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# FOUNDATIONS OF ITERATIVE LEARNING CONTROL

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## Abstract

Iterative Learning Control (ILC) is a technique for adaptive feed forward control of electro-mechanical plant that either performs programmed periodic behavior or rejects quasi-periodic disturbances. For example, ILC can suppress particle-beam RF-loading transients in RF cavities for acceleration. This paper, for the first time, explains the structural causes of "bad learning transients" for causal and noncausal learning in terms of their eigen-system properties. This paper underscores the fundamental importance of the linear weighted-sums of the column elements of the iteration matrix in determining convergence, and the relation to the convergence of sum of squares. This paper explains how to apply the z-transform convergence criteria to causal and noncausal learning. These criteria have an enormous advantage over the matrix formulation because the algorithm scales as $N^2$ (or smaller) versus $N^3$, where $N$ is the length of the column vector containing the time series. Finally, the paper reminds readers that there are also wave-like (soliton) solutions of the ILC equations that may occur even when all convergence criteria are satisfied.


## INTRODUCTION

Iterative Learning Control (ILC) is a method to train robots to perform repetitive tasks, or train a system to reject quasi-periodic disturbances. ILC is concerned with iterations of a trial. A trial consists of a plant operator **P** generating a time-series of values in response to an input vector **d**. The series is processed by a learning function **L**. The vector $\mathbf{e} = (\mathbf{I} - \mathbf{PL})\mathbf{d} \equiv \mathbf{Fd}$ becomes the input for the next trial, and so on. So ILC is concerned with a sequence of series, and the convergence of that sequence. If **L** delays (lowers) or advances (lifts) the data record, learning is called causal or noncausal, respectively. "Advances" serve to pre-empt the disturbance. In the limit of infinite vectors and matrices, there is an equivalent z-operator equation $e(z) = F(z)d(z)$ if **L** is causal, and a recursion if $L(z)$ is noncausal.

The ILC concept dates back to the 1980's and achieved some degree of maturity circa 2006 as outlined in the inspirational review [1], which recounts conditions for asymptotic convergence (AC) based on the eigenvalues of **F**, and monotonic convergence (MC) of the error-vector norm based on the eigenvalues of $\mathbf{S} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$. And for causal learning only, the review gives z-operator conditions for iteration-stability and monotonic convergence of the error-norm (that are identical). Thus it may be surprising to see "Foundations..." in the title of this work. However, the asymptotic (and similar geometric) convergence conditions are ineffective. For plant operating-points in the domain between the AC and MC conditions, extremely large transients may occur before ultimate convergence; so large that the plant will certainly be damaged. The review [1] acknowledges these transients, but does not explain them. Ref. [2] offers an explanation of the transients, but it is unconvincing. In subsequent decades, ever more elaborate and sophisticated (and successful) schemes have been used to avoid the learning transients. But "work arounds" are not fundamental; various authors [3-5] lament the incompleteness of ILC convergence theory.

*This work presents structural explanations of causal and non-causal learning transients*, and demonstrates why geometric convergence of their eigen-systems of **F** does not imply monotonic convergence of the error-vector norm. *This work presents z-domain MC conditions for noncausal learning*, and explains how these tests may be performed using experimental data from the plant. Further, we stress the stunning computational advantage of z-domain over eigenvalues. Illustrative examples are provided in Refs. [6-8].

### Toeplitz Matrices

Elements of Toeplitz matrices obey the rule $F_{i,j} = F_{i+1,j+1} = f_{i-j}$. Sums of these matrices are also Toeplitz. Special cases are the triangular forms: "lower" $F_{i,j} = 0$ when $j > i$, and "upper" $F_{i,j} = 0$ when $i > j$. Pure causal/noncausal learning matrices **L** are lower/upper, respectively. The product of upper and lower Toeplitz matrices is *not* Toeplitz. The response of physical, linear systems can be described by a convolution integral with the impulse response as kernel. The exact analogue of convolution for physical plant in discrete time is a lower Toeplitz matrix **P**, where the first column is the sampled impulse response. The iteration matrix **F** is (is not) Toeplitz for causal (nocausal) learning.

