent with the Ith-subsystem.

Note that in [5], we made use of I-local actions in order to show two other equivalent possible characterizations of 'observational equivalence' as a relation on Σ. More specifically, for $I \subseteq N$, and $s, s' \in \Sigma$, the following statements are equivalent:

(1). $tr_{N\setminus I}(\rho_s) = tr_{N\setminus I}(\rho_{s'})$;
(2). For every I-local test, the probability of obtaining any given result is the same in state s as in state s';
(3). $s' = U(s)$ for some I-remote unitary map U.

Properties of Observational Equivalence. One can observe that these equivalent expressions of observational equivalence will all satisfy the following properties:

1. All $\sim_I$ are equivalence relations (i.e., they are reflexive, transitive, and symmetric relations), labeled by sets $I \subseteq N$;
2. *Information is Monotonic with respect to groups* [4]: if $I \subseteq J$ then $\sim_J \subseteq \sim_I$;
3. *Observability Principle*: if $s \sim_N s'$ then $s = s'$.
4. *Vacuous Information*: $s \sim_\emptyset s'$ for all $s, s' \in \Sigma$.

The given properties (1–4) of $\sim_I$ seem natural when thinking of the relations $\sim_I$ in terms of observational equivalence. The first property captures that we see these states as equivalent from the perspective of one subsystem. The second property captures the fact that the states that are observationally equivalent for a system/group I are also observationally equivalent for any subgroup $J \subseteq I$. The third property states that if two states are indistinguishable with respect to the global system N, then they are the same. The last property captures the fact that the empty group cannot distinguish between any two states.

Semantics. The semantic structure of the quantum epistemic fragment of our logic will extend $\mathbf{\Sigma}$ with the family of binary relations $\{\sim_I\}_{I\subseteq N} \subseteq \Sigma \times \Sigma$, one for every subsystem $I \subseteq N$. As explained above, the relation $\sim_I$ is used to capture a notion of "observational equivalence" between possible states of the system. Below, this relation of observational equivalence is employed to give an interpretation to the K_I-operator.

Interpretation of the Epistemic Operator. The epistemic modality $K_I\varphi$ in our language is used to express the fact that the subsystem I carries the information that φ holds (as a property of the global system). As already mentioned, the quantum information carried by a subsystem I (when the global system is in state s) is given by its local state $s_I = tr_{N\setminus I}(\rho_s)$. So, we say that I 'knows' φ whenever φ is entailed by I's local state; i.e., we set:

$$s \in \|K_I\varphi\| \text{ iff } s_I \subseteq \|\varphi\|$$

This amounts to defining

$$\|K_I\varphi\| \ := \ \{s \in \Sigma : t \in \|\varphi\| \text{ for every } t \sim_I s\},$$

which matches the usual definition of knowledge K_I in epistemic logic as the Kripke modality for the corresponding indistinguishability relation $\sim_I$:

$$s \in \|K_I\varphi\| \ \text{ iff } \ t \in \|\varphi\| \text{ for all } t \sim_I s.$$

Properties of Quantum 'Knowledge' Using standard results on correspondence theory in Modal Logic, we can see that the above-mentioned four properties of observational equivalence correspond to the following properties of our epistemic operators:

1. K_I satisfies the *axioms of the standard modal-epistemic system S*5:
$K_I\varphi \Rightarrow \varphi$ (Factivity of Knowledge); $K_I\varphi \Rightarrow K_IK_I\varphi$ (Positive Introspection); $\neg K_I\varphi \Rightarrow K_I\neg K_I\varphi$ (Negative Introspection);
2. *Monotonicity of Knowledge*: if $I \subseteq J$, then $K_I\varphi \Rightarrow K_J\varphi$ (every system knows everything 'known' by its subsystems);
3. *Observability*: $\varphi \Rightarrow K_N\varphi$ (the global system 'knows' everything):
4. *Vacuity*: $K_\emptyset$ is the global modality, quantifying universally over all states (so, in particular, we have $K_\emptyset\varphi \Rightarrow [\pi]\varphi$).

In the remainder of this section, we illustrate how our formalism can be used to model entangled quantum systems that contain non-local information.

Epistemic Characterization of Entanglement and Separability. As explained in [5], we can provide an epistemic–informational characterization of entanglement and separability. To formalize this, we need to be given a formula φ_0 that denotes the state $s_0 = |00\cdots0\rangle = |0\rangle \otimes \cdots \otimes |0\rangle$ (consisting of n separated bits). [5]

Let $I, J \subseteq N$ be disjoint subsystems (with $I \cap J = \emptyset$). We say that I and J are *mutually separable* in a (global) state s if we have $s = s_I \otimes s_J \otimes s_{N-(I\cup J)}$ (where s_I is some state of subsystem I, and the same for s_J, etc.); and say that I and J are *mutually entangled* in s if they are not mutually separable. We say that state s is *I-separable* if I and its environment $N - I$ are mutually separable in s, i.e., $s = s_I \otimes s_{N-I}$; finally, s is *I-entangled* if it is not I-separable (or equivalently, if I and $N - I$ are mutually entangled).