## MATRIX EIGEN-SYSTEMS

We abbreviate eigenvector/eigenvalue to e-vector/e-value. Let $\lambda$ and $\sigma$ be the e-values of **F** and **S**, respectively. Underlying the "mystery" of learning transients is that authors have focused on e-values, but not paid attention to e-vectors. The sum of squares iterates according to: $\mathbf{x}_{n+1}^{\mathrm{T}}\mathbf{x}_{n+1} = \mathbf{x}_n^{\mathrm{T}}\mathbf{S}\mathbf{x}_n$. Modulus of all e-values $< 1$ is a sufficient condition for monotonic convergence of the vector norm, *only if* the e-values and e-vectors are real and distinct.[1] $\lambda$ are complex. $\sigma$ are real and distinct. This is the root cause of transients for noncausal learning: from a complex vector basis and a spectrum of e-vals, it is possible to synthesize functions that initially grow and then decay. This is the analogue of the Laplace inversion integral wherein an almost arbitrary (single-sided) time function is synthesized from a spectrum of decaying exponentials. Nevertheless, the condition largest value $|\hat{\lambda}| \leq 1$ has some utility: it cuts down the domain of operating points and it's computational cost is $O(< N^2)$.

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[1] This follows from specifics of similarity transform between vector bases.

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Now is the time for a revelation: a triangular Toeplitz matrix does not have an eigen-system! The putative eigen-value equation $(\mathbf{F} - \lambda \mathbf{I})\mathbf{e} = \mathbf{0}$ has an infinite set of trivial solutions $\mathbf{e} = \mathbf{0}$ satisfied by any value of $\lambda$. All but one of the e-vectors of a triangular matrix are trivial zero vectors; therefore, the usual results for complete eigen-systems (that have a full set of non-zero e-vectors) do not apply. For example, the matrix power $\mathbf{F}^n$ resulting from $n$ iterations cannot be found in terms of e-vectors and e-values. As important as the condition $|\lambda| \leq 1$, is the region $\lambda \to 0$ which gives super-convergence for causal learning.

### *Causal Learning*

$\mathbf{F}$ is lower Toeplitz. If the z-operator $F(z)$ is known, the elements of $\mathbf{F}^n$ can be found from the inverse z-transform:

$$F^n_{i,j} = \frac{1}{2\pi\sqrt{-1}} \oint F(z)^n \frac{z^i}{z^j} \frac{dz}{z} .$$

Alternatively, working directly with the iteration equation $\mathbf{x}_{n+1} = \mathbf{F}\mathbf{x}_n$, the system is solved row-by row by the method of forward-substitution and solving a recurrence equation for each row. The number of terms required to represent the last matrix element $F^n_{N,1}$ grows exponentially with matrix dimension $N$. Explicitly for $N = 4$, the first column is:

$$\begin{array}{c} F_1^n \\ nF_1^{n-1}F_2 \\ \frac{1}{2}nF_1^{n-2}\left((n-1)F_2^2 + 2F_1F_3\right) \\ \frac{1}{6}nF_1^{n-3}\left(\left(2-3n+n^2\right)F_2^3 + 6(n-1)F_1F_2F_3 + 6F_1^2F_4\right) . \end{array}$$

Assuming the integer power $n$ is large, the largest single term within $F^n_{i,1}$ is $n^i F_1^{n-i} F_2^i/(i!)$. The competition between high powers of $n$ and the eigenvalue $F_1$ may induce apparently divergent behavior. However, the factorial in the denominator, which eventually grows faster than any single power, guarantees ultimate convergence of the series $F^n_{i,1}$ provided that $|F_1| < 1$. Thus the asymptotic behaviour depends only on $F_{i,i} = F_1$, whereas the short time-term is influenced [1] strongly by the other elements $F_j$ with $j > 1$.

## Z-OPERATORS

The (unilateral) z-transform is the discrete-time version of the Laplace transform, with $z \equiv \exp(s\tau)$ and $s, z$ complex, and $\tau$ is the sampling period. It converts an infinite time-series into a weighted sum. z-operators manipulate infinite sums, and they provide insights to the properties of very large matrices. The operators have interesting properties, some of which we write for $F(z)$ causal. (Modifications are required for the noncausal case).

**Linear sums property** This property is less well known. Let $F$ and $d$ be operator and data, respectively. Let $a$ be some particular value of $z$ larger than the circle of convergence.

$$\begin{aligned} d_1(z) &= F^1(z)d_0(z) \\ \sum_{i=0}^{\infty} d_1[i]/a^i &= F(a)d_0(a) = F(a)\sum_{i=0}^{\infty} d_0[i]/a^i \\ d_n(z) &= F^n(z)d_0(z) \\ \sum_{i=0}^{\infty} d_n[i]/a^i &= F^n(a)d_0(a) = F^n(a)\sum_{i=0}^{\infty} d_0[i]/a^i \\ \sum_{i=0}^{\infty} d_n[i](\pm 1)^i &= F^n(\pm 1)d_0(\pm 1) = F^n(\pm 1)\sum_{i=0}^{\infty} d_0[i](\pm 1)^i . \end{aligned}$$

Here $F(\ldots)$ is continuous function; and $F[\ldots]$ is discrete function. Evidently, the ratio of consecutive sums is $F(a)$; and if $|F(a)| < 1$ all of these sequences decay as $n$ increases. We may wonder what is the consequence of $|F(z)| < 1$ for all $z = \exp(i\theta)$ on the unit circle, and it is answered by Parseval's theorem:

$$\sum_{i=0}^{\infty} d_n[i]^2 = \frac{1}{\pi}\int_0^{\pi} \{F^n(e^{j\theta})F^n(e^{-j\theta})\}\{d_0(e^{j\theta})d_0(e^{-j\theta})\}d\theta.$$

So the geometric convergence of all possible weighted sums $|F(e^{i\theta})| \leq 1$ implies monotonic convergence of the sum of squares (MCSS). Notably, $F(z)$ is the transform of the first column of the causal iteration matrix. Further, if $\theta_m = m \times 2\pi/N$, where integer $m = 0, 1, 2, \ldots N$, then $F(z)$ is the Fourier series decomposition of the first column of $\mathbf{F}$; so all quantities needed for the MC test are physically accessible given the measured impulse response. Further, it should be note that the two bracketing conditions $s^{\pm} = -1 \leq F(\pm 1) \leq 1$ are trivial to compute, and serve as preconditions: if either of them fail, there is no feed for deeper analysis.

### *Noncausal Learning*

Noncausal operators are those which attempt to generate a time series that begins before the time origin ($t = 0$). The physical plant $\mathbf{P}$ is incapable of such an operation, and neither is the real-time control system that runs within an iteration. However between the iterations, the stored digital data record may be manipulated at will - which is solely the domain of the learning function $\mathbf{L}$. So a noncausal operation is made by manipulating data. This being so, the order of actions is important: manipulate the record, then let the plant operate on the data. The matrices operate in the order $\mathbf{P}.\mathbf{L}$, and do not commute. Noncausal learning is made by including time-advances (lifts) in the learning function. These lifts are instituted by an upper-Toeplitz matrix. Hence, the product $\mathbf{P}.\mathbf{L}$ is lower-Hessenberg, not Toeplitz.