Proposition 1. *Let $I, J \subseteq N$ be disjoint subsystems and φ_0 be a formula denoting the state $s_0 = |00\cdots0\rangle$. Then, I and J are mutually separable in a global state s if we have $s \sim_I \sim_J s_0$.* [6] *As a consequence, I and J are mutually separable in s if s satisfies $\neg K_IK_J\neg\varphi_0$; and hence dually, I and J are mutually entangled in s if s satisfies $K_IK_J\neg\varphi_0$. In particular, s is I-entangled if it satisfies $K_IK_{N\setminus I}\neg\varphi_0$. Finally, s is I-separable if it is not I-entangled, i.e., s satisfies $\neg K_IK_{N\setminus I}\neg\varphi_0$.*

The state $s_0 = |00\cdots0\rangle$ is not playing any special role here except for being a fully separable state: any other such state will do. Essentially, this result captures the fact that two physical (sub-)systems are mutually entangled if and only if they potentially carry (non-trivial) information about each other (without any communication between them).

3.2. Adding Probabilistic Features

As already mentioned, our notion of quantum test $\varphi?$ captures the *successful* test of a property φ: a binary measurement is being made, and the outcome is positive, indicating that the property φ *does hold* (after the measurement). However, of course, binary measurements may also yield a negative outcome, indicating that the quantum negation $\sim \varphi$ holds after that. In general, the (positive or negative) outcome of a quantum measurement is *not* determined in advance: the initial quantum states typically only determines the *probability* of each of the two outcomes.

To capture this probabilistic ingredient, we now extend our language with a probabilistic operator $P^{\geq r}$ to obtain the full logic QDL as follows:

$$\begin{array}{lllllll} \varphi & ::= & p & \mid \neg\varphi & \mid \varphi \wedge \varphi & \mid [\pi]\varphi & \mid K_I\varphi \mid P^{\geq r}\varphi \\ \pi & ::= & U & \mid \varphi? & \mid \pi \cup \pi & \mid \pi;\pi & \end{array}$$

The expression $P^{\geq r}\varphi$ is here used to express the fact that testing property φ (on the current state of the system) will succeed with probability $\geq r$ if we perform a binary measurement of φ on the system. Here, r is a rational number in the interval $[0,1]$. Having this probabilistic modality $P^{\geq r}$ in our language does turn this logic into a probabilistic logic as explained in [7], which enhances the expressive power of the logic and makes it fit for the verification of probabilistic quantum algorithms.

Note that using both the quantum and classical negation in our language, one can express different strict versions of the probabilistic modality as done in [7]:

$$P^{\leq r}\varphi := P^{\geq(1-r)} \sim \varphi$$

$$P^{<r}\varphi := \neg P^{\geq r}\varphi$$

$$P^{>r}\varphi := \neg P^{\leq r}\varphi$$

We can further plug in the probabilistic formula inside the dynamic modality as follows

$$[\varphi?^{\geq r}]\psi := P^{\leq r}\varphi \Rightarrow [\varphi?]\psi$$

This construct has been used in [7] to express that "if the test of φ will succeed with probability $\geq r$, then ψ will be the case after any successful execution of the test".

Semantics of the Probabilistic Operator. To calculate the probabilistic expectation values of a measurement of φ in quantum mechanics, we adopt the Born rule. The Born rule states that the probability that a test of property φ (or a projective measurement of φ) is successful for a system in state s represented by any of the vectors $x \in s$ is given by

$$< x|Proj_{\varphi}(x) >= |Proj_{\varphi}(x)|^2,$$

where we denoted by $Proj_{\varphi}$ the projector onto the closed linear subspace $\overline{\bigcup \|\varphi\|} = \bigcup \|\varphi\|^{\perp\perp}$ generated by $\|\varphi\|$, while $< x|y >$ is as usual the inner product of vectors x and y, and $|x| = \sqrt{< x|x >}$ is the norm of vector x.

The interpretation that we give to our probabilistic modality will make use of the here-stated Born rule. In standard dirac notation, we express that the expectation value is equal or bigger than r for the test of property φ to be successful for a system in state s:

$$s \in \|P^{\geq r}\varphi\| \text{ iff } |Proj_{\varphi}(x)|^2 \geq r \text{ for all unit vectors } x \in s$$

Decidability of *QDL*. The full quantum dynamic logic QDL includes the three fragments of the logic that we have explained, starting with the logic of quantum tests and adding the epistemic features and the probabilistic features. The obtained logical system QDL has been shown to be a decidable logic [7]. It is important to note that this result shows that quantum logic has a great computational advantage over its classical first-order (and higher-order) variants, which are known to be undecidable. Decidability is an important feature for a logical system to have, as it opens up the road for implementations in the area of quantum computing. We have proven the important decidability result via a translation into the first-order logic of real numbers in [7]. We refer to this paper for more details about the general proof method we constructed to show the decidability result, which extends upon an idea employed in the decidability proof in [33].

In the next section, we illustrate how the full logic QDL can now be used to specify and verify a range of specific quantum protocols.

4. Application to Quantum Communication Protocols

As an example to illustrate the usefulness of our formalism, we look at two basic examples of such protocols, Quantum Teleportation and Quantum Secret Sharing. For our analysis of these protocols we follow the work in [3]; after that, it will be easy to see how our formal tools can be used to analyze a wide range of other protocols for quantum communication.

Quantum Teleportation. To start with the standard teleportation scenario, we are working in a 3-qubit space $\Sigma_1 \otimes \Sigma_2 \otimes \Sigma_3$ and consider a system consisting of three components hence $N = \{1, 2, 3\}$. In addition, we assume that there are two classical agents, Alice and Bob. Alice and Bob are assumed to be separated in space, and each holds one qubit of an "entangled EPR pair", i.e., a pair of qubits 2, 3 prepared in the Bell state $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ denoted in our formalism as $\beta_{00}^{2,3}$. Alice also holds, in addition to her part of the EPR pair, a qubit 1, in an unknown 1-local state. Alice now wants to "teleport" this unknown state of her qubit 1 to Bob; i.e., she will perform a program that will output a state satisfying id_{13}, the identity map. In order to do this, Alice first entangles qubit 1 with her part (qubit 2) of the EPR pair. She does this by performing a series of I-local actions: she applies a $CNOT_{1,2}$ logical gate on the two qubits followed by a Hadamard transformation H_1 on the first component. Next, Alice measures her qubits in the standard basis, thus destroying the entanglement of the EPR pair that she shared with Bob. As a result, Bob's qubit 3 will collapse to a state q_3. Moreover, the result that Alice obtains from the two measurements indicate the actions that Bob has to perform in order to transfer his qubit from its state q_3 into the same state as Alice's original qubit, i.e., the state $id_{13}(q_1)$ (which corresponds to the qubit Alice had before the protocol).

Note that it is enough for Alice to communicate to Bob the result of her measurement, i.e., send him two classical bits $(x, y) \in \{0, 1\} \times \{0, 1\}$, encoding the result x_1 of the first measurement and the result y_2 of the second measurement. This tells Bob that he will have to apply y times the X-gate followed by x times the Z gate if he wants to force his qubit q_3 into the state $id_{13}(q_1)$.

The above description of the "abstract teleportation protocol", as one finds it in most textbooks, can be encoded into our dynamic quantum logic QDL. To do so, we have to capture the passing of the first two qubits through some quantum gates, measuring them in the standard basis, then perform some correction on the third qubit, i.e., another quantum gate which depends on the result of the previous measurement outcome. Encoded as the quantum program, we obtain:

$$\pi = \bigcup_{x,y \in \{0,1\}} CNOT_{12}; H_1; (x_1 \wedge y_2)?; X_3^y; Z_3^x$$

The correctness of the teleportation-protocol specified in our logic can be stated via the following logical formula:

$$\vdash (q_1 \wedge \beta_{00}^{2,3}) \Rightarrow [\pi] q_3$$

This formula expresses the fact that after the protocol, the local state of qubit 3 is the same as the initial local state of qubit 1. Given the appropriate dynamic quantum logic, one can easily proof the correctness of this protocol, and indeed, such a proof has been provided in [3].

Quantum Secret Sharing. In [3], we provided the correctness of the protocol of quantum secret sharing. This protocol refers to a scenario in which we split quantum information into a number m of 'shares' to be distributed among m agents in such a way that the original information, i.e., the 'secret', can only be recovered when one pools together the information in all the m shares. Consider the scenario where we work with three shares. We start from a given physical system that has four components, i.e., $N = \{1, 2, 3, 4\}$, it has a single qubit 1 and three more entangled qubits prepared in a GHZ triple state $\beta_{000}^{2,3,4}$. We distribute the GSZ state over the three agents Alice, Bob, and Charles so that Alice holds qubit 2, Bob has qubit 3, and Charles has qubit 4. In addition, Alice also has the qubit 1 in the unknown state q_1. Alice wants to split the information q_1 into three shares, so she measures her two qubits 1 and 2 in the Bell basis, obtaining two classical bits x, y (corresponding to which of the four Bell states β_{12}^{xy} she obtained). After that, Bob measures his qubit 3 in the dual basis $\{|+\rangle, |-\rangle\}$, obtaining another bit $|-\rangle^z$ as result, with $z = \{0, 1\}$. Finally, Charles is given qubit 4, which is now in one of 8 possible states (depending on the results x, y, z obtained by Alice and Bob). To recover the original secret q from his qubit 4, Charles can now apply a local unitary transformation $Z_4^z; X_4^y; Z_4^x$ for which he needs to know the values x, y, and z. Hence, the three agents have to share their information in order to recover the secret q. Expressed in our language, the protocol expresses the following quantum program