The z-operators discussed thus far were commuting, but what is needed are operators where the multiplication order is important. The place to begin is with the rule for lifts on the data. The general $k$-lift operation $\mathbf{e} =\uparrow^k \mathbf{d}$ has the unilateral transform:

$$\begin{aligned} e(z) &= \mathcal{Z}\{e[i]\} = \mathcal{Z}\{[d[i+k]\} \\ &= \mathcal{Z}\{\uparrow^k d[i]\} = z^k \left[ d(z) - \sum_{j=0}^{k-1} \frac{d[j]}{z^j} \right] . \end{aligned}$$

### *M-Term Learning With a Lift Power Series*

Suppose the learning operator is $\mathbf{L} = \sum_{p=0}^{M} \alpha_p \Uparrow^p$, and iterants are related by $\mathbf{x}_{n+1} = \left[\mathbf{I} - \mathbf{P}\sum_{p=0}^{M} \alpha_p \Uparrow^p\right] \mathbf{x}_n$.

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The corresponding z-domain iteration is:

$$d_{n+1}(z) = F_M(z)d_n(z) + P(z)\sum_{p=0}^{M}\alpha_p z^p \sum_{k=0}^{p-1}\frac{d_n[k]}{z^k} \quad (1)$$

with $F_M(z) = \left[1 - P(z)\sum_{p=0}^{M}\alpha_p z^p\right]$. Starting with $n = 0$, let us write the effect of two iterations:

$$\begin{aligned} d_2(z) &= F_M(z)^2 d_0(z) + P(z)\sum_{q=0}^{M}\alpha_q z^q \sum_{k=0}^{q-1} z^{-k} d_1[k] \\ &+ F(z)_M P(z)\sum_{p=0}^{M}\alpha_p z^p \sum_{k=0}^{p-1} z^{-k} d_0[k]. \end{aligned}$$

The first term, in $F^2$, is the same as for causal learning. The second and third terms in $P$ and $FP$, respectively, are the cumulant effect of data loss. Fortunately, we do not need to consider further iterations. All the information required to construct a convergence test is contained in the single iteration Eq. (1). As a general principle, the iterations do not convergence unless the sequence initiated by any single data impulse alone converges. The data impulse $\delta(t - j\tau)$ corresponds to the sum $d(z) = d_n[j]/z^j$ and collateral $d_n[k] \rightarrow d_n[k]\delta_{k,j}$. Performing the summation leads to

$$\begin{aligned} &d_{n+1}(z, j) = F_M(z)\frac{d_n[j]}{z^j} \\ &+P(z)\sum_{p=0}^{M}\alpha_p z^p \left\{ \begin{array}{c} z^{-j} d_n[j] \ \text{ if } \ j \geq 0 \ \& \ p - j \geq 1 \\ 0 \ \text{ otherwise} \end{array} \right\}. \end{aligned}$$

From this equation we may either (i) find the elements $F_{i,j}$ of column $j$ of matrix $\mathbf{F}$ by performing the inverse z-transform; or (ii) investigate the recursion as a function of $j$ and $p$; we do the latter. For example, when $M = 0$ (i.e. no lift) then $d_{n+1} = F_0(z)d_n[j]/z^j$ for all $j$; in which case every sequence converges if $|F_0(z)| \leq 1$.

For example, when $M = 1$ then $d_{n+1} = F_0(z)d_n[0]$ if $j = 0$, and $d_{n+1} = F_1(z)d_n[j]/z^j$ if $j > 0$. Hence there are two simultaneous conditions for MC: $|F_0(z)| \leq 1$ and $|F_1(z)| \leq 1$ for all $z = e^{i\theta}$. And the equipment operating point must satisfy them both!

Similarly, for $M = 2$ there are three MC conditions for all $z = e^{i\theta}$: $|F_0(z)| \leq 1$ for $j = 0$ and $|F_1(z)| \leq 1$ for $j = 1$ and $F_2(z)| \leq 1$ for $j \geq 2$. The contraction to causal learning for $j < M$ is typical, and has the following interpretation and implication when $M = 2$. On the first iteration, matrix columns #1,2 behave like a causal operation; and the remainder behave according to the double lift $\uparrow^2$. On the second iteration, columns #1,2,3,4 behave causally; and the remainder behave like $\uparrow^2$. The effect slowly sweeps across the matrix; until after $N/M$ iterations the entire matrix operator behaves as $\mathbf{F} = \mathbf{I} - \mathbf{P}$. Thus the character of the matrix changes as the iterations progress. For general case, ILC system must satisfy $M$ simultaneous z-domain monotonic convergence conditions: $|F_p(z)| \leq 1$ for $p = 0, 1, 2 \ldots M$.