$$\pi = \bigcup_{x,y\in\{0,1\}} \beta_{xy}^{12}?; |-\rangle_3^z?; Z_4^z; X_4^y; Z_4^x$$

The statement of the correctness of the protocol is the following expression in our logic DQL:

$$\vdash q_1 \wedge \beta_{000}^{234} \Rightarrow [\pi] q_4$$

The proof of correctness was given in [3], and a generalization to the case of an m-share split among m agents has been provided in [34].

Probabilistic Protocols. As our logic has the expressive power to handle probabilistic statements, it can be applied to a wider range of protocols that contain essential probabilistic elements. Examples include Grover's quantum search algorithm [35] (see also [36]) and the quantum leader election protocol [37], which have been verified using this logic in [7]. The verification of the main BB84 quantum key distribution protocol has been provided in a follow-up paper in [38].

5. Conclusions

In this paper, we provided an overview of the development of dynamic quantum logic, explaining its main syntactic constructs and quantum interpretation. We highlighted how to handle both global and local quantum information and how epistemic constructs in the language can express notions of separability and entanglement. The expressive power of this logic has been increased by adding a probabilistic operator to the language, which makes the logic fit not only to specify and verify a variety of discrete quantum protocols but also probabilistic quantum protocols. We illustrated how one can specify and verify the protocols of quantum teleportation and quantum secret sharing in the last section and gave references where the proofs of these statements can be obtained.

On the philosophical side, we mention our work in [4,39–42], highlighting the role played by logical dynamics in this framework to explain how it advances our understanding of the logical foundations of quantum mechanics. In [4,41,42], we compared the formalisms to model informational interactions in two very different areas of application: namely, in quantum theory and formal epistemology. More specifically, we zoomed in on the informational aspects of quantum measurements and compared them with other types of observation, testing, belief revision, announcements, and counterfactual conditionals in [4,42].

We finish this paper with a final note about our currently ongoing research on QDL concerning an extension of our logic to incorporate besides the flow of quantum information also the flow of classical information. In [42], we have proposed a first unified framework to model both quantum tests and classical observations by bringing together the tools from quantum dynamic logic and dynamic epistemic logic. Further work in this direction is ongoing and aims to model both the flow of information in complex quantum systems as well as public and private forms of classical communication between the agents who exchange information about their quantum resources and observations. The envisioned logical setting will be more expressive than QDL and can improve on the obtained results about the specification and verification of different multi-agent quantum protocols by expressing besides the flow of quantum information also the agents' epistemic states, their quantum and classical actions, as well as their control over the available quantum systems.

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Notes

1 We should stress here that when we refer to the notion of quantum information in this paper, we are mainly concerned with the "qualitative" ("logical", or "semantic") aspects of information, as used within logic and parts of computer science (see e.g., [16,17]). This approach to information stands in contrast with the syntactic (or quantitative) approach, which focuses on quantitative measures of information. While the quantitative aspects of information are of interest in their own right, they fall outside the focus of our paper.

2 Formally, this language is just a variant of the well-known Propositional Dynamic Logic (PDL) [27]. But our formal semantics is different, being tailored to a quantum interpretation.

3 The notion of a density operator, represented by a density matrix in quantum theory, generalizes the previous representation of pure states as vectors by making it possible to express also weighted combination of vectors, i.e., so-called 'mixed states'. We use the notion of trace as standard in mathematics for the sum of the diagonal elements of the matrix representing the density operator.

4 At first sight, this may look as an anti-monotonicity property, since inclusion between relations goes in reverse order, but as we will see, this principle corresponds to the monotonicity of information/knowledge K_I.

5 One way to do this is to simply assume as given a constant atomic sentence $\vec{0}$ denoting this state, i.e., its valuation $||\vec{0}|| = \{s_0\}$ is the singleton consisting of state s_0. Such an atomic sentence that is true by definition in only one state is called a "nominal" in modal logic. Another way is to assume as given n atomic sentences $|0\rangle_1, \ldots, |0\rangle_n$, with any global state, s satisfies $|0\rangle_i$ if in state s, the subsystem i is in the bit state $|0\rangle$.

6 This means that there exists some state w s.t. $s \sim_I w \sim_J s_0$.

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