### *Two Convergence Test Paradigms*

The starting point is the measured impulse response of the physical plant for a particular operating point of the equipment. The Zero-Order-Hold effect of the sampling has to be compensated by a lift. From this data, we may construct the matrix operator $\mathbf{P}$ or samples of the z-operator $P(z = \exp[i\theta_n])$ in order to perform the MC test. Let $0 < \mu < 1$ be an adjustable scalar gain.

Matrix Operators

Construct $\mathbf{P}$ and $\mathbf{F} = \mathbf{I} - \mu\mathbf{PL}$. Construct $\mathbf{S} = \mathbf{F}^T\mathbf{F}$. Find the largest eigenvalue $\sigma$ of $\mathbf{S}$. If $\sigma > 1$, ILC is unstable. Consider to repeat with lower learning gain $\mu$. Making $\mathbf{P}$ takes Order($N$) operations, and finding the eigenvalue takes Order($N^3$) operations.

Z-operators

Let $i = \sqrt{-1}$. Construct $P(z = \exp[i\theta_n]) = p + iq$ from the data. $p = \mathrm{Re}[P]$ and $q = \mathrm{Im}[P]$. Construct $L(z = \exp[i\theta_n]) = a + ib$ from the analytic expression for the learning scheme. $a = \mathrm{Re}[L]$ and $b = \mathrm{Im}[L]$. Construct $F(z) = 1 - \mu P(z)L(z)$. For all values of $\theta_m$ evaluate $S(\theta_m)$

$$= |F(z)F(z^*)| = 1 + 2\mu(b.q - a.p) + \mu^2(a^2 + b^2)(p^2 + q^2) .$$

If $S(\theta_m) > 1$, ILC is unstable. Consider to repeat with lower learning gain $\mu$. Making $N$ values of $P(\theta_n)$ takes Order($N^2$) operations, and performing the test takes Order($N$) operations. If $L(z)$ is noncausal, then the entire procedure has to be repeated for each lowered learning function until the residual $L(z)$ is causal.

The z-operator offers the advantage over matrix operators of $N^2$ versus $N^3$ computational steps. In either case, it is important to think at the outset think about an appropriate sampling period and matrix size. There is a huge cost to the stability analysis of choosing more samples than is necessary.

## SOLITONS

At the outset, the eigen-system analysis of ILC presumes that the iteration index $n$ and within-trial sample-time index $k$ are the arguments of separate functions; and this implicitly excludes wave-like solutions $W(n - ck)$ where $c$ is the wave speed. Given that the ILC gain parameters are tuned for decay, ordinary waves are excluded; but not wave-packets with high-frequency carriers. To be clear, these disturbances do not appear to travel within a single trial; it is only when they are plotted in the 2-dimensional space $(n, k)$ that their motion becomes manifest. They satisfy the usual definition of a soliton wave: a self-reinforcing wave packet that maintains its (unique) shape while it propagates at constant speed; and they persist long after all disturbances should have decayed practically to zero. The shape and group velocity must be found [6] self-consistently. The presence of the high-frequency carrier implies they probably can be eliminated by pre-pending a low pass filter $\mathbf{Q}$ to the iteration matrix: $(\mathbf{I} - \mathbf{PL}) \rightarrow \mathbf{Q}(\mathbf{I} - \mathbf{PL})$, but at the cost of displacing the fixed point of the mapping from zero - leading to residual error. (In graphic terms, the robot arm losses its tremor but misses the target.)

